Lecture 5

Orbit Description & Determination

In this lesson, classical orbital elements of an orbiting body and obtaining them using position and velocity vectors are discussed. Then, one of the several known methods of preliminary orbit determination in the absence of complete position and velocity information, namely using two position vectors and time of flight, is briefly reviewed.

Orbit Description

The vectors $\underrightarrow{r}(t)$ and $\underrightarrow{v}(t)$, with a total of 6 components, would be sufficient for a complete solution of the equations of motion:

$$\underrightarrow{r}^{\bullet} = \underrightarrow{v} \quad , \quad \underrightarrow{r}(0) = \underrightarrow{r}_0 \tag{5.1a}$$

$$\underrightarrow{v}^{\bullet} = \frac{-\mu}{r^3}\underrightarrow{r} \quad , \quad \underrightarrow{v}(0) = \underrightarrow{v}_0 \tag{5.1b}$$

which form a coupled system of second order ODEs. However, for convenience, spacecraft dynamicists and control engineers typically uses other element to describe orbits.

Classical Orbital Elements

Recalling the Earth-centred inertial (ECI, also known as geocentric-equatorial) reference frame defined in FUNDAMENTALS, the following set of 6 orbital elements (among which the angular ones are depicted in Figure 5.1) are called the "classical orbital elements":

- a semi-major axis
- e eccentricity
- i inclination
- Ω right ascension of the ascending node (RAAN)
- ω argument of periapsis
- t_0 time of periapsis passage

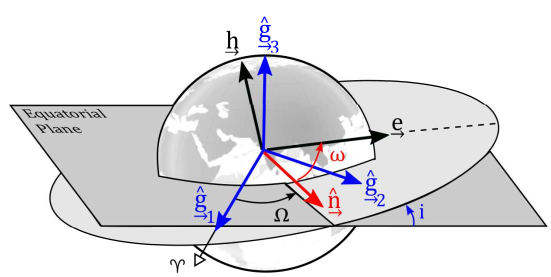

Figure 5.1: Orbital Elements and $\mathscr{F}_G$

Note: The *ascending node* is one of the two points on the orbit at which the orbiting spacecraft passes the reference plane (the equatorial plane in this case).

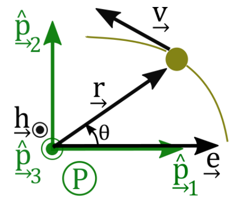

Figure 5.2: Perifocal

As mentioned in ORBITAL MECHANICS, a and e describe the shape and size of the spacecraft's orbit, respectively. The parameters $0 \leqslant i \leqslant \pi$, $0 \leqslant \Omega < 2\pi$ and $0 \leqslant \omega < 2\pi$ describe its orientation in space (with respect to the equatorial plane and the 1-axis pointing in the direction of vernal equinox, ♈). A 3-1-3 transformation could be used to move from the ECI frame, $\mathscr{F}_G$, to the perifocal frame, $\mathscr{F}_P$ (shown in Figure 5.2) :

$$\boldsymbol{C}_{PG} = \boldsymbol{C}_3(\omega)\boldsymbol{C}_1(i)\boldsymbol{C}_3(\Omega) \tag{5.2}$$

Position and Velocity using Orbital Elements

The following procedure could be followed for determining the position and velocity vectors at any time t, using the classical orbital elements:

1. Solve for eccentric anomaly, E, using Kepler's equation, and then solve for θ using E based on the following from ORBITAL MECHANICS:

$$E - e\sin(E) = \sqrt{\frac{\mu}{a^3}}(t - t_0) = M \quad , \quad \tan\left(\frac{\theta}{2}\right) = \sqrt{\frac{1+e}{1-e}}\tan\left(\frac{E}{2}\right) \tag{5.3}$$

2. Find the radial distance, r, employing the polar equation, and use it to determine the position vector from its perifocal components:

$$\underrightarrow{\boldsymbol{r}} = \underrightarrow{\mathscr{F}}_G^{\mathsf{T}}\boldsymbol{C}_{GP}\begin{bmatrix} r\cos(\theta) \\ r\sin(\theta) \\ 0 \end{bmatrix} \quad , \quad r = \frac{l}{1 + e\cos(\theta)} \quad , \quad l \triangleq \frac{h^2}{\mu} = a(1 - e^2) \tag{5.4}$$

3. Find the time derivatives of radial distance and true anomaly, $\dot{r}$ and $\dot{\theta}$, from differentiating the polar equation above with respect to time, and using the following angular momentum relationship from ORBITAL MECHANICS:

$$\dot{r} = \frac{a(1-e^2)e\dot{\theta}\sin(\theta)}{\big(1 + e\cos(\theta)\big)^2} \quad , \quad \dot{\theta} = \frac{h}{r^2} = \frac{\sqrt{\mu a(1-e^2)}}{r^2} \tag{5.5}$$

4. Use $\dot{r}$ and $\dot{\theta}$ to determine the velocity vector from its perifocal components:

$$\underrightarrow{\boldsymbol{v}} = \underrightarrow{\boldsymbol{r}}^{\bullet} = \underrightarrow{\mathscr{F}}_G^{\mathsf{T}}\boldsymbol{C}_{GP}\begin{bmatrix} \dot{r}\cos(\theta) - r\dot{\theta}\sin(\theta) \\ \dot{r}\sin(\theta) + r\dot{\theta}\cos(\theta) \\ 0 \end{bmatrix} = \underrightarrow{\mathscr{F}}_G^{\mathsf{T}}\boldsymbol{C}_{GP}\begin{bmatrix} -\sin(\theta) \\ \cos(\theta) + e \\ 0 \end{bmatrix}\sqrt{\frac{\mu}{a(1-e^2)}} \tag{5.6}$$

Similarly but in the reverse direction, the knowledge of $\underrightarrow{\boldsymbol{r}}$ and $\underrightarrow{\boldsymbol{v}}$ at any given time could be used to determine the classical orbital parameters. This will follow next.

Orbital Elements using Position and Velocity

First, we look at how to determine the classical orbital elements, assuming *both* $\underset{\rightarrow}{r}(t)$ and $\underset{\rightarrow}{v}(t)$ are specified at a given time, t. Once again, many of the relationships used in the following steps are from ORBITAL MECHANICS:

1. Find the specific angular momentum, a constant of motion, and express its components in the ECI frame, $\mathscr{F}_G$:
$$\underset{\rightarrow}{h} = \underset{\rightarrow}{r} \times \underset{\rightarrow}{v} \quad \Rightarrow \quad \boldsymbol{h}_G = \boldsymbol{r}_G^{\times}\boldsymbol{v}_G \tag{5.7}$$

2. Calculate the specific energy, another constant of motion, and use it to determine the orbit's size:
$$\epsilon = \frac{\underset{\rightarrow}{v} \cdot \underset{\rightarrow}{v}}{2} - \frac{\mu}{\sqrt{\underset{\rightarrow}{r} \cdot \underset{\rightarrow}{r}}} \quad , \quad a = \frac{-\mu}{2\epsilon} \tag{5.8}$$

3. Find the eccentricity vector, the third constant of motion, and use it to determine the orbit's shape:
$$\underset{\rightarrow}{e} = \frac{\underset{\rightarrow}{v} \times \underset{\rightarrow}{h}}{\mu} - \frac{\underset{\rightarrow}{r}}{r} \quad \Rightarrow \quad \boldsymbol{e}_G = \frac{\boldsymbol{v}_G^{\times}\boldsymbol{h}_G}{\mu} - \frac{\boldsymbol{r}_G}{r} \quad , \quad e = |\underset{\rightarrow}{e}| \tag{5.9}$$
Alternatively, eccentricity can also be obtained by rearranging $l \triangleq h^2/\mu = a(1-e^2)$, where h and a are now known from Eqs. (5.7) and (5.8), respectively.

4. Use the scalar product of the angular momentum vector (which is normal to the orbital plane) and the 3-axis of $\mathscr{F}_G$, both shown in Figure 5.1, to find the orbit's inclination:
$$\underset{\rightarrow}{h} \cdot \underset{\rightarrow}{\hat{g}}_3 = h\cos(i) \quad \Rightarrow \quad i = \cos^{-1}\left(\frac{1}{h}\begin{bmatrix}0 & 0 & 1\end{bmatrix}^{\mathsf{T}}\boldsymbol{h}_G\right) \tag{5.10}$$
Note: Since $0 \leqslant i \leqslant \pi$, Eq. (5.10) provides a unique solution.

5. Find the unit vector pointing from the common origin $O_P \equiv O_G$ of $\mathscr{F}_P$ and $\mathscr{F}_G$ to the ascending node, and resolve its components in $\mathscr{F}_G$:
$$\underset{\rightarrow}{\hat{n}} = \frac{\underset{\rightarrow}{\hat{g}}_3 \times \underset{\rightarrow}{h}}{|\underset{\rightarrow}{\hat{g}}_3 \times \underset{\rightarrow}{h}|} \quad \Rightarrow \quad \hat{\boldsymbol{n}}_G = \frac{\begin{bmatrix}0 & 0 & 1\end{bmatrix}^{\mathsf{T}\times}\boldsymbol{h}_G}{n} \tag{5.11}$$

6. Use the scalar product of the ascending node vector (which specifies the intersection line) with the 1-axis of $\mathscr{F}_G$ and the eccentricity vector (determined by Eq. (5.9)), all depicted in Figure 5.1, to determine the orbit's orientation with respect to the reference equatorial plane:
$$\underset{\rightarrow}{\hat{n}} \cdot \underset{\rightarrow}{\hat{g}}_1 = \cos(\Omega) \quad \Rightarrow \Omega = \cos^{-1}\left(\begin{bmatrix}1 & 0 & 0\end{bmatrix}^{\mathsf{T}}\hat{\boldsymbol{n}}_G\right) \tag{5.12a}$$
$$\underset{\rightarrow}{\hat{n}} \cdot \underset{\rightarrow}{e} \;= e\cos(\omega) \quad \Rightarrow \omega = \cos^{-1}\left(\frac{1}{e}\boldsymbol{e}_G^{\mathsf{T}}\hat{\boldsymbol{n}}_G\right) \tag{5.12b}$$
Note: Since $0 \leqslant \Omega < 2\pi$, the 2-axis is used to determine the appropriate quadrant: if $\underset{\rightarrow}{\hat{n}} \cdot \underset{\rightarrow}{\hat{g}}_2 \geqslant 0$, then $\Omega \in [0, \pi]$; otherwise, $\Omega \in (\pi, 2\pi)$. Similarly, using the 3-axis (normal to the equatorial plane), if $\underset{\rightarrow}{\hat{e}} \cdot \underset{\rightarrow}{\hat{g}}_3 \geqslant 0$, then $\omega \in [0, \pi]$; otherwise, $\omega \in (\pi, 2\pi)$.

7. Noting that both ascending node and position vectors are on the orbital plane, use their scalar product to determine the true anomaly:

$$\underset{\rightarrow}{\hat{\boldsymbol{n}}} \cdot \underset{\rightarrow}{\boldsymbol{r}} = r\cos(\theta+\omega) \quad \Rightarrow \quad \theta = \cos^{-1}\left(\frac{1}{r}\hat{\boldsymbol{n}}_G^{\intercal}\boldsymbol{r}_G\right) - \omega \tag{5.13}$$

   Note: Since $0 \leqslant \theta < 2\pi$, the 3-axis (normal to the equatorial plane) is used to determine the appropriate quadrant: if $\underset{\rightarrow}{\boldsymbol{r}} \cdot \underset{\rightarrow}{\hat{\boldsymbol{g}}}_3 \geqslant 0$, then the *argument of latitude*, $\theta + \omega \in [0, \pi]$.

8. Find the eccentric anomaly, then use Kepler's equation to determine the time of perigee passage (using re-arranged forms of Eq. (5.3)):

$$\tan\left(\frac{E}{2}\right) = \sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right) \quad , \quad t_0 = t - \left[E - e\sin(E)\right]\sqrt{\frac{a^3}{\mu}} \tag{5.14}$$

Notice that once the constants of motion, namely $\underset{\rightarrow}{\boldsymbol{h}}$, ϵ, and $\underset{\rightarrow}{\boldsymbol{e}}$ are determined, the rest of the procedure simply involves taking scalar products of the known vectors (one fixed to $\mathscr{F}_G$ and one lying on the orbit) to evaluate the angular relations. To determine the appropriate quadrant, scalar product with an additional $\mathscr{F}_G$-fixed vector is used.

Orbit Determination

If we do not know both $\underset{\rightarrow}{\boldsymbol{r}}(t)$ and $\underset{\rightarrow}{\boldsymbol{v}}(t)$ at a given time, t, it is still possible to estimate a preliminary orbit using other sources of information, such as using two position vectors, $\underset{\rightarrow}{\boldsymbol{r}}_1 \triangleq \underset{\rightarrow}{\boldsymbol{r}}(t_1)$ and $\underset{\rightarrow}{\boldsymbol{r}}_2 \triangleq \underset{\rightarrow}{\boldsymbol{r}}(t_2)$ and the travel time between the two, $\Delta t = t_2 - t_1$. This is known as "Lambert's Problem", and is illustrated in Figure 5.3.

Note: For other methods of orbit determination, such as using three position vectors (with at least one of the associated times known), or three line of sight vectors (known direction but unknown range), refer to Sections 4.1 and 4.2, respectively, of *Spacecraft Dynamics and Control: an Introduction*.

Since a pair of $\underset{\rightarrow}{\boldsymbol{r}}(t)$ and $\underset{\rightarrow}{\boldsymbol{v}}(t)$ at a given t is enough to determine the orbital elements, as discussed in the previous section of this lesson, it suffices to find either $\underset{\rightarrow}{\boldsymbol{v}}_1$ or $\underset{\rightarrow}{\boldsymbol{v}}_2$. We will focus on finding the former, because it tells us how much initial velocity change is required for a spacecraft to travel from Earth, starting at time t_1, to another planet, the position of which at t_2 is known. Assume the orbit is elliptic and the angle between the two position vectors does not exceed $90°$.

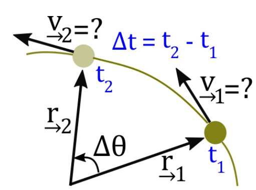

Figure 5.3: Lambert's Problem

Since an orbit remains planar, any vector on the orbital plane (including $\underset{\rightarrow}{\boldsymbol{r}}_2$) can be written as a linear combination of any two non-parallel vectors on the same plane, which we take to be $\underset{\rightarrow}{\boldsymbol{r}}_1$ and $\underset{\rightarrow}{\boldsymbol{v}}_1 = \underset{\rightarrow}{\boldsymbol{r}}_1^{\bullet}$:

$$\underset{\rightarrow}{\boldsymbol{r}}_2 = \alpha\underset{\rightarrow}{\boldsymbol{r}}_1 + \beta\underset{\rightarrow}{\boldsymbol{v}}_1 \quad \Rightarrow \quad \underset{\rightarrow}{\boldsymbol{v}}_1 = \frac{\underset{\rightarrow}{\boldsymbol{r}}_2 - \alpha\underset{\rightarrow}{\boldsymbol{r}}_1}{\beta} \tag{5.15}$$

where α and β are the so-called Lagrangian scalar coefficients that we need to find. Recall, from Eqs. (5.4) and (5.6) and as shown in Figure 5.2, that the $\underset{\rightarrow}{\boldsymbol{r}}$ and $\underset{\rightarrow}{\boldsymbol{v}}$ expressions in $\mathscr{F}_P$ can be written as:

$$\begin{bmatrix}\underset{\rightarrow}{\boldsymbol{r}}\\ \underset{\rightarrow}{\boldsymbol{v}}\end{bmatrix} = \begin{bmatrix} r\cos(\theta) & r\sin(\theta) \\ -\sqrt{\frac{\mu}{l}}\sin(\theta) & \sqrt{\frac{\mu}{l}}\left[e+\cos(\theta)\right]\end{bmatrix}\begin{bmatrix}\underset{\rightarrow}{\hat{\boldsymbol{p}}}_1\\ \underset{\rightarrow}{\hat{\boldsymbol{p}}}_2\end{bmatrix} \triangleq \boldsymbol{A}(\theta)\begin{bmatrix}\underset{\rightarrow}{\hat{\boldsymbol{p}}}_1\\ \underset{\rightarrow}{\hat{\boldsymbol{p}}}_2\end{bmatrix} \tag{5.16}$$

where we define $\boldsymbol{A}(\theta)$ to represent the coordinates, and we have:

$$\det(\boldsymbol{A}) = \sqrt{\frac{\mu}{l}}\, r\big[1 + e\cos(\theta)\big] = \sqrt{\frac{\mu}{l}}\, \overset{h^2/\mu}{\uparrow l} \;\Rightarrow \det(\boldsymbol{A}) = h \neq 0 \tag{5.17}$$

which shows that $\boldsymbol{A}(\theta)$ is invertible. Inverting the relationship in Eq. (5.16) yields:

$$\begin{bmatrix} \underset{\rightarrow}{\hat{\boldsymbol{p}}}_1 \\ \underset{\rightarrow}{\hat{\boldsymbol{p}}}_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{l}\big[e + \cos(\theta)\big] & \frac{-r}{\sqrt{\mu l}}\sin(\theta) \\ \frac{1}{l}\sin(\theta) & \frac{r}{\sqrt{\mu l}}\cos(\theta) \end{bmatrix} \begin{bmatrix} \underset{\rightarrow}{\boldsymbol{r}} \\ \underset{\rightarrow}{\boldsymbol{v}} \end{bmatrix} \triangleq \boldsymbol{B}(\theta) \begin{bmatrix} \underset{\rightarrow}{\boldsymbol{r}} \\ \underset{\rightarrow}{\boldsymbol{v}} \end{bmatrix} \tag{5.18}$$

where $\boldsymbol{B}(\theta) = \boldsymbol{A}^{-1}(\theta)$. To express $\underset{\rightarrow}{\boldsymbol{r}}_2$ and $\underset{\rightarrow}{\boldsymbol{v}}_2$ in terms of $\underset{\rightarrow}{\boldsymbol{r}}_1$ and $\underset{\rightarrow}{\boldsymbol{v}}_1$, we have:

$$\begin{bmatrix} \underset{\rightarrow}{\boldsymbol{r}}_2 \\ \underset{\rightarrow}{\boldsymbol{v}}_2 \end{bmatrix} = \boldsymbol{A}(\theta_2) \begin{bmatrix} \underset{\rightarrow}{\hat{\boldsymbol{p}}}_1 \\ \underset{\rightarrow}{\hat{\boldsymbol{p}}}_2 \end{bmatrix} = \boldsymbol{A}(\theta_2)\boldsymbol{B}(\theta_1) \begin{bmatrix} \underset{\rightarrow}{\boldsymbol{r}}_1 \\ \underset{\rightarrow}{\boldsymbol{v}}_1 \end{bmatrix} \tag{5.19}$$

But we also have Eq. (5.15) for $\underset{\rightarrow}{\boldsymbol{r}}_2$, equating the coefficients of which to those of the expanded form of $\boldsymbol{A}(\theta_2)\boldsymbol{B}(\theta_1)$, using the polar equation for $e\cos(\theta_2) = \frac{l}{r_2} - 1$, and some trigonometric identities for $\Delta\theta \triangleq \theta_2 - \theta_1$ eventually results in:

$$\alpha = 1 - \frac{r_2}{l}\big[1 - \cos(\Delta\theta)\big] \tag{5.20a}$$

$$\beta = \frac{r_1 r_2}{\sqrt{\mu l}}\sin(\Delta\theta) \tag{5.20b}$$

where $r_1 = |\underset{\rightarrow}{\boldsymbol{r}}_1|$, $r_2 = |\underset{\rightarrow}{\boldsymbol{r}}_2|$, and $\Delta\theta$ are known, so the only missing variable is the semilatus rectum, l. From its definition, $l \triangleq h^2/\mu$, so finding h would suffice to solve the problem. To this end, we define the *sector-to-triangle area ratio* as the ratio of the area of the curved segment shaped by the orbit (the area swept by the orbiting body, shown in Figure 5.4), S, to the area of the triangle formed by joining the initial and final positions of the orbiting body, A, shown in Figure 5.5. We have:

$$\eta \triangleq \frac{S}{A} = \frac{\frac{h}{\cancel{2}}(t_2 - t_1)}{\frac{1}{\cancel{2}}|\underset{\rightarrow}{\boldsymbol{r}}_1 \times \underset{\rightarrow}{\boldsymbol{r}}_2|} \tag{5.21}$$

where the numerator comes from ORBITAL MECHANICS, where we derived Kepler's 2nd law using Newton's law of gravitation, and the denominator follows directly from $|\underset{\rightarrow}{\boldsymbol{r}}_1 \times \underset{\rightarrow}{\boldsymbol{r}}_2| = r_1 r_2 \sin(\Delta\theta)$. The effects of Δt are evident from Eq. (5.21): longer travel times would result in larger sector-triangle area ratios as a result of more deviations of the curve from a straight line.

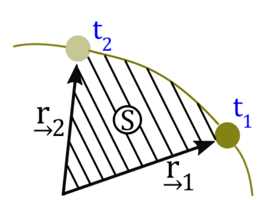

Figure 5.4: Sector

Rearranging Eq. (5.21) for h and substituting the result in $l \triangleq h^2/\mu$ yields:

$$l = \frac{\eta^2 |\underset{\rightarrow}{\boldsymbol{r}}_1 \times \underset{\rightarrow}{\boldsymbol{r}}_2|^2}{\mu(\Delta t)^2} \tag{5.22}$$

The ratio η does not have a closed form solution, as it is the solution of a transcendental equation (involving non-algebraic functions), namely $\eta = 1 + (m/\eta^2) f(m/\eta^2 - n)$, where m, n, and $f(\cdot)$ are all functions of r_1, r_2, Δt, or $\Delta\theta$. This equation can be solved iteratively, starting from an initial guess and improving upon it until $\eta_{i+1} \approx \eta_i$. Refer to pages 112 and 113 of *Spacecraft Dynamics and Control: an Introduction*, and the references therein, for more details on this iteration.

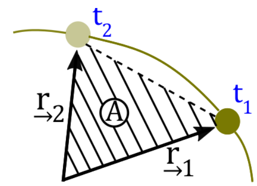

Figure 5.5: Triangle

Porkchop Plots

In practical space mission design, some charts known as "porkchop plots" are used to determine the launch windows that meet the mission requirements and are compatible with the spacecraft capabilities. These charts show contours of equal characteristic energy, with the launch date on the x-axis and the arrival date on the y-axis.

Note: Characteristic energy, c_3, is a measure of the excess specific energy, left over upon escaping the primary body's gravitational influence. For the two-body problem with a hyperbolic escape trajectory, $c_3 = v_\infty^2 = 2\epsilon$.

The centre of a porkchop plot, surrounded by the contours of constant characteristic energy, represents the energy-optimal (lowest c_3) trajectory. In addition, parallel oblique lines could be drawn to represent lines of constant time of flight (TOF), such that the higher a line is, the longer a mission corresponding to an intersecting energy contour will take. Figure 5.6 shows an illustrative porkchop plot, for example for an Earth-Venus mission.

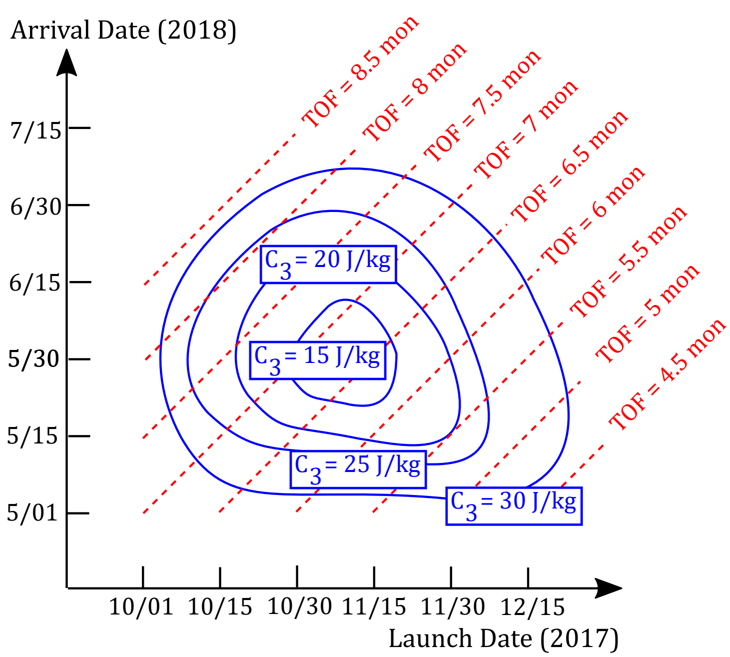

Figure 5.6: Sample Porkchop Plot