Design and Analysis of Algorithms

L23: Huffman Codes
Optimal Tree Subproblem

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Resources

- Text book 1: Sec 9.1-5.4 - Levitin
- R1: Introduction to Algorithms
    - Cormen et al.

Data Compression

- Consider saving a text file consisting of alphabets
  - It uses ASCII encoding,
  - Each character uses 7 bits, e.g.
    - 'A'— '100001', 'B'— '1000010'
  - Thus, if file has 1000 characters,
    - file size is 7000 bits
- We know in english text certain characters appear more often than others e.g. a, e, i, t, h, s etc.
  - Other characters appear less often e.g. z, x, q etc.
- Can we use a different representation than ASCII
  - Assign shorter codes to chars occurring frequently
  - Assign longer codes to chars occurring less times.
  - Will we save disk space?

Data Compression: Communication

- Consider choosing electives with percentage of students
  - `17CS561`: Java Programming `(25%)`
  - `17CS562`: Artificial Intelligence `(12.5%)`
  - `17CS563`: Embedded Systems `(6.25%)`
  - `17CS564`: Dot Net framework `(6.25%)`
  - `17CS565`: Cloud computing `(50%)`
- A general coding would require 3 bits, e.g.

  `000` - Java Programming, `001` - AI

  `010` - Embedded Systems, `011` - DotNet framework

  `100` - Cloud Computing
- For `100` students, we need 3 bits. e.g. for first 5 students

  001000000010100—(AI,Java,Java,Embedded,Cloud)
- Can we employ better encoding so that average bits becomes less than 3.

Data Compression: Communication

- Consider choosing electives with percentage of students
  - Java Programming (25%), 17CS562:AI (12.5%)
  - Embedded (6.25%), Dot Net (6.25%)
  - Cloud computing (50%)
- Consider following encoding

  0 - Cloud Computing, 1 - Java Programming

  01 - AI, 11 - Embedded Systems

  10 - DotNet framework
- Coding scheme for (AI,Java,Java,Embedded,Cloud)

  0111110
  - Can it be read as : Cloud, Embedded,Embedded, DotNet
  - Need a separator between two encodings
    - increases encoding space
- What is the average number of bits for encoding these?

  3*1/2+2*1/4+3*1/8+3*1/16+3*1/16=2.75

Data Compression: Communication

- Consider choosing electives with percentage of students
  - Java Programming (25%), 17CS562:AI (12.5%)
  - Embedded (6.25%), Dot Net (6.25%)
  - Cloud computing (50%)
- Consider following encoding (prefix-free)

  0 - Cloud Computing, 10 - Java Programming

  110 - AI, 1110 - Embedded Systems

  1111 - DotNet framework
- Coding scheme for (AI,Java,Java,Embedded,Cloud)

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  - Can it be misinterpreted i.e. ambiguous. No.
- What is the average number of bits for encoding these?

  1*1/2+2*1/4+3*1/8+4*1/16+4*1/16

  =15/8=1.875

What is a Coding Problem…

- Coding: assignment of bit strings to alphabets
- Codewords: bit strings assigned for characters of alphabet. Example:
  - if `P(a)=0.4,P(b)=0.3,P(c)=0.2,P(d)=0.1`
  - `ASCII` codes can be assigned as
    `a:00, b:01, c:10, d:11`
    Number of bits for each code is 2 (avg is 2 bits too)
  - Codes can be assigned as
    `a:0, b:10, c:110, d:111`
  - Average length of this coding scheme is
    = `1*0.4 + 2*0.3 + 3*0.2 + 3*0.1 = 1.9` bits

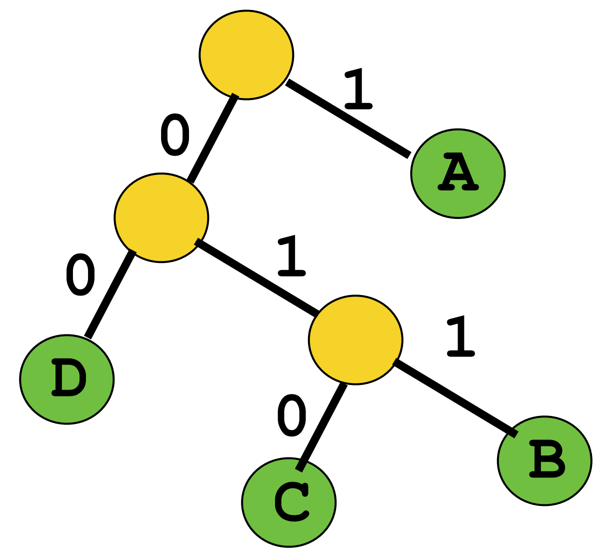
What is a Coding Problem…

- Two kind of encodings:
  - Fixed encoding e.g. `ASCII`
  - Variable length encoding: Morse encoding (dots, dashes)
    - Morse encoding is not prefix free, need separator
- Prefix free codes
  - No codeword is prefix of another code
  - Allows for efficient decoding.
- Problem: If the frequency of character occurrences are known, what is the best binary prefix code?
  - Best: Shortest average code length
  - Average code lengths represents expected number of bits required to transmit/store a character.

Huffman Codes

- Any binary tree with edges labeled as `0, 1`
  - Provides a prefix code for characters assigned to leaves
  - Just concatenate the label of edges on the path from root to a vertex
  - Example:

```
A: 1
B: 011
C: 010
D: 00
```



- Optimal binary tree can be constructed using Huffman's algorithm

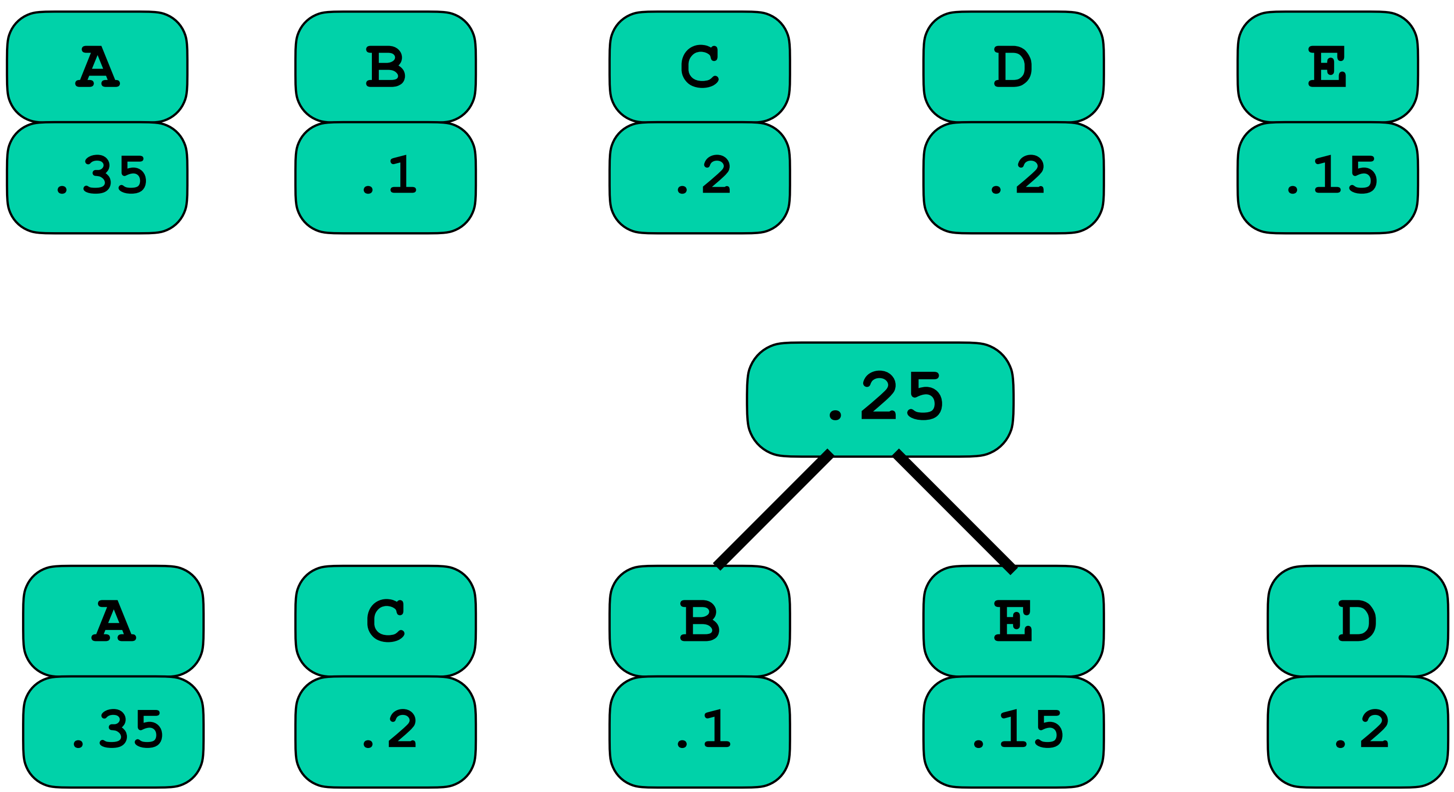
Huffman's Algorithm

- Initialize `n` one node trees with alphabet characters
  – Assign tree weights as character frequencies
- Repeat the following steps `n-1` times
  – Join two binary trees with smallest weights into one binary tree
    - One tree would become left subtree
    - Other tree would become right sub-tree
  - Make the weight of new binary tree (after joining) as equal to sum of weights of its sub trees.
    - Mark the edge joining left subtree with label `0`
    - Mark the edge joining right subtree with label `1`

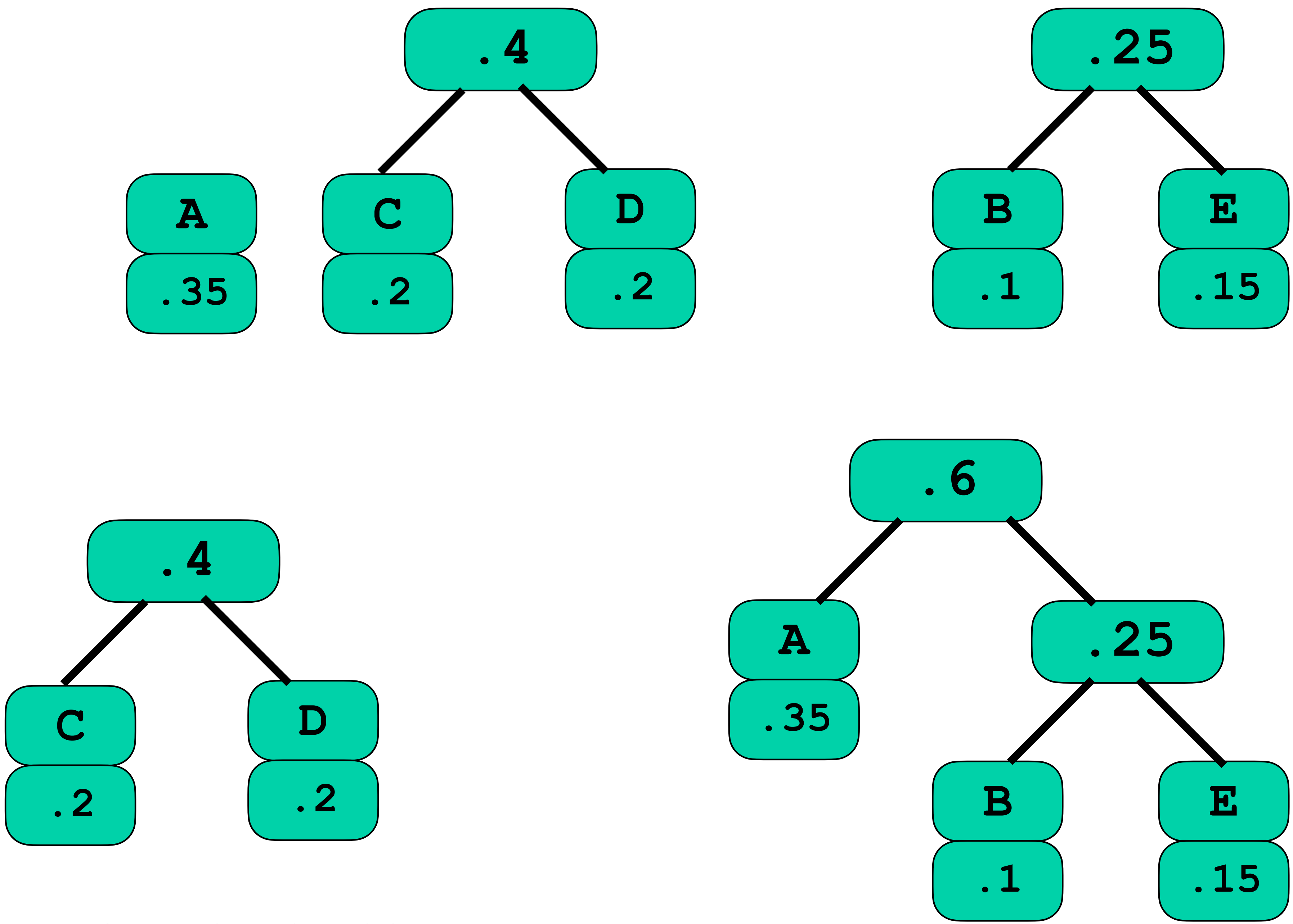
Example: Huffman Tree

- Character frequences (probabilities)

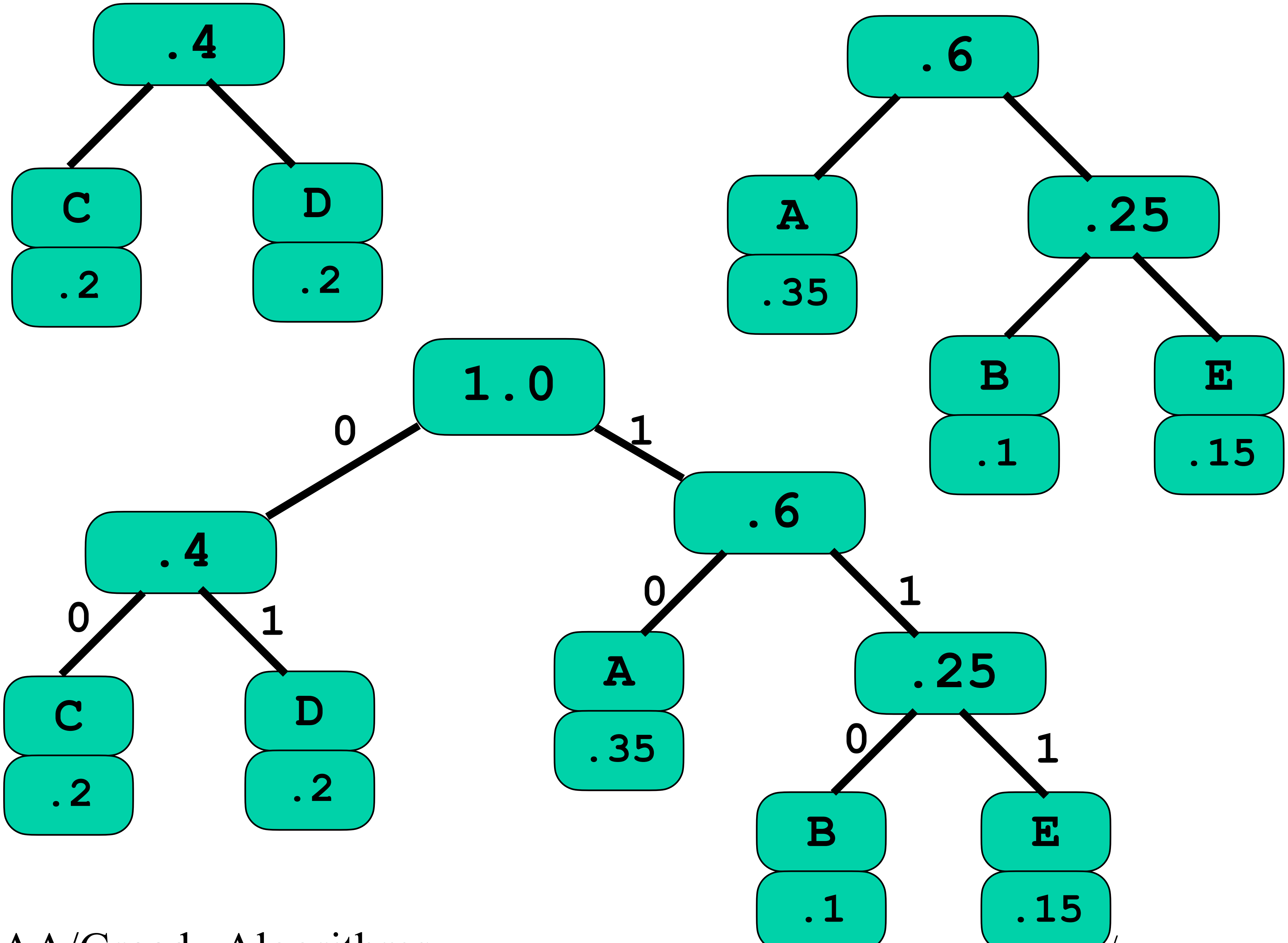
  `A:0.35, B:0.1, C:0.2, D:0.2, E:0.15`



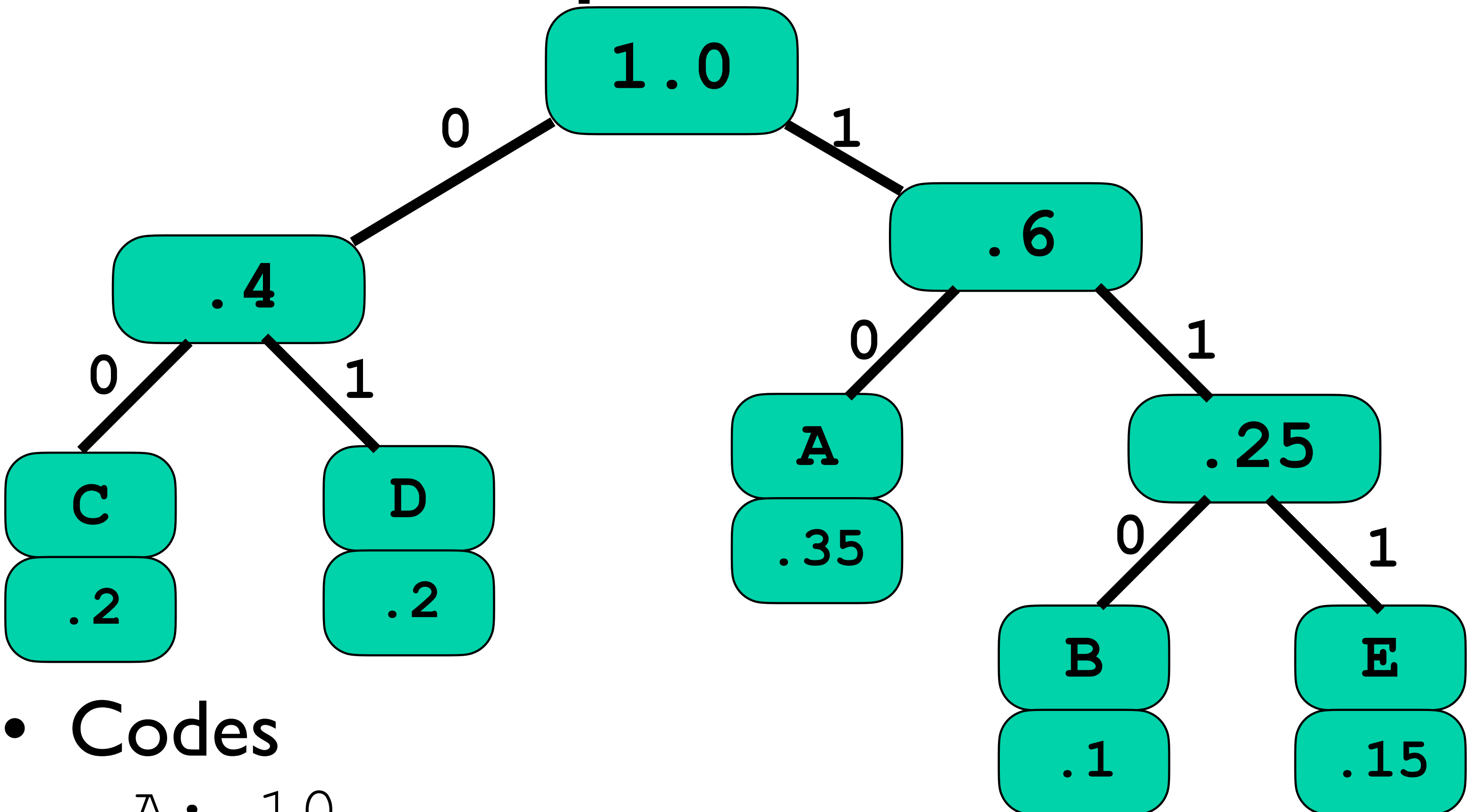
Example: Huffman Tree



Example: Huffman Tree



Example: Huffman Tree



- Codes

```
A: 10
B: 110
C: 00
D: 01
E: 111
```

Note: Characters with

Higher freq. (prob.) have shorter codes

Shorter freq.(prob.) have longer codes.

Huffman Tree: Observations

- Obs 1: Code tree is a full tree i.e. each non-leaf node has exactly two children.
  - – If a node has only 1 child, then promote the child to up level and thus create a shorter tree.
- Obs 2: In an optimal tree, if a leaf labelled as x is at a smaller depth than a leaf labelled y, then following holds for their frequencies (occurrence) f(x), f(y)
  - f(x) >= f(fy)
    - – if f(y) > f(y), then exchange the nodes.
- Obs 3: If a leaf at maximum depth labelled as x, then its sibling is also a leaf
  - – if it is not, then sibling has children and therefor x can not be at maximum depth.

Building a Solution

- From obs 3, leaves at maximum depth occurs in pairs.
- From obs 2, the leaves at maximum depth must have smallest frequencies
- Solution:
  - Pick the nodes with smallest frequencies and start building up.

Example: Huffman Tree

- Character frequences (probabilities)

  `A:0.35, B:0.1, C:0.2, D:0.2, E:0.15`
- Codes

  `A: 10, B: 110, C: 00, D: 01, E: 111`
- Average code length

  `2*.35 + 3*.1 + 2*.2 + 2*.2 + 3*.15`

  `= 0.70 + 0.3 + 0.4 + 0.4 + 0.45 = 2.25`
- Code length for fixed length coding : 3
  - 5 characters, require 3 bits
- Compression ratio:
  - `(3.0 - 2.25)/3.0*100% = 25%`
- Q: Represent character squence ACDBA

Huffman Tree/Codes

- Some characteristics
  - Codewords size of two least freq characters is same
    - They are at same level
    - What happens when more than 2 least frequent chars?
  - Codeword length of a more frequent character is always smaller than codeword length of less frequent characters. Proof by contradiction.
  - If alphabet's frequency is sorted,
    - Huffman tree can be constructed in Linear time
  - The max length of a codeword in huffman encoding of `n` characters can be `n-1`. Consider when each frequency is different.

Algorithm: Huffman Code

Algo `Huffman(W[0:n-1])`

// i/p: an array `W[0:n-1]` of weights

// o/p: A Huffman tree with leaves having weights of W

Initialize Priority Queue `Q` (heap) of size `n` with `1`−node trees
and weights equal to elements of `W[0:n-1]`

while `Q` has more than `1` element *do*

T_1 ← minimum weight tree in `Q`

delete T_1 from `Q`

T_2 ← minimum weight tree in `Q`

delete T_2 from `Q`

create a new tree `T` with T_1/T_2 as left/right subtree of `T`
with $weight(T)=weight(T_1)+weight(T_2)$

Insert `T` into `Q`

Complexity Analysis (General)

- In each iteration
  - Removing two tree (each removal is *deletemin*)
  - Adding one tree (*insert*)
  - Effectively reducing number of trees by `1`.
- Total iteration = `n-1`
  - Removing min weight node takes (`lg n`)
  - Insertion of a node takes (`lg n`)
  - Thus, time complexity `O(n*lg n)`

Huffman Coding

- Is this greedy approach? Why?
  – We take min weight node each time (being greedy)
  –

Quiz (NPTel)

- Consider huffman code for four letters `a`, `b`, `c`, and `d` with `f(a)>f(b)>f(c)>f(d)`. Which of the followings are `True`?
  a. The Huffman code will assign two bits to every letter always.
  b. If `f(c)+f(d) > f(b)`, then Huffman code will assign two bits to every letter, otherwise the letters will be encoded with varying numbers of bits.
  c. If `f(c)+f(d) > f(a)`, then Huffman code will assign two bits to every letter, otherwise the letters will be encoded with varying numbers of bits
  d. If `f(b) > f(c)+f(d)`, then Huffman code will assign two bits to every letter, otherwise the letters will be encoded with varying numbers of bits
- Answer: option c.

Summary

- Huffman codes
- Huffman tree
- Algo
- Complexity analysis