Lecture 13

Spin Stabilization

Stability of rotational motion with respect to perturbations of angular velocity is considered, and conclusions are drawn about the stability of pure spinning about each of the three principal axes. A geometric interpretation of spin stability and another one of torque-free motion are provided, and the effects of internal energy dissipation on spin stability are considered.

Stability of Torque-Free Pure Spin

Consider, similarly to Torque-Free Motion, Euler's equations in the absence of external torques, but this time, for the general "tri-inertial" case (not necessarily "axisymmetrical") with distinct I_1, I_2, and I_3 as the principal moments of inertia. Once again, we have:

$$I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 = \cancelto{0}{T_1} \tag{13.1a}$$

$$I_2\dot{\omega}_2 + (I_1 - I_3)\omega_3\omega_1 = \cancelto{0}{T_2} \tag{13.1b}$$

$$I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 = \cancelto{0}{T_3} \tag{13.1c}$$

where ω_1, ω_2, and ω_3 are the components of angular velocity in the body-fixed principal axes frame, $\mathscr{F}_P$, namely $\boldsymbol{\omega}_P$.

Linear Stability Analysis

Consider a "pure spin" case in which the rotation occurs only about one of the three principal axes of the rigid body, and when the spin rate ν is constant, equilibrium is achieved with $\overset{\bullet}{\boldsymbol{\omega}} = \mathbf{0}$. Without loss of generality (since the I's can be in any order in terms of their relative magnitude), let the axis of nominal rotation be the 2-axis, and consider an infinitesimal deviation of the angular velocity from this reference:

$$\boldsymbol{\omega}_0 \triangleq \begin{bmatrix} 0 \\ \nu \\ 0 \end{bmatrix} \quad , \quad \boldsymbol{\omega} = \boldsymbol{\omega}_0 + \delta\boldsymbol{\omega} = \begin{bmatrix} \delta\omega_1 \\ \nu + \delta\omega_2 \\ \delta\omega_3 \end{bmatrix} \tag{13.2}$$

where $\boldsymbol{\omega}_0$ is the nominal angular velocity and $\delta\boldsymbol{\omega}$ represents a small perturbation, with $\delta\omega_i \ll \nu$. With the perturbed angular velocity, the equations of motion in Eq. (13.1) become:

$$I_1\delta\dot{\omega}_1 + (I_3 - I_2)(\nu\delta\omega_3 + \overbrace{\cancel{\delta\omega_2\delta\omega_3}}^{\approx 0}) = 0 \quad\Rightarrow\quad 0 = I_1\delta\dot{\omega}_1 + (I_3 - I_2)\nu\delta\omega_3 \tag{13.3a}$$

$$I_2\delta\dot{\omega}_2 + (I_1 - I_3)(\overbrace{\cancel{\delta\omega_3\delta\omega_1}}^{\approx 0}) = 0 \quad\Rightarrow\quad 0 = I_2\delta\dot{\omega}_2 \tag{13.3b}$$

$$I_3\delta\dot{\omega}_3 + (I_2 - I_1)(\nu\delta\omega_1 + \overbrace{\cancel{\delta\omega_1\delta\omega_2}}^{\approx 0}) = 0 \quad\Rightarrow\quad 0 = I_3\delta\dot{\omega}_3 + (I_2 - I_1)\nu\delta\omega_1 \tag{13.3c}$$

where a linearity assumption is used. Now $\delta\omega_2(t)$ can be readily obtained by integrating Eq. (13.3b):

$$I_2\delta\omega_2(t) = \text{constant} \quad\Rightarrow\quad \delta\omega_2 = \delta\omega_2(0) \tag{13.4}$$

which implies that any non-zero initial perturbation of ω_2 will result in unbounded growth of the 2-component of the attitude, hence eliminating the possibility of *attitude stability*; however, we are more interested in $\boldsymbol{\omega}$-*stability* (in the Lyapunov sense): will $\boldsymbol{\omega}(t)$ remain arbitrarily close to $\boldsymbol{\omega}_0$ if $\delta\boldsymbol{\omega}$ is sufficiently small? So far, from Eq. (13.4) and boundedness of $\delta\omega_2$, the answer may be yes, as far as the 2-component is concerned.

Differentiating Eqs. (13.3a) and (13.3c) with respect to time, and using their re-arranged form again to replace $\delta\dot{\omega}_3$ and $\delta\dot{\omega}_1$ in the resulting relationships yields:

$$I_1\delta\ddot{\omega}_1 + (I_3 - I_2)\nu\delta\dot{\omega}_3 = 0 \quad\Rightarrow\quad I_1\delta\ddot{\omega}_1 + (I_3 - I_2)\Big(\frac{I_1 - I_2}{I_3}\Big)\nu^2\delta\omega_1 = 0 \quad\Rightarrow\quad \delta\ddot{\omega}_1 = \beta^2\delta\omega_1 \tag{13.5a}$$

$$I_3\delta\ddot{\omega}_3 + (I_2 - I_1)\nu\delta\dot{\omega}_1 = 0 \quad\Rightarrow\quad I_3\delta\ddot{\omega}_3 + (I_2 - I_1)\Big(\frac{I_2 - I_3}{I_1}\Big)\nu^2\delta\omega_3 = 0 \quad\Rightarrow\quad \delta\ddot{\omega}_3 = \beta^2\delta\omega_3 \tag{13.5b}$$

where the scalar (and potentially imaginary) variable β is defined for convenience as follows:

$$\beta^2 \triangleq \frac{(I_2 - I_3)(I_1 - I_2)}{I_1 I_3}\nu^2$$

and can be used to represent the general solution of the $\delta\omega$'s in Eq. (13.5):

$$\delta\omega(t) = Ae^{\beta t} + Be^{-\beta t} \quad , \quad \beta \neq 0 \tag{13.6a}$$

$$\delta\omega(t) = A + Bt \quad , \quad \beta = 0 \tag{13.6b}$$

The boundedness of the 1- and 3-components of angular velocity, therefore, depends upon the sign of β^2:

- if $\beta^2 > 0$: $e^{\beta t} \to \infty$ as $t \to \infty$, so $\delta\omega_1$ and $\delta\omega_3$ are *unbounded*.
- if $\beta^2 = 0$: $Bt \to \infty$ as $t \to \infty$, so $\delta\omega_1$ and $\delta\omega_3$ are *unbounded*.
- if $\beta^2 < 0$: $e^{\pm\beta t} = \cos(\mp i\beta t) + i\sin(\mp i\beta t)$ as $t \to \infty$, so $\delta\omega_1$ and $\delta\omega_3$ are *bounded*.

Thus, $\boldsymbol{\omega}$-*stability* occurs only when $\beta^2 \triangleq (I_2 - I_3)(I_1 - I_2)/(I_1 I_3) < 0$, which requires one of two cases:

- $I_1 > I_2$ and $I_3 > I_2$: pure spin about the "minor axis" with the smallest principal moment of inertia
- $I_1 < I_2$ and $I_3 < I_2$: pure spin about the "major axis" with the largest principal moment of inertia

Nominal pure spin of a torque-free rigid body about its "intermediate axis" is, however, *unstable*.

Note: The following inertia ratios are defined and used in literature to graphically represent stability:

$$k_1 \triangleq \frac{I_2 - I_3}{I_1} \quad , \quad k_3 \triangleq \frac{I_2 - I_1}{I_3}$$

which conveniently reduce the required parameters from 3 to 2, and are constrained as $|k_1| < 1$ and $|k_3| < 1$ for physically meaningful bodies (owing to the properties of $\boldsymbol{I}$). The three regimes of simple spin stability associated with each (k_1, k_3) pair are depicted in Figure 13.1.

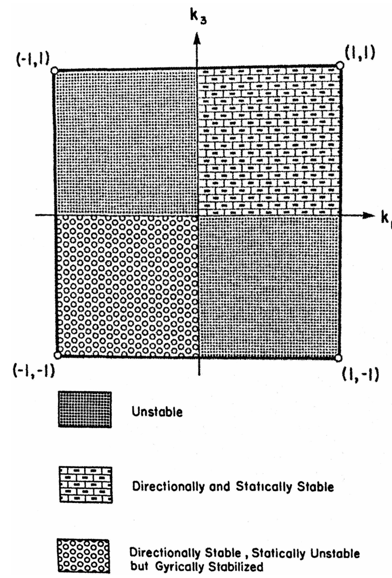

Figure 13.1: Stability Diagram [Hughes] (used with permission)

It can also be shown that the major and minor axis spins are also *directionally stable* (closely related to $\boldsymbol{\omega}$-*stability*), such that deviation of the spin axis, $\underset{\rightarrow}{\hat{\boldsymbol{p}}}_2$, from the angular momentum vector, $\underset{\rightarrow}{\boldsymbol{h}}$, can be made to remain arbitrarily small for $t > t_0$ by making it sufficiently small at $t = t_0$. The intermediate axis spin is, however, *directionally unstable*, as evident from Eq. (13.4) that shows unbounded growth of the 2-component of the attitude deviation.

Attitude Stability from Linear Mechanical Systems Perspective

Having looked at $\boldsymbol{\omega}$-*stability* (closely related to *directional stability*), we now briefly look at *attitude stability* using infinitesimal perturbations and adopting a mechanical systems view introduced in STABILITY.

Recall, from KINEMATICS, that for three reference frames $\mathscr{F}_I$, $\mathscr{F}_N$, and $\mathscr{F}_P$, angular velocity is additive as follows:

$$\boldsymbol{\omega}_P^{PI} = \boldsymbol{\omega}_P^{PN} + \boldsymbol{\omega}_P^{NI} = \boldsymbol{\omega}_P^{PN} + \boldsymbol{C}_{PN}\boldsymbol{\omega}_N^{NI} \tag{13.7}$$

Taking $\boldsymbol{\omega}_N^{NI} = \boldsymbol{\omega}_0$ (as given in Eq. (13.2)) as the nominal angular velocity of the body with respect to the inertial $\mathscr{F}_I$, and $\boldsymbol{\omega}_P^{PN} \approx \delta\dot{\boldsymbol{\theta}}$ (with infinitesimally small angular variations) as the angular velocity of the perturbed body with respect to the intermediate nominal frame, $\mathscr{F}_N$, Eq. (13.7) simplifies to:

$$\boldsymbol{\omega} \approx \delta\dot{\boldsymbol{\theta}} + \left(\mathbf{1} - \delta\boldsymbol{\theta}^{\times}\right)\boldsymbol{\omega}_0 = \begin{bmatrix} \delta\dot{\theta}_1 + \nu\delta\theta_3 \\ \delta\dot{\theta}_2 + \nu \\ \delta\dot{\theta}_3 - \nu\delta\theta_1 \end{bmatrix} \tag{13.8}$$

where the infinitesimal angle approximation $\boldsymbol{C}_{PN} \approx \mathbf{1} - \delta\boldsymbol{\theta}^{\times}$ is used. Note that unlike Eq. (13.2), Eq. (13.8) accounts for all 6 state variables, $\delta\theta$'s and $\delta\dot{\theta}$'s. Substituting this expression into the torque-free Euler's equation in Eq. (13.1) and eliminating the 2$^{\text{nd}}$ order terms results in:

$$I_1(\delta\ddot{\theta}_1 + \nu\delta\dot{\theta}_3) + (I_3 - I_2)(\cancelto{\approx 0}{\delta\dot{\theta}_2\delta\dot{\theta}_3} + \nu\delta\dot{\theta}_3 - \cancelto{\approx 0}{\nu\delta\dot{\theta}_2\delta\theta_1} - \nu^2\delta\theta_1) = 0 \tag{13.9a}$$

$$I_2\delta\ddot{\theta}_2 + (I_1 - I_3)(\cancelto{\approx 0}{\delta\dot{\theta}_1\delta\dot{\theta}_3} + \cancelto{\approx 0}{\nu\delta\theta_3\delta\dot{\theta}_3} - \cancelto{\approx 0}{\nu\delta\dot{\theta}_1\delta\theta_1} - \cancelto{\approx 0}{\nu^2\delta\theta_1\delta\theta_3}) = 0 \tag{13.9b}$$

$$I_3(\delta\ddot{\theta}_3 - \nu\delta\dot{\theta}_1) + (I_2 - I_1)(\cancelto{\approx 0}{\delta\dot{\theta}_1\delta\dot{\theta}_2} + \cancelto{\approx 0}{\nu\delta\theta_3\delta\dot{\theta}_2} + \nu\delta\dot{\theta}_1 + \nu^2\delta\theta_3) = 0 \tag{13.9c}$$

which, upon factoring the like terms, can be rewritten as:

$$I_1\delta\ddot{\theta}_1 + (I_1 + I_3 - I_2)\nu\delta\dot{\theta}_3 + (I_2 - I_3)\nu^2\delta\theta_1 = 0 \tag{13.10a}$$
$$I_2\delta\ddot{\theta}_2 = 0 \quad\Rightarrow\quad \delta\dot{\theta}_2(t) = \delta\dot{\theta}_2(0) \tag{13.10b}$$
$$I_3\delta\ddot{\theta}_3 - (I_1 + I_3 - I_2)\nu\delta\dot{\theta}_1 + (I_2 - I_1)\nu^2\delta\theta_3 = 0 \tag{13.10c}$$

where Eq. (13.10b), similarly to Eq. (13.4), implies an *attitude unstable* system, since any initial attitude perturbation persists for all times. In an attempt to draw conclusions about *directional stability*, we rewrite Eqs. (13.10a) and (13.10c) (that are decoupled from $\delta\theta_2$ and its rates) in the following linear mechanical system form:

$$\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{G}\dot{\boldsymbol{q}} + \boldsymbol{K}\boldsymbol{q} = \boldsymbol{0} \tag{13.11}$$

$$\boldsymbol{q} \triangleq \begin{bmatrix}\delta\theta_1\\ \delta\theta_3\end{bmatrix} \;,\; \boldsymbol{M} \triangleq \begin{bmatrix}I_1 & \boldsymbol{0}\\ 0 & I_3\end{bmatrix} \;,\; \boldsymbol{G} \triangleq \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}(I_1 + I_3 - I_2)\nu \;,\; \boldsymbol{K} \triangleq \begin{bmatrix}I_2 - I_3 & 0\\ 0 & I_2 - I_1\end{bmatrix}\nu^2$$

where $\boldsymbol{M} = \boldsymbol{M}^{\mathsf{T}} > 0$, $\boldsymbol{G} = -\boldsymbol{G}^{\mathsf{T}}$, and $\boldsymbol{K} = \boldsymbol{K}^{\mathsf{T}}$, consistent with our definition of a conservative gyric system from STABILITY. From the stability theory associated with such systems, the results of which were summarized in STABILITY, the system in Eq. (13.11):

- is *statically stable* if $\boldsymbol{K} > 0$, satisfied if and only if $I_2 > I_1$, $I_2 > I_3$ (major axis spin).
- may be *gyrically stable* even if $\boldsymbol{K} \not> 0$, satisfied if and only if $I_2 < I_1$, $I_2 < I_3$ (minor axis spin).

Note: To see where the second result comes from, the characteristic equation, $\det\big(\boldsymbol{M}\mathrm{r}^2 + \boldsymbol{G}\mathrm{r} + \boldsymbol{K}\big) = 0$, can be studied, which results in:

$$b_0 r^4 + b_1 r^2 + b_2 = 0 \tag{13.12}$$

$$b_0 \triangleq I_1 I_3 \;,\; b_1 \triangleq (I_2^2 + 2I_1I_3 - I_1I_2 - I_3I_2)\nu^2 \;,\; b_2 \triangleq (I_2 - I_1)(I_2 - I_3)\nu^4$$

The necessary and sufficient conditions for the s^2 roots of the characteristic polynomial above to be on the negative real axis (indicating stability) are $b_0 > 0$ (already satisfied), $b_1 > 0$, $b_2 > 0$, and $b_1^2 - 4b_0b_2 > 0$. Further examination of these conditions implies only a major or a minor axis spin satisfies all conditions, the former of which is not possible for $\boldsymbol{K} \not> 0$.

Geometrical Interpretation

Recall, from TORQUE-FREE MOTION, that both rotational kinetic energy and angular momentum are constants of motion when there is no external torque. We obtain the "energy ellipsoid" and the "momentum ellipsoid", both shown in Figure 13.2, as follows:

$$T = \frac{1}{2}\boldsymbol{\omega}^{\mathsf{T}}\boldsymbol{I}\boldsymbol{\omega} = \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2) \;\Rightarrow\; \frac{\omega_1^2}{2T/I_1} + \frac{\omega_2^2}{2T/I_2} + \frac{\omega_3^2}{2T/I_3} = 1 \;\rightarrow\; \mathscr{T}\text{ ellipsoid} \tag{13.13a}$$
$$h^2 = |\boldsymbol{I}\boldsymbol{\omega}|^2 = I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2 \;\Rightarrow\; \frac{\omega_1^2}{h^2/I_1^2} + \frac{\omega_2^2}{h^2/I_2^2} + \frac{\omega_3^2}{h^2/I_3^2} = 1 \;\rightarrow\; \mathscr{H}\text{ ellipsoid} \tag{13.13b}$$

both of which are fixed to $\mathscr{F}_P$ and centred at its origin. The tip of $\underset{\rightarrow}{\omega}$ is constrained to lie on the intersection of the $\mathscr{T}$ and $\mathscr{H}$ ellipsoids, known as a *polhode* and illustrated in Figure 13.2. If, for a given moment of inertia matrix, T is held constant and h is varied (by appropriate changes in $\underset{\rightarrow}{\omega}$), the loci created by the ellipsoids'

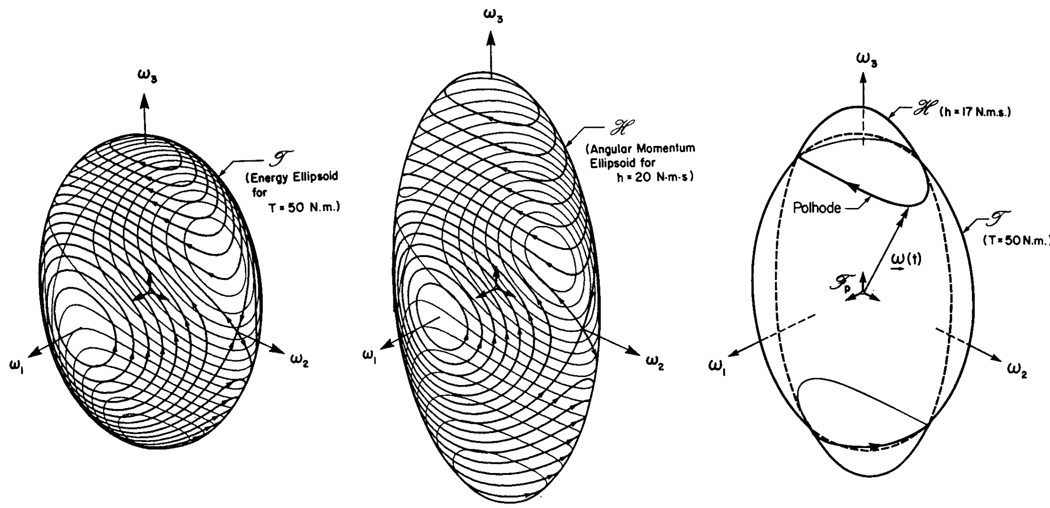

Figure 13.2: Energy and Momentum Ellipsoids, and Polhodes [Hughes] (used with permission)

intersection represent a family of polhodes. Stability of the major and minor axes, and instability of the intermediate 2-axis owing to the "separatrix" of the polhodes passing through it, can be visualized using such a family.

Another interpretation of torque-free motion can be obtained using the energy ellipsoid. We have:

$$T = \frac{1}{2}\boldsymbol{\omega}^{\mathsf{T}}\boldsymbol{I}\boldsymbol{\omega} \quad \Rightarrow \quad \nabla_{\boldsymbol{\omega}} T = \boldsymbol{I}\boldsymbol{\omega} = \boldsymbol{h} \tag{13.14}$$

where $\nabla_{\boldsymbol{\omega}}(\cdot)$ represents the gradient with respect to $\boldsymbol{\omega}$. The result in Eq. (13.14) implies that the normal of a plane tangent to the $\mathscr{T}$ ellipsoid at any point is parallel to $\underset{\rightarrow}{\boldsymbol{h}}$.

Since $2T = \boldsymbol{\omega}^{\mathsf{T}}\boldsymbol{h} = \underset{\rightarrow}{\boldsymbol{\omega}} \cdot \underset{\rightarrow}{\boldsymbol{h}}$, for constant T the projection of $\underset{\rightarrow}{\boldsymbol{\omega}}$ onto $\underset{\rightarrow}{\boldsymbol{h}}$ (which is fixed in $\mathscr{F}_I$ for torque-free motion) remains constant, suggesting that the tip of $\underset{\rightarrow}{\boldsymbol{\omega}}$ remains on an "invariable plane" normal to $\underset{\rightarrow}{\boldsymbol{h}}$. But we also know that the tip of $\underset{\rightarrow}{\boldsymbol{\omega}}$ follows a polhode on the $\mathscr{T}$ ellipsoid (which is fixed to $\mathscr{F}_P$). The overall conclusion is that the invariable plane is tangent to the $\mathscr{T}$ ellipsoid, as shown in Figure 13.3, and the torque-free motion of a rigid body is described by the $\mathscr{T}$ ellipsoid's rolling without slipping on an invariable plane normal to $\underset{\rightarrow}{\boldsymbol{h}}$. This is known as "Poinsot's Geometrical Construction", and the trace of $\underset{\rightarrow}{\boldsymbol{\omega}}$'s tip on the invariable plane is known as a *herpolhode*.

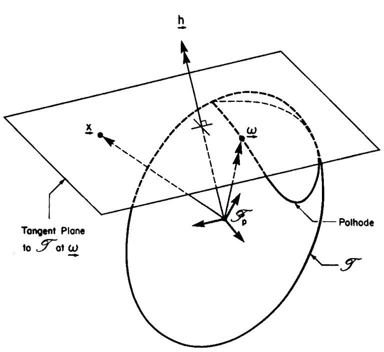

Figure 13.3: Poinsot's Construction [Hughes] (used with permission)

Internal Energy Dissipation

A "quasi-rigid body", one that is not too flexible to require additional degrees of freedom to describe its deformations, undergoes structural deformations that result in loss of kinetic energy, for example in the form of heat. We accept the "energy sink hypothesis", which states that during quasi-rigid body rotation, kinetic energy is *slowly* dissipated until a state of minimum kinetic energy is reached, and recall that $h = |\underset{\rightarrow}{\boldsymbol{h}}|$ is assumed to be constant as a consequence of zero external torques.

Assuming, without loss of generality, that $I_1 > I_2 > I_3$, and letting $\dot{T} < 0$ as a consequence of quasi-rigidity, we have:

- pure major axis spin: $T = \frac{1}{2} I_1 \nu_1^2 = \frac{h^2}{2I_1}$
- pure minor axis spin: $T = \frac{1}{2} I_3 \nu_3^2 = \frac{h^2}{2I_3}$

so for constant h, minimum $T = h^2/(2I_1)$ is achieved by pure major axis spin. Therefore, regardless of the initial conditions, all torque-free rotational motions of a quasi-rigid body will eventually reach a pure spin about the body's major axis. This is known as the "major axis rule", and can be visualized as in Figure 13.4 by tracking the spiralling polhode resulting from the intersection of the shrinking $\mathscr{T}$ ellipsoid and the constant $\mathscr{H}$ ellipsoid.

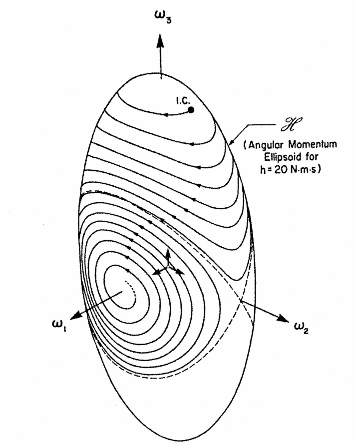

Figure 13.4: Energy Dissipation [Hughes] (used with permission)

References

[Hughes] Hughes, P. C., *Spacecraft Attitude Dynamics*, Dover Publications Inc., New York, Chaps. 4-5.