Lecture 11

Stability

This section provides some of the many definitions of stability. Linear stability is defined first, and is then followed by the notion of Lyapunov stability. One method due to Lyapunov used for assessing the stability of nonlinear systems is discussed. The stability definitions provided in this section will be applied to attitude dynamics and control in the subsequent lectures

Overview

Many definitions of *stability* exist in literature. A system that is stable typically has a behaviour that is not adversely affected by external disturbances. The notions of *stability* and *boundedness* are, in general, independent concepts, but for linear systems of order n (such that $\boldsymbol{x}(t) \in \mathbb{R}^n$) they are closely related:

- for a homogeneous system, $\dot{\boldsymbol{x}}(t) = \boldsymbol{A}(t)\boldsymbol{x}(t)$ with the initial conditions (ICs) $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$, all solutions $\boldsymbol{x}(t)$ are stable for $t > t_0$ if an only if they are bounded (not growing infinitely large) for all $t > t_0$.
- for an inhomogeneous system, $\dot{\boldsymbol{x}}(t) = \boldsymbol{A}(t)\boldsymbol{x}(t) + \boldsymbol{h}(t)$ with the initial conditions (ICs) $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$, all solutions $\boldsymbol{x}(t)$ are stable for $t > t_0$ if they are bounded for all $t > t_0$, and they are bounded for all $t > t_0$ if they are stable for all $t > t_0$ and *at least one* solution $\boldsymbol{x}(t)$ is bounded for all $t > t_0$.

This intimate tie between stability and boundedness for linear systems naturally leads to the concept of linear stability.

Linear Stability

Consider a scalar linear ordinary differential equation (ODE) of order n, with the following form:

$$\frac{d^n x}{dt^n} + a_{n-1}\frac{d^{n-1}x}{dt^{n-1}} + \cdots + a_2\frac{d^2 x}{dt^2} + a_1\frac{dx}{dt} + a_0 x = 0 \tag{11.1}$$

where $x(t)$ is a scalar variable and a_i's are constant scalar coefficients. Analogously to our study of stability of Lagrange points, we let $x(t) = \bar{x}e^{\lambda t}$ represent the solution of Eq. (11.1), where $\bar{x}$ is a constant such that $x(0) = \bar{x}$. With this form of the solution, Eq. (11.1) becomes:

$$\big(\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_2\lambda^2 + a_1\lambda + a_0\big)\bar{x}e^{\lambda t} = 0 \tag{11.2}$$

which, for a non-trivial solution, $x(t)$, requires the polynomial in the brackets to be identically zero, hence yielding the following so-called characteristic equation:

$$\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_2\lambda^2 + a_1\lambda + a_0 = 0 \tag{11.3}$$

Assuming this equation has n distinct roots, λ_i, $i \in \{1, \cdots, n\}$, the solution can be written as:

$$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \cdots + c_n e^{\lambda_n t} \tag{11.4}$$

where the constant coefficients c_i are obtained using the ICs (for x and its derivatives at time 0) of Eq. (11.1).

The equilibrium solution, $x(t) \equiv 0$, of Eq. (11.1) is:

- *stable* if λ_i's are distinct and $\mathrm{Re}\{\lambda_\mathrm{i}\} \leqslant 0$, as it corresponds to a bounded (possibly periodic) solution.
- *asymptotically stable* if $\mathrm{Re}\{\lambda_\mathrm{i}\} < 0$, as it corresponds to a decaying exponential solution.
- *unstable* if $\mathrm{Re}\{\lambda_\mathrm{i}\} > 0$ for *at least* one λ_i, as it corresponds to a growing exponential.

Note: To see where the above results come from, consider $\alpha = \mathrm{Re}\{\lambda_\mathrm{i}\}$ and $\beta = \mathrm{Im}\{\lambda_\mathrm{i}\}$, and recall that $e^{(\alpha+i\beta)t} = e^{\alpha t}\cos(\beta t) + ie^{\alpha t}\sin(\beta t)$. The bounded periodic case results from imaginary λ_i's (with $\alpha = 0$), while $\alpha > 0$ and $\alpha < 0$ lead to a dominating exponential growth or decay pattern, superimposed with oscillatory changes resulting from the trigonometric terms.

Note: Recall, also, "Routh's Stability Criterion" (to determine stability given a transfer function) from your introductory control background. Refer to pages 321 and 322 of *Spacecraft Dynamics and Control: an Introduction* for a description of the criterion.

Linear Mechanical Systems

Many real systems of mechanical or other nature can be modelled using a second order system of ODEs, sometimes with physically significant matrices. The following form of such systems will be frequently encountered in this course:

$$\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{G}\dot{\boldsymbol{q}} + \boldsymbol{K}\boldsymbol{q} = \boldsymbol{f} \tag{11.5}$$

where $\boldsymbol{M} = \boldsymbol{M}^\mathsf{T} > 0$ is the "mass matrix", $\boldsymbol{G} = -\boldsymbol{G}^\mathsf{T}$ is the "gyricity matrix", and $\boldsymbol{K} = \boldsymbol{K}^\mathsf{T}$ is the "stiffness matrix". The terms on the left-hand side describe the inertial, gyric, and stiffness forces, while the vector $\boldsymbol{f}$ contains any remaining generalized forces.

Note: The system in Eq. (11.5) can also be written in the following familiar linear form:

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{h} \;\; ; \;\; \boldsymbol{x} \triangleq \begin{bmatrix} \boldsymbol{q} \\ \dot{\boldsymbol{q}} \end{bmatrix} \;\; , \;\; \boldsymbol{A} \triangleq \begin{bmatrix} \boldsymbol{0} & \boldsymbol{1} \\ -\boldsymbol{M}^{-1}\boldsymbol{K} & -\boldsymbol{M}^{-1}\boldsymbol{G} \end{bmatrix} \;\; , \;\; \boldsymbol{h} \triangleq \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{M}^{-1}\boldsymbol{f} \end{bmatrix} \tag{11.6}$$

The following subclasses of the linear mechanical system in Eq. (11.5) and their stability will be considered:

- A "gyric system", $\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{G}\dot{\boldsymbol{q}} = \boldsymbol{0}$, is *stable* if and only if $\det(\boldsymbol{G}) \neq 0$.

- A “conservative system”, $\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{K}\boldsymbol{q} = \boldsymbol{0}$, is *statically stable* if and only if $\boldsymbol{K} > 0$, and it is *stable* if and only if it is *statically stable*.
- A “conservative gyric system”, $\boldsymbol{M}\ddot{\boldsymbol{q}}+\boldsymbol{G}\dot{\boldsymbol{q}}+\boldsymbol{K}\boldsymbol{q} = \boldsymbol{0}$ is *stable* if it is *statically stable*. A *stable* conservative gyric system is:
  - *statically stable* if $\boldsymbol{K} > 0$.
  - *gyrically stable* if $\boldsymbol{K} \ngtr 0$.

Note: In the presence of gyric effects, a positive-definite $\boldsymbol{K}$ is sufficient but no longer necessary for stability.

Note: If a damping term, $\boldsymbol{D}\dot{\boldsymbol{q}}$ with $\boldsymbol{D} = \boldsymbol{D}^{\mathsf{T}} \geqslant 0$, is also introduced into a stable conservative gyric system: if it were *statically stable*, it becomes *asymptotically stable*; if it were *gyrically stable*, it becomes *unstable*.

Input/Output Stability

Consider a linear control system, with the state vector $\boldsymbol{x}(t)$. For attitude control purposes, the state vector could be defined as $\boldsymbol{x}(t) \triangleq [\boldsymbol{\omega}^{\mathsf{T}}(t)\ \boldsymbol{\epsilon}^{\mathsf{T}}(t)\ \eta(t)]^{\mathsf{T}}$ or $\boldsymbol{x}(t) \triangleq [\boldsymbol{\omega}^{\mathsf{T}}(t)\ \boldsymbol{\theta}^{\mathsf{T}}(t)]^{\mathsf{T}}$ when using quaternions or Euler angles, respectively, to represent attitude. This system can be written in the following state-space form:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}(t)\boldsymbol{x}(t) + \boldsymbol{B}(t)\boldsymbol{u}(t) \tag{11.7a}$$

$$\dot{\boldsymbol{y}}(t) = \boldsymbol{C}(t)\boldsymbol{x}(t) + \boldsymbol{D}(t)\boldsymbol{u}(t) \tag{11.7b}$$

where $\boldsymbol{u}(t)$ is the control input vector (such as those to control the forces applied by thrusters) and $\boldsymbol{y}(t)$ is the output vector (such as the resulting acceleration measured by on-board accelerometers). The state-space matrices $\boldsymbol{A}(t), \boldsymbol{B}(t), \boldsymbol{C}(t)$, and $\boldsymbol{D}(t)$ map the state and control vectors to the rate of change of the state and output vectors.

Such a system is considered:

- *stable* if for all $\boldsymbol{u}(t) \in \mathcal{L}_2$, we have $\boldsymbol{y}(t) \in \mathcal{L}_2$ as well.
- *unstable* if for some $\boldsymbol{u}(t) \in \mathcal{L}_2$, we have $\boldsymbol{y}(t) \notin \mathcal{L}_2$.

Note: The $\mathcal{L}_2$ space is defined as the space of all square-integrable vector functions:

$$\mathcal{L}_2 \triangleq \{\boldsymbol{v}(t) \in \mathbb{R}^{n\times 1} | \int_0^\infty \boldsymbol{v}^{\mathsf{T}}(t)\boldsymbol{v}(t)\ dt < \infty\}$$

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Lyapunov Stability

Now consider a more general case of an autonomous nonlinear system, not necessarily representable using a matrix $\boldsymbol{A}$ that linearly maps the state vector $\boldsymbol{x}(t) \in \mathbb{R}^{n\times 1}$ to its derivative:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}) \ \ , \ \ \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \tag{11.8}$$

where the term "autonomous" refers to the fact that the vector function $\boldsymbol{f}$ containing the nonlinear equations corresponding to each state $x_i(t)$ (components of $\boldsymbol{x}(t)$) is assumed to not be a direct function of time.

Definition 1. The *equilibirium* solution of the system represented by Eq. (11.8) is any $\boldsymbol{x}(t)$ for which $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) \equiv \mathbf{0}$, which implies that the state remains constant over time.

We can, without loss of generality and for the sake of clarity, take $\boldsymbol{x} \equiv \mathbf{0}$ as the equilibrium solution, the stability corresponding to which will be called "the stability of the origin".

The equilibrium solution, $\boldsymbol{x}(t) \equiv \mathbf{0}$, of Eq. (11.8) is:

- *L-stable* if, for any given $\epsilon > 0$, one can find $\delta(\epsilon) > 0$ such that for any $|\boldsymbol{x}_0| < \delta$ we have $|\boldsymbol{x}(t)| < \epsilon$ at all times $t \geqslant t_0$.
- *asymptotically stable* if it is L-stable *and* for any $|\boldsymbol{x}(t)| < \delta$, we have $|\boldsymbol{x}(t)| \to 0$ as $t \to \infty$.
- *globally asymptotically stable* if it is asymptotically stable for *any* $\boldsymbol{x}_0$.
- *unstable* if it is not L-stable.

Directional Stability

Specific to attitude dynamics, this is a special case of *L-stability*, and is a lesser (more relaxed) version of *attitude stability*. A body that possesses *directional stability* has a body-fixed axis that can be made to remain arbitrarily close to an inertially-fixed axis, provided that the disturbances are small. In other words, the *direction* of a body-fixed axis is Lyapunov stable.

Let a body-fixed unit vector be represented by:

$$\underset{\rightarrow}{\hat{\boldsymbol{u}}}(t) = \underset{\rightarrow}{\hat{\boldsymbol{u}}}_0(t) + \Delta\underset{\rightarrow}{\boldsymbol{u}}(t) \quad , \quad \Delta\underset{\rightarrow}{\boldsymbol{u}}(t_0) = \Delta\underset{\rightarrow}{\boldsymbol{u}}_0 \tag{11.9}$$

where $\underset{\rightarrow}{\hat{\boldsymbol{u}}}_0$ is a unit vector representing the reference direction and $\Delta\underset{\rightarrow}{\boldsymbol{u}}(t)$ is a directional perturbation about this reference.

The equilibrium solution, $\underset{\rightarrow}{\hat{\boldsymbol{u}}}(t) \equiv \underset{\rightarrow}{\hat{\boldsymbol{u}}}_0$, is:

- *directionally stable* if we can make $|\Delta\underset{\rightarrow}{\boldsymbol{u}}(t)|$ arbitrarily bounded for $t > t_0$ with sufficiently small $|\Delta\underset{\rightarrow}{\boldsymbol{u}}_0|$.
- *asymptotically directionally stable* if $|\Delta\underset{\rightarrow}{\boldsymbol{u}}(t)| \to 0$ as $t \to \infty$.

Stability and Linearization (Lyapunov's First Method)

Consider a solution of Eq. (11.8) of the form $\boldsymbol{x}(t) = \boldsymbol{x}_e + \delta\boldsymbol{x}(t)$, with $\boldsymbol{x}_e$ is constant. Substituting this into Eq. (11.8) and using Taylor's series of $\boldsymbol{f}(\boldsymbol{x})$ about $\boldsymbol{x}_e$ yields:

$$\delta\dot{\boldsymbol{x}} = \dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}_e + \delta\boldsymbol{x}) = \cancelto{\mathbf{0}}{\boldsymbol{f}(\boldsymbol{x}_e)} + \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}^{\mathsf{T}}}\bigg|_{\boldsymbol{x}=\boldsymbol{x}_e} \delta\boldsymbol{x} + \cdots \quad \Rightarrow \quad \delta\dot{\boldsymbol{x}} \approx \boldsymbol{A}\delta\boldsymbol{x} \quad , \quad \boldsymbol{A} \triangleq \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}^{\mathsf{T}}}\bigg|_{\boldsymbol{x}=\boldsymbol{x}_e} \tag{11.10}$$

The Jacobian matrix evaluated at the equilibrum, $\boldsymbol{A}$, can be expanded as:

$$\boldsymbol{A} \triangleq \left.\begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial f_n}{\partial x_1} & \dfrac{\partial f_n}{\partial x_2} & \cdots & \dfrac{\partial f_n}{\partial x_n} \end{bmatrix}\right|_{\boldsymbol{x}=\boldsymbol{x}_e}$$

Assuming the solutions of Eq. (11.10) have the form $\delta\boldsymbol{x}(t) = \delta\bar{\boldsymbol{x}}e^{\lambda t}$, an eigenvalue problem is arrived at:

$$\delta\dot{\boldsymbol{x}}(t) = \delta\bar{\boldsymbol{x}}\lambda e^{\lambda t} = \lambda\delta\boldsymbol{x} \quad \Rightarrow \quad \boldsymbol{A}\delta\boldsymbol{x} = \lambda\delta\boldsymbol{x} \tag{11.11}$$

which yields an n^{th}-order characteristic equation (similar to Eq. (11.3)) arising from $\det(\boldsymbol{A} - \lambda\mathbf{1}) = 0$, which should hold in order for Eq. (11.11) to be satisfied for a non-trivial $\delta\boldsymbol{x} \neq \mathbf{0}$. Then, the general solution is:

$$\delta\boldsymbol{x}(t) = c_1\delta\bar{\boldsymbol{x}}_1 e^{\lambda_1 t} + \cdots + c_n\delta\bar{\boldsymbol{x}}_n e^{\lambda_n t} \tag{11.12}$$

the growing, decaying, or periodic nature of the exponential terms of which allow for conclusions to be made (similarly to linear stability) in the vicinity of the equilibrium, where linearization is relatively accurate.

The equilibrium solution, $\boldsymbol{x}_e(t) \equiv \mathbf{0}$, of the nonlinear system in Eq. (11.8) is:

- *asymptotically stable* if the linearized solution $\delta\boldsymbol{x}(t) \equiv \mathbf{0}$ is asymptotically stable.
- *unstable* if the linearized solution $\delta\boldsymbol{x}(t) \equiv \mathbf{0}$ is unstable.

Note: If the linearized solution $\delta\boldsymbol{x}(t) \equiv \mathbf{0}$ is merely stable (and not asymptotically stable), then no conclusion can be drawn about the stability of the nonlinear equilibrium solution, $\boldsymbol{x}_e(t) \equiv \mathbf{0}$.