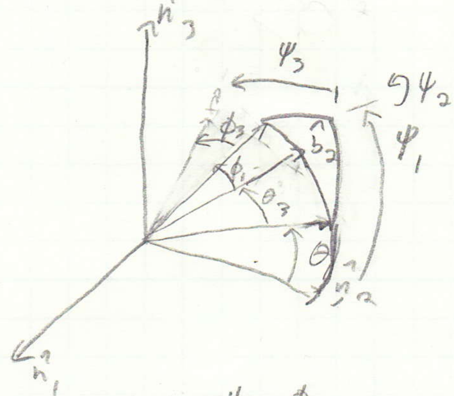
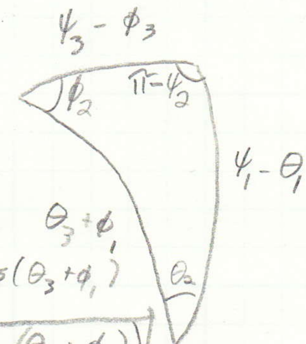
ASEN 5010 | John Clouse

Problem 2: Analytically solve for direct 1-2-1 Euler angle addition Formula.

Total rotation : $\vec{\psi}$
1st rotation : $\vec{\theta}$
2nd rotation : $\vec{\phi}$





Law of Cosines:

$$\cos(\pi - \psi_2) = -\cos\theta_2 \cos\phi_2 + \sin\theta_2 \sin\phi_2 \cos(\theta_3 + \phi_1)$$
$$= -\cos\psi_2$$

$$\Rightarrow \boxed{\psi_2 = \cos^{-1}\left(+\cos\theta_2 \cos\phi_2 - \sin\theta_2 \sin\phi_2 \cos(\theta_3+\phi_1)\right)}$$

$$\frac{\sin(\psi_1 - \theta_1)}{\sin\phi_2} = \frac{\sin(\theta_3 + \phi_1)}{\sin(\pi - \psi_2)} = \frac{\sin(\theta_3+\phi_1)}{\sin\psi_2}$$

$$\sin(\psi_1 - \theta_1) = \frac{\sin\phi_2}{\sin\psi_2}\sin(\theta_3 + \phi_1)$$

Need tangent for proper quadrant representation...

$$\cos\phi_2 = +\cos\theta_2 \underbrace{\cos(\pi - \psi_2)}_{+\cos\psi_2} + \sin\theta_2 \underbrace{\sin(\pi - \psi_2)}_{\sin\psi_2} \cos(\psi_1 - \theta_1)$$

$$\Rightarrow \cos(\psi_1 - \theta_1) = \frac{\cos\phi_2 - \cos\theta_2 \cos\psi_2}{\sin\theta_2 \sin\psi_2}$$

$$\tan(\psi_1 - \theta_1) = \frac{\sin\phi_2 \sin(\theta_3 + \phi_1) \sin\theta_2 \cancel{\sin\psi_2}}{\cancel{\sin\psi_2}(\cos\phi_2 - \cos\theta_2 \cos\psi_2)}$$

$$\Rightarrow \boxed{\psi_1 = \theta_1 + \tan^{-1}\left(\frac{\sin\phi_2 \sin\theta_2 \sin(\theta_3 + \phi_1)}{\cos\phi_2 - \cos\theta_2 \cos\psi_2}\right)}$$

continued →

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Problem 2 cont:

$$\frac{\sin(\psi_3 - \phi_3)}{\sin\theta_2} = \frac{\sin(\theta_3 + \phi_1)}{\underbrace{\sin(\cancel{\pi} - \psi_2)}_{\sin\psi_2}}$$

$$\Rightarrow \sin(\psi_3 - \phi_3) = \frac{\sin\theta_2}{\sin\psi_2}\sin(\theta_3 + \phi_1)$$

$$\cos\theta_2 = +\cos\phi_2 \underbrace{\cos(\cancel{\pi} - \psi_2)}_{+\cos\psi_2} + \sin\phi_2 \underbrace{\sin(\cancel{\pi} - \psi_2)}_{\sin\psi_2} \cos(\psi_3 - \phi_3)$$

$$\Rightarrow \cos(\psi_3 - \phi_3) = \frac{\cos\theta_2 - \cos\phi_2\cos\psi_2}{\sin\phi_2 \sin\psi_2}$$

$$\tan(\psi_3 - \phi_3) = \frac{\sin\theta_2 \sin(\theta_3 + \phi_1)\sin\phi_2 \cancel{\sin\psi_2}}{\cancel{\sin\psi_2}(\cos\theta_2 - \cos\phi_2\cos\psi_2)}$$

$$\Rightarrow \boxed{\psi_3 = \phi_3 + \tan^{-1}\left(\frac{\sin\theta_2 \sin\phi_2 \sin(\theta_3 + \phi_1)}{\cos\theta_2 - \cos\phi_2\cos\psi_2}\right)}$$

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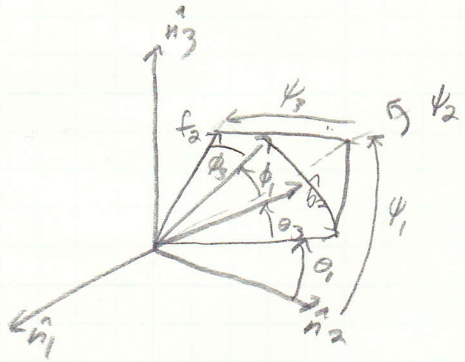
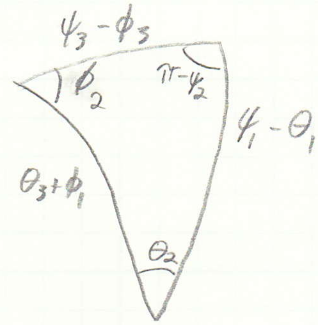
Problem 3: Analytically solve for direct 1-2-1 Euler angle addition Formula





given $\vec{\theta}$ & $\vec{\psi}$, solve for $\vec{\phi}$ $\quad \vec{\theta}$ "−" $\vec{\psi} = \vec{\phi}$

$$\cos\phi_2 = +\cos\theta_2 \underbrace{\cos(\cancel{\pi}-\psi_2)}_{+\cos\psi_2} + \sin\theta_2 \underbrace{\sin(\cancel{\pi}-\psi_2)}_{\sin\psi_2}\cos(\psi_1-\theta_1)$$

$$\boxed{\phi_2 = \cos^{-1}\left(\cos\theta_2\cos\psi_2 + \sin\theta_2\sin\psi_2\cos(\psi_1-\theta_1)\right)}$$

$$\frac{\sin(\theta_3+\phi_1)}{\underbrace{\sin(\cancel{\pi}-\psi_2)}_{\sin\psi_2}} = \frac{\sin(\psi_1-\theta_1)}{\sin\phi_2} \Rightarrow \sin(\theta_3+\phi_1) = \frac{\sin\psi_2}{\sin\phi_2}\sin(\psi_1-\theta_1)$$

$$\underbrace{\cos(\cancel{\pi}-\psi_2)}_{-\cos\psi_2} = -\cos\theta_2\cos\phi_2 + \sin\theta_2\sin\phi_2\cos(\theta_3+\phi_1)$$

$$\cos(\theta_3+\phi_1) = \frac{-\cos\psi_2 + \cos\theta_2\cos\phi_2}{\sin\theta_2\sin\phi_2}$$

$$\tan(\theta_3+\phi_1) = \frac{\sin\psi_2\sin(\psi_1-\theta_1)\sin\theta_2\cancel{\sin\phi_2}}{\cancel{\sin\phi_2}(-\cos\psi_2+\cos\theta_2\cos\phi_2)}$$

$$\boxed{\phi_1 = -\theta_3 + \tan^{-1}\left(\frac{\sin\psi_2\sin\theta_2\sin(\psi_1-\theta_1)}{\cos\theta_2\cos\phi_2 - \cos\psi_2}\right)}$$

$$\frac{\sin(\psi_3-\phi_3)}{\sin\theta_2} = \frac{\sin(\psi_1-\theta_1)}{\sin\phi_2} \Rightarrow \sin(\psi_3-\phi_3) = \frac{\sin\theta_2}{\sin\phi_2}\sin(\psi_1-\theta_1)$$

$$\cos\theta_2 = +\cos\phi_2\underbrace{\cos(\cancel{\pi}-\psi_2)}_{+\cos\psi_2} + \sin\phi_2\underbrace{\sin(\cancel{\pi}-\psi_2)}_{\sin\psi_2}\cos(\psi_3-\phi_3)$$

$$\cos(\psi_3-\phi_3) = \frac{\cos\theta_2 - \cos\phi_2\cos\psi_2}{\sin\phi_2\sin\psi_2}$$

$$\tan(\psi_3-\phi_3) = \frac{\sin\theta_2\sin(\psi_1-\theta_1)\cancel{\sin\phi_2}\sin\psi_2}{\cancel{\sin\phi_2}(\cos\theta_2-\cos\phi_2\cos\psi_2)}$$

$$\boxed{\phi_3 = \psi_3 - \tan^{-1}\left(\frac{\sin\theta_2\sin\psi_2\sin(\psi_1-\theta_1)}{\cos\theta_2-\cos\phi_2\cos\psi_2}\right)}$$