Clouse
ASEN 5014
HW3

Problem 1

a) for positive x, $y=3x+2$

for negative x, $y=-3x+2 \neq -1(3x+2)$

not for all vectors $\exists$ a unique vector s.t. their sum is $\vec{0}$, $\boxed{\text{not a VS}}$

b) $\boxed{\text{This is a VS}}$

c) for positive x, $y=x^3-2x$

for negative x, $y=-x^3+2x=-1(x^3-2x)$

$$y_1 = x_1^3 - 2x_1$$
$$+\; y_2 = x_2^3 - 2x_2$$
$$y_3 = (x_1^3 + x_2^3) - 2(x_1+x_2)$$

$$y_3 = x_3^3 - 2x_3$$
$$= (x_1+x_2)^3 - 2(x_1+x_2)$$
$$= x_1^3 + 3x_1^2x_2 + 3x_1^1x_2^2 + x_3^3 - 2x_1 - 2x_2$$

1
1 1
1 2 1
1 3 3 1

These quantities are not equal, so you cannot add any two candidate vectors to get a valid third vector. $\boxed{\text{Not a VS}}$

d) $\boxed{\text{This is a vector space}}$ $(\mathbb{R}^2)$

e) for $x \in G$ + $y \in G$, $z \in G$

✓ $x+y=z$

✓ $x+y=y+x \Rightarrow g_1 \Delta g_2 = (p_1+p_2, q_1+q_2) = (p_2+p_1, q_2+q_1) = g_2 \Delta g_1$

✓ $(x+y)+z \Rightarrow (g_1 \Delta g_2)\Delta g_3 = ((p_1+p_2)+p_3, (q_1+q_2)+q_3)$

$$= (p_1+(p_2+p_3), q_1+(q_2+q_3)) = g_1 \Delta (g_2 \Delta g_3)$$

✓ $g_1 = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$, $g_2 = \begin{pmatrix} -p_1 \\ -p_2 \end{pmatrix}$, $g_1 \Delta g_2 = (p_1 - p_1, q_1 - q_1) = \underline{(0,0)}$

✓ $a(p_1, q_1) = (ap_1, aq_1)$

✓ $a(bg_1) = a(b(p_1, q_1)) = a(bp_1, bq_1) = (abp_1, abq_1) = ab(p_1, q_1)$

✓ $(a+b)(p_1, q_1) = ((a+b)p_1, (a+b)q_1) = (ap_1+bp_1, aq_1+bq_1)$

✓ $a(p_1, q_1) \Delta b(p_1, q_1) = (ap_1, q_1)\Delta(bp_1, q_1) = (ap_1+bp_1, aq_1+bq_1)$

$\boxed{\text{This is a VS}}$

Problem 2: Brogan 5.39 a)

show that $\{x_1, x_2, x_3\}$ are LI

$$x_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad x_2 = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}, \quad x_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} a_1 \\ 2a_1 \\ 3a_1 \end{pmatrix} + \begin{pmatrix} a_2 \\ -2a_2 \\ 3a_2 \end{pmatrix} + \begin{pmatrix} 0 \\ a_3 \\ a_3 \end{pmatrix} = 0 \Rightarrow \quad a_1 = -a_2$$

$$\Rightarrow \begin{pmatrix} -a_2 \\ -2a_2 \\ 3a_2 \end{pmatrix} + \begin{pmatrix} a_2 \\ -2a_2 \\ 3a_2 \end{pmatrix} + \begin{pmatrix} 0 \\ a_3 \\ a_3 \end{pmatrix} = 0 \Rightarrow \begin{array}{l} -4a_2 + a_3 = 0 \\ 6a_2 + a_3 = 0 \end{array} \Rightarrow a_2 = a_3 = 0 = a_1$$

$$\boxed{\therefore \{x_i\} \text{ is LI}}$$

Problem 3, Brogan 5.40

if $\{\vec{x}_i\}$ is a basis set, find reciprocal basis set $\{\vec{r}_i\}$

$$\vec{x}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \vec{x}_2 = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} \quad \vec{x}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$R = B^{-1} = [\vec{r}_1 \ \vec{r}_2 \ \vec{r}_3] = \begin{bmatrix} 1 & 1 & 0 \\ 2 & -2 & 1 \\ 3 & 3 & 1 \end{bmatrix}^{-1} = \boxed{\begin{bmatrix} 5/4 & 1/4 & -\frac{1}{4} \\ -1/4 & -1/4 & 1/4 \\ -3 & 0 & 1 \end{bmatrix}}$$

Problem 4: Brogan 5.42

express $\vec{z}$ in terms of $\{x_i\}$ using $\{\vec{r}_i\}$

$$\vec{z} = \begin{pmatrix} 6 \\ 4 \\ -3 \end{pmatrix} = \begin{pmatrix} a_1 \vec{x}_1 \\ a_2 \vec{x}_2 \\ a_3 \vec{x}_3 \end{pmatrix} = \begin{pmatrix} \langle \vec{r}_1, \vec{z} \rangle \vec{x}_1 \\ \langle \vec{r}_2, \vec{z} \rangle \vec{x}_2 \\ \langle \vec{r}_3, \vec{z} \rangle \vec{x}_3 \end{pmatrix} = \begin{bmatrix} (30/4 + 4/4 + 3/4)\vec{x}_1 \\ (-6/4 + -4/4 - 3/4)\vec{x}_2 \\ (-18 + 0 - 3)\vec{x}_3 \end{bmatrix}$$

$$\boxed{\vec{z} = \begin{pmatrix} 9\,1/4\, \vec{x}_1 \\ -3\,1/4\, \vec{x}_2 \\ -21\, \vec{x}_3 \end{pmatrix}}$$

Problem 5: Brogan 5.47

Compute the Grammian for $\{\vec{y}_i\}$, determine linear independence.

$$\vec{y}_1 = \begin{pmatrix} 1.0 \\ 1.0 \\ 1.0 \\ 1.0 \end{pmatrix}, \quad \vec{y}_2 = \begin{pmatrix} 1.0001 \\ 9.9989998e\text{-}1 \\ 1.0 \\ 1.0 \end{pmatrix}, \quad \vec{y}_3 = \begin{pmatrix} -2.0 \\ -1.9999 \\ -2.0 \\ -2.0 \end{pmatrix}$$

$G = M^T M$, $M = [\vec{y}_1 \ \vec{y}_2 \ \vec{y}_3]$

$$\begin{pmatrix} 1.0 & 1.0001 & -2.0 \\ 1.0 & 0.99989998 & -1.9999 \\ 1.0 & 1.0 & -2.0 \\ 1.0 & 1.0 & -2.0 \end{pmatrix}$$

$(\vec{y}_{21})^2 = 1.00020001$

$(\vec{y}_{22})^2 = 0.9997999700004$

$$G = \begin{pmatrix} 1.0 & 1.0 & 1.0 & 1.0 \\ 1.0001 & 0.99989998 & 1.0 & 1.0 \\ -2.0 & -1.9999 & -2.0 & -2.0 \end{pmatrix} \begin{pmatrix} 4.0 & 3.99999998 & -7.9999 \\ 3.99999998 & 3.999999960004 & -7.99989997 \\ -7.9999 & -7.99989997 & 15.99960001 \end{pmatrix}$$

$\det(G) = 2e\text{-}16$, linearly independent. depending on computer precision, could be seen as dependent in a software check.

Problem 6: Brogan 5.49

determine dim of VS spanned by

$$\vec{x}_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \end{pmatrix}, \quad \vec{x}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \vec{x}_3 = \begin{pmatrix} 3 \\ 4 \\ 4 \\ 3 \end{pmatrix} = \underline{2\vec{x}_1 + \vec{x}_2}$$

dim(VS) = # basis vectors. $\vec{x}_3$ is a linear combo of the others. It can be removed, and $\vec{x}_1$ & $\vec{x}_2$ are the basis vectors. $\boxed{\therefore \dim = 2}$

Challenge 1

a) $\underline{A} = \{A_1, A_2, \dots A_n\}$

$\underline{A}' = \{A_1^T, A_2^T, \dots A_n^T\}$

$\underline{A}$ is linearly independent, so

$$\alpha_1 A_1 + \alpha_2 A_2 + \dots \alpha_n A_n = 0 \text{ iff } \alpha_1 = \alpha_2 = \alpha_n = 0$$

$$= \begin{pmatrix} \alpha_1 A_{111} & \alpha_1 A_{112} & \cdots & \alpha_1 A_{11n} \\ \alpha_1 A_{121} & \ddots & & \\ & & & \alpha_1 A_{1nn} \end{pmatrix} + \begin{pmatrix} \alpha_2 A_{211} & \alpha_2 A_{212} & \cdots & \alpha_2 A_{21n} \\ \alpha_2 A_{212} & \ddots & & \\ & & & \alpha_2 A_{2nn} \end{pmatrix} + \begin{pmatrix} \alpha_n A_{n11} & \alpha_n A_{n12} & \cdots & \alpha_n A_{n1n} \\ & & & \\ & & & \alpha_n A_{nnn} \end{pmatrix} = 0$$

$$= \sum_{k=1}^{n} \alpha_k \sum_{i=1}^{n} \sum_{j=1}^{n} A_{kij}$$

$\rightarrow \underline{A}'$'s LI can be shown:

$$\begin{pmatrix} \alpha_1 A_{111} & \alpha_1 A_{121} & \cdots & \alpha_1 A_{1n1} \\ & \ddots & & \\ & & & \alpha_1 A_{1nn} \end{pmatrix} + \begin{pmatrix} \alpha_2 A_{211} & \alpha_2 A_{221} & \cdots & \alpha_2 A_{2n1} \\ & \ddots & & \\ & & & \alpha_2 A_{2nn} \end{pmatrix} + \begin{pmatrix} \alpha_n A_{n11} & \alpha_n A_{n21} & \cdots & \alpha_n A_{nn1} \\ & \ddots & & \\ & & & \alpha_n A_{nnn} \end{pmatrix}$$

$$= \sum_{k=1}^{n} \alpha_k \sum_{j=1}^{n} \sum_{n=1}^{n} A_{kij} = \sum_{k=1}^{n} \alpha_k \sum_{i=1}^{n} \sum_{j=1}^{n} A_{kij}$$

$\therefore \underline{A}'$ is LI

Challenge 1b)

Let $A = [f \;\; g]^T = [e^{rt} \;\; e^{st}]^T$

$G = A^T A = [f \;\; g]\begin{bmatrix} f \\ g \end{bmatrix} = f^2 + g^2$

$\boxed{\det(G) \neq 0, \therefore \text{ LI in } F}$

Challenge 1c)

$\dim(M_{n \times n}(\mathbb{R})) = n^2$, $\underline{M} = \left\{ \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ \vdots & & & \cdots & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & & & \\ 0 & & & 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & & & \end{pmatrix} \right\}$

$\underline{I} = \left\{ \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ \vdots & & & \\ 0 & & \cdots & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ \vdots & 1 & 0 & \cdots & \vdots \\ \vdots & & & & \\ 0 & & & & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{pmatrix} \right\}$

$\underline{I}$ contains subsets of basises $\underline{M}$, the dim of this set is n

Challenge 2. for $x, y, z \in M_{m \times n}(\mathbb{Z}_2)$

$x + y = z$

communicative addition is fulfilled: $1+0=0+1=1$
$0+0=0$
$1+1=0$

associative addition is fulfilled: $(1+0)+0 = 1+(0+0)$
$(1+1)+0 = 1+(1+0)$
$\vdots$

zero-vector exists, $x+x=0$

$0x = 0$, $1x = x$

associative multiplication: $0(1x) = (0 \cdot 1)x$, same for $(0,0)(1,0),(1,1)$

Distributive mult: $(0+0)x = 0x + 0x$
$(0+1)x = 0x + 1x$
$(1+1)x = 1x + 1x$
$\Rightarrow 0x = x + x$

$\boxed{\text{Is a vector Space}}$

for $m = n = 2$: $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \underset{\times 4}{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}, \underset{\times 4}{\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}} \underset{\times 4}{\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

for $m = 2, n = 3$: $\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \underset{\times 6}{\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}}, \underset{\times 7}{\begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}}$