# Design and Analysis of Algorithms

# L24: Heapsort

## Transform and Conquer Approach


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# Resources

- Text book 1: Sec 9.1-5.4 - Levitin
- R1: Introduction to Algorithms
    - Cormen et al.

# Transform and Conquer

- Secret to life: Replace one worry with another.
  - American cartoonist Charles M Shultz (1922-2000)
- Transform and conquer approach
  - A two stage process
    - Transformation stage: change the problem instance to another form, more amenable to solution
    - Conquering stage: Solve the problem
- Transformation can be done in 3 ways
  - Instance simplification: to a simpler or more convenient instance of the problem: presorted lists
  - Different representation: Heaps, Horner's rule
  - Problem reduction: transform to a different problem for which solution is available.

# Priority Queue

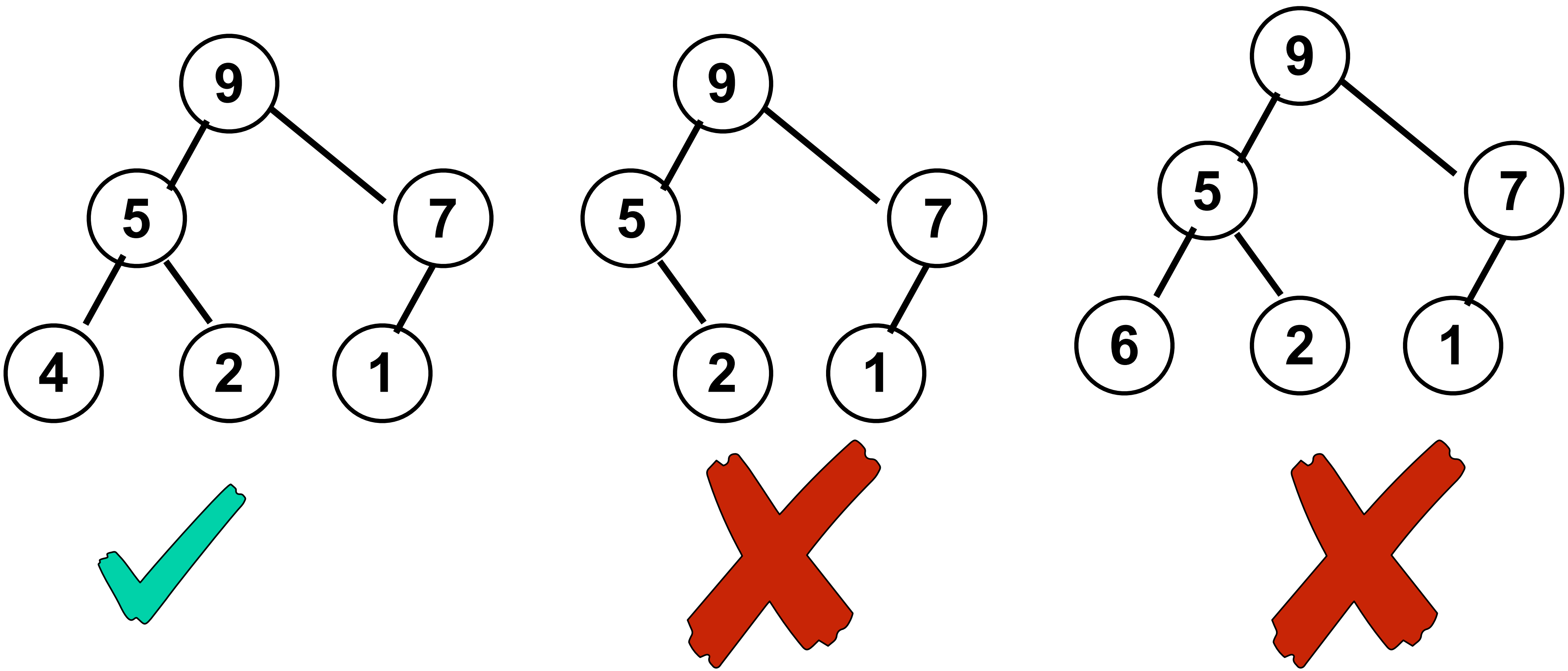
- Priority Queue:
  - A data structure with an orderable (called priority) characteristic on set of elements maintained by it
  - Allows 3 operations in an efficient way
    - FindMin (or even FindMax):
      - Find an item with highest priority (e.g. max, min)
  - DeleteMin:
      - Delete an item with highest priority
    - Insert:
      - Add a new item to the data structure
- Heaps makes these 3 operations interesting and useful
- Heapsort: a cornerstone of theoretical sorting problem

# Heap

- Definition:
  – Heap is defined as binary tree with keys assigned to nodes (one key per node) with following conditions
    - Binary tree is a a complete tree except possibly at the last level
      – Few rightmost leaves may be missing
    - The key of a parent is greater than or equal to keys of its children and hence descendants
      – Also, known as parental dominance.

# Examples

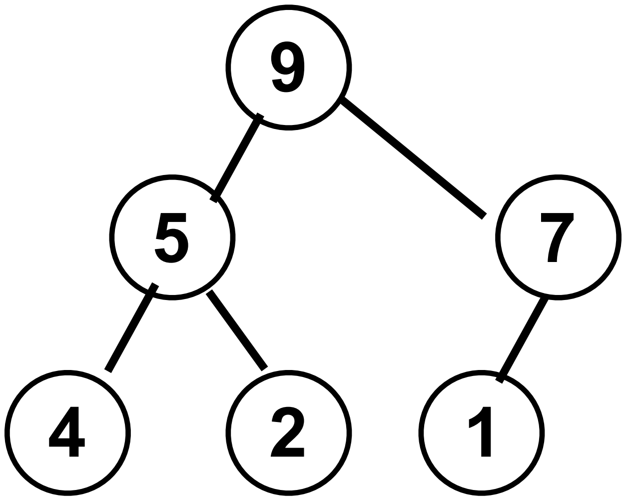
Q: Identify if given binary tree is a heap?



# Heap Properties

- There exists only `1` complete binary tree with `n` nodes.
- The root of the heap is always the largest element
- A node of heap taken together with all its descendants is also a heap
- Heap implementation
  - Can be an array `H[]` with top-down and left to right
  - Store heap elements in positions thru `1` to `n`.
  - Element `H[0]` can either be unused or a sentinel
    - Its value can be greater than every element of heap
  - Parental nodes are in first $\lfloor n/2 \rfloor$ positions of the array
  - Leaf nodes will be last $\lceil n/2 \rceil$ positions of the array

# Example: Heap Implementation




- Left child of node at `j` is at `2j`
- Right child (if exists) of node at `j` is at `2j+1`
- Parent of node at `j` is at $\lfloor$`j/2`$\rfloor$
- Parental nodes are in first $\lfloor$`n/2`$\rfloor$ positions of the array
- Leaf nodes are in last $\lceil$`n/2`$\rceil$ positions of the array

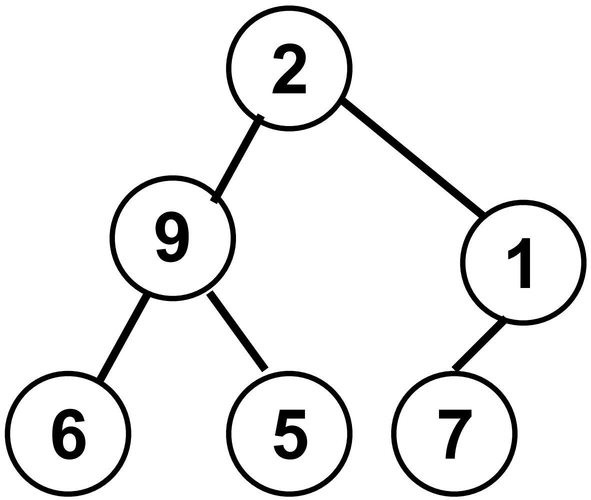
| 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| | 9 | 5 | 7 | 4 | 2 | 1 |

# Heap Construction

- S0: Initiaize heap structure with keys in order
- S1: Start with the last (right most) parental node
  - Fix the heap rooted at it.
  - If it fails the heap condition, then exchange with larger child
  - Repeat the process till heap condition satisfies
- S2: Repeat the previous step (s1) for preceding parental nodes.

# Heap Construction

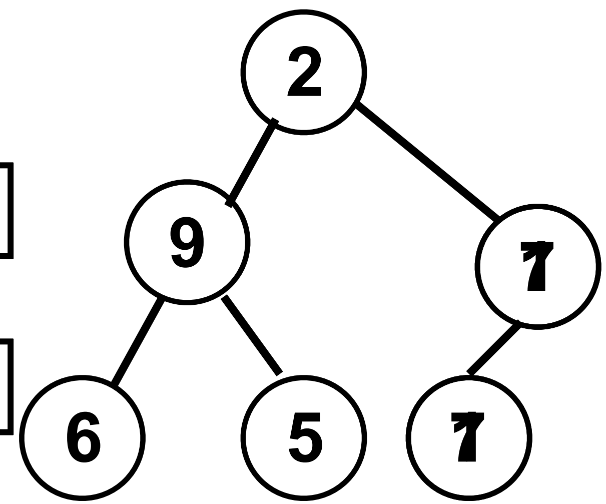
- Consider the data: `2,9,1,6,5,7`
- Construct the heap in order.




- Let us heapify
- Last parental node (at $\lfloor 6/2 \rfloor = 3$ is 1
  - Smaller than child node 7 at pos 6
  - Exchange it
  - Heap property satisfies

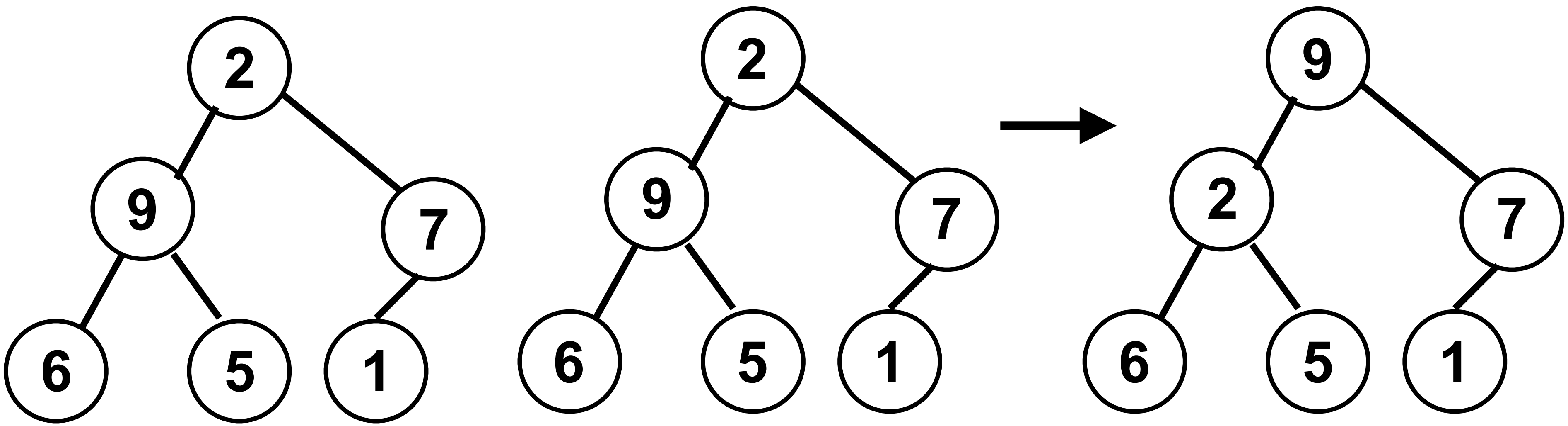
| | | | | | | |
|---|---|---|---|---|---|---|
| - | 2 | 9 | 1 | 6 | 5 | 7 |

| | | | | | | |
|---|---|---|---|---|---|---|
| - | 2 | 9 | 7 | 6 | 5 | 1 |



# Heap Construction

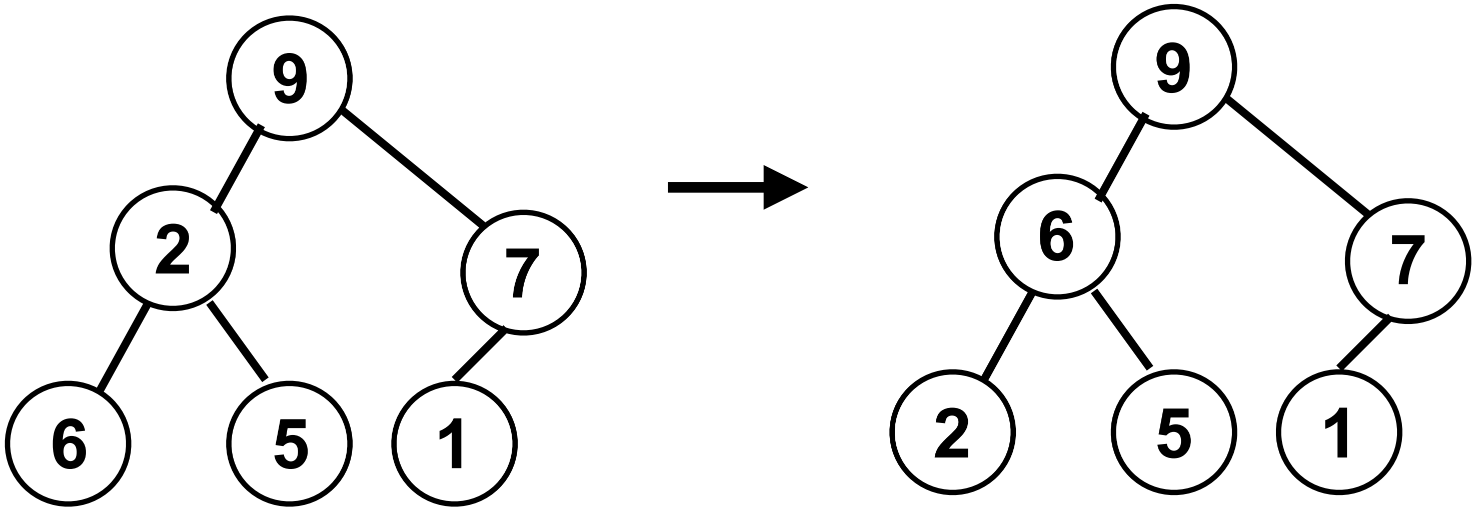
- Consider preceding parental node 9 (at pos `3-1=2`)
- It is in order. No exchange required
- Next parental node 2 (at pos `1`). Needs heapification.
- Exchange it with 9, and repeat the process




keys: `2,9,7,6,5,1`

# Heap Construction

- Since exchanged node 2 is not in heap order
- This needs to be exchanged with 6.

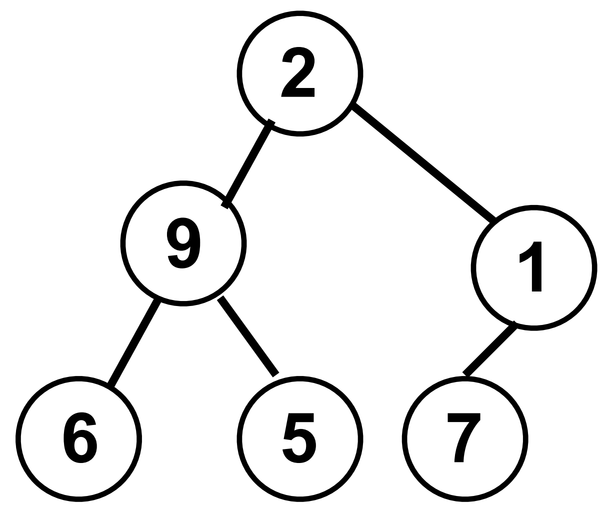



- Now heap is in order.

# Heap Algorithm

- Algo `HeapBottomUp(H[1:n])`

```
// i/p: an array H[1:n] of items to be ordered
// o/p: Heap H[1:n] of ordered items
for i ← ⌊n/2⌋ to 1 do
   k←i; v←H[k]; heap←False
   while not heap and 2*k≤n do
      j ←2*k
      if j<n // there are two children
         if H[j] < H[j+1]
            j ←j+1
      if v≥H[j]
         heap←True
      else
         H[k] ← H[j]; k←j
   H[k]←v
```



# Complexity Analysis

- Consider the tree height
  - height of a node: length of the path from it to leaf
  - $n$-element heap has height $\lfloor \lg_2 n \rfloor$
  - Number of nodes at height $h$ is $\lceil n/2^{h+1} \rceil$
    - e.g. $n=15,\ h=3,$
    - nodes at $h=0$ is $8$, at $h=1$ is $4$, at $h=2$ is $2$
- Generalized analysis
  - Moving $\lfloor n/2 \rfloor$ nodes i.e. considering parent nodes
  - Each node may move $h= \lfloor \lg_2 n \rfloor$ times.
  - Thus complexity for heapifying array is $\Theta(n \lg_2 n)$

# Complexity Analysis: Improved

- Node at height `1` moves at most `1` times
- Node at height `2` moves at most `2` times
- i.e. Node at height `h` moves at most `h` times.
- Total number of moves are

$$\sum_{i=0}^{h} \lceil \frac{n}{2^{i+1}} \rceil * i = O(n \sum_{i=0}^{lg_2 n} \lceil \frac{i}{2^i} \rceil) \qquad (1)$$

Some basic mathematics

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \quad for\ x < 1$$

Differentiating both sides

$$\sum_{k=1}^{\infty} k x^{k-1} = \frac{1}{(1-x)^2} \Rightarrow \sum_{k=0}^{\infty} k x^k = \frac{x}{(1-x)^2} \qquad (2)$$

# Complexity Analysis: Improved

Taking `x=1/2` in eqn (2) gives

$$\sum_{k=0}^{\infty} k(\frac{1}{2})^k = \frac{\frac{1}{2}}{(1-\frac{1}{2})^2}$$

$$\Rightarrow \sum_{k=0}^{\infty} \frac{k}{2^k} = \frac{\frac{1}{2}}{\frac{1}{4}}$$

$$\Rightarrow \sum_{i=0}^{\infty} \frac{i}{2^i} = 2$$

Thus eqn (`1`) becomes $O(n \sum_{i=0}^{lg_2 n} \lceil \frac{i}{2^i} \rceil) \leq O(n.2) = O(n)$

That is heap from the array can be built in `O(n)` time

# Exercises:

- Consider the array
  - 3,5,6,7,20,8,2,9,12,15,30,17
  - Draw the Complete Binary Tree
  - Heapify the tree.
    - Workout the upates in array when heapifying.
- In the above heap, insert the following items, one at a time.
  - 16, 21, 45
- Perform 3 DeleteMax operations
  - Show the heap structure after each delete

# Summary

- Priority queue
- 3 Operations
  - FindMin
  - DeleteMin
  - Add
- Heap
- Heapification (building an heap)
- Time complexity analysis