Problem 1: Brogan 7.30

$$A = \begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix} \qquad A - I\lambda = \begin{bmatrix} 2-\lambda & -2 & 3 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{bmatrix}$$

Characteristic polynomial

$$|A - I\lambda| = (2-\lambda)(1-\lambda)(-1-\lambda) - 3(2-\lambda) + 2(-1-\lambda-1) + 3(3-1+\lambda) = 0$$

$$= (2-\lambda)(-1+0\lambda+\lambda^2) - 6 + 3\lambda - 4 - 2\lambda + 6 + 3\lambda$$

$$= -2 + 2\lambda^2 + \lambda - \lambda^3 - 6 + 3\lambda - 4 - 2\lambda + 6 + 3\lambda = -\lambda^3 + 2\lambda^2 + 5\lambda - 6$$

$\xrightarrow[\text{Matlab roots.m}]{} \boxed{\lambda_i = \{3, -2, 1\}}$

$\lambda = 3$: $A - I \cdot 3 = \begin{bmatrix} -1 & -2 & 3 \\ 1 & -2 & 1 \\ 1 & 3 & -4 \end{bmatrix} \xrightarrow[QR]{} Q = \left[\, q_1 \;\middle|\; q_2 \;\middle|\; \begin{matrix} 0.5774 \\ 0.5774 \\ 0.5774 \end{matrix} \,\right]$

e'vec for $\lambda = 3$

Because R shows rank = 2

$\lambda = -2$: $A - I \cdot 2 = \begin{bmatrix} 4 & -2 & 3 \\ 1 & 3 & 1 \\ 1 & 3 & 1 \end{bmatrix} \xrightarrow[QR]{} Q = \left[\, q_1 \;\middle|\; q_2 \;\middle|\; \begin{matrix} -0.6168 \\ -0.0561 \\ 0.7851 \end{matrix} \,\right]$

e'vec for $\lambda = -2$

Because R shows rank = 2

$\lambda = 1$: $A - I = \begin{bmatrix} 1 & -2 & 3 \\ 1 & 0 & 1 \\ 1 & 3 & -2 \end{bmatrix} \xrightarrow[QR]{} Q = \left[\, q_1 \;\middle|\; q_2 \;\middle|\; \begin{matrix} -0.5774 \\ 0.5774 \\ 0.5774 \end{matrix} \,\right]$

e'vec for $\lambda = 1$

Because R shows rank = 2

$g = m$, $\therefore$ diagonalizable

Jordan form: $\begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Eigenspaces

$\lambda = 3$: $\begin{bmatrix} 0.5774 \\ 0.5774 \\ 0.5774 \end{bmatrix}$ $\quad \lambda = -2$: $\begin{bmatrix} -0.6168 \\ -0.0561 \\ 0.7851 \end{bmatrix}$ $\quad \lambda = 1$: $\begin{bmatrix} -0.5774 \\ 0.5774 \\ 0.5774 \end{bmatrix}$

$\boxed{\text{A is Simple}}$, since eigen vecs are real.

Problem 1 cont:

if $E = \{ \underset{\substack{\uparrow \\ \text{e'vec for } \lambda_i}}{v_i} \}$

$$x(t) = E e^{\Lambda t} E^{-1} x(t_0 = 0)$$

$$= E e^{\Lambda t} E^{-1} x_0$$

where $e^{\Lambda t} = \begin{bmatrix} e^{3t} & 0 & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & e^{1t} \end{bmatrix}$, $E = \begin{bmatrix} 0.5774 & -0.6168 & -0.5774 \\ 0.5774 & -0.0561 & 0.5774 \\ 0.5774 & 0.7851 & 0.5774 \end{bmatrix}$

$$\boxed{x(t) = \begin{bmatrix} 0.5e^{3t} + 0.5e^{t} & 0.1e^{3t} + 0.7333e^{-2t} - 0.8333e^{t} & 0.4e^{3t} - 0.7333e^{-2t} + 0.3333e^{t} \\ 0.5e^{3t} - 0.5e^{t} & 0.1e^{3t} + 0.0667e^{-2t} \;\; 0.8333e^{t} & 0.4e^{3t} - 0.0667e^{-2t} - 0.3333e^{t} \\ 0.5e^{3t} - 0.5e^{t} & 0.1e^{3t} - 0.9333e^{-2t} \;\; 0.8333e^{t} & 0.4e^{3t} + 0.9333e^{-2t} - 0.3333e^{t} \end{bmatrix} \vec{X}_0}$$

Problem 2: $A=\begin{bmatrix}1 & 2 & 3\\ -2 & 3 & -4\\ 1 & 1 & -4\end{bmatrix}$ $\quad A-I\lambda=\begin{bmatrix}1-\lambda & 2 & 3\\ -2 & 3-\lambda & -4\\ 1 & 1 & -4-\lambda\end{bmatrix}$

$$|A-I\lambda|=(1-\lambda)(3-\lambda)(-4-\lambda)+4-4\lambda-2(+8+2\lambda+4)+3(-2-3+\lambda)=0$$
$$=(1-\lambda)(-12+\lambda+\lambda^2)+4-4\lambda-24-4\lambda-15+3\lambda=0$$
$$=-12+\lambda+\lambda^2+12\lambda-\lambda^2-\lambda^3+4-4\lambda-24-4\lambda-15+3\lambda=-\lambda^3+0\lambda^2+8\lambda-47$$

$\xrightarrow[\text{Matlab roots.m}]{}$ $\boxed{\lambda_i=\{4.3395,\ 2.1697+2.4745j,\ 2.1697-2.4745j\}}$

$\lambda=4.3395$: $A-I\cdot 4.3395=\begin{bmatrix}5.3395 & 2 & 3\\ -2 & 7.3395 & -4\\ 1 & 1 & 0.3395\end{bmatrix}\xrightarrow[QR(^T)]{}Q=\left[q_1 \;\middle|\; q_2 \;\middle|\; \boxed{\begin{matrix}0.5479\\ 0.2803\\ 0.7882\end{matrix}}\right]$ Because R shows rank=2

e'vec

$\lambda=2.1697+2.4745j$:

$$A-I\cdot\lambda=\begin{bmatrix}-1.1697-2.4745j & 2 & 3\\ -2 & 0.8303-2.4745j & -4\\ 1 & 1 & -6.1697-2.4745j\end{bmatrix}$$

$\xrightarrow[QR([A-I\lambda]^T)]{}$ $Q=\left[q_1 \;\middle|\; q_2 \;\middle|\; \boxed{\begin{matrix}-0.6529+0.0038j\\ -0.1566-0.7229j\\ -0.1533-0.0551j\end{matrix}}\right]$ Because R shows rank=2

e'vec

$\lambda=2.1697-2.4745j$: e'vec is the complement of the e'vec of the complementary e'val:

$$\boxed{\text{e'vec}=\begin{bmatrix}-0.6529-0.0038j\\ -0.1566+0.7229j\\ -0.1533+0.0551j\end{bmatrix}}\qquad \boxed{J=\begin{bmatrix}4.3395 & 0 & 0\\ 0 & 2.1697+2.4745j & 0\\ 0 & 0 & 2.1697-2.4745j\end{bmatrix}}$$

b) $\boxed{\text{The eigenspaces of } \lambda_i \text{ are their unique eigenvectors}}$

c) $\boxed{\text{A is semisimple}}$

$$\boxed{F=G^{-1}AG=\begin{bmatrix}4.3395 & 0 & 0\\ 0 & 2.1697 & 2.4745\\ 0 & -2.4745 & 2.1697\end{bmatrix}}$$

d) $x(t)=Ee^{Jt}Ex_0$, where $e^{Jt}=\begin{bmatrix}4.3395t & 0 & 0\\ 0 & (2.1697+2.4745j)t & 0\\ 0 & 0 & (2.1697-2.4745j)t\end{bmatrix}$

$$E=\begin{bmatrix}-0.5479 & -0.6529+0.0038j & -0.6529-0.0038j\\ 0.2803 & 0.1566-0.7229j & -0.1566+0.7229j\\ 0.7882 & -0.1533-0.0551j & -0.1533+0.0551j\end{bmatrix}$$

cont. $\longrightarrow$

Problem 2 cont.

$$
X(t)=\left[\begin{array}{l}
0.1339e^{4.3395t}+(0.4331-0.0603j)e^{(2.1697+2.4745j)t}\ (0.4331+0.0603)e^{(2.1697-2.4745j)t} \\
-0.0685e^{4.3395t}+(0.0342+0.4942j)e^{(2.1697+2.4745j)t}+(0.0342-0.4942j)e^{(2.1697-2.4745j)t} \\
-0.1026e^{4.3395t}+(0.0963+0.0513j)e^{(2.1697+2.4745j)t}+(0.0963-0.0513j)e^{(2.1697-2.4745j)t}
\end{array}\right. \quad \ldots
$$

$$
\begin{array}{l}
0.0479e^{4.3395t}+(-0.0239-0.4671j)e^{(2.1697+2.4745j)t}+(-0.0239+0.4671j)e^{(2.1697-2.4745j)t} \\
-0.0245e^{4.3395t}+(0.5122-0.1356j)e^{(2.1697+2.4745j)t}+(0.5122+0.1356j)e^{(2.1697-2.4745j)t} \\
-0.0689e^{4.3395t}+(0.0344-0.1115j)e^{(2.1697+2.4745j)t}+(0.0344+0.1115j)e^{(2.1697-2.4745j)t}
\end{array} \quad \ldots
$$

$$
\left.\begin{array}{l}
-0.6190e^{4.3395t}+(0.3095+0.2080j)e^{(2.1697+2.4745j)t}+(0.3095-0.2080j)e^{(2.1697-2.4745j)t} \\
0.3167e^{4.3395t}+(-0.1584+0.3917j)e^{(2.1697+2.4745j)t}+(-0.1584-0.3917j)e^{(2.1697-2.4745)t} \\
0.8906e^{4.3395t}+(0.0547+0.0753j)e^{(2.1697+2.4745j)t}+(0.0547-0.0753j)e^{(2.1697-2.4745)t}
\end{array}\right] X_0
$$

Problem 3: $A=\begin{bmatrix}3&-1&0&1\\0&2&0&0\\0&0&2&0\\1&-1&0&3\end{bmatrix}$ $A-I\lambda=\begin{bmatrix}3-\lambda&-1&0&1\\0&2-\lambda&0&0\\0&0&2-\lambda&0\\1&-1&0&3-\lambda\end{bmatrix}$

$$|A-I\lambda|=(3-\lambda)(2-\lambda)(2-\lambda)(3-\lambda)--1(0)+0+1(+(2-\lambda)(-2+\lambda)=0$$
$$=(9-6\lambda+\lambda^2)(4-4\lambda+\lambda^2)+-4+4\lambda-\lambda^2=36-24\lambda+4\lambda^2-36\lambda+24\lambda^2-4\lambda^3+9\lambda^2-6\lambda^3+\lambda^4$$
$$-4+4\lambda-\lambda^2=0$$
$$=\lambda^4-10\lambda^3+36\lambda^2-56\lambda+32$$

Matlab roots.m $\rightarrow$ $\boxed{\lambda_i=\{4,2,2,2\}}$

$\lambda=4$: $A-I\cdot4=\begin{bmatrix}-1&-1&0&1\\0&-2&0&0\\0&0&-2&0\\1&-1&0&-1\end{bmatrix}\xrightarrow[QR]{}Q=\left[q_1\;\middle|\;q_2\;\middle|\;q_3\;\middle|\;\boxed{\begin{matrix}0.7071\\0\\0\\0.7071\end{matrix}}\right]$ Because R shows rank = 3

$\uparrow v_1$

$\lambda=2$: $A-I\cdot2=\begin{bmatrix}1&-1&0&1\\0&0&0&0\\0&0&0&0\\1&-1&0&1\end{bmatrix}\rightarrow Q=\left[q_1\;\middle|\;\begin{matrix}0.5774\\0.7887\\0\\0.2113\end{matrix}\;\middle|\;\begin{matrix}0\\0\\1\\0\end{matrix}\;\middle|\;\begin{matrix}-0.5774\\0.2113\\0\\0.7887\end{matrix}\right]$ Because R shows rank = 1

$\uparrow v_2$ $\uparrow v_3$ $\uparrow v_4$

$\boxed{N_{\lambda=4}=\{v_1\},\; N_{\lambda=2}=\{v_2,v_3,v_4\}}$

$\boxed{J=\begin{bmatrix}4&0&0&0\\0&2&1&0\\0&0&2&1\\0&0&0&2\end{bmatrix}}$

$\boxed{A \text{ is simple}}$

$x(t)=Ee^{\Lambda t}E^{-1}$ where: $e^{\Lambda t}=\begin{bmatrix}e^{4t}&0&0&0\\0&e^{2t}&0&0\\0&0&e^{2t}&0\\0&0&0&e^{2t}\end{bmatrix}$ $\Rightarrow X(t)=0.5e^{4t}+0.5e^{2t}$

$E=[v_1\;v_2\;v_3\;v_4]$

$$\boxed{x(t)=\begin{bmatrix}0.5e^{4t}+0.5e^{2t}&-0.5e^{4t}+0.5e^{2t}&0&0.5e^{4t}-0.5e^{2t}\\0&e^{2t}&0&0\\0&0&e^{2t}&0\\0.5e^{4t}-0.5e^{2t}&-0.5e^{4t}+0.5e^{4t}&0&0.5e^{4t}+0.5e^{2t}\end{bmatrix}X_0}$$

Problem 4a: if $A^r = 0$ for $r > 1$, all λ_i's $= 0$

$$Av = \lambda v$$

$$A^2 v = AAv = A\lambda v = \lambda A v = \underline{\lambda^2 v}$$

$$A^3 v = \lambda^3 v$$

$$\vdots$$

$A^r v = \lambda^r v$, so all v's are shared for all powers.

for non-zero eigenvec, $0v = \lambda^r v \Rightarrow \lambda^r = 0$

$\Rightarrow \underline{\lambda = 0}$ only

Problem 4b) Prove A is invertable iff $\lambda = 0$ is not an eigenvalue of A.

A is only invertable if $\det(A) \neq 0$

Eigen vals satisfy the characteristic eqn $\det(A - \lambda I) = 0$

if $\lambda = 0$, $\det(A - \lambda I) = \underline{\det(A) = 0}$

$\therefore \lambda \neq 0$ for A to be invertable.

Challenge I) $Ax = \lambda x$

Try letting λ be complex. it has a complex conjugate $\bar{\lambda}$ & conjugate e'vec.

$A\bar{x} = \bar{\lambda}\bar{x}$

$\Rightarrow x^T A \bar{x} = \bar{\lambda} x^T \bar{x}$

$\bar{x}^T A x = \lambda \bar{x}^T x$ $\Rightarrow x^T A \bar{x} - \bar{x}^T A x = \bar{\lambda} x^T \bar{x} - \lambda \bar{x}^T x = (\bar{\lambda} - \lambda)\bar{x}^T x = 0$

if $x \neq 0$, $\bar{\lambda} = \lambda$, $\therefore$ λ <u>cannot be complex</u>