Problem 21 $y'' + 2\alpha y' + \omega^2 y = 0$

$\Rightarrow y'' = -2\alpha y' - \omega^2 y$

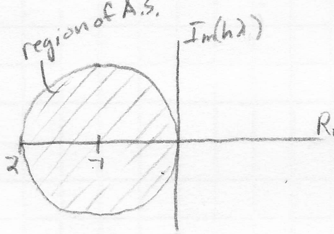
let $\vec{y} = \begin{pmatrix} y \\ y' \end{pmatrix} \Rightarrow \vec{y}\,' = \begin{pmatrix} y' \\ y'' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -2\alpha \end{pmatrix} \vec{y} = A\vec{y}$

Eigenvalues, Eigenvectors of A:

$\det(A - \lambda I) = 0 = \begin{vmatrix} -\lambda & 1 \\ -\omega^2 & -\lambda - 2\alpha \end{vmatrix} = \lambda^2 + 2\alpha\lambda + \omega^2$

$\lambda = \dfrac{-2\alpha \pm \sqrt{4\alpha^2 - 4\cdot\omega^2}}{2} = -\alpha \pm \sqrt{\alpha^2 - \omega^2}$

a) $\alpha = 2,\ \omega = 1 \Rightarrow \lambda = -2 \pm \sqrt{4-1} = -2 \pm \sqrt{3}$



$\left.\begin{matrix} |1 + Re(h\lambda)| \le 1 \\ \sqrt{Re(\lambda h)^2 + Im(\lambda h)^2} \le 1 \end{matrix}\right\}$ For absolute stability.

$\lambda_+ : |1 + (-2+\sqrt{3})h| \Rightarrow -1 \le 1 + (-2+\sqrt{3})h \Rightarrow -2 \le (-2+\sqrt{3})h$

$\Rightarrow h_+ \le \dfrac{2}{2-\sqrt{3}}$

$\lambda_- : |1 - (2+\sqrt{3})h| \Rightarrow -1 \le 1 - (2+\sqrt{3})h \Rightarrow -2 \le -(2+\sqrt{3})h$

$\Rightarrow h_- \le \dfrac{2}{2+\sqrt{3}}$

$\max(h_-) < \max(h_+)$, so $\boxed{h \le \dfrac{2}{2+\sqrt{3}}}$ to maintain A.S.

b) $\alpha = 1,\ \omega = 2 \Rightarrow \lambda = -1 \pm \sqrt{1-4} = -1 \pm \sqrt{3}\, i$

$\lambda_+ : |1 + (-1+\sqrt{3})h| \le 1 \Rightarrow -1 \le 1 - (1-\sqrt{3})h \Rightarrow -2 \le -(1-\sqrt{3})h$

$\Rightarrow h \le \dfrac{2}{1-\sqrt{3}}$

$|\sqrt{1h^2 + 3h^2}| \le 1 \Rightarrow |\sqrt{4h^2}| \le 1 \Rightarrow 2h \le 1 \Rightarrow h \le 1/2$

$\lambda_- : |1 - (1+\sqrt{3})h| \le 1 \Rightarrow -1 \le 1 - (1+\sqrt{3})h \Rightarrow -2 \le -(1+\sqrt{3})h$

$\Rightarrow h \le \dfrac{2}{1+\sqrt{3}}$

$|\sqrt{1h^2 + 3h^2}| \le 1 \Rightarrow h \le 1/2$

For absolute stability, $\boxed{h \le 1/2}$