Lecture 8

Orbital Manoeuvres

HAVING studied the dynamics of orbital motion and how to describe an orbit, we will now look at how to control and modify spacecraft's orbits. In particular, we will study the effects of applying thrusts in various directions, and discuss a few specific manoeuvre types.

Overview

Recall, from ORBIT DESCRIPTION AND DETERMINATION, that an orbit is completely determined by the knowledge of $\vec{\boldsymbol{r}}(t)$ and $\vec{\boldsymbol{v}}(t)$ at any given time, t. To move the spacecraft from one orbit to another, thruster force is applied, which results in a change in $\vec{\boldsymbol{v}}$. Therefore, if a single-impulse manoeuvre is desired, the initial and final orbits shall have at least one intersection point where the thrust will be applied, as depicted in Figure 8.1.

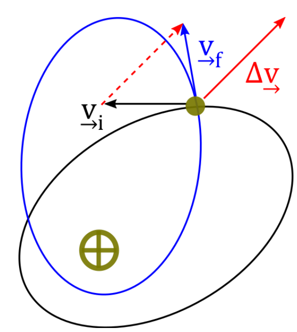


Figure 8.1: General Manoeuvre

It is assumed that an instantaneous change in the velocity vector, $\Delta\vec{\boldsymbol{v}}$, is applied via an impulsive thrust. As a result, the position vector, $\vec{\boldsymbol{r}}$, is assumed to be unaltered after the burn. This is not completely accurate for real missions with long thruster on/off times, but it is sufficient for our purposes. The overall objective of most orbital manoeuvres is to minimize the fuel consumption by minimizing $\Delta v = |\Delta\vec{\boldsymbol{v}}|$, and/or to minimize the time taken to complete the manoeuvre.

General Manoeuvres

Recall the constants of orbital motion from ORBITAL MECHANICS, namely:

$$\epsilon \triangleq \frac{\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{v}}}{2} - \frac{\mu}{r} = -\frac{\mu}{2a} \quad , \quad \vec{\boldsymbol{h}} \triangleq \vec{\boldsymbol{r}} \times \vec{\boldsymbol{v}} \quad , \quad \vec{\boldsymbol{e}} \triangleq \frac{\vec{\boldsymbol{v}} \times \vec{\boldsymbol{h}}}{\mu} - \frac{\vec{\boldsymbol{r}}}{r}$$

The general approach to determining the changes in the orbital parameters as a result of a known $\Delta\vec{\boldsymbol{v}}$, applied when the spacecraft is at $\vec{\boldsymbol{r}}$ with an initial velocity of $\vec{\boldsymbol{v}}_i$, involves computing the resultant velocity vector, $\vec{\boldsymbol{v}}_f = \vec{\boldsymbol{v}}_i + \Delta\vec{\boldsymbol{v}}$, and determining the constants of the new orbit. Using $\vec{\boldsymbol{r}}$ and $\vec{\boldsymbol{v}}_f$ at the manoeuvre node, the orbital parameters of the resultant orbit can be evaluated using the procedure outlined in ORBIT DESCRIPTION AND DETERMINATION.

If, as is often the case, the unknown variable is the $\Delta\underrightarrow{\boldsymbol{v}}$ required in order to place the spacecraft into a new orbit that satisfies some requirements, such as passing through a known inertial point, $\underrightarrow{\boldsymbol{r}}_2$, then orbit determination may be required in order to obtain the parameters of the new orbit, and compute $\underrightarrow{\boldsymbol{v}}_f$, the spacecraft's velocity after the burn at the intersection point. The required $\Delta\underrightarrow{\boldsymbol{v}} = \underrightarrow{\boldsymbol{v}}_f - \underrightarrow{\boldsymbol{v}}_i$ can then be computed. Refer back to ORBIT DESCRIPTION AND DETERMINATION for an overview of Lambert's problem, a common orbit determination problem.

Special Manoeuvres

Simplifications arise when the manoeuvre to be applied is in certain directions. The types of manoeuvres considered in this section involve special cases of the general manoeuvres discussed above, in which $\Delta\underrightarrow{\boldsymbol{v}}$ is no longer a general vector with an arbitrary direction.

Radial Thrust

If the thrust is applied along the radial vector, as in Figure 8.2, it results in no change in angular momentum:

$$\underrightarrow{\boldsymbol{h}}_f = \underrightarrow{\boldsymbol{r}} \times \underrightarrow{\boldsymbol{v}}_f = \underrightarrow{\boldsymbol{r}} \times \underrightarrow{\boldsymbol{v}}_i + \cancelto{\mathbf{0}}{\underrightarrow{\boldsymbol{r}} \times \Delta\underrightarrow{\boldsymbol{v}}} \Rightarrow \underrightarrow{\boldsymbol{h}}_f = \underrightarrow{\boldsymbol{h}}_i = \underrightarrow{\boldsymbol{h}} \;\;,\;\; \Delta h = 0 \tag{8.1}$$

as a result of which the change in the eccentricity vector can be simplified as follows:

$$\Delta\underrightarrow{\boldsymbol{e}} = \underrightarrow{\boldsymbol{e}}_f - \underrightarrow{\boldsymbol{e}}_i = \frac{\underrightarrow{\boldsymbol{v}}_f \times \underrightarrow{\boldsymbol{h}}}{\mu} - \frac{\cancel{\underrightarrow{\boldsymbol{r}}}}{\cancel{r}} - \frac{\underrightarrow{\boldsymbol{v}}_i \times \underrightarrow{\boldsymbol{h}}}{\mu} + \frac{\cancel{\underrightarrow{\boldsymbol{r}}}}{\cancel{r}} = \frac{\Delta\underrightarrow{\boldsymbol{v}} \times \underrightarrow{\boldsymbol{h}}}{\mu} \tag{8.2}$$

using which the eccentricity of the new orbit can be found more easily than that corresponding to a general $\Delta\underrightarrow{\boldsymbol{v}}$. In addition, the direction in which the periapsis rotates as a result of an increasing radial thrust can also be visualized by simply taking the cross product of $\Delta\underrightarrow{\boldsymbol{v}}$ and $\underrightarrow{\boldsymbol{h}}$ (which is constant and normal to the orbital plane).

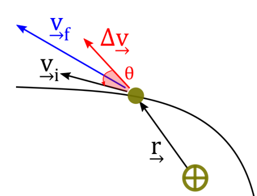


Figure 8.2: Radial

Note: In other resources, the term "radial" thrust may refer to one applied locally normal to the orbit path (normal to the tangent vector). Care must be taken not to mistakenly apply this subsection's relationships to the thrusts applied in that manner.

The change in the specific orbital energy is obtained, using the cosine law, by:

$$\Delta\epsilon = \frac{\underrightarrow{\boldsymbol{v}}_f \cdot \underrightarrow{\boldsymbol{v}}_f}{2} - \frac{\cancel{\mu}}{\cancel{r}} - \frac{\underrightarrow{\boldsymbol{v}}_i \cdot \underrightarrow{\boldsymbol{v}}_i}{2} + \frac{\cancel{\mu}}{\cancel{r}} = \frac{v_f^2 - v_i^2}{2} \quad\Rightarrow\quad \Delta\epsilon = \frac{(\Delta v)^2}{2} - v_i \Delta v \cos(\pi - \theta) = \frac{(\Delta v)^2}{2} + \frac{\underrightarrow{\boldsymbol{r}} \cdot \underrightarrow{\boldsymbol{v}}_i}{r}\Delta v \tag{8.3}$$

where the fact that $\underrightarrow{\boldsymbol{r}}$ and $\Delta\underrightarrow{\boldsymbol{v}}$ are parallel is used. In the special case that the manoeuvre corresponds to a point at which $\underrightarrow{\boldsymbol{r}}$ and $\underrightarrow{\boldsymbol{v}}$ are perpendicular (which is always the case for a circular orbit), Eq. (8.3) reduces to $\Delta\epsilon = (\Delta v)^2/2$. As we will soon see, this change is much smaller than that achieved using a tangential burn.

Tangential Thrust

If the thrust is applied tangentially to the spacecraft's path and along its velocity vector, as shown in Figure 8.3, the change in the specific orbital energy can be computed using only the magnitudes involved:

$$v_f = v_i + \Delta v \quad \Rightarrow \quad \Delta\epsilon = \frac{v_f^2 - v_i^2}{2} = \frac{(\Delta v)^2}{2} + v_i \Delta v \tag{8.4}$$

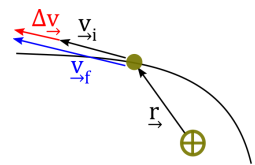


Figure 8.3: Tangential

Considering the correspondence between an orbit's energy and its size ($a = -\mu/(2\epsilon)$), we conclude that a tangential burn is much more efficient in changing the size of an orbit than a radial burn. Considering the relationship for $\dot{r}$ from ORBIT DESCRIPTION AND DETERMINATION, which is in turn obtained by differentiating the polar equation, it can be observed that $\dot{r} = 0$ occurs when $\sin(\theta) = 0$, namely at the periapsis and apoapsis of an elliptic orbit. This makes sense, as these two points represent the minimum and maximum radial distances. Since $\dot{r} = 0$ at every point of a circular orbit, in order to circularize an elliptic orbit, the tangential burn should be applied at either the periapsis or the apoapsis.

Hohmann Transfer A sequence of two tangential burns, both *prograde* (in the same direction as the motion) or both *retrograde* (in the opposite direction) can be used to transfer between two concentric circular orbits. The first impulse changes the spacecraft's orbit to an elliptic transfer orbit that is tangent to the circular orbit at its periapsis (or apoapsis, if the final circular orbit is to be smaller than the initial one), and the second impulse, applied at the apoapsis (or periapsis) circularizes the transfer orbit. A Hohmann transfer is known as the most fuel-efficient two-impulse transfer between circular orbits, but it is rarely used in real interplanetary missions because of its long time of flight.

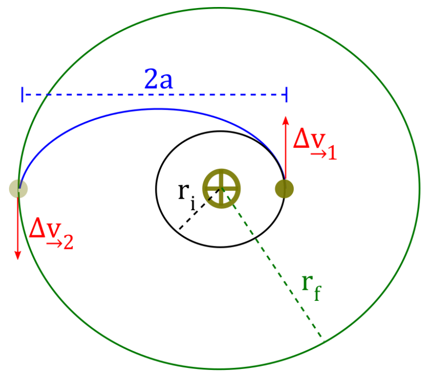
Consider two circular orbits of radii r_i and r_f, and an elliptic Hohmann transfer orbit with $a = (r_i + r_f)/2$, as shown in Figure 8.4a). The Δv's associated with the two necessary impulses are computed by finding the difference between the spacecraft's velocity in one of the circular obits and that in the elliptic orbit (at the manoeuvre node), both of which are found using the vis-visa equation from ORBITAL MECHANICS:

$$\Delta v_1 = \left| v_{i_e} - v_{i_c} \right| = \left| \sqrt{\mu\left(\frac{2}{r_i} - \frac{1}{a}\right)} - \sqrt{\frac{\mu}{r_i}} \right| \tag{8.5a}$$

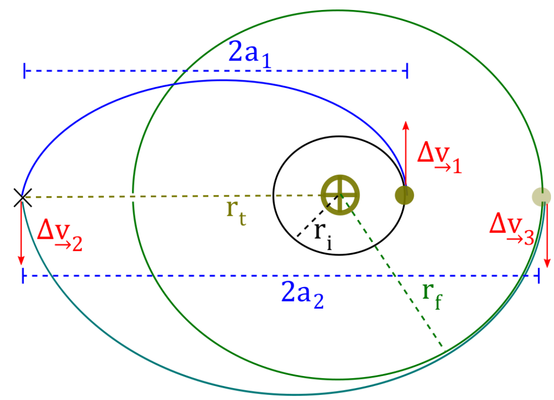
$$\Delta v_2 = \left| v_{f_c} - v_{f_e} \right| = \left| \sqrt{\frac{\mu}{r_f}} - \sqrt{\mu\left(\frac{2}{r_f} - \frac{1}{a}\right)} \right| \tag{8.5b}$$

where the subscripts 'c' and 'e' denote 'circular' and 'elliptic', respectively. The direction in which these Δv's should be applied (prograde or retrograde) is determined be the relative size of the two circular orbits: prograde if $r_f > r_i$, retrograde otherwise.

Bi-Elliptic Transfer Making use of a sequence of three tangential burns, this type of manoeuvre is similar to a Hohmann transfer, but involves two elliptic transfers to change one circular obit to another (with an intermediate circular orbit that is larger or smaller than both). For a bi-elliptic transfer between two circular orbits of radii r_i and r_f, depicted in Figure 8.4b, the two transfer ellipses will have $a_1 = (r_i + r_t)/2$ and $a_2 = (r_t + r_f)/2$, where r_t is the radius of the intermediate circular orbit that is tangent to both ellipses at the 2$^{\text{nd}}$ manoeuvre node. In this case, the final burn is applied in the opposite direction to the first two, relative



(a) Hohmann Transfer



(b) Bi-Elliptic Transfer

Figure 8.4: Use of Tangential Manoeuvre for Different Orbit Transfer Strategies

to the orbital motion.

Note: As the size of the intermediate circle increases, the total Δv decreases while the TOF increases. As $r_t \to \infty$, the transfer ellipses tend towards parabolae.

Note: It can be shown that when $r_f/r_i < 11.94$ a Hohmann transfer is more efficient than a bi-elliptic one, but when $r_f/r_i > 15.58$, the latter is superior is terms of fuel efficiency. This improved efficiency, however, comes at the cost of a more complicated and time-consuming mission architecture.

Rendezvous (Phasing) Consider two spacecraft, orbiting in the same direction in the same circular orbit, that need to rendezvous with each other (for docking purposes, for instance). One way to achieve this goal is for the spacecraft that is further behind to temporarily modify its orbit into a smaller elliptic orbit, and by doing so, move faster in order to catch up with the other spacecraft. This type of manoeuvre, known as a phasing manoeuvre (since it can also be used to change a sing spacecraft's position in its orbit, without another vehicle involved), is depicted in Figure 8.5.

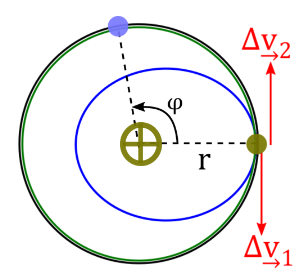


Figure 8.5: Rendezvous

For a circular orbit with a period of T_c, if the phase angle between the two spacecraft is ϕ as shown in Figure 8.5, we have:

$$T_c = 2\pi\sqrt{\frac{r^3}{\mu}} \quad , \quad T_e = 2\pi\sqrt{\frac{a_e^3}{\mu}} = T_c\left(1 - \frac{\phi}{2\pi}\right) \quad \Rightarrow \quad a_e = \sqrt[3]{\mu\left(\frac{T_e}{2\pi}\right)^2} \tag{8.6}$$

where a_e and T_e are the semi-major axis and orbital period of the transfer ellipse. The Δv of the 1st tangential burn on the lagging spacecraft is then provided by Eq. (8.5a) (with $r_i = r$, the circular orbit's radius), and the second impulse is applied at the same point (apoapsis of the transfer orbit) with the same magnitude, $\Delta v_2 = \Delta v_1$, but in the opposite direction to recircularize the ellipse.

Normal Thrust

The transfer between non-coplanar orbits is achieved using out-of-plane manoeuvres that change i, possibly Ω (depending on where on the orbit the thrust is applied), and maybe even a and e as well, depending on the in-plane component of $\Delta\underset{\rightarrow}{v}$.

For a change of α in the inclination of an orbit with no change in its shape and size, as shown in Figure 8.6a, $\Delta\underset{\rightarrow}{v}_o$ should have an angle of $(\pi+\alpha)/2$ relative to $\underset{\rightarrow}{v}_i$, and from the resulting isosceles vector triangle, we have:

$$\Delta v_o = 2v\sin\left(\frac{\alpha}{2}\right) \tag{8.7}$$

and for a general out-of-plane manoeuvre, as depicted in Figure 8.6b with $\Delta\underset{\rightarrow}{v} = \Delta\underset{\rightarrow}{v}_{out} + \Delta\underset{\rightarrow}{v}_{in}$, where $\Delta\underset{\rightarrow}{v}_{in}$ is the in-plane component that acts according to the tangential and radial thrusts previously considered, we have:

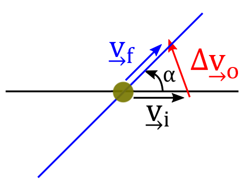
$$(\Delta v)^2 = v_i^2 + v_f^2 - 2v_i v_f \cos(\alpha) \tag{8.8}$$

Note: To change i but not Ω, the out-of-plane manoeuvre should be applied at the ascending node; otherwise, Ω will also change, as illustrated in Figure 8.6c and governed by the following relationships:

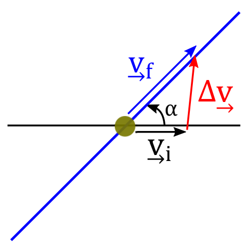
$$\sin(A_{la}) = \frac{\sin(i_f)\cdot\sin(\Delta\Omega)}{\sin(\alpha)} \tag{8.9a}$$

$$\cos(\alpha) \quad = \cos(i_i)\cdot\cos(i_f) + \sin(i_i)\cdot\sin(i_f)\cdot\cos(\Delta\Omega) \tag{8.9b}$$

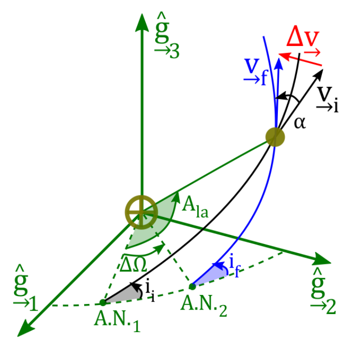
where $A_{la} \triangleq \theta + \omega$ is the *argument of latitude of intersection point.*



(a) Sideview: Change of i



(b) Sideview: i and size



(c) Change of i and Ω, Thrust Not at A.N.

Figure 8.6: Use of Normal Manoeuvre to Change the Orbit's Orientation (and Size)