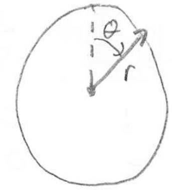
Clouse
ASEN 5014
Midterm 1

1. Given: $\ddot{r} = r\dot{\theta}^2 - \frac{k}{r^2} + u_1(t)$ $\qquad r(0)=r_0$

$\qquad \dot{r}(0)=0$

$\ddot{\theta} = \frac{-2\dot{\theta}\dot{r}}{r} + \frac{1}{r}u_2(t)$ $\qquad \theta(0)=0$

$\qquad \dot{\theta}(0)=\omega_0$



a)

Let $\vec{x} = \begin{pmatrix} r \\ \dot{r} \\ \theta \\ \dot{\theta} \end{pmatrix}$, $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

$$\boxed{\dot{\vec{x}} = \begin{pmatrix} \dot{r} \\ \ddot{r} \\ \dot{\theta} \\ \ddot{\theta} \end{pmatrix} = \begin{pmatrix} \dot{r} \\ r\dot{\theta}^2 - \frac{k}{r^2} + \\ \dot{\theta} \\ \frac{-2\dot{\theta}\dot{r}}{r} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & \frac{1}{r} \end{pmatrix} \vec{u}}$$

b)

$$\delta\dot{\vec{x}} = \frac{\delta \vec{f}}{\delta \vec{x}}\delta\vec{x} + \cancel{\frac{\delta \vec{f}}{\delta \vec{u}}\delta\vec{u}}^{\,0}, \quad x_n = \begin{pmatrix} r_0 \\ 0 \\ \omega_0 t + c \\ \omega_0 \end{pmatrix}$$

$$\frac{\delta f_1}{\delta \vec{x}} = [0 \;\; 1 \;\; 0 \;\; 0]$$

$$\frac{\delta f_2}{\delta \vec{x}} = \left[\dot{\theta}^2 + 2\frac{k}{r^3} \quad 0 \quad 0 \quad 2r\dot{\theta}\right]_n$$

$$= \left[\omega_0^2 + 2\frac{k}{r_0^3} \quad 0 \quad 0 \quad 2r_0\omega_0\right]$$

$$\frac{\delta f_3}{\delta \vec{x}} = [0 \;\; 0 \;\; 0 \;\; 1]$$

$$\frac{\delta f_4}{\delta \vec{x}} = \left[\frac{2\dot{\theta}\dot{r}}{r^2} - \frac{u_2}{r^2} \quad -\frac{2\dot{\theta}}{r} \quad 0 \quad -\frac{2\dot{r}}{r}\right]_n$$

$$= \left[0 \quad -\frac{2\omega_0}{r_0} \quad 0 \quad 0\right]$$

$$\Rightarrow \boxed{\delta\dot{\vec{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \omega_0^2 + 2\frac{k}{r_0^3} & 0 & 0 & 2r_0\omega_0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{-2\omega_0}{r_0} & 0 & 0 \end{bmatrix} \delta\vec{x}}$$

1. cont

c) $$\left.\frac{\delta \vec{f}}{\delta \vec{u}}\right|_{\begin{subarray}{l}u_n\\x_n\end{subarray}} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & \frac{1}{r} \end{bmatrix}_n = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & \frac{1}{r_0} \end{bmatrix}$$

$$\boxed{\begin{aligned} \dot{\vec{X}} &= \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ \omega_0^2 + 2\frac{k}{r_0^3} & 0 & 0 & 2r_0\omega_0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{2\omega_0}{r_0} & 0 & 0 \end{bmatrix}}_{A} \vec{X} + \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & \frac{1}{r_0} \end{bmatrix}}_{B} \vec{u} \\ \vec{y} &= \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{C} \vec{X} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{D} \vec{u} \end{aligned}}$$

2. Using the axioms found on pp 159-160 of Brogan

a) for even fcns, $f(-t)=f(t)$, so symmetric about the y-axis

1. $\vec{x} \in G, \vec{y} \in G, \vec{z} \in G \Rightarrow \vec{x}+\vec{y}=\vec{z}$

   True, because even functions add to be other even functions

2. $\vec{x}+\vec{y}=\vec{y}+\vec{x} \Rightarrow$ True
3. Associative Addition $\Rightarrow$ True
4. $\exists$ zero-vector $\Rightarrow$ True, 0 is an even fcn
5. $-\vec{x}$ exist for all $\vec{x} \Rightarrow$ True, all $-f(t)$ is even for even $f(t)$
6. scalar Mult $\Rightarrow$ True
7. Associative Mult. of scalars $\Rightarrow$ True
8. Distributive Mult. of scalars $\Rightarrow$ True

$\boxed{\text{Is a vector space}}$

b) $G=\{$all real pairs$\}$, $g_1+g_2=(a_1,a_2)\#(b_1,b_2)=(a_1+2b_1, a_2+3b_2)$

1. True, see addition definition ↑
2. $g_1+g_2=(a_1+2b_1, a_2+3b_2)$

   $g_2+g_1=(b_1+2a_1, b_2+3a_2)$ $\Rightarrow a_1+2b_1=b_1+2a_1 \Rightarrow b_1=a_1$

   False, only true if $a_1=b_1$, $\therefore$ cannot add any two vectors commutatively

$\boxed{\therefore \text{Not a vector space}}$

c) $G=\{$all real pairs$\}$, $c(a_1,a_2)=\begin{cases}(0,0) & \text{if } c=0 \\ (ca_1, \frac{a_2}{c}) & \text{if } c\neq 0\end{cases}$

1. True
2. Commutative Addition: True
3. Associative Addition: True
4. $\exists$ zero vector: True
5. $-\vec{x}$ exist for all $\vec{x}$: True
6. Scalar mult, + unit scalar Mult: True, $1\cdot\vec{x}=1\cdot(a_1,a_2)=(1a_1, \frac{a_2}{1})$
7. Assoc. Mult: $b(c\vec{x})=b\cdot(c\cdot a_1, \frac{a_2}{c})=(b\cdot c\cdot a_1, \frac{a_2}{c\cdot b})$

   $(bc)\vec{x}=(b\cdot c\cdot a_1, \frac{a_2}{bc})$ $\Rightarrow$ True

8. Dist. Mult: $(c+b)\vec{x}=((c+b)a_1, \frac{a_2}{c+b})$

   $c\vec{x}+b\vec{x}=(ca_1, \frac{a_2}{c})+(ba_1, \frac{a_2}{b})=((c+b)a_1, a_2\cdot(\frac{1}{b}+\frac{1}{c}))$ $\neq$ False

$\boxed{\therefore \text{Not a vector space}}$

3. Find basis for $N(A)$, $R(A)$, & dim for both

a) Square matrix, so if $\det(A) \neq 0$ it is L.I. and full rank.
Used Matlab for determinant calculation, see Attached result
$\det(A) = 8.456 \times 10^3$

$\Rightarrow \dim(N(A)) = 0 \Rightarrow N(A) = \vec{0}$
$\dim(R(A)) = 5$

$$R(A) = \text{columns of } A = \left\{ \begin{bmatrix} 1.0000 \\ -10.5092 \\ 4.5354 \\ 0.9451 \\ 1.0000 \end{bmatrix}, \begin{bmatrix} -10.5092 \\ 1.0000 \\ -0.1899 \\ 0.2517 \\ 1.0000 \end{bmatrix}, \begin{bmatrix} 4.5354 \\ -0.1999 \\ 1.0000 \\ -5.3037 \\ 1.0000 \end{bmatrix}, \begin{bmatrix} 0.9451 \\ 0.2517 \\ -5.3037 \\ 1.0000 \\ 1.0000 \end{bmatrix}, \begin{bmatrix} 1.0000 \\ 1.0000 \\ 1.0000 \\ 1.0000 \\ 2.0000 \end{bmatrix} \right\}$$

b) $\dim(R(A))$ = number of pivots in R matrix, derived from QR composition.
$\dim(N(A)) = n - r$ if A is $m \times n$

$A = QR$

See attached Matlab for Q & R matrices
$\text{rank}(A) = 5$, $\therefore$ $\boxed{\dim(R(A)) = 5, \ \dim(N(A)) = 2}$

basis of $R(A)$ = columns of Q. See Attached Matlab result.
basis of $N(A)$: if $A^T = Q_{A^T} R_{A^T}$, Last two columns of Q_{A^T} are the basis of $N(A)$.
See attached Matlab result.

Problem 3a)

```
A = [1.0000   -10.5092    4.5354    0.9451    1.0000
    -10.5092    1.0000   -0.1899    0.2517    1.0000
    4.5354   -0.1899    1.0000   -5.3037      1.0000
    0.9451    0.2517   -5.3037    1.0000    1.0000
    1.0000    1.0000    1.0000    1.0000    2.0000];

determinant_A = det(A)
```

```
determinant_A =

     8.455510510026739e+03
```

Published with MATLAB® R2013b

Problem 3b)

```
Ab = [8     2     6     7     2     3     7
    8     2     3     9     3    10     5
    3     5   8    10     9     4     4
    7    10     3     6     3     2     9
    7     4     6     2     9     3     6];

[Q,R] = qr(Ab)
rank_A = 5;
range_A_basis = Q
[Q_AT, R_AT] = qr(Ab')
null_A_basis = Q_AT(:,rank_A+1:end)
```

```
Q =

  Columns 1 through 3

  -0.521862458442754  -0.380545247324821  -0.193649306515763
  -0.521862458442754  -0.380545247324821   0.262898350739971
  -0.195698421916033   0.408120989884590  -0.796722981003681
  -0.456629651137410   0.736272326345849   0.465978841883819
  -0.456629651137410  -0.041363613839654  -0.203667900567051

  Columns 4 through 5

   0.040645231017152  -0.737355274266145
  -0.543972405989053   0.466715510677822
  -0.383299558592901   0.115988470109281
  -0.005809507187083  -0.179505965645316
   0.745311803694785   0.439099208270851


R =

  Columns 1 through 3

 -15.329709716755893  -9.458757059274912 -10.372016361549733
                   0   7.715692768223541   1.800695989152955
                   0                   0  -6.571055512654960
                   0                   0                   0
                   0                   0                   0

  Columns 4 through 6

 -13.959820763343661  -9.850153903106975  -9.850153903106977
   2.327392672044557   3.606907126817866  -1.966150444511572
  -4.568152548818599  -7.204184969296739  -0.817902354095818
  -6.988464058443356   1.690054928422847  -4.626670204500490
                   0   4.382707191986418   3.877328857938831
```

```
  Column 7

 -13.894587956038318
   3.444210245715226
  -0.256143142373142
   0.451041610288154
  -1.344913927219521


range_A_basis =

  Columns 1 through 3

  -0.521862458442754  -0.380545247324821  -0.193649306515763
  -0.521862458442754  -0.380545247324821   0.262898350739971
  -0.195698421916033   0.408120989884590  -0.796722981003681
  -0.456629651137410   0.736272326345849   0.465978841883819
  -0.456629651137410  -0.041363613839654  -0.203667900567051

  Columns 4 through 5

   0.040645231017152  -0.737355274266145
  -0.543972405989053   0.466715510677822
  -0.383299558592901   0.115988470109281
  -0.005809507187083  -0.179505965645316
   0.745311803694785   0.439099208270851


Q_AT =

  Columns 1 through 3

  -0.545595471576379  -0.022748948018772   0.476196441497620
  -0.136398867894095  -0.005687237004693  -0.306305252708422
  -0.409196603682284  -0.383888497816780  -0.308636819617246
  -0.477396037629331   0.224645861685375  -0.261239892306536
  -0.136398867894095   0.116588358596207  -0.676537198125930
  -0.204598301841142   0.847398313699263   0.101544958805178
  -0.477396037629331  -0.264456520718226   0.218854093874491

  Columns 4 through 6

  -0.015000040151501  -0.344588559642402  -0.230628358575208
  -0.845991224443723   0.107434881549417   0.276429581772348
   0.419300961997411  -0.030223205808153   0.634363698284307
   0.152461244789753   0.698494091515857  -0.374244995506417
   0.037559363258159  -0.594035356643606  -0.386929361997940
   0.049374660876777  -0.149305037519217   0.418703482742359
  -0.284899203508563  -0.075756409410977  -0.053793034333592

  Column 7

  -0.550353794794153
  -0.289626271335198
```

```
  -0.103860645834654
  -0.046725100793018
   0.078082527762489
   0.172342091024591
   0.751304450426275


R_AT =

  Columns 1 through 3

 -14.662878298615180 -15.003875468350419 -14.321881128879944
                   0   8.178246812748597   2.459730004529740
                   0                   0  -9.991668991467277
                   0                   0                   0
                   0                   0                   0
                   0                   0                   0
                   0                   0                   0

  Columns 4 through 5

 -14.390080562826991 -12.480496412309666
  -0.355452312793314  -0.031279803525811
  -2.079862081188851  -4.737221966137789
  -8.744907590816009  -1.891473886369035
                   0  -7.015503221932307
                   0                   0
                   0                   0


null_A_basis =

  -0.230628358575208  -0.550353794794153
   0.276429581772348  -0.289626271335198
   0.634363698284307  -0.103860645834654
  -0.374244995506417  -0.046725100793018
  -0.386929361997940   0.078082527762489
   0.418703482742359   0.172342091024591
  -0.053793034333592   0.751304450426275
```

Published with MATLAB® R2013b

Challenge 1

$M_{m \times n}, m > n, \quad y \in \{\mathbb{R}^m = R(M) \oplus N(M^T)\}$

$x \in \{\mathbb{R}^n = R(M^T) \oplus N(M)\} \quad \dim(N(M)) = 0$

$w \in N(M^T)$

w is a member of a subspace that is not contained by $\mathbb{R}^n$, which x is a member of. The only component of y that can affect x is contained within $R(M)$

Challenge 2

a) Let $\vec{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $A = I_{2\times2} \Rightarrow b = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \Rightarrow [A \mid b] = \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \end{array}\right]$

let $W' = \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 3 & 1 \end{array}\right] \Rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 8 \\ 9 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\boxed{\text{False}}$

b) $\boxed{\text{False}}$ the system of eqns. produces the same solution if any rows are scaled or added, but swapping rows swaps elements of $\vec{x}$

c) $\boxed{\text{True}}$, $\text{Rank}(A_{n\times n}) = n$ iff all columns are L.I., Reduced-REF requires that only the pivot $\underline{1}$ be in the pivot column, and there can only be one pivot per row. A pivot represents an LI column.

d) $\boxed{\text{True}}$, elementary row operations are used to create RREF. A finite # of operations are used.

e) consistent → at least one soln. According to Rouche-Capelli Thm, inconsistent if $\text{Rank}([A \mid b]) > \text{Rank}(A)$. if $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $[A \mid b]$ is RREF but $\text{Rank}(A) = 2$ and $\text{Rank}([A \mid b]) = 3$, $\therefore \boxed{\text{False}}$

f) $A_{m\times n}$, $x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $r = \text{rank}(A) \to Ax = 0$ solns are in $N(A)$. $\dim(N(A)) = n - r$ $\boxed{\text{True}}$

g) $\boxed{\text{True}}$, pivots only exist in non-zero rows, and # pivots = Rank