University of Toronto

Spacecraft Dynamics and Control

AER506H1F

Course Notes

Prepared by: Behrad Vatankhahghadim

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Prologue

These course notes are developed for the students of Spacecraft Dynamics and Control (AER506) based on the following resources:

- AER506 Course Notes
  C. J. Damaren, University of Toronto, Fall 2015.
- AER506 Lecture Notes
  based on lectures by G. M. T. D'Eleuterio, University of Toronto, Fall 2013.
- *Spacecraft Dynamics and Control: an Introduction*
  A. H. J. de Ruiter, C. J. Damaren, J. R. Forbes; Wiley, 2013.
- *Spacecraft Attitude Dynamics*
  P. C. Hughes; Dover, 2004.
- *Space Vehicle Dynamics and Control*
  B. Wie; American Institute of Aeronautics and Astronautics, 1998.

These notes are meant as a supplement and *not* a substitute for the lectures. They do not include examples or tutorial problems discussed in class, nor do they elaborate on the concepts as much as the lectures are intended to do. They also make occasional references to the course's required and recommended textbooks (the third and fourth items in the list above, respectively), and hence are not self-contained. It is hoped, nevertheless, that the students will find them useful for preparing for each lecture beforehand, for facilitating their in-class note-taking, and for reviewing the key concepts as the course progresses.

Lecture 1

Fundamentals

THIS lesson reviews some basic concepts, such as vectors and reference frames, that will be frequently used in the following lectures. The “vectrix” notation for vectors is introduced, and change of frames using rotation matrices is discussed. Finally, various representations of an object’s attitude are studied, and simplifications resulting from infinitesimally small angles are considered.

Vectors and Reference Frames

Consider a vector, a mathematical quantity with both magnitude and direction, represented in reference frame $\mathscr{F}_A$:

$$\underset{\rightarrow}{\boldsymbol{v}} = v_1\underset{\rightarrow}{\hat{\boldsymbol{a}}}_1 + v_2\underset{\rightarrow}{\hat{\boldsymbol{a}}}_2 + v_3\underset{\rightarrow}{\hat{\boldsymbol{a}}}_3 \tag{1.1}$$

where $\underset{\rightarrow}{\hat{\boldsymbol{a}}}_1$, $\underset{\rightarrow}{\hat{\boldsymbol{a}}}_2$, and $\underset{\rightarrow}{\hat{\boldsymbol{a}}}_3$ are dextral (right-handed) and orthonormal (mutually-perpendicular unit) set of vectors that define $\mathscr{F}_A$.

Note: A vector, by itself, is independent of a reference frame, but its representation using scalar components (such as v_1, v_2, and v_3 above) requires choosing a reference frame.

Definition. A reference frame is an *inertial frame* if Newton’s laws of motion hold in it. Any other frame that is moving with a constant velocity (but *not* rotating) with respect to an inertial frame is also inertial.

Presented below, and shown in Figures 1.1 and 1.2, are only some of the important reference frames that are frequently encountered in the field of spacecraft dynamics and control, and illustrate the importance of being able to change mathematical representations from one frame to another, and to relate different frames to each other. Most of these frames will be revisited later in the course.

- $\mathscr{F}_H$: Heliocentric-Ecliptic
  - Origin, O_H, at Sun’s centre of mass
  - $\underset{\rightarrow}{\hat{\boldsymbol{h}}}_1$ in direction of vernal equinox, ♈
  - $\underset{\rightarrow}{\hat{\boldsymbol{h}}}_3$ normal to the ecliptic plane (on which Earth’s orbit about Sun lies)

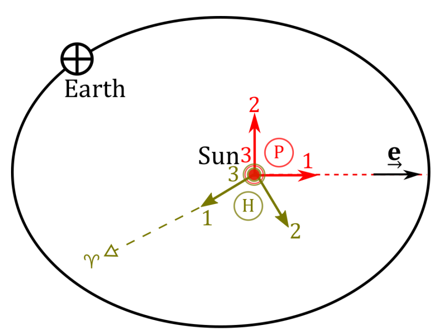
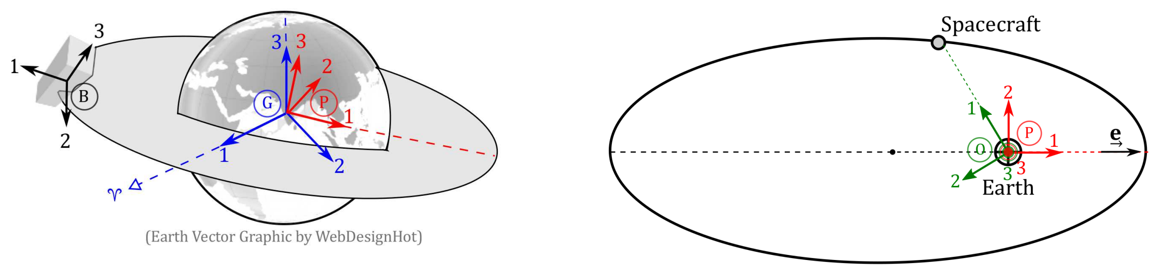


Figure 1.1: Top View of (H) Heliocentric and (P) Perifocal (for Earth’s orbit) Reference Frames

- $\mathscr{F}_G$: Geocentric-Equatorial (or Earth-Centred Inertial (ECI))
  - Origin, O_G, at Earth's centre of mass
  - $\underset{\rightarrow}{\hat{\boldsymbol{g}}}_1$ in direction of vernal equinox, ♈
  - $\underset{\rightarrow}{\hat{\boldsymbol{g}}}_3$ towards Earth's North pole
- $\mathscr{F}_P$: Perifocal (defined for an arbitrary orbit)
  - Origin at the centre of mass of the primary body (Ex. Earth for Earth-orbiting satellites)
  - $\underset{\rightarrow}{\hat{\boldsymbol{p}}}_1$ towards the orbit's periapsis (parallel to its eccentricity vector, $\underset{\rightarrow}{\boldsymbol{e}}$)
  - $\underset{\rightarrow}{\hat{\boldsymbol{p}}}_3$ normal to the orbit's plane (parallel to its angular momentum vector, $\underset{\rightarrow}{\boldsymbol{h}}$)
- $\mathscr{F}_O$: Orbiting (defined for an arbitrary orbit)
  - Origin at the centre of mass of the primary body (Ex. Earth for the Earth-orbiting satellites)
  - $\underset{\rightarrow}{\hat{\boldsymbol{o}}}_1$ towards the orbiting body (Ex. Earth-orbiting satellites)
  - $\underset{\rightarrow}{\hat{\boldsymbol{o}}}_3$ normal to the orbit's plane (parallel to its angular momentum vector, $\underset{\rightarrow}{\boldsymbol{h}}$)
- $\mathscr{F}_B$: Body-Fixed (defined for an arbitrary rigid body)
  - Origin, O_B, at the body's centre of mass
  - Each unit vector towards a fixed point on the body (Ex. along the spacecraft's principal axes that form a frame in which the moment of inertia matrix, $\boldsymbol{I}$, is diagonal).



(a) Side View:
(B) Body-Fixed, (G) ECI, and (P) Perifocal

(b) Top View:
(P) Perifocal, (O) Orbiting

Figure 1.2: Illustration of Earth-Related Reference Frames

Note: Vernal equinox, ♈, is an inertially-fixed direction towards which the Earth-Sun vector points on the first day of spring (Northern hemisphere) represents the Vernal equinox (♈) direction. See Fig. 3.14 of *Spacecraft Dynamics and Control: an Introduction.*

Note: The ECI frame is not truly inertial, as Earth's centre of mass accelerates in its orbit; however, Earth's movement with respect to the Sun is slow enough (compared to the time scales of our studies) that ECI could be considered as inertial for all practical purposes.

In addition to the reference frames listed above, we also have the Earth-Centred Earth-Fixed (ECEF) frame, $\mathscr{F}_E$, that is similar to ECI, but its 1-axis rotates with Earth. Also, note that $\mathscr{F}_O$ is similar to $\mathscr{F}_P$, but its 1-axis rotates together with the orbiting body (Ex. pointing from Earth towards an Earth-orbiting satellite). To move from $\mathscr{F}_G$ to $\mathscr{F}_E$, or from $\mathscr{F}_P$ to $\mathscr{F}_O$, one needs a rotation about their mutual 3-axis. Change of frames will be revisited later in this section.

Vectrix Notation

Let us rewrite the frame $\mathscr{F}_A$ representation of $\underset{\rightarrow}{v}$, given by Eq. (1.1), as follows:

$$\underset{\rightarrow}{v} = v_1 \underset{\rightarrow}{\hat{\boldsymbol{a}}}_1 + v_2 \underset{\rightarrow}{\hat{\boldsymbol{a}}}_2 + v_3 \underset{\rightarrow}{\hat{\boldsymbol{a}}}_3 = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} \underset{\rightarrow}{\hat{\boldsymbol{a}}}_1 \\ \underset{\rightarrow}{\hat{\boldsymbol{a}}}_2 \\ \underset{\rightarrow}{\hat{\boldsymbol{a}}}_3 \end{bmatrix} = \begin{bmatrix} \underset{\rightarrow}{\hat{\boldsymbol{a}}}_1 & \underset{\rightarrow}{\hat{\boldsymbol{a}}}_2 & \underset{\rightarrow}{\hat{\boldsymbol{a}}}_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \tag{1.2}$$

where $\boldsymbol{v}_A \triangleq [v_1 \ v_2 \ v_3]^\mathsf{T}$ is a column matrix with the components of the vector $\underset{\rightarrow}{v}$, as expressed in $\mathscr{F}_A$.

Definition. We define $\underset{\rightarrow}{\mathscr{F}}_A \triangleq [\underset{\rightarrow}{\hat{\boldsymbol{a}}}_1 \ \underset{\rightarrow}{\hat{\boldsymbol{a}}}_2 \ \underset{\rightarrow}{\hat{\boldsymbol{a}}}_3]^\mathsf{T}$ as the *vectrix* (matrix of vectors) corresponding to the frame $\mathscr{F}_A$. The elements of a vectrix form a set of basis vectors.

When multiple reference frames are considered, the same vector can be represented in any of them using their corresponding vectrices. For example, with two frames $\mathscr{F}_A$ and $\mathscr{F}_B$, we have:

$$\underset{\rightarrow}{v} = \underset{\rightarrow}{\mathscr{F}}_A^\mathsf{T} \boldsymbol{v}_A = \underset{\rightarrow}{\mathscr{F}}_B^\mathsf{T} \boldsymbol{v}_B \tag{1.3}$$

Note: The elements of $\boldsymbol{v}_A$ and $\boldsymbol{v}_B$ are scalar quantities. These column matrices no longer have a sense of direction or orientation, unlike the vector $\underset{\rightarrow}{v}$, since the directional nature of the entity is extracted using the vectrices $\underset{\rightarrow}{\mathscr{F}}_A$ and $\underset{\rightarrow}{\mathscr{F}}_B$. This is why we choose to adopt this notation: it allows us to separate the effects of the reference frame's orientation on the mathematical representation of a vector, and once all vectors being considered are expressed in the *same* frame, we can work only with their scalar representations.

Scalar Product

Consider the scalar (dot) product of two vectors, $\underset{\rightarrow}{u}$ and $\underset{\rightarrow}{v}$, with their representations in frame $\mathscr{F}_A$:

$$\underset{\rightarrow}{u} \cdot \underset{\rightarrow}{v} = \boldsymbol{u}_A^\mathsf{T} \underset{\rightarrow}{\mathscr{F}}_A \cdot \underset{\rightarrow}{\mathscr{F}}_A^\mathsf{T} \boldsymbol{v}_A = \boldsymbol{u}_A^\mathsf{T} \begin{bmatrix} \underset{\rightarrow}{\hat{\boldsymbol{a}}}_1 \\ \underset{\rightarrow}{\hat{\boldsymbol{a}}}_2 \\ \underset{\rightarrow}{\hat{\boldsymbol{a}}}_3 \end{bmatrix} \cdot \begin{bmatrix} \underset{\rightarrow}{\hat{\boldsymbol{a}}}_1 & \underset{\rightarrow}{\hat{\boldsymbol{a}}}_2 & \underset{\rightarrow}{\hat{\boldsymbol{a}}}_3 \end{bmatrix} \boldsymbol{v}_A = \boldsymbol{u}_A^\mathsf{T} \begin{bmatrix} \cancelto{1}{\underset{\rightarrow}{\hat{\boldsymbol{a}}}_1 \cdot \underset{\rightarrow}{\hat{\boldsymbol{a}}}_1} & \cancelto{0}{\underset{\rightarrow}{\hat{\boldsymbol{a}}}_1 \cdot \underset{\rightarrow}{\hat{\boldsymbol{a}}}_2} & \cancelto{0}{\underset{\rightarrow}{\hat{\boldsymbol{a}}}_1 \cdot \underset{\rightarrow}{\hat{\boldsymbol{a}}}_3} \\ \cancelto{0}{\underset{\rightarrow}{\hat{\boldsymbol{a}}}_2 \cdot \underset{\rightarrow}{\hat{\boldsymbol{a}}}_1} & \cancelto{1}{\underset{\rightarrow}{\hat{\boldsymbol{a}}}_2 \cdot \underset{\rightarrow}{\hat{\boldsymbol{a}}}_2} & \cancelto{0}{\underset{\rightarrow}{\hat{\boldsymbol{a}}}_2 \cdot \underset{\rightarrow}{\hat{\boldsymbol{a}}}_3} \\ \cancelto{0}{\underset{\rightarrow}{\hat{\boldsymbol{a}}}_3 \cdot \underset{\rightarrow}{\hat{\boldsymbol{a}}}_1} & \cancelto{0}{\underset{\rightarrow}{\hat{\boldsymbol{a}}}_3 \cdot \underset{\rightarrow}{\hat{\boldsymbol{a}}}_2} & \cancelto{1}{\underset{\rightarrow}{\hat{\boldsymbol{a}}}_3 \cdot \underset{\rightarrow}{\hat{\boldsymbol{a}}}_3} \end{bmatrix} \boldsymbol{v}_A = \boldsymbol{u}_A^\mathsf{T} \boldsymbol{v}_A \tag{1.4}$$

where, using the orthonormality of the unit vectors defining $\mathscr{F}_A$, we have $\underset{\rightarrow}{\mathscr{F}}_A \cdot \underset{\rightarrow}{\mathscr{F}}_A^\mathsf{T} = \mathbf{1}$. Repeating the same operation in another frame, $\mathscr{F}_B$, we have:

$$\underset{\rightarrow}{u} \cdot \underset{\rightarrow}{v} = \boldsymbol{u}_A^\mathsf{T} \boldsymbol{v}_A = \boldsymbol{u}_B^\mathsf{T} \boldsymbol{v}_B \tag{1.5}$$

Cross Product

Consider the cross product of two vectors, $\underset{\rightarrow}{u}$ and $\underset{\rightarrow}{v}$, with their representations in frame $\mathscr{F}_A$:

$$\underset{\rightarrow}{u} \times \underset{\rightarrow}{v} = \boldsymbol{u}_A^\mathsf{T} \underset{\rightarrow}{\mathscr{F}}_A \times \underset{\rightarrow}{\mathscr{F}}_A^\mathsf{T} \boldsymbol{v}_A = \boldsymbol{u}_A^\mathsf{T} \begin{bmatrix} \underset{\rightarrow}{\hat{\boldsymbol{a}}}_1 \\ \underset{\rightarrow}{\hat{\boldsymbol{a}}}_2 \\ \underset{\rightarrow}{\hat{\boldsymbol{a}}}_3 \end{bmatrix} \times \begin{bmatrix} \underset{\rightarrow}{\hat{\boldsymbol{a}}}_1 & \underset{\rightarrow}{\hat{\boldsymbol{a}}}_2 & \underset{\rightarrow}{\hat{\boldsymbol{a}}}_3 \end{bmatrix} \boldsymbol{v}_A = \boldsymbol{u}_A^\mathsf{T} \begin{bmatrix} \cancelto{\underset{\rightarrow}{\mathbf{0}}}{\underset{\rightarrow}{\hat{\boldsymbol{a}}}_1 \times \underset{\rightarrow}{\hat{\boldsymbol{a}}}_1} & \cancelto{\underset{\rightarrow}{\hat{\boldsymbol{a}}}_3}{\underset{\rightarrow}{\hat{\boldsymbol{a}}}_1 \times \underset{\rightarrow}{\hat{\boldsymbol{a}}}_2} & \cancelto{-\underset{\rightarrow}{\hat{\boldsymbol{a}}}_2}{\underset{\rightarrow}{\hat{\boldsymbol{a}}}_1 \times \underset{\rightarrow}{\hat{\boldsymbol{a}}}_3} \\ \cancelto{-\underset{\rightarrow}{\hat{\boldsymbol{a}}}_3}{\underset{\rightarrow}{\hat{\boldsymbol{a}}}_2 \times \underset{\rightarrow}{\hat{\boldsymbol{a}}}_1} & \cancelto{\underset{\rightarrow}{\mathbf{0}}}{\underset{\rightarrow}{\hat{\boldsymbol{a}}}_2 \times \underset{\rightarrow}{\hat{\boldsymbol{a}}}_2} & \cancelto{\underset{\rightarrow}{\hat{\boldsymbol{a}}}_1}{\underset{\rightarrow}{\hat{\boldsymbol{a}}}_2 \times \underset{\rightarrow}{\hat{\boldsymbol{a}}}_3} \\ \cancelto{\underset{\rightarrow}{\hat{\boldsymbol{a}}}_2}{\underset{\rightarrow}{\hat{\boldsymbol{a}}}_3 \times \underset{\rightarrow}{\hat{\boldsymbol{a}}}_1} & \cancelto{-\underset{\rightarrow}{\hat{\boldsymbol{a}}}_1}{\underset{\rightarrow}{\hat{\boldsymbol{a}}}_3 \times \underset{\rightarrow}{\hat{\boldsymbol{a}}}_2} & \cancelto{\underset{\rightarrow}{\mathbf{0}}}{\underset{\rightarrow}{\hat{\boldsymbol{a}}}_3 \times \underset{\rightarrow}{\hat{\boldsymbol{a}}}_3} \end{bmatrix} \boldsymbol{v}_A \tag{1.6}$$

where, once again, orthonormality of the basis vectors is used. Manipulating Eq. (1.6) further yields:

$$\underrightarrow{\boldsymbol{u}}\times\underrightarrow{\boldsymbol{v}} = \boldsymbol{u}_A^\mathsf{T}\begin{bmatrix} \underrightarrow{\hat{\boldsymbol{a}}}_3 v_2 - \underrightarrow{\hat{\boldsymbol{a}}}_2 v_3 \\ -\underrightarrow{\hat{\boldsymbol{a}}}_3 v_1 + \underrightarrow{\hat{\boldsymbol{a}}}_1 v_3 \\ \underrightarrow{\hat{\boldsymbol{a}}}_2 v_1 - \underrightarrow{\hat{\boldsymbol{a}}}_1 v_2 \end{bmatrix} = \begin{bmatrix} \underrightarrow{\hat{\boldsymbol{a}}}_1 & \underrightarrow{\hat{\boldsymbol{a}}}_2 & \underrightarrow{\hat{\boldsymbol{a}}}_3 \end{bmatrix}\begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix} = \underrightarrow{\mathscr{F}}_A^\mathsf{T}\begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix}\boldsymbol{v}_A = \underrightarrow{\mathscr{F}}_A^\mathsf{T}\boldsymbol{u}_A^\times\boldsymbol{v}_A \tag{1.7}$$

where we define the skew-symmetric operator, $(\cdot)^\times$, to act on a generic column matrix, $\boldsymbol{w}$, as follows:

$$\boldsymbol{w}^\times \triangleq \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix}$$

and among its properties are $\boldsymbol{w}^\times = -\boldsymbol{w}^{\times\mathsf{T}}$ and $\boldsymbol{u}^\times\boldsymbol{v} = -\boldsymbol{v}^\times\boldsymbol{u}$. Repeating the operations of Eqs. (1.6) and (1.7) in another frame, $\mathscr{F}_B$, we have:

$$\underrightarrow{\boldsymbol{u}}\times\underrightarrow{\boldsymbol{v}} = \underrightarrow{\mathscr{F}}_A^\mathsf{T}\boldsymbol{u}_A^\times\boldsymbol{v}_A = \underrightarrow{\mathscr{F}}_B^\mathsf{T}\boldsymbol{u}_B^\times\boldsymbol{v}_B \tag{1.8}$$

Note: All of the well-known vector identities can be written in referential form (using column matrices as above) via relationships in Eqs. (1.5) and (1.8). For example:

$$\underrightarrow{\boldsymbol{a}}\cdot(\underrightarrow{\boldsymbol{b}}\times\underrightarrow{\boldsymbol{c}}) = \underrightarrow{\boldsymbol{b}}\cdot(\underrightarrow{\boldsymbol{c}}\times\underrightarrow{\boldsymbol{a}}) = \underrightarrow{\boldsymbol{c}}\cdot(\underrightarrow{\boldsymbol{a}}\times\underrightarrow{\boldsymbol{b}}) \quad\Rightarrow\quad \boldsymbol{a}^\mathsf{T}\boldsymbol{b}^\times\boldsymbol{c} = \boldsymbol{b}^\mathsf{T}\boldsymbol{c}^\times\boldsymbol{a} = \boldsymbol{c}^\mathsf{T}\boldsymbol{a}^\times\boldsymbol{b} \tag{1.9}$$

that are known as the vector and referential forms of the “scalar triple product”. Another useful identity (in referential form) is:

$$\boldsymbol{a}^\times\boldsymbol{b}^\times = \boldsymbol{b}\boldsymbol{a}^\mathsf{T} - (\boldsymbol{a}^\mathsf{T}\boldsymbol{b})\mathbf{1} \tag{1.10}$$

Refer to pages 6 and 11 of *Spacecraft Dynamics and Control: and Introduction* for some more examples of such identities.

Change of Frames

Recall that a vector can be written in any reference frame using the appropriate vectrices and column matrices of components, as shown in Eq. (1.3) for vector $\underrightarrow{\boldsymbol{v}}$ and frames $\mathscr{F}_A$ and $\mathscr{F}_B$. Pre-multiplying (scalar) Eq. (1.3) by $\underrightarrow{\mathscr{F}}_A$ yields:

$$\cancelto{\mathbf{1}}{\underrightarrow{\mathscr{F}}_A\cdot\underrightarrow{\mathscr{F}}_A^\mathsf{T}}\boldsymbol{v}_A = \underrightarrow{\mathscr{F}}_A\cdot\underrightarrow{\mathscr{F}}_B^\mathsf{T}\boldsymbol{v}_B \quad\Rightarrow \boldsymbol{v}_A = \underrightarrow{\mathscr{F}}_A\cdot\underrightarrow{\mathscr{F}}_B^\mathsf{T}\boldsymbol{v}_B \tag{1.11}$$

Definition. The *rotation matrix* from frame $\mathscr{F}_B$ to frame $\mathscr{F}_A$ is defined as:

$$\boldsymbol{C}_{AB} \triangleq \underrightarrow{\mathscr{F}}_A\cdot\underrightarrow{\mathscr{F}}_B^\mathsf{T} = \begin{bmatrix} \underrightarrow{\hat{\boldsymbol{a}}}_1\cdot\underrightarrow{\hat{\boldsymbol{b}}}_1 & \underrightarrow{\hat{\boldsymbol{a}}}_1\cdot\underrightarrow{\hat{\boldsymbol{b}}}_2 & \underrightarrow{\hat{\boldsymbol{a}}}_1\cdot\underrightarrow{\hat{\boldsymbol{b}}}_3 \\ \underrightarrow{\hat{\boldsymbol{a}}}_2\cdot\underrightarrow{\hat{\boldsymbol{b}}}_1 & \underrightarrow{\hat{\boldsymbol{a}}}_2\cdot\underrightarrow{\hat{\boldsymbol{b}}}_2 & \underrightarrow{\hat{\boldsymbol{a}}}_2\cdot\underrightarrow{\hat{\boldsymbol{b}}}_3 \\ \underrightarrow{\hat{\boldsymbol{a}}}_3\cdot\underrightarrow{\hat{\boldsymbol{b}}}_1 & \underrightarrow{\hat{\boldsymbol{a}}}_3\cdot\underrightarrow{\hat{\boldsymbol{b}}}_2 & \underrightarrow{\hat{\boldsymbol{a}}}_3\cdot\underrightarrow{\hat{\boldsymbol{b}}}_3 \end{bmatrix} = \begin{bmatrix} \cos\theta_{11} & \cos\theta_{12} & \cos\theta_{13} \\ \cos\theta_{21} & \cos\theta_{22} & \cos\theta_{23} \\ \cos\theta_{31} & \cos\theta_{32} & \cos\theta_{33} \end{bmatrix}$$

where θ_{ij} is the angle between $\underrightarrow{\hat{\boldsymbol{a}}}_i$ and $\underrightarrow{\hat{\boldsymbol{b}}}_j$, and $\cos\theta_{ij}$ are known as direction cosines. We thus have:

$$\boldsymbol{v}_A = \boldsymbol{C}_{AB}\boldsymbol{v}_B \quad , \quad \boldsymbol{v}_B = \boldsymbol{C}_{BA}\boldsymbol{v}_A \tag{1.12}$$

Note: Rotation matrices could also be related to their associated vectrices:

$$\underrightarrow{\mathscr{F}}_A = \boldsymbol{C}_{AB}\underrightarrow{\mathscr{F}}_B \quad , \quad \underrightarrow{\mathscr{F}}_A^{\mathsf{T}} = \underrightarrow{\mathscr{F}}_B^{\mathsf{T}}\boldsymbol{C}_{BA} \tag{1.13}$$

In addition, rotation matrices enjoy the following important and useful properties:

$$\boldsymbol{C}_{BA} = \boldsymbol{C}_{AB}^{-1} = \boldsymbol{C}_{AB}^{\mathsf{T}} \tag{1.14}$$

$$(\boldsymbol{C}_{BA}\boldsymbol{v}_A)^{\times} = \boldsymbol{C}_{BA}\boldsymbol{v}_A^{\times}\boldsymbol{C}_{AB} \tag{1.15}$$

Multiple Reference Frames

Now, consider more than two reference frames, namely $\underrightarrow{\mathscr{F}}_A$, $\underrightarrow{\mathscr{F}}_B$, and $\underrightarrow{\mathscr{F}}_C$:

$$\underrightarrow{v} = \underrightarrow{\mathscr{F}}_A^{\mathsf{T}}\boldsymbol{v}_A = \underrightarrow{\mathscr{F}}_B^{\mathsf{T}}\boldsymbol{v}_B = \underrightarrow{\mathscr{F}}_C^{\mathsf{T}}\boldsymbol{v}_C \tag{1.16}$$

pre-multiplying (scalar) which with $\underrightarrow{\mathscr{F}}_C$ and $\underrightarrow{\mathscr{F}}_B$ separately yields:

$$\boldsymbol{v}_C = \underrightarrow{\mathscr{F}}_C \cdot \underrightarrow{\mathscr{F}}_B^{\mathsf{T}}\boldsymbol{v}_B = \boldsymbol{C}_{CB}\boldsymbol{v}_B \tag{1.17a}$$

$$\boldsymbol{v}_B = \underrightarrow{\mathscr{F}}_B \cdot \underrightarrow{\mathscr{F}}_A^{\mathsf{T}}\boldsymbol{v}_A = \boldsymbol{C}_{BA}\boldsymbol{v}_A \tag{1.17b}$$

which can be combined to result in:

$$\boldsymbol{v}_C = \boldsymbol{C}_{CB}\boldsymbol{C}_{BA}\boldsymbol{v}_A \tag{1.18}$$

But Eq. (1.16) could also be revisited by pre-multiplying its far right-hand side with $\underrightarrow{\mathscr{F}}_C$:

$$\boldsymbol{v}_C = \underrightarrow{\mathscr{F}}_C \cdot \underrightarrow{\mathscr{F}}_A^{\mathsf{T}}\boldsymbol{v}_A = \boldsymbol{C}_{CA}\boldsymbol{v}_A \tag{1.19}$$

Equating Eqs. (1.18) and (1.19) shows that successive rotations of multiple frames can be simply represented by multiplication of the rotation matrices involved:

$$\boldsymbol{C}_{CA} = \boldsymbol{C}_{CB}\boldsymbol{C}_{BA} \tag{1.20}$$

Principal Rotations

As special rotation matrices, *principal* rotation matrices describe the change of frames obtained by rotating one frame about only one of its coordinate axes. For example, if $\underrightarrow{\mathscr{F}}_B$ is obtained by rotating $\underrightarrow{\mathscr{F}}_A$ about its 1-axis by angle θ, the two frames will share a common basis vector ($\underrightarrow{\hat{a}}_1 \equiv \underrightarrow{\hat{b}}_1$), and we have:

$$\boldsymbol{C}_1(\theta) \triangleq \boldsymbol{C}_{BA_1}(\theta) = \underrightarrow{\mathscr{F}}_B \cdot \underrightarrow{\mathscr{F}}_A^{\mathsf{T}} = \begin{bmatrix} \cancelto{1}{\underrightarrow{\hat{b}}_1 \cdot \underrightarrow{\hat{a}}_1} & \cancelto{0}{\underrightarrow{\hat{b}}_1 \cdot \underrightarrow{\hat{a}}_2} & \cancelto{0}{\underrightarrow{\hat{b}}_1 \cdot \underrightarrow{\hat{a}}_3} \\ \cancelto{0}{\underrightarrow{\hat{b}}_2 \cdot \underrightarrow{\hat{a}}_1} & \underrightarrow{\hat{b}}_2 \cdot \underrightarrow{\hat{a}}_2 & \underrightarrow{\hat{b}}_2 \cdot \underrightarrow{\hat{a}}_3 \\ \cancelto{0}{\underrightarrow{\hat{b}}_3 \cdot \underrightarrow{\hat{a}}_1} & \underrightarrow{\hat{b}}_3 \cdot \underrightarrow{\hat{a}}_2 & \underrightarrow{\hat{b}}_3 \cdot \underrightarrow{\hat{a}}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \cancelto{\sin\theta}{\cos(\frac{\pi}{2}-\theta)} \\ 0 & \cancelto{-\sin\theta}{\cos(\frac{\pi}{2}+\theta)} & \cos\theta \end{bmatrix} \tag{1.21}$$

A similar procedure can be followed for the special case of rotating about the 2- or 3-axis of $\underrightarrow{\mathscr{F}}_A$. The three principal rotation matrices (each corresponding to the rotation about the axis in its subscript as shown in

Figure 1.3) are, therefore, given by:

$$\boldsymbol{C}_1(\theta) \triangleq \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \quad , \quad \boldsymbol{C}_2(\theta) \triangleq \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \quad , \quad \boldsymbol{C}_3(\theta) \triangleq \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.22}$$

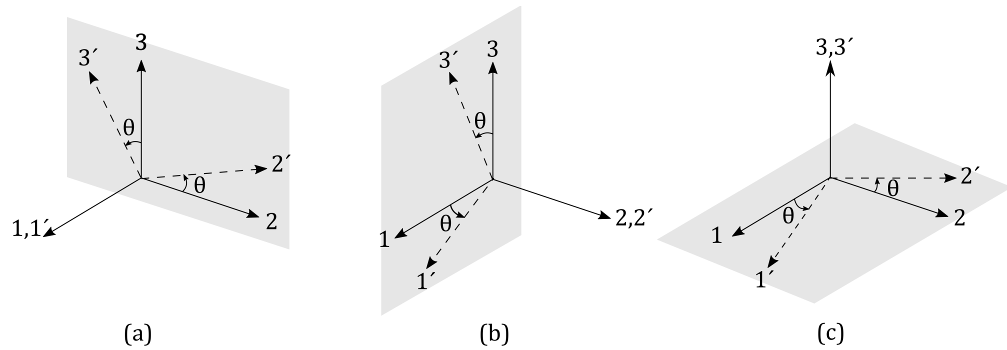


Figure 1.3: Principal Rotations about the (a) 1-axis, (b) 2-axis, and (c) 3-axis

Attitude Representations

There are several ways one can parameterize angular displacement, and describe the orientation of a body (or, rather, a reference frame attached to it) with respect to other frames:

Euler Angles $(\theta_1, \theta_2, \theta_3)$

The principal rotations described above have only 1 degree of freedom (about one of the axes). In a three-dimensional space, 3 degrees of freedom are required to describe a general rotation. Therefore, a set of 3 angles (known as "Euler Angles") could be used in conjunction with 3 principal rotation matrices to describe attitude:

$$\boldsymbol{C} = \boldsymbol{C}_\gamma(\theta_3)\boldsymbol{C}_\beta(\theta_2)\boldsymbol{C}_\alpha(\theta_1) \tag{1.23}$$

where α, β, and γ, representing the axis of rotation, can be any combination of $\{1, 2, 3\}$, as long as $\alpha \neq \beta$ and $\beta \neq \gamma$ to avoid consecutive rotations about the same axis. Note that rotations do not, in general, commute, so it is important to specify the correct order of principal rotations.

For example, a rotation of θ_1 about the 3-axis of $\mathscr{F}_A$, followed by a rotation of θ_2 about the 1-axis of the resulting frame, and an additional rotation of θ_3 about the 3-axis of the previous intermediate frame to eventually obtain $\mathscr{F}_B$ could be described by $\boldsymbol{C}_{BA}(\theta_3, \theta_2, \theta_1) = \boldsymbol{C}_3(\theta_3)\boldsymbol{C}_1(\theta_2)\boldsymbol{C}_3(\theta_1)$. This is known as a 3-1-3 rotation matrix.

Note: A drawback of using Euler angles for attitude representation is their singularity issue that might occur at certain angles. For instance, if $\theta_2 = 0$ in the 3-1-3 rotation described above, $\boldsymbol{C}_1(0) = \mathbf{1}$ results in $\boldsymbol{C}_{BA} = \boldsymbol{C}_3(\theta_3)\boldsymbol{C}_3(\theta_1) = \boldsymbol{C}_3(\theta_3 + \theta_1)$, which implies that the first and third rotations collapse into one. As a result, the rotation matrix cannot be uniquely determined from the object's attitude in this case.

Note: Refer to Table 2.1 of *Spacecraft Attitude Dynamics* for all 12 compound rotation matrices obtained using Euler angles.

Euler Axis/Angle Variables $(\hat{\underrightarrow{a}}, \phi)$

According to Euler's rotation theorem, the most general displacement of a rigid body with one point fixed is a rotation of ϕ about an axis, along $\hat{\boldsymbol{a}}$, through that point;

$$\boldsymbol{C} = \cos(\phi)\mathbf{1} + \left[1 - \cos(\phi)\right]\hat{\boldsymbol{a}}\hat{\boldsymbol{a}}^{\mathsf{T}} - \sin(\phi)\hat{\boldsymbol{a}}^{\times} \tag{1.24}$$

Refer to pages 19 and 20 of *Spacecraft Dynamics and Control: an Introduction* for a geometric study that yields this result, and pages 11 and 12 of *Spacecraft Attitude Dynamics* that considers the eigenvalue problem $\boldsymbol{C}\boldsymbol{e} = \lambda_\alpha \boldsymbol{e}$, with $\boldsymbol{C}\hat{\boldsymbol{a}} = \hat{\boldsymbol{a}}$, in order to show that the right-hand side of Eq. (1.24) is, indeed, a rotation matrix.

Euler Parameters (Quaternions) $(\boldsymbol{\epsilon}, \eta)$

Euler parameters (also known as "Quaternions") consist of a 4-parameter set, defined as:

$$\boldsymbol{\epsilon} \triangleq \hat{\boldsymbol{a}}\sin(\frac{\phi}{2}) \quad , \quad \eta \triangleq \cos(\frac{\phi}{2})$$

using which the rotation matrix of Eq. (1.24) can be represented as:

$$\boldsymbol{C} = (\eta^2 - \boldsymbol{\epsilon}^{\mathsf{T}}\boldsymbol{\epsilon})\mathbf{1} + 2\boldsymbol{\epsilon}\boldsymbol{\epsilon}^{\mathsf{T}} - 2\eta\boldsymbol{\epsilon}^{\times} \tag{1.25}$$

To see the motivation behind defining quaternions as above, use Eq. (1.24) twice to expand $\boldsymbol{C}(\hat{\boldsymbol{a}}_3, \phi_3) = \boldsymbol{C}(\hat{\boldsymbol{a}}_2, \phi_2)\boldsymbol{C}(\hat{\boldsymbol{a}}_1, \phi_1)$ for two consecutive rotations, resulting in:

$$\cos(\frac{\phi_3}{2}) = \cos(\frac{\phi_1}{2})\cos(\frac{\phi_2}{2}) - \sin(\frac{\phi_1}{2})\sin(\frac{\phi_2}{2})\hat{\boldsymbol{a}}_1^{\mathsf{T}}\hat{\boldsymbol{a}}_2 \tag{1.26a}$$

$$\sin(\frac{\phi_3}{2})\hat{\boldsymbol{a}}_3 = \hat{\boldsymbol{a}}_1\sin(\frac{\phi_1}{2})\cos(\frac{\phi_2}{2}) + \hat{\boldsymbol{a}}_2\cos(\frac{\phi_1}{2})\sin(\frac{\phi_2}{2}) + \hat{\boldsymbol{a}}_1^{\times}\hat{\boldsymbol{a}}_2\sin(\frac{\phi_1}{2})\cos(\frac{\phi_2}{2}) \tag{1.26b}$$

An examination of Eq. (1.26) suggests simplifications resulting from defining quaternions as indicated.

Note: The 4 Euler parameters are *not* linearly independent, since they are, by definition, subject to the unity constraint that $\boldsymbol{\epsilon}^{\mathsf{T}}\boldsymbol{\epsilon} + \eta^2 = 1$. Regardless, as a result of using 4 parameters, they circumvent the singularity issues arising from using only 3 Euler angles to represent attitude.

Infinitesimal Rotations

The attitude parameterization sets described above are general enough for angular displacements of arbitrary magnitude. However, if the rotations are relatively small, such that small angle approximations $\cos\theta_i \approx 1, \sin\theta_i \approx \theta_i$, and $\theta_i\theta_j \approx 0$ hold for Euler angles, or $\cos\phi \approx 1$, $\sin\phi \approx \phi$, and $\phi_i^2 \approx 0$ hold for Euler axis/angle variables or quaternions, then the attitude can be represented using simplified forms of Eqs. (1.23), (1.24), or (1.25):

$$\boldsymbol{C} \approx \mathbf{1} - \boldsymbol{\theta}^{\times} \approx \mathbf{1} - \phi\hat{\boldsymbol{a}}^{\times} \approx \mathbf{1} - 2\boldsymbol{\epsilon}^{\times} \tag{1.27}$$

where $\boldsymbol{\theta} \triangleq [\theta_3 \ \theta_2 \ \theta_1]$. We conclude that the various attitude parameterizations lose their differences when the rotation angles are small. In addition, unlike arbitrarily large rotations, the order of compound rotations no longer matters for infinitesimally small angles; i.e. the constituent rotation matrices can commute: $\boldsymbol{C}_{CA} = \boldsymbol{C}_{CB}\boldsymbol{C}_{BA} = \boldsymbol{C}_{BA}\boldsymbol{C}_{CB}$ for small angles.