Design and Analysis of Algorithms

L03: Analysis Framework

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Resources

- Text book 1: Levitin
- https://brainly.com/

Algorithm Analysis

- Analysis
  – Detailed examination of the elements or structure of something, typically as a basis for discussion or interpretation.
  – *Mathematics:* the part of mathematics concerned with the theory of functions and the use of limits, continuity, and the operations of calculus.
- Analysis of algorithms:
  – Investigation of an algorithm's efficiency w.r.t. running time and memory space.

Algorithm Analysis

- Issues:
  - Correctness
  - Time efficiency
  - Space efficiency
  - Optimality
- Approaches:
  - Theoretical analysis
  - Empirical analysis
- From practical point of view:
  - Efficiency concerns are primary
  - Space (memory) is no more an issue today
    - Difference: secondary, primary, cache
  - We will mostly study time efficiency

Time v/s Space Efficiency

- Consider multiplication of any two digit numbers
  - Example: Multiply 79 * 67
- Time efficiency requires computation
  - Number of multiplications (single digit): 4
  - Number of additions (single digit): 5 (or 4)
  - Total mathematical operations: 9
  - Total space requirement: 14 digits
- Space (memory) efficiency
  - Store all multiplication values in an 100x100 array.
  - Then do a lookup.
  - Time requirement: 2 lookup
  - Space requirement: 10000 memory locations

Measuring Input Size

- Representing a polynomial:
  - $P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$
  - I/p size: $n+1$
    - or n ($+1$ is fixed and ignored, inconsequential)
- At times choice of parameter specifying the input size is important
  - Consider $n \times n$ matrix multiplication
  - Is input size n or n^2?
  - If considering matrix order: size is n
  - If consider number of elements: size is n^2
  - Latter is preferred as it works for $n \times m$ matrix too

Input Size and Operation Examples

Problem	**Input size measure**	**Basic operation**
Searching for key in a list of n items	Number of list's items, i.e. n	Key comparison
Multiplication of two matrices	Matrix dimensions or total number of elements	Multiplication of two numbers
Checking primality of a given integer n	number of digits (in binary representation)	Division
Typical graph problem	number of vertices and/or edges	Visiting a vertex or traversing an edge

Empirical Analysis of Time Efficiency

- Select a specific (typical) sample of inputs
- Choices:
  - Physical unit of time (e.g., milliseconds), or
  - Count actual number of basic operation's executions
- Physical unit of time varies depending upon computer being used, operations available
  - Count of basic operations is considered.
    - This is further approximated, need not be actual value
- Analyze the empirical data

Theoretical Analysis: Time Efficiency

- Time efficiency is analyzed by determining the count of number of the *basic operation* as a function of *input size*
- Basic operation: the operation that contributes the most towards the running time of the algorithm
  - $T(n) \approx c_{op} C(n)$
    - c_{op} is cost of operation
    - $C(n)$ represents number of operations

Time Efficiency

- Consider sum of n numbers
  - Total sum operations $C(n) = n-1 \approx n$
  - $T(n) = c_{op}C(n) = c_{op}n$
  - $T(2n) = c_{op}C(2n) = c_{op}(2n) = c_{op}2n$
  - $T(2n)/T(n) = 2$
  - $T(10n)/T(n) = 10$
- Summary: c_{op} does not play a role when comparing the performance on two inputs.
  - **Order of growth** is more important.
    - Lower order terms and constants multiple are not important
    - These are subsumed in **order of growth.**

Best, Worst and Average case

- For some algorithms, efficiency depends on form of input:
- Worst case: $C_{worst}(n)$ – max over inputs size n
- Best case: $C_{best}(n)$ – min over inputs of size n
- Avg case: $C_{avg}(n)$ – "average" over inputs of size n
  - Number of times the basic operation will be executed on typical input
  - NOT the average of worst and best case
  - Expected number of basic operations considered as a random variable under some assumption about the probability distribution of all possible inputs.
  - So, avg = expected under uniform distribution.

Example: Sequential Search

- Algorithm: SequentialSearch(`A[0..n-1]`,`K`)

```
// searches for a value K in a given array by sequential searching
// Input:  an array A[0..n-1], and key K
// Output: Index of the element that matches K
//           -1, if element is not found
i←0
while  i < n  and  A[i] ≠ K  do
    i ← i+1
if i < n  then
    return i
else
  return -1
```

- What is Worst, Best and Average case analysis?

Sequential Search: Avg Case Analysis

- Let `p` be the probability of finding key `K`
  – Thus, probability of not finding `K` is `(1-p)`
- Probability of finding `K` at position `i` is same i.e. `p/n`
- The exepected number of searches $C_{avg}(n)$ is given by

$$1*p/n+2*p/n+...+n*p/n+(1-p)*n$$

$$= (p/n)(1+2+...+n) + (1-p)*n$$

$$= (p/n)*n(n+1)/2 + (1-p)*n$$

$$= p*(n+1)/2 + (1-p)*n$$

- When `p=1` i.e. search is always successful

$$C_{avg}(n) = (n+1)/2 \approx n/2$$

- When `p=0` i.e. search is failure

$$C_{avg}(n) = n$$

Sequential Search Analysis

- Computation of average case analysis is lot more complex than best case and worst case.
- For this specific problem

  $C_{avg}(n) = (C_{worst}(n) + C_{best}(n))/2$

  - This is not true in general and can't be simplified this way. Not a legitimate way.

Order of Growth

n	$\log_2 n$	$\sqrt{n}$	n	$n\log_2 n$	n^2	n^3	2^n	$n!$
10	3.3	3.16	10	33.2	**10^2**	**10^3**	**10^3**	$3.6*10^6$
10^2	6.6	10	**10^2**	664	**10^4**	**10^6**	$1.3*10^{30}$	$9.3*10^{157}$
10^3	10	31.6	**10^3**	9965	**10^6**	**10^9**		
10^4	13.2	100	**10^4**	$1.3*10^5$	**10^8**	**10^{12}**		
10^5	16.5	316.2	**10^5**	$1.6*10^6$	**10^{10}**	**10^{15}**		
10^6	20	1000	**10^6**	$2.0*10^7$	**10^{12}**	**10^{18}**		

Amortised Efficiency

- So far, efficiency is related to a single run of algorithm.
- In some cases, single run can be very expensive, but subsequent run can be much cheaper
  - Real life example:
    - To drink water, dig a well.
      - First time very costly, subsequently minimal cost
    - Giving a lecture first time on new topic
      - Subsequent lectures on same topic much easier
- Thus, amortise the cost over `n` operations

Exercises - A

- For the following algorithms (problems), identify
  - Natural input size metric
  - Basic operation
  - Count of basic operation (average case)

- P01: Computing sum of n numbers
- P02: Computing factorial(n)
- P03: Finding largest element of n numbers
- P04: List all prime numbers $<n$ using Sieve method
- P05: Multiplying 2 numbers each of n digits

Exercises-B

- Glove selection: There are 24 gloves in a drawer: 5 pairs of red gloves, 4 pairs of yellow, and 3pairs of green.
- You select two gloves in the dark and can check them only after a selection has been made. What is the smallest number of gloves you need to select to have (guarantee) the following:

  (Hint: Gloves are left and right side)

  - At least one matching pair? (worst case)
  - At least one matching pair in the best case?
  - At least one matching pair of each color? (worse case)
- Ans:
  - one matching pair: worst case: 13, best case: 2
  - one matching pair of each color: 22

Exercises-C

Missing socks: Imagine that after washing 5 distinct pairs of socks, you discover that two socks are missing. Of course, you would like to have the largest number of complete pairs remaining. Thus, you are left with 4 complete pairs in the best-case scenario and with 3 complete pairs in the worst case. Assuming that the probability of disappearance for each of the 10 socks is the same, find the probability of

- Q C_1: The best-case scenario;
- Q C_2: The worst-case scenario;
- Q C_3: The number of pairs you should expect in the average case.

- Answers:
  - C_1: 5/45 = 1/9
  - C_2: 40/45 = 8/9
  - C_3: 4*1/9 + 3*8/9 = 28/9 = $3^1/_9$

Summary : Analysis Framework

- Both time and space efficiencies are measured as functions of the algorithm's input size
- Time efficiency is measured by counting the number of times the base operation of algorithm is executed.
- Space efficiency is measured by counting the number of extra memory units consumed by algo.
- Efficiency for same algorithm may vary significantly for inputs of same size.
  – Worst case, Best case, and Average case
- Framework primary interest in order of growth
  – Running time of algorithm