Problem 8: SSJ 3.17

$$[C] = \begin{bmatrix} e_1^2\Sigma + c\phi & e_1e_2\Sigma + e_3 s\phi & e_1e_3\Sigma - e_2 s\phi \\ e_2e_1\Sigma - e_3 s\phi & e_2^2\Sigma + c\phi & e_2e_3\Sigma + e_1 s\phi \\ e_3e_1\Sigma + e_2 s\phi & e_3e_2\Sigma - e_1 s\phi & e_3^2\Sigma + c\phi \end{bmatrix}$$

where $\Sigma = 1 - c\phi$

$$[C] = \begin{bmatrix} \beta_0^2+\beta_1^2-\beta_2^2-\beta_3^2 & 2(\beta_1\beta_2+\beta_0\beta_3) & 2(\beta_1\beta_3-\beta_0\beta_2) \\ 2(\beta_1\beta_2-\beta_0\beta_3) & \beta_0^2-\beta_1^2+\beta_2^2-\beta_3^2 & 2(\beta_2\beta_3+\beta_0\beta_1) \\ 2(\beta_1\beta_3+\beta_0\beta_2) & 2(\beta_2\beta_3-\beta_0\beta_1) & \beta_0^2-\beta_1^2-\beta_2^2+\beta_3^2 \end{bmatrix}$$

where $\beta_0 = \cos\frac{\phi}{2}$, $\beta_i = e_i \sin\frac{\phi}{2}$

$s\phi = 2\sin(\frac{\phi}{2})\cos(\frac{\phi}{2})$

$c\phi = 2\cos^2(\frac{\phi}{2}) - 1$

$$C_{ii} = e_i^2 - e_i^2 c\phi + c\phi = e_i^2 - e_i^2(2\cos^2(\tfrac{\phi}{2}) - 1) + 2\cos^2(\tfrac{\phi}{2}) - 1$$

$$= e_i^2 - 2e_i^2\cos^2(\tfrac{\phi}{2}) + e_i^2 + 2\cos^2(\tfrac{\phi}{2}) - 1$$

$$= \cancel{2e_i^2} - \cancel{2e_i^2} + 2e_i^2\underbrace{\sin^2\tfrac{\phi}{2}}_{\beta_i^2} + \underbrace{2\cos^2\tfrac{\phi}{2}}_{\beta_0^2} + (-e_1^2 - e_2^2 - e_3^2)\underbrace{(\sin^2\tfrac{\phi}{2} + \cos^2\tfrac{\phi}{2})}_{1\,\beta_0^2}$$

$$= 2\beta_i^2 + 2\beta_0^2 - \beta_1^2 - \beta_2^2 - \beta_3^2 \underbrace{- e_1^2\beta_0^2 - e_2^2\beta_0^2 - e_3^2\beta_0^2}_{-1\,\beta_0^2}$$

$$= 2\beta_i^2 + \beta_0^2 - \beta_1^2 - \beta_2^2 - \beta_3^2$$

$\Rightarrow$

$C_{11} = 2\beta_1^2 + \beta_0^2 - \beta_1^2 - \beta_2^2 - \beta_3^2 = \beta_0^2 + \beta_1^2 - \beta_2^2 - \beta_3^2$

$C_{22} = 2\beta_2^2 + \beta_0^2 - \beta_1^2 - \beta_2^2 - \beta_3^2 = \beta_0^2 - \beta_1^2 + \beta_2^2 - \beta_3^2$

$C_{33} = 2\beta_3^2 + \beta_0^2 - \beta_1^2 - \beta_2^2 - \beta_3^2 = \beta_0^2 - \beta_1^2 - \beta_2^2 + \beta_3^2$

opposite sign as corresponding element from vector

$$C_{ij} = e_i e_j \Sigma \pm e_k s\phi = e_i e_j - e_i e_j c\phi \pm e_k s\phi = e_i e_j - 2e_i e_j \cos^2(\tfrac{\phi}{2}) + e_i e_j \pm e_k 2\sin\tfrac{\phi}{2}\cos\tfrac{\phi}{2}$$

$$= \cancel{2e_i e_j} - \cancel{2e_i e_j} + 2e_i e_j \sin^2(\tfrac{\phi}{2}) \pm 2e_k \sin\tfrac{\phi}{2}\cos\tfrac{\phi}{2} = \underline{2(\beta_i\beta_j \pm \beta_0\beta_k)}$$

$C_{12} = 2(\beta_1\beta_2 + \beta_0\beta_3)$
$C_{13} = 2(\beta_1\beta_3 - \beta_0\beta_2)$
$C_{21} = 2(\beta_2\beta_1 - \beta_0\beta_3)$
$C_{23} = 2(\beta_2\beta_3 + \beta_0\beta_1)$
$C_{31} = 2(\beta_3\beta_1 + \beta_0\beta_2)$
$C_{32} = 2(\beta_3\beta_2 - \beta_0\beta_1)$