Design and Analysis of Algorithms

L10: MaxMin (Divide & Conquer)

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Resources

- Text book 2: Horowitz (MaxMin)
- Text book 1: Levitin (Mergesort)
- https://www.tutorialspoint.com/design_and_analysis_of_algorithms/design_and_analysis_of_algorithms_max_min_problem.htm

MaxMin

- Problem: Given a set of N elements, find the max and min of these elements
  - Assume that these elements are in an array of size N
  - Element could be complex and comparison cost is not negligible e.g. address matching, URL name etc.
- Approaches
  - Naive approach
    - Just using simple looping and iterate over all elements
    - Each iteration, compares curr min/max and update
  - Optimization to Naive approach
    - If num is less than Min, then max does not change
    - If num is greater than Max, Min does not change.
  - Divide and Conquer approach

Finding Max: Naive approach

- Algo `Max(A[])`

```
max=A[1]
for i←2 to N do
  if A[i] > Max, then
    max=A[i]
return(max)
```

- Time complexity analysis
  - `N-1` comparisons (cost contributing operation)
  - `N` assignments.
  - Time Complexity: `C(n)=n`

Finding Min: Naive approach

- Algo `Min(A[])`

```
min=A[1]
for i←2 to N do
 if A[i] < Min, then
  min=A[i]
return(min)
```

- Time complexity analysis
  - `N-1` comparisons (cost contributing operation)
  - `N` assignments. (non contributing)
  - Time Complexity: `C(n)=n`

MaxMin: Naive approach

- Algo `MaxMin(A[])`

```
max=A[1]
min=A[1]
for i←2 to N do
 if A[i] < min, then
  min=A[i]
 if A[i] > Max, then
  max=A[i]
return(min, max)
```

- Time complexity analysis
  - `2(N-1)` comparisons (contributing operation)
  - `N+1` assignments (non contributing)
  - Time Complexity: `C(n) = 2(n-1)`

MaxMin: Optimized Naive approach

- Algo `MaxMin(A[])`

```
max=A[1]
min=A[1]
for i←2 to N do
 if A[i] < min, then
  min=A[i]
 else if A[i] > Max, then
  max=A[i]
return(min, max)
```

- Time complexity analysis
  - Comparisons vary for best case and worst case

MaxMin: Optimized Naive approach

- Time complexity analysis
- Best case:
  - Elements are sorted in descending order.
  - Comparison of **Min** always succeeds
  - Comparison of **Max** never occurs
  - Time complexity `C(n) = (n-1)`
- Worst case:
  - Elements are sorted in ascending order.
  - Comparison of **Min** always fails
  - Comparison of **Max** invoked every time
  - Time complexity `C(n) = 2(n-1)`

MaxMin

- Approach : Divide and Conquer
  - Divide the array into two halves
  - Find maximum for each half
  - Find minimum for each half
  - Compare the max values of two halves
    - Larger `max` will be desired max
  - Compare the min values of two halves
    - Smaller `min` will be the desired min

MaxMin: Divide and Conquer

- Algo `MaxMin(i,j,A[])`
  - `i`: lower array index on the left
  - `j`: higher index on the right

```
if (i==j) # small input array, recursion ends
  max=min=A[i]
elif (i=j-1) #small input array, recursion ends
  if A[i] < A[j]
    max = A[j]
    min = A[i]
  else
    max = A[i]
    min = A[j]
  fi
fi
# input array is not small, divide and conquer
```

MaxMin: Divide and Conquer…

```
# input array is not small, divide and conquer
mid = (i + j)/2
(max1, min1) ← MaxMin(i,mid, A[])
(max2, min2) ← MaxMin(mid+1, j, A[])
if (max1 < max2) then
   max = max2
else
   max = max1
if (min1 < min2) then
   min = min1
else
   min = min2
return (max, min)
```

MaxMin: Divide and Conquer…

- Complexity analysis
  - When input size N is 1, no comparison i.e. $C_N=0$
  - When input size N is 2, one comparison i.e. $C_N=1$
  - When input size N is >2,
    - Two invocations of algo with input size N/2
    - Two comparisons (one of max, one for min)
- Thus,

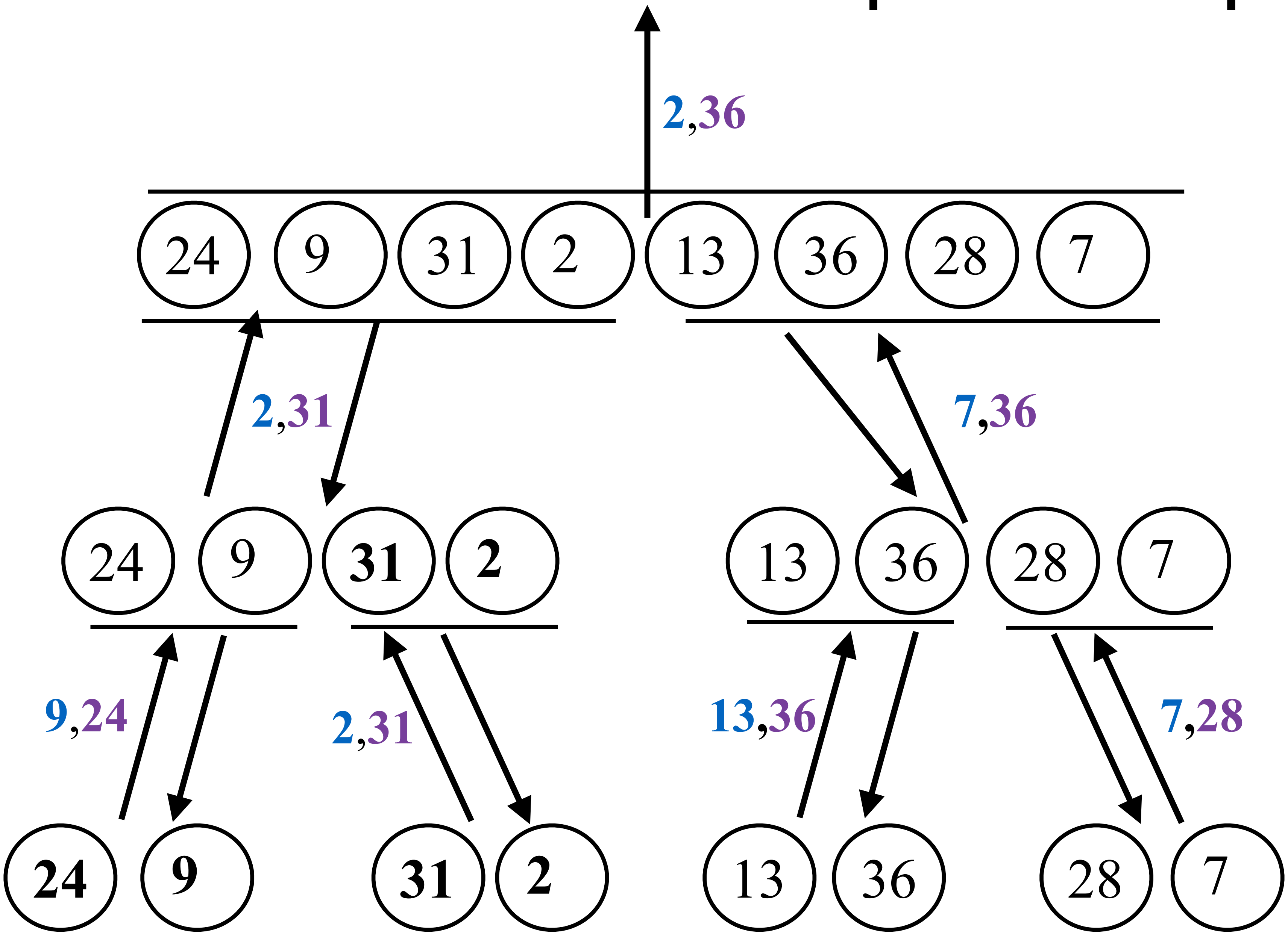
$$T(n) = \begin{cases} 0, & n = 1 \\ 1, & n = 2 \\ T(n/2) + T(n/2) + 2, & n > 2 \end{cases}$$

MaxMin :Complexity Analysis

- Let $n = 2^k$
- Then,

$$
\begin{aligned}
T(n) &= 2T(n/2)+2 \\
&= 2[2T(n/4)+2]+2 \\
&= 2^2T(n/2^2)+2^2+2 \\
&= \ldots \\
&= 2^{k-1}T(n/2^{k-1})+2^{k-1}+\ldots+2^2+2 \\
&= 2^{k-1}T(2)+2^{k-1}+\ldots+2^2+2 \\
&= 2^{k-1}*1+(2^{k-1}+\ldots+2^2+2+1)-1 \\
&= 2^{k-1}+(2^k-1)-1 \\
&= (n/2) + n - 2 = 3n/2 - 2
\end{aligned}
$$

MaxMin: Divide and Conquer Example



Analysis: MaxMin Div & Conq

- Consider when comparison of indices `i` and `j` are of equal cost to that comparing elements.
  - Then, for small input size (`i=j`, or `i=j-1`), then

  $C(n) = 2$

  - And for larger input size

$$
\begin{aligned}
C(n) &= 2C(n/2)+3 \\
&= 2^2C(n/2^2)+6+3 \\
&= 2^2C(n/2^2)+(2^1+2^0)3 \\
&= \ldots \\
&= 2^{k-1}C(n/2^{k-1})+3(2^{k-2}+2^1+2^0) \\
&= 2^{k-1}C(2)+3(2^{k-1}-1) \\
&= n/2*2+3(n/2-1) \\
&= 5n/2 - 3
\end{aligned}
$$

- For naive approach (using loop comparison): $3(n-1)$

Summary: MaxMin

- When elements comparison is much more costly than integer comparison (loop variables)
  - Divide & Conquer is more efficient
    - Actually, it is an optimal strategy
- When elements comparisons are of similar cost, then overhead of recursion overheads (stacking of variables etc) will not yield much benefits.
- Use Divide and Conquer as a guide to develop better algorithm,
  - but it is not necessarily true always.