Design and Analysis of Algorithms

L27: Warshall & Floyd Algo

Dynamic Programming

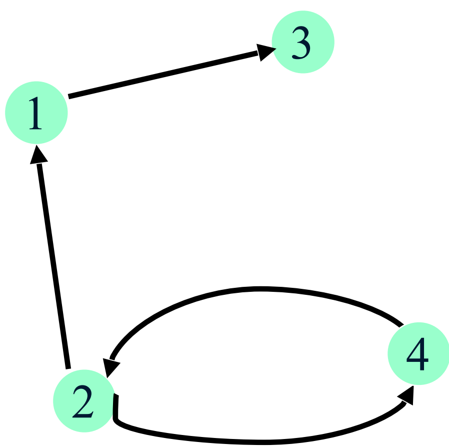
Dr. Ram P Rustagi
Sem IV (2020-Even)
Dept of CSE, KSIT
rprustagi@ksit.edu.in

Resources

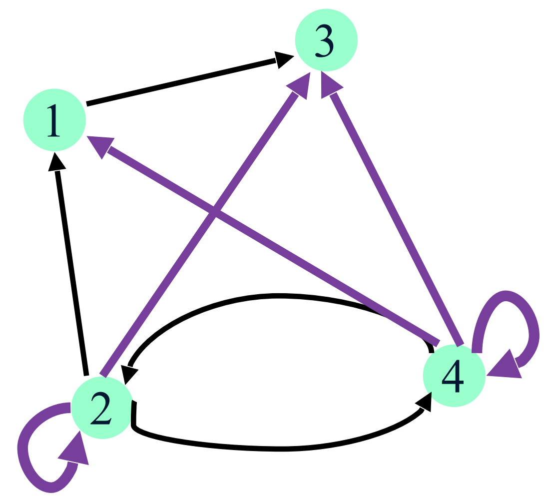
- Text book 1: Levitin
  – Sec **8 . 2**
- R1: Introduction to Algorithms
  - Cormen et al.

Transitive Closure

- Computes the transitive closure of a relation
- Alternatively:
  - Existence of all nontrivial paths in a digraph
- Example of transitive closure:



	1	2	3	4
1	0	0	1	0
2	1	0	0	1
3	0	0	0	0
4	0	1	0	0



	1	2	3	4
1	0	0	1	0
2	1	**1**	**1**	1
3	0	0	0	0
4	**1**	1	**1**	**1**

Warshall's Approach

- Constructs transitive closure T as the last matrix in the sequence of n-by-n matrices

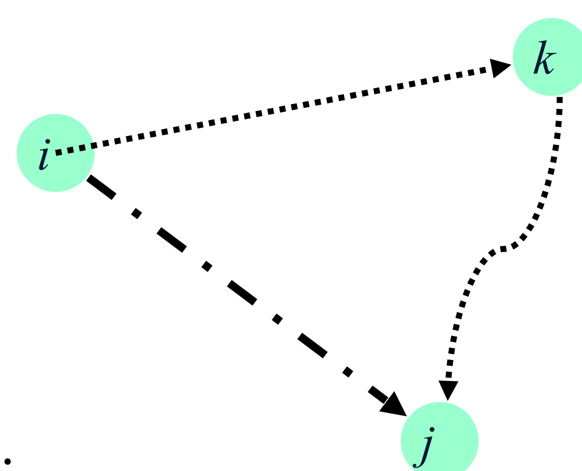
  $R^{(0)}$, … , $R^{(k)}$, … , $R^{(n)}$ where

- $R^{(k)}[i,j]=1$ iff
  - There is nontrivial path from i to j
  - Only the first k vertices (numbered from 1 to k) are allowed as intermediate
- Note that
  - $R^{(0)} = A$ (adjacency matrix),
  - $R^{(n)} = T$ (transitive closure)

Warshall's algo: Recurrence

- On the k^{th} iteration,
  - Algo determines for every pair of vertices i, j
  - If a path exists from `i` to `j`
    - Using vertices `1`, …, k only as intermediate

$$R^{(k)}[i,j] = \begin{cases} R^{(k-1)}[i,j] & \text{(path using just } \mathtt{1}, \ldots, \mathtt{k-1}\text{)} \\ \textbf{or} & \\ R^{(k-1)}[i,k] \ \textbf{and}\ R^{(k-1)}[k,j] & \\ \text{(path from } \mathtt{i} \text{ to } \mathtt{k} \text{ and from } \mathtt{k} \text{ to } \mathtt{j}\text{, using just } \mathtt{1}, \ldots, \mathtt{k-1}\text{)} & \end{cases}$$



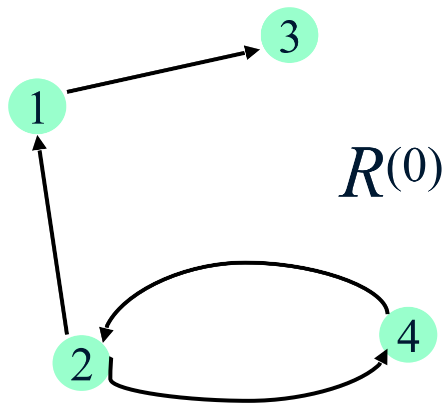
Warshall's algo: Matrix Generation

- Recurrence equation relating elements $R^{(k)}$ to elements of $R^{(k-1)}$ is:

  R(k)[i,j]=R(k-1)[i,j] **or** (R(k-1)[i,k] and R(k-1)[k,j])

- It implies the following rules for generating $R^{(k)}$ from $R^{(k-1)}$:
  - Rule 1: If an element in row i and column j is 1 in $R^{(k-1)}$,
    it remains 1 in $R^{(k)}$
  - Rule 2: If an element in row i and column j is 0 in $R^{(k-1)}$,
    - It is set to 1 in $R^{(k)}$ iff both below are 1's in $R^{(k-1)}$
      - element in its row i and column k, and
      - element in its row k and column j
- Use of Dynamic Programming
  – Computation of matrix R(i)
    - makes use of matrix R(i-1)

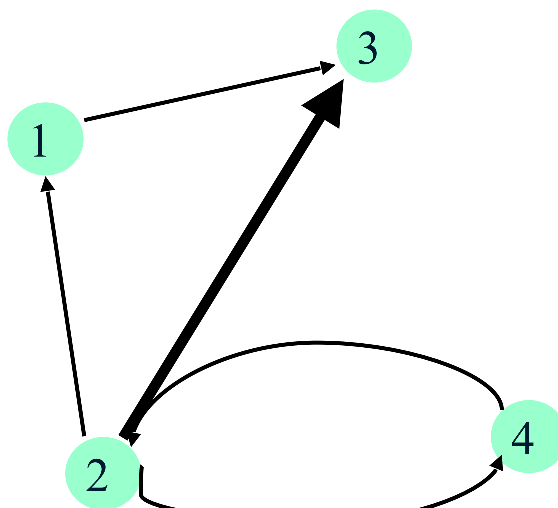
Warshall's algo: Example



$$R^{(0)} = \begin{array}{|c|ccc|} \hline 0 & 0 & 1 & 0 \\ \hline 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline \end{array}$$

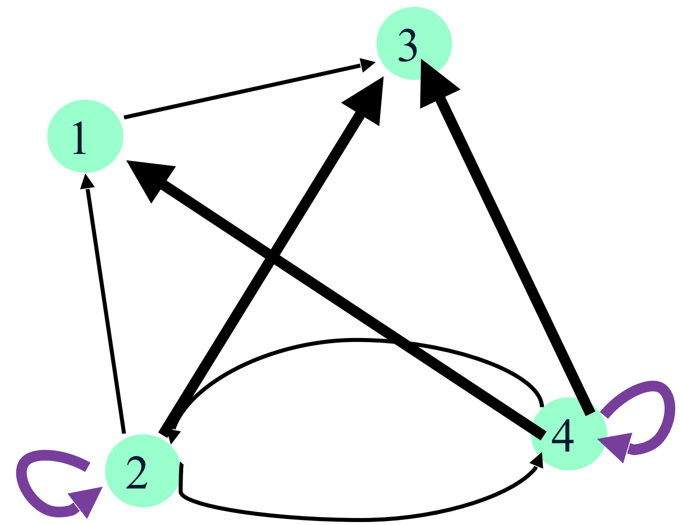
$$R^{(1)} = \begin{matrix} 0 & 0 & 1 & 0 \\ 1 & 0 & \mathbf{1} & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{matrix}$$

$R^0(2,1)=1, R^0(1,3)=1$



Warshall's algo: Example

$$R^{(1)} = \begin{matrix} 0 & 0 & 1 & 0 \\ 1 & 0 & \mathbf{1} & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{matrix}$$



$R^2(4,1)=R^1(4,2)=1, R^1(2,1)=1$
$R^2(4,3)=R^1(4,2)=1, R^1(2,3)=1$
$R^2(4,4)=R^1(4,2)=1, R^1(2,4)=1$

$$R^{(2)} = \begin{matrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ \mathbf{1} & 1 & \mathbf{1} & \mathbf{1} \end{matrix}$$

$$R^{(3)} = \begin{matrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{matrix}$$

No Change

$$R^{(4)} = \begin{matrix} 0 & 0 & 1 & 0 \\ 1 & \mathbf{1} & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{matrix}$$

$R^4(2,2)=R^3(2,4)=1, R^3(4,2)=1$

Warshall's Algo: Analysis

```
Algo Warshall(A[1..n,1..n])
// i/p: Adjacency matrix A of a diagraph with n vertices
// o/p: Transitive closure of diagraph
R(0) ← A
for k←1 to n do
    for i←1 to n  do
        for j←1 to n do
            R(k)[i,j]←R(k-1)[i,j] OR
                (R(k-1)[i,k] AND R(k-1)[k,j])
return R(n)
```

Time efficiency: $\Theta(n^3)$

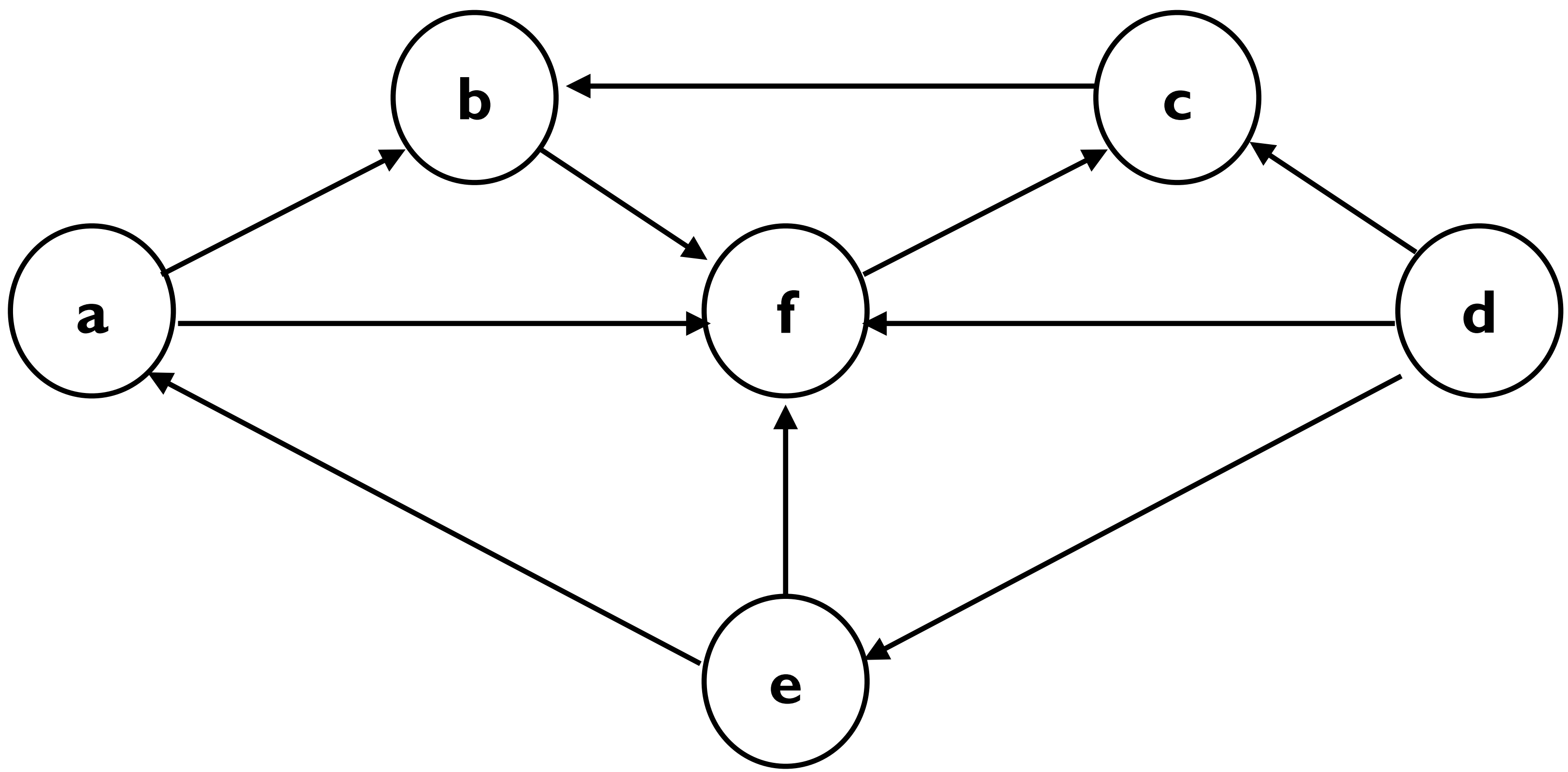
Space efficiency: General analysis - $\Theta(n^3)$

Matrices can be written over their predecessors (with some care), so it's $\Theta(n^2)$.

Warshall’s Algo

Exercise:

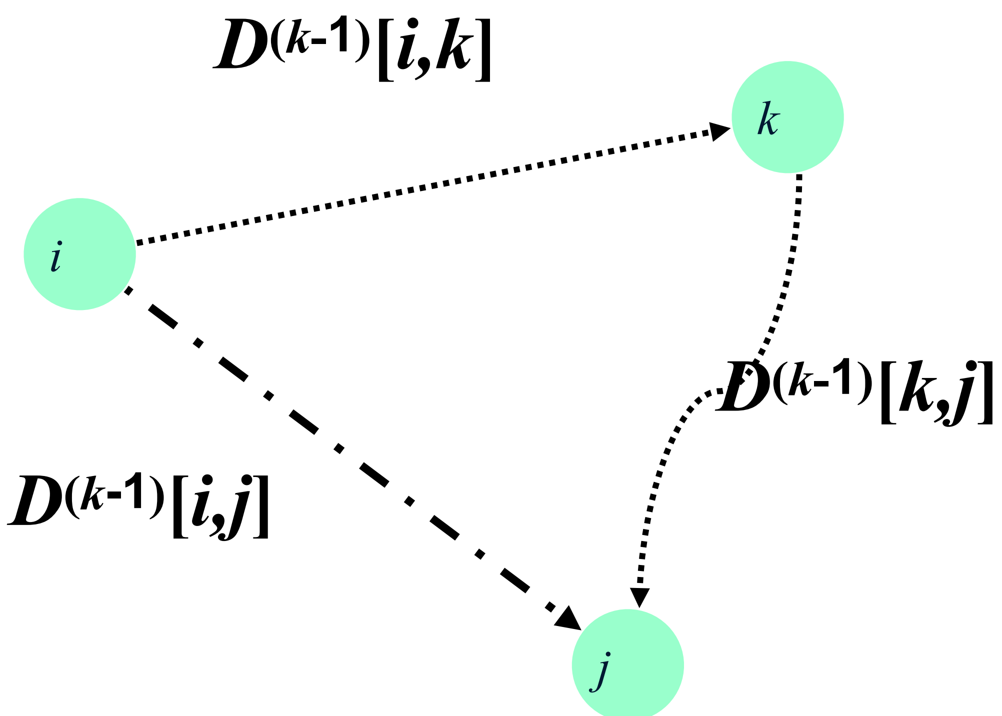
- Ex: Construct transitive closure for below graph



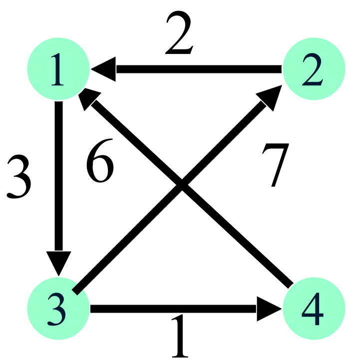
Floyd's Algorithm: Matrix Generation

- On the k^{th} iteration,
  - The algorithm determines shortest paths between every pair of vertices i, j that use only vertices among $1, \ldots, k$ as intermediate

$$D^{(k)}[i,j] = \min\{D^{(k-1)}[i,j],\ D^{(k-1)}[i,k] + D^{(k-1)}[k,j]\}$$



Example: Floyd Algo



$$D^{(0)} = \begin{bmatrix} 0 & \infty & 3 & \infty \\ 2 & 0 & \infty & \infty \\ \infty & 7 & 0 & 1 \\ 6 & \infty & \infty & 0 \end{bmatrix} \qquad D^{(1)} = \begin{bmatrix} 0 & \infty & 3 & \infty \\ 2 & 0 & \mathbf{5} & \infty \\ \infty & 7 & 0 & 1 \\ 6 & \infty & \mathbf{9} & 0 \end{bmatrix}$$

$D^1(2,3)=D^0(2,1)+D^0(1,3)=5$

$D^1(4,3)=D^0(4,1)+D^0(1,3)=9$

$D^2(3,1)=D^1(3,2)+D^1(2,1)=9$

$$D^{(2)} = \begin{bmatrix} 0 & \infty & 3 & \infty \\ 2 & 0 & 5 & \infty \\ \mathbf{9} & 7 & 0 & 1 \\ 6 & \infty & 9 & 0 \end{bmatrix} \qquad D^{(3)} = \begin{bmatrix} 0 & \mathbf{10} & 3 & \mathbf{4} \\ 2 & 0 & 5 & \mathbf{6} \\ 9 & 7 & 0 & 1 \\ 6 & \mathbf{16} & 9 & 0 \end{bmatrix} \qquad D^{(4)} = \begin{bmatrix} 0 & 10 & 3 & 4 \\ 2 & 0 & 5 & 6 \\ \mathbf{7} & 7 & 0 & 1 \\ 6 & 16 & 9 & 0 \end{bmatrix}$$

$D^3(1,2)=D^2(1,3)+D^2(3,2)=10$

$D^3(2,4)=D^2(2,3)+D^2(3,4)=6$

$D^3(4,2)=D^2(4,3)+D^2(3,2)=16$

$D^4(3,1)=D^3(3,4)+D^3(4,1)=7$

Floyd Algo: Analysis

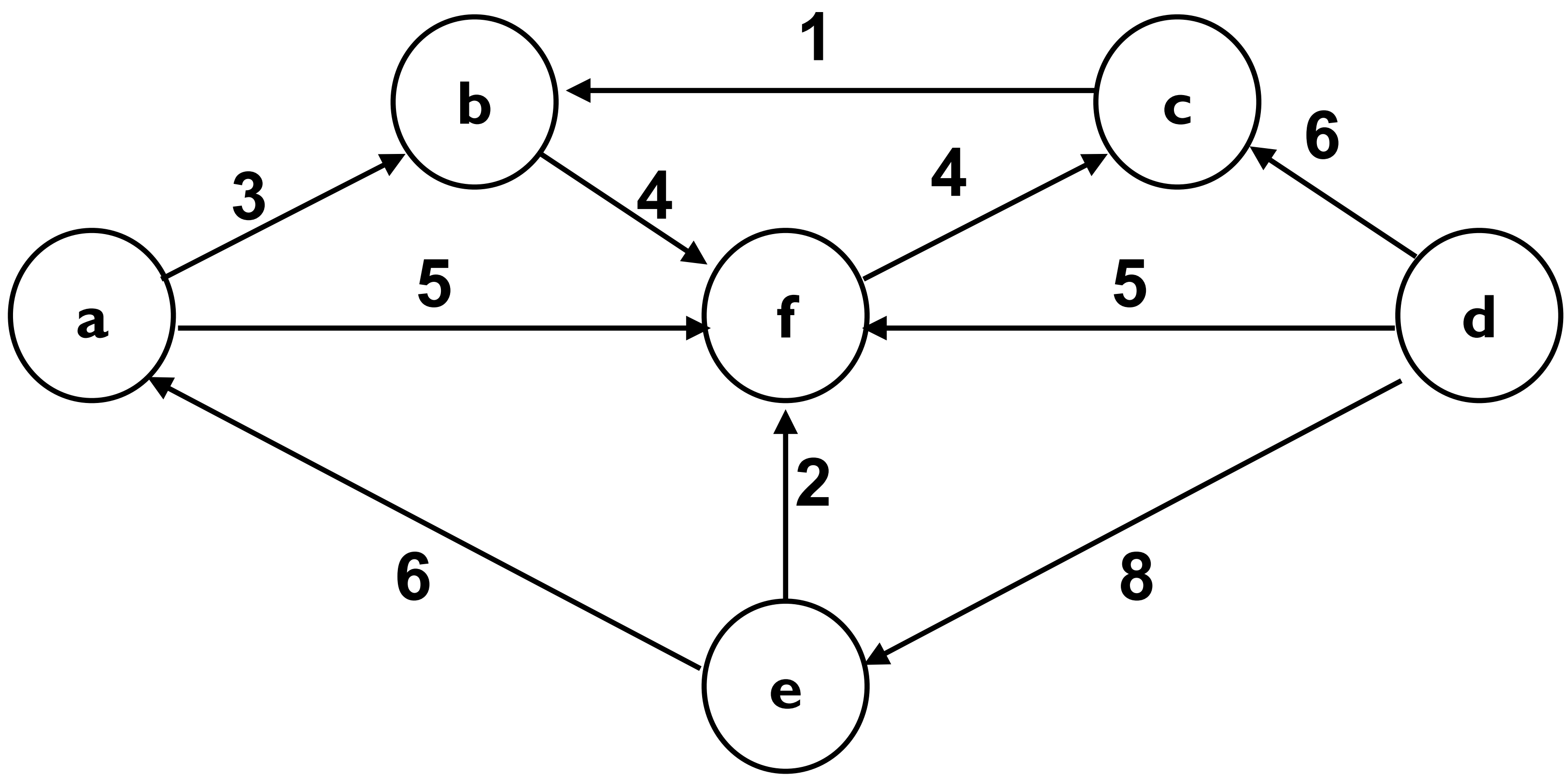
```
Algo Floyd(A[1..n,1..n])
// i/p: Weight matrix W of a diagraph A with n vertices
// o/p: Distance matrix D of shortest path lengths
//      Precedence matrix P to know predecessor vertex
D ← W  // not necessary, if W can be overwritten.
for k←1 to n do
    for i←1 to n  do
        for j←1 to n do
            if D[i,k]+D[k,j] < D[i,j] then
                P[i,j] ← k
                D[i,j]←D[i,k]+D[k,j]
return D
```

Time efficiency: $\Theta(n^3)$

Space efficiency: $\Theta(n^2)$.

Exercise:

- Ex: Find all pair shortest distance for below graph



Summary

- Transitive closure
- Warshall Algorithm
- Floyd Algorithm