Model Discrimination Design Generation for Accelerated Life Testing Experiments via Hybridized Optimization Algorithms

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Outline

Approximation design

Two models:

$$f_1(y \mid x, \theta_1, \sigma_1^2) = \beta_0 + \beta_1 x$$

$$f_2(y \mid x, \theta_2, \sigma_2^2) = \beta_0 + \beta_1 x + \beta_2 x^2$$

- ex.Advertising expenses (x):\$10,000 $\sim$ \$500,000 VS. Sales volume (y)
- Approximation design:$\xi = \begin{bmatrix} x_1 & x_2 & \dots & x_n \\ w_1 & w_2 & \dots & w_n \end{bmatrix}$, x_i are support point and $w_i \in [0, 1], \sum_{i=1}^{n} w_i = 1$ with $i = 1, 2, \dots, n$.
  - If experimental budget allows for 100 runs.

KL-optimal design

- Chen et al. (2020)
- When models do not have homoscedastic or normally distributed errors.
- KL divergence: $\mathcal{L}(f_{tr}, f_2, x, \theta_2) = \int f_{tr}(y \mid x, \sigma_1^2) \log\left\{\frac{f_{tr}(y|x,\sigma_1^2)}{f_2(y|x,\theta_2,\sigma_2^2)}\right\} dy$

KL-optimal criterion:

$$\max_{\xi\in\Xi} KL_{2,tr}(\xi) = \overbrace{\max_{\xi\in\Xi} \overbrace{\min_{\theta_2\in\Theta_2} \mathcal{L}(f_{tr}, f_2, x, \theta_2)}^{L-BFGS}}^{PSO}$$

- Equivalence theorem:

$$\psi_{KL}(x, \xi_{KL}^*) = \mathcal{L}(f_{tr}, f_2, x, \hat{\theta}_2(\xi_{KL}^*)) - KL_{2,tr}(\xi_{KL}^*) \leq 0$$

T-optimal design

- Atkinson and Fedorov(1975)
- Suppose we have 2 homoscedastic Gaussian models.

T-optimal criterion:

$$\max_{\xi\in\Xi} T_{2,tr}(\xi) = \overbrace{\max_{\xi\in\Xi} \underbrace{\min_{\theta_2\in\Theta_2} \int_X [\eta_{tr}(x) - \eta_2(x,\theta_2)]^2\, \xi(dx)}_{\text{Inner Loop}}}^{\text{Outer Loop}}$$

- Equivalence theorem:

$$\psi_T(x,\xi_T^*) = \Delta_{2,tr}(x,\hat{\theta}_2(\xi_T^*)) - T_{2,tr}(\xi_T^*) \le 0$$

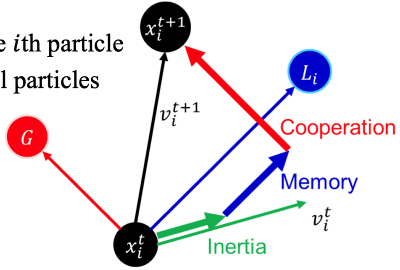
PSO(Particle Swarm Optimization)

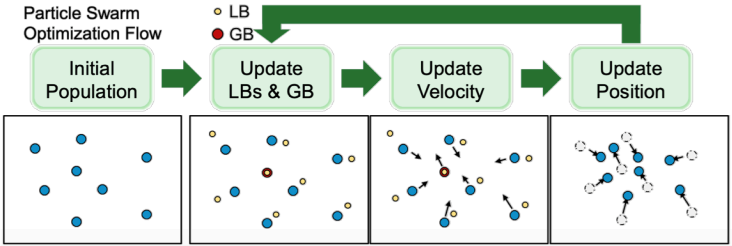
- Local Best L_i: historical best solution of the ith particle
- Global Best G: historical best solution of all particles

$$v_i^{t+1} = \underline{w \cdot v_i^t} + \underline{c_1 \cdot r_1(L_i - x_i^t)} + \underline{c_2 \cdot r_2(G - x_i^t)}$$

$$x_i^{t+1} = x_i^t + v_i^{t+1}$$

- w, c_1, c_2 are tuning parameters.
- r_1, r_2 are uniform random vectors.





Fidalgo Case

- López-Fidalgo et al.(2007)

Pharmacokinetic models are often assumed to be Log-Normally distributed.

- Modified Michaelis–Menten (MMM) model.
- Michaelis–Menten (MM) model.

$$\text{Pharmacokinetic model} \Rightarrow \begin{cases} \text{MMM}: y = \frac{x}{1+x} + x \\ \text{MM}: \quad y = \frac{Vx}{K+x} \end{cases}, X = [0.1, 5]$$

- x is the substrate concentration.
- y is the velocity rate of product formation in a chemical reaction.
- V is the maximum velocity rate.
- K is the concentration at which half of the maximum velocity rate is reached.

Fidalgo Case

Log-Normal

$$
\begin{aligned}
D_{KL}(P \parallel Q) &= \int_{-\infty}^{\infty} p(y) \log\left(\frac{p(y)}{q(y)}\right) dy \\
&= \int_{0}^{\infty} \frac{1}{y\sigma_1\sqrt{2\pi}} e^{-\frac{(\log y-\mu_1)^2}{2\sigma_1^2}} \log\left(\frac{\frac{1}{y\sigma_1\sqrt{2\pi}} e^{-\frac{(\log y-\mu_1)^2}{2\sigma_1^2}}}{\frac{1}{y\sigma_2\sqrt{2\pi}} e^{-\frac{(\log y-\mu_2)^2}{2\sigma_2^2}}}\right) dy \\
&= \int_{0}^{\infty} \frac{1}{y\sigma_1\sqrt{2\pi}} e^{-\frac{(\log y-\mu_1)^2}{2\sigma_1^2}} \left(\log\left(\frac{\sigma_2}{\sigma_1}\right) + \frac{(\log y-\mu_2)^2}{2\sigma_2^2} - \frac{(\log y-\mu_1)^2}{2\sigma_1^2}\right) dy \\
&= \log\left(\frac{\sigma_2}{\sigma_1}\right) \int_{0}^{\infty} \frac{1}{y\sigma_1\sqrt{2\pi}} e^{-\frac{(\log y-\mu_1)^2}{2\sigma_1^2}} dy + \frac{1}{2\sigma_2^2} \int_{0}^{\infty} \frac{1}{y\sigma_1\sqrt{2\pi}} e^{-\frac{(\log y-\mu_1)^2}{2\sigma_1^2}} (\log y-\mu_2)^2 \, dy \\
&- \frac{1}{2\sigma_1^2} \int_{0}^{\infty} \frac{1}{y\sigma_1\sqrt{2\pi}} e^{-\frac{(\log y-\mu_1)^2}{2\sigma_1^2}} (\log y-\mu_1)^2 \, dy \\
&= \log\left(\frac{\sigma_2}{\sigma_1}\right) + \frac{\sigma_1^2 + (\mu_1-\mu_2)^2}{2\sigma_2^2} - \frac{1}{2} \\
&= \log\left(\frac{\sigma_2}{\sigma_1}\right) + \frac{\sigma_1^2 - \sigma_2^2 + (\mu_1-\mu_2)^2}{2\sigma_2^2} \\
&= \log\left(\frac{\sigma_2}{\sigma_1}\right) - \frac{\sigma_2^2 - \sigma_1^2 + (\mu_2-\mu_1)^2}{2\sigma_2^2}
\end{aligned}
$$

Fidalgo Case

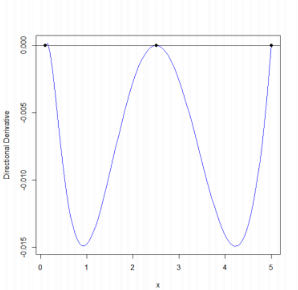
Assumption:

- homoscedasticity and models are assumed to be Log-Normal distributed.

Closed-Form (ξ^*_{KL-c})

$$\xi^*_{KL-c} = \left\{ \begin{array}{ccc} 0.1 & 2.5 & 5 \\ 0.538 & 0.329 & 0.133 \end{array} \right\}$$

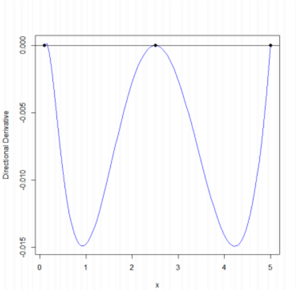
Computation time is 39 seconds.



Numerical Integration (ξ^*_{KL-n})

$$\xi^*_{KL-n} = \left\{ \begin{array}{ccc} 0.1 & 2.5 & 5 \\ 0.538 & 0.329 & 0.133 \end{array} \right\}$$

Computation time is 5165 seconds.



Fidalgo Case

Weibull

$$
\begin{aligned}
D_{KL}(P \parallel Q) &= \int_{-\infty}^{\infty} p(y) \log\left(\frac{p(y)}{q(y)}\right) dy \\
&= \int_0^{\infty} \frac{k_1}{\lambda_1}\left(\frac{y}{\lambda_1}\right)^{k_1-1} e^{-\left(\frac{y}{\lambda_1}\right)^{k_1}} \log\left(\frac{\frac{k_1}{\lambda_1}\left(\frac{y}{\lambda_1}\right)^{k_1-1} e^{-\left(\frac{y}{\lambda_1}\right)^{k_1}}}{\frac{k_2}{\lambda_2}\left(\frac{y}{\lambda_2}\right)^{k_2-1} e^{-\left(\frac{y}{\lambda_2}\right)^{k_2}}}\right) dy \\
&= \log\left(\frac{k_1}{k_2}\right) \int_0^{\infty} \frac{k_1}{\lambda_1}\left(\frac{y}{\lambda_1}\right)^{k_1-1} e^{-\left(\frac{y}{\lambda_1}\right)^{k_1}} dy + \log\left(\frac{\lambda_2}{\lambda_1}\right) \int_0^{\infty} \frac{k_1}{\lambda_1}\left(\frac{y}{\lambda_1}\right)^{k_1-1} e^{-\left(\frac{y}{\lambda_1}\right)^{k_1}} dy \\
&+ (k_1-1) \int_0^{\infty} \frac{k_1}{\lambda_1}\left(\frac{y}{\lambda_1}\right)^{k_1-1} e^{-\left(\frac{y}{\lambda_1}\right)^{k_1}} \log\left(\frac{y}{\lambda_1}\right) dy - \int_0^{\infty} \frac{k_1}{\lambda_1}\left(\frac{y}{\lambda_1}\right)^{k_1-1} e^{-\left(\frac{y}{\lambda_1}\right)^{k_1}} \left(\frac{y}{\lambda_1}\right)^{k_1} dy \\
&- (k_2-1) \int_0^{\infty} \frac{k_1}{\lambda_1}\left(\frac{y}{\lambda_1}\right)^{k_1-1} e^{-\left(\frac{y}{\lambda_1}\right)^{k_1}} \log\left(\frac{y}{\lambda_2}\right) dy - \frac{\gamma}{k_1} + \int_0^{\infty} \frac{k_1}{\lambda_1}\left(\frac{y}{\lambda_1}\right)^{k_1-1} e^{-\left(\frac{y}{\lambda_1}\right)^{k_1}} \left(\frac{y}{\lambda_2}\right)^{k_2} dy \\
&= \log\left(\frac{k_1}{k_2}\right) + \log\left(\frac{\lambda_2}{\lambda_1}\right) - \frac{k_1-1}{k_1}\gamma - 1 - (k_2-1)\log\left(\frac{\lambda_1}{\lambda_2}\right) + \frac{k_2-1}{k_1}\gamma + \left(\frac{\lambda_1}{\lambda_2}\right)^{k_2} \Gamma\left(\frac{k_2}{k_1}+1\right)
\end{aligned}
$$

Fidalgo Case

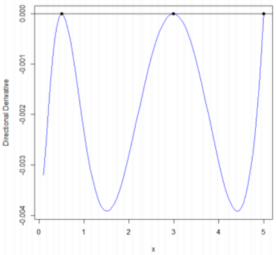
Assumption:

- homoscedasticity and models are assumed to be Weibull distributed.

Closed-Form (ξ^*_{KL-c})

$$\xi^*_{KL-c} = \begin{Bmatrix} 0.504 & 2.989 & 5 \\ 0.570 & 0.310 & 0.120 \end{Bmatrix}$$

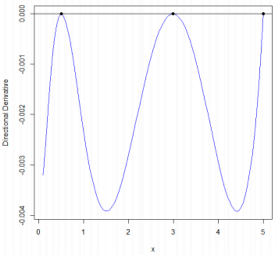
Computation time is 40 seconds.



Numerical Integration (ξ^*_{KL-n})

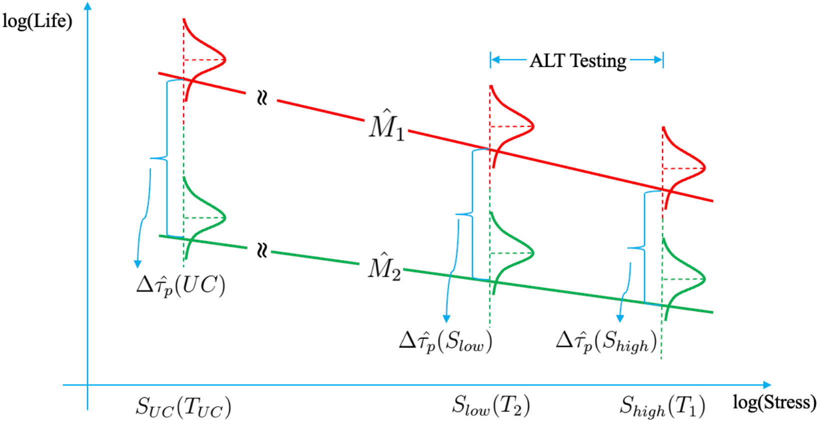
$$\xi^*_{KL-n} = \begin{Bmatrix} 0.507 & 2.991 & 5 \\ 0.570 & 0.310 & 0.120 \end{Bmatrix}$$

Computation time is 102768 seconds.



Accelerated Life Testing

- Nasir and Pan(2014)



Accelerated Life Testing

- Nasir and Pan(2014)

The Arrhenius life-temperature:

$$t(T) = Aexp\left(\frac{E_a}{K \times Temp}\right)$$

Type I censored data (time censoring):

$$Pr(t > C) = 1 - F(C), C > 0$$

Disadvantages of using bayesian approach:

- Computation time is too long.
- Results are not reproducible.

Accelerated Life Testing

- Park and Shin(2012)

Consider the Type I censored variable $min(X, C)$, where C is a fixed censoring point. The density function becomes:

$$f_C(x) = \begin{cases} f(x) & \textit{if } x < C, \\ 1 - F(C) & \textit{if } x = C, \\ 0 & \textit{if elsewhere}. \end{cases}$$

The Kullback–Leibler (KL) divergence between two censored densities f_{tr} and f_r is:

$$D_{CKL}(f_{tr}, f_r) = \int_{-\infty}^{C} f_{tr} \log\left\{\frac{f_{tr}}{f_r}\right\} dy + \bar{F}_{tr}(C) \log\left\{\frac{\bar{F}_{tr}(C)}{\bar{F}_r(C)}\right\}$$

Divergence Measures

- A.Pakgohar et al.(2019) proposed another three divergence measures under Type I censored data.
  - Lin-Wong (LW) divergence:

$$D_{CLW}(f_{tr}, f_r) = \int_{-\infty}^{C} f_{tr} \log\left\{\frac{2f_{tr}}{f_{tr}+f_r}\right\} dy + \bar{F}_{tr}(C)\log\left\{\frac{2\bar{F}_{tr}(C)}{\bar{F}_{tr}(C)+\bar{F}_r(C)}\right\}$$

  - Bhattacharyya (B) distance measure:

$$D_{CB}(f_{tr}, f_r) = \int_{-\infty}^{C} \sqrt{f_{tr}\cdot f_r}\, dy + \sqrt{\bar{F}_{tr}(C)\cdot \bar{F}_r(C)}$$

  - Chi-Square (χ^2) distance measure:

$$D_{C\chi^2}(f_{tr}, f_r) = \int_{-\infty}^{C} \frac{(f_{tr})^2}{f_r} dy + \frac{\left(\bar{F}_{tr}(C)\right)^2}{\bar{F}_r(C)} - 1$$

- So we propose four optimal design:CKL-,CLW-,CB-,Cχ^2-optimal design

ALT Case

- Arrhnius model

$$t(T) = Aexp\left(\frac{E_a}{K \times Temp}\right)$$
$$\Rightarrow \log(t(T)) = \log(A) + \frac{E_a}{K \times Temp}$$
$$\Rightarrow \underbrace{\log(t(T))}_{\mu} = \underbrace{\log(A)}_{\beta_0} + \underbrace{\frac{E_a}{K}}_{\beta_1} \times \underbrace{\frac{1}{Temp}}_{x}.$$

- The true model M_1 is a quadratic form:

$$\eta_{tr}(x, \theta_1) = \zeta_1 + \zeta_2 x + \zeta_3 x^2.$$

- The rival model M_2 is a linear form:

$$\eta_2(x, \theta_2) = \delta_1 + \delta_2 x.$$

ALT Case

- 18 simulation cases were conducted (fixed the variance of true model and rival model)
- Setting:
  - Censoring time:5000
  - The true model parameters:$\theta_{tr} = (\zeta_1, \zeta_2, \zeta_3) = (-5.0, -1.5, 0.05)$
  - The parameter space of the rival model:$\theta_2 = (\delta_1, \delta_2) \in [-100, -10] \times [0.1, 5.0]$
  - The design space:$x \in [10, 80]$
- CKL-optimal design: 12 $\surd$, 3 $\triangle$, 3 $\times$.
- CLW-optimal design: 4 $\surd$, 6 $\triangle$, 8 $\times$.
- CB-optimal design: 0 $\surd$, 14 $\triangle$, 4 $\times$.
- Cχ^2-optimal design: 2 $\surd$, 7 $\triangle$, 9 $\times$.

ALT Case

Parameterized Variance:

- The true model parameters:$\theta_{tr} = (\zeta_1, \zeta_2, \zeta_3) = (-5.0, -1.5, 0.05)$, with the variance fixed 0.9780103 and 1.4780103.
- The rival model parameters:$(\delta_1, \delta_2) \in [-100, -10] \times [0.1, 5.0]$, with an additional unknown constant variance parameter $\sigma_2 \in [0.4780103, 4.9780103]$.
- The design space is $x \in [10, 80]$.
- $\theta_2 = (\delta_1, \delta_2, \sigma_2)$
- The true model M_1 is a quadratic form:

$$\eta_{tr}(x, \theta_1) = \zeta_1 + \zeta_2 x + \zeta_3 x^2.$$

- The rival model M_2 is a linear form:

$$\eta_2(x, \theta_2) = \delta_1 + \delta_2 x.$$

ALT Case

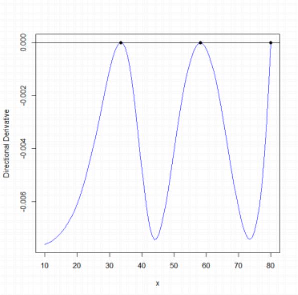
- Both models follow the Weibull distribution:

$$\sigma_1 = 0.9780103$$

$$\xi^*_{CKL-c} = \left\{ \begin{array}{ccc} 33.531 & 58.185 & 80 \\ 0.340 & 0.434 & 0.226 \end{array} \right\}$$

$$\hat{\theta}_2(\xi^*_{CKL}) = (-66.207, 2.002, 0.957)$$

Computation time is 134408 seconds.

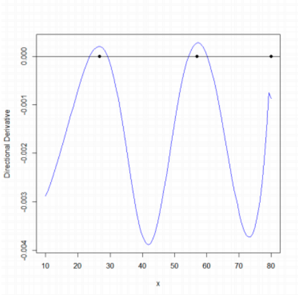


$$\sigma_1 = 1.4780103$$

$$\xi^*_{CKL-d} = \left\{ \begin{array}{ccc} 26.713 & 57.032 & 80 \\ 0.396 & 0.382 & 0.222 \end{array} \right\}$$

$$\hat{\theta}_2(\xi^*_{CKL}) = (-66.560, 2.013, 1.441)$$

Computation time is 85551 seconds.



Meeker Case

Stress-Dependent Variance

- Pascual and Meeker (1997)
- Their study estimated the parameter $\gamma = 75.71$
- Censoring time:1000
- The true model M_1 is:

$$\eta_{tr}(x, \theta) = \zeta_1 + \zeta_2 \log(x - \gamma)$$
$$\sigma_1 = exp\left\{\phi_1 + \phi_2 \log(x - \gamma)\right\}$$

- The rival model M_2:

$$\eta_2(x, \theta) = \delta_1 + \delta_2 \log(x - \gamma)$$
$$\sigma_2 = exp\left\{\kappa_1 + \kappa_2 \log(x - \gamma)\right\}$$

- $\theta_2(\xi^*_{CKL}) = (\delta_1, \delta_2, \kappa_1, \kappa_2)$

Meeker Case

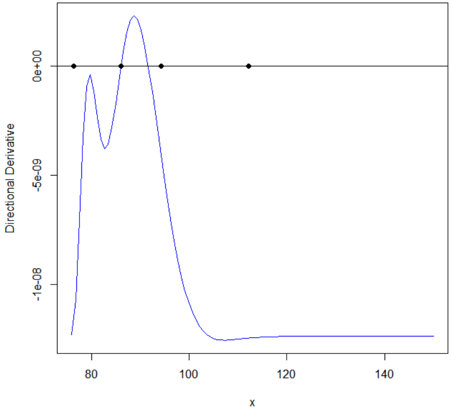
Stress-Dependent Variance

	Dis.		M_1				M_2			
Case	M_1	M_2	ζ_1	ζ_2	ϕ_1	ϕ_2	δ_1	δ_2	κ_1	κ_2
(1)	LN	WB	14.75	-1.39	10.97	-2.5	[12.06,17.44]	[-2.02,-0.76]	[10,20]	[-3,-0.01]
(2)			14.75	-1.39	10.97	-2.5	[15.9,21.45]	[-2.81,-0.92]	[10,20]	[-3,-0.01]
(3)			10	-2	0.63	-0.91	[9.5,15]	[-2.1,-1]	[0.5,1]	[-1,-0.81]
(4)			43	-0.63	4.32	-0.88	[5,50]	[-1,-0.05]	[3.12,5.32]	[-1,-0.5]
(5)			458	-53	4.32	-0.88	[432,480]	[-100,-1]	[3.12,5.32]	[-1,-0.5]
(6)			53.39	-7.81	4.32	-0.88	[50,60]	[-10,-5]	[3.12,5.32]	[-1,-0.5]
(7)	WB	LN	14.75	-1.39	10.97	-2.5	[12.06,17.44]	[-2.02,-0.76]	[10,20]	[-3,-0.01]
(8)			14.75	-1.39	10.97	-2.5	[15.9,21.45]	[-2.81,-0.92]	[10,20]	[-3,-0.01]
(9)			10	-2	0.63	-0.91	[9.5,15]	[-2.1,-1]	[0.5,1]	[-1,-0.81]
(10)			43	-0.63	4.32	-0.88	[5,50]	[-1,-0.05]	[3.12,5.32]	[-1,-0.5]
(11)			458	-53	4.32	-0.88	[432,480]	[-100,-1]	[3.12,5.32]	[-1,-0.5]
(12)			53.39	-7.81	4.32	-0.88	[50,60]	[-10,-5]	[3.12,5.32]	[-1,-0.5]

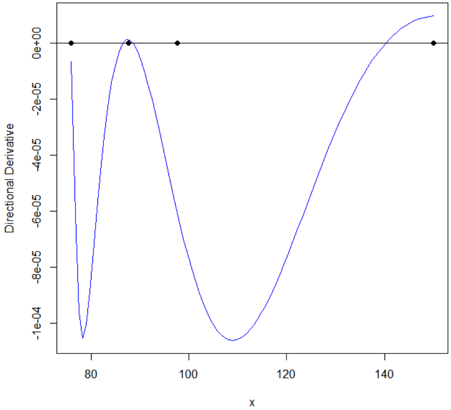
Meeker Case

Case	ξ^*_{CKL}	$C^*(\hat{C})$	$\hat{\theta}_2(\xi^*_{CKL})$	Eqv.	Opt?	Time
(1)	$\left\{\begin{array}{cccc} 88.245 & 115.191 & 122.132 & 123.571 \\ 1.000 & 0.000 & 0.000 & 0.000 \end{array}\right\}$	2.023×10^{-8} (4.956×10^{-9})	$(14.918, -1.350, 11.824, -2.942)$	A.13a	△	4083
(2)	$\left\{\begin{array}{cccc} 76.389 & 86.024 & 94.240 & 112.132 \\ 0.000 & 1.000 & 0.000 & 0.000 \end{array}\right\}$	6.725×10^{-8} (2.136×10^{-8})	$(20.968, -2.056, 10, -2.397)$	A.13b	△	5043
(3)	$\left\{\begin{array}{cccc} 111.009 & 112.296 & 113.75 & 150 \\ 0.000 & 0.000 & 1.000 & 0.000 \end{array}\right\}$	27.167 (6.275×10^{-5})	$(10.597, -1.873, 0.723, -0.922)$	A.13c	×	4245
(4)	$\left\{\begin{array}{cccc} 77.297 & 78.038 & 109.932 & 132.334 \\ 1.000 & 0.000 & 0.000 & 0.000 \end{array}\right\}$	2.449×10^{-12} (5.462×10^{-14})	$(36.931, -0.523, 3.791, -0.878)$	A.13d	×	3897
(5)	$\left\{\begin{array}{cccc} 90.99 & 98.775 & 108.941 & 116.822 \\ 0.464 & 0.231 & 0.271 & 0.035 \end{array}\right\}$	0 (0)	$(455.318, -49.116, 4.219, -0.533)$	A.13e	△	1376
(6)	$\left\{\begin{array}{cccc} 123.238 & 123.969 & 125.026 & 127.373 \\ 0.022 & 0.977 & 0.000 & 0.000 \end{array}\right\}$	2.265×10^{-14} (-4.305×10^{-49})	$(55.951, -6.358, 3.846, -0.928)$	A.13f	△	7386
(7)	$\left\{\begin{array}{cccc} 76 & 86.748 & 126.815 & 150 \\ 0.476 & 0.247 & 0.000 & 0.277 \end{array}\right\}$	1.235×10^{-4} (1.208×10^{-4})	$(17.44, -1.638, 10, -2.028)$	A.14a	△	48506
(8)	$\left\{\begin{array}{cccc} 76 & 87.744 & 97.631 & 150 \\ 0.469 & 0.242 & 0.000 & 0.289 \end{array}\right\}$	1.173×10^{-4} (1.156×10^{-4})	$(18.350, -1.781, 10.011, -2.012)$	A.14b	△	54794
(9)	$\left\{\begin{array}{cccc} 80.622 & 94.153 & 94.67 & 127.625 \\ 0.259 & 0.519 & 0.000 & 0.221 \end{array}\right\}$	0.0907 (0.0872)	$(9.972, -1.995, 0.882, -0.839)$	A.14c	×	217023
(10)	$\left\{\begin{array}{cccc} 76 & 76.879 & 76.950 & 118.382 \\ 0.000 & 0.000 & 1.000 & 0.000 \end{array}\right\}$	3.197×10^{-9} (3.808×10^{-10})	$(27.531, -0.658, 4.374, -0.728)$	A.14d	△	1487
(11)	$\left\{\begin{array}{cccc} 79.235 & 94.878 & 110.173 & 150 \\ 0.000 & 1.000 & 0.000 & 0.000 \end{array}\right\}$	1.808×10^{-91} (1.534×10^{-92})	$(473.842, -9.458, 4.621, -0.364)$	A.14e	△	1436
(12)	$\left\{\begin{array}{cccc} 104.273 & 113.865 & 130.105 & 150 \\ 0.982 & 0.000 & 0.000 & 0.018 \end{array}\right\}$	8.222×10^{-8} (4.220×10^{-8})	$(54.974, -7.423, 4.200, -0.586)$	A.14f	△	6743

Meeker Case



(a) Meeker Case (2)



(b) Meeker Case (8)

Conclusion and Limitations

- This study considers censored data, a common feature in the field of reliability and survival analysis.
- We propose four divergence measures-based model discrimination criteria: CKL-, CLW-, CB-,and Cχ^2-optimal designs.
- The numerical optimization was implemented using a hybrid method: PSO + L-BFGS.
- The algorithm is refined to accommodate both constant and stress-dependent variance estimation.
- In addition to models with different mean functions, we also explore cases with identical means but different distributional assumptions.

Conclusion and Limitations

- Among the four criteria, the CKL-optimal design consistently outperformed the others.
- The Meeker case shows strong potential for CKL-optimal design.
- The inner objective function, may exhibit multiple local optima or strong non-convexity, limiting the effectiveness of L-BFGS.
- Numerical instability can occur during the integration process.

Future Works

- Improved the integration implementation in the code.
- Revised mathematical form of the censoring term.
- Further investigate the landscape of the inner objective function and explore global or hybrid optimization strategies beyond traditional gradient-based methods.
- Explored a compound CKL + D-optimal design strategy.

R Implementation

- R Code: GitHub/GPLIN514
- R shiny: shinyapps.io/MSGPLIN

Thank You!

Questions and Discussion are welcome.

Example

- Suppose the true model is the quadratic model of the form

$$\eta_{tr}(x) = x^2 + 2x + 1$$

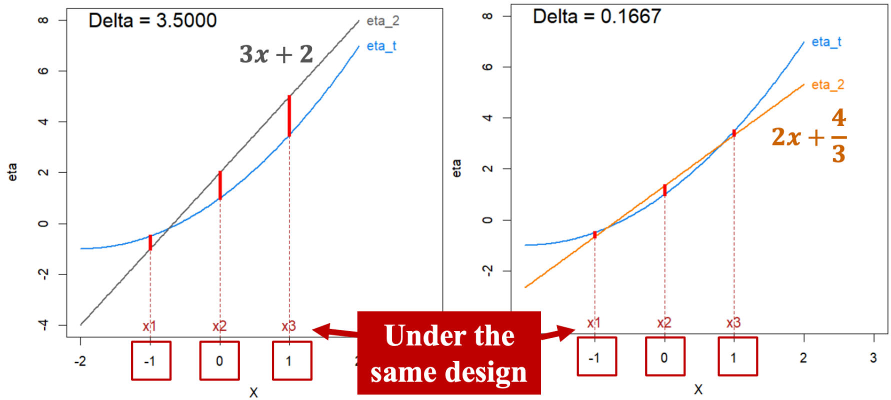
- The rival model is of linear form with unknown parameter $\theta_2 = (\beta_0, \beta_1)$

$$\eta_2(x, \theta_2) = \beta_1 x + \beta_0$$

- Consider the exact design $\xi = x_1, x_2, x_3$
- Find the best ξ that discriminate $\eta_{tr}(x)$ and $\eta_2(x, \theta_2)$ the most.

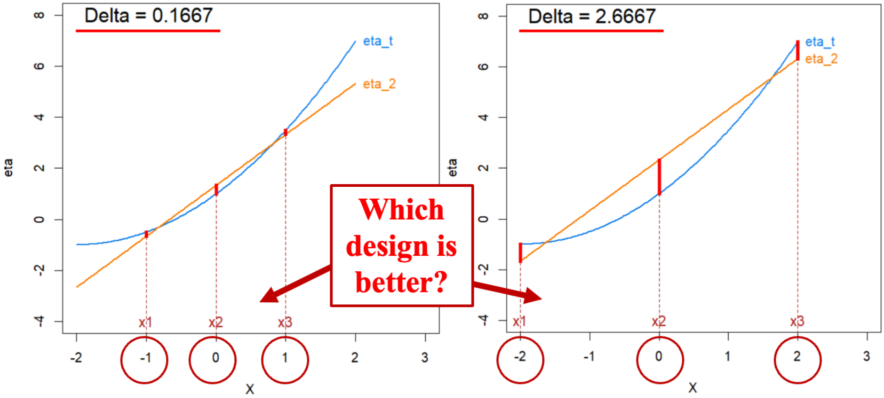
Example

$$\xi_T^* = \max_{\xi} \left\{ \boxed{\min_{\boldsymbol{\theta}_2 \in \boldsymbol{\Theta}_2} \int_X \Delta_{2,t}(x, \boldsymbol{\theta}_2)\, \xi(dx)} \right\}$$



Example

$$\xi_T^* = \boxed{\max_{\xi}\left\{\min_{\boldsymbol{\theta}_2\in\boldsymbol{\Theta}_2}\int_X \Delta_{2,t}(x,\boldsymbol{\theta}_2)\,\xi(dx)\right\}}$$



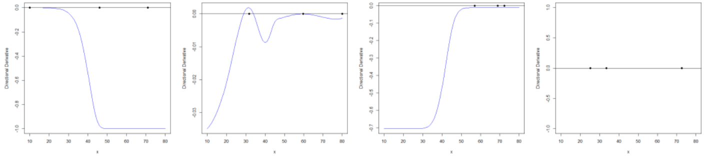
ALT Case

The optimality check results are classified into three levels:

- $\surd$ indicates full satisfaction of the optimality conditions:

  each support point has a non-zero weight; the directional derivative function lies entirely below zero; the design points correspond to local maxima of the function with value exactly zero; and the curve is smooth and continuous in shape.

- $\triangle$ indicates partial satisfaction. Typically arising in the following scenarios:



- $\times$ indicates that the optimality conditions are not satisfied.

ALT Case

Dis.	σ	ξ^*_{CKL}	$C^*(\hat{C})$	$\hat{\theta}_2(\xi^*_{CKL})$	Eqv.	Opt?	Time
LN	0.98	$\left\{\begin{array}{ccc} 33.557 & 56.829 & 80 \\ 0.330 & 0.436 & 0.234 \end{array}\right\}$	0.00927 (0.00927)	$(-66.712, 2.017)$	A.1a	$\surd$	18386
LN	1.48	$\left\{\begin{array}{ccc} 29.699 & 55.482 & 80 \\ 0.360 & 0.413 & 0.227 \end{array}\right\}$	0.00517 (0.00517)	$(-67.204, 2.031)$	A.1c	$\surd$	58458
LN	1.98	$\left\{\begin{array}{ccc} 25.854 & 55.196 & 80 \\ 0.366 & 0.407 & 0.227 \end{array}\right\}$	0.00385 (0.00380)	$(-66.268, 2.006)$	A.1e	$\triangle$	49577
WB	0.98	$\left\{\begin{array}{ccc} 32.545 & 57.686 & 80 \\ 0.368 & 0.415 & 0.217 \end{array}\right\}$	0.00822 (0.00822)	$(-66.459, 2.010)$	A.1b	$\triangle$	59919
WB	1.48	$\left\{\begin{array}{ccc} 25.576 & 55.791 & 80 \\ 0.441 & 0.359 & 0.200 \end{array}\right\}$	0.00489 (0.00489)	$(-67.083, 2.028)$	A.1d	$\surd$	68091
WB	1.98	$\left\{\begin{array}{ccc} 18.386 & 53.830 & 80 \\ 0.484 & 0.325 & 0.191 \end{array}\right\}$	0.00386 (0.00386)	$(-63.987, 1.944)$	A.1f	$\times$	62045

ALT Case

Dis.	σ	$\xi^*_{C\chi^2}$	$C^*(\hat{C})$	$\hat{\theta}_2(\xi^*_{C\chi^2})$	Eqv.	Opt?	Time
LN	0.98	$\left\{\begin{array}{ccc} 80 & 80 & 80 \\ 0.014 & 0.449 & 0.537 \end{array}\right\}$	4.938×10^{11} (-1.698×10^{-10})	$(-75.507, 2.289)$	A.10a	$\times$	3394
LN	1.48	$\left\{\begin{array}{ccc} 41.953 & 46.694 & 47.929 \\ 0.000 & 0.000 & 1.000 \end{array}\right\}$	726.370 (-9.216×10^{-6})	$(-65.563, 1.983)$	A.10c	$\times$	2277
LN	1.98	$\left\{\begin{array}{ccc} 22.557 & 28.732 & 34.66 \\ 0.000 & 1.000 & 0.000 \end{array}\right\}$	0.0230 (0.0233)	$(-64.951, 1.967)$	A.10e	$\triangle$	1353
WB	0.98	$\left\{\begin{array}{ccc} 11.049 & 16.362 & 34.388 \\ 0.000 & 0.000 & 1.000 \end{array}\right\}$	0.0296 (-4.966×10^{-9})	$(-59.045, 1.819)$	A.10b	$\triangle$	25836
WB	1.48	$\left\{\begin{array}{ccc} 29.181 & 46.269 & 61.937 \\ 0.999 & 0.000 & 0.001 \end{array}\right\}$	0.0196 (-4.332×10^{-9})	$(-71.474, 2.151)$	A.10d	$\times$	65818
WB	1.98	$\left\{\begin{array}{ccc} 12.069 & 23.116 & 31.719 \\ 0.000 & 1.000 & 0.000 \end{array}\right\}$	0.0146 (-4.941×10^{-9})	$(-62.198, 1.919)$	A.10f	$\times$	47360

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