Lecture 4

Orbital Mechanics

THIS lesson combines the foundations of kinematics and dynamics to focus on the specific problem of describing the motion of a celestial body, orbiting about a massive primary body as a result of the gravitational force between the two. Conic sections that geometrically represent different types of orbits are also discussed.

Kepler's Laws

Based on the observational data obtained by Tycho, Kepler's laws describe the motion of planets about Sun:

I) The orbit of a planet is an ellipse, with Sun at one of its two foci.

II) The radius vector from Sun to the planet sweeps out equal areas in equal time: $dA/dt = \text{constant}$.

III) The square of the planet's orbital period is proportional to the cube of its semi-major axis: $T^2 \propto a^3$.

These laws provided corrections to Copernicus' model that suggested circular orbits.

As shown in the subsequent sections, Kepler's laws can be derived analytically from Newton's gravitational law for two bodies of mass m_1 and m_2:

$$\underset{\rightarrow}{\boldsymbol{f}} = \frac{-Gm_1m_2}{r^3}\underset{\rightarrow}{\boldsymbol{r}} \tag{4.1}$$

where $\underset{\rightarrow}{\boldsymbol{r}}$ is the position vector between the two bodies, pointing to the body that is experiencing the force $\underset{\rightarrow}{\boldsymbol{f}}$ (in the opposite direction of $\underset{\rightarrow}{\boldsymbol{r}}$), and $r \triangleq |\underset{\rightarrow}{\boldsymbol{r}}|$. The constant $G \approx 6.67 \times 10^{-11}\ \mathrm{m}^3/(\mathrm{kg}\cdot\mathrm{s}^2)$ is known as the universal gravitational constant.

The Two-Body Problem

Consider two point masses, m_1 and m_2 (with $m_1 > m_2$), exerting gravitational forces on each other as shown in Figure 4.1a. Consider an inertial frame, $\mathscr{F}_I$, and define $\underset{\rightarrow}{\boldsymbol{r}} \triangleq \underset{\rightarrow}{\boldsymbol{r}}_2 - \underset{\rightarrow}{\boldsymbol{r}}_1$ as the relative position vector of m_2

with respect to m_1, and let r denote its magnitude. From DYNAMICS, we have:

$$m_1 \ddot{\underset{\to}{\boldsymbol{r}}}_1 = \underset{\to}{\boldsymbol{f}}_{12} = \frac{+Gm_1m_2}{r^3}\underset{\to}{\boldsymbol{r}} \;\Rightarrow\; \ddot{\underset{\to}{\boldsymbol{r}}}_1 = \frac{+Gm_2}{r^3}\underset{\to}{\boldsymbol{r}} \tag{4.2a}$$

$$m_2 \ddot{\underset{\to}{\boldsymbol{r}}}_2 = \underset{\to}{\boldsymbol{f}}_{21} = \frac{-Gm_1m_2}{r^3}\underset{\to}{\boldsymbol{r}} \;\Rightarrow\; \ddot{\underset{\to}{\boldsymbol{r}}}_2 = \frac{-Gm_1}{r^3}\underset{\to}{\boldsymbol{r}} \tag{4.2b}$$

Subtracting Eq. (4.2a) from Eq. (4.2b) yields:

$$\ddot{\underset{\to}{\boldsymbol{r}}} = \frac{-G(m_1+m_2)}{r^3}\underset{\to}{\boldsymbol{r}} = \frac{-\mu}{r^3}\underset{\to}{\boldsymbol{r}} \tag{4.3}$$

where $\mu \triangleq G(m_1+m_2)$, and if $m_1 \gg m_2$ (as in Sun/planet, or planet/spacecraft pairs), $\mu \approx Gm_1$ is known is the primary body's standard gravitational parameter.

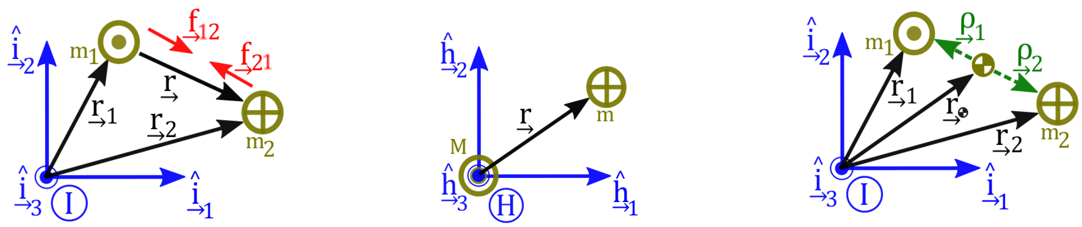
Note: An equivalent one-body problem can also be considered as in Figure 4.1b, in which a primary body of mass $M \triangleq m_1 + m_2$ is assumed to be fixed, and the motion of a *reduced mass*, $m \triangleq m_1m_2/(m_1+m_2)$, is studied as the unknown. The relationship in Eq. (4.3) still holds (because of its relative nature), multiplying which by the reduced mass yields:

$$m\ddot{\underset{\to}{\boldsymbol{r}}} = \underset{\to}{\boldsymbol{f}}_{21} = \frac{-GMm}{r^3}\underset{\to}{\boldsymbol{r}} \;\;,\;\; M \triangleq m_1+m_2 \;\;,\;\; m \triangleq \frac{m_1m_2}{m_1+m_2} \tag{4.4}$$

Let us study the motion of the centre of mass of the bodies, illustrated in Figure 4.1c, starting from its definition and differentiating twice with respect to time:

$$(m_1+m_2)\underset{\to}{\boldsymbol{r}}_{\circledast} = m_1\underset{\to}{\boldsymbol{r}}_1 + m_2\underset{\to}{\boldsymbol{r}}_2 \;\Rightarrow\; M\ddot{\underset{\to}{\boldsymbol{r}}}_{\circledast} = m_1\ddot{\underset{\to}{\boldsymbol{r}}}_1 + m_2\ddot{\underset{\to}{\boldsymbol{r}}}_2 = \underset{\to}{\boldsymbol{f}}_{12} + \underset{\to}{\boldsymbol{f}}_{21} = \underset{\to}{\boldsymbol{0}} \;\Rightarrow\; \ddot{\underset{\to}{\boldsymbol{r}}}_{\circledast} = \underset{\to}{\boldsymbol{0}} \tag{4.5}$$

where the left-hand side relationships in Eqs. (4.2a) and (4.2b) are used to relate forces to accelerations, and Newton's 3rd law is used to equate $\underset{\to}{\boldsymbol{f}}_{12} = -\underset{\to}{\boldsymbol{f}}_{21}$. The result in Eq. (4.5) implies that the centre of mass in the two-body problem moves at a constant velocity relative to O_I, the origin of $\mathscr{F}_I$, and since $\mathscr{F}_I$ is an inertial frame, a frame attached to the centre of mass would also be inertial.



(a) Two-Body Problem (b) One-Body Problem (c) Centre of Mass

Figure 4.1: Treating Two Bodies under Gravitational Forces of Each Other

Constants of Orbital Motion

When the orbit of m_2 about m_1 is considered, with $\underset{\to}{\boldsymbol{r}}$ denoting the relative position vector from the latter to the former, there are three parameters related to the motion that remain constant: orbital angular momentum, orbital energy, and eccentricity vector. These constants will prove useful for determining the shape, size, and orientation of orbits, and for predicting the effects of orbital manoeuvres on these properties.

Angular Momentum

Definition. The *specific angular momentum* associated with an orbit is defined as:

$$\underset{\rightarrow}{\boldsymbol{h}} \triangleq \underset{\rightarrow}{\boldsymbol{r}} \times \underset{\rightarrow}{\boldsymbol{r}}^{\bullet} \tag{4.6}$$

Differentiating both sides of Eq. (4.6) with respect to time, as measured in $\mathscr{F}_I$, and using Eq. (4.3) yield:

$$\underset{\rightarrow}{\boldsymbol{h}}^{\bullet} = \cancelto{\underset{\rightarrow}{\boldsymbol{0}}}{\underset{\rightarrow}{\boldsymbol{r}}^{\bullet} \times \underset{\rightarrow}{\boldsymbol{r}}^{\bullet}} + \underset{\rightarrow}{\boldsymbol{r}} \times \underset{\rightarrow}{\boldsymbol{r}}^{\bullet\bullet} = \underset{\rightarrow}{\boldsymbol{r}} \times \left(\frac{-\mu}{r^3}\underset{\rightarrow}{\boldsymbol{r}}\right) \quad \Rightarrow \quad \underset{\rightarrow}{\boldsymbol{h}}^{\bullet} = \frac{-\mu}{r^3}\cancelto{\underset{\rightarrow}{\boldsymbol{0}}}{\underset{\rightarrow}{\boldsymbol{r}} \times \underset{\rightarrow}{\boldsymbol{r}}} = \underset{\rightarrow}{\boldsymbol{0}} \tag{4.7}$$

which implies that $\underset{\rightarrow}{\boldsymbol{h}}$ remains constant in the absence of external forces. In addition, $\underset{\rightarrow}{\boldsymbol{h}}$ is normal to $\underset{\rightarrow}{\boldsymbol{r}}$ by definition, so $\underset{\rightarrow}{\boldsymbol{r}}$ is confined to the plane that has the constant $\underset{\rightarrow}{\boldsymbol{h}}$ as its normal.

Energy

Definition. The *specific energy* associated with an orbit is defined as:

$$\epsilon \triangleq \frac{1}{2}\underset{\rightarrow}{\boldsymbol{r}}^{\bullet} \cdot \underset{\rightarrow}{\boldsymbol{r}}^{\bullet} - \frac{\mu}{r} \tag{4.8}$$

where the first term on the right-hand side is kinetic energy, while the second term represents potential energy, both per unit reduced mass.

Note: To see where the potential term comes from, consider a potential field, the gradient of which is negative of the gravitational force (which is conservative):

$$V \triangleq \frac{-\mu m}{r} \quad \Rightarrow \quad -\underset{\rightarrow}{\boldsymbol{\nabla}}V = -\frac{\partial V}{\partial r}\underset{\rightarrow}{\hat{\boldsymbol{r}}} = \frac{-\mu m}{r^3}\underset{\rightarrow}{\boldsymbol{r}} = \underset{\rightarrow}{\boldsymbol{f}}_{21} \tag{4.9}$$

Differentiating both sides of Eq. (4.8) with respect to time, as measured in $\mathscr{F}_I$, yields:

$$\dot{\epsilon} = \underset{\rightarrow}{\boldsymbol{r}}^{\bullet} \cdot \underset{\rightarrow}{\boldsymbol{r}}^{\bullet\bullet} + \frac{\mu}{r^2}\dot{r} = \frac{-\mu}{r^3}\cancelto{\frac{1}{2}\frac{d}{dt}\left(\underset{\rightarrow}{\boldsymbol{r}} \cdot \underset{\rightarrow}{\boldsymbol{r}}\right) = r\dot{r}}{\underset{\rightarrow}{\boldsymbol{r}}^{\bullet} \cdot \underset{\rightarrow}{\boldsymbol{r}}} + \frac{\mu}{r^2}\underset{\rightarrow}{\boldsymbol{r}}^{\bullet} \quad \Rightarrow \quad -\frac{\mu}{r^2}\dot{r} + \frac{\mu}{r^2}\dot{r} = 0 \tag{4.10}$$

which implies that ϵ is constant. As we will see later, the energy of an orbit determines which type of conic section governs the geometry of motion.

Eccentricity Vector

Definition. The *eccentricity vector* (the magnitude of which is known simply as *eccentricity*) associated with an orbit is defined as:

$$\underset{\rightarrow}{\boldsymbol{e}} \triangleq \frac{\underset{\rightarrow}{\boldsymbol{r}}^{\bullet} \times \underset{\rightarrow}{\boldsymbol{h}}}{\mu} - \frac{\underset{\rightarrow}{\boldsymbol{r}}}{r} \tag{4.11}$$

where both terms in the right-hand side lie on the orbit's plane.

Differentiating both sides of Eq. (4.11) with respect to time, as measured in $\mathscr{F}_I$, yields:

$$\underset{\rightarrow}{\boldsymbol{e}}^{\bullet} = \left(\frac{\underset{\rightarrow}{\boldsymbol{r}}^{\bullet\bullet}\times\cancelto{\triangleq \underset{\rightarrow}{\boldsymbol{r}}\times\underset{\rightarrow}{\boldsymbol{r}}^{\bullet}}{\underset{\rightarrow}{\boldsymbol{h}}}}{\mu} + \frac{\underset{\rightarrow}{\boldsymbol{r}}^{\bullet}\times\cancelto{\underset{\rightarrow}{\boldsymbol{0}}}{\underset{\rightarrow}{\boldsymbol{h}}^{\bullet}}}{\mu}\right) - \left(\frac{\underset{\rightarrow}{\boldsymbol{r}}^{\bullet}}{r} - \frac{\dot{r}}{r^2}\underset{\rightarrow}{\boldsymbol{r}}\right) = \frac{1}{\mu}\left[(\underset{\rightarrow}{\boldsymbol{r}}^{\bullet\bullet}\cdot\underset{\rightarrow}{\boldsymbol{r}}^{\bullet})\underset{\rightarrow}{\boldsymbol{r}} - (\underset{\rightarrow}{\boldsymbol{r}}^{\bullet\bullet}\cdot\underset{\rightarrow}{\boldsymbol{r}})\underset{\rightarrow}{\boldsymbol{r}}^{\bullet}\right] - \left(\frac{\underset{\rightarrow}{\boldsymbol{r}}^{\bullet}}{r} - \frac{\dot{r}}{r^2}\underset{\rightarrow}{\boldsymbol{r}}\right) \tag{4.12}$$

where the vector triple product identity $\underset{\rightarrow}{\boldsymbol{a}}\times(\underset{\rightarrow}{\boldsymbol{b}}\times\underset{\rightarrow}{\boldsymbol{c}}) = (\underset{\rightarrow}{\boldsymbol{a}}\cdot\underset{\rightarrow}{\boldsymbol{c}})\underset{\rightarrow}{\boldsymbol{b}} - (\underset{\rightarrow}{\boldsymbol{a}}\cdot\underset{\rightarrow}{\boldsymbol{b}})\underset{\rightarrow}{\boldsymbol{c}}$ is used. Lastly, using Eq. (4.3) for $\underset{\rightarrow}{\boldsymbol{r}}^{\bullet\bullet}$, we obtain:

$$\underset{\rightarrow}{\boldsymbol{e}}^{\bullet} = \frac{1}{\mu}\left(\frac{-\mu}{r^3}\cancelto{\frac{1}{2}\frac{d}{dt}(\underset{\rightarrow}{\boldsymbol{r}}\cdot\underset{\rightarrow}{\boldsymbol{r}}) = r\dot{r}}{(\underset{\rightarrow}{\boldsymbol{r}}\cdot\underset{\rightarrow}{\boldsymbol{r}}^{\bullet})}\underset{\rightarrow}{\boldsymbol{r}} + \frac{\mu}{r^3}\cancelto{r^2}{(\underset{\rightarrow}{\boldsymbol{r}}\cdot\underset{\rightarrow}{\boldsymbol{r}})}\underset{\rightarrow}{\boldsymbol{r}}^{\bullet}\right) - \left(\frac{\underset{\rightarrow}{\boldsymbol{r}}^{\bullet}}{r} - \frac{\dot{r}}{r^2}\underset{\rightarrow}{\boldsymbol{r}}\right) = \underset{\rightarrow}{\boldsymbol{0}} \tag{4.13}$$

which implies that $\underset{\rightarrow}{\boldsymbol{e}}$ is constant. This is another parameter that lies on the orbital plane (similar to $\underset{\rightarrow}{\boldsymbol{r}}$) and is used to describe the orientation of the orbit in the plane.

Describing the Orbit

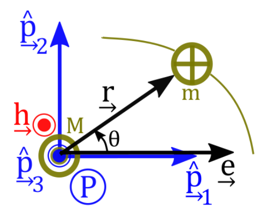


Figure 4.2: Perifocal

As shown in Figure 4.2, we have the constant $\underset{\rightarrow}{\boldsymbol{h}}$, normal to the orbital plane, that specifies the plane of motion, as well as the constant $\underset{\rightarrow}{\boldsymbol{e}}$ that is fixed on the plane. We now would like to find r, the orbiting body's radial distance from the primary, and θ, the angle between $\underset{\rightarrow}{\boldsymbol{r}}$ and $\underset{\rightarrow}{\boldsymbol{e}}$ (known as the *true anomaly*).

With the objective of finding θ in mind, we find the scalar product of $\underset{\rightarrow}{\boldsymbol{r}}$ and $\underset{\rightarrow}{\boldsymbol{e}}$:

$$\underset{\rightarrow}{\boldsymbol{r}}\cdot\underset{\rightarrow}{\boldsymbol{e}} = \underset{\rightarrow}{\boldsymbol{r}}\cdot\frac{\underset{\rightarrow}{\boldsymbol{r}}^{\bullet}\times\underset{\rightarrow}{\boldsymbol{h}}}{\mu} - \frac{\cancelto{r^2}{\underset{\rightarrow}{\boldsymbol{r}}\cdot\underset{\rightarrow}{\boldsymbol{r}}}}{r} \;\Rightarrow\; re\cos(\theta) = \underset{\rightarrow}{\boldsymbol{h}}\cdot\frac{\cancelto{\underset{\rightarrow}{\boldsymbol{h}}}{\underset{\rightarrow}{\boldsymbol{r}}\times\underset{\rightarrow}{\boldsymbol{r}}^{\bullet}}}{\mu} - r = \frac{h^2}{\mu} - r \tag{4.14}$$

where the scalar triple product identity from FUNDAMENTALS is used. Rearranging Eq. (4.14) yields the well-known "polar equation":

$$r = \frac{l}{1 + e\cos(\theta)} \quad , \quad l \triangleq \frac{h^2}{\mu} \tag{4.15}$$

where l is known as the *semilatus rectum*. Therefore, the radial distance of an orbiting planet can be determined using the knowledge of true anomaly, and vice versa. Together, these two parameters provide a polar description of the orbital motion in its fixed plane.

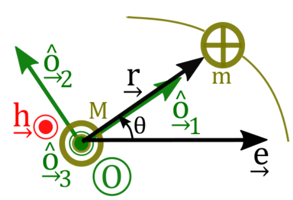


Figure 4.3: Orbiting

With the aim of deriving two other useful relationships, one for energy and one for speed, we define a rotating reference frame, $\mathscr{F}_O$ shown in Figure 4.3, that has its origin on the primary, and its 1-axis is always pointing to the orbiting body. Consider, also, the perifocal frame from FUNDAMENTALS, $\mathscr{F}_P$ shown in Figure 4.2, assumed to be inertial and with its origin on the primary ($O_P \equiv O_O$ for the two frames), and 1- and 3-axes parallel to $\underset{\rightarrow}{\boldsymbol{e}}$ and $\underset{\rightarrow}{\boldsymbol{h}}$, respectively.

Resolving position $\underset{\rightarrow}{\boldsymbol{r}}$, velocity $\underset{\rightarrow}{\boldsymbol{v}} \triangleq \underset{\rightarrow}{\boldsymbol{r}}^{\bullet}$ (time-derivative as measured in inertial $\mathscr{F}_P$), and angular velocity $\underset{\rightarrow}{\boldsymbol{\omega}}$ (of $\mathscr{F}_O$ relative to $\mathscr{F}_P$) in $\mathscr{F}_O$ yields:

$$\underset{\rightarrow}{\boldsymbol{r}} = \mathscr{F}_O^{\mathsf{T}}\begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix} \quad , \quad \underset{\rightarrow}{\boldsymbol{\omega}} = \mathscr{F}_O^{\mathsf{T}}\begin{bmatrix} 0 \\ 0 \\ \dot{\theta} \end{bmatrix} \quad , \quad \underset{\rightarrow}{\boldsymbol{v}} = \mathscr{F}_O^{\mathsf{T}}\left(\overset{\bullet}{\boldsymbol{r}}_O + \boldsymbol{\omega}_O^{\times}\boldsymbol{r}_O\right) = \mathscr{F}_O^{\mathsf{T}}\begin{bmatrix} \dot{r} \\ \dot{\theta}r \\ 0 \end{bmatrix} \tag{4.16}$$

where the transport theorem from KINEMATICS is used to determine the velocity vector. From Eqs. (4.8) and (4.6), and using the referential forms of $\underset{\rightarrow}{r}$ and $\underset{\rightarrow}{r}^{\bullet}$ presented in Eq. (4.16), we have:

$$\epsilon = \frac{1}{2}\boldsymbol{v}_O^{\mathsf{T}}\boldsymbol{v}_O - \frac{\mu}{r} = \frac{v^2}{2} - \frac{\mu}{r} = \frac{1}{2}\left(\dot{r}^2 + \dot{\theta}^2 r^2\right) - \frac{\mu}{r} \tag{4.17a}$$

$$h = |\boldsymbol{r}_O^{\times}\boldsymbol{v}_O| = r^2\dot{\theta} \tag{4.17b}$$

where $v \triangleq |\underset{\rightarrow}{r}^{\bullet}|$ represents speed. Setting the derivative of r from Eq. (4.15) with respect to θ to 0 shows that the orbiting body is at the "peripasis" of the orbit, the closest distance to the primary, when $\theta = 0$. Substituting this result into Eq. (4.15), we have:

$$r_\pi = \frac{l}{1+e} = \frac{h^2/\mu}{1+e} \quad , \quad \dot{r}_\pi = 0 \tag{4.18}$$

where the subscript 'π' refers to periapsis. Substituting the zero time-derivative relationship from Eq. (4.18) into Eq. (4.17a) and making use of Eq. (4.17b) and r_π from Eq. (4.18) yields, upon some algebraic manipulations:

$$\epsilon_\pi = \frac{1}{2}\dot{\theta}_\pi^2 r_\pi^2 - \frac{\mu}{r_\pi} \quad \Rightarrow \quad \epsilon = \frac{\mu^2}{2h^2}\left(e^2 - 1\right) \tag{4.19}$$

where the fact that the orbital energy remains constant is used to omit its periapsis subscript. Rearranging Eq. (4.19) provides a useful relationship for determining the orbit's shape:

$$e = \sqrt{1 + \frac{2\epsilon h^2}{\mu^2}} = \sqrt{1 + \frac{2\epsilon l}{\mu}} \tag{4.20}$$

We also define the so-called *semi-major axis*, which is another important constant parameter that determines the size of the orbit, and we relate it to energy using Eq. (4.20):

$$a \triangleq \frac{l}{1-e^2} \quad \Rightarrow \quad \epsilon = -\frac{\mu}{2a} \tag{4.21}$$

Lastly, combining the right-hand side relationship in Eq. (4.21) with Eq. (4.17a), we obtain the "vis-viva equation", which provides a *very* useful relationship between radial distance and speed:

$$v = \sqrt{\mu\left(\frac{2}{r} - \frac{1}{a}\right)} \tag{4.22}$$

Note: For a parabola, semi-major axis is not defined. Since, as discussed below, $\epsilon = 0$ for this type of orbit, the relevant speed equation is simply $v = \sqrt{\mu/r}$.

Conic Sections

The relationship in Eq. (4.15) is a polar coordinates description of a conic section, a curve obtained from a plane's intersection with a double cone. Therefore, the orbit of a body about another can adopt one of three general shapes, determined by its eccentricity, e: an ellipse, a parabola, or a hyperbola. The properties associated with each category are summarized in Table 4.1. The parameter v_∞ is the *excess speed*, reached as $r \to \infty$, the "left-over" speed upon escaping the orbit.

Note: Refer to Section 3.3.1 of *Spacecraft Dynamics and Control: an Introduction* for a perifocal frame-based

Table 4.1: Properties of Each Type of Orbit

Conic Section	e	ϵ	a	r_{min}	r_{max}	v_∞
Ellipse	<1	<0	>0	$=\dfrac{l}{1+e}>\dfrac{l}{2}$	$\dfrac{l}{1-e}$	N/A
Parabola	$=1$	$=0$	N/A	$=\dfrac{l}{1+1}=\dfrac{l}{2}$	N/A	0
Hyperbola	>1	>0	<0	$=\dfrac{l}{1+e}<\dfrac{l}{2}$	N/A	$\sqrt{2\epsilon}=\sqrt{\dfrac{-\mu}{a}}$

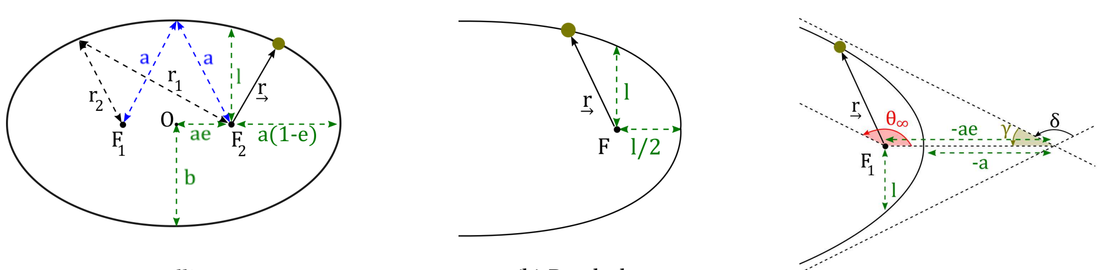
approach of studying the position that naturally leads to the Cartesian equations of conic sections.

Some notes about the geometry and physics of each type of orbit are in order:

- Elliptic orbits: Sum of the orbiting body's distance from the two foci is always constant, $r_1+r_2=2a$ as shown in Figure 4.4a, so we have $a=\frac{1}{2}(r_{min}+r_{max})=l/(1-e^2)$, with r_{min} measuring the *periapsis* distance, and r_{max} known as the *apoapsis* distance. The line connecting the two extrema is known as the *line of apsides.* An elliptic orbit has a negative energy, so an object in such an orbit is bound to remain there, unless its energy is increased via external sources (such as spacecraft thrusters).

- Parabolic orbits: As a transition point between elliptic and hyperbolic orbits, parabolic orbits have zero energy, and the orbiting body will have no excess speed upon "escaping" from the gravitational pull of the primary body. The minimum speed that an object in an elliptic orbit requires in order to embark on a parabolic orbit is known as the escape velocity, and is obtained by setting $\epsilon=0$ in Eq. (4.17a) to yield $v_{esc}=\sqrt{2\mu/r_\pi}$, where $r_\pi=l/2$ is the periapsis distance. This type of orbit is depicted in Figure 4.4b.

- Hyperbolic orbits: As $r\to\infty$, from Eq. (4.15) we have $\theta_\infty\to\cos^{-1}\big(-1/e\big)$ for the true anomaly, from which one can compute the half-angle between the hyperbola's asymptotes, $\gamma=\pi-\theta_\infty$. An object entering a hyperbolic orbit will eventually, upon exiting the branch, be deflected by $\delta=\pi-2\gamma$ from its original path, as shown in Figure 4.4c. Because of its positive energy and non-zero excess speed, a hyperbolic orbit is particularly useful for interplanetary trajectories and planetary fly-by (or "slingshot") both of which will be visited later in the course.



(a) Ellipse (b) Parabola (c) Hyperbola

Figure 4.4: Illustration of Conic Sections

Revisiting Kepler's Laws

We now show the validity of Kepler's laws, originally based on empirical data, using Newton's law of gravitation that we have focused on thus far:

I Since an ellipse is a conic section with $e<1$, this result has already been derived.

II Approximating the differential area swept by the orbit as a triangle, we have:

$$dA \approx \frac{1}{2}r(rd\theta) \quad \Rightarrow \quad \frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} = \frac{h}{2} \tag{4.23}$$

where Eq. (4.17b) is used. Since $h = \underset{\rightarrow}{\boldsymbol{h}}$ is a constant of motion, the area is swept at a constant rate.

III Integrating Eq. (4.23) over one complete orbital period, and equating the result with the known area of an ellipse yields:

$$\int_0^A dA = \int_{t_0}^{T+t_0} \frac{h}{2}\,dt \quad \Rightarrow \quad A = \frac{h}{2}T = \pi ab \tag{4.24}$$

where a and b are the semi-major and semi-minor axis of the ellipse, respectively. We use geometry to find b, for which we first write the periapsis distance in terms of a and e:

$$r_\pi = r_{min} = \frac{l}{1+e} \quad \Rightarrow r_\pi = a(1-e) \tag{4.25}$$

where Eq. (4.18) and the definition of a from Eq. (4.21) is used. The semi-minor axis is then obtained as follows, using the Pythagorean theorem in Figure 4.4a:

$$b^2 + (a - r_\pi)^2 = b^2 + (ae)^2 = a^2 \quad \Rightarrow \quad b = a\sqrt{1-e^2} \tag{4.26}$$

Also, from the definition of semilatus rectum in Eq. (4.15) and from Eq. (4.21), we have:

$$l \triangleq \frac{h^2}{\mu} \quad \Rightarrow \quad h^2 = l\mu = a\mu(1-e^2) \tag{4.27}$$

Lastly, substituting Eqs. (4.25) and (4.26) back into the squared form of Eq. (4.24) yields the desired result:

$$T^2 = \frac{4\pi^2 a^2 \left[a^2 \cancel{(1-e^2)}\right]}{a\mu\cancel{(1-e^2)}} = \frac{4\pi^2}{\mu}a^3 \quad \Rightarrow T^2 \propto a^3 \tag{4.28}$$

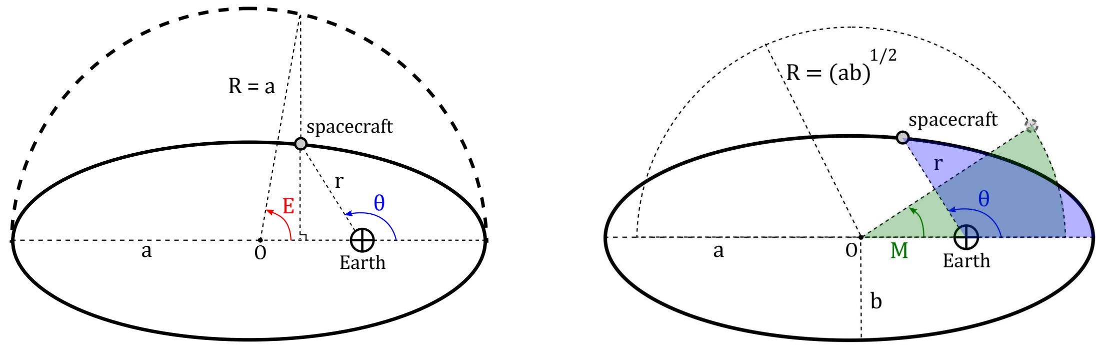
We also have the following useful relationship for orbital period, using which we define the *mean orbital motion*:

$$T = 2\pi\sqrt{\frac{a^3}{\mu}} \quad , \quad n \triangleq \frac{2\pi}{T} = \sqrt{\frac{\mu}{a^3}} \tag{4.29}$$

The mean motion, n, is measured in $\mathrm{rad/s}$, and represents the constant angular rate that would correspond to a circular motion of the same period as the original elliptic orbit.

Definitions of Anomalies

Three different angular measures, each with a different purpose and geometric significance, are commonly used in orbital mechanics:



(a) True Anomaly, θ and Eccentric Anomaly, E

(b) True Anomaly, θ, and Mean Anomaly, M

Figure 4.5: Angular Measures Used in Orbital Mechanics

θ	true anomaly	between line of apsides and $\underset{\rightarrow}{r}$ (measured from primary body)
E	eccentric anomaly	between line of apsides and orbiting body's vertical projection on a circle of radius $R = a$ (measured from centre point)
M	Mean Anomaly	between line of apsides and an *imaginary* replica of the orbiting body that moves uniformly on a circle of the same period (measured from centre point)

Note: The geometric interpretation of mean anomaly $M = n(t - t_0) = (2\pi/T)(t - t_0)$, is not intuitive, since it requires a "hypothetical" scenario of a spacecraft moving in a circular orbit, concentric with the ellipse (hence uniformly changing its angle as measured from the centre point). If the radius of this hypothetical circle is taken to be $R = \sqrt{ab}$, then the circle and the ellipse will have equal areas. Mean anomaly, M, would then correspond to the angle of the sector that has the same area as swept by the real orbiting body, starting from t_0. In Figure 4.5b, the blue and green areas are equal. Since, according to Kepler's second law, this area increases at a constant rate, the imaginary body that we have pictured will also move at a constant rate, hence resulting in a linear (in time) increase in M from 0 to 2π.

Relating Time and Orbital Position

Since an ellipse is a projection of a circle, such that all vertical lines are shortened by an equal ratio, we have:

$$r\sin(\theta) = \frac{b}{\cancel{a}}\cancel{a}\sin(E) \quad \Rightarrow \quad \frac{\sqrt{1-e^2}\sin(\theta)}{1+e\cos(\theta)} = \sin(E) \tag{4.30}$$

where Eqs. (4.26) and (4.15), and $l = a(1-e^2)$ are used. Using some trigonometric identities and further manipulations, a relationship between the true and the eccentric anomaly is obtained:

$$\tan\left(\frac{\theta}{2}\right) = \sqrt{\frac{1+e}{1-e}}\tan\left(\frac{E}{2}\right) \tag{4.31}$$

differentiating which with respect to E and rearranging yields:

$$\frac{d\theta}{dE} = \frac{\cancel{2}\cos^2(\theta/2)}{\cancel{2}\cos^2(E/2)}\sqrt{\frac{1+e}{1-e}} \tag{4.32}$$

Rearranging Eq. (4.17b) and substituting Eq. (4.32) into it yields:

$$dt = \frac{r^2}{h}\,d\theta = \frac{r^2}{h}\frac{\cos^2\left(\theta/2\right)}{\cos^2\left(E/2\right)}\sqrt{\frac{1+e}{1-e}}\,dE \tag{4.33}$$

applying the half-angle formula and simplifying eventually yields:

$$dt = \frac{r^2}{h}\left(\frac{a\sqrt{1-e^2}}{r}\right)dE \quad\Rightarrow\quad h\int_{t_0}^{t} d\tau = a\sqrt{1-e^2}\int_{0}^{E} r\,dE \tag{4.34}$$

But it can be shown, from geometry and Cartesian coordinates, that $r = a\big(1 - e\cos(E)\big)$, substituting which into Eq. (4.34) and carrying out the integration produces:

$$h(t-t_0) = a^2\sqrt{1-e^2}\big(E - e\sin(E)\big) \quad\Rightarrow E - e\sin(E) = \sqrt{\frac{\mu}{a^3}}(t-t_0) = M \tag{4.35}$$

where Eq. (4.27) is used to replace h. This result is known as "Kepler's equation". By solving Eqs. (4.35) and (4.31) together, one can relate time to true anomaly.

Note: Refer to Section 3.5 of *Spacecraft Dynamics and Control: and Introduction* for some of the skipped steps, and for more details on the derivation of Kepler's equation.