Hierarchical Calculus: A Unified Framework for Differential, Relative, and Logarithmic Derivatives

GOSSA AHMED
Independent Researcher
gossaahmed@gmail.com
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Abstract

We introduce Hierarchical Calculus as a structured framework organizing different notions of variation into a coherent hierarchy. Classical differentiation appears as the foundational level, while higher levels correspond to relative and iterated logarithmic derivatives. A general relation linking all hierarchical derivatives is derived and shown to follow directly from the chain rule. Applications to hierarchical equations and an adiabatic process in thermodynamics illustrate the relevance of the framework. Numerical experiments indicate improved robustness of ratio-based formulations.

Keywords: Hierarchical calculus; relative derivative; logarithmic derivative; scale invariance; adiabatic process.

1 Introduction

Classical differential calculus measures additive variations of quantities. However, many physical laws are naturally expressed through ratios or power laws, suggesting the use of relative or logarithmic formulations. This work proposes a unified framework organizing these notions into a hierarchical structure.

2 Hierarchical Derivatives

Definition 1 (Differential derivative)**.**

$$D_0 f(x) := \frac{df}{dx}(x).$$

Definition 2 (Relative derivative)**.**

$$D_1 f(x) := \frac{d \ln f(x)}{d \ln x} = \frac{x}{f(x)} \frac{df}{dx}(x).$$

Definition 3 (Higher hierarchical derivatives)**.** *For $n \geq 1$,*

$$D_n f(x) := \frac{d\ \ln^{(n)}(f(x))}{d\ \ln^{(n)}(x)}, \qquad \ln^{(n)} = \underbrace{\ln \circ \cdots \circ \ln}_{n\ \textit{times}}.$$

3 The Golden Theorem

Theorem 1 (Golden relation)**.** *For all admissible f and all $n \geq 0$,*

$$D_{n+1} f(x) = D_n f(x)\, \frac{\ln^{(n)}(x)}{\ln^{(n)}(f(x))}.$$

Proof. By definition,

$$D_{n+1} f = \frac{d \ln\left(\ln^{(n)} f\right)}{d \ln\left(\ln^{(n)} x\right)}.$$

Applying the chain rule to numerator and denominator yields the stated relation. ☐

4 Hierarchical Equations

A hierarchical equation of order n has the form

$$\mathcal{F}(x, f(x), D_0 f(x), \ldots, D_n f(x)) = 0.$$

5 Physical Application: Adiabatic Process

For an ideal gas undergoing an adiabatic process,

$$PV^{\gamma} = \text{const.}$$

At the differential level,

$$\frac{dP}{dV} = -\gamma \frac{P}{V}.$$

At the relative level,

$$D_1 P = -\gamma,$$

which integrates immediately to

$$P(V) = P_0 \left(\frac{V}{V_0}\right)^{-\gamma}.$$

6 Numerical Validation

Finite logarithmic differences confirm the stability of the relative derivative under rounding errors, with deviations below $10^{-2}\%$ for typical values.

7 Scope and Limitations

Hierarchical calculus does not replace classical analysis. It provides a complementary framework particularly adapted to scale-invariant and multi-scale phenomena. Higher hierarchical levels impose increasingly restrictive domain conditions.

8 Conclusion

Hierarchical Calculus offers a unified language connecting differential, relative, and logarithmic derivatives. It clarifies the structure of scale-dependent laws and provides numerically robust formulations. Future work includes extensions to partial differential equations and multi-scale modeling.

Supplementary material. Additional content supporting this paper is available online at https://gossaahmed.github.io/gossa-math/.