# Damping

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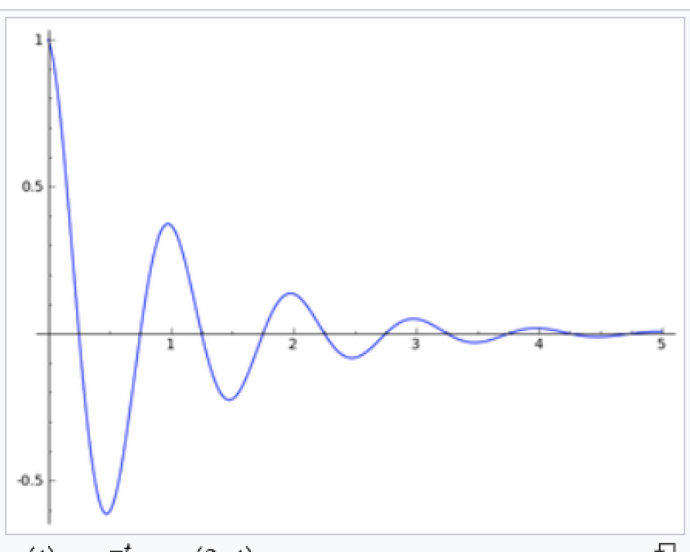
*For other uses, see Damping (disambiguation).*

**Damping** is an influence within or upon an oscillatory system that has the effect of reducing or preventing its oscillation. In physical systems, damping is produced by processes that dissipate the energy stored in the oscillation.[1] Examples include viscous drag in mechanical systems, resistance in electronic oscillators, and absorption and scattering of light in optical oscillators. Damping not based on energy loss can be important in other oscillating systems such as those that occur in biological systems and bikes.[2]

## Damped sine wave [ edit ]

*Not to be confused with Damped wave (radio transmission).*

A **damped sine wave** or **damped sinusoid** is a sinusoidal function whose amplitude approaches zero as time increases, corresponding to the **underdamped** case of damped second-order systems, or underdamped second-order differential equations.[3] Damped sine waves are commonly seen in science and engineering, wherever a harmonic oscillator is losing energy faster than it is being supplied. A true sine wave starting at time = 0 begins at the origin (amplitude = 0). A cosine wave begins at its maximum value due to its phase difference from the sine wave. A given sinusoidal waveform may be of intermediate phase, having both sine and cosine components. The term "damped sine wave" describes all such damped waveforms, whatever their initial phase.




$y(t) = e^{-t} \cdot \cos(2\pi t)$

The most common form of damping, and that usually assumed, is the form found in linear systems, which is exponential damping, in which the outer envelope of the successive peaks is an exponential decay curve. The general equation for an exponentially damped sinusoid may be represented as:

## 13.1 Position Functions

A *position function* provides the 3D position of an object as a function of time. Time is usually measured relative to some starting point when the position of an object is known. For instance, suppose that an object is traveling in a straight line with a constant velocity $\mathbf{v}_0$. If the position of the object at time $t = 0$ is known to be $\mathbf{x}_0$, then its position $\mathbf{x}(t)$ at any time afterward is given by

$$\mathbf{x}(t) = \mathbf{x}_0 + \mathbf{v}_0 t. \tag{13.1}$$

A *velocity function* describes the 3D velocity of an object as a function of time. The velocity function $\mathbf{v}(t)$ of an object is given by the derivative of the position function with respect to time. The time derivative is commonly denoted by placing a dot above the function being differentiated:

$$\mathbf{v}(t) = \dot{\mathbf{x}}(t) = \frac{d}{dt}\mathbf{x}(t). \tag{13.2}$$

Since the velocity of the object whose position is given by Equation (13.1) is constant, its velocity function $\mathbf{v}(t)$ is simply given by

$$\mathbf{v}(t) = \mathbf{v}_0. \tag{13.3}$$

An object undergoing a constant acceleration $\mathbf{a}_0$ has the velocity function

$$\mathbf{v}(t) = \mathbf{v}_0 + \mathbf{a}_0 t. \tag{13.4}$$

The *acceleration function* $\mathbf{a}(t)$ of an object, which describes the object's 3D acceleration as a function of time, is given by the derivative of the velocity function:

$$\mathbf{a}(t) = \dot{\mathbf{v}}(t) = \ddot{\mathbf{x}}(t) = \frac{d^2}{dt^2}\mathbf{x}(t). \tag{13.5}$$

We can integrate any velocity function to determine the distance $d$ that an object has traveled between times $t_1$ and $t_2$ as follows.

$$d = \int_{t_1}^{t_2} \mathbf{v}(t)\,dt \tag{13.6}$$

Integrating Equation (13.4) from time zero to time $t$, we have

$$\begin{aligned} d &= \int_0^t (\mathbf{v}_0 + \mathbf{a}_0 t)\,dt \\ &= \mathbf{v}_0 t + \tfrac{1}{2}\mathbf{a}_0 t^2. \end{aligned} \tag{13.7}$$

Adding the distance $d$ to an initial position $\mathbf{x}_0$, the position function $\mathbf{x}(t)$ of a uniformly accelerating object is given by

$$\mathbf{x}(t) = \mathbf{x}_0 + \mathbf{v}_0 t + \tfrac{1}{2}\mathbf{a}_0 t^2. \tag{13.8}$$

$$\sum_{i=1}^{N} \mathbf{F}_i(t) = m\mathbf{a}(t) = m\ddot{\mathbf{x}}(t). \tag{13.9}$$

Each force $\mathbf{F}_i(t)$ may be a constant, a function of the object's position, or a function of the object's velocity. Equation (13.9) is a second-order differential equation whose solution $\mathbf{x}(t)$ is the object's position function. The next section reviews the general solutions to second-order differential equations, and solutions to specific force equations are discussed at various places throughout this chapter and Chapter 14.

## 13.2 Second-Order Differential Equations

A second-order linear ordinary differential equation in the function $x(t)$ is one of the following form.

$$\frac{d^2}{dt^2}x(t) + a\frac{d}{dt}x(t) + bx(t) = f(t) \tag{13.10}$$

Using prime symbols to denote derivatives, we can write this in a slightly more compact form as

$$x''(t) + ax'(t) + bx(t) = f(t). \tag{13.11}$$

In this chapter, $a$ and $b$ are always constants; but in general, they may be functions of $t$.

### 13.2.1 Homogeneous Equations

The function $f(t)$ is identically zero in many situations, in which case the differential equation is called *homogeneous*. Before attempting to find a solution $x(t)$ to the equation

$$x''(t) + ax'(t) + bx(t) = 0, \tag{13.12}$$

we make a couple of important observations. First, suppose that the functions $x_1(t)$ and $x_2(t)$ are solutions to Equation (13.12). Then the functions $Ax_1(t)$ and $Bx_2(t)$ are also solutions, where $A$ and $B$ are arbitrary constants. Furthermore, the function $Ax_1(t) + Bx_2(t)$ is also a solution to Equation (13.12) since we can write

$$\begin{aligned} &Ax_1''(t) + Bx_2''(t) + a[Ax_1'(t) + Bx_2'(t)] + b[Ax_1(t) + Bx_2(t)] \\ &\quad = A[x_1''(t) + ax_1'(t) + bx_1(t)] + B[x_2''(t) + ax_2'(t) + bx_2(t)] \\ &\quad = A \cdot 0 + B \cdot 0 = 0. \end{aligned} \tag{13.13}$$

A general solution $x(t)$ to Equation (13.12) becomes evident upon making the substitution

$$x(t) = e^{rt}. \tag{13.14}$$

Derivatives of exponential and logarithmic functions [ edit ]

$$\frac{d}{dx}(c^{ax}) = ac^{ax}\ln c, \qquad c > 0$$

the equation above is true for all $c$, but the derivative for $c < 0$ yields a complex number.

$$\frac{d}{dx}(e^{ax}) = ae^{ax}$$

$$\frac{d}{dx}(\log_c x) = \frac{1}{x\ln c}, \qquad c > 0, c \neq 1$$

$$\begin{aligned} x'(t) &= re^{rt} \\ x''(t) &= r^2 e^{rt}, \end{aligned} \tag{13.15}$$

and substitution into Equation (13.12) yields

$$r^2 e^{rt} + are^{rt} + be^{rt} = 0. \tag{13.16}$$

Multiplying both sides by $e^{-rt}$ eliminates the exponentials, and we have

$$r^2 + ar + b = 0. \tag{13.17}$$

Equation (13.17) is called the *auxiliary equation* and has the solutions

$$\begin{aligned} r_1 &= -\frac{a}{2} + \frac{1}{2}\sqrt{a^2 - 4b} \\ r_2 &= -\frac{a}{2} - \frac{1}{2}\sqrt{a^2 - 4b}. \end{aligned} \tag{13.18}$$

Unless $r_1 = r_2$, the general solution to Equation (13.12) is thus given by

$$x(t) = Ae^{r_1 t} + Be^{r_2 t}. \tag{13.19}$$

**Example 13.1.** Solve the differential equation

$$x''(t) - 5x'(t) + 6x(t) = 0. \tag{13.20}$$

**Solution.** The auxiliary equation is

$$r^2 - 5r + 6 = 0, \tag{13.21}$$

which has the solutions $r_1 = 2$ and $r_2 = 3$. The general solution to Equation (13.20) is therefore given by

$$x(t) = Ae^{2t} + Be^{3t}, \tag{13.22}$$

where $A$ and $B$ are arbitrary constants. ■

# Damping in Physics Engine

ODE:
$a v'(t) + bv(t) = 0$
$v'(t) + cv(t) = 0$

The auxiliary equation
$r^2 + ar + b = 0$
$v(t) = Ae^{r_1 t} + Be^{r_2 t}$

$0r^2 + 1r + c = 0$
$r_1 = -c, r_2 = 0$
$v_2(t) = v_1(t)e^{-ct}$

```
void IntegrateForces(KRigidbody *b, float dt)
{
    if (b->m_invMass == 0.0f)
        return;

    b->velocity += (b->force * b->m_invMass + g_gravity) * dt;
    b->angularVelocity += b->torque * b->m_invI * dt;

    // Apply damping.
    // ODE: dv/dt + c * v = 0
    // Solution: v(t) = v0 * exp(-c * t)
    // Time step: v(t + dt) = v0 * exp(-c * (t + dt))
    //                      = v0 * exp(-c * t) * exp(-c * dt) = v * exp(-c * dt)
    // v2 = exp(-c * dt) * v1
    b->velocity *= std::exp(-(b->m_linearDamping)*dt);
    b->angularVelocity *= std::exp(-(b->m_linearDamping)*dt);

    //// Pade approximation:
    //// v2 = v1 * 1 / (1 + c * dt)
    //// https://en.wikipedia.org/wiki/Pad%C3%A9_approximant
    //b->velocity *= 1.0f / (1.0f + dt * b->m_linearDamping);
    //b->angularVelocity *= 1.0f / (1.0f + dt * b->m_angularDamping);
}
```