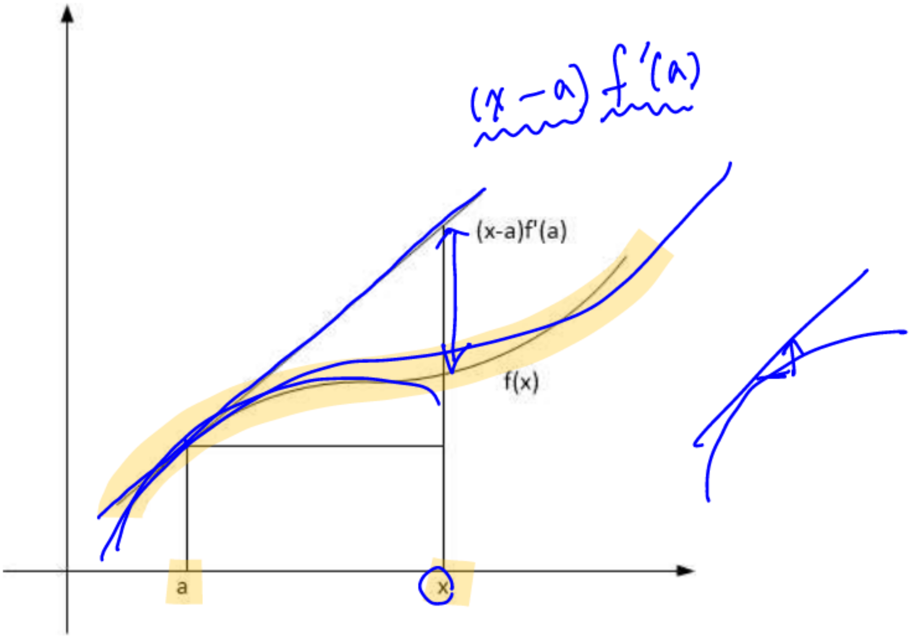
Taylor's Formula

Meaning



$$f(x) = f(a) + (x - a)f'(a)$$

$$f(x) = c_0 + c_1 x + c_2 x^2$$

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$$

Taylor's series

Taylor's polynomial

Definition

$$f(x) = \cancel{f(a)} + \frac{f'(a)}{1!}(x-a) + \cancel{\frac{f''(a)}{2!}(x-a)^2} + \frac{f'''(a)}{3!}(x-a)^3 + \dots$$

$$+ \frac{f^k(a)}{k!}(x-a)^k + \dots$$

x

$\frac{x^3}{3!}$

$$\frac{f'''(a)}{3!}(x-a)^3$$

$$\frac{3\cdot f'''(a)}{3!}(x-a)^2$$

$$\frac{3\cdot 2\cdot f'''(a)}{3!}(x-a)^1$$

$$\cancel{3!}.\quad \frac{3\cdot 2\cdot 1\cdot f'''(a)}{\cancel{3!}}(x-a)$$

Examples

$a = \phi$

$$\sin(0) = 0$$
$$\sin'(0) = \cos(0) = 1$$
$$\cos'(0) = -\sin(0) = 0$$
$$-\sin'(0) = -\cos(0) = -1$$
$$-\cos'(0) = --\sin(0) = 0$$
$$\sin'(0) = \cos(0) = 1$$
$$\dots$$

$f(x) = \sin(x)$ - - >

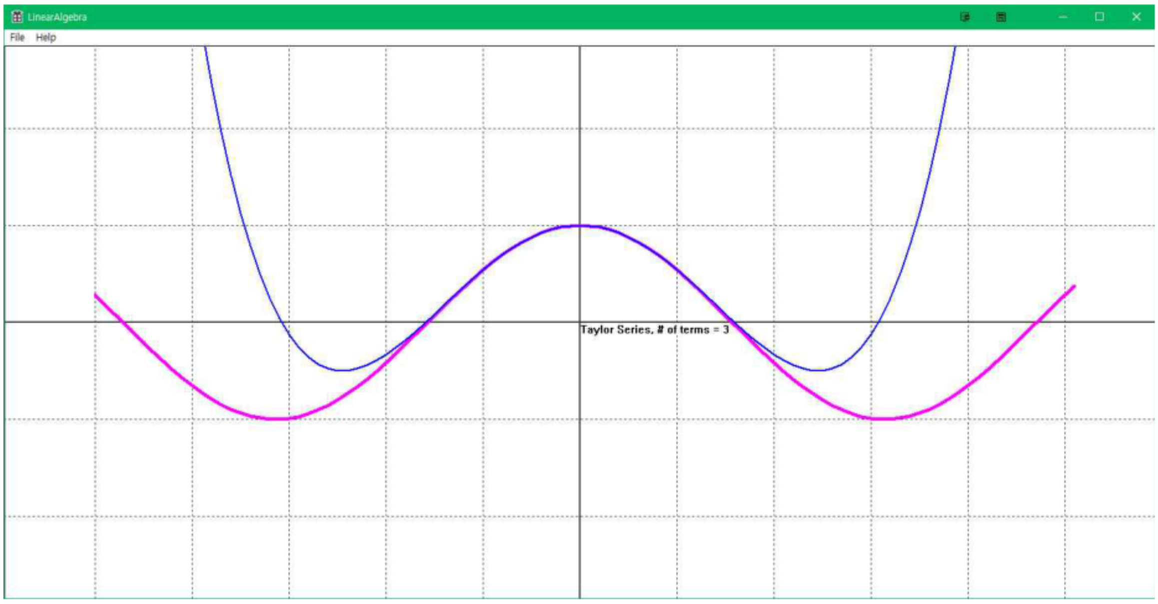
polynomial

$$\sin(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

$$\cos(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$$

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

Implementation




```
double Cosine(double x)
{
    return cos(x);
}

long long factorial(long long x, long long result = 1)
{
    if (x == 0)
        return result;
    else
        return factorial(x - 1, x * result);
}

long long numberOfTaylorSeriesTerms = 3;
double TaylorCosine(double x)
{
    double result = 0;
    for (int n = 0; n < numberOfTaylorSeriesTerms; ++n)
    {
        result += (std::pow(-1, n) / factorial(2 * n))*std::pow(x, 2 * n);
    }
```

```
    return result;
    //return 1 - (x*x) / (2*1) + (x*x*x*x) / (4 * 3 * 2 * 1);
}
```

Proof Idea

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$f'(a) \simeq \frac{f(x) - f(a)}{x - a}$$

$$f(x) = f(a) + (x-a)f'(a)$$

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$$f(x) - f(a) = \int_a^x f'(\xi)\, d\xi$$

$$(f(x))' = \left(f(a) + \int_a^x f'(\xi)\, d\xi\right)'$$

product rule.

$$\begin{aligned}(f(x))' &= (f(a) + (x-a)f'(a))' \\ f'(x) &= 0 + f'(a) + (x-a)f''(a) \\ &= f'(a) + (x-a)f''(a)\end{aligned}$$

$$\begin{aligned}f(x) &= f(a) + \int_a^x \{f'(a) + (x-a)f''(a)\}\, d\xi \\ &= f(a) + f'(a)[\xi]_a^x + f''(a)\left[\frac{1}{2}(\xi - a)^2\right]_a^x \\ &= f(a) + (x-a)f'(a) + \frac{1}{2}(x-a)^2 f''(a)\end{aligned}$$

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