Lecture 12

Torque-Free Motion

ATTITUDE dynamics in the absence of external torques is considered, as it is a good approximation for situations with weak torques. The special case of an axisymmetric object, which leads to an analytic solution to the equations of motion, is studied in detail, and geometric interpretations of the motion for this special torque-free case are provided.

Overview

Recall Euler's equations of motion from DYNAMICS, in particular for the case with a diagonal moment of inertia matrix, $\boldsymbol{I}$, that leads to a set of three scalar equations. Assuming there is no torque, $\boldsymbol{\tau} \equiv \mathbf{0}$:

$$I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 = \cancelto{0}{\tau_1} \tag{12.1a}$$

$$I_2\dot{\omega}_2 + (I_1 - I_3)\omega_3\omega_1 = \cancelto{0}{\tau_2} \tag{12.1b}$$

$$I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 = \cancelto{0}{\tau_3} \tag{12.1c}$$

where I_1, I_2, and I_3 are the body's principal moments of inertia, and ω_1, ω_2, and ω_3 are the components of its angular velocity $\boldsymbol{\omega}_P$ in the body-fixed principal axes frame, $\mathscr{F}_P$. This torque-free case is an important one, because a general rotational motion can be considered as a deviation from this reference motion. Before proceeding to study the torque-free motion in more detail, we note that rotational kinetic energy and angular momentum are constant for this type of motion:

$$T = \frac{1}{2}\boldsymbol{\omega}^\mathsf{T}\boldsymbol{I}\boldsymbol{\omega} = \frac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right) \;\Rightarrow\; \dot{T} = I_1\omega_1\dot{\omega}_1 + I_2\omega_2\dot{\omega}_2 + I_3\omega_3\dot{\omega}_3 = \boldsymbol{\omega}^\mathsf{T}\cancelto{= -\boldsymbol{\omega}^\times\boldsymbol{I}\boldsymbol{\omega}}{\left(\boldsymbol{I}\overset{\bullet}{\boldsymbol{\omega}}\right)} = 0 \tag{12.2a}$$

$$h = |\boldsymbol{I}\boldsymbol{\omega}| = \sqrt{I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2} \;\Rightarrow\; \dot{h} = \frac{1}{h}\left(I_1^2\omega_1\dot{\omega}_1 + I_2^2\omega_2\dot{\omega}_2 + I_3^2\omega_3\dot{\omega}_3\right) = \frac{\left(\boldsymbol{I}\boldsymbol{\omega}\right)^\mathsf{T}}{h}\cancelto{= -\boldsymbol{\omega}^\times\boldsymbol{I}\boldsymbol{\omega}}{\left(\boldsymbol{I}\overset{\bullet}{\boldsymbol{\omega}}\right)} = \mathbf{0} \tag{12.2b}$$

where the triple product identity is used to simplify both equations to zero. In fact, we observe that, in addition to its magnitude, the direction of the angular momentum vector *in the inertial frame*, $\mathscr{F}_I$, is constant too:

$$\underset{\rightarrow}{h}^\bullet = \underset{\rightarrow}{\tau} = \underset{\rightarrow}{0} \;\Rightarrow\; \underset{\rightarrow}{h} : \text{constant} \tag{12.3}$$

Because of the nonlinearity of Eq. (12.1) owing to the presence of the $\omega_i\omega_j$ terms, there are numerical difficulties associated with integrating these equations. Analytical solutions of Eq. (12.1) exist for a limited number of special cases. We choose to focus on the special case of an inertially axisymmetrical body.

Definition. A rigid body is called *inertially axisymmetrical* if two of its principal moments of inertia are equal: $I_1 = I_2 = I_t$ (transverse) and $I_3 = I_a$ (axial). Similarly, an *isoinertial* body has the same moment of inertia about all three of its principal axes: $I_1 = I_2 = I_3 = I$.

Note: An inertially axisymmetrical body need not be a body of revolution, like a circular shaft. For example, a rectangular prism of uniform density with side lengths a, a, and b is inertially axisymmetrical as well.

Note: The torque-free motion of an isoinertial body has a trivial solution, because Eq. (12.1) reduces to $I\dot{\omega}_1 = I\dot{\omega}_2 = I\dot{\omega}_3 = 0$, which implies a constant $\boldsymbol{\omega}$. The result is a constant rate spinning of the body about an axis fixed in itself and in an inertial frame, $\mathscr{F}_I$.

Angular Velocity Vector

For an inertially axisymmetric body with moments of inertia $I_1 = I_2 = I_t$ and $I_3 = I_a$, Eq. (12.1) becomes:

$$I_t\dot{\omega}_1 = (I_t - I_a)\omega_2\omega_3 \tag{12.4a}$$

$$I_t\dot{\omega}_2 = (I_a - I_t)\omega_3\omega_1 \tag{12.4b}$$

$$I_a\dot{\omega}_3 = 0 \qquad \Rightarrow \omega_3 = \omega_{3_0} \triangleq \nu \tag{12.4c}$$

where we let ν represent the constant 3-component of the angular velocity. We then define the *relative spin rate* as:

$$\Omega \triangleq \left(\frac{I_t - I_a}{I_t}\right)\nu$$

using which in Eq. (12.4) and differentiating the result yields:

$$\dot{\omega}_1 - \Omega\omega_2 = 0 \quad \Rightarrow \quad \ddot{\omega}_1 - \Omega\dot{\omega}_2 = \ddot{\omega}_1 + \Omega^2\omega_1 = 0 \tag{12.5a}$$

$$\dot{\omega}_2 + \Omega\omega_1 = 0 \quad \Rightarrow \quad \ddot{\omega}_2 + \Omega\dot{\omega}_1 = \ddot{\omega}_2 + \Omega^2\omega_2 = 0 \tag{12.5b}$$

where Eqs. (12.4b) and (12.4a) are used for $\dot{\omega}_2$ and $\dot{\omega}_1$ in the right-hand side relationship. The general solutions of these ODEs are provided by:

$$\omega_1(t) = A\cos(\Omega t) + B\sin(\Omega t) \tag{12.6a}$$

$$\omega_2(t) = C\cos(\Omega t) + D\sin(\Omega t) \tag{12.6b}$$

differentiating which with respect to time and using the ICs of Eq. (12.5) in which yields:

$$\dot{\omega}_1(0) = B\Omega = \Omega\omega_2(0) \quad \Rightarrow \quad B = \omega_{2_0} = C \tag{12.7a}$$

$$\dot{\omega}_2(0) = D\Omega = -\Omega\omega_1(0) \quad \Rightarrow \quad D = -\omega_{1_0} = -A \tag{12.7b}$$

With the coefficients of the particular solution determined and substituted back into Eq. (12.6), it can easily be shown that:

$$\omega_t(t) \triangleq \sqrt{\omega_1^2(t)+\omega_2^2(t)} = \sqrt{\omega_{1_0}^2+\omega_{2_0}^2} \quad \Rightarrow \quad \omega_t : \text{constant} \tag{12.8a}$$

$$\omega(t) \triangleq \sqrt{\omega_1^2(t)+\omega_2^2(t)+\omega_3^2(t)} = \sqrt{\omega_t^2+\nu^2} \quad \Rightarrow \quad \omega : \text{constant} \tag{12.8b}$$

where we introduce ω_t, the transverse component of $\underset{\rightarrow}{\boldsymbol{\omega}}$. The fact that it remains constant implies the angular velocity vector's projection onto the 1-2 plane rotates in a circle. With this definition, the angular velocity solution becomes:

$$\omega_1(t) = \omega_{1_0}\cos(\Omega t) + \omega_{2_0}\sin(\Omega t) = \omega_t \sin\left[\Omega t + \tan^{-1}\left(\frac{\omega_{1_0}}{\omega_{2_0}}\right)\right] \tag{12.9a}$$

$$\omega_2(t) = \omega_{2_0}\cos(\Omega t) - \omega_{1_0}\sin(\Omega t) = \omega_t \cos\left[\Omega t + \tan^{-1}\left(\frac{\omega_{1_0}}{\omega_{2_0}}\right)\right] \tag{12.9b}$$

which follows from the following general trigonometric identity:

$$A\sin(\theta) + B\cos(\theta) = \sqrt{A^2+B^2}\sin\left[\theta + \tan^{-1}\left(\frac{B}{A}\right)\right] = \sqrt{A^2+B^2}\cos\left[\theta + \tan^{-1}\left(\frac{-A}{B}\right)\right] \tag{12.10}$$

Lastly, defining t_0 such that $\omega_1(t_0) = 0$ and $\omega_2(t_0) = \omega_t$, we introduce the *spin angle* as follows:

$$\mu(t) \triangleq \Omega(t - t_0)$$

making note that $\dot{\mu} = \Omega$ is the relative spin rate. With this definition, the particular angular velocity solution, resolved in $\mathscr{F}_P$ and given by Eq. (12.6), can be rewritten as:

$$\omega_1(t) = \omega_t \sin\big(\mu(t)\big) \tag{12.11a}$$

$$\omega_2(t) = \omega_t \cos\big(\mu(t)\big) \tag{12.11b}$$

$$\omega_3(t) = \nu \tag{12.11c}$$

Since ν and ω_t (and ω) are constant, we obtain an angular velocity vector, $\underset{\rightarrow}{\boldsymbol{\omega}}$, that is spinning about $\underset{\rightarrow}{\hat{\boldsymbol{p}}}_3$ of $\mathscr{F}_P$ (the symmetry direction that has I_a as the principal moment of inertia about itself) as shown in Figure 12.1a, with its tip tracing a circle and its 1- and 2-components changing periodically. We denote the constant angle between $\underset{\rightarrow}{\boldsymbol{\omega}}$ and $\underset{\rightarrow}{\hat{\boldsymbol{p}}}_3$ as β. Thus far, we have a description of the motion of $\underset{\rightarrow}{\boldsymbol{\omega}}$ with respect to $\mathscr{F}_P$, but our ultimate goal is to describe the motion of $\mathscr{F}_P$ with respect to $\mathscr{F}_I$ in this special case.

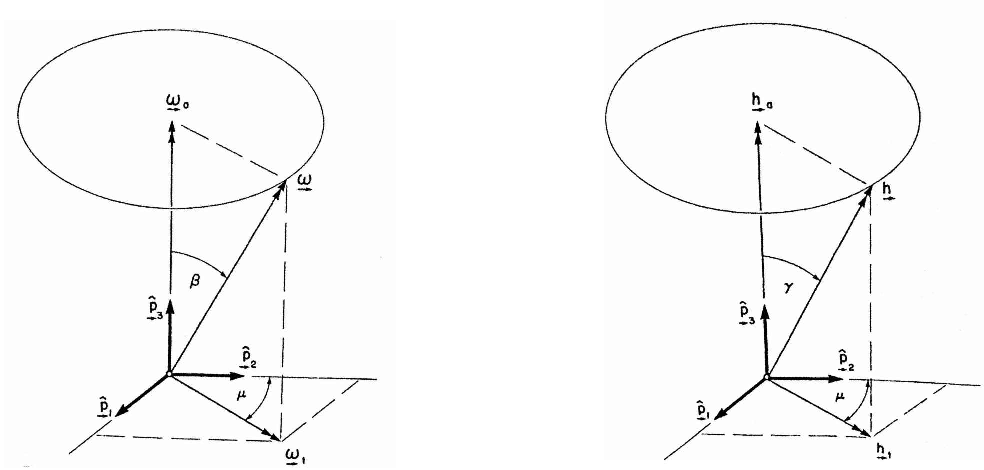
Angular Momentum Vector

Using $\boldsymbol{h}_P = \boldsymbol{I}\boldsymbol{\omega}_P$, the inertial axisymmetry assumption on the body, and the velocity vector components obtained in Eq. (12.11), the components of angular momentum in $\mathscr{F}_P$ corresponding to this special case of torque-free motion can be written as

$$h_1(t) = I_1\omega_1(t) = I_t\omega_t \sin\big(\mu(t)\big) \quad \Rightarrow \quad h_1(t) = h_t \sin\big(\mu(t)\big) \tag{12.12a}$$

$$h_2(t) = I_2\omega_2(t) = I_t\omega_t \cos\big(\mu(t)\big) \quad \Rightarrow \quad h_2(t) = h_t \cos\big(\mu(t)\big) \tag{12.12b}$$

$$h_3(t) = I_3\omega_3(t) = I_a\nu \quad \Rightarrow \quad h_3(t) = I_a\nu = h_a \tag{12.12c}$$



(a) Angular Velocity Vector

(b) Angular Momentum Vector

Figure 12.1: Spinning of Vectors as Observed in Principal Axes Frame, $\mathscr{F}_P$ [Hughes] (used with permission)

from which, similarly to the angular velocity vector, the following constant magnitudes can be deduced:

$$h_t(t) \triangleq \sqrt{h_1^2(t)+h_2^2(t)} = \sqrt{h_t^2\left[\sin^2\big(\mu(t)\big)+\cos^2\big(\mu(t)\big)\right]} = h_t \quad \Rightarrow \quad h_t : \text{constant} \tag{12.13a}$$

$$h(t) \triangleq \sqrt{h_1^2(t)+h_2^2(t)+h_3^2(t)} = \sqrt{h_t^2+h_a^2} \quad \Rightarrow \quad h : \text{constant} \tag{12.13b}$$

which is consistent with Eq. (12.2b). Therefore, the angular momentum vector also rotates about $\hat{\underrightarrow{p}}_3$ of $\mathscr{F}_P$ with its tip on a circle of radius $h_t = I_t\omega_t$, as shown in Fig. 12.1b. The constant angle between $\underrightarrow{h}$ and $\hat{\underrightarrow{p}}_3$, represented by γ, is known as the *nutation angle*.

Note: For real "nearly" axisymmetrical bodies, γ is not constant and oscillates about a fixed mean value. This oscillatory process is known as *nutation*.

Since this spinning and that of the angular velocity vector are both dictated by the angle $\mu(t)$, we conclude that $\underrightarrow{h}$, $\underrightarrow{\omega}$, and $\hat{\underrightarrow{p}}_3$ always lie on the same plane during the torque-free rotation of an axisymmetrical rigid body, as illustrated in Figure 12.2. Recall, also, from Eq. (12.3) that $\underrightarrow{h}$ is fixed (both in magnitude and direction) in the inertial space, $\mathscr{F}_I$, even though it is rotating in the body space, $\mathscr{F}_P$, as just described. Without loss of generality, we let $\underrightarrow{h}$ be aligned with $\hat{\underrightarrow{i}}_3$ of $\mathscr{F}_I$, and proceed to study the body's attitude over time.

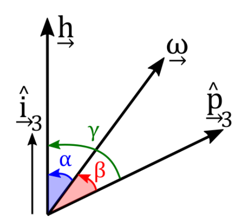


Figure 12.2: Coplanar Vectors

Attitude History

Using the following sequence of rotations, $\mathscr{F}_P$ may be obtained starting from $\mathscr{F}_I$ as depicted in Figure 12.3:

- rotating about $\hat{\underrightarrow{i}}_3 = \hat{\underrightarrow{i}}'_3$ by *precession angle*, ψ
- rotating about $\hat{\underrightarrow{i}}'_1 = \hat{\underrightarrow{i}}''_1$ by *nutation angle*, γ
- rotating about $\hat{\underrightarrow{i}}''_3 = \hat{\underrightarrow{p}}'_3$ by *spin angle*, μ

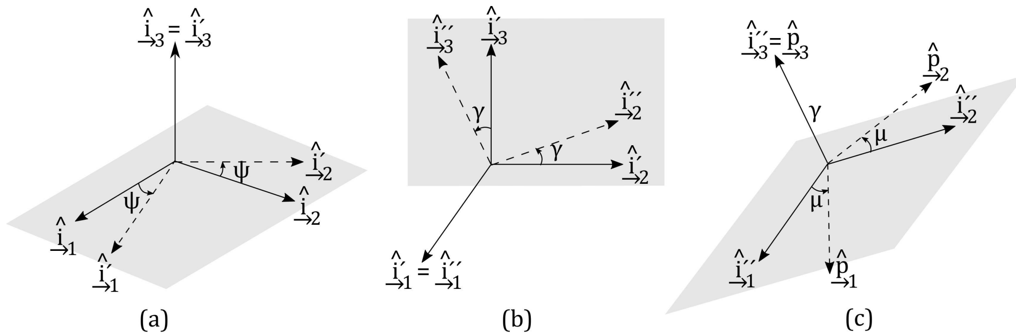


Figure 12.3: Principal Rotation Sequence to Transform $\mathscr{F}_I$ to $\mathscr{F}_P$

where $(\cdot)'$ and $(\cdot)''$ represent basis vectors of intermediate reference frames resulting from the first and the second rotation, respectively. Therefore, from FUNDAMENTALS, the following rotation matrix describes the attitude of $\mathscr{F}_P$ with respect to $\mathscr{F}_I$:

$$\boldsymbol{C}_{PI} = \boldsymbol{C}_3(\mu)\boldsymbol{C}_1(\gamma)\boldsymbol{C}_3(\psi) \tag{12.14}$$

and recalling the relationship between $\boldsymbol{\omega}_P^{PI}(t)$ and $\boldsymbol{\theta}(t) = [\mu\ \gamma\ \psi]^\mathsf{T}$ from KINEMATICS, we have:

$$\boldsymbol{\omega}_P^{PI} = \boldsymbol{S}(\mu,\gamma)\overset{\bullet}{\boldsymbol{\theta}} = \begin{bmatrix}\mathbf{1}_3 & \boldsymbol{C}_3(\mu)\mathbf{1}_1 & \boldsymbol{C}_3(\mu)\boldsymbol{C}_1(\gamma)\mathbf{1}_3\end{bmatrix}\begin{bmatrix}\cancelto{\Omega}{\dot{\mu}} \\ \cancelto{0}{\dot{\gamma}} \\ \dot{\psi}\end{bmatrix} \Rightarrow \boldsymbol{\omega} = \begin{bmatrix}\dot{\psi}\sin(\gamma)\sin\left(\mu(t)\right) \\ \dot{\psi}\sin(\gamma)\cos\left(\mu(t)\right) \\ \Omega + \dot{\psi}\cos(\gamma)\end{bmatrix} \tag{12.15}$$

where the fact that γ is constant (since $\underset{\rightarrow}{\boldsymbol{h}}$ rotates about $\underset{\rightarrow}{\hat{\boldsymbol{p}}}_3$ with a constant nutation angle) and the definition $\mu \triangleq \Omega(t - t_0)$ are used. The *precession rate* is, thus, obtained by rearranging the last row of Eq. (12.15) as follows:

$$\Omega_p \triangleq \dot{\psi} = \frac{\cancelto{\nu}{\omega_3} - \Omega}{\cos(\gamma)} = \frac{\nu - (I_t - I_a)\nu/I_t}{\cos\gamma} = \frac{\cancelto{h_a}{I_a\nu}}{I_t\cos(\gamma)} \Rightarrow \Omega_p = \frac{h}{I_t} \tag{12.16}$$

where $h_a \triangleq h\cos(\gamma)$ is used. This is the rate at which the body-fixed $\underset{\rightarrow}{\hat{\boldsymbol{p}}}_3$ rotates about the inertial $\underset{\rightarrow}{\hat{\boldsymbol{\imath}}}_3$, and it can be related to the previously-defined relative spin rate as follows:

$$\Omega_p = \frac{\nu - \Omega}{\cos(\gamma)} = \frac{I_a\Omega}{(I_t - I_a)\cos(\gamma)} \tag{12.17}$$

In summary, given the principal moments of inertia, I_t and I_a, and the ICs, $\boldsymbol{\omega}_P(0)$ and ψ_0, the time history of an axisymmetrical rigid body with respect to the inertial frame can be obtained using Eq. (12.14), with the angles given by:

$$\psi(t) = \Omega_p t + \psi_0 \ \ , \ \ \gamma(t) = \gamma_0 \ \ , \ \ \mu(t) = \Omega(t - t_0) = \Omega t - \mu_0 \tag{12.18}$$

where it should be emphasized that the ICs γ_0 and μ_0 are not arbitrary, and are dictated by our initial choice of aligning $\underset{\rightarrow}{\boldsymbol{h}}$ with $\underset{\rightarrow}{\hat{\boldsymbol{\imath}}}_3$. For a more general case that avoids this assumption, two rotational transformations on the above results can be used.

Geometrical Interpretation

We use the angles β, γ, and $\alpha \triangleq \gamma - \beta$ (all shown in Figure 12.2) to geometrically describe the rotational motion of an axisymmetrical rigid body. From their definition, we have:

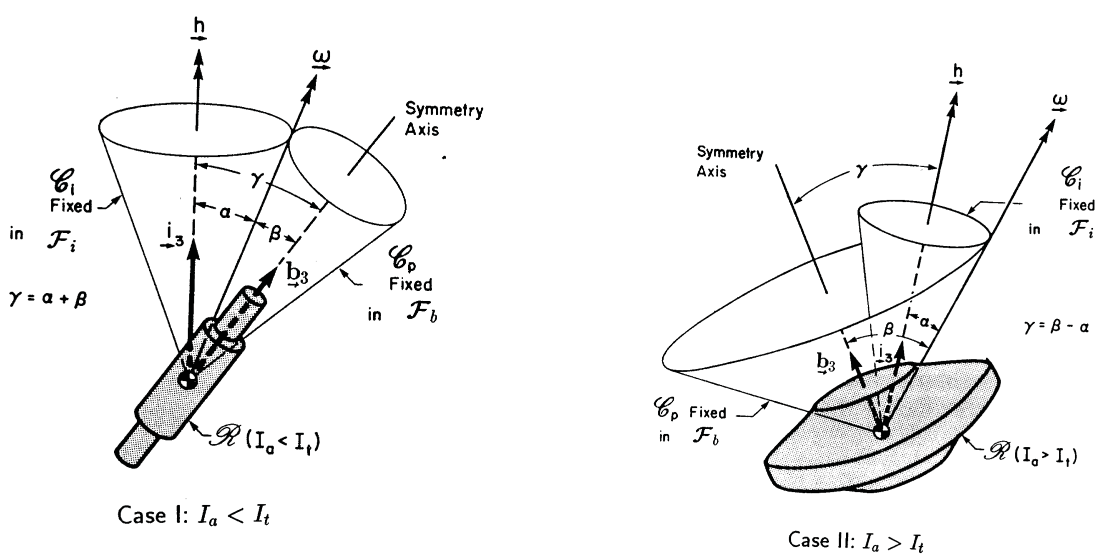
$$\tan(\gamma) = \frac{h_t}{h_a} = \frac{I_t \omega_t}{I_a \omega_3} \quad , \quad \tan(\beta) = \frac{\omega_t}{\omega_3} \quad \Rightarrow \quad \tan(\gamma) = \frac{I_t}{I_a} \tan(\beta) \tag{12.19}$$

which provides a comparison between γ and β depending on the ratio of the principal moments of inertia. Consider two cones: the "space cone" , $\mathscr{C}_{\mathrm{s}}$, with its axis of symmetry along $\hat{\underset{\rightarrow}{\boldsymbol{i}}}_3$ and half-angle α; and the "body cone", $\mathscr{C}_{\mathrm{p}}$, with its axis of symmetry along $\hat{\underset{\rightarrow}{\boldsymbol{p}}}_3$ and half-angle β. To describe the motion:

- for a "prolate" body (like a pencil) with $I_t > I_a$, we have $\gamma > \beta$, so the motion can be described by the rolling without slipping motion of the body cone on the outside surface of the inertially-fixed space cone, as shown in Figure 12.4a. Since Ω and Ω_p have equal signs (from Eq. (12.17)), this is a "prograde precession".

- for an "oblate" body (like a disk) with $I_t < I_a$, we have $\gamma < \beta$, so the motion can be described by the rolling without slipping motion of the body cone, while the inertially-fixed space cone is inside the body cone, as shown in Figure 12.4b. Since Ω and Ω_p have opposite signs (from Eq. (12.17)), this is a "retrograde precession".

To see where the above interpretation comes from, consider the rolling without slipping of the body cone on the space cone, with its instantaneous angular velocity, $\underset{\rightarrow}{\boldsymbol{\omega}}'$, along the contact line of the cones. Owing to our construction, $\boldsymbol{\omega}$ of the body's rotation is also along this line, since $\gamma = \alpha + \beta$ represents the angle between the cones' axes of symmetry, so we have $\underset{\rightarrow}{\boldsymbol{\omega}} \parallel \underset{\rightarrow}{\boldsymbol{\omega}}'$. But using the definition of angular velocity from KINEMATICS, we have:



(a) Prolate: Prograde Precession

(b) Oblate: Retrograde Precession

Figure 12.4: Torque-Free Rotation of (a) Prolate and (b) Oblate Bodies [Hughes] (used with permission)

- $\hat{\underset{\to}{p}}_3$ is fixed in the body cone, so $\hat{\underset{\to}{p}}_3^{\bullet} = \underset{\to}{\omega}' \times \hat{\underset{\to}{p}}_3$.
- $\hat{\underset{\to}{p}}_3$ is fixed in the body-fixed frame, so $\hat{\underset{\to}{p}}_3^{\bullet} = \underset{\to}{\omega} \times \hat{\underset{\to}{p}}_3$.

Therefore, since the two vectors $\underset{\to}{\omega}$ and $\underset{\to}{\omega}'$ are parallel and have the same cross product with the vector $\hat{\underset{\to}{p}}_3$, we must have $\underset{\to}{\omega} \equiv \underset{\to}{\omega}'$. This establishes the validity of the above interpretation, namely using the cones' rolling to describe the rigid body's rotational motion.

References

[Hughes] Hughes, P. C., *Spacecraft Attitude Dynamics*, Dover Publications Inc., New York, Chap. 4.