Lecture 17

Gravity Gradient Stabilization

ATTITUDE motion and its stability under the influence of external torques resulting from gravity gradient are studied. Euler's rigid body equations are applied, in conjunction with the gravity gradient disturbance torques, and linearized stability analysis is performed, assuming small roll, pitch, and yaw displacements of the spacecraft.

Overview

SPIN STABILIZATION and DUAL-SPIN STABILIZATION focused on torque-free motion of a spinning spacecraft, and that of a platform with a spinning wheel, respectively. In contrast, this lesson takes gravity gradient torques into account, and aims to perform a stability analysis similar to those previously seen. Via judicious design, naturally-occurring force fields can be exploited for the purpose of passive attitude stabilization that requires no power, control laws, sensing, etc.

The following reference frames, shown in Figure 17.1, are used for the purpose of this study:

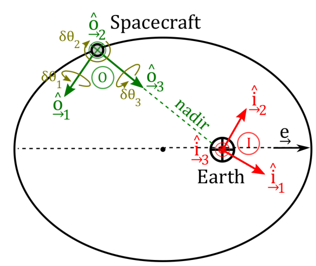

Figure 17.1: Orbiting (O) and Inertial (I) Frames

- $\mathscr{F}_I$: inertial frame fixed to (but not rotating with) Earth
- $\mathscr{F}_O$: orbiting frame, with origin fixed to spacecraft, 3-axis towards Earth's centre, 2-axis anti-parallel to orbital angular momentum, $\underrightarrow{\boldsymbol{h}}$
- $\mathscr{F}_B$: body-fixed frame, with origin at spacecraft centre of mass

A circular orbit is assumed, with a mean motion of $\omega_0 = \sqrt{\mu/r_{\oplus}^3}$.

Based on FUNDAMENTALS, the spacecraft's attitude with respect to the nominal orbiting frame can be described using a 3-2-1 rotation matrix:

$$\boldsymbol{C}_{BO} = \boldsymbol{C}_1(\delta\theta_1)\boldsymbol{C}_2(\delta\theta_2)\boldsymbol{C}_3(\delta\theta_3) \quad \Rightarrow \quad \boldsymbol{C}_{BO} \approx \mathbf{1} - \delta\boldsymbol{\theta}^\times \approx \begin{bmatrix} 1 & \delta\theta_3 & -\delta\theta_2 \\ -\delta\theta_3 & 1 & \delta\theta_1 \\ \delta\theta_2 & -\delta\theta_1 & 1 \end{bmatrix} \quad , \quad \delta\boldsymbol{\theta} \triangleq \begin{bmatrix} \delta\theta_1 \\ \delta\theta_2 \\ \delta\theta_3 \end{bmatrix} \qquad (17.1)$$

where $\delta\theta_1$, $\delta\theta_2$, and $\delta\theta_3$ are the infinitesimal roll, pitch, and yaw angles, respectively. Small Euler angle approximations are used in Eq. (17.1).

The spacecraft's position and angular velocity (with respect to $\mathscr{F}_I$) can be resolved in $\mathscr{F}_B$ as:

$$\boldsymbol{r}_{\oplus} = \boldsymbol{C}_{BO} \begin{bmatrix} 0 \\ 0 \\ -r_{\oplus} \end{bmatrix} = \begin{bmatrix} r_{\oplus}\delta\theta_2 \\ -r_{\oplus}\delta\theta_1 \\ -r_{\oplus} \end{bmatrix} \quad , \quad \boldsymbol{\omega}^{BI} = \cancelto{\delta\dot{\boldsymbol{\theta}}}{\boldsymbol{\omega}^{BO}} + \boldsymbol{C}_{BO} \begin{bmatrix} 0 \\ -\omega_0 \\ 0 \end{bmatrix} = \begin{bmatrix} \delta\dot{\theta}_1 - \omega_0\delta\theta_3 \\ \delta\dot{\theta}_2 - \omega_0 \\ \delta\dot{\theta}_3 + \omega_0\delta\theta_1 \end{bmatrix} \tag{17.2}$$

where Eq. (17.1) is used to replace $\boldsymbol{C}_{BO}$, and assuming a circular orbit, $\mathscr{F}_O$'s rotation about its 3-axis at a constant rate of ω_0 (but in the negative $\underset{\rightarrow}{\hat{\boldsymbol{o}}}_3$ direction) and the constant distance of $r_{\oplus}$ between O_I and O_O (but in the negative $\underset{\rightarrow}{\hat{\boldsymbol{o}}}_1$ direction) are used to express $\boldsymbol{\omega}_O^{OI}$ and $\boldsymbol{r}_O$, the components of the spacecraft's angular velocity and position as expressed in $\mathscr{F}_O$.

Equations of Motion

Assume the selected body-fixed frame is a principal axes frame; that is, $\mathscr{F}_B \equiv \mathscr{F}_P$. Using the diagonal moment of inertia matrix, $\boldsymbol{I}$, associated with such a frame, the gravity gradient disturbance torques can be determined as derived in DISTURBANCE TORQUES:

$$\boldsymbol{\tau}_{gg} = \frac{3\mu}{r_{\oplus}^5}\boldsymbol{r}_{\oplus}^{\times}\boldsymbol{I}\boldsymbol{r}_{\oplus} \tag{17.3}$$

in which the left-hand side relationship in Eq (17.2) can be substituted to yield:

$$\boldsymbol{\tau}_{gg} = \frac{3\mu}{r_{\oplus}^5}\left(r_{\oplus}\begin{bmatrix} \delta\theta_2 \\ -\delta\theta_1 \\ -1 \end{bmatrix}\right)^{\times}\boldsymbol{I}\left(r_{\oplus}\begin{bmatrix} \delta\theta_2 \\ -\delta\theta_1 \\ -1 \end{bmatrix}\right) = 3\omega_0^2\begin{bmatrix} \delta\theta_2 \\ -\delta\theta_1 \\ -1 \end{bmatrix}^{\times}\boldsymbol{I}\begin{bmatrix} \delta\theta_2 \\ -\delta\theta_1 \\ -1 \end{bmatrix} = 3\omega_0^2\begin{bmatrix} (I_3 - I_2)\delta\theta_1 \\ (I_3 - I_1)\delta\theta_2 \\ (I_1 - I_2)\cancelto{\approx 0}{\delta\theta_1\delta\theta_2} \end{bmatrix} \tag{17.4}$$

where $\omega_0 = \sqrt{\mu/r_{\oplus}^3}$ owing to the orbit's circularity assumption is used, and the second order term $\delta\theta_1\delta\theta_2$ is neglected assuming small angles. As usual, I_1, I_2, and I_3 are the spacecraft's principal moments of inertia, the components of its diagonal $\boldsymbol{I}$ determined for $\mathscr{F}_P$.

The following observations could readily be made about the gravity gradient torque, $\boldsymbol{\tau}_{gg}$:

- It has no (or negligible) 3-component (yaw), which seems logical since the gravitational force acts along $\underset{\rightarrow}{\hat{\boldsymbol{o}}}_3$ and its resulting torque will be perpendicular to it.

- Its 1- and 2-components (roll and pitch) are proportional to the roll and pitch angles. Having $I_3 < I_2$ or $I_3 < I_1$ will, therefore, result in a restoring torque about the roll or pitch axis, respectively.

Neglecting all external torques other than $\boldsymbol{\tau}_{gg}$, Euler's rigid body equations can be written in the following scalar form upon expanding $\boldsymbol{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega}^{\times}\boldsymbol{I}\boldsymbol{\omega} = \boldsymbol{\tau}_{gg}$ and simplifying by assuming small angles and rate:

$$I_1\delta\ddot{\theta}_1 - (I_1 - I_2 + I_3)\omega_0\delta\dot{\theta}_3 + 4\omega_0^2(I_2 - I_3)\delta\theta_1 = 0 \tag{17.5a}$$

$$I_2\delta\ddot{\theta}_2 + 3\omega_0^2(I_1 - I_3)\delta\theta_2 = 0 \tag{17.5b}$$

$$I_3\delta\ddot{\theta}_3 + (I_1 - I_2 + I_3)\omega_0\delta\dot{\theta}_1 + \omega_0^2(I_2 - I_1)\delta\theta_3 = 0 \tag{17.5c}$$

which are the roll, pitch, and yaw equations of motion, respectively.

Attitude stability

The pitch equation is observed to be decoupled from the other two, while the roll-yaw equations are coupled and should be studied together. We now apply our familiar tools of stability analysis to assess the spacecraft's attitude stability in each direction.

Pitch Stability

From Eq. (17.5b), we have:

$$\delta\ddot{\theta}_2 = \beta^2 \delta\theta_2 \quad , \quad \beta^2 = \frac{-3\omega_0^2(I_1 - I_3)}{I_2} \tag{17.6}$$

which was encountered (at times for $\delta\omega_i$) in SPIN STABILIZATION and DUAL-SPIN STABILIZATION as well. The general solution is:

$$\delta\theta_2(t) = Ae^{\beta t} + Be^{-\beta t} \quad , \quad \beta \neq 0 \tag{17.7a}$$

$$\delta\theta_2(t) = A + Bt \quad , \quad \beta = 0 \tag{17.7b}$$

from which the boundedness of $\delta\theta_2$ can be deduced:

- if $\beta^2 > 0$: $e^{\beta t} \to \infty$ as $t \to \infty$, so $\delta\theta_2$ is *unbounded*.
- if $\beta^2 = 0$: $Bt \to \infty$ as $t \to \infty$, so $\delta\theta_2$ is *unbounded*.
- if $\beta^2 < 0$: $e^{\pm\beta t} = \cos(\mp i\beta t) + i\sin(\mp i\beta t)$ as $t \to \infty$, so $\delta\theta_2$ is *bounded*.

Thus, *pitch stability* occurs only when $\beta^2 \triangleq -3\omega_0^2(I_1 - I_3)/I_2 < 0$, which is satisfied if and only if $I_1 > I_3$. The *pitch oscillation frequency* is given by:

$$\Omega^2 \triangleq -\beta^2 \quad \Rightarrow \quad \Omega = \omega_0\sqrt{3\frac{I_1 - I_3}{I_2}} \tag{17.8}$$

where ω_0 depends on the size of the orbit, and the square root term is determined by the spacecraft's moment of inertia matrix, $\boldsymbol{I}$.

Roll/Yaw Stability

From Eqs. (17.5a) and (17.5c), we have the following (reduced) linear mechanical system:

$$\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{G}\dot{\boldsymbol{q}} + \boldsymbol{K}\boldsymbol{q} = \boldsymbol{0} \tag{17.9}$$

$$\boldsymbol{q} \triangleq \begin{bmatrix} \delta\theta_1 \\ \delta\theta_3 \end{bmatrix} \quad , \quad \boldsymbol{M} \triangleq \begin{bmatrix} I_1 & \boldsymbol{0} \\ 0 & I_3 \end{bmatrix} \quad , \quad \boldsymbol{G} \triangleq \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}(I_1 - I_2 + I_3)\omega_0 \quad , \quad \boldsymbol{K} \triangleq \begin{bmatrix} 4(I_2 - I_3) & 0 \\ 0 & I_2 - I_1 \end{bmatrix}\omega_0^2$$

where $\boldsymbol{M} = \boldsymbol{M}^\mathsf{T} > 0$, $\boldsymbol{G} = -\boldsymbol{G}^\mathsf{T}$, and $\boldsymbol{K} = \boldsymbol{K}^\mathsf{T}$. As discussed previously, such a conservative gyric system:

- is *statically stable* if $\boldsymbol{K} > 0$, satisfied if and only if $I_2 > I_1$, $I_2 > I_3$.
- may be *gyrically stable* even if $\boldsymbol{K} \ngtr 0$, satisfied under certain conditions for I_1 and I_3.

Combining these results with the $I_1 > I_3$ condition for pitch stability, we observe that three-axis *attitude stability* is guaranteed for $I_2 > I_1 > I_3$. In other words, a spacecraft with its major axis along the orbital plane's normal and its minor axis in the nadir direction is gravity gradient-stabilized.

Note: Even though $\boldsymbol{\tau}_{gg}$ only acts on pitch and roll, yaw is also influenced owing to the roll/yaw coupling.

To determine the conditions under which *gyric stability* is achieved (for $\boldsymbol{K} \not> 0$), we study the reduced system's characteristic equation:

$$b_0 r^4 + b_1 r^2 + b_2 = 0 \tag{17.10}$$

$$b_0 \triangleq I_1 I_3 \quad , \quad b_1 \triangleq \big[I_1 I_3 + 3(I_2 - I_3)I_3 + (I_2 - I_3)(I_2 - I_1)\big]\omega_0^2 \quad , \quad b_2 \triangleq 4(I_2 - I_3)(I_2 - I_1)\omega_0^4$$

We define $\lambda \triangleq r^2/\omega_0^2$, and requiring the poles to not be on the right half plane, arrive at the following necessary and sufficient conditions for stability: $b_0 > 0$, $b_1 > 0$, $b_2 > 0$, and $b_1^2 - 4b_0 b_2 > 0$.

To better understand this result and also convert it into some conditions for $k_1 \triangleq (I_2 - I_3)/I_1$ and $k_3 \triangleq (I_2 - I_1)/I_3$, we can divide Eq. (17.10) by $I_1 I_3$ and make use of the definition of λ above to arrive at:

$$r^4 + p\omega_0^2 r^2 + q\omega_0^4 = 0 \quad , \quad p \triangleq 1 + 3k_1 + k_1 k_3 \quad , q \triangleq 4k_1 k_3 \quad \Rightarrow \quad \lambda^2 + p\lambda + q = 0 \tag{17.11}$$

the right-hand side relationship of which is obtained by dividing the left-hand side one by ω_0^4. The solutions of this equation can be categorized as those with real or complex values.
For $\lambda \in \mathbb{R}$:

- if $\lambda > 0$, r is real, $r = \pm\omega_0\sqrt{\lambda}$, resulting in unbounded exponential growth of $\delta\theta_1$ or $\delta\theta_3$.
- if $\lambda = 0$, $r = 0$, resulting in unbounded linear growth of $\delta\theta_1$ and $\delta\theta_3$.
- if $\lambda < 0$, r is purely imaginary, $r = \pm i\omega_0\sqrt{-\lambda}$, resulting in oscillations of $\delta\theta_1$ and $\delta\theta_3$ for $\lambda_1 \neq \lambda_2$.

For $\lambda \in \mathbb{C}$, $\lambda \notin \mathbb{R}$: we have $\lambda = \rho e^{i\theta}$, implying $\sqrt{\lambda} = \sqrt{\rho}e^{i\theta/2}$, but $-\pi/2 < \theta/2 < \pi/2$ (considering $\theta \notin \{-\pi, 0, \pi\}$, as they would correspond to real values of λ), which implies $\sqrt{\lambda}$ has a positive real part, and $r = \pm\omega_0\sqrt{\lambda}$, resulting in unbounded exponential growth of $\delta\theta_1$ or $\delta\theta_3$.

Among the above possibilities, the only stable case corresponds to $\lambda < 0$ with $\lambda_1 \neq \lambda_2$ (since, otherwise, repeated imaginary poles would result in oscillatory behaviour with unbounded linear growth). This yields the following stability conditions:

$$r^2 = \omega_0^2\Big(\frac{-p \pm \sqrt{p^2 - 4q}}{2}\Big) < 0 \quad \Rightarrow \quad p^2 - 4q > 0 \quad , \quad p > 0 \quad , \quad q > 0 \tag{17.12}$$

where the first condition ensures non-repeated poles, while the second and third conditions guarantee $-p + \sqrt{p^2 - 4q}$ is not positive. Returning to the definition of p and q in Eq. (17.11), the following necessary and sufficient conditions for roll/yaw stability are obtained:

$$1 + 3k_1 + k_1 k_3 > 0 \text{ and } k_1 k_3 > 0 \text{ and } (1 + 3k_1 + k_1 k_3)^2 - 16k_1 k_3 > 0 \tag{17.13}$$

When combined with $k_1 > k_3$ required for pitch stability (implying $I_1 > I_3$), these conditions provide the so-called "Lagrange region" on the $k_1 - k_3$ stability diagram (with $k_1 > k_3 > 0$, the same region associated with *static stability*), as well as an additional part known as the "DeBra-Delp region" for *gyric stability*. Such diagrams may prove useful for inertia augmentation, which involves designing spacecraft that possess desirable gravity gradient stabilization properties via judicious placement and size of booms with tip masses in order to modify $\boldsymbol{I}$ as required.

Note: Similarly to the previous cases of *gyric stability*, the DeBra-Delp region becomes *unstable* when damping is introduced, while the Lagrange region becomes *asymptotically stable*.

We can summarize the roll/yaw stability regions as follow:

- Lagrange region: $I_2 > I_1 > I_3$, a combination of major axis spin (about $\underset{\rightarrow}{\hat{\boldsymbol{p}}}_2$, which is almost along $\underset{\rightarrow}{\hat{o}}_2$) with a favourable (restoring) $\boldsymbol{\tau}_{gg}$ owing to $k_1 > 0$ and $k_3 > 0$
- DeBra-Delp region: a combination of minor axis spin (stable in the absence of damping) dominating over unfavourable $\boldsymbol{\tau}_{gg}$ owing to $k_1 < 0$

Note: Gravity gradient stabilization is very coarse, some times to only within $\pm 20°$. Active attitude control, to be discussed next, is required for higher accuracy.

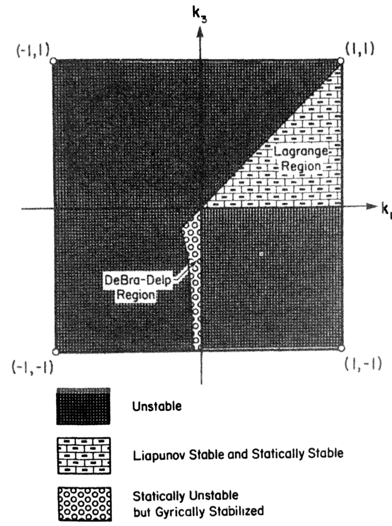

Figure 17.2: Gravity Gradient [Hughes] (used with permission)

References

[Hughes] Hughes, P. C., *Spacecraft Attitude Dynamics*, Dover Publications Inc., New York, Chap. 9.