# Design and Analysis of Algorithms

# L04: Complexity Analysis

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# Resources

- Text Book 1: Levitin
- Text Book 2: Horowitz, Sahani, Rajsekharan

# Amortized Cost

- Amortized cost:
  - An accounting artifact that often may not have any direct relationship to actual complexity of the operation.
- Requirement for amortized cost
  - *Sum of amortized complexities of all operations in any sequence of operations be greater than or equal to sum of the actual complexities, i.e*

$$\sum_{1 \le i \le n} amortized(i) \ge \sum_{1 \le i \le n} actual(i) \qquad (1)$$

- Amortized cost can be viewed as the amount being charged for the operation rather than actual cost of the operation.

# Examples: Amortized Cost

- Amortized cost of eating out (per day)
  - Consider that you eat out dinner every day and actual cost of dinner on daily basis is typically as follows.
    - Mon-Thu: Rs 50
    - Fri: Rs Rs 100
    - Sat-Sun: Rs 150
  - Consider that you need to have the needed money in the pocket before you proceed for dinner. The restaurant requires full payment in advance. You have fixed source of income per day (e.g. you can get only X amount per day from the source e.g. parents?).
  - What should be the value of X?
    - Rs 80?, Rs 100?
    - X is considered as amortized cost (potential func)

# Amortized Cost : Potential Function

- To arrive at amortized cost relative to actual cost, define a Potential function `P(i)` for the `i`th operation

  `P(i)=amortized(i)-actual(i)+P(i-1)`.....(2)

- `i`th operation causes the potential function to change by the difference between the amortized and actual cost

$$\sum_{1 \le i \le n} P(i) = \sum_{1 \le i \le n} (amortized(i) - actual(i) + P(i-1))$$

$$\Rightarrow \sum_{1 \le i \le n} (P(i) - P(i-1)) = \sum_{1 \le i \le n} (amortized(i) - actual(i))$$

$$\Rightarrow P_n - P0 = \sum_{1 \le i \le n} (amortized(i) - actual(i))$$

$$\Rightarrow P_n - P_0 \ge 0 \qquad (3)$$

# Amortized Cost : Potential Function

$$P(n) - P(0) \geq 0 \qquad (3)$$

- Potential function:
  - Assuming $P_0 \geq 0$, the potential `P(i)` represents the amount by which the first `i` operations have been over charged.
- Methods for computing amortized costs
  - Aggregate method
  - Accounting method
  - Potential method

# Computing Amortized Costs

- Aggregate method:
  - Determine an upper bound for the sum of `n` operations.
  - Divide the amount by `n` to get amortized cost
  - Example: eating out dinner weekly limit of Rs `1000`
    - Amortized cost per day = `1000/7`= Rs.`142.85`...
- Accouting method:
  - Guess an amount, show that P(i) satisfies eqn 2 & 3 i.e. `P(i)=amortized(i)-actual(i)+P(i-1)`, and `P(n) - P(0) ≥ 0`
- Potential method:
  - Guess a potential function that satisfied eqn (3), and
  - Compute amortized cost using eqn (2)

# Computing Amortized Costs

- Aggregate method appears to be most comfortable
  - Hardest to use because
  - How to get upper bound on worst case behaviour
- Accouting  method is intuitive to use
  - Just need to verify eqn (2) and (3)
  - Often provides a tight bound on complexity of sequence of operations
- Potential method is again hardest to use
  - Guessing a proper potential function is difficult
  - For some applications, this is the only way

# Example: Subset Generation

- Problem: Given a set of `n` elements, generate all of its subsets i.e.
  - subset of set of `n` elements is defined by $2^n$ vectors `x[1:n]`, where each `x[i]` is either `0` or `1`.
  - `x[i]` is `1`, iff $i^{th}$ element of the set is member of the subset.
  - Subsets of a set of 3 elements given by 8 vectors
    - `000,100,010,110,001,101,011,111`
- Enumeration strategy:
  - Start with subset `00...0`, and generate remaining subsets one at a time by invoking a fn *NextSubset*.
  - Scans current subset from left to right, change every `1` to `0` till it encounters `0` which is changed to `1`, and return.
  - Stops when no `0` is encounted.

# Subset Generation: Time Complexity

- Problem: Determine time complexity of generating first `m` subsets (excluding first subset `00...0`).
- Worst case method:
  - In any invocation of *NextSubset* function, max `n` bits can be changed. Thus, worst case cost is `n`. Since there are m subsets, the total aggregate cost is `m*n`.
- Aggregate method:
  - When function *NextSubset* is invoked `m` times, the vector `x[n]` changes `m` times, `x[n-1]` changes `m/2` times, `x[n-2]` changed $m/2^2$ times, `x[n-i]` changes $m/2^i$ times. Thus, the total cost is

$$\sum_{0 \le i \le \lfloor log_2 m \rfloor} \left\lfloor \frac{m}{2^i} \right\rfloor < 2m$$

# Subset Generation: Accounting Method

- Accounting method:
  - First guess the amortized complexity and then verify it
  - Intuitively, we think on the average 2 vectors will change
  - Let us guess the amortized cost as 2.
    - To show that $P(n) - P(0) \geq 0$
  - Approach: use the credit method and distributed the excess charge when needed.
  - Initially, each `x[i]` is zero and a credit of `0`.
  - On first invocation of *NextSubset* `1` cost is used for changing `x[n]`, `1` cost goes to credit on changing vector `x[n]`.
  - On 2nd invocation of *NextSubset*, the credit of `x[n]` is used to change `1` to `0`, `1` cost (from amortized cost of 2) is used to change `x[n-1]`, `1` credit goes to `x[n-1]`

# Subset Generation: Accounting Method

- Accounting method…
  - On 3rd invocation of *NextSubset*, the credit of `x[n]` is changed to `1` from `0` using `1` cost (from amortized cost of 2), `1` credit is assigned to `x[n]`, `1` previous credit remains with `x[n-1]`
  - On 4`th` invocation, `1` credit of `x[n]` is used to change it to `0`, `1` credit of `x[n-1]` is used to change it `0`, `1` credit from amortized cost (of 2) is used to change `x[n-2]` and `1` credit is assigned to `x[n-2]`
  - Continuing this way, each `x[i]` that is `1`, has a credit of `1` unit with it. This credit is used to pay for changing it from `1` to `0`. From amortized cost of `2`, cost of `1` unit is used to change last `x[i]` and `1` credit is assigned to `x[i]`
  - The credit on each `x[i]` which is `0` is zero.

# Subset Generation: Accounting Method

- Accounting method…
  - Thus, on any $j^{th}$ invocation, $P(j)-P(0)$ equals number of `x[i]`s that are `1`. This number is always non-negative.

$$P(i)=amortized(i)-actual(i)+P(i-1) \quad \ldots\ldots (2)$$

  - Thus equation $P(m)-P(0) \geq 0$ holds true for all `m`.

$$P_n - P_0 \geq 0 \quad \ldots\ldots\ldots\ldots\ldots (3)$$

  - Thus, the complexity of Accounting method is $2m$, since amortized cost is taken as $2$ units.

# Subset Generation: Potential Method

- Approach for Potential method:
  - Guess (postulate) a potential function that satisfies equation `P(n)-P(0)≥0`, and then obtain amortized costs.
  - Define `P(i)` as equal to number of `x[i]`'s that are `1` in `i`$^{th}$ subset.
  - By definition, then P(0) = 0
  - There for `P(i)-P(0)` is always `≥0`
  - Computing amortized cost
  - Consider any subset `x[1:n]`. Let q be the number of `1`s at the left end of `x[]`.
    - On next invocation potential changes by `1-q`, since q vectors change from `1` to `0`, and one `0` is replaced by `1`.
    - Actual cost is `q+1`, since `q+1` vectors are changed.

# Subset Generation: Potential Method

- Thus, the amortized cost is given by (eqn (2))

```
amortized(i) = actual(i)+P(i)-P(i-1)
             = actual(i) + change in potential
             = q+1+1-q
             = 2
```

# Summary

- Amortised costs
  - Aggregate method
  - Accounting method
  - Potential function method