Design and Analysis of Algorithms

L06:
Recursive and Non-Recursive Algo

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Resources

- T1: Levitin
- T2: Horowitz

Efficiency of Non-Recursive Algos

- Generic plan for non-recursive algorithms
  - Decide on parameter n indicating *input size*
  - Identify algorithm's *basic operation*
    - The operation that is most executed
  - Determine *worst*, *average*, and *best* cases for input of size n
    - Does input data affects the performance of algo
  - Set up an expression for the number of times the basic operation is executed
  - Simplify the expression using standard formulas and rules

Ex 01: Finding Maximum Element

- **Prog:** `FindMax(A[1..n])`

```
// Input: array A
// Output: The value of largest element
max ← A[1]
for i = 2 to n, do
 if A[i] > max, then
  max ← A[i]
 fi
end // for
return max
```

- **Efficiency:** $\Theta(n)$, $O(n)$
  - The operation `A[i]>max` is executed `n−1` times
    - $\Sigma_{2 \le i \le n} 1 = n-1 = \Theta(n)$

Ex 02: Uniqueness problem

- Verify if input array `A[1..n]` has all unique elements
- Output: `True` if all elements in array are unique, `False` otherwise.
- Algo

```
for i =1 to n-1, do
  for j = i+1 to n, do
    if A[i] == A[j], then
      return False
return True
```

- Efficiency: basic operation `A[i] == A[j]`

$T(n) = \Sigma_{1 \le i \le n-1} \ \Sigma_{i+1 \le j \le n} \ 1)$

$= (n-1) + (n-2) + \ldots + 2 + 1 = (n-1)n/2$

$= \Theta(n^2)$ comparisons

Ex 03: Matrix Multiplication

- Multiply two `nxn` matrices `A` and `B`
- Output: Matrix `C=AB`.
- Algo

```
for i=1 to n, do
  for j=i to n, do
    C[i,j]=0
    for k=i to n, do
      C[i,j] = C[i,j] + A[i,k] * B[k,j]
return C
```

- Efficiency: basic operation `C[i,j]+A[i,k]*B[k,j]`

$T(n) = \Sigma_{1\le i\le n}\ \Sigma_{1\le j\le n}\ \Sigma_{1\le k\le n\ 2})$

$= 2n^3$

$= \Theta(n^3)$

Ex 04: Binary Digits in a Number

- Find the number of binary digits in a +ve decimal integer
- Input: a positive decimal integer n
- Output: number of binary digits
- Algo : //we can't use for loop anymore

```
count ← 0
while n>1, do
  count++
  n ← ⌊n/2⌋
return count
```

- Efficiency (basic operations): comparison `n>1`

  division: $n \leftarrow \lfloor n/2 \rfloor$

  = Each iteration, number is halved. total iterations $\log_2 n$

  $T_n = \Theta(\log n)$

Efficiency of Recursive Algos

- Generic plan for recursive algorithms
  - Identify the similar sub problem
  - Decide on parameter n indicating *input size*
  - Identify algorithm's *basic operation*
    - The operation that is most executed
  - Compute *worst*, *avg*, and *best* cases for i/p of size n
    - Does input data affects the performance of algo
    - Investigate the three cases separately
  - Set up a recurrence relation
    - How many times the number basic op. is executed.
  - Solve the recurrence (or, at the very least, establish its solution's order of growth) by backward substitutions or another method

Ex 05: Computation of Factorial `n`

- General Defintion `n! = n*(n-1)*...*2*1`
- Recursive definition `F(n) = n * F(n-1)`
  – Recursion exit on `n=1`
- Algorithm F(n)

```
if n equals 0 or n equal 1, then
  return 1
else
  return n*F(n-1)
```

- Efficiency: Basic operation : multiplication
  - Number of recursion invocations : `n`

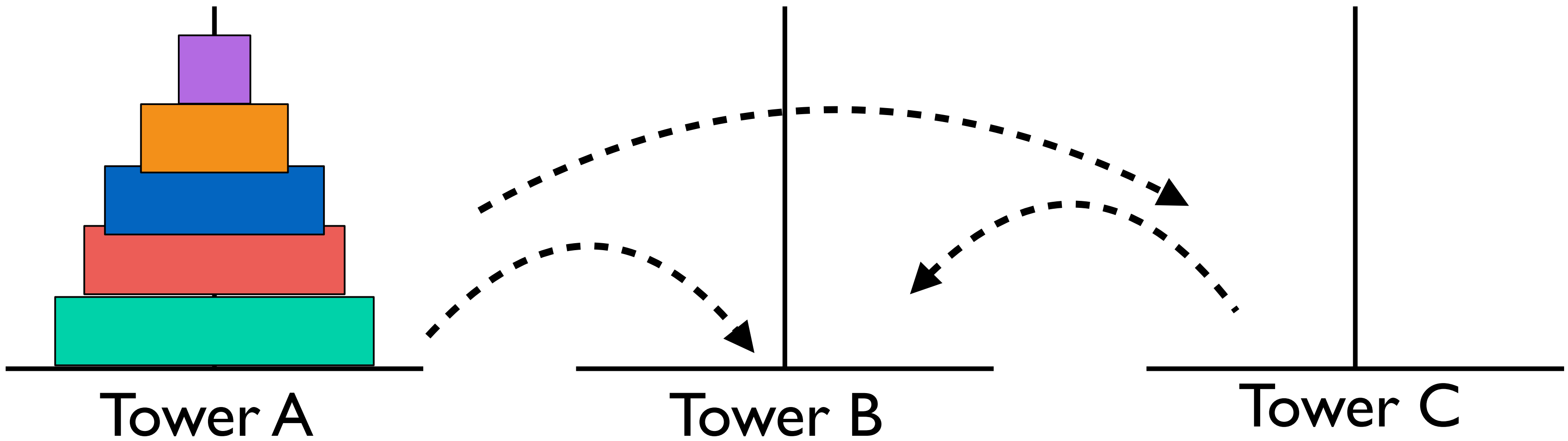
$T(n)=1+T(n-1)$

$\quad = 1+1+T(n-2) = n$

$T(n)= \Theta(n)$

Ex 06: Tower of Hanoi

- Task: Transfer `n` discs from tower A to tower B using tower C while following the rule of discs placement



- Efficiency: Basic operations: Move `(n-1),1,(n-1)`

```
T(n)=T(n-1)+1+T(n-1)
     = 1+2*T(n-1)
     =1+2(1+2*T(n-2))
```

$$=2^0+2^1+2^2+\ldots+2^{n-1} = 2^n - 1$$

$$= \Theta(2^n)$$

Ex 07: Binary Digits in a Number

- Find the number of binary digits in a +ve decimal integer
- Input: a positive decimal integer `n`
- Output: number of binary digits
- Algo : `BinDigits(n)`

```
if n equals 1
  return 1
else
  return 1 + BinDigits(⌊n/2⌋)
```

- Efficiency: Basic operations: Halving the value

$$
\begin{aligned}
T(n) &= 1+T(\lfloor n/2 \rfloor) \\
&= 1+1+T(\lfloor n/2^2 \rfloor) \\
&= 1+1+\ldots+1 \quad (\log_2 n \text{ times}) \\
&= \log_2 n \\
&= \Theta(\log_2 n)
\end{aligned}
$$

Solving Recursion Relations

- Method of forward substitution

  `T(n) = aT(n-1) + 1`

- Method of backward substitution

  `T(n) = T(n-1) + n`

- Decrease by `1`

  `T(n) = T(n-1) + f(n)`

- Decrease by a constant factor

  `T(n) = T(n/b) + f(n)`

- Divide and conquer

  `T(n) = aT(n/b) + f(n)`

- …

Method of Forward Substitution

- Generate first few terms
- Guess/Identify the closed form (expression)
- Prove by induction or direct substitution
- Example: $T(n) = 2T(n-1)+1,\ T(0)=1$

$T(0) = 1$

$T(1) = 3$

$T(2) = 7$

$T(3) = 15$

$T(4) = 31$

$T(n) = 2^{n+1}-1$

Method of Backward Substitution

$T(n)=T(n-1)+n$, **and** $T(0)=1$

$=T(n-2)+(n-1)+n$

$=T(n-3)+(n-2)+(n-1)+n$

$=T(0)+1+2+\ldots+n$

:

:

$= n(n+1)/2$

$=\Theta(n^2)$

Decrease by 1

$$
\begin{aligned}
T(n) &= T(n-1) + f(n), \text{ and } T(0) = 1 \\
&= T(n-2) + f(n-1) + f(n) \\
&= T(n-3) + f(n-2) + f(n-1) + f(n) \\
&\vdots \\
&\vdots \\
&= T(0) + \Sigma_{1 \le i \le n} f(i)
\end{aligned}
$$

- Growth dependes upon how $f(n)$ behaves.
  - For $f(n)=1, \; T(n)=n$
  - For $f(n)=\log_2 n, \; T(n)=n\log_2 n$
  - For $f(n)=n, \; T(n)=n(n+1)2 = \Theta(n^{2)}$
  - For $f(n)=n^k, \; T(n)= \Theta(n^{k+1})$

Decrease by Constant Factor

$T(n) = T(n/b) + f(n)$, **and** $T(0) = 1$

$= T(n/b^2) + f(n/b) + f(n)$

$= T(n/b^3) + f(n/b^2) + f(n/b) + f(n)$

$\vdots$

$\vdots$

$= T(1) + \Sigma_{1 \le i \le k} f(i)$, **where** $n = b^k$

- Growth dependes upon how $f(n)$ behaves.
  - For $f(n)=1, \ k=\log_b n, \ T(n)=\log_b n$
  - For $f(n)=n, k=\log_b n; \ T(n)=\Sigma_{1 \le i \le k} f(b^i)$

    $T(n)=\Sigma_{1 \le i \le k} b^i = (b^k-1)/(b-1)=\Theta(b^k)$

    $=\Theta(n)$

Solving Recurrence Relation

$$T(n) = aT(n/b) + f(n)$$

- Let `n=b`^{k}, then

$$T(b^k) = aT(b^{k-1}) + f(b^k)$$

$$= a[aT(b^{k-2}) + f(b^{k-1})] + f(b^k)$$

$$= a^2T(b^{k-2}) + af(b^{k-1}) + f(b^k)$$

$$= a^3T(b^{k-3}) + a^2f(b^{k-2}) + af(b^{k-1}) + a^0f(b^k)$$

$$\vdots$$

$$= a^kT(b^{k-k}) + a^{k-1}f(b^{k-(k-1)} + a^2f(b^{k-2}) + af(b^{k-1}) + a^0f(b^k)$$

$$= a^kT(1) + a^{k-1}f(b^1) + a^{k-2}f(b^2) + \ldots + a^0f(b^k)$$

$$= a^k[T(1) + f(b^1)/a^1 + f(b^2)/a^2 + \ldots + f(b^k)/a^k]$$

Solving Recurrence Relation

$$T(n) = aT(n/b) + f(n)$$

$$T(b^k) = aT(b^{k-1}) + f(b^k)$$

$$= a^k[T(1) + f(b^1)/a^1 + f(b^2)/a^2 + \ldots + f(b^k)/a^k]$$

$$= a^k[T(1) + \sum_{j=1}^{k} \frac{f(b^j)}{a^j}]$$

- Thus, `T(n)` depends upon `a`, `b`, and `f()`

As $n=b^k$, then $k=\log_b n$, thus

$a^k=a^{\log_b n} = n^{\log_b a}$, the recursion equation becomes

$$T(n) = n^{log_b a}[T(1) + \sum_{j=1}^{log_b n} \frac{f(b^j)}{a^j}] \qquad (1)$$

Recurrence Relation: Examples

- Example 01: $T(n)=2T(n/2)+n$
- $a=2,\ b=2, T(1)=1, f(n)=n$

$T(n)=2T(n/2)+n$

$= 2[2T(n/2^2)+n/2]=2^2T(n/2^2)+n+n$

$= 2^3T(n/2^3)+n+n+n$

$= 2^kT(1)+n+...+n$ ($\log_2 n$ times)

$= 2^k+n.\log_2 n$

$= n + n\log_2 n = \Theta(n\log_2 n)$

Using the eqn (1)

$\log_b a=\log_2 2=1,\ b/a=1 \rightarrow f(b^j)/a^j=b^j/a^j=1$

$$T(n) = n^{\log_b a}[T(1) + \sum_{j=1}^{\log_b n} \frac{f(b^j)}{a^j}]$$

$= n[1+(1+1+\ldots(\log_2 n \text{ times})+1)]=n\log_2 n$

$=\Theta(n\log_2 n)$

Recurrence Relation: Examples

- **Example 02:** $T(n)=9T(n/3)+4n^6$

  $= 3^2T(n/3)+4n^6$

- $a=9,\ b=3, T(1)=4, f(n)=4n^6$

  Given $\log_b a=\log_3 9=2$, and

  $f(b^j)/a^j=4(b^j)^6/a^j=4*3^{6j}/3^{2j}=4*3^{4j}$

$$T(n) = n^{\log_b a}[T(1) + \sum_{j=1}^{\log_b n} \frac{f(b^j)}{a^j}]$$

$= n^2[4+(4*3^4+4*3^{4*2}+\ldots+4*3^{4*\log_3 n})]$

$= n^2[4+4(3^4+3^{4*2}+\ldots+3^{4*\log_3 n})]$

$= n^2*4[(3^4)0+3^4+3^{4*2}+\ldots+3^{4*\log_3 n})]$

$= n^2*4(3^{4*(\log_3 n+1)}-1)/(3^4-1)$

$= c*n^2*3^{4*(\log_3 n)}+d=c*n^2*n^4+d$

$=\Theta(n^6)$

Summary

- Analysis of Non Recursive algorithms
- Analysis of recursive algorithms
- Recurrence relation examples