Lecture 2

Kinematics

This lesson focuses on how motion and time variations of vectors can be described geometrically, disregarding any external forces and physical laws of nature. Angular velocity is defined and used to relate derivatives of vectors as measured in different frames that are rotating with respect to each other. The relationship between angular velocity and various attitude parameterizations is also studied.

Angular Velocity

Consider two reference frames, $\mathscr{F}_A$ and $\mathscr{F}_B$, rotating with respect to each other as shown in Figure 2.1a. Any generic vector, $\underrightarrow{\boldsymbol{v}}$, could be fixed in either $\mathscr{F}_A$ or $\mathscr{F}_B$, or it could change in both frames. In each of these cases, how can we describe the change over time, as seen in either $\mathscr{F}_A$ or $\mathscr{F}_B$?

Define the time derivative operators $(\cdot)^\bullet$ and $(\cdot)^\circ$, corresponding to the time derivatives as measured in $\mathscr{F}_A$ and $\mathscr{F}_B$, respectively, to act on the vector $\underrightarrow{\boldsymbol{v}}$ as follows:

$$\underrightarrow{\boldsymbol{v}}^\bullet \triangleq \frac{d}{dt}\underrightarrow{\boldsymbol{v}}\Big|_{\mathscr{F}_A} \quad , \quad \underrightarrow{\boldsymbol{v}}^\circ \triangleq \frac{d}{dt}\underrightarrow{\boldsymbol{v}}\Big|_{\mathscr{F}_B}$$

Since all basis vectors defining the coordinate axes are fixed in their own reference frames, we have:

$$\underrightarrow{\mathscr{F}}_A^\bullet = \frac{d}{dt}\begin{bmatrix} \hat{\underrightarrow{\boldsymbol{a}}}_1 \\ \hat{\underrightarrow{\boldsymbol{a}}}_2 \\ \hat{\underrightarrow{\boldsymbol{a}}}_3 \end{bmatrix}\Bigg|_{\mathscr{F}_A} = \underrightarrow{\boldsymbol{0}} \quad , \quad \underrightarrow{\mathscr{F}}_B^\circ = \frac{d}{dt}\begin{bmatrix} \hat{\underrightarrow{\boldsymbol{b}}}_1 \\ \hat{\underrightarrow{\boldsymbol{b}}}_2 \\ \hat{\underrightarrow{\boldsymbol{b}}}_3 \end{bmatrix}\Bigg|_{\mathscr{F}_B} = \underrightarrow{\boldsymbol{0}} \tag{2.1}$$

(a) A Vector as Seen in Two Rotating Frames

(b) Differential Change of a basis vector of $\mathscr{F}_B$

Figure 2.1: Relating Angular Velocity to Differential Changes of a Vector in Rotating Frames

We now study how one of these basis vectors, $\underrightarrow{\hat{\boldsymbol{b}}}_1$ for example, which is fixed in length, changes as seen in $\mathscr{F}_A$: Since $\mathscr{F}_B$ is rotating with respect to $\mathscr{F}_A$, assume the rotation can be described with a constant rate of $\dot{\phi}$ about the rotation axis, $\underrightarrow{\hat{\boldsymbol{a}}}$ (recall Euler axis/angle variables from FUNDAMENTALS). Define $\underrightarrow{\boldsymbol{\omega}} \triangleq \dot{\phi}\underrightarrow{\hat{\boldsymbol{a}}}$.

As illustrated in Figure 2.1b, we have, for the norm of a differential change in $\underrightarrow{\hat{\boldsymbol{b}}}_1$:

$$|d\underrightarrow{\hat{\boldsymbol{b}}}_1| = |\underrightarrow{\hat{\boldsymbol{b}}}_{1\perp}|\ d\phi = |\underrightarrow{\hat{\boldsymbol{b}}}_1| \sin\theta\ d\phi \tag{2.2}$$

where $\underrightarrow{\hat{\boldsymbol{b}}}_{1\perp}$ denotes the component of $\underrightarrow{\hat{\boldsymbol{b}}}_1$ normal to $\underrightarrow{\hat{\boldsymbol{a}}}$, and θ represents the angle between $\underrightarrow{\hat{\boldsymbol{b}}}_1$ and $\underrightarrow{\hat{\boldsymbol{a}}}$. From the definition of $\underrightarrow{\boldsymbol{\omega}}$ above and noting the unit magnitude of $\underrightarrow{\hat{\boldsymbol{a}}}$, we have $d\phi = |\underrightarrow{\boldsymbol{\omega}}|dt$, substituting which into Eq. (2.2) yields:

$$|d\underrightarrow{\hat{\boldsymbol{b}}}_1| = |\underrightarrow{\hat{\boldsymbol{b}}}_1|\ |\underrightarrow{\boldsymbol{\omega}}| \sin\theta\ dt = |\underrightarrow{\boldsymbol{\omega}} \times \underrightarrow{\hat{\boldsymbol{b}}}_1|\ dt \tag{2.3}$$

Lastly, since $d\underrightarrow{\hat{\boldsymbol{b}}}_1$ is normal to both $\underrightarrow{\hat{\boldsymbol{b}}}_1$ and $\underrightarrow{\boldsymbol{\omega}}$, it is parallel to their cross product, and from Eq. (2.3) we obtain:

$$\frac{d\underrightarrow{\hat{\boldsymbol{b}}}_1}{dt} = \underrightarrow{\hat{\boldsymbol{b}}}_1^{\bullet} = \underrightarrow{\boldsymbol{\omega}} \times \underrightarrow{\hat{\boldsymbol{b}}}_1 \tag{2.4}$$

By repeating the same procedure for $\underrightarrow{\hat{\boldsymbol{b}}}_2$ and $\underrightarrow{\hat{\boldsymbol{b}}}_3$, we conclude:

$$\underrightarrow{\mathscr{F}}_B^{\bullet} = \begin{bmatrix} \underrightarrow{\hat{\boldsymbol{b}}}_1^{\bullet} \\ \underrightarrow{\hat{\boldsymbol{b}}}_2^{\bullet} \\ \underrightarrow{\hat{\boldsymbol{b}}}_3^{\bullet} \end{bmatrix} = \begin{bmatrix} \underrightarrow{\boldsymbol{\omega}} \times \underrightarrow{\hat{\boldsymbol{b}}}_1 \\ \underrightarrow{\boldsymbol{\omega}} \times \underrightarrow{\hat{\boldsymbol{b}}}_2 \\ \underrightarrow{\boldsymbol{\omega}} \times \underrightarrow{\hat{\boldsymbol{b}}}_3 \end{bmatrix} \Rightarrow \underrightarrow{\mathscr{F}}_B{}^{\bullet} = \underrightarrow{\boldsymbol{\omega}} \times \underrightarrow{\mathscr{F}}_B \Rightarrow \underrightarrow{\mathscr{F}}_B{}^{\mathsf{T}^{\bullet}} = \underrightarrow{\boldsymbol{\omega}} \times \underrightarrow{\mathscr{F}}_B^{\mathsf{T}} \tag{2.5}$$

which is the rate of change of $\underrightarrow{\mathscr{F}}_B$, as observed in $\mathscr{F}_A$. Expressing $\underrightarrow{\boldsymbol{\omega}}$ in $\mathscr{F}_B$, $\underrightarrow{\boldsymbol{\omega}} = \boldsymbol{\omega}_B^{\mathsf{T}}\underrightarrow{\mathscr{F}}_B$, and using $\boldsymbol{\omega}_B^{\mathsf{T}}\underrightarrow{\mathscr{F}}_B \times \underrightarrow{\mathscr{F}}_B^{\mathsf{T}} = \underrightarrow{\mathscr{F}}_B^{\mathsf{T}}\boldsymbol{\omega}_B^{\times}$ from the definition of cross product from FUNDAMENTALS, Eq. (2.5) provides a relationship for angular velocity, as expressed in $\mathscr{F}_B$, in terms of the vectrix $\underrightarrow{\mathscr{F}}_B$ and its derivative:

$$\underrightarrow{\mathscr{F}}_B^{\mathsf{T}^{\bullet}} = \underrightarrow{\boldsymbol{\omega}} \times \underrightarrow{\mathscr{F}}_B^{\mathsf{T}} = \underrightarrow{\mathscr{F}}_B^{\mathsf{T}}\boldsymbol{\omega}_B^{\times} \tag{2.6}$$

Angular Velocity and Rotation Matrix

Recall, from the definition of a rotation matrix and its symmetry in FUNDAMENTALS, that:

$$\underrightarrow{\mathscr{F}}_A = \boldsymbol{C}_{AB}\underrightarrow{\mathscr{F}}_B \Rightarrow \underrightarrow{\mathscr{F}}_A^{\mathsf{T}} = \underrightarrow{\mathscr{F}}_B^{\mathsf{T}}\boldsymbol{C}_{BA} \tag{2.7}$$

differentiating both sides of which with respect to time, as measured in $\mathscr{F}_A$, results in (upon evoking 2.1):

$$\cancelto{\underrightarrow{\boldsymbol{0}}}{\underrightarrow{\mathscr{F}}_A^{\mathsf{T}^{\bullet}}} = \underrightarrow{\mathscr{F}}_B^{\mathsf{T}^{\bullet}}\boldsymbol{C}_{BA} + \underrightarrow{\mathscr{F}}_B^{\mathsf{T}}\overset{\bullet}{\boldsymbol{C}}_{BA} = \underrightarrow{\mathscr{F}}_B^{\mathsf{T}}\boldsymbol{\omega}_B^{\times}\boldsymbol{C}_{BA} + \underrightarrow{\mathscr{F}}_B^{\mathsf{T}}\overset{\bullet}{\boldsymbol{C}}_{BA} \tag{2.8}$$

where Eq. (2.6) is used and $\overset{\bullet}{(\cdot)}$ is the time derivative of a scalar-valued matrix (which is the same in any frame). We thus obtain "Poisson's kinematical equation":

$$\overset{\bullet}{\boldsymbol{C}}_{BA} + \boldsymbol{\omega}_B^{\times}\boldsymbol{C}_{BA} = \boldsymbol{0} \quad \text{or} \quad \boldsymbol{\omega}_{\mathrm{B}}^{\times} = -\overset{\bullet}{\boldsymbol{C}}_{\mathrm{BA}}\boldsymbol{C}_{\mathrm{AB}} = \boldsymbol{C}_{\mathrm{BA}}\overset{\bullet}{\boldsymbol{C}}{}_{\mathrm{BA}}^{\mathsf{T}} \tag{2.9}$$

which can be numerically integrated to find $\boldsymbol{C}(t)$ upon measuring $\boldsymbol{\omega}(t)$, or to determine $\boldsymbol{\omega}(t)$ given $\boldsymbol{C}(t)$.

Note: Recall, from FUNDAMENTALS, that $\boldsymbol{C} \approx \mathbf{1} - \boldsymbol{\theta}^\times \approx \mathbf{1} - \phi\hat{\boldsymbol{a}}^\times$ for infinitesimal angles. Using this simplification and also assuming $\boldsymbol{\theta}^\times\dot{\boldsymbol{\theta}}^\times \approx \mathbf{0}$, Eq. (2.9) reduces to $\boldsymbol{\omega}_B \approx \dot{\boldsymbol{\theta}}$, which can be integrated directly to obtain $\boldsymbol{\omega}(t)$. However, this does not hold for general rotations.

Additivity of Angular Velocity

We now consider three reference frames, namely $\mathscr{F}_A$, $\mathscr{F}_B$, and $\mathscr{F}_C$, and would like to show that $\underrightarrow{\boldsymbol{\omega}}^{BA} + \underrightarrow{\boldsymbol{\omega}}^{CB} = \underrightarrow{\boldsymbol{\omega}}^{CA}$, where $\underrightarrow{\boldsymbol{\omega}}^{BA}$ denotes angular velocity of $\mathscr{F}_B$ with respect to $\mathscr{F}_A$, and so on. Each of these vectors can be represented in their respective frame:

$$\underrightarrow{\boldsymbol{\omega}}^{CA} = \underrightarrow{\mathscr{F}}_C^{\mathsf{T}}\boldsymbol{\omega}_C^{CA} \quad , \quad \underrightarrow{\boldsymbol{\omega}}^{CB} = \underrightarrow{\mathscr{F}}_C^{\mathsf{T}}\boldsymbol{\omega}_C^{CB} \quad , \quad \underrightarrow{\boldsymbol{\omega}}^{BA} = \underrightarrow{\mathscr{F}}_B^{\mathsf{T}}\boldsymbol{\omega}_B^{BA} \tag{2.10}$$

From Eq. (2.9), we have:

$$\left(\boldsymbol{\omega}_C^{CA}\right)^\times = \boldsymbol{C}_{CA}\dot{\boldsymbol{C}}_{AC} = \left(\boldsymbol{C}_{CB}\boldsymbol{C}_{BA}\right)\left(\dot{\boldsymbol{C}}_{AB}\boldsymbol{C}_{BC} + \boldsymbol{C}_{AB}\dot{\boldsymbol{C}}_{BC}\right) \tag{2.11}$$

where compound rotation relationships from FUNDAMENTALS are used. Expanding Eq. (2.11) yields:

$$\left(\boldsymbol{\omega}_C^{CA}\right)^\times = \boldsymbol{C}_{CB}\cancelto{\left(\boldsymbol{\omega}_B^{BA}\right)^\times}{\boldsymbol{C}_{BA}\dot{\boldsymbol{C}}_{AB}}\boldsymbol{C}_{BC} + \boldsymbol{C}_{CB}\cancelto{\mathbf{1}}{\boldsymbol{C}_{BA}\boldsymbol{C}_{AB}}\dot{\boldsymbol{C}}_{BC} = \boldsymbol{C}_{CB}\left(\boldsymbol{\omega}_B^{BA}\right)^\times\boldsymbol{C}_{BC} + \cancelto{\left(\boldsymbol{\omega}_C^{CB}\right)^\times}{\boldsymbol{C}_{CB}\dot{\boldsymbol{C}}_{BC}} \tag{2.12}$$

where Eq. (2.9) is used twice. But we have the following identity mentioned in FUNDAMENTALS:

$$\boldsymbol{C}_{CB}\left(\boldsymbol{\omega}_B^{BA}\right)^\times\boldsymbol{C}_{BC} = \left(\boldsymbol{C}_{CB}\boldsymbol{\omega}_B^{BA}\right)^\times = \left(\boldsymbol{\omega}_C^{BA}\right)^\times \tag{2.13}$$

substituting which into Eq. (2.12) produces:

$$\left(\boldsymbol{\omega}_C^{CA}\right)^\times = \left(\boldsymbol{\omega}_C^{BA}\right)^\times + \left(\boldsymbol{\omega}_C^{CB}\right)^\times \Rightarrow \boldsymbol{\omega}_C^{CA} = \boldsymbol{\omega}_C^{BA} + \boldsymbol{\omega}_C^{CB} \tag{2.14}$$

Lastly, pre-multiplying Eq. (2.14) by $\underrightarrow{\mathscr{F}}_C^{\mathsf{T}}$ or expressing each $\boldsymbol{\omega}$ in its own frame results in:

$$\underrightarrow{\boldsymbol{\omega}}^{CA} = \underrightarrow{\boldsymbol{\omega}}^{CB} + \underrightarrow{\boldsymbol{\omega}}^{BA} \tag{2.15a}$$

$$\boldsymbol{\omega}_C^{CA} = \boldsymbol{\omega}_C^{CB} + \boldsymbol{C}_{CB}\boldsymbol{\omega}_B^{BA} \tag{2.15b}$$

The desired additivity result is achieved in both vectorial and referential forms, expressed via Eqs. (2.15a) and (2.15b), respectively.

Vector Derivatives in Different Frames

Recall that when $\mathscr{F}_B$ has an angular velocity of $\underrightarrow{\boldsymbol{\omega}}$ with respect to $\mathscr{F}_A$, the time derivative of each of its basis vectors, $\underrightarrow{\hat{\boldsymbol{b}}}_i$, as measured in $\mathscr{F}_A$ is given by Eq. (2.4), and the derivative of the associated vectrix, also measured in $\mathscr{F}_A$, is provided by Eq. (2.5). We now consider a general vector, $\underrightarrow{\boldsymbol{r}}$, that is not necessarily fixed in $\mathscr{F}_B$ (or $\mathscr{F}_A$), unlike $\underrightarrow{\hat{\boldsymbol{b}}}_i$. The goal of this section is to relate the derivatives of this vector, as seen in $\mathscr{F}_B$ and $\mathscr{F}_A$.

First Derivative

We have $\underset{\rightarrow}{\boldsymbol{r}} = \underset{\rightarrow}{\mathscr{F}}_A^{\mathsf{T}} \boldsymbol{r}_A = \underset{\rightarrow}{\mathscr{F}}_B^{\mathsf{T}} \boldsymbol{r}_B$. Upon differentiating each expression with respect to time in its own frame, we obtain:

$$\underset{\rightarrow}{\boldsymbol{r}}^{\bullet} = \cancelto{\underset{\rightarrow}{\mathbf{0}}}{\overset{\bullet}{\underset{\rightarrow}{\mathscr{F}}}_A{}^{\mathsf{T}} \boldsymbol{r}_A} + \underset{\rightarrow}{\mathscr{F}}_A{}^{\mathsf{T}} \overset{\bullet}{\boldsymbol{r}}_A \quad , \quad \underset{\rightarrow}{\boldsymbol{r}}^{\circ} = \cancelto{\underset{\rightarrow}{\mathbf{0}}}{\overset{\circ}{\underset{\rightarrow}{\mathscr{F}}}_B{}^{\mathsf{T}} \boldsymbol{r}_B} + \underset{\rightarrow}{\mathscr{F}}_B{}^{\mathsf{T}} \overset{\circ}{\boldsymbol{r}}_B \tag{2.16}$$

where the operators $\overset{\bullet}{(\cdot)}$ and $\overset{\circ}{(\cdot)}$ represent (analogously to $(\cdot)^{\bullet}$ and $(\cdot)^{\circ}$ for vectors) the derivatives of column matrices, as observed in $\mathscr{F}_A$ and $\mathscr{F}_B$, respectively. But $\overset{\bullet}{\boldsymbol{r}}_A = \overset{\circ}{\boldsymbol{r}}_A$ and $\overset{\bullet}{\boldsymbol{r}}_B = \overset{\circ}{\boldsymbol{r}}_B$, because they have scalar components and the effects of reference frame have already been extracted via the vectrices $\underset{\rightarrow}{\mathscr{F}}_A$ and $\underset{\rightarrow}{\mathscr{F}}_B$. We can, thus, rewrite Eq. (2.16) as:

$$\underset{\rightarrow}{\boldsymbol{r}}^{\bullet} = \underset{\rightarrow}{\mathscr{F}}_A{}^{\mathsf{T}} \overset{\bullet}{\boldsymbol{r}}_A \quad , \quad \underset{\rightarrow}{\boldsymbol{r}}^{\circ} = \underset{\rightarrow}{\mathscr{F}}_B{}^{\mathsf{T}} \overset{\circ}{\boldsymbol{r}}_B = \underset{\rightarrow}{\mathscr{F}}_B{}^{\mathsf{T}} \overset{\bullet}{\boldsymbol{r}}_B \tag{2.17}$$

Starting from $\underset{\rightarrow}{\boldsymbol{r}} = \underset{\rightarrow}{\mathscr{F}}_B^{\mathsf{T}} \boldsymbol{r}_B$, we differentiate with respect to time once again, but now *as observed in* $\underset{\rightarrow}{\mathscr{F}}_A$:

$$\underset{\rightarrow}{\boldsymbol{r}}^{\bullet} = \cancelto{\underset{\rightarrow}{\boldsymbol{\omega}} \times \underset{\rightarrow}{\mathscr{F}}_B^{\mathsf{T}}}{\overset{\bullet}{\underset{\rightarrow}{\mathscr{F}}}_B{}^{\mathsf{T}}} \boldsymbol{r}_B + \cancelto{\underset{\rightarrow}{\boldsymbol{r}}^{\circ}}{\underset{\rightarrow}{\mathscr{F}}_B{}^{\mathsf{T}} \overset{\bullet}{\boldsymbol{r}}_B} = \underset{\rightarrow}{\boldsymbol{\omega}} \times \cancelto{\underset{\rightarrow}{\boldsymbol{r}}}{\underset{\rightarrow}{\mathscr{F}}_B^{\mathsf{T}} \boldsymbol{r}_B} + \underset{\rightarrow}{\boldsymbol{r}}^{\circ} \tag{2.18}$$

where Eq. (2.5) is used for $\overset{\bullet}{\underset{\rightarrow}{\mathscr{F}}}_B{}^{\mathsf{T}}$, and the right-hand side equality of Eq. (2.17) is used to replace the second term, also noting that $\underset{\rightarrow}{\mathscr{F}}_B^{\mathsf{T}} \boldsymbol{r}_B = \underset{\rightarrow}{\boldsymbol{r}}$ holds by definition.

To obtain a similar result to Eq. (2.18) in referential (as opposed to vectorial) form, we make use of the left-hand side equality of Eq. (2.17) and recall $\underset{\rightarrow}{\boldsymbol{\omega}} = \boldsymbol{\omega}_B^{\mathsf{T}} \underset{\rightarrow}{\mathscr{F}}_B$ to rewrite Eq. (2.18) as:

$$\underset{\rightarrow}{\mathscr{F}}_A{}^{\mathsf{T}} \overset{\bullet}{\boldsymbol{r}}_A = \boldsymbol{\omega}_B^{\mathsf{T}} \underset{\rightarrow}{\mathscr{F}}_B \times \underset{\rightarrow}{\mathscr{F}}_B^{\mathsf{T}} \boldsymbol{r}_B + \underset{\rightarrow}{\mathscr{F}}_B^{\mathsf{T}} \overset{\bullet}{\boldsymbol{r}}_B = \underset{\rightarrow}{\mathscr{F}}_B^{\mathsf{T}} \left(\overset{\bullet}{\boldsymbol{r}}_B + \boldsymbol{\omega}_B^{\times} \boldsymbol{r}_B \right) \tag{2.19}$$

where the definition of cross product from FUNDAMENTALS is used. Lastly, pre-multiplying (scalar) Eq. (2.19) by $\underset{\rightarrow}{\mathscr{F}}_A$ yields:

$$\cancelto{\mathbf{1}}{\underset{\rightarrow}{\mathscr{F}}_A \cdot \underset{\rightarrow}{\mathscr{F}}_A{}^{\mathsf{T}}} \overset{\bullet}{\boldsymbol{r}}_A = \cancelto{\boldsymbol{C}_{AB}}{\underset{\rightarrow}{\mathscr{F}}_A \cdot \underset{\rightarrow}{\mathscr{F}}_B^{\mathsf{T}}} \left(\overset{\bullet}{\boldsymbol{r}}_B + \boldsymbol{\omega}_B^{\times} \boldsymbol{r}_B \right) \tag{2.20}$$

where the definition of rotation matrices from FUNDAMENTALS is used. We thus conclude from Eqs. (2.18) and (2.20) that the derivative of a vector (and that of its scalar representation) as measured in $\mathscr{F}_A$ is related to its derivative (and that of its scalar representation) as seen in $\mathscr{F}_B$ (which is rotating at $\underset{\rightarrow}{\boldsymbol{\omega}}$ with respect to $\mathscr{F}_A$) as follow:

$$\underset{\rightarrow}{\boldsymbol{r}}^{\bullet} = \underset{\rightarrow}{\boldsymbol{r}}^{\circ} + \underset{\rightarrow}{\boldsymbol{\omega}} \times \underset{\rightarrow}{\boldsymbol{r}} \tag{2.21a}$$

$$\overset{\bullet}{\boldsymbol{r}}_A = \boldsymbol{C}_{AB} \left(\overset{\bullet}{\boldsymbol{r}}_B + \boldsymbol{\omega}_B^{\times} \boldsymbol{r}_B \right) \tag{2.21b}$$

The relationships in Eqs. (2.21a) and (2.21b) are known as the vectorial and referential forms of the "transport theorem", respectively. If, for example, $\underset{\rightarrow}{\boldsymbol{r}}$ represents a position vector and $\underset{\rightarrow}{\mathscr{F}}_A$ is a non-rotating inertial frame, these relationships describe the corresponding velocity vector in the inertial frame, *in terms of* the motion observed in the rotating non-inertial frame, $\mathscr{F}_B$.

Note: if $\underset{\rightarrow}{\boldsymbol{\omega}} = \underset{\rightarrow}{\mathbf{0}}$, which would imply $\mathscr{F}_B$ is also an inertial frame, Eq. (2.21a) simplifies to $\underset{\rightarrow}{\boldsymbol{r}}^{\bullet} = \underset{\rightarrow}{\boldsymbol{r}}^{\circ}$. In other words, it would no longer matter in which one of the two frames the changes are measured.

Second Derivative

Recursive application of the first derivative described above could be used to determine how to relate the second (or higher) derivatives as seen in two frames, $\mathscr{F}_A$ and $\mathscr{F}_B$, where the latter is rotating with respect to the former with angular velocity $\underrightarrow{\boldsymbol{\omega}}$. We have:

$$\underrightarrow{\boldsymbol{r}}^{\bullet\bullet} = (\underrightarrow{\boldsymbol{r}}^{\bullet})^{\bullet} = (\underrightarrow{\boldsymbol{r}}^{\circ} + \underrightarrow{\boldsymbol{\omega}} \times \underrightarrow{\boldsymbol{r}})^{\bullet} = (\underrightarrow{\boldsymbol{r}}^{\circ})^{\bullet} + \underrightarrow{\boldsymbol{\omega}}^{\bullet} \times \underrightarrow{\boldsymbol{r}} + \underrightarrow{\boldsymbol{\omega}} \times \underrightarrow{\boldsymbol{r}}^{\bullet} \tag{2.22}$$

where Eq. (2.21a) is used once. Using the same equation two more times (once for $\underrightarrow{\boldsymbol{r}}^{\bullet}$ and once for $(\underrightarrow{\boldsymbol{r}}^{\circ})^{\bullet}$), and noting that $\underrightarrow{\boldsymbol{\omega}}^{\bullet} = \underrightarrow{\boldsymbol{\omega}}^{\circ} + \cancelto{\mathbf{0}}{\underrightarrow{\boldsymbol{\omega}} \times \underrightarrow{\boldsymbol{\omega}}} = \underrightarrow{\boldsymbol{\omega}}^{\circ}$ results in:

$$\begin{aligned} \underrightarrow{\boldsymbol{r}}^{\bullet\bullet} &= \underrightarrow{\boldsymbol{r}}^{\circ\circ} + \underrightarrow{\boldsymbol{\omega}}^{\circ} \times \underrightarrow{\boldsymbol{r}} + 2\underrightarrow{\boldsymbol{\omega}} \times \underrightarrow{\boldsymbol{r}}^{\circ} + \underrightarrow{\boldsymbol{\omega}} \times (\underrightarrow{\boldsymbol{\omega}} \times \underrightarrow{\boldsymbol{r}}) && \text{(2.23a)} \\ \ddot{\boldsymbol{r}}_A &= \boldsymbol{C}_{AB}\left(\ddot{\boldsymbol{r}}_B + \dot{\boldsymbol{\omega}}_B^{\times}\boldsymbol{r}_B + 2\boldsymbol{\omega}_B^{\times}\dot{\boldsymbol{r}}_B + \boldsymbol{\omega}_B^{\times}\boldsymbol{\omega}_B^{\times}\boldsymbol{r}_B\right) && \text{(2.23b)} \end{aligned}$$

which are the vectorial and referential forms of the second derivative as measured in $\mathscr{F}_A$, *in terms of* the motion in $\mathscr{F}_B$. Once again, if $\underrightarrow{\boldsymbol{r}}$ represents a position vector, these relationships describe the associated acceleration. Following the first term on the right-hand of Eq. (2.23a), the second, third, and fourth terms are called "tangential", "coriolis", and "centripetal" accelerations, respectively.

Relating Angular Velocity to Attitude

With a definition of angular velocity at hand, we now examine how to determine the attitude of an object using the various representations discussed in FUNDAMENTALS in terms of the $\boldsymbol{\omega}(t)$ history, or vice versa. To this end, we consider, once again, $\mathscr{F}_B$ rotating with respect to $\mathscr{F}_A$ with an angular velocity of $\underrightarrow{\boldsymbol{\omega}}$, and revisit the different parameterizations from which $\boldsymbol{C} = \boldsymbol{C}_{BA}$ could be determined.

Angular Velocity and Euler Angles $(\theta_1, \theta_2, \theta_3)$

Recall that using Euler angles, $\boldsymbol{C} = \boldsymbol{C}_\gamma(\theta_3)\boldsymbol{C}_\beta(\theta_2)\boldsymbol{C}_\alpha(\theta_1)$ for $\alpha, \beta, \gamma \in \{1, 2, 3\}$. Using Eq. (2.9), we have:

$$\boldsymbol{\omega}^{\times} = -\dot{\boldsymbol{C}}\boldsymbol{C}^{\mathsf{T}} = -\left(\dot{\boldsymbol{C}}_\gamma\boldsymbol{C}_\beta\boldsymbol{C}_\alpha + \boldsymbol{C}_\gamma\dot{\boldsymbol{C}}_\beta\boldsymbol{C}_\alpha + \boldsymbol{C}_\gamma\boldsymbol{C}_\beta\dot{\boldsymbol{C}}_\alpha\right)\boldsymbol{C}_\alpha^{\mathsf{T}}\boldsymbol{C}_\beta^{\mathsf{T}}\boldsymbol{C}_\gamma^{\mathsf{T}} \tag{2.24}$$

which, using orthonormality of a rotation matrix, $\boldsymbol{C}\boldsymbol{C}^{\mathsf{T}} = \mathbf{1}$, can be expanded and simplified as:

$$\boldsymbol{\omega}^{\times} = -\left(\cancelto{-(\dot{\theta}_3\mathbf{1}_\gamma)^{\times}}{\dot{\boldsymbol{C}}_\gamma\boldsymbol{C}_\gamma^{\mathsf{T}}}\right) - \boldsymbol{C}_\gamma\left(\cancelto{-(\dot{\theta}_2\mathbf{1}_\beta)^{\times}}{\dot{\boldsymbol{C}}_\beta\boldsymbol{C}_\beta^{\mathsf{T}}}\right)\boldsymbol{C}_\gamma^{\mathsf{T}} - \boldsymbol{C}_\gamma\boldsymbol{C}_\beta\left(\cancelto{-(\dot{\theta}_1\mathbf{1}_\alpha)^{\times}}{\dot{\boldsymbol{C}}_\alpha\boldsymbol{C}_\alpha^{\mathsf{T}}}\right)\boldsymbol{C}_\beta^{\mathsf{T}}\boldsymbol{C}_\gamma^{\mathsf{T}} \tag{2.25}$$

where $\mathbf{1}_1 = [1\ 0\ 0]^{\mathsf{T}}$, $\mathbf{1}_2 = [0\ 1\ 0]^{\mathsf{T}}$, and $\mathbf{1}_3 = [0\ 0\ 1]^{\mathsf{T}}$, and the fact that the angular velocity about a fixed axis, $\underrightarrow{\hat{\boldsymbol{a}}}$, is given by $\boldsymbol{\omega} = \dot{\theta}\hat{\boldsymbol{a}}$ is used. Lastly, making use of the identity $(\boldsymbol{C}\boldsymbol{u})^{\times} = \boldsymbol{C}\boldsymbol{u}^{\times}\boldsymbol{C}^{\mathsf{T}}$ from FUNDAMENTALS, and equating the terms inside the $(\cdot)^{\times}$ in both sides, Eq. (2.25) implies:

$$\boldsymbol{\omega} = \dot{\theta}_3\mathbf{1}_\gamma + \boldsymbol{C}_\gamma\dot{\theta}_2\mathbf{1}_\beta + \boldsymbol{C}_\gamma\boldsymbol{C}_\beta\dot{\theta}_1\mathbf{1}_\alpha = \boldsymbol{S}(\theta_3, \theta_2)\dot{\boldsymbol{\theta}} \tag{2.26}$$

where $\boldsymbol{\theta} \triangleq [\theta_3\ \theta_2\ \theta_1]^{\mathsf{T}}$ and $\boldsymbol{S} \triangleq [\mathbf{1}_\gamma\ \ \boldsymbol{C}_\gamma\mathbf{1}_\beta\ \ \boldsymbol{C}_\gamma\boldsymbol{C}_\beta\mathbf{1}_\alpha]$. Inverting Eq. (2.26) provides $\dot{\boldsymbol{\theta}} = \boldsymbol{S}^{-1}(\theta_3, \theta_2)\boldsymbol{\omega}$ that can be numerically integrated to determine the attitude at each time.

Note: The singularity issue with Euler angles manifests itself, once again, in singularity of $\boldsymbol{S}$. For example, for a 3-2-1 rotation, $\theta_2 = \pi/2$ would result in a singular $\boldsymbol{S}$ for which $\boldsymbol{S}^{-1}$ is no longer defined.

Note: Refer to Table 2.2 of *Spacecraft Attitude Dynamics* for all 12 inverse matrices, $\boldsymbol{S}^{-1}$, each corresponding to a different Euler angle sequence.

Angular Velocity and Euler Axis/Angle Variables $(\underset{\rightarrow}{\hat{\boldsymbol{a}}}, \phi)$

Recall that using Euler axis/angle variables, $\boldsymbol{C} = \cos\phi\boldsymbol{1} + (1-\cos\phi)\hat{\boldsymbol{a}}\hat{\boldsymbol{a}}^{\mathsf{T}} - \sin\phi\hat{\boldsymbol{a}}^{\times}$. Using Eq. (2.9) and after algebraic manipulations that are omitted here for brevity, we will eventually obtain:

$$\boldsymbol{\omega} = \dot{\phi}\hat{\boldsymbol{a}} - \big[1-\cos(\phi)\big]\hat{\boldsymbol{a}}^{\times}\dot{\hat{\boldsymbol{a}}} + \sin(\phi)\dot{\hat{\boldsymbol{a}}} \tag{2.27}$$

which provides $\boldsymbol{\omega}$ in terms of $\dot{\hat{\boldsymbol{a}}}$ and $\dot{\phi}$. Pre-multiplying Eq. (2.27) by $\hat{\boldsymbol{a}}^{\mathsf{T}}$ yields:

$$\hat{\boldsymbol{a}}^{\mathsf{T}}\boldsymbol{\omega} = \dot{\phi}(\cancelto{1}{\hat{\boldsymbol{a}}^{\mathsf{T}}\hat{\boldsymbol{a}}}) - \big[1-\cos(\phi)\big](\cancelto{\dot{\hat{\boldsymbol{a}}}^{\mathsf{T}}\hat{\boldsymbol{a}}^{\times}\hat{\boldsymbol{a}}=0}{\hat{\boldsymbol{a}}^{\mathsf{T}}\hat{\boldsymbol{a}}^{\times}\dot{\hat{\boldsymbol{a}}}}) + \sin(\phi)(\cancelto{\frac{1}{2}\frac{d}{dt}\hat{\boldsymbol{a}}^{\mathsf{T}}\hat{\boldsymbol{a}}=0}{\hat{\boldsymbol{a}}^{\mathsf{T}}\dot{\hat{\boldsymbol{a}}}}) \quad\Rightarrow\quad \dot{\phi} = \hat{\boldsymbol{a}}^{\mathsf{T}}\boldsymbol{\omega} \tag{2.28}$$

where the scalar triple product identity and the constant unit magnitude of $\hat{\boldsymbol{a}}$ are used for simplifications. Pre-multiplying Eq. (2.27) once again, but this time by $\hat{\boldsymbol{a}}^{\times}\hat{\boldsymbol{a}}^{\times}$ results in:

$$\hat{\boldsymbol{a}}^{\times}\hat{\boldsymbol{a}}^{\times}\boldsymbol{\omega} = \dot{\phi}\hat{\boldsymbol{a}}^{\times}\hat{\boldsymbol{a}}^{\times}\hat{\boldsymbol{a}} - \big[1-\cos(\phi)\big]\hat{\boldsymbol{a}}^{\times}\hat{\boldsymbol{a}}^{\times}\hat{\boldsymbol{a}}^{\times}\dot{\hat{\boldsymbol{a}}} + \sin(\phi)\hat{\boldsymbol{a}}^{\times}\hat{\boldsymbol{a}}^{\times}\dot{\hat{\boldsymbol{a}}} \tag{2.29}$$

but recalling the identity for $\boldsymbol{a}^{\times}\boldsymbol{b}^{\times}$ (for generic $\boldsymbol{a}$ and $\boldsymbol{b}$) from FUNDAMENTALS, we have:

$$\hat{\boldsymbol{a}}^{\times}\hat{\boldsymbol{a}}^{\times}\dot{\hat{\boldsymbol{a}}} = \Big[\hat{\boldsymbol{a}}\hat{\boldsymbol{a}}^{\mathsf{T}} - (\cancelto{1}{\hat{\boldsymbol{a}}^{\mathsf{T}}\hat{\boldsymbol{a}}})\boldsymbol{1}\Big]\dot{\hat{\boldsymbol{a}}} = \hat{\boldsymbol{a}}\cancelto{\frac{1}{2}\frac{d}{dt}\hat{\boldsymbol{a}}^{\mathsf{T}}\hat{\boldsymbol{a}}=0}{\hat{\boldsymbol{a}}^{\mathsf{T}}\dot{\hat{\boldsymbol{a}}}} - \dot{\hat{\boldsymbol{a}}} = -\dot{\hat{\boldsymbol{a}}} \tag{2.30}$$

substituting which back into Eq. (2.29), as well as Eq. (2.27) premultiplied by $\hat{\boldsymbol{a}}^{\times}$, yields:

$$\hat{\boldsymbol{a}}^{\times}\hat{\boldsymbol{a}}^{\times}\boldsymbol{\omega} = \big[1-\cos(\phi)\big]\hat{\boldsymbol{a}}^{\times}\dot{\hat{\boldsymbol{a}}} - \sin(\phi)\dot{\hat{\boldsymbol{a}}} \tag{2.31a}$$

$$\hat{\boldsymbol{a}}^{\times}\boldsymbol{\omega} = \big[1-\cos(\phi)\big]\dot{\hat{\boldsymbol{a}}} + \sin(\phi)\hat{\boldsymbol{a}}^{\times}\dot{\hat{\boldsymbol{a}}} \tag{2.31b}$$

subtracting the second one of which from the first one, and multiplying the result by $\sin(\phi)/\big[1-\cos(\phi)\big] = \cot(\phi/2)$ yields $\dot{\hat{\boldsymbol{a}}}$ upon rearranging. Given $\boldsymbol{\omega}$, this result and Eq. (2.28) provide the relationships necessary for determining the derivatives of the Euler axis/angle variables:

$$\dot{\hat{\boldsymbol{a}}} = \frac{1}{2}\big(\hat{\boldsymbol{a}}^{\times} - \cot\frac{\phi}{2}\hat{\boldsymbol{a}}^{\times}\hat{\boldsymbol{a}}^{\times}\big)\boldsymbol{\omega} \quad , \quad \dot{\phi} = \hat{\boldsymbol{a}}^{\mathsf{T}}\boldsymbol{\omega} \tag{2.32}$$

Velocity and Euler Parameters (Quaternions) $(\boldsymbol{\epsilon}, \eta)$

Recall that using quaternions, $\boldsymbol{\epsilon} \triangleq \hat{\boldsymbol{a}} \sin(\frac{\phi}{2})$ and $\eta = \cos(\frac{\phi}{2})$, we have $\boldsymbol{C} = (\eta^2 - \boldsymbol{\epsilon}^{\mathsf{T}}\boldsymbol{\epsilon})\mathbf{1} + 2\boldsymbol{\epsilon}\boldsymbol{\epsilon}^{\mathsf{T}} - 2\eta\boldsymbol{\epsilon}^{\times}$. Having previously looked at $\dot{\hat{\boldsymbol{a}}}$ and $\dot{\phi}$ above, we can use the quaternions' definition to find their derivatives:

$$\dot{\boldsymbol{\epsilon}} = \dot{\hat{\boldsymbol{a}}} \sin(\frac{\phi}{2}) + \frac{1}{2}\hat{\boldsymbol{a}} \cos(\frac{\phi}{2})\dot{\phi} = \frac{1}{2}\Big[\hat{\boldsymbol{a}}^{\times} \sin(\frac{\phi}{2}) - \hat{\boldsymbol{a}}^{\times}\hat{\boldsymbol{a}}^{\times} \cancelto{\cos(\frac{\phi}{2})}{\cot(\frac{\phi}{2}) \sin(\frac{\phi}{2})}\Big]\boldsymbol{\omega} + \frac{1}{2}\cos(\frac{\phi}{2})\hat{\boldsymbol{a}}\hat{\boldsymbol{a}}^{\mathsf{T}}\boldsymbol{\omega} \tag{2.33}$$

where both parts of Eq. (2.32) are used. Upon gathering the like terms, Eq. (2.33) becomes:

$$\dot{\boldsymbol{\epsilon}} = \frac{1}{2}\Big[\big(\cancelto{\hat{\boldsymbol{\epsilon}}}{\sin(\frac{\phi}{2})\hat{\boldsymbol{a}}}\big)^{\times} + \cos(\frac{\phi}{2})\big(\cancelto{(\hat{\boldsymbol{a}}^{\mathsf{T}}\hat{\boldsymbol{a}})\mathbf{1} = \mathbf{1}}{\hat{\boldsymbol{a}}\hat{\boldsymbol{a}}^{\mathsf{T}} - \hat{\boldsymbol{a}}^{\times}\hat{\boldsymbol{a}}^{\times}}\big)\Big]\boldsymbol{\omega} \;\;\Rightarrow\;\; \dot{\boldsymbol{\epsilon}} = \frac{1}{2}(\boldsymbol{\epsilon}^{\times} + \eta\mathbf{1})\boldsymbol{\omega} \tag{2.34}$$

where the identity from FUNDAMENTALS for $\hat{\boldsymbol{a}}^{\times}\hat{\boldsymbol{a}}^{\times}$ (also used in Eq. (2.30)) is employed. Repeating the differentiation for η and making use of right-hand side of Eq. (2.32) results in:

$$\dot{\eta} = -\frac{1}{2}\sin(\frac{\phi}{2})\dot{\phi} = -\frac{1}{2}\sin(\frac{\phi}{2})\hat{\boldsymbol{a}}^{\mathsf{T}}\boldsymbol{\omega} \;\;\Rightarrow\;\; \dot{\eta} = -\frac{1}{2}\boldsymbol{\epsilon}^{\mathsf{T}}\boldsymbol{\omega} \tag{2.35}$$

The derivatives of the quaternions in terms of $\boldsymbol{\omega}$ are provided by Eqs. (2.34) and (2.35). To obtain $\boldsymbol{\omega}$ in terms of $\boldsymbol{\epsilon}$ and η and their derivatives, one approach is to invert Eq. (2.34) (using Cramer's rule for an explicit inverse), which eventually yields:

$$\boldsymbol{\omega} = 2\Big[\frac{\eta^2\mathbf{1} - \eta\boldsymbol{\epsilon}^{\times} + \boldsymbol{\epsilon}\boldsymbol{\epsilon}^{\mathsf{T}}}{\eta}\Big]\dot{\boldsymbol{\epsilon}} \tag{2.36}$$

Note: For an alternative approach of determining $\dot{\boldsymbol{\epsilon}}$ and $\dot{\eta}$ in terms of $\boldsymbol{\omega}$ and vice versa, using $\boldsymbol{\omega} = -\dot{\boldsymbol{C}}\boldsymbol{C}^{\mathsf{T}}$, refer to Section 1.4.3 of *Spacecraft Dynamics and Control: an Introduction*.

Angular Velocity for Infinitesimal Rotations

As discussed in FUNDAMENTALS, for very small rotation angles and rates $\boldsymbol{C} \approx \mathbf{1} - \boldsymbol{\theta}^{\times} \approx \mathbf{1} - \phi\hat{\boldsymbol{a}}^{\times} \approx \mathbf{1} - 2\boldsymbol{\epsilon}^{\times}$. Generalizing this relationship with the notation $\boldsymbol{C} \approx \mathbf{1} - \boldsymbol{\alpha}^{\times}$, where $\boldsymbol{\alpha} \in \{\boldsymbol{\theta}, \phi\hat{\boldsymbol{a}}, 2\boldsymbol{\epsilon}\}$, the angular velocity relationship from Eq. (2.9) becomes:

$$\boldsymbol{\omega}^{\times} = -\dot{\boldsymbol{C}}\boldsymbol{C}^{\mathsf{T}} = -\big(-\dot{\boldsymbol{\alpha}}^{\times}\big)\Big(\mathbf{1} - \boldsymbol{\alpha}^{\times^{\mathsf{T}}}\Big) = \dot{\boldsymbol{\alpha}}^{\times} + \cancelto{\approx \mathbf{0}}{\dot{\boldsymbol{\alpha}}^{\times}\boldsymbol{\alpha}^{\times}} \tag{2.37}$$

from which we conclude that, for infinitesimally small angles and rates, $\boldsymbol{\omega} \approx \dot{\boldsymbol{\theta}} \approx \phi\dot{\hat{\boldsymbol{a}}} \approx 2\dot{\boldsymbol{\epsilon}}$.

Note: A summary of attitude parameterizations, including alternative sets that are not discussed in this course, is provided in Table 2.3 of *Spacecraft Attitude Dynamics*, along with the $\boldsymbol{C}$ and $\boldsymbol{\omega}$ associated with each parameterization.