Lecture 19

Bias-Momentum Stabilization

DYNAMICS of spacecraft under gravity gradient with actively-controlled wheels are considered. Linearized pitch and roll/yaw equations of motion are used in conjunction with modified P- and PD-type feedback control laws, and the system's resulting stability properties and steady-state behaviour are studied.

Overview

Bias-momentum-stabilized spacecraft are similar to gyrostats considered in DUAL-SPIN STABILIZATION, but instead of large external rotors, they have relatively small rapidly-spinning internal wheels that provide gyricity beneficial to attitude stabilization. They also tend to rely more heavily on active control than gyrostats typically do, and bias-momentum stabilization can generally be considered as an amalgam of passive dual-spin stabilization and active attitude control in the presence of gravity gradient.

The following advantages motivate bias-momentum stabilization:

- providing short term stability against disturbances, similarly to spin stabilization
- increasing roll/yaw coupling, hence reducing the need for yaw sensing, which is difficult because the body-fixed frame measurements of the spacecraft's position vector relative to Earth do not depend on yaw
- enhancing gravity gradient stabilization as a consequence of having a wheel nominally aligned with the orbiting frame's 2-axis (pitch)

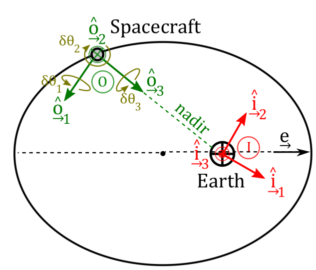

Figure 19.1: Orbiting (O) and Inertial (I) Frames

Note: As mentioned before, a bias-momentum wheel that remains aligned with the pitch axis and only changes rate is known as a "reaction wheel", while one that changes direction from roll/yaw to pitch is a "control moment gyro" (CMG).

Analogously to GRAVITY GRADIENT STABILIZATION, The following reference frames, shown in Figure 19.1, are used for the purpose of this study:

- $\mathscr{F}_I$: inertial frame fixed to (but not rotating with) Earth

- $\mathscr{F}_O$: orbiting frame, with origin fixed to spacecraft, 3-axis towards Earth's centre, 2-axis anti-parallel to orbital angular momentum, $\underrightarrow{\boldsymbol{h}}$

- $\mathscr{F}_B$: body-fixed frame, with origin at spacecraft centre of mass

A circular orbit is assumed, with a mean motion of $\omega_0 = \sqrt{\mu/r_\oplus^3}$. In addition, the wheel's spin axis is taken to be along the negative pitch axis, $\hat{\boldsymbol{a}} = -\hat{\boldsymbol{o}}_2$, and its angular momentum is given by $\boldsymbol{h}_s = I_s\omega_s\hat{\boldsymbol{a}}$. The wheel is assumed to be spinning rapidly enough to justify taking $h_s \gg I_i\omega_0$, $i \in \{1, 2, 3\}$.

Based on FUNDAMENTALS, the spacecraft's attitude with respect to the nominal orbiting frame can be described using a 3-2-1 rotation matrix:

$$\boldsymbol{C}_{BO} = \boldsymbol{C}_1(\delta\theta_1)\boldsymbol{C}_2(\delta\theta_2)\boldsymbol{C}_3(\delta\theta_3) \quad \Rightarrow \quad \boldsymbol{C}_{BO} \approx \mathbf{1} - \delta\boldsymbol{\theta}^\times \approx \begin{bmatrix} 1 & \delta\theta_3 & -\delta\theta_2 \\ -\delta\theta_3 & 1 & \delta\theta_1 \\ \delta\theta_2 & -\delta\theta_1 & 1 \end{bmatrix} \;,\; \delta\boldsymbol{\theta} \triangleq \begin{bmatrix} \delta\theta_1 \\ \delta\theta_2 \\ \delta\theta_3 \end{bmatrix} \tag{19.1}$$

where $\delta\theta_1$, $\delta\theta_2$, and $\delta\theta_3$ are the infinitesimal roll, pitch, and yaw angles, respectively. Small Euler angle approximations are used in Eq. (19.1).

Equations of Motion

We recall the following results (all expressed in $\mathscr{F}_B$) from GRAVITY GRADIENT STABILIZATION, with $\delta\boldsymbol{\theta}$ representing the small roll, pitch, and yaw angles:

$$\boldsymbol{\omega}^{BI} = \cancelto{\approx \delta\dot{\boldsymbol{\theta}}}{\boldsymbol{\omega}^{BO}} + \boldsymbol{C}_{BO}\begin{bmatrix} 0 \\ -\omega_0 \\ 0 \end{bmatrix} = \begin{bmatrix} \delta\dot{\theta}_1 - \omega_0\delta\theta_3 \\ \delta\dot{\theta}_2 - \omega_0 \\ \delta\dot{\theta}_3 + \omega_0\delta\theta_1 \end{bmatrix} \;,\; \boldsymbol{\tau}_{gg} = 3\omega_0^2 \begin{bmatrix} (I_3 - I_2)\delta\theta_1 \\ (I_3 - I_1)\delta\theta_2 \\ (I_1 - I_2)\cancelto{\approx 0}{\delta\theta_1\delta\theta_2} \end{bmatrix} \tag{19.2}$$

Euler's equations of motion in the presence of gravity gradient, control, and disturbance torques (namely $\boldsymbol{\tau}_{gg}$, $\boldsymbol{\tau}_c$, and $\boldsymbol{\tau}_d$, respectively) are given by:

$$\dot{\boldsymbol{h}} + \boldsymbol{\omega}^\times\boldsymbol{h} = \boldsymbol{\tau}_{gg} + \boldsymbol{\tau}_c + \boldsymbol{\tau}_d \;\;;\;\; \boldsymbol{h} = \boldsymbol{I}\boldsymbol{\omega}^{BI} + \boldsymbol{h}_s \;,\; \boldsymbol{h}_s = \begin{bmatrix} 0 \\ -I_s\omega_s \\ 0 \end{bmatrix} \tag{19.3}$$

where, for now, no active torque about the pitch axis is assumed; that is, $\tau_{c_2} = 0$.

Expanding and rearranging Eq. (19.3) yields the equations of motion:

$$I_1\delta\ddot{\theta}_1 - \big[(I_1 - I_2 + I_3)\omega_0 - h_s\big]\delta\dot{\theta}_3 + \big[4\omega_0^2(I_2 - I_3) + h_s\omega_0\big]\delta\theta_1 = \tau_{c_1} + \tau_{d_1} \tag{19.4a}$$

$$I_2\delta\ddot{\theta}_2 + 3\omega_0^2(I_1 - I_3)\delta\theta_2 = \dot{h}_s + \tau_{d_2} \tag{19.4b}$$

$$I_3\delta\ddot{\theta}_3 + \big[(I_1 - I_2 + I_3)\omega_0 - h_s\big]\delta\dot{\theta}_1 + \big[\omega_0^2(I_2 - I_1) + h_s\omega_0\big]\delta\theta_3 = \tau_{c_3} + \tau_{d_3} \tag{19.4c}$$

which resemble the equations of motion involved in gravity gradient stabilization, but with the h_s effects. The $\dot{h}_s$ term behaves as pitch control provided by gyric effects of the wheel spinning in the pitch direction.

Assuming a rapidly-spinning wheel, we let $h_s = I_s\omega_s \gg I_i\omega_0$ for $i \in \{1, 2, 3\}$, which simplifies the equations of motion in Eq. (19.4) for the h_s term inside the brackets dominate:

$$I_1\delta\ddot{\theta}_1 + h_s\delta\dot{\theta}_3 + h_s\omega_0\delta\theta_1 = \tau_{c_1} + \tau_{d_1} \tag{19.5a}$$

$$I_2\delta\ddot{\theta}_2 + 3\omega_0^2(I_1 - I_3)\delta\theta_2 = \dot{h}_s + \tau_{d_2} \tag{19.5b}$$

$$I_3\delta\ddot{\theta}_3 - h_s\delta\dot{\theta}_1 + h_s\omega_0\delta\theta_3 = \tau_{c_3} + \tau_{d_3} \tag{19.5c}$$

which has its pitch equation uncoupled, and its roll/yaw equations coupled together. Similarly to GRAVITY GRADIENT STABILIZATION, we now treat the control about pitch and roll/yaw axes separately.

Pitch Control

Let us treat $\dot{h}_s$ as pitch control torque, $\tau_{c_2} \triangleq \dot{h}_s$, which can be modified by the wheel's rotation. For simplicity of notation, we replace $\delta\theta_2$ with θ_2, and considering a stabilization problem in the presence of disturbances, we let $\theta_{2_{ref}} = 0$ and $\theta_2(0) = \dot{\theta}_2(0) = 0$. Taking the Laplace transform of the motion equation in Eq. (19.5b) results in:

$$(I_2 s^2 + C)\Theta_2 = \mathbb{T}_{c_2} + \mathbb{T}_{d_2} \;\;,\;\; C \triangleq 3\omega_0^2(I_1 - I_3) \tag{19.6}$$

Consider the following modified PD control law and its Laplace transform:

$$\tau_{c_2} = K_p(\cancelto{0}{\theta_{2_{ref}}} - \theta_2) + K_d(\cancelto{0}{\dot{\theta}_{2_{ref}}} - \dot{\theta}_2) + 3\omega_0^2(I_1 - I_3)\theta_2 \;\; \overset{\mathscr{L}}{\Longrightarrow} \;\; \mathbb{T}_{c_2} = -\big[(K_p - C)\Theta_2 + K_d s\Theta_2\big] \tag{19.7}$$

substituting which into Eq. (19.6) and rearranging yields the following input/output relationship mediated by the system's close-loop transfer function:

$$\Theta_2 = \frac{1}{I_2 s^2 + \cancel{C} + (K_p - \cancel{C}) + K_d s}\mathbb{T}_{d_2} = \frac{1/I_2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}\mathbb{T}_d \;\;,\;\; \zeta \triangleq \frac{K_d}{2}\sqrt{\frac{1}{K_p I_2}} \;,\; \omega_0 \triangleq \sqrt{\frac{K_p}{I_2}} \tag{19.8}$$

where ζ and ω_0 are the damping ratio and undamped natural frequency previously encountered in ACTIVE ATTITUDE CONTROL.

Steady-State Performance

Let there be a constant disturbance torque of magnitude D_2 (step input), $\boldsymbol{\tau}_{d_2} = D_2 H(t)$, with Laplace transform $\mathbb{T}_{d_2} = D_2/s$. The steady-state value of the pitch output subject to the modified PD law in Eq. (19.7) is found by:

$$\theta_{ss} = \lim_{t\to\infty}\theta_2(t) = \lim_{s\to 0} s\theta_2(s) = \lim_{s\to 0}\cancel{s}\frac{1}{I_2 s^2 + K_d s + K_p}\cdot\frac{D_2}{\cancel{s}} = \frac{D_2}{K_p} \tag{19.9}$$

where the final value theorem of Laplace transform, mentioned in ACTIVE ATTITUDE CONTROL, is used. This result implies that the steady-state error in pitch can be reduced by increasing the control gain (which would require more fuel or power), and gain selection can be performed keeping the maximum acceptable θ_{ss} in mind. In order to find the wheel's angular speed required to achieve the desired control torque, $\tau_{c_2}(t) = \dot{h}_s(t) = I_s\dot{\omega}_s$, we have:

$$I_s\big(\omega_s(t) - \omega_{s_0}\big) = \int_0^t \dot{h}_s\,dt \;\;\Rightarrow\;\; \omega_s = \omega_{s_0} + \frac{\int_0^t \tau_{c_2}\,dt}{I_s} \tag{19.10}$$

Roll/Yaw Control

For simplicity of notation, we replace $\delta\theta_1$ and $\delta\theta_3$ with θ_1 and θ_3, respectively, and considering a stabilization problem in the presence of disturbances, we let $\theta_{1_{ref}} = \theta_{3_{ref}} = 0$, $\theta_1(0) = \dot{\theta}_1(0) = 0$, and $\theta_3(0) = \dot{\theta}_3(0) = 0$. Taking the Laplace transform of the motion equations in Eqs. (19.5a) and (19.5c) results in:

$$I_1 s^2 \Theta_1 + h_s s \Theta_3 + h_s \omega_0 \Theta_1 = \tau_{c_1} + \tau_{d_1} \tag{19.11a}$$

$$I_3 s^2 \Theta_3 - h_s s \Theta_1 + h_s \omega_0 \Theta_3 = \tau_{c_3} + \tau_{d_3} \tag{19.11b}$$

Consider the following modified P control laws and their Laplace transforms:

$$\tau_{c_1} = K_p(\cancelto{0}{\theta_{1_{ref}}} - \theta_1) + h_s\omega_0\theta_1 \quad \overset{\mathscr{L}}{\Longrightarrow} \quad \tau_{c_1} = -K_p\Theta_1 + h_s\omega_0\Theta_1 \tag{19.12a}$$

$$\tau_{c_3} = -K_r K_p(\cancelto{0}{\theta_{1_{ref}}} - \theta_1) - h_s\dot{\theta}_1 \quad \overset{\mathscr{L}}{\Longrightarrow} \quad \tau_{c_3} = K_r K_p\Theta_1 - h_s s\Theta_1 \tag{19.12b}$$

which benefit from using only θ_1 and $\dot{\theta}_1$ measurements of roll angle and rate, hence circumventing the difficulties associated with yaw measurement.

To see one motivation behind such a selection of control laws, consider the closed-loop roll/yaw dynamics, and assume $h_s\dot{\theta}_1 \gg I_3\ddot{\theta}_3$ and $h_s\dot{\theta}_3 \gg I_1\ddot{\theta}_1$ (which are reasonable for a rapidly-spinning wheel). Assume, also, no disturbances:

$$h_s\dot{\theta}_3 + \cancel{h_s\omega_0\theta_1} \approx -K_p\theta_1 + \cancel{h_s\omega_0\theta_1} \quad \Rightarrow \quad h_s\dot{\theta}_3 \approx -K_p\frac{h_s\omega_0}{K_r K_p}\theta_3 \quad \Rightarrow \quad \theta_3 \approx \theta_3(0)e^{-\omega_0 t/K_r} \tag{19.13a}$$

$$\cancel{-h_s\dot{\theta}_1} + h_s\omega_0\theta_3 \approx K_r K_p\theta_1 - \cancel{h_s\dot{\theta}_1} \quad \Rightarrow \quad \theta_1 \approx \theta_3(0)\frac{h_s\omega_0}{K_r K_p}e^{-\omega_0 t/K_r} \tag{19.13b}$$

where Eq. (19.13b) is substituted into Eq. (19.13a), and the final result of Eq. (19.13a) is substituted back into Eq. (19.13b). Both roll and yaw angles asymptotically reach 0, suggesting *asymptotic stability*.

Returning to the original roll/yaw equations of motion in Eq. (19.11) using the control inputs in Eq. (19.12), and putting the closed-loop system in matrix form yields:

$$\begin{bmatrix} I_1 s^2 + K_p & h_s s \\ -K_r K_p & I_3 s^2 + h_s\omega_0 \end{bmatrix} \begin{bmatrix} \Theta_1 \\ \Theta_3 \end{bmatrix} = \begin{bmatrix} \tau_{d_1} \\ \tau_{d_3} \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} \Theta_1 \\ \Theta_3 \end{bmatrix} = \mathbb{G}\begin{bmatrix} \tau_{d_1} \\ \tau_{d_3} \end{bmatrix} \tag{19.14}$$

where $\mathbb{G}$ is the system's closed-loop transfer matrix. The characteristic equation is, thus, given by:

$$\det(\mathbb{G}^{-1}) = I_1 I_3 s^4 + (K_p I_3 + h_s\omega_0 I_1)s^2 + K_r K_p h_s s + K_p h_s \omega_0 = 0 \tag{19.15}$$

According to Routh-Hurwitz stability criterion, the system is *not asymptotically stable* owing to the presence of a zero coefficient (of s^3). We can use the following modified PD law instead, with derivative terms added:

$$\tau_{c_1} = K_p(\cancelto{0}{\theta_{1_{ref}}} - \theta_1) + K_d(\cancelto{0}{\dot{\theta}_{1_{ref}}} - \dot{\theta}_1) + h_s\omega_0\theta_1 \quad \overset{\mathscr{L}}{\Longrightarrow} \quad \tau_{c_1} = -(K_p + K_d s)\Theta_1 + h_s\omega_0\Theta_1 \tag{19.16a}$$

$$\tau_{c_3} = -K_r K_p(\cancelto{0}{\theta_{1_{ref}}} - \theta_1) - K_r K_d(\cancelto{0}{\dot{\theta}_{1_{ref}}} - \dot{\theta}_1) - h_s\dot{\theta}_1 \quad \overset{\mathscr{L}}{\Longrightarrow} \quad \tau_{c_3} = K_r(K_p + K_d s)\Theta_1 - h_s s\Theta_1 \tag{19.16b}$$

which result in the following closed-loop dynamics when substituted into Eq. (19.11):

$$\begin{bmatrix} I_1 s^2 + K_d s + K_p & h_s s \\ -K_r(K_d s + K_p) & I_3 s^2 + h_s\omega_0 \end{bmatrix} \begin{bmatrix} \Theta_1 \\ \Theta_3 \end{bmatrix} = \begin{bmatrix} \mathbb{T}_{d_1} \\ \mathbb{T}_{d_3} \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} \Theta_1 \\ \Theta_3 \end{bmatrix} = \mathbb{H} \begin{bmatrix} \mathbb{T}_{d_1} \\ \mathbb{T}_{d_3} \end{bmatrix} \tag{19.17}$$

where $\mathbb{H}$ is the system's closed-loop transfer matrix.

Steady-State Performance

Let there be constant disturbance torques of magnitude D_i (step input), $\boldsymbol{\tau}_{d_1} = D_1 H(t)$ and $\boldsymbol{\tau}_{d_3} = D_3 H(t)$, with Laplace transforms $\mathbb{T}_{d_1} = D_1/s$ and $\mathbb{T}_{d_3} = D_3/s$. The steady-state values of the roll and yaw outputs subject to the modified PD law in Eq. (19.16) are found by:

$$\boldsymbol{\theta}_{ss} = \lim_{t\to\infty} \boldsymbol{\theta}(t) = \lim_{s\to 0} s\Theta(s) = \lim_{s\to 0} \cancel{s}\mathbb{H} \begin{bmatrix} \frac{D_1}{\cancel{s}} \\ \frac{D_3}{\cancel{s}} \end{bmatrix} \tag{19.18}$$

where $\boldsymbol{\theta}(t) \triangleq [\theta_1(t)\ \theta_3(t)]^\intercal$, and $\Theta(s) \triangleq [\Theta_1(s)\ \Theta_3(s)]^\intercal$, and the final value theorem of Laplace transform is used again. We thus have:

$$\boldsymbol{\theta}_{ss} = \lim_{s\to 0} \frac{\begin{bmatrix} I_3 s^2 + h_s\omega_0 & -h_s s \\ K_r(K_d s + K_p) & I_1 s^2 + K_d s + K_p \end{bmatrix}}{(I_1 s^2 + K_d s + K_p)(I_3 s^2 + \omega_0 h_s) + h_s s(K_r K_d s + K_r K_p)} \begin{bmatrix} D_1 \\ D_3 \end{bmatrix} \tag{19.19}$$

which simplifies as follows:

$$\boldsymbol{\theta}_{ss} = \frac{1}{K_p \omega_0 h_s} \begin{bmatrix} \omega_0 h_s & 0 \\ K_r K_p & K_P \end{bmatrix} \begin{bmatrix} D_1 \\ D_3 \end{bmatrix} = \begin{bmatrix} \frac{D_1}{K_p} \\ \frac{K_r D_1}{\omega_0 h_s} + \frac{D_3}{\omega_0 h_s} \end{bmatrix} \tag{19.20}$$

As expected based on our choice of control laws in Eq. (19.16) (using θ_1 and $\dot{\theta}_1$ only), a step disturbance torque about the yaw axis does not affect roll, while that about the roll axis does influence yaw.