Design and Analysis of Algorithms

L13: Strassen Multiplication

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Resources

- Text book 2: Sec 3.8 - Horowitz
- Text book 1: Sec 4.5 - Levitin

Matrix Multiplication

- Conventional matrix multiplication

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & & & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \vdots & & & \vdots \\ c_{n1} & c_{n2} & \dots & c_{nn} \end{bmatrix}$$

where the element c_{ij} is computed as

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \dots + a_{in}b_{nj}$$

- computations required for c_{ij}
  - Multiplications: n
  - additions: n
- Total computations required for matrix multiplication:
  - $2n^3$ i.e.
  - $\Theta(n^3)$

Matrix Multiplication

- Conventional matrix multiplication

$$\begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} * \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}$$

$$= \begin{bmatrix} A_1B_1+A_2B_3 & A_1B_2+A_2B_4 \\ A_3B_1+A_4B_3 & A_3B_2+A_4B_4 \end{bmatrix}$$

- Multiplications: 8 (=2^3), Additions: 4
- Recurrence relation:

$$T(n)=8T(n/2)+4(n/2)^2 = 2^3T(n/2)+n^2$$

$$= \Theta(n^3)$$

- Master theorem: $a=8, \; b=2, \; d=2$

Thus, $b^d=2^2=4 \;\Rightarrow a>b^1$ #3^{rd} case in Master Theorem

$$T(n)=\Theta(n^{\log_b a}) = \Theta(n^{\log_2 8}) = \Theta(n^3)$$

Multiplication Example

- Multiply 2 numbers (of 2 digits) e.g. 98, 76

  A= 98 = 10*9+8

  B= 76 = 10*7+6

  C = A*B = $10^2(9*7)+10(8*7+9*6)+8*6$

  $=6300+10(56+54)+48 = 7448$

  – Multiplications : **4** , additions : **3**

- Alternatively

  C = $10^2(9*7) + 8*6$

  $+ 10((9+8)*(7+6)-\underline{9*7}-\underline{8*6}))$

  $= 10^2*9*7+10*(8*7+9*6)+8*6=7448$

  – Multiplications : **3** , Additions : **6**

Multiplication Example

- Multiply 2 numbers (of 2 digits)
  - A = a_2a_1 $(=10a_2+a_1)$, B=b_2b_1 $(=10b_2+b_1)$
  - C = A * B

    = $100c_3 + 10c_2 + c_1$

  where

  $c_3=a_2b_2$, $c_2=(a_2b_1+a_1b_2)$, $c_1=a_1b_1$
  - Multiplications : 4 Additions : 3
- Alternatively

  $c_3=a_2b_2$,

  $c_1=a_1b_1$

  $c_2=(a_2+a_1)*(b_2+b_1)-c_3-c_1$
  - Multiplications : 3 Additions : 6

Multiplication Example

- Multiply 2 numbers (of 4 digits) e.g. 9876, 5432

  A= $98*10^2+76$, B= $54*10^2+32$

  C = A*B = $10^4(98*54)$

  $+\ 10^2(98*32 + 76*54)$

  $+\ 76*32$

  – Multiplications : **4**, additions : **3**
- Alternatively

  C = $10^4(98*54 + 76*32) +$

  $10^2((98+76)*(54+32)-98*54-76*32))$

  – Multiplications : **3**, additions : **6**

Large Numbers Multiplication

- Problem:
  Given two large numbers with N digits, Multiply these in efficient way
- Solution : traditional way (high school mathematics)

$$
\begin{array}{r}
a_n \ a_{n-1} \ \ldots \ a_2 \ a_1 \\
b_n \ b_{n-1} \ \ldots \ b_2 \ b_1 \\
\hline
c_{1n+1} \ c_{1n} c_{1n-1} \ \ldots c_{12} \ c_{11} \quad \\
c_{2n+1} \ c_{2n} c_{1n-1} \ldots c_{22} \ c_{21} \quad\quad\quad \\
: \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \\
c_{nn+1} c_{nn} c_{nn-1} \ldots c_{n2} c_{n1} \quad\quad\quad\quad \\
\hline
\end{array}
$$

- Efficiency : Multiplications: n^2, Additions: $O(n^2)$,
- Complexity analysis: $\Theta(n^2)$

Large Numbers Multiplication

- Given two large numbers A, B with n digits,
  - multiply these in efficient way
- Let $A = A_1A_2$, and $B=B_1B_2$,
  - where A and B are n-digit numbers, and
  - A_1, A_2, B_1, and B_2 are $(n/2)$-digit numbers, then

$$A*B = 10^n(A_1*B_1)+10^{n/2}(A_1*B_2+A_2*B_1)+A_2*B_2$$

- Efficiency : Multiplications: 4 each of $n/2$
- Recurrence relation

$$\begin{aligned}T(n) &= 4T(n/2)+O(n)=2^2T(n/2)+n\\ &=2^2[4T(n/4)+O(n/2)]+n\\ &=2^4T(n/2^2)+2^2(n/2)+n=2^4T(n/2^2)+2n+n\\ &= \Theta(n^2)\end{aligned}$$

Large Numbers Multiplication

- Let $A=A_1A_2$, and $B=B_1B_2$, are n-digit numbers, and
  - A_1, A_2, B_1, and B_2 are $(n/2)$-digit numbers, then

$$A*B = 10^n(A_1*B_1) + A_2*B_2$$
$$+10^{n/2}[(A_1+A_2)*(B_1+B_2)-\underline{A_1*B_1}-\underline{A_2*B_2}]$$

- Recurrence relation using 3 multiplications

$$
\begin{aligned}
T(n) &= 3T(n/2)+O(n) \\
&= 3^1T(n/2^1)+n \\
&= 3^1[3T(n/4)+n/2]+n \\
&= 3^2T(n/2^2)+[3^1n/2^1+n] \\
&= \ldots \\
&= 3^kT(n/2^k)+[3^{k-1}n/2^{k-1}+\ldots+n] \\
&= \Theta(3^{\log_2 n}) = \Theta(n^{\log_2 3}) \\
&\approx \Theta(n^{1.585})
\end{aligned}
$$

Number Multiplication: Master Theorem

$T(n)=aT(n/b)+\Theta(n^d)$ for $n=b^k$, $k=1,2,$

$T(1) = c$, where, $a\geq 1$, $b\geq 2$, $c>0$

$$T(n) \in \begin{cases} \Theta(n^d) & \text{if } a < b^d \\ \Theta(n^d logn) & \text{if } a = b^d \\ \Theta(n^{log_b a}) & \text{if } a > b^d \end{cases}$$

$T(n)=3T(n/2) + n$

$a=3$ $(a\geq 1)$, $b=2(b\geq 2)$, $c=T(1)=1$, and

$f(n)=n \in \Theta(n^d) \Rightarrow f(n) \in \Theta(n^1) \Rightarrow d=1$

Thus, $b^d=b^1=b \Rightarrow a>b^1$ #3^{rd} case in Master Theorem

$T(n)=\Theta(n^{log_b a}) = \Theta(n^{log_2 3}) = \Theta(n^{1.585})$

Strassen's Matrix Multiplication

- Strassen discovered that product of two matrices can be computed as follows

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & c_{21} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} * \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$= \begin{bmatrix} M_1+M_4-M_5+M_7 & M_3+M_5 \\ M_2+M_4 & M_1+M_3-M_2+M_6 \end{bmatrix}$$

- $M_1=(A_{11}+A_{22})*(B_{11}+B_{22})$
- $M_2=(A_{21}+A_{22})*B_{11}$
- $M_3=A_{11}*(B_{12}-B_{22})$
- $M_4=A_{22}*(B_{21}-B_{11})$
- $M_5=(A_{11}+A_{12})*B_{22}$
- $M_6=(A_{21}-A_{11})*(B_{11}+B_{12})$
- $M_7=(A_{12}-A_{22})*(B_{21}+B_{22})$

Strassen's Matrix Multiplication

- $M_1 = (A_{11}+A_{22}) * (B_{11}+B_{22})$
- $M_2 = (A_{21}+A_{22}) * B_{11}$
- $M_3 = A_{11} * (B_{12}-B_{22})$
- $M_4 = A_{22} * (B_{21}-B_{11})$
- $M_5 = (A_{11}+A_{12}) * B_{22}$
- $M_6 = (A_{21}-A_{11}) * (B_{11}+B_{12})$
- $M_7 = (A_{12}-A_{22}) * (B_{21}+B_{22})$

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} * \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$\begin{bmatrix} M_1+M_4-M_5+M_7 & M_3+M_5 \\ M_2+M_4 & M_1+M_3-M_2+M_6 \end{bmatrix}$$

$$\begin{aligned} C_{11} = M_1+M_4-M_5+M_7 &= (A_{11}+A_{22}) * (B_{11}+B_{22}) + A_{22} * (B_{21}-B_{11}) \\ &\quad - (A_{11}+A_{12}) * B_{22} + (A_{12}-A_{22}) * (B_{21}+B_{22}) \\ &= A_{11}B_{11} + \cancel{A_{22}B_{11}} + \cancel{A_{11}B_{22}} + \cancel{A_{22}B_{22}} + \cancel{A_{22}B_{21}} \\ &\quad - \cancel{A_{22}B_{11}} - \cancel{A_{11}B_{22}} - \cancel{A_{12}B_{22}} + A_{12}B_{21} \\ &\quad - \cancel{A_{22}B_{21}} + \cancel{A_{12}B_{22}} - \cancel{A_{22}B_{22}} \\ &= A_{11}B_{11} + A_{12}B_{21} \end{aligned}$$

Strassen's Matrix Multiplication

- $M_1 = (A_{11}+A_{22}) * (B_{11}+B_{22})$
- $M_2 = (A_{21}+A_{22}) * B_{11}$
- $M_3 = A_{11} * (B_{12}-B_{22})$
- $M_4 = A_{22} * (B_{21}-B_{11})$
- $M_5 = (A_{11}+A_{12}) * B_{22}$
- $M_6 = (A_{21}-A_{11}) * (B_{11}+B_{12})$
- $M_7 = (A_{12}-A_{22}) * (B_{21}+B_{22})$

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} * \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$\begin{bmatrix} M_1+M_4-M_5+M_7 & M_3+M_5 \\ M_2+M_4 & M_1+M_3-M_2+M_6 \end{bmatrix}$$

$$\begin{aligned}
C_{12} = M_3+M_5 &= A_{11} * (B_{12}-B_{22}) + (A_{11}+A_{12}) * B_{22} \\
&= A_{11}B_{12} - \cancel{A_{11}B_{22}} + \cancel{A_{11}B_{22}} + A_{12}B_{22} \\
&= A_{11}B_{12} + A_{12}B_{22}
\end{aligned}$$

$$\begin{aligned}
C_{21} = M_2+M_4 &= (A_{21}+A_{22}) * B_{11} + A_{22} * (B_{21}-B_{11}) \\
&= A_{21}B_{11} + \cancel{A_{22}B_{11}} + A_{22}B_{21} - \cancel{A_{22}B_{11}} \\
&= A_{21}B_{11} + A_{22}B_{21}
\end{aligned}$$

Strassen's Matrix Multiplication

- $M_1 = (A_{11}+A_{22}) * (B_{11}+B_{22})$
- $M_2 = (A_{21}+A_{22}) * B_{11}$
- $M_3 = A_{11} * (B_{12}-B_{22})$
- $M_4 = A_{22} * (B_{21}-B_{11})$
- $M_5 = (A_{11}+A_{12}) * B_{22}$
- $M_6 = (A_{21}-A_{11}) * (B_{11}+B_{12})$
- $M_7 = (A_{12}-A_{22}) * (B_{21}+B_{22})$

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} * \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$\begin{bmatrix} M_1+M_4-M_5+M_7 & M_3+M_5 \\ M_2+M_4 & M_1+M_3-M_2+M_6 \end{bmatrix}$$

$$\begin{aligned} C_{22} = M_1+M_3-M_2+M_6 &= (A_{11}+A_{22}) * (B_{11}+B_{22}) + A_{11} * (B_{12}-B_{22}) \\ &\quad - (A_{21}+A_{22}) * B_{11} + (A_{21}-A_{11}) * (B_{11}+B_{12}) \\ &= \cancel{A_{11}B_{11}} + \cancel{A_{22}B_{11}} + \cancel{A_{11}B_{22}} + A_{22}B_{22} + \cancel{A_{11}B_{12}} - \cancel{A_{11}B_{22}} \\ &\quad - \cancel{A_{21}B_{11}} - \cancel{A_{22}B_{11}} + \cancel{A_{21}B_{11}} - \cancel{A_{11}B_{11}} + A_{21}B_{12} - \cancel{A_{11}B_{12}} \\ &= A_{22}B_{22} + A_{21}B_{12} \\ &= A_{21}B_{12} + A_{22}B_{22} \end{aligned}$$

Strassen's Matrix Multiplication

- $M_1 = (A_{11}+A_{22}) * (B_{11}+B_{22})$
- $M_2 = (A_{21}+A_{22}) * B_{11}$
- $M_3 = A_{11} * (B_{12}-B_{22})$
- $M_4 = A_{22} * (B_{21}-B_{11})$
- $M_5 = (A_{11}+A_{12}) * B_{22}$
- $M_6 = (A_{21}-A_{11}) * (B_{11}+B_{12})$
- $M_7 = (A_{12}-A_{22}) * (B_{21}+B_{22})$

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} * \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$\begin{bmatrix} M_1+M_4-M_5+M_7 & M_3+M_5 \\ M_2+M_4 & M_1+M_3-M_2+M_6 \end{bmatrix}$$

- Count number of Multiplications and Additions
  - Multiplications : 7, Additions : 18
- Recurrence equation

$$\begin{aligned} T(n) &= 7T(n/2)+18(n/2);\ T(1) = 1 \\ &= \Theta(7^{\log_2 n}) \\ &= \Theta(n^{\log_2 7}) \\ &= \Theta(n^{2.807}) \text{ vs. } \Theta(n^3) \text{ of brute force} \end{aligned}$$

- There exist algos with better efficiency, but more complexity

Matrix Multiplication: Master Theorem

$T(n)=aT(n/b)+\Theta(n^d)$ for $n=b^k$, $k=1,2,$

$T(1) = c$, where, $a\geq 1$, $b\geq 2$, $c>0$

$$T(n) \in \begin{cases} \Theta(n^d) & if\ a < b^d \\ \Theta(n^d logn) & if\ a = b^d \\ \Theta(n^{log_b a}) & if\ a > b^d \end{cases}$$

$T(n)=7T(n/2)+18(n/2)=7T(n/2)+O(n)$

$a=7$ $(a\geq 1)$, $b=2(b\geq 2)$, $c=T(1)=1$, and

$f(n)=n \in \Theta(n^d) \Rightarrow f(n) \in \Theta(n^1) \Rightarrow d=1$

Thus, $b^d=2^1=2 \Rightarrow a>b^1$ #3^{rd} case in Master Theorem

$T(n)=\Theta(n^{\log_b a}) = \Theta(n^{\log_2 7}) = \Theta(n^{2.807})$

Summary

- Large number multiplication
- Matrix Multiplication