Design and Analysis of Algorithms

L11: MergeSort

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Resources

- Text book 1: Levitin (Mergesort)
- NPTel - DAA (Prof Madhavan Mukund)
  - https://onlinecourses.nptel.ac.in/noc20_cs27/unit?unit=12&lesson=16
  - https://onlinecourses.nptel.ac.in/noc20_cs27/unit?unit=12&lesson=17

MergeSort

- Problem: Given a set of N elements, sort the elements in ascending (or descending) order
  - Assume that these elements are in an array of size N
- Approaches
  - Divide and Conquer approach

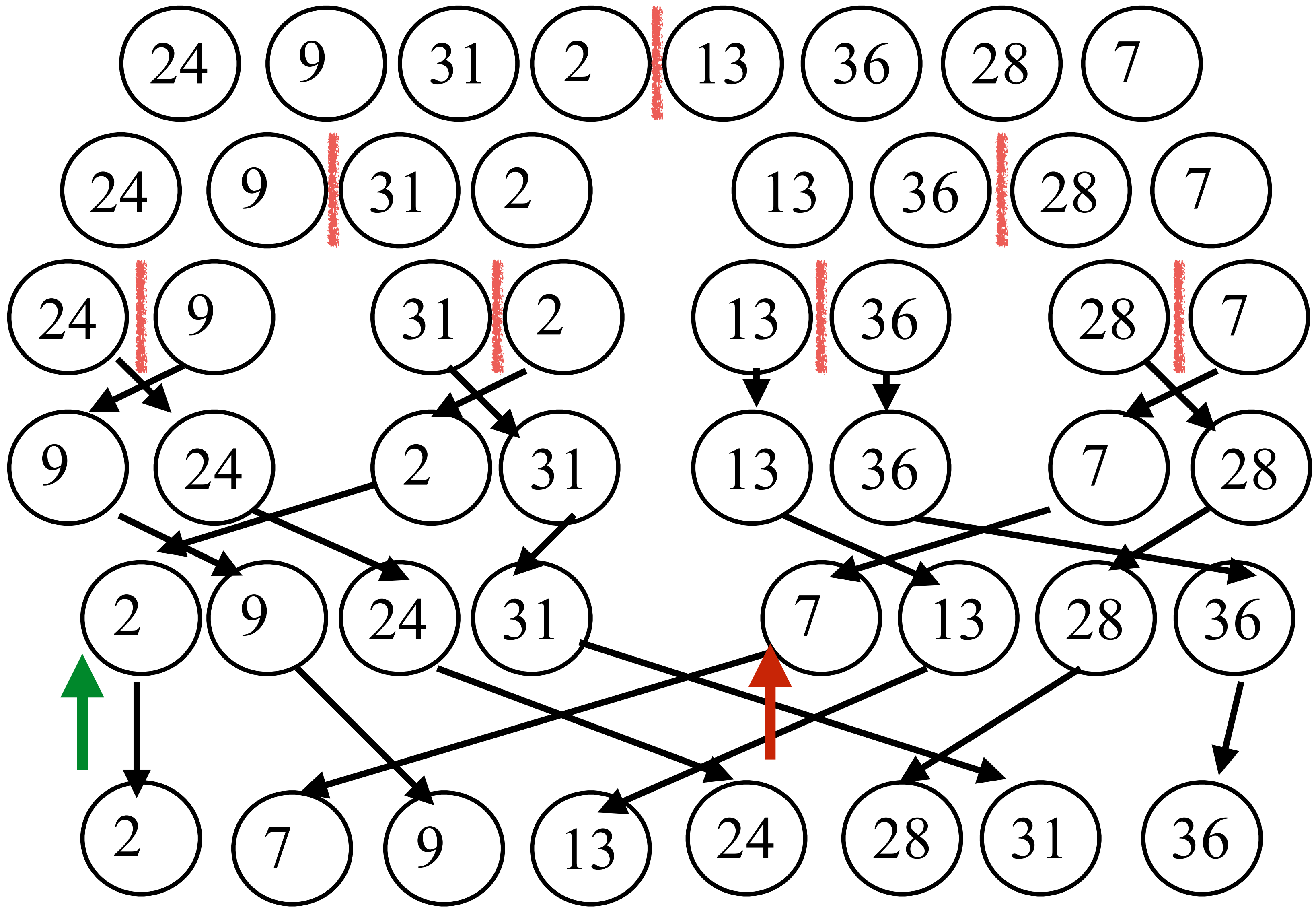
Sort Algorithms

- Bubble sort
- Selection sort
- Insertion sort
- **Mergesort**
- Quicksort
- Shell sort
- Heap sort
- Radix sort

MergeSort

- Basic idea
  - Take two sorted list and merge them into a single sorted list.
- Approach
  - Keep dividing the elements into (almost) equal half size (recursively) till sublist becomes of size 1
  - List of size `1` is sorted by default
  - Merge the sorted lists and keep repeating (recursively back)
  - When all the lists are merged, all elements are sorted.

MergeSort Example



MergeSort

- Split array A[1:n] into about equal halves
  - Make copies of each half in arrays B and C
- Sort arrays B and C recursively
- Merge sorted arrays B and C into A as follows:
  - Repeat until one of the arrays becomes empty
    - Compare the first elements of the remaining unprocessed portions of the arrays
    - Copy the smaller of the two into A,
      - Increment the index of the array (smaller)
  - Once all elements in one of the arrays are copied
    - Copy the remaining unprocessed elements from the other array into A.

Algo: MergeSort

- Algo `MergeSort(1,n,A[])`

```
#Sort array A recursive by merging
#i/p: unsorted array A[1:n]
#o/p: sorted array A[1:n]
if n>1, then
  copy A[1:n/2] to B[1:n/2]
  copy A[n/2+1:n] to C[1:n/2]
  Mergesort(1,n/2,B) #recursive
  Mergesort(1,n/2,C) #recursive
  Merge(B,C,A) # merge two arrays
# else part not required, why?
```

Algo: MergeSort

- **Algo** `Merge(B[1:p],C[1:q],A[1:p+q])`

```
#maintain one index for each array
i←1; j←1; k←1;
while (i<p+1) and (j<q+1) do
    if (B[i]≤C[j]), then
        A[k] ← B[i]
        i ← i+1
    else
        A[k] ← C[j]
        j ← j+1
    k ← k+1
if (i > p) then #B has been fully copied to A
    copy C[j:q] to A[k:p+q]
else
    copy B[i:p] to A[k:p+q]
```

MergeSort: Analysis

- Each step of Mergesort
  - Two recursive invocations of size $n/2$: $2T(n/2)$
  - Merging of two $n/2$ array into one array of size n
    - Time complexity: n
- Recurrence relation for time complexity becomes

$$
\begin{aligned}
T(n) &= 2T(n/2) + n \\
&= 2(2T(n/4)+n/2)+n = 2^2T(n/2^2)+n+n \\
&= \ldots \\
&= 2^kT(n/2^k)+n+\ldots(\log_2 n \text{ times}) \\
&= n*T(1)+n\log_2 n = n + n\log_2 n \\
&= \Theta(n\log_2 n)
\end{aligned}
$$

  - Better than Θ(n) for Insertion, Selection sort
- Space complexity $= \Theta(n)$

MergeSort: Master Theorem

$T(n)=aT(n/b)+\Theta(n^d)$ for **$n=b^k$, $k=1,2,$**

$T(1) = c$, where, **$a \geq 1$, $b \geq 2$, $c>0$**

$$T(n) \in \begin{cases} \Theta(n^d) & \text{if } a < b^d \\ \Theta(n^d \log n) & \text{if } a = b^d \\ \Theta(n^{\log_b a}) & \text{if } a > b^d \end{cases}$$

$T(n)=2T(n/2) + n$

$a=2$ $(a \geq 1)$, $b=2$ $(b \geq 2)$, $c=T(1)=1$, **and**

$f(n)=n \in \Theta(n^d) \Rightarrow f(n) \in \Theta(n^1) \Rightarrow d=1$

Thus, $b^d=b^1=b \Rightarrow a=b^1$ **#2nd case in Master Theorem**

$T(n)=\Theta(n^d \log_b n) = \Theta(n^1 \log_2 n) = \Theta(n \log_2 n)$

Mergesort Shortcomings

- Creates a new array i.e. requires additional O(n) space
  - No obvious way to merge in place in linear time.
- It is inherently recursive.
  - Recursive implemenation requires function invocation and return, a costly operation.
- Thus, Generally, not used in pratice.
- Alternative approaches
  - Can we ensure that left part is always less than the rigth part.
    - Thus, no need to merge the two.
    - Approach taken by **QuickSort.**

MergeSort (Inplace)

- If we need to merge in place, what is time and space complexity
  - Space: `O(1)`
  - Time: $O(n^2)$

	6	10	15	20		3	4	5	19	Moves
S1	3	10	15	20		4	5	6	19	4
S2	3	4	15	20		5	6	10	19	4
S3	3	4	5	20		6	10	15	19	4
S4	3	4	5	6		10	15	19	20	5

3-way MergeSort

- Divide into 3 parts
- Mergesort each part separately
- Merge the parts.
- Time complexity

$$T(n) = 3T(n) + O(n)$$

$$= O(\log_3 n)$$

Summary

- Mergesort
  - Not in place sort
  - Stable sort
-

Count Inversions

- Applications of Mergesort
- An inversion is defined as `A[i]>A[j]` if `i<j`.
- Consider the array

```
24, 9, 31, 2, 13, 36, 28, 7
```

- Inversions are

```
24-2, 24-2, 24-13, 24-7
9-2, 9-7
31-2, 31-13, 31-28, 31-7
13-7
36-28, 36-7
28-7
```

- Total inversions: `14`
- Use Merge and count approach (Divide & Conq)

Count Inversions

- Consider the array
  24, 9, 31, 2, 13, 36, 28, 7
- Inversions in Left half: 24, 9, 31, 2
  (4): 24-31, 24-2, 9-2, 31-2
  Sorted subarray: 2, 9, 24, 31
- Inversions in Right half: 13, 36, 28, 7
  (4): 13-7, 36-28, 36-7, 28-7
  Sorted subarray: 7, 13, 28, 36
- Inversions from left half to right half
  2,9,24,31 —— 7,13,28,36
  (6): 9-7,24-7,24-13,31-7,31-13,31-28
  Total inversions: 4+4+6=14
- Use Merge and count approach (Divide & Conq)