Challenge Problem

$$X_a = \begin{bmatrix} X \\ u \end{bmatrix} \Rightarrow \dot{X}_a = \begin{bmatrix} \dot{X} \\ \dot{u} \end{bmatrix} = \underbrace{\left[\begin{array}{c|c} A & B \\ \hline 0 & 0 \end{array}\right]}_{A_a} \begin{bmatrix} X \\ u \end{bmatrix} \quad \Leftarrow \text{ZOH} : \dot{u} = 0$$

$$X_a(k+T) = e^{A_a T} x_a(k) + 0$$

$$e^{A_a T} = \sum_{r=0}^{\infty} \frac{A_a^r T^r}{r!}$$

$$A_a^0 = I$$

$$A_a^1 = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}$$

$$A_a^2 = \begin{bmatrix} A^2 & AB \\ 0 & 0 \end{bmatrix} \Rightarrow e^{A_a T} = \left[\begin{array}{c|c} \sum_{r=0}^{\infty} \frac{A^r T^r}{r!} & \sum_{r=0}^{\infty} \frac{A^r B T^{r+1}}{(r+1)!} \\ \hline 0 & 0 \end{array}\right]$$

$$A_a^3 = \begin{bmatrix} A^3 & A^2 B \\ 0 & 0 \end{bmatrix}$$

$$= \left[\begin{array}{c|c} F & G \\ \hline 0 & 0 \end{array}\right]$$

$$\Rightarrow F = \sum_{r=0}^{\infty} \frac{A^r T^r}{r!}$$

$$G = \sum_{r=0}^{\infty} \frac{A^r B T^{r+1}}{(r+1)!}$$