Design and Analysis of Algorithms

L28: Optimal Binary Search

Dynamic Programming

Dr. Ram P Rustagi
Sem IV (2020-Even)
Dept of CSE, KSIT
rprustagi@ksit.edu.in

Resources

- Text book 1: Levitin
  - Sec ***8.3***
- R1: Introduction to Algorithms
    - Cormen et al.

Binary Search

- Binary search tree
  - Key value of left child is smaller than parent
  - Key value of right child is greater than the parent
- Balanced binary search tree- Height : `O(log n)`
  - **Red** **Black** tree - faster insertion and deletion
    - Root is always **black**
    - Both children of **Red** node are **black**
    - Any path from any node to leaf descendant contains same number of **black** nodes.
    - For any node, height of one subtree is at most twice the height of other subtree
  - AVL (Adelson-Velskii and Landis) Tree: faster search
    - Difference of height of two subtree is at most `1`
- For a random binary search tree, height is `O(log n)`
  - Worst case height can be `O(n)`

Optimal Binary Search Tree

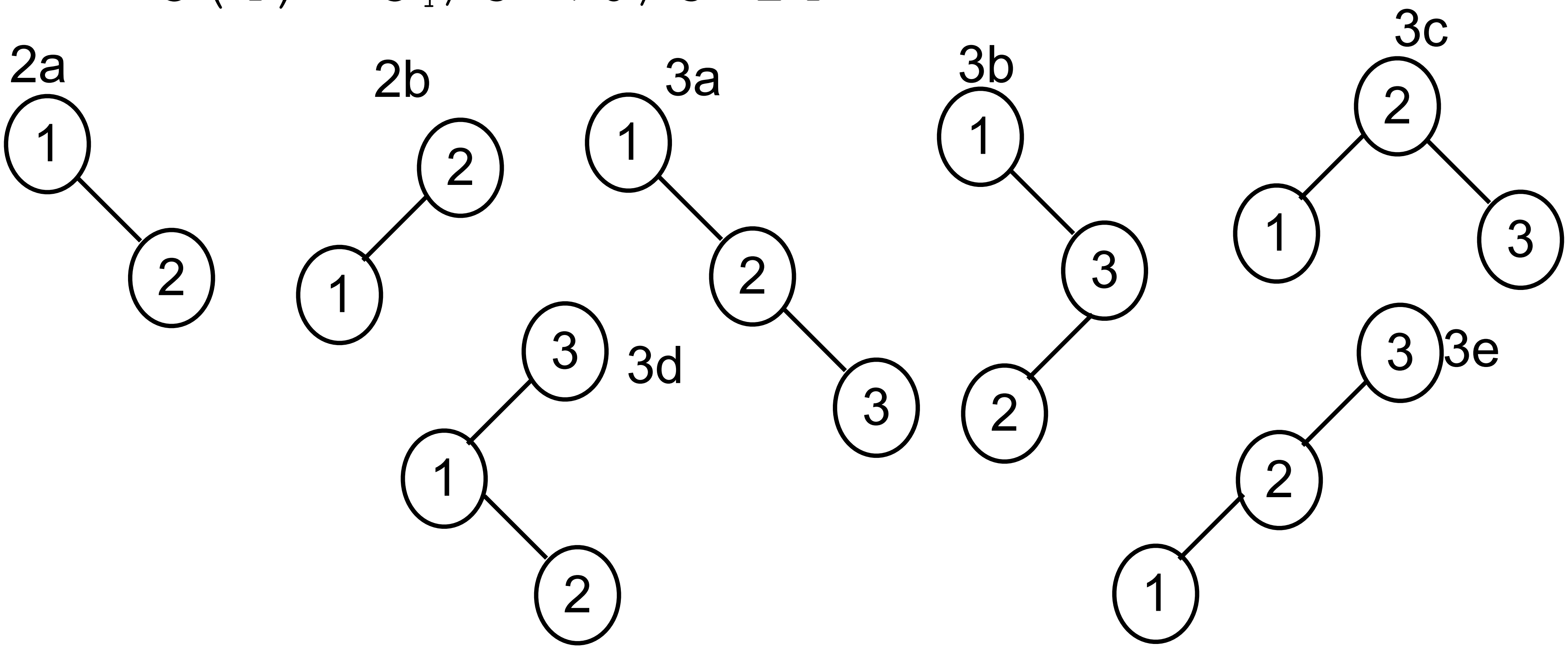
- Use case `1`:
  - You need to translate a english document containing (n words) to Kannada.
  - You have a dictionary providing kannada translation for each english word.
  - Translation process:
    - Consider each word of english document, search in the english-kannada dictionary and use the same
    - Using a generic balanced binary search tree, average translation would take $O(n.\log_2 n)$ time.
  - If we know the frequency of occurrence of each word in english document, can we do better?
    - How to optimally organize binary search tree?

Optimal Binary Search Tree

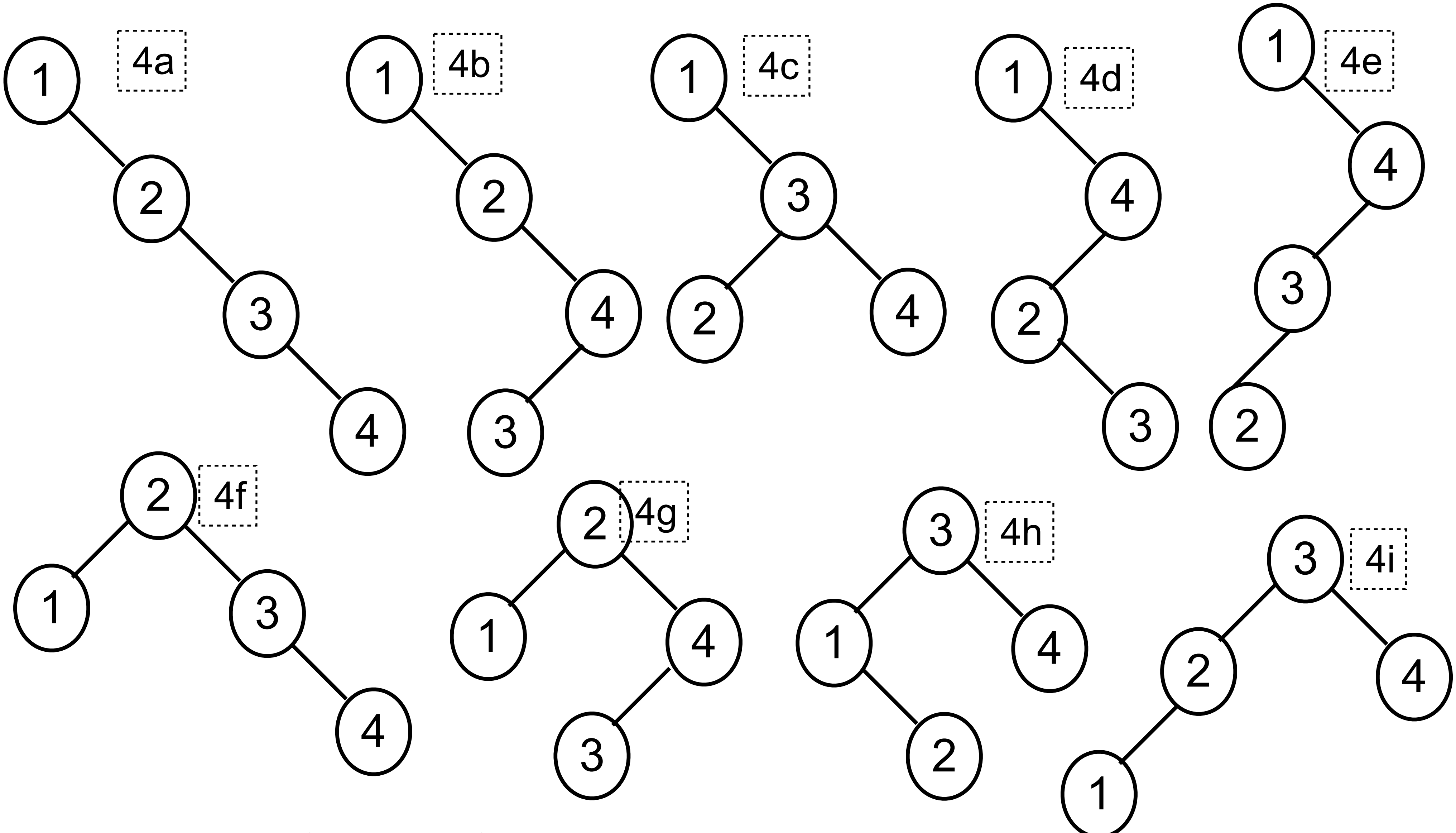
- Use case 2:
  - As an e-tailor, you are selling phones.
    - Total `n` types of brands/models etc.
  - Different customer will choose different brand/ models
  - You organize the product details (price etc) in a binary search tree.
  - In general, each search $O(n.\log_2 n)$ takes time.
  - If we know the purchase frequency of each brand/ model, can we improve upon the search time
    - How to optimally organize binary search tree?
- Objective: organize binary search tree in such a way to reduce average look up time.

Binary Search Tree

- Given `n` nodes, how many possible binary trees
  - Catalan number $C(n): {}^{2n}C_n/(n+1)$
  - $C(2)={}^{4}C_2/3=6/3=2$
  - $C(3)={}^{6}C_3/4=20/4=5$
  - $C(4)={}^{8}C_4/5=70/5=14$

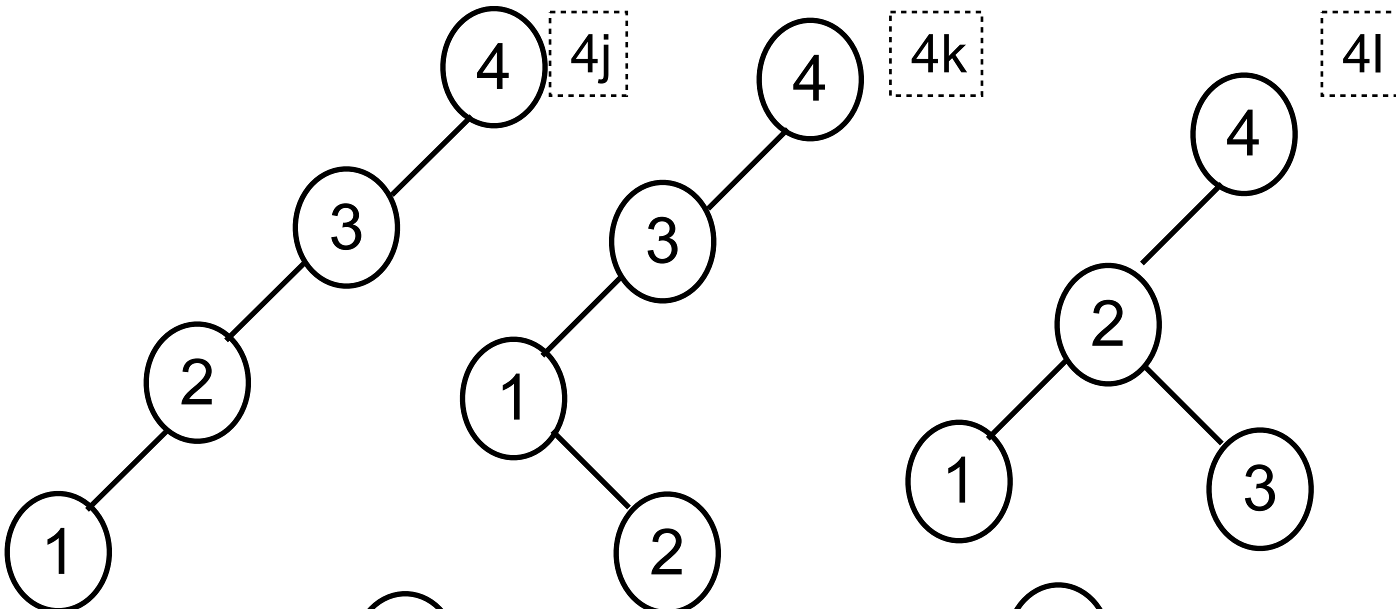


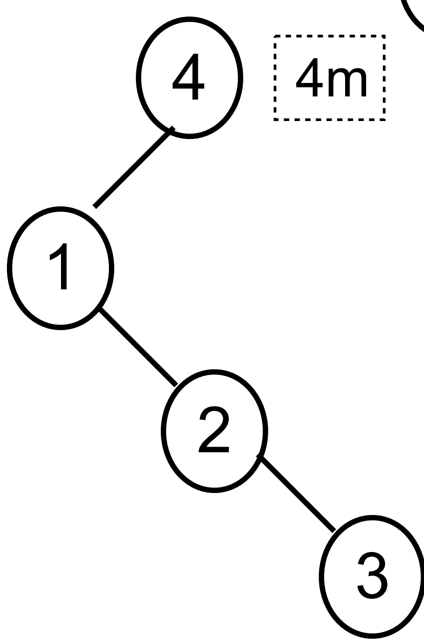
Binary Search Tree : 4 nodes…

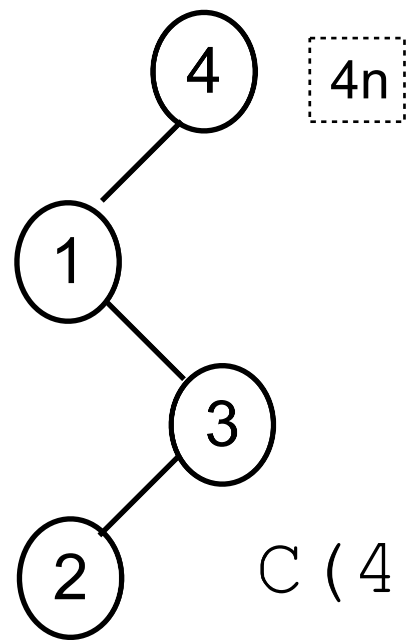


$C(4) = {}^{8}C_{4}/5 = 70/5 = 14$

Binary Search Tree : 4 nodes







$C(4) = {}^{8}C_{4}/5 = 70/5 = 14$

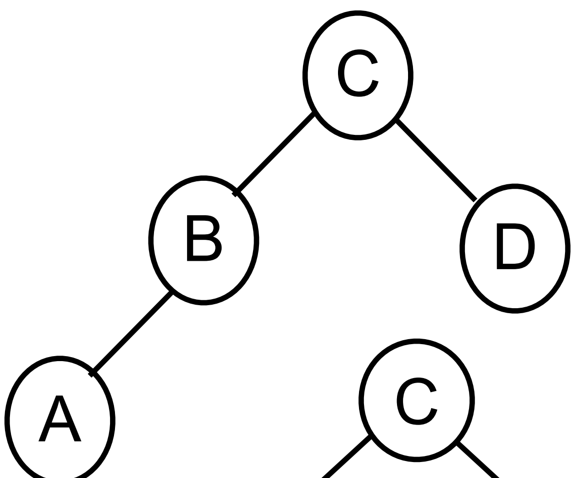
Optimal Binary Search Tree

- Problem:
  - Given n keys $a_1 \le a_2 \le \ldots \le a_n$, with
    - Probabilities of occurrences $p_1, p_2, \ldots, p_n$
  - Find a Binary Search Tree (BST) with
    - Minimum average number of comparisons in successful search
- Brute force methods
  - Total number of BST: $C(n) = {}^{2n}C_n / (n+1)$

    $= \Omega(4^n / n^{1.5})$
  - Requires exponential number of searches
    - An impractical approach

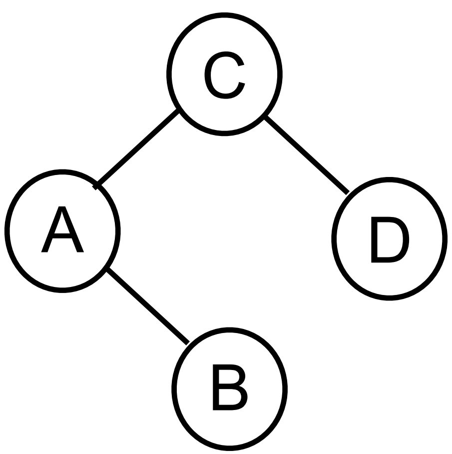
Example: BSTs

- Consider 4 keys `A,B,C,D` with their probabilities as
  – $p_A=0.1, p_B=0.2, p_C=0.4$ and $p_D=0.3$
- Compute the average number of comparisons for BSTs given below



- Average number of comparisons

=0.1*3+0.2*2+0.4*1+0.3*2
=1.7



- Average number of comparisons

=0.1*2+0.2*3+0.4*1+0.3*2
=1.8

Finding Optimal BST

- For 4 nodes, possible BSTs : `14`
  - Finding the optimal BST requires evalutaion of `14` trees
  - When probability values changes, need recomputation to find a new BST
  - With inreasing `n`, it becomes challenging
    - Requires exponential computing.
- Use of dynamic programming helps solve this issue in polynomial time.

Optimal BST: DP Approach

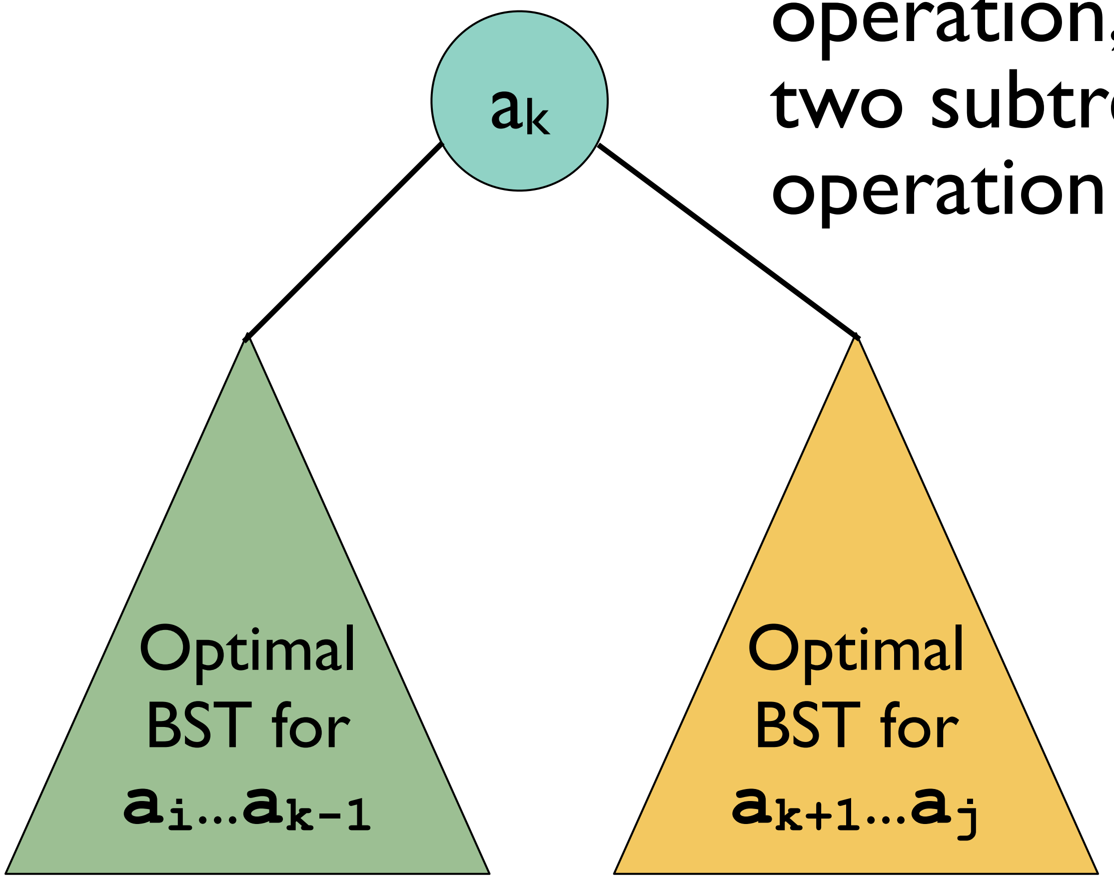
- Given n keys: $a_1 \leq a_2 \leq \ldots \leq a_n$, with
  – Respective prob. of occurrences $p_1, p_2, \ldots, p_n$
- Let T_i^j is a tree consisting of keys $a_i \leq a_{i+1} \leq \ldots \leq a_j$,
    - i, j are some integer indices $1 \leq i \leq j \leq n$.
- Let $C(i,j)$ denote the smallest number of average comparisons in a successful search for BST T_i^j.
- Thus, desired answer for our n keys would be $C(1,n)$
- Dynamic Programming approach:
  - Find smaller instances corresponding to $C(i,j)$
    - With the aim to solve $C(1,n)$

Optimal BST: DP Approach

- Solving $C(i,j)$ for T_i^j, $a_i \le a_{i+1} \le \ldots \le a_j$, $1 \le i \le j \le n$
- Derive a recurrence for $C(i,j)$.
  - Need to find the root $a_k (i \le k \le j)$ for T_i^j,
  - Consider all possible ways of choosing root a_k
    - a_k could be any node between a_i and a_j
  - To find an optimal BST with root a_k,
    - Use principle of optimality
    - Left subtree will have keys $a_i \le \ldots \le a_{k-1}$ arranged optimally
    - Right subtree will have keys $a_{k+1} \le \ldots \le a_j$ arranged optimally.

DP for Optimal BST

- Trees T_i^{k-1}, and T_{k+1}^{j}, are 1 level below the root node a_k.
- Comparison with a_k require 1 operation, comparions of keys in two subtrees need to count this operation of comparison at root a_k



DP for Optimal BST

- Recurrence for BST using DP

$$C(i,j) = \min_{i \le k \le j} \left\{ p_k.1 + \sum_{s=i}^{k-1} p_s.(level\ of\ a_s \in T_i^{k-1} + 1) + \sum_{s=k+1}^{j} p_s.(level\ of\ a_s \in T_{k+1}^{j} + 1) \right\}$$

$$= \min_{i \le k \le j} \left\{ p_k + \sum_{s=i}^{k-1} p_s.level\ of\ a_s \in T_i^{k-1} + \sum_{s=i}^{k-1} p_s + \sum_{s=k+1}^{j} p_s.level\ of\ a_s \in T_{k+1}^{j} + \sum_{s=k+1}^{j} p_s \right\}$$

$$= \min_{i \le k \le j} \left\{ \sum_{s=1}^{k-1} p_s + p_k + \sum_{s=k+1}^{j} p_s + \sum_{s=i}^{k-1} p_s.level\ of\ a_s \in T_i^{k-1} + \sum_{s=k+1}^{j} p_s.level\ of\ a_s \in T_{k+1}^{j} \right\}$$

$$= \min_{i \le k \le j} \left\{ \sum_{s=i}^{j} p_s + \sum_{s=i}^{k-1} p_s.level\ of\ a_s \in T_i^{k-1} + \sum_{s=k+1}^{j} p_s.level\ of\ a_s \in T_{k+1}^{j} \right\}$$

$$= \sum_{s=i}^{j} p_s + \min_{i \le k \le j} \left\{ C(i,k-1) + C(k+1,\ j) \right\} \qquad (1)$$

DP for Optimal BST

$$C(i,j) = \sum_{s=i}^{j} p_s + \min_{i \le k \le j} \left\{ C(i,k-1) + C(k+1,j) \right\} \qquad (1)$$

- Recurrence for BST using DP:
  - `C(i,i-1)=0` (number of comparisons in empty tree)
  - $C(i,i)=p_i*1=p_i$ since tree has only one key a_i
- Example: computation for `C(2,4)` using eqn `(1)`

$$C(2,4)=\Sigma_{2\le s\le 4}\ p_s + \min\{C(2,1)+C(3,4),\ C(2,2)+C(4,4),\ C(2,3)+C(5,4)\}$$

$$=\Sigma_{2\le s\le 4}\ p_s + \min\{0+C(3,4),\ p2+p4,\ C(2,3)+0\}$$

DP for Optimal BST

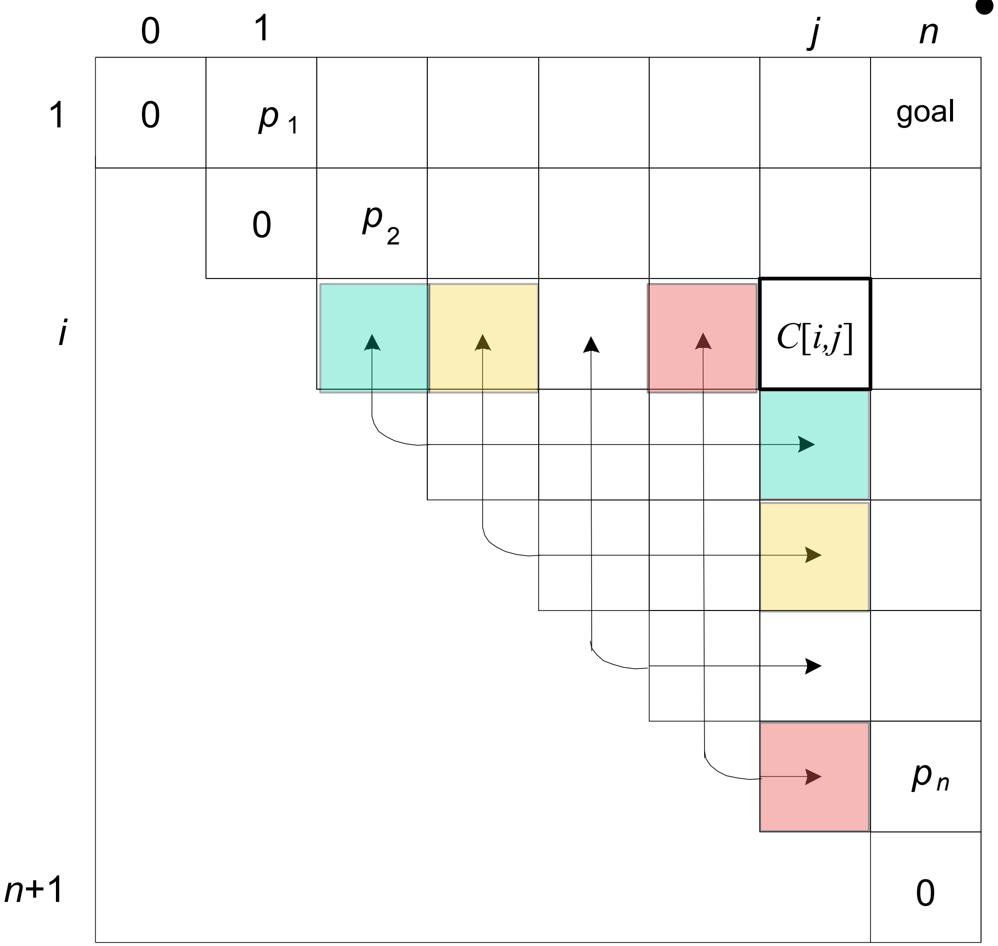
$$C(2,4) = \Sigma_{2 \le s \le 4}\, p_s + \min\{C(2,1)+C(3,4),\; C(2,2)+C(4,4),\; C(2,3)+C(5,4)\}$$

$$= \Sigma_{2 \le s \le 4}\, p_s + \min\{0+C(3,4),\; p2+p4,\; C(2,3)+0\}$$

	0	1	2	3	4	5	
1							
2		0	p_2	C(2,3)	**C(2,4)**		
3			0	p_3	C(3,4)		
4				0	p_4		
5					0		

DP for Optimal BST

$$C(i,j) = \sum_{s=i}^{j} p_s + \min_{i \le k \le j} \left\{ C(i,k-1) + C(k+1,\, j) \right\} \qquad (1)$$



- Contribution for `C(i,j)` is from

```
C(i,i-1)+C(i+1,j)
C(i,i)+C(i+2,j)
C(i,i+1)+C(i+3,j)
:
:
C(i,j-1)+C(j+1,j)
```

Exercise: Optimal BST

- Consider 4 keys : `A(1),B(2),C(3),D(4)` with their prob. as
  - $p_A=0.1,\ p_B=0.2,\ p_C=0.4,\ p_D=0.3,$

$$C(1,4) = \sum_{s=1}^{4} p_s + \min_{1 \le k \le 4} \left\{ C(1,k-1) + C(k+1,\, 4) \right\}$$

- Note: `0=C(1,0)=C(2,1)+C(3,2)=C(4,3)=C(5,4)`

Table: Computation of `C(i,j)`

	0	1	2	3	4
1	0	0.1			
2		0	0.2		
3			0	0.4	
4				0	0.3
5					0

Table: Optimal `k` for `C(i,j)`

	0	1	2	3	4
1		1			
2			2		
3				3	
4					4
5					

Exercise: Optimal BST

$C(1,2) = \Sigma_{1 \le s \le 2} p_s + \min\{C(1,0)+C(2,2), C(1,1)+C(3,2)\}$
$= 0.3 + \min\{0+0.2, 0.1+0) = 0.4$, optimal $k=2$

$C(2,3) = \Sigma_{2 \le s \le 3} p_s + \min\{C(2,1)+C(3,3), C(2,2)+C(4,3)\}$
$= 0.6 + \min\{0+0.4, 0.2+0) = 0.8$, optimal $k=3$

$C(3,4) = \Sigma_{3 \le s \le 4} p_s + \min\{C(3,2)+C(4,4), C(3,3)+C(5,4)\}$
$= 0.7 + \min\{0+0.3, 0.4+0) = 1.0$, optimal $k=3$

Table: Computation of C(i,j)

	0	1	2	3	4
1	0	0.1	0.4		
2		0	0.2	0.8	
3			0	0.4	1.0
4				0	0.3
5					0

Table: Optimal k for C(i,j)

	0	1	2	3	4
1		1	2		
2			2	3	
3				3	3
4					4
5					

Exercise: Optimal BST

$$C(1,3)=\Sigma_{1\le s\le 3}p_s+\min\{C(1,0)+C(2,3),\ C(1,1)+C(3,3),\ C(1,2)+C(4,3)$$

$$=0.7+\min\{0+0.8,0.1+0.4,0.4+0)=1.1,\ \text{opt } k=3$$

$$C(2,4)=\Sigma_{2\le s\le 4}p_s+\min\{C(2,1)+C(3,4),\ C(2,2)+C(4,4),\ C(2,3)+C(5,4)\}$$

$$=0.9+\min\{0+1.0,0.2+0.3,0.8+0)=1.4,\text{opt } k=3$$

Table: Computation of `C(i,j)`

	0	1	2	3	4
1	0	0.1	0.4	1.1	
2		0	0.2	0.8	1.4
3			0	0.4	1.0
4				0	0.3
5					0

Table: Optimal `k` for `C(i,j)`

	0	1	2	3	4
1		1	2	3	
2			2	3	3
3				3	3
4					4
5					

Exercise: Optimal BST

```
C(1,4)=Σ1≤s≤4ps+min{C(1,0)+C(2,4),
                    C(1,1)+C(3,4),
                    C(1,2)+C(4,4),
                    C(1,3)+C(5,4)}
        =1.0+min{0+1.4,0.1+1.0,0.4+0.3,1.1+0)
        =1.7, optimal k=3
```

Table: Computation of `C(i,j)`

	0	1	2	3	4
1	0	0.1	0.4	1.1	1.7
2		0	0.2	0.8	1.4
3			0	0.4	1.0
4				0	0.3
5					0

Table: Optimal `k` for `C(i,j)`

	0	1	2	3	4
1		1	2	3	3
2			2	3	3
3				3	3
4					4
5					

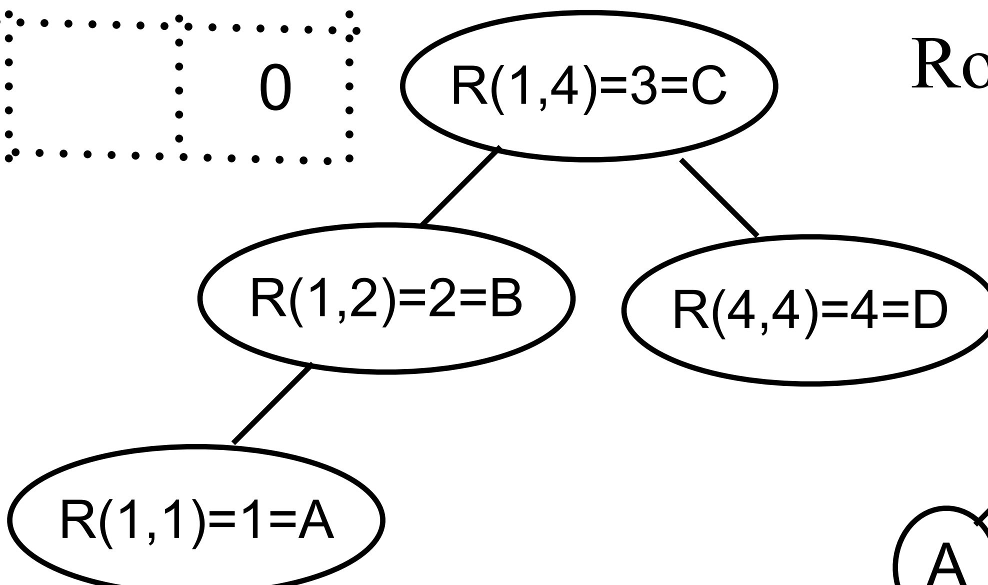
Ex: Optimal BST Construction

Table: Computation of `C(i,j)`

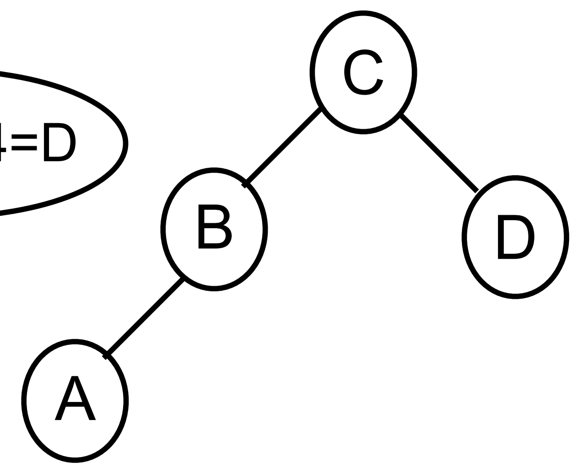
	0	1	2	3	4
1	0	0.1	0.4	1.1	1.7
2		0	0.2	0.8	1.4
3			0	0.4	1.0
4				0	0.3
5					0

Table: Optimal `k` for `C(i,j)`

	0	1	2	3	4
1		1	2	3	3
2			2	3	3
3				3	3
4					4



Root Tree



Algorithm

ALGORITHM *OptimalBST*$(P[1..n])$

```
//Finds an optimal binary search tree by dynamic programming
//Input: An array P[1..n] of search probabilities for a sorted list of n keys
//Output: Average number of comparisons in successful searches in the
//           optimal BST and table R of subtrees' roots in the optimal BST
for i ← 1 to n do
      C[i, i − 1] ← 0
      C[i, i] ← P[i]
      R[i, i] ← i
C[n + 1, n] ← 0
for d ← 1 to n − 1 do //diagonal count
      for i ← 1 to n − d do
            j ← i + d
            minval ← ∞
            for k ← i to j do
                  if C[i, k − 1] + C[k + 1, j] < minval
                        minval ← C[i, k − 1] + C[k + 1, j]; kmin ← k
            R[i, j] ← kmin
            sum ← P[i]; for s ← i + 1 to j do sum ← sum + P[s]
            C[i, j] ← minval + sum
return C[1, n], R
```

Time Efficiency: Optimal BST

- From general analysis of algo,
  - 3 nested loops, each running n times
- Thus time efficiency: $O(n^3)$
  - Space Efficiency: $O(n^2)$
- Time Efficiency: Accounting time smartly.
  - Entries in `root` (2^{nd}) table are always non-decreasing
    - Along each row and column
    - Value of `root` table entry `R[i,j]` is limited to the range `R[i,j-1]`, …, `R[i+1,j]`
    - This reduces the time complexity to $O(n^2)$

Summary

- Binary search tree
- Optimal binary search tree
- Dynamic programming for BST
- Algo: DP for BST
- Evaluation of `C(i,j)` and Tree construction