Design and Analysis of Algorithms

L09: Divide and Conquer

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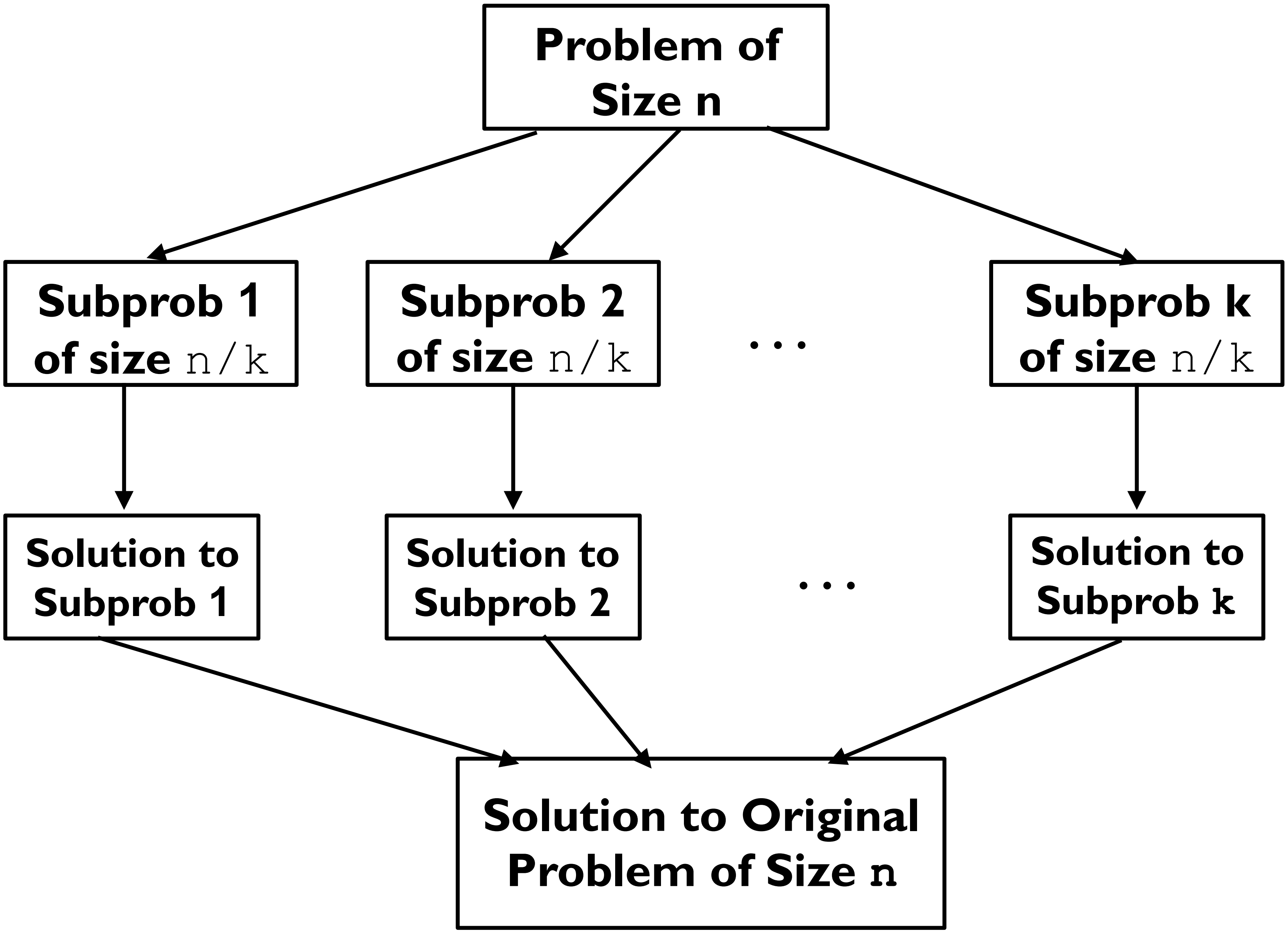
Resources

- Text book 2: Horowitz
- Text book 1: Levitin
- https://visualgo.net/en

Divide and Conquer Algo

- Divide (break) the problem (size n) into similar sub problems
  - Size of sub problems should be some factor of original e.g. `n/c`
    - When small enough, solve by brute force
- Conquer (Solve) the sub-problem
  - Use recursion to solve small problem
- Combine (Merge) the solution of sub-parts
- The cost is
  - Cost of breaking
  - Cost of solving subproblem
  - Cost of combining

Divide and Conquer Approach

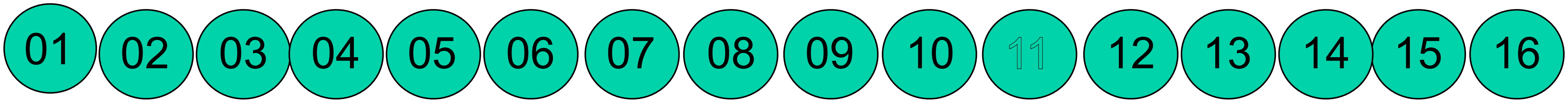


Divide and Conquer Examples

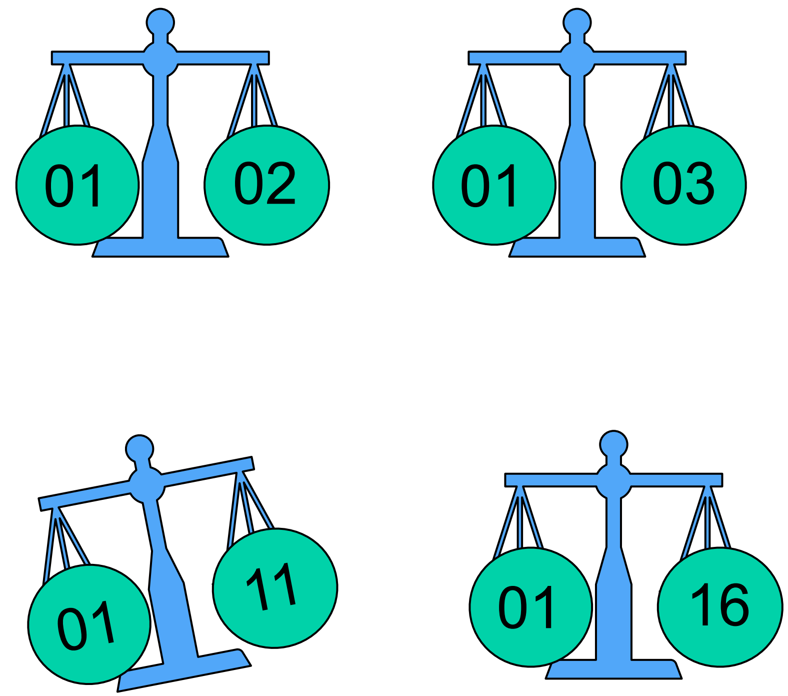
- Sorting and Searching
- Binary Tree traversals
- Binary search
- Multiplication of large numbers (Karatsuba Algo)
- Matrix multiplicatin - Strassen's algorithm
- Closest pair problem
- Convex Hull problem

Simple Example

- Given 16 balls with one defective (say lighter)
  – Identify the defective ball.

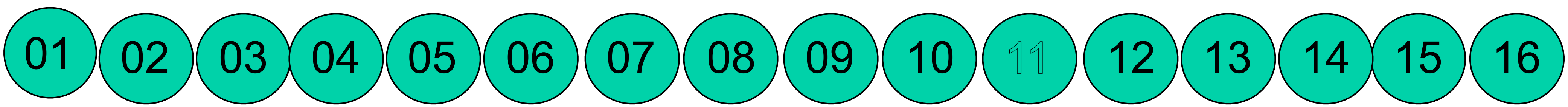


- Solution 1:
  - Compare 1 with 2
  - Compare 1 with 3
  - :
  - Compare 1 with 16
- Time taken:
  - 15 comparisons (worst case)

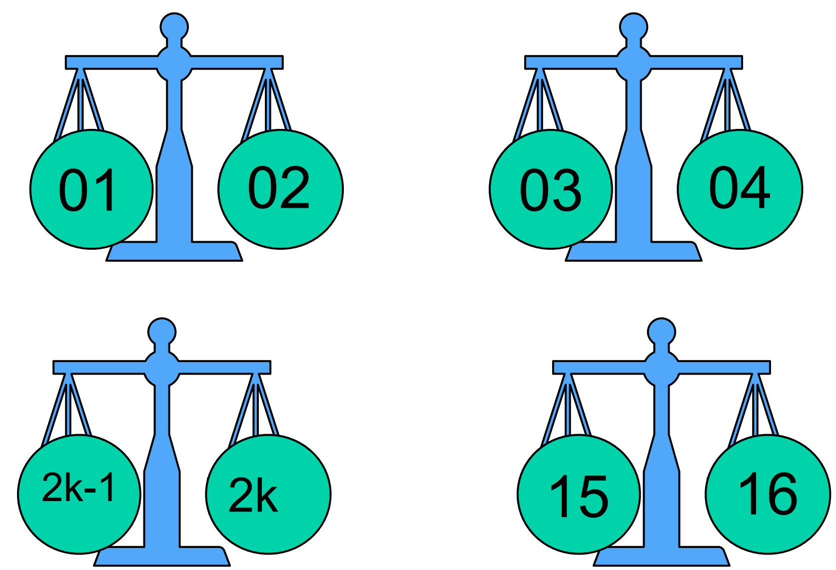


Simple Example

- Given 16 balls with one defective (say lighter)
  - – Identify the defective ball.



- Solution 2:
  - Compare 1 with 2
  - Compare 3 with 4
  - :
  - Compare 15 with 16
- Time taken:
  - 8 comparisons (worst case)



Simple Example

- Given 16 balls with one defective (say lighter)
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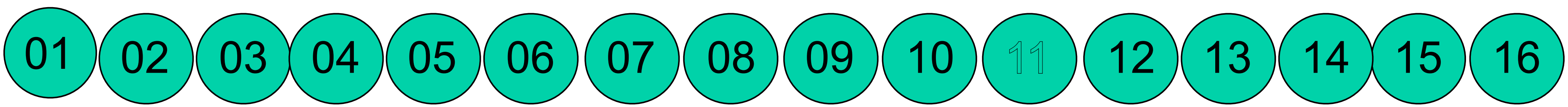
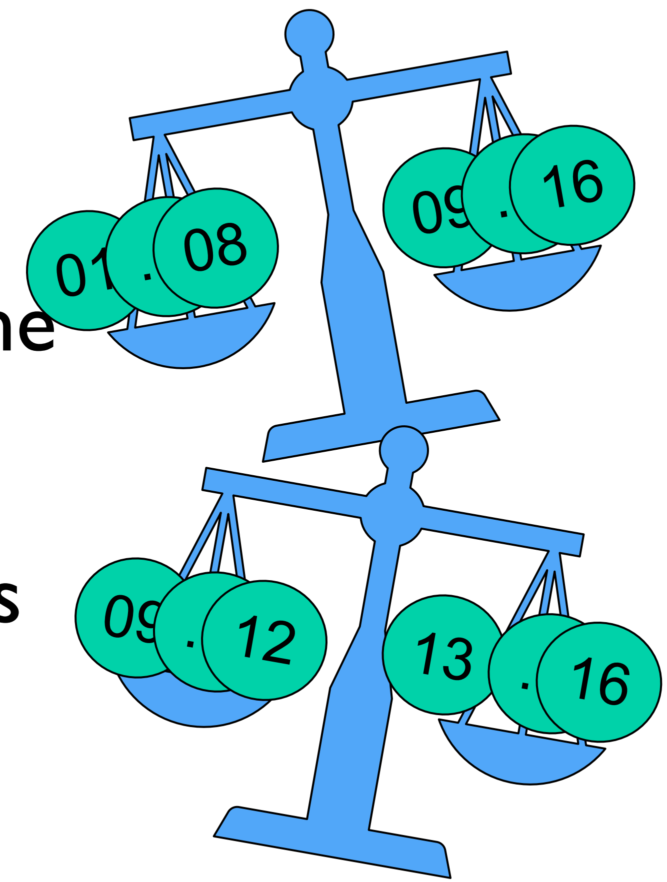


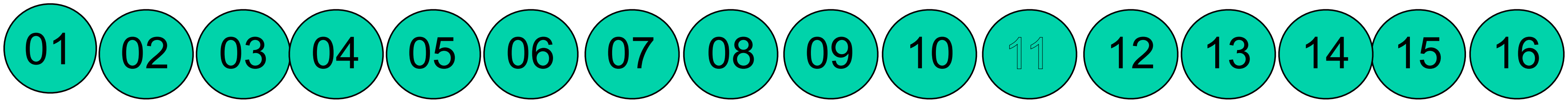


- Soltion3: Divide and Conquer
  - Divide into 2 sets, each of 8 balls
  - Compare 1-8 with 9-16, and divide the lighter set into two parts each of 4.

    :
  - Continue the process till lighter ball is found
- Time taken: 4 comparisons ($\log_2 16$)

Simple Example

- Given 16 balls with one defective (say lighter)
  - Identify the defective ball.



- Soltion3: Time complexity

$T(n) = T(n/2) + 1$ #1 comparison reduces it by half

$= T(n/4) + 1 + 1 = T(n/2^2)+2$

$= T(n/2^3) + 3$

:

$= T(n/2^i) + i$

$= \log_2 n$

Divide & Conquer: Control Abstraction

```
Algo D_And_C(P) {
   if Small(P)
      return S(P)
   else {
      Divide P into smaller sets P1, …,Pk
      Apply D_And_C to each subproblem
      return Combine(D_And_C(P1),
                     …,
                     D_And_C(Pk))
   }
}
```

Divide and Conquer: Recurrence Relation

$$T(n) = \begin{cases} g(n) & n \ small \\ T(n_1) + T(n_2) + \ldots + T(n_k) + f(n) & otherwise \end{cases}$$

- `T(n)`: time complexity for a problem of input size n
- `g(n)`: time complexity for solving directly for small inputs
- `f(n)`: Time complexity for dividing the problem into `k` subproblems and combining again from the solutions of `k` sub problems.
- `k` would vary depending upon the problem
  - Generally, $n_1 = n_2 = \ldots = n_k$
  - Assuming `a` instances, each of size `n/b`

$$T(n) = \begin{cases} T(1) & n = 1 \\ aT(n/b) + f(n) & n > 1 \end{cases}$$

Solving Recurrence Relation

$$T(n) = aT(n/b) + f(n)$$

- Let `n=b`^{k}, then

$$T(b^k) = aT(b^{k-1}) + f(b^k)$$

$$= a[aT(b^{k-2}) + f(b^{k-1})] + f(b^k)$$

$$= a^2T(b^{k-2}) + af(b^{k-1}) + f(b^k)$$

$$= a^3T(b^{k-3}) + a^2f(b^{k-2}) + af(b^{k-1}) + a^0f(b^k)$$

$$\vdots$$

$$= a^kT(b^{k-k}) + a^{k-1}f(b^{k-(k-1)} + a^2f(b^{k-2}) + af(b^{k-1}) + a^0f(b^k)$$

$$= a^kT(1) + a^{k-1}f(b^1) + a^{k-2}f(b^2) + \ldots + a^0f(b^k)$$

$$= a^k[T(1) + f(b^1)/a^1 + f(b^2)/a^2 + \ldots + f(b^k)/a^k]$$

Solving Recurrence Relation

$$T(n) = aT(n/b) + f(n)$$

$$T(b^k) = aT(b^{k-1}) + f(b^k)$$

$$= a^k[T(1) + f(b^1)/a^1 + f(b^2)/a^2 + \ldots + f(b^k)/a^k]$$

$$= a^k[T(1) + \sum_{j=1}^{k} \frac{f(b^j)}{a^j}]$$

- Thus, `T(n)` depends upon `a`, `b`, and `f()`

As $n=b^k$, then $k=\log_b n$, thus

$a^k=a^{\log_b n} = n^{\log_b a}$, the recursion equation becomes

$$T(n) = n^{log_b a}[T(1) + \sum_{j=1}^{log_b n} \frac{f(b^j)}{a^j}] \qquad (1)$$

Recurrence Relation: Examples

- **Example 01:** $a=2,\ b=2, T(1)=1, f(n)=n$

$T(n)=2T(n/2)+n$

$= 2[2T(n/2^2)+n/2]=2^2T(n/2^2)+n+n$

$= 2^3T(n/2^3)+n+n+n$

$= 2^kT(1)+n+...+n$ ($\log_2 n$ times)

$= 2^k+n.\log_2 n$

$= n + n.(\log_2 n)$

$= n + n\log_2 n = \Theta(n\log_2 n)$

Using the eqn (1)

$\log_b a=\log_2 2=1,\ b/a=1 \rightarrow f(b^j)/a^j=b^j/a^j=1$

$$T(n) = n^{log_b a}[T(1) + \sum_{j=1}^{log_b n} \frac{f(b^j)}{a^j}]$$

$= n[1+(1+1+\ldots(\log_2 n \text{ times})+1)]=n\log_2 n$

$=\Theta(n\log_2 n)$

Recurrence Relation: Examples

- **Example 02:** $a=9,\ b=3, T(1)=4, f(n)=4n^6$

  Given

  $\log_b a=\log_3 9=2,$

  $f(b^j)/a^j=4b^{6j}/a^j=4*3^{6j}/3^{2j}=4*3^{4j}$

$$T(n) = n^{\log_b a}[T(1) + \sum_{j=1}^{\log_b n} \frac{f(b^j)}{a^j}]$$

$= n^2[4+(4*3^4+4*3^{4*2}+\ldots+4*3^{4*\log_3 n})]$

$= n^2*4(3^{4*(\log_3 n+1)}-1)/(3^4-1)$

$= c*n^2*3^{4*(\log_3 n)}+d=c*n^2*n^4+d$

$=\Theta(n^6)$

Fun Exercise of Game of 128 numbers

- A practical fun example of Data structures and Algorithm

1	2	3	4	5	6	7	8
9	10	11	12	13	14	15	16
17	18	19	20	21	22	23	24
25	26	27	28	29	30	31	32
33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48
49	50	51	52	53	54	55	56
57	58	59	60	61	62	63	64
65	66	67	68	69	70	71	72
73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88
89	90	91	92	93	94	95	96
97	98	99	100	101	102	103	104
105	106	107	108	109	110	111	112
113	114	115	116	117	118	119	120
121	122	123	124	125	126	127	128

Game:

. Go thru a set of cards

. Say Y/N if present or not

. You will get your number graphically displayed to you

Q?:

Which algorithm we are discussing?

Aim: Can we find more such examples

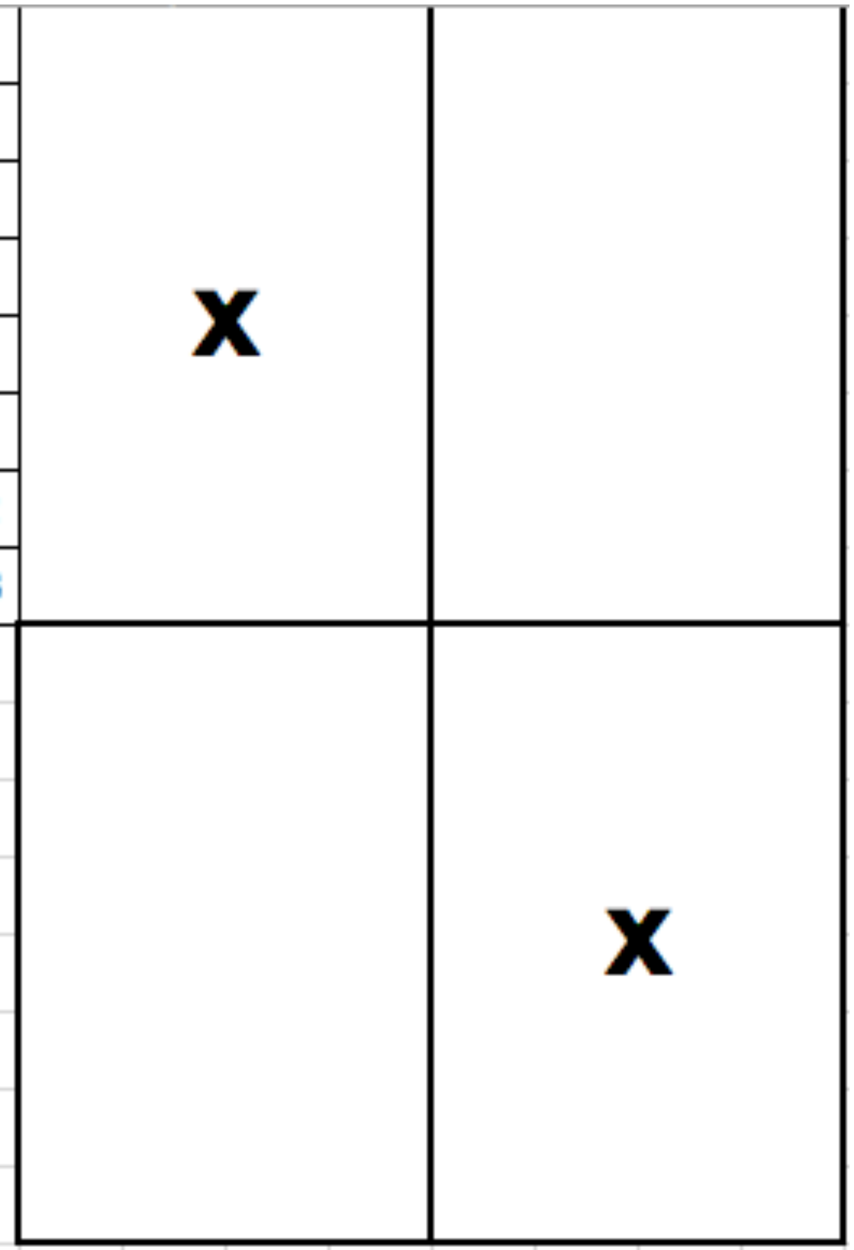
Game of 128 numbers - b

1	2	3	4	5	6	7	8
9	10	11	12	13	14	15	16
17	18	19	20	21	22	23	24
25	26	27	28	29	30	31	32
33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48
49	50	51	52	53	54	55	56
57	58	59	60	61	62	63	64

x

Game of 128 numbers - c

1	2	3	4	9	10	11	12
17	18	19	20	25	26	27	28
33	34	35	36	41	42	43	44
49	50	51	52	57	58	59	60
69	70	71	72	77	78	79	80
85	86	87	88	93	94	95	96
101	102	103	104	109	110	111	112
117	118	119	120	125	126	127	128



Game of 128 numbers - d

1	2	3	4	5	6	7	8	
9	10	11	12	13	14	15	16	x
17	18	19	20	21	22	23	24	
25	26	27	28	29	30	31	32	
65	66	67	68	69	70	71	72	
73	74	75	76	77	78	79	80	
81	82	83	84	85	86	87	88	
89	90	91	92	93	94	95	96	
								x

Game of 128 numbers - d

1	2	5	6	9	10	13	14
17	18	21	22	25	26	29	30
33	34	37	38	41	42	45	46
49	50	53	54	57	58	61	62
67	68	71	72	75	76	79	80
83	84	87	88	91	92	95	96
99	100	103	104	107	108	111	112
115	116	119	120	123	124	127	128

x		x	
	x		x

Exercise G

- Exercise G
  - Work out the remaining 3 cards

Summary: Divide and Conquer

- Break the problem into smaller subsets
  - By a factor `c` i.e. `n`—> `n/c`
- Conquer (Solve) the sub-problem
- Combine (Merge) the solution of sub-parts
- Example cases
  - Sorting and Searching
  - Binary Tree traversals
  - Binary search
  - Multiplication of large numbers (Karatsuba Algo)
  - Matrix multiplicatin - Strassen’s algorithm
  - Closest pair problem
  - Convex Hull problem

Summary

- Divide and Conquer approach
- Cost efficiency:
  - Define recurrence relation
  - Solve the recurrance equation
- Example of binary search (visual)