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Problem 1: $f(x,y) = \begin{cases} k(x^2+y^2) & 0 \le x \le 2,\ 1 \le y \le 3, \\ 0 & \text{elsewhere} \end{cases}$

a) find k.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k(x^2+y^2)\,dx\,dy = 1 = k\int_0^2\int_1^3 (x^2+y^2)\,dy\,dx$$

$$= k\int_0^2 \left(x^2y + \tfrac{1}{3}y^3\right)\Big|_{y=1}^{y=3} dx = k\int_0^2 \left((3-1)x^2 + \tfrac{27-1}{3}\right)dx$$

$$= k\left(\tfrac{2}{3}x^3 + \tfrac{26}{3}x\right)\Big|_0^2 = \left(\tfrac{2}{3}(8-0) + \tfrac{26}{3}(2-0)\right)k = \tfrac{68}{3}k$$

$$\Rightarrow \boxed{k = \tfrac{3}{68}}$$

b) $p(1 < x \le 2,\ 2 < y \le 3) = ?$

$$= k\int_1^2\int_2^3 (x^2+y^2)\,dy\,dx = k\int_1^2 \left((3-2)x^2 + \tfrac{27-8}{3}\right)dx$$

$$= k\left(\tfrac{1}{3}x^3 + \tfrac{19}{3}x\right)\Big|_1^2 = k\left(\tfrac{1}{3}(8-1) + \tfrac{19}{3}(2-1)\right) = k\left(\tfrac{7}{3} + \tfrac{19}{3}\right)$$

$$= \tfrac{36}{68} = \boxed{\tfrac{13}{34}}$$

c) $p(1 \le x \le 2) = ?$

$$= k\int_1^2\int_1^3 (x^2+y^2)\,dy\,dx = k\left(\tfrac{2}{3}x^3 + \tfrac{26}{3}x\right)\Big|_1^2 = k\left(\tfrac{2}{3}(8-1) + \tfrac{26}{3}(2-1)\right)$$

$$= \left(\tfrac{14}{3} + \tfrac{26}{3}\right)k = \tfrac{40}{3}k = \tfrac{40}{68} = \boxed{\tfrac{10}{17}}$$

d) $p(x+y \ge 4) = ?$

$y \ge 4-x$
$1 \ge x \ge 2$ $\Rightarrow$ $k\int_1^2\int_{4-x}^3 (x^2+y^2)\,dy\,dx = k\int_1^2 \left((3-4+x)x^2 + \tfrac{1}{3}\left(27 + (-64 + 3\cdot 16x + -3\cdot 4x^2 + x^3)\right)\right)dx$

$$= k\int_1^2 \left(-x^2 + x^3 - \tfrac{37}{3} + 16x - 4x^2 + \tfrac{1}{3}x^3\right)dx = k\int_1^2 \left(\tfrac{4}{3}x^3 - 5x^2 + 16x - \tfrac{37}{3}\right)dx$$

$$= k\left(\tfrac{1}{3}x^4 - \tfrac{5}{3}x^3 + 8x^2 - \tfrac{37}{3}x\right)\Big|_1^2 = k\left(\tfrac{16-1}{3} - \tfrac{5(8-1)}{3} + 8(4-1) - \tfrac{37}{3}\right)$$

$$= \boxed{\tfrac{15}{68}}$$

cont →

Problem 1 cont:

e) $p(x+y=4)=?$

$$\int_1^2\int_{4-x}^{4-x} f(x,y)\,dy\,dx = \boxed{0}$$

f) $p(x\leq 1/y=3)$

$$k\int_0^1 g(x/y)\,dx = \frac{\int_0^1 f(x,y)\,dx}{h(y)}$$

$$h(y)=\int_0^2 k(x^2+y^2)\,dx = k\left(\tfrac{1}{3}x^3+y^2x\right)_0^2 = k\left(\tfrac{8}{3}+2y^2\right)$$

$$=\frac{k\int_0^1(x^2+y^2)\,dx}{k\left(\frac{8}{3}+2y^2\right)} = \frac{\frac{1}{3}+y^2}{\frac{8}{3}+2y^2} = \frac{\frac{1}{3}+\frac{9\cdot 3}{3}}{\frac{8}{3}+\frac{18\cdot 3}{3}} = \frac{28}{8+54} = \boxed{\frac{14}{31}}$$

g) $\sigma_x=?$

$$\sigma_x^2=\mu_{20}=E[(X-\lambda_{10})^2]=\lambda_{20}-\lambda_{10}^2$$

$$\lambda_{10}=k\int_0^2\int_1^3 x(x^2+y^2)\,dy\,dx = k\int_0^2\left(x^3y+\tfrac{1}{3}y^3x\right)_{y=1}^{y=3}dx = k\int_0^2\left(2x^3+\tfrac{26}{3}x\right)dx = k\left(\tfrac{2}{4}x^4+\tfrac{\overset{13}{\cancel{26}}}{\cancel{2}\cdot 3}x^2\right)_0^2$$

$$= k\left(\tfrac{\overset{8}{\cancel{16}}}{\cancel{2}}+\tfrac{52}{3}\right)=\frac{76}{68}$$

$$\lambda_{20}=k\int_0^2\int_1^3 x^2(x^2+y^2)\,dy\,dx = k\int_0^2\left(x^4y+\tfrac{1}{3}y^3x^2\right)_{y=1}^{y=3}dx = k\int_0^2\left(2x^4+\tfrac{26}{3}x^2\right)dx$$

$$= k\left(\tfrac{2}{5}x^5+\tfrac{26}{9}x^3\right)_0^2 = k\left(\tfrac{64}{5}+\tfrac{\overset{208}{\cancel{26\cdot 8}}}{9}\right)=\frac{404}{255}$$

$$\sigma_x=\sqrt{\frac{404}{255}-\frac{76^2}{68^2}}$$

$$\boxed{\sigma_x=\sqrt{\frac{1453}{4335}}\approx 0.58}$$

h) $p(1<x<2/1<y<2)=?$

$$=\frac{k\int_1^2\int_1^2(x^2+y^2)\,dy\,dx}{k\int_1^2\left(8/3+2y^2\right)dy} = \frac{\int_1^2\left(x^2y+\frac{1}{3}y^3\right)_{y=1}^{y=2}dx}{\left(\frac{8}{3}y+\frac{2y^3}{3}\right)_1^2} = \frac{\int_1^2\left(x^2+\frac{7}{3}\right)dx}{\frac{8}{3}+\frac{14}{3}} = \frac{\left(\frac{x^3}{3}+\frac{7}{3}x\right)_1^2}{\frac{22}{3}}$$

$$=\frac{\frac{7}{3}+\frac{7}{3}}{\frac{22}{3}}=\frac{14}{22}=\boxed{\frac{7}{11}}$$

Problem 2: Show that

$$f(x) = \frac{1}{b\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-a}{b}\right)^2} \qquad -\infty \le x \le \infty$$

is given by

$$M_x(\theta) = e^{\frac{\theta^2 b^2}{2} + a\theta}$$

$$M_x(\theta) = E[e^{\theta x}] = \int_{-\infty}^{\infty} e^{\theta x} \cdot \frac{1}{b\sqrt{2\pi}} e^{-\frac{1}{2b^2}(x-a)^2}$$

$$e^{\theta x} e^{-\frac{1}{2b^2}(x-a)^2} = e^{\theta x - \frac{1}{2b^2}(x^2 - 2ax + a^2)}$$

$$= e^{-\frac{1}{2b^2}(x^2 - 2ax - 2b^2\theta x + a^2)}$$

$$= e^{-\frac{1}{2b^2}(\underbrace{x^2 - 2(a+b^2\theta)x + (a+b^2\theta)^2} - (a+b^2\theta)^2 + a^2)}$$

$$= e^{-\frac{1}{2b^2}(x-(a+b^2\theta))^2 + \frac{1}{2b^2}(\cancel{+a^2} + 2ab^2\theta + b^4\theta^2) \cancel{+a^2})}$$

$$= e^{-\frac{1}{2b^2}(x-(a+b^2\theta))^2 + (a\theta + \frac{b^2\theta^2}{2})}$$

$$= e^{-\frac{1}{2b^2}(x-(a+b^2\theta))^2} \underbrace{e^{(a\theta + \frac{b^2\theta^2}{2})}}_{\text{const}}$$

$$M_x(\theta) = e^{\frac{b^2\theta^2}{2} + a\theta} \underbrace{\int_{-\infty}^{\infty} \frac{1}{b\sqrt{2\pi}} e^{-\frac{1}{2b^2}(x-(a+b^2\theta))^2}}_{N(a+b^2\theta,\, b) \Rightarrow \int_{-\infty}^{\infty} f_N(x) = 1}$$

$$\boxed{M_x(\theta) = e^{\frac{b^2\theta^2}{2} + a\theta}}$$

Problem 3: if x & y are independent, random variables, show that

$$\sigma_{xy}^2 = \sigma_x^2 \sigma_y^2 + \lambda_{10}^2 \sigma_y^2 + \lambda_{01}^2 \sigma_x^2$$

Independence: $f(x,y) = g(x)h(x)$, $g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$, $h(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$

$$\underline{\int_{-\infty}^{\infty} g(x)\,dx = 1,\ \int_{-\infty}^{\infty} h(y)\,dy = 1}$$

$$\sigma_{xy}^2 = E[(xy - E[xy])^2]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(xy - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f(x,y)\,dy\,dx\right)^2 f(x,y)\,dy\,dx$$

$$\lambda_{10} = E[x^1 y^0] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x g(x) h(y)\,dy\,dx$$

$$\underline{= \int_{-\infty}^{\infty} x g(x)\,dx}$$

$$\underline{\lambda_{01} = \int_{-\infty}^{\infty} y h(x)\,dy}$$

$$\sigma_x^2 = E[(x-\lambda_{10})^2] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\lambda_{10})^2 g(x) \underbrace{h(y)\,dy}_{=1}\,dx = \int_{-\infty}^{\infty} (x^2 - 2x\lambda_{10} + \lambda_{10}^2) g(x)\,dx$$

$$= \int_{-\infty}^{\infty} x^2 g(x)\,dx - 2\lambda_{10}\int_{-\infty}^{\infty} x g(x)\,dx + \lambda_{10}^2 \int_{-\infty}^{\infty} g(x)\,dx$$

$$\sigma_x^2 = \int_{-\infty}^{\infty} x^2 g(x)\,dx - 2\lambda_{10}^2 + \lambda_{10}^2 = \underline{\int_{-\infty}^{\infty} x^2 g(x)\,dx - \lambda_{10}^2}$$

$$\underline{\sigma_y^2 = \int_{-\infty}^{\infty} y^2 h(y)\,dy - \lambda_{01}^2}$$

$$E[xy] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, g(x) h(y)\,dy\,dx = \int_{-\infty}^{\infty} x g(x) \int_{-\infty}^{\infty} y h(y)\,dy\,dx = \int_{-\infty}^{\infty} x g(x) \cdot \lambda_{01}\,dx = \underline{\lambda_{10}\lambda_{01}}$$

$$\sigma_{xy}^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x^2y^2 - 2\lambda_{10}\lambda_{01} xy + \lambda_{10}^2\lambda_{01}^2)\, g(x) h(y)\,dy\,dx$$

$$= \int_{-\infty}^{\infty} x^2 g(x) \int_{-\infty}^{\infty} y^2 h(y)\,dy\,dx - 2\lambda_{10}\lambda_{01} \int_{-\infty}^{\infty} x g(x) \int_{-\infty}^{\infty} y h(y)\,dy\,dx + \lambda_{10}^2\lambda_{01}^2 \underbrace{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx}_{1}$$

$$= (\sigma_x^2 + \lambda_{10}^2)(\sigma_y^2 + \lambda_{01}^2) - 2\lambda_{10}\lambda_{01} \cdot \lambda_{10} \cdot \lambda_{01} + \lambda_{10}^2 \lambda_{01}^2$$

$$= \sigma_x^2\sigma_y^2 + \sigma_x^2\lambda_{01}^2 + \sigma_y^2\lambda_{10}^2 + \lambda_{10}^2\lambda_{01}^2 - \lambda_{10}^2\lambda_{01}^2 \Rightarrow \underline{\sigma_{xy}^2 = \sigma_x^2\sigma_y^2 + \lambda_{10}^2\sigma_y^2 + \lambda_{01}^2\sigma_x^2}$$