Advanced Astrodynamics HW4

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Contents

I. Problem 16: Problem Formulation

For a Newtonian formulation of the CR3BP, one may take the N-body problem in an inertial frame and set N = 3:

$$
\begin{aligned}
m_i \ddot{\mathbf{r}}_i &= -\sum_{j=0, j\neq i}^{3} \frac{G m_i m_j}{r_{ji}^3} \mathbf{r}_{ji} \\
n &= 1, 2, 3 \\
\mathbf{r}_{ji} &= \mathbf{r}_i - \mathbf{r}_j
\end{aligned}
\tag{1}
$$

where G is the gravitational constant, m_i is the mass of the ith body, and $\mathbf{r}_i$ is the position vector of body i in an inertial frame. The primary body is $i = 1$, the smaller, secondary body is $i = 2$, and the yet-smaller orbiter is $i = 3$.By using the convention $m_1 > m_2 >> m_3$, the following equations of motion can be found:

$$\ddot{\mathbf{r}}_1 = -\frac{G m_2}{r_{21}^3} \mathbf{r}_{21} \tag{2}$$

$$\ddot{\mathbf{r}}_2 = -\frac{G m_1}{r_{12}^3} \mathbf{r}_{12} \tag{3}$$

$$\ddot{\mathbf{r}}_3 = -\frac{G m_1}{r_{13}^3} \mathbf{r}_{13} - \frac{G m_2}{r_{23}^3} \mathbf{r}_{23} \tag{4}$$

Using this formulation, it is required to integrate the primary and secondary bodies' movement in order to solve for the third body's motion.

The Lagrantian of a system is the sum of the kinetic energy and force potential U:

$$L = \frac{1}{2} \dot{\mathbf{r}}_I \cdot \dot{\mathbf{r}}_I + U \tag{5}$$

where U for a barycentric system is

$$U = \frac{G M_1}{|\mathbf{r} + \nu \mathbf{R}|} + \frac{G M_2}{|\mathbf{r} - (1-\nu)\mathbf{R}|} \tag{6}$$

$$\nu = \frac{M_2}{M_1 + M_2} \tag{7}$$

and $\mathbf{R}$ is the vector from the primary body to the secondary body.

A coordinate system is chosen to be a constantly-rotating frame where the line between the primary and secondary bodies lies only on the x axis; the origin is the system barycenter. For a circular orbit of the secondary body, both the primary and secondary bodies are stationary in this rotating frame. The velocity can be transformed to inertial coordinates by

$$\dot{\mathbf{r}}_I = \dot{\mathbf{r}}_R + \boldsymbol{\omega} \times \mathbf{r}_R \tag{8}$$

and $\mathbf{R}$ lies solely in x.

With the new rotating coordinate frame, the Lagrangian is now

$$L = \frac{1}{2} \left[\dot{\mathbf{r}}_R \cdot \dot{\mathbf{r}}_R + (\boldsymbol{\omega} \times \mathbf{r}) \cdot (\boldsymbol{\omega} \times \mathbf{r}) + 2 \dot{\mathbf{r}}_R \cdot (\boldsymbol{\omega} \times \mathbf{r}) \right] + U \tag{9}$$

In scalar form, the Lagrangian is represented as

$$
\begin{aligned}
L =& \frac{1}{2} \left[\dot{x}^2 + \dot{y}^2 + \dot{z}^2 + n^2 y^2 + n^2 x^2 + 2(-n y \dot{x} + n x \dot{y}) \right] \\
&+ \frac{G M_1}{\sqrt{x^2 + 2\nu R x + \nu^2 R^2 + y^2 + z^2}} + \frac{G M_2}{\sqrt{x^2 - 2(1-\nu) R x + (1-\nu)^2 R^2 + y^2 + z^2}}
\end{aligned}
\tag{10}
$$

The Lagrangian equations of motion are given by

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{\mathbf{q}}} \right) = \frac{\partial L}{\partial \mathbf{q}} \tag{11}$$

where the general coordinates are

$$\mathbf{q} = \mathbf{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{12}$$

$$\dot{\mathbf{q}} = \dot{\mathbf{r}} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} \tag{13}$$

Taking the partials with respect to $\mathbf{q}$ and $\dot{\mathbf{q}}$,

$$\frac{\partial L}{\partial \mathbf{q}} = \begin{matrix} \frac{\partial L}{\partial x} \\ \frac{\partial L}{\partial y} \\ \frac{\partial L}{\partial z} \end{matrix} = \begin{matrix} xn^2 + n\dot{y} - \frac{GM_1}{|\mathbf{r} + \nu\mathbf{R}|^3}(x + \nu R) - \frac{GM_2}{|\mathbf{r} - (1-\nu)\,\mathbf{R}|^3}(x - (1-\nu)R) \\ yn^2 - n\dot{x} - \frac{GM_1}{|\mathbf{r} + \nu\mathbf{R}|^3}y - \frac{GM_2}{|\mathbf{r} - (1-\nu)\,\mathbf{R}|^3}y \\ -\frac{GM_1}{|\mathbf{r} + \nu\mathbf{R}|^3}z - \frac{GM_2}{|\mathbf{r} - (1-\nu)\,\mathbf{R}|^3}z \end{matrix} \tag{14}$$

$$\frac{\partial L}{\partial \dot{\mathbf{q}}} = \begin{matrix} \frac{\partial L}{\partial \dot{x}} \\ \frac{\partial L}{\partial \dot{y}} \\ \frac{\partial L}{\partial \dot{z}} \end{matrix} = \begin{matrix} \dot{x} - ny \\ \dot{y} + nx \\ \dot{z} \end{matrix} \tag{15}$$

The equations of motion in the barycentric rotating frame using the Lagrangian formulation is:

$$\ddot{q} = \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} xn^2 + 2n\dot{y} - \frac{GM_1}{|\mathbf{r}+\nu\mathbf{R}|^3}(x + \nu R) - \frac{GM_2}{|\mathbf{r}-(1-\nu)\mathbf{R}|^3}(x - (1-\nu)R) \\ yn^2 - 2n\dot{x} - \frac{GM_1}{|\mathbf{r}+\nu\mathbf{R}|^3}y - \frac{GM_2}{|\mathbf{r}-(1-\nu)\mathbf{R}|^3}y \\ -\frac{GM_1}{|\mathbf{r}+\nu\mathbf{R}|^3}z - \frac{GM_2}{|\mathbf{r}-(1-\nu)\mathbf{R}|^3}z \end{bmatrix} \tag{16}$$

The Jacobi integral for the Lagrangian is given by

$$J = \frac{\partial L}{\partial \dot{\mathbf{q}}} \cdot \dot{\mathbf{q}} - L \tag{17}$$

For this circular, rotating, barycentric system, the constant Jacobi integral is

$$\begin{aligned} J &= \dot{x}^2 - ny\dot{x} + \dot{y}^2 + nx\dot{y} + \dot{z}^2 - L \\ &= \dot{\mathbf{r}}_R \cdot \dot{\mathbf{r}}_R + 2\dot{\mathbf{r}}_R \cdot (\boldsymbol{\omega} \times \mathbf{r}) - \frac{1}{2}\left[\dot{\mathbf{r}}_R \cdot \dot{\mathbf{r}}_R + (\boldsymbol{\omega} \times \mathbf{r}) \cdot (\boldsymbol{\omega} \times \mathbf{r}) + 2\dot{\mathbf{r}}_R \cdot (\boldsymbol{\omega} \times \mathbf{r})\right] - U \\ &= \frac{1}{2}\dot{\mathbf{r}}_R \cdot \dot{\mathbf{r}}_R - \frac{(\boldsymbol{\omega} \times \mathbf{r}) \cdot (\boldsymbol{\omega} \times \mathbf{r})}{2} - U \end{aligned} \tag{18}$$

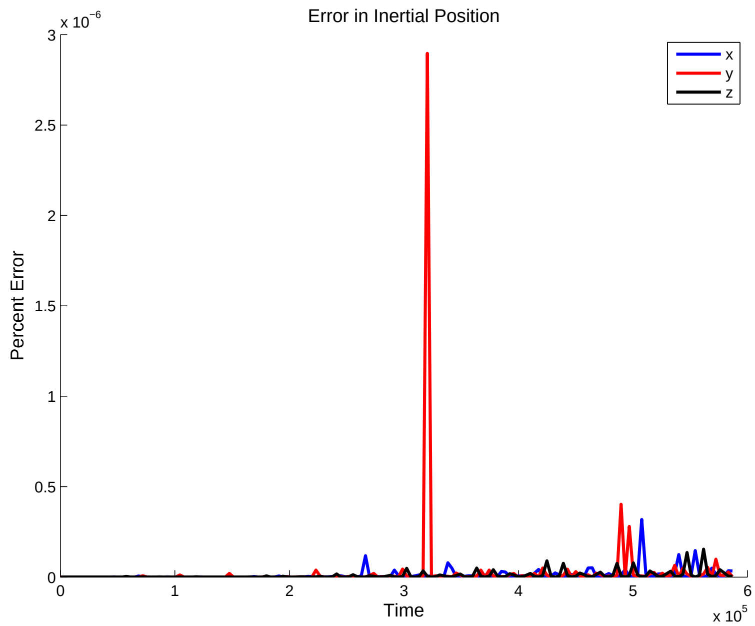
II. Problem 16: Simulation verification

Numerical integration routines using the Newton and Lagrange formulations were created to study the CR3BP. Some test cases were performed to ensure that the integrators had similar solutions, that the 2-body orbit elements were constant when $M_2 = 0$ (except true anomaly), and that the Jacobi integral was conserved. Table 1

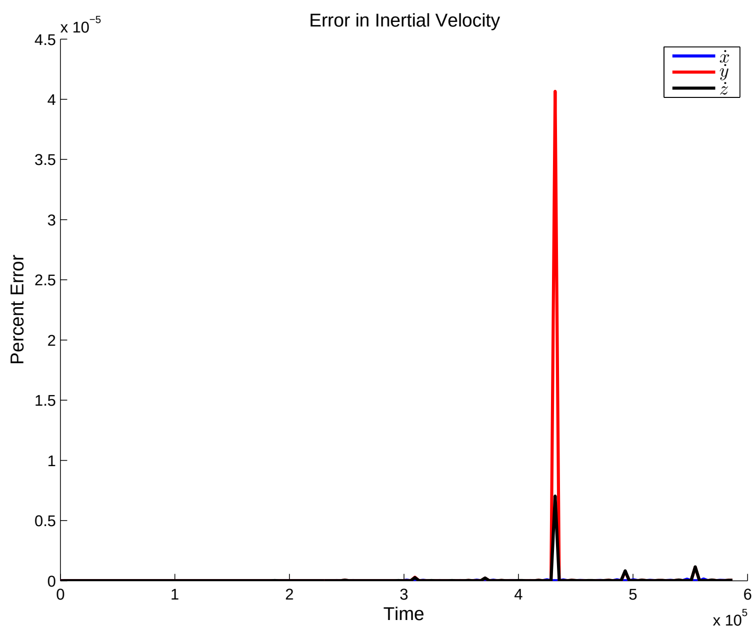
Table 1. Orbit characteristics

Parameter	Value
M_1	5.9742e+24 kg
M_2	7.3483e+22 kg
R	384400 km
a	7000 km
e	0.01
i	30°
Ω	0°
ω	45°

The percent error between the Newton and Lagrange formulations are shown in Figure 1 below:



(a) Position percent error



(b) Velocity percent error

Figure 1. Different integration routines compared.

The Jacobi integral conservation can be seen in Figure 2 below:

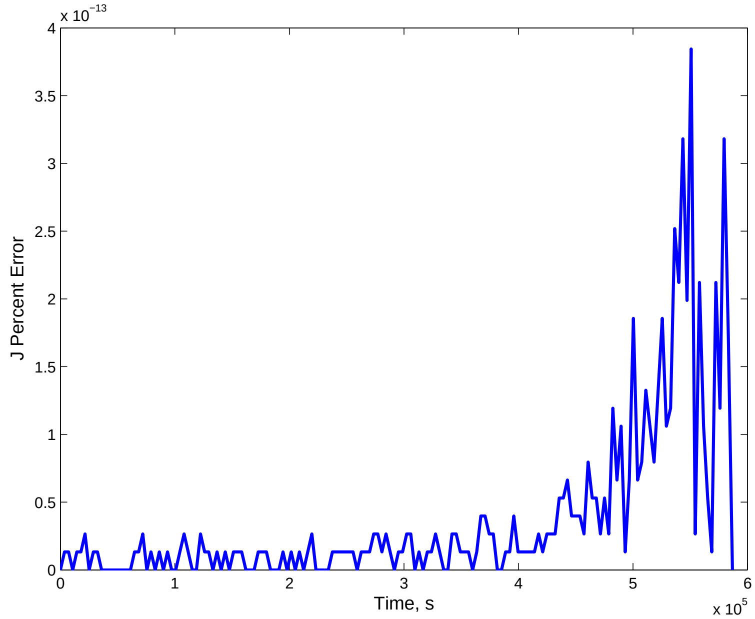
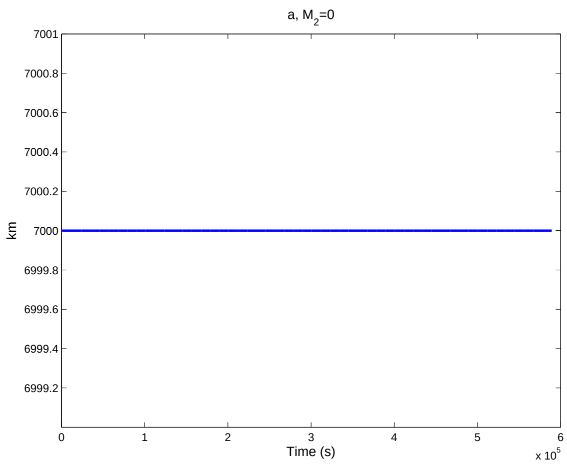
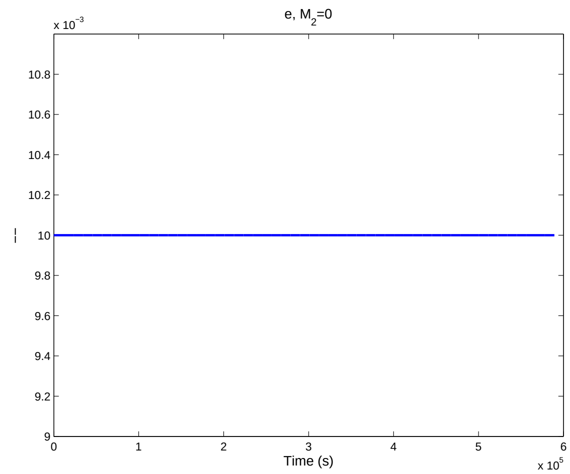


Figure 2. Conservation of the Jacobi integral.

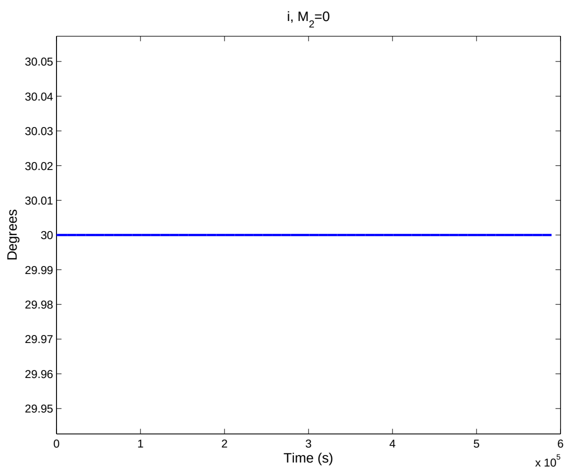
Figure 3 shows the orbital elements when $M_2 = 0$. Figure 4 shows the error from the initial conditions.



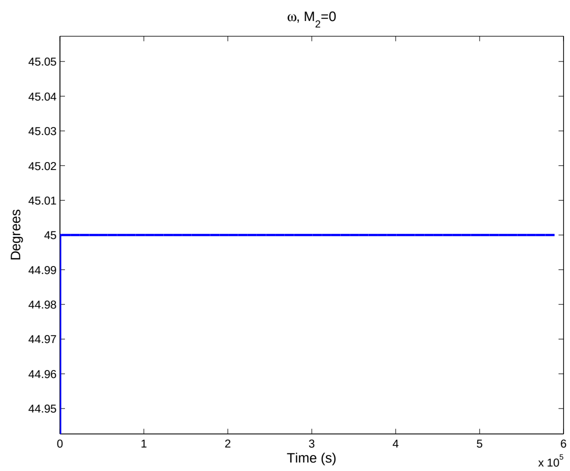
(a) a



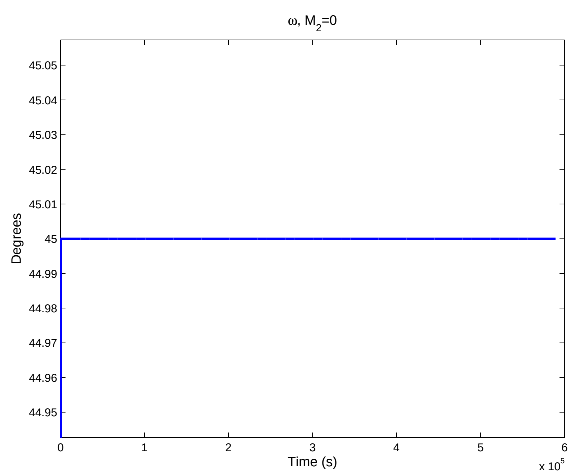
(b) e



(c) i

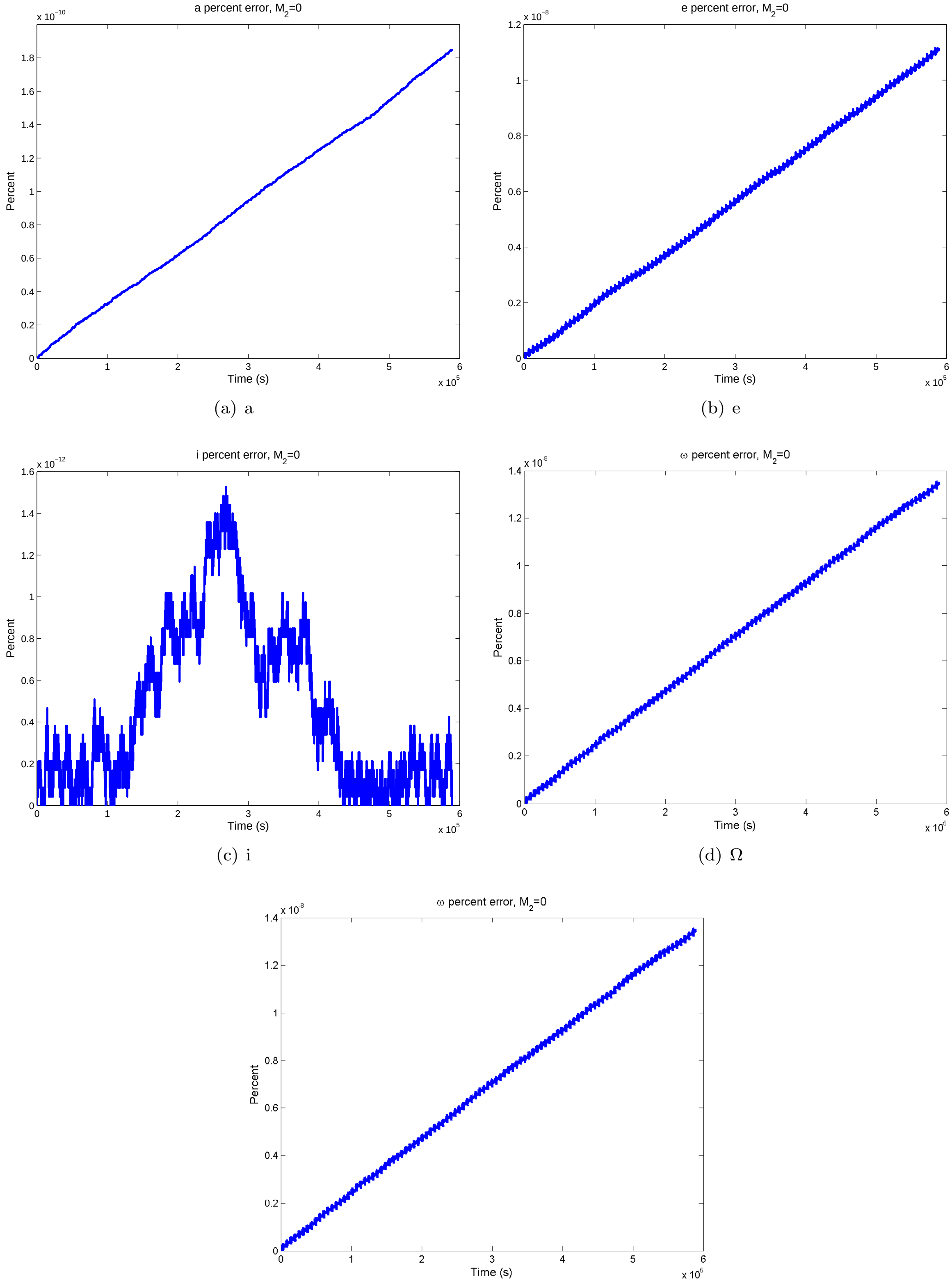


(d) Ω



(e) ω

Figure 3. Numerical integration results when $M_2 = 0$.



(a) a

(b) e

(c) i

(d) Ω

(e) ω

Figure 4. Numerical integration errors with initial conditions when $M_2 = 0$.

References