Design and Analysis of Algorithms

L30: Knapsack problem

Dynamic Programming

Dr. Ram P Rustagi
Sem IV (2020-Even)
Dept of CSE, KSIT
rprustagi@ksit.edu.in

Resources

- Text book 1: Levitin
  - Sec **8 . 4**
- Text book 2: Horowitz
  - Sec 5 . 7

- R1: Introduction to Algorithms
    - Cormen et al.

Knapsack problem

- Knapsack problem:
  - Given n items of known weights $w_1, \ldots, w_n$, and
  - Having values $v_1, \ldots, v_n$ and a capacity W
  - Find the most valuable subset of items that fit into the knapsack.
    - Items are to be considered in full, not partially
  - Note: All the weights w_i's and knapsack capacity W are integers, but values can be real numbers.
- Goal: solve the knapsack problem using dynamic programming.

DP Approach: Knapsack

- To solve knapsack problem using DP,
  - Need to design a recurrence relation
    - Express a solution to an instance in terms of smaller instances.
- Consider an instance defined by first i items with
  - Weights $w_1, w_2, \ldots, w_i; \ 1 \le i \le n$
  - Values $v_1, v_2, \ldots, v_i; \ 1 \le i \le n$
  - Knapsack capacity $j, \ 1 \le j \le w$
- Let $V[i, j]$ be the optimal solution to this instance
  - i.e. the value of most valuable subsets of first i items that fit knapsack of capacity j.
- Approach: divide first i items into two categories:
  - Those that include i^{th} item, and
  - Those that don't include i^{th} item

DP Approach: Knapsack

	0		j-w_i		j		W
0	0	0	0	0	0	0	0
	0						
i-1	0		**V[i-1,j-w_i]**		**V[i-1,j]**		
i	0				**V[i,j]**		
	0						
n	0						

Table for solving knapsack problem using dynamic programming

- Category 1: subsets that do not include i^{th} item.
  - Value of optimal subset is V[i-1,j]
- Category 2: subsets that do include i^{th} item.
  - Thus $j>w_i$ i.e. $j-w_i \geq 0$.
  - Value of optimal subset is v_i+V[i-1,j-w_i]

DP Approach: Knapsack

- Possible cases:
  - $j < w_i$ (i.e. $j - w_i < 0$), i.e. weight of i^{th} item is more than j and thus can't be included
  - $j \geq w_i$ (i.e $j - w_i \geq 0$) weight of i^{th} item is less than or equal to j, and thus i^{th} item may included or execluded.
- Thus,

$$V[i,j] = \begin{cases} max\{V[i-1,j], v_i + V[i-1,\ j-w_i]\} & if\ j - w_i \geq 0 \\ V[i-1,\ j] & if\ j - w_i < 0 \end{cases} \quad (1)$$

- The initial conditions can be defined as

  $V[i,0]=0$ for $i \geq 0$

  i.e. i items and 0 capacity

  $V[0,j]=0$ for $j \geq 0$

  i.e. 0 items and j capacity

Example: Knapsack

$$V[i,j] = \begin{cases} max\{V[i-1,j], v_i + V[i-1,\ j-w_i]\} & if\ j - w_i \geq 0 \\ V[i-1,\ j] & if\ j - w_i < 0 \end{cases} \quad (1)$$

- Example: consider knapsack of size 5
  - (i.e. max weight it can hold is 5),
  - Weights as
    $w_1=2, w_2=1, w_3=3, w_4=2$
  - Values as
    $v_1=\$12, v_2=\$10, v_3=\$20, v_4=\$15$
  - Need to compute `V[4,5]`
    - Max value with 4 items with knapsack capacity 5

Example Knapsack

Capacity→ wts, values↓		0	1	2	3	4	5
	0	**0**	**0**	**0**	**0**	**0**	**0**
$w_1=2$ $v_1=12$	1	**0**					
$w_2=1$ $v_2=10$	2	**0**					
$w_3=3$ $v_3=20$	3	**0**					
$w_4=2$ $v_4=15$	4	**0**					

$V[0,j]=0$ **for** $0 \le j \le 5$
$V[i,0]=0$ **for** $0 \le i \le 4$

Capacity→ wts, values↓		0	1	2	3	4	5
	0	**0**	**0**	**0**	**0**	**0**	**0**
$w_1=2$ $v_1=12$	1	**0**	**0**	**12**	**12**	**12**	**12**
$w_2=1$ $v_2=10$	2	**0**					
$w_3=3$ $v_3=20$	3	**0**					
$w_4=2$ $v_4=15$	4	**0**					

$V[1,1]=V[1-1,1]$ **since** $j=1<w_1=2$

$=0$

$V[1,2]=\max\{V[0,2],12+V[0,2-2]\};j=2\geq w_1=2$

$=\max\{0,12+V[0,0]\} = 12$

$V[1,3]=\max\{V[0,3],12+V[0,3-2]\};j=3\geq w_1=2$

$=\max\{0,12+V[0,1]\}=12$

$V[1,4]=\max\{V[0,4],12+V[0,4-2]\};j=4\geq w_1=2$

$=\max\{0,12+V[0,2]\}=12$

$V[1,5]=\max\{V[0,5],12+V[0,5-2]\};j=5\geq w_1=2$

$=\max\{0, 12+V[0,3]\}=12$

Capacity→ wts, values↓		0	1	2	3	4	5
	0	**0**	**0**	**0**	**0**	**0**	**0**
$w_1=2$ $v_1=12$	1	**0**	**0**	**12**	**12**	**12**	**12**
$w_2=1$ $v_2=10$	2	**0**	**10**	**12**	**22**	**22**	**22**
$w_3=3$ $v_3=20$	3	**0**					
$w_4=2$ $v_4=15$	4	**0**					

$V[2,1]=\max\{V[1,1],10+V[1,1-1]\};\ j=1\geq w_2=1$
$=\max\{0,\ 10+V[1,0]\} = 10$
$V[2,2]=\max\{V[1,2],\ 10+V[1,2-1]\};\ j=2\geq w_2=1$
$=\max\{12,\ 10+0\} = 12$
$V[2,3]=\max\{V[1,3],\ 10+V[1,3-1]\};\ j=3\geq w_2=1$
$=\max\{12,\ 10+V[1,2]\}=\max\{12,22\}=22$
$V[2,4]=\max\{V[1,4],\ 10+V[1,4-1]\};\ j=4\geq w_2=1$
$=\max\{12,\ 10+V[1,3]\}=\max\{12,22\}=22$
$V[2,5]=\max\{V[1,5],\ 10+V[1,5-1]\};\ j=5\geq w_2=1$
$=\max\{12,\ 10+V[1,4]\}=\max\{12,22\}=22$

Capacity→ wts, values↓		0	1	2	3	4	5
	0	**0**	**0**	**0**	**0**	**0**	**0**
$w_1=2$ $v_1=12$	1	**0**	**0**	**12**	**12**	**12**	**12**
$w_2=1$ $v_2=10$	2	**0**	**10**	**12**	**22**	**22**	**22**
$w_3=3$ $v_3=20$	3	**0**	**10**	**12**	**22**	**30**	**32**
$w_4=2$ $v_4=15$	4	**0**					

$V[3,1]=V[2,1] = 10;$ $(j=1<w_3=3)$

$V[3,2]=V[2,2] = 12;$ $(j=2<w_3=3)$

$V[3,3]=\max\{V[2,3],\ 20+V[2,3-3]\};$ $(j=3\geq w_3=3)$

$\quad =\max\{22,\ 20+0\} = 22$

$V[3,4]=\max\{V[2,4],\ 20+V[2,4-3]\};$ $(j=4\geq w_3=3)$

$\quad =\max\{22,\ 20+V[2,1]\}=\max\{22,30\}=30$

$V[3,5]=\max\{V[2,5],\ 20+V[2,5-3]\};$ $(j=5\geq w_3=3)$

$\quad =\max\{12,\ 20+V[2,2]\}=\max\{12,20+12\}=32$

Capacity→ wts, values↓		0	1	2	3	4	5
	0	**0**	**0**	**0**	**0**	**0**	**0**
$w_1=2$ $v_1=12$	1	**0**	**0**	**12**	**12**	**12**	**12**
$w_2=1$ $v_2=10$	2	**0**	**10**	**12**	**22**	**22**	**22**
$w_3=3$ $v_3=20$	3	**0**	**10**	**12**	**22**	**30**	**32**
$w_4=2$ $v_4=15$	4	**0**	**10**	**15**	**25**	**30**	**37**

$V[4,1]=V[3,1] = 10;$ $(j=1<w_4=2)$

$V[4,2]=\max\{V[3,2],15+V[3,2-2]\};$ $(j=2\geq w_4=2)$

$=\max\{12,15+V[3,0]\}=\max\{12,15+0\}=15$

$V[4,3]=\max\{V[3,3],\ 15+V[3,3-2]\};$ $(j=3\geq w_4=2)$

$=\max\{22,15+V[3,1]\}=\max\{22,15+10\}=25$

$V[4,4]=\max\{V[3,4],\ 15+V[3,4-2]\};$ $(j=4\geq w_4=2)$

$=\max\{30,\ 15+V[3,2]\}=\max\{30,15+12\}=30$

$V[4,5]=\max\{V[3,5],\ 15+V[3,5-2]\};$ $(j=5\geq w_4=2)$

$=\max\{32,\ 15+V[3,3]\}=\max\{32,15+22\}=37$

Example Knapsack: Optimal Subset

Capacity→ wts, values↓		0	1	2	3	4	5
	0	0	0	0	0	0	0
$w_1=2$ $v_1=12$	1	0	0	12	12	12	12
$w_2=1$ $v_2=10$	2	0	10	12	22	22	22
$w_3=3$ $v_3=20$	3	0	10	12	22	30	32
$w_4=2$ $v_4=15$	4	0	10	15	25	30	37

- Optimal subset
  - Backtrack from maximal value V[4,5] to prev. rows.
  - Thus, optimal subsets are

V[4,5]=37 (≠V[3,5]) implies $\underline{w_4=2}$ is included

V[3,3]=22 (=V[2,3]) implies $w_3=3$ is not included

V[2,3]=22 (≠V[1,3] implies $\underline{w_2=1}$ is included

V[1,2]=12 (≠V[0,2] implies $w_1=2$ is included

Algorithm: Knapsack using DP

```
Algo DPKnapsack(w[1..n], v[1..n], W)
  int V[0..n,0..W],  P[1..n,1..W];
  for j=0 to W do
      V[0,j]=0
  for i=0 to n do
     V[i,0]= 0
  for i=1 to n do
     for j= 1 to W do
       if w[i]≤j and (v[i]+V[i-1,j-w[i]]) > V[i-1,j] then
         V[i,j]= v[i] + V[i-1,j-w[i]];
      else
         V[i,j]= V[i-1,j]
  return V[n,W] (and the optimal subset by backtracing)
```

Efficiency of Knapsack

- Time Efficiency: Θ(nW)
- Space efficiency: Θ(nW)

Summary

- Knapsack algorithm using dynamic programming
- Efficiency
- Optimal subsets using backtracking