Advanced Astrodynamics HW3

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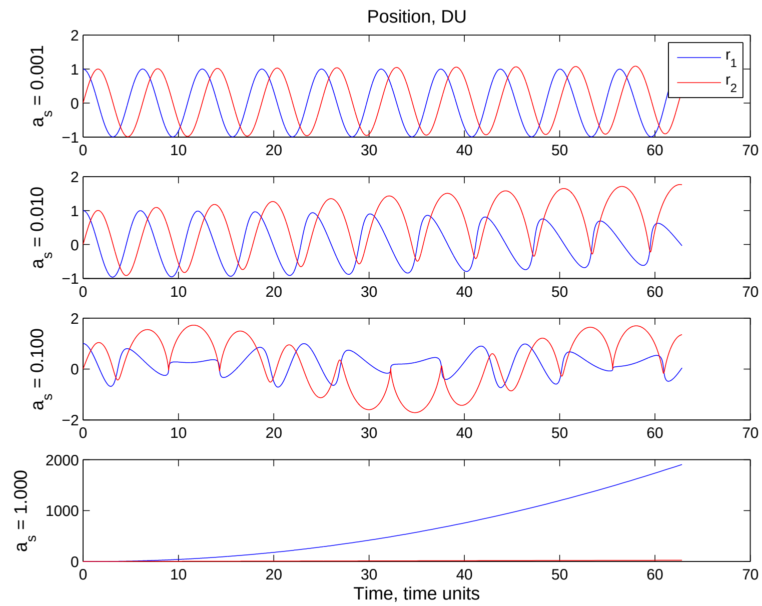
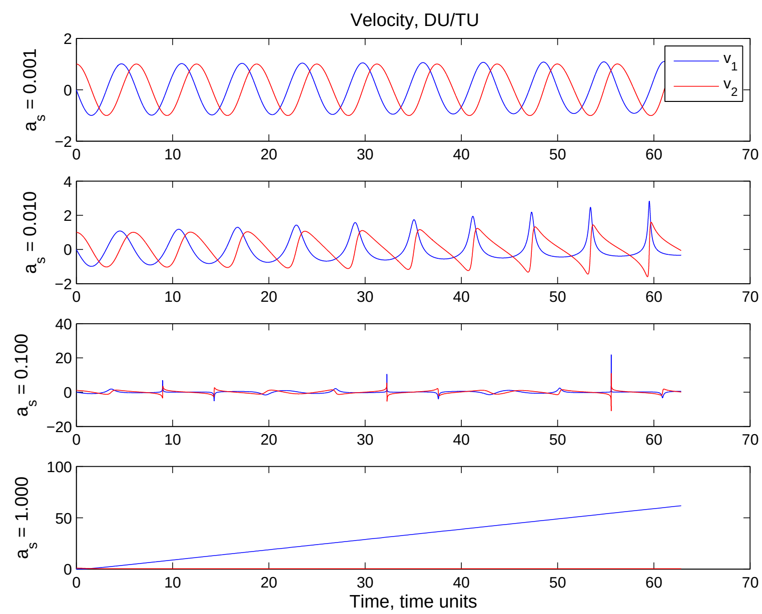
Contents

I. Problem 14

Due to the given initial conditions, there will be no z-component to the motion. This means that the right-ascention, the argument of periapsis, and true anomaly are not well-defined; they are combined into the longitude of periapsis.

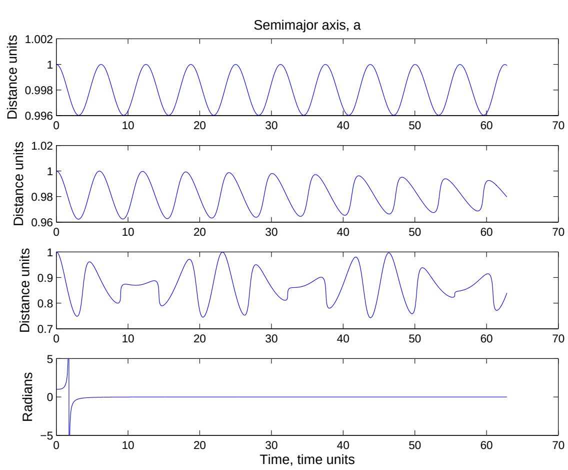


(a) Position

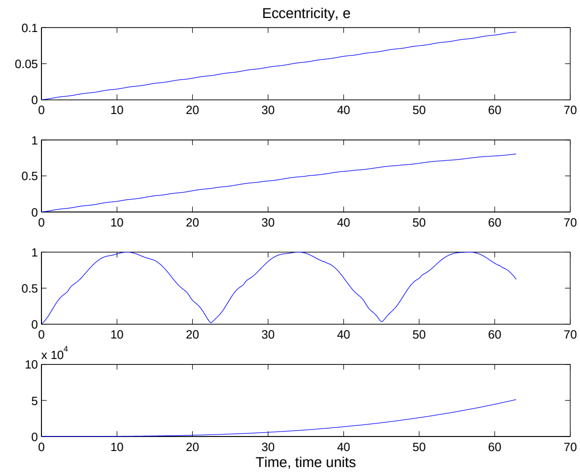


(b) Velocity

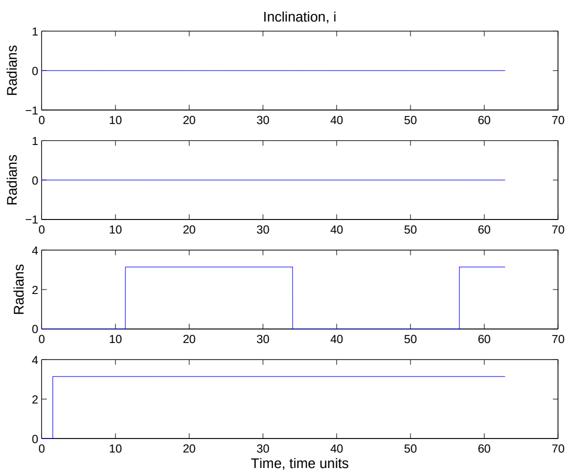
Figure 1. Simulation results, cartesian elements



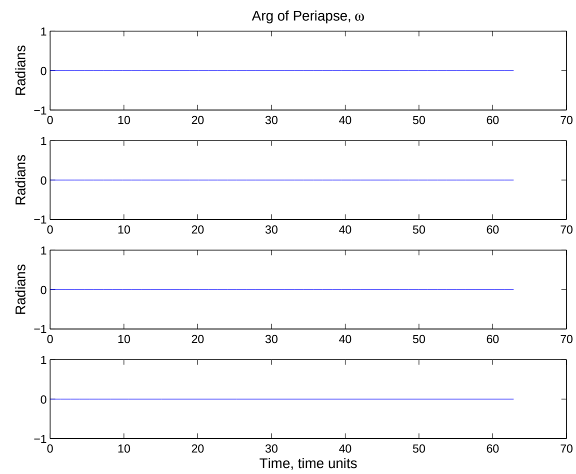
(a) Semimajor axis



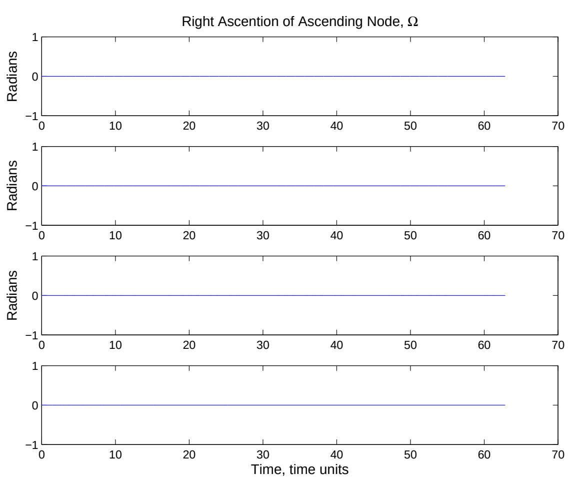
(b) Eccentricity



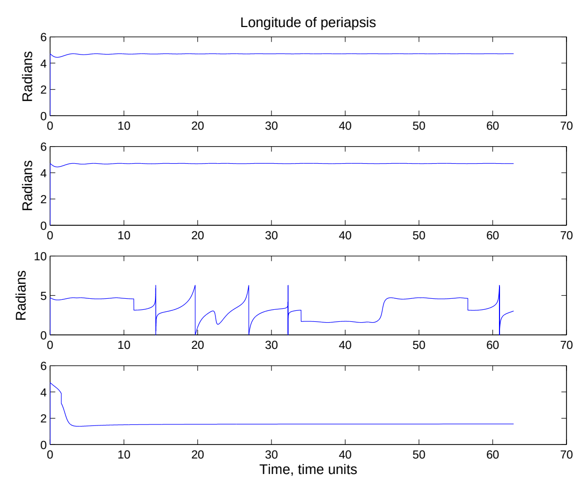
(c) Inclination



(d) Argument of periapsis



(e) RAAN



(f) Longitude of periapsis

Figure 2. Simulation results, orbital elements

For the cartesian elements, the third components were zero due to the initial conditions and the perturbation acceleration. In the lowest acceleration case, the case stays close to circular and is evident to the components being about 90° out of phase. The second case has even more pronounced effects that take the motion from sinusoidal to almost a sine wave. The third case has an acceleration large enough to take the orbiter close to the origin; the inclination plot shows in this case that the orbit flips between retrograde and prograde. In the final case, the orbiter escapes and the x component grows significantly due to the perturbation acceleration in that direction.

For the orbital elements, one can see the first two cases are prograde with increasing eccentricity. Interestingly, the osculating longitude of periapsis for both cases converges to some value, instead of varying sinusoidally between 0 and 2π. This means the osculating orbit is rotating with the orbiter. For the third case, the orbit switches between prograde and retrograde when the eccentricy hits 1. The perturbation acceleration is just enough to reverse to orbit direction, but not enough to cancel out the acceleration due to the primary body. Finally, the last case shows a transition to a hyperbolic orbit. There is a discontinuity in a when the transition happens, going to infinity (parabolic trajectory) and then negative (hyperbolic). The $e < 1$ confirms this result.

II. Problem 15

Many of the derivations in this problem include integrals of trigonomic functions of the eccentric anomaly E. They are derived here to make subsequent derivations cleaner.

$$\int_0^{2\pi} \cos E dE = \sin E|_0^{2\pi} = \sin 2\pi - \sin 0 = 0 - 0 = 0 \tag{1}$$

$$\int_0^{2\pi} \sin E dE = -\cos E|_0^{2\pi} = -(\cos 2\pi - \cos 0) = -1 + 1 = 0 \tag{2}$$

For the integral of $\cos^2 E$, the substitution $u = 2E$ is used:

$$\int_0^{2\pi} \cos^2 E dE = \int_0^{2\pi} \frac{1 + \cos 2E}{2} dE = \int_0^{2\pi} \frac{1}{2} dE + \int_0^{\pi} \frac{1}{4} \cos u du = \left[\frac{E}{2} + \frac{\sin 2E}{4}\right]_0^{2\pi} = \pi \tag{3}$$

For the integral of $\cos E \sin E$, the substitution $u = \cos E$ is used:

$$\int_0^{2\pi} \cos E \sin E dE = -\int_{E=0}^{E=2\pi} u du = -\frac{u^2}{2}\bigg|_{E=0}^{E=2\pi} = -\frac{\cos^2 E}{2}\bigg|_{E=0}^{E=2\pi} = -\frac{(1-1)}{2} = 0 \tag{4}$$

And finally, the simplest integral of them all:

$$\int_0^{2\pi} dE = E\bigg|_0^{2\pi} = 2\pi \tag{5}$$

A.

For R in terms of E, separate ω and f and convert to the eccentric anomaly:

$$R = \frac{a(1-e^2)}{1 + e\cos f} \sin i(\sin\omega \cos f + \cos\omega \sin f) a_s \tag{6}$$

$$R = a(1 - e\cos E)\sin i(\sin\omega \frac{\cos E - e}{(1 - e\cos E)} + \cos\omega \frac{\sqrt{1-e^2}\sin E}{(1 - e\cos E)}) a_s \tag{7}$$

$$R = a \sin i(-e\sin\omega + \sin\omega \cos E + \sqrt{1-e^2}\cos\omega \sin E) a_s \tag{8}$$

The averaging operator is defined as:

$$\bar{g} = \frac{1}{2\pi} \int_0^{2\pi} g(M) dM = \frac{1}{2\pi} \int_0^{2\pi} g(E)(1 - e\cos E) dE \tag{9}$$

Applying the operator to the potential function for the mean potential over one orbit:

$$\bar{R} = \frac{1}{2\pi} \int_0^{2\pi} a \sin i(-e\sin\omega + \sin\omega \cos E + \sqrt{1-e^2}\cos\omega \sin E) a_s (1 - e\cos E) dE \tag{10}$$

$$\bar{R} = \frac{a a_s \text{sin}i}{2\pi} \int_0^{2\pi} (-e\text{sin}\omega + \text{sin}\omega\text{cos}E + \sqrt{1-e^2}\text{cos}\omega\text{sin}E + e^2\text{sin}\omega\text{cos}E - e\text{sin}\omega\text{cos}^2 E - e\sqrt{1-e^2}\text{cos}\omega\text{sin}E\text{cos}E)dE \tag{11}$$

Substituting the integrals from Eqns 1, 2, 3, 4, and 5:

$$\bar{R} = \frac{a a_s \text{sin}i}{2\pi} \left(-e\text{sin}\omega[2\pi + \pi]\right) = -\frac{3}{2} ae\text{sin}\omega\text{sin}i\, a_s \tag{12}$$

B.

The Lagrange equations for a and e are generally:

$$\dot{a} = \frac{2}{na}\frac{\partial R}{\partial \sigma} \tag{13}$$

$$\dot{e} = \frac{1}{na^2 e}\left[(1-e^2)\frac{\partial R}{\partial \sigma} - \sqrt{1-e^2}\frac{\partial R}{\partial \omega}\right] \tag{14}$$

The quantity $\frac{\partial R}{\partial \sigma}$ needs to be found by use of the chain rule, so that $\frac{\partial R}{\partial \sigma} = \frac{\partial R}{\partial f}\frac{\partial f}{\partial E}\frac{\partial E}{\partial \sigma}$. Start with $\frac{\partial E}{\partial \sigma}$ by taking the relation between M and E

$$M = nt + \sigma = E - e\text{sin}E \tag{15}$$

and taking the partial derivative with respect to σ:

$$1 = \frac{\partial E}{\partial \sigma} - e\frac{\partial E}{\partial \sigma}\text{cos}E \tag{16}$$

$$\frac{\partial E}{\partial \sigma} = \frac{1}{1 - e\text{cos}E} = \frac{a}{r} = \frac{1 + e\text{cos}f}{1-e^2} \tag{17}$$

For $\frac{\partial f}{\partial E}$ take the relation between E and f

$$\text{tan}\frac{1}{2}E = \sqrt{\frac{1-e}{1+e}}\text{tan}\frac{1}{2}f \tag{18}$$

and take the partial derivative with respect to E:

$$\frac{1}{2}\text{sec}^2\frac{1}{2}E = \frac{\partial f}{\partial E}\sqrt{\frac{1-e}{1+e}}\frac{1}{2}\text{sec}^2\frac{1}{2}f \tag{19}$$

Solve for $\frac{\partial f}{\partial E}$:

$$\frac{\partial f}{\partial E} = \sqrt{\frac{1+e}{1-e}}\frac{\text{sec}^2\frac{1}{2}E}{\text{sec}^2\frac{1}{2}f} = \sqrt{\frac{1+e}{1-e}}\frac{\text{cos}^2\frac{1}{2}f}{\text{cos}^2\frac{1}{2}E} \tag{20}$$

Using the identity $\text{cos}^2\frac{1}{2}x = \frac{1+cosx}{2}$:

$$\frac{\partial f}{\partial E} = \sqrt{\frac{1+e}{1-e}}\frac{1+\text{cos}f}{1+\text{cos}E}\sqrt{\frac{1+e}{1+e}} = \frac{(1+e)\,(1+\text{cos}f)}{\sqrt{1-e^2}\,(1+\text{cos}E)} = \frac{1 + e\text{cos}f + e + \text{cos}f}{\sqrt{1-e^2}\,(1+\text{cos}E)} \tag{21}$$

$$\frac{\partial f}{\partial E} = \frac{\left[\frac{1+e\text{cos}f}{1+e\text{cos}f} + \frac{e+\text{cos}f}{1+e\text{cos}f}\right](1+e\text{cos}f)}{\sqrt{1-e^2}\,(1+\text{cos}E)} \tag{22}$$

$$\frac{\partial f}{\partial E} = \frac{(1+\text{cos}E)\,(1+e\text{cos}f)}{\sqrt{1-e^2}\,(1+\text{cos}E)} = \frac{1+e\text{cos}f}{\sqrt{1-e^2}} \tag{23}$$

$\frac{\partial R}{\partial f}$ is found through the derivative quotient rule and simplifying:

$$\frac{\partial R}{\partial f} = \frac{\partial}{\partial f}\left(\frac{a(1-e^2)}{1+e\text{cos}f}\text{sin}i\text{sin}(\omega+f)\,a_s\right) = a(1-e^2)\text{sin}i\,a_s\left[\frac{\text{cos}(\omega+f)\,(1+e\text{cos}f) - (-e\text{sin}f)\text{sin}(\omega+f)}{(1+e\text{cos}f)^2}\right] \tag{24}$$

Simplifying the bracketed quantity, assigned to A:

$$\begin{aligned} A &= \frac{\cos(\omega + f)\,(1 + e\cos f) - (-e\sin f)\sin(\omega + f)}{(1 + e\cos f)^2} \\ &= \frac{\cos(\omega + f) + e\cos f\cos(\omega + f) + e\sin f\sin(\omega + f)}{(1 + e\cos f)^2} \\ &= \frac{\cos(\omega + f) + e\cos f(\cos\omega\cos f - \sin\omega\sin f) + e\sin f(\sin\omega\cos f + \cos\omega\sin f)}{(1 + e\cos f)^2} \\ &= \frac{\cos(\omega + f) + e\cos^2 f\cos\omega - e\sin\omega\sin f\cos f + e\sin\omega\sin f\cos f + e\cos\omega\sin^2 f)}{(1 + e\cos f)^2} \\ &= \frac{\cos(\omega + f) + e\cos\omega}{(1 + e\cos f)^2} \end{aligned} \tag{25}$$

Which makes the desired partial

$$\frac{\partial R}{\partial f} = a(1 - e^2)\sin i\, a_s \left[\frac{\cos(\omega + f) + e\cos\omega}{(1 + e\cos f)^2}\right] \tag{26}$$

Now we have $\frac{\partial R}{\partial \sigma}$ through Equations 17, 23, and 26:

$$\begin{aligned} \frac{\partial R}{\partial \sigma} &= \frac{\partial R}{\partial f}\frac{\partial f}{\partial E}\frac{\partial E}{\partial \sigma} = a(1 - e^2)\sin i\, a_s \left[\frac{\cos(\omega + f) + e\cos\omega}{(1 + e\cos f)^2}\right]\frac{1 + e\cos f}{\sqrt{1 - e^2}}\frac{1 + e\cos f}{1 - e^2} \\ &= \frac{a}{\sqrt{1 - e^2}}\sin i\,[\cos(\omega + f) + e\cos\omega]\,a_s \end{aligned} \tag{27}$$

Equation 13 becomes

$$\begin{aligned} \dot{a} &= \frac{2}{na}\frac{a}{\sqrt{1 - e^2}}\sin i\,[\cos(\omega + f) + e\cos\omega]\,a_s \\ &= \frac{2}{n\sqrt{1 - e^2}}\sin i\,[\cos(\omega + f) + e\cos\omega]\,a_s \end{aligned} \tag{28}$$

Finding $\frac{\partial R}{\partial \omega}$ to be

$$\frac{\partial R}{\partial \omega} = \frac{\partial}{\partial \omega}\left(r a_s \sin i \sin(\omega + f)\right)) = r a_s \sin i \cos(\omega + f)) \tag{29}$$

equation 14 becomes

$$\begin{aligned} \dot{e} &= \frac{1}{na^2 e}\left[(1 - e^2)\left[\frac{a}{\sqrt{1 - e^2}}\sin i\,[\cos(\omega + f) + e\cos\omega]\,a_s\right] - \sqrt{1 - e^2}\,[r a_s \sin i \cos(\omega + f))]\right] \\ &= \frac{\sqrt{1 - e^2}\sin i}{na}\,a_s\left[\frac{\cos(\omega + f)}{e} + \cos\omega - \frac{\cos(\omega + f)}{e}\frac{1 - e^2}{1 + e\cos f}\right] \\ &= \frac{\sqrt{1 - e^2}\sin i}{na}\,a_s\left[\cos\omega + \frac{\cos(\omega + f)}{e}\left[1 - \frac{1 - e^2}{1 + e\cos f}\right]\right] \\ &= \frac{\sqrt{1 - e^2}\sin i}{na}\,a_s\left[\cos\omega + \frac{\cos(\omega + f)}{e}\left[\frac{1 + e\cos f}{1 + e\cos f} - \frac{1 - e^2}{1 + e\cos f}\right]\right] \\ &= \frac{\sqrt{1 - e^2}\sin i}{na}\,a_s\left[\cos\omega + \frac{\cos(\omega + f)}{e}\frac{e(e + \cos f)}{1 + e\cos f}\right] \end{aligned} \tag{30}$$

$$\dot{e} = \frac{\sqrt{1 - e^2}\sin i}{na}\,a_s\left[\cos\omega + \cos(\omega + f)\frac{(e + \cos f)}{1 + e\cos f}\right] \tag{31}$$

C.

Integrating $\dot{a}$ and $\dot{e}$ to find the change over an orbit. First, find Δa.

$$
\begin{aligned}
\Delta a &= \frac{1}{n}\int_0^E \dot{a}(1-e\text{cos}E)dE = \frac{1}{n}\int_0^E \frac{2}{n\sqrt{1-e^2}}\text{sin}i\left[\cos(\omega+f)+e\text{cos}\omega\right]a_s(1-e\text{cos}E)dE \\
&= \frac{2}{n^2\sqrt{1-e^2}}\text{sin}i\, a_s\int_0^E \left[\text{cos}\omega\text{cos}f-\text{sin}\omega\text{sin}f+e\text{cos}\omega\right](1-e\text{cos}E)dE \\
&= \frac{2}{n^2\sqrt{1-e^2}}\text{sin}i\, a_s\int_0^E \left[\frac{\text{cos}\omega(\text{cos}E-e)-\text{sin}\omega\sqrt{1-e^2}\text{sin}E}{1-e\text{cos}E}+e\text{cos}\omega\right](1-e\text{cos}E)dE \\
&= \frac{2}{n^2\sqrt{1-e^2}}\text{sin}i\, a_s\int_0^E \left[\text{cos}\omega(\text{cos}E-e)-\text{sin}\omega\sqrt{1-e^2}\text{sin}E+e\text{cos}\omega-e^2\text{cos}\omega\text{cos}E\right]dE
\end{aligned} \tag{32}
$$

Continuing:

$$
\begin{aligned}
\Delta a &= \frac{2}{n^2\sqrt{1-e^2}}\text{sin}i\, a_s\int_0^E \left[(1-e^2)\text{cos}\omega\text{cos}E-\sqrt{1-e^2}\text{sin}\omega\text{sin}E\right]dE \\
&= \frac{2}{n^2\sqrt{1-e^2}}\text{sin}i\, a_s\int_0^E \left[(1-e^2)\text{cos}\omega\text{cos}E-\sqrt{1-e^2}\text{sin}\omega\text{sin}E\right]dE \\
&= \frac{2\text{sin}i}{n^2}\left[\sqrt{1-e^2}\text{cos}\omega\text{sin}E+\text{sin}\omega\text{cos}E\right]_0^E a_s \\
&= \frac{2\text{sin}i}{n^2}\left[\sqrt{1-e^2}\text{cos}\omega(\text{sin}E-0)+\text{sin}\omega(\text{cos}E-1)\right]a_s \\
&= \frac{2\text{sin}i}{n^2}\left[\sqrt{1-e^2}\text{cos}\omega\text{sin}E-\text{sin}\omega(1-\text{cos}E)\right]a_s
\end{aligned} \tag{33}
$$

Next, Δe:

$$
\begin{aligned}
\Delta e &= \frac{1}{n}\int_0^E \frac{\sqrt{1-e^2}\text{sin}i}{na}a_s\left[\text{cos}\omega+\cos(\omega+f)\frac{(e+\text{cos}f)}{1+e\text{cos}f}\right](1-e\text{cos}E)dE \\
&= \frac{\sqrt{1-e^2}\text{sin}i}{n^2a}a_s\int_0^E \left[\text{cos}\omega+\cos(\omega+f)\text{cos}E\right](1-e\text{cos}E)dE \\
&= \frac{\sqrt{1-e^2}\text{sin}i}{n^2a}a_s\int_0^E \left[\text{cos}\omega+(\text{cos}\omega\text{cos}f-\text{sin}\omega\text{sin}f)\text{cos}E\right](1-e\text{cos}E)dE \\
&= \frac{\sqrt{1-e^2}\text{sin}i}{n^2a}a_s\int_0^E \left[\text{cos}\omega+\frac{\text{cos}\omega(\text{cos}E-e)-\text{sin}\omega\sqrt{1-e^2}\text{sin}E}{1-e\text{cos}E}\text{cos}E\right](1-e\text{cos}E)dE
\end{aligned} \tag{34}
$$

Continuing, using Equations 3 and **??** where appropriate:

$$
\begin{aligned}
\Delta e &= \frac{\sqrt{1-e^2}\text{sin}i}{n^2a}a_s\int_0^E \left[\text{cos}\omega+\frac{\text{cos}\omega(\text{cos}E-e)-\text{sin}\omega\sqrt{1-e^2}\text{sin}E}{1-e\text{cos}E}\text{cos}E\right](1-e\text{cos}E)dE \\
&= \frac{\sqrt{1-e^2}\text{sin}i}{n^2a}a_s\int_0^E \left[\text{cos}\omega-e\text{cos}\omega\text{cos}E+\text{cos}\omega\text{cos}^2E-e\text{cos}\omega\text{cos}E-\sqrt{1-e^2}\text{sin}\omega\text{sin}E\text{cos}E\right] \\
&= \frac{\sqrt{1-e^2}\text{sin}i}{n^2a}\left[\text{cos}\omega\, E-2e\text{cos}\omega\text{sin}E+\text{cos}\omega\left[\frac{E}{2}+\frac{\text{sin}2E}{4}\right]+\sqrt{1-e^2}\text{sin}\omega\frac{\text{cos}^2E}{2}\right]_0^E a_s \\
&= \frac{\sqrt{1-e^2}\text{sin}i}{n^2a}\left[-2e\text{cos}\omega\text{sin}E+\text{cos}\omega\left[\frac{3E}{2}+\frac{\text{sin}2E}{4}\right]+\sqrt{1-e^2}\text{sin}\omega\frac{\text{cos}^2E}{2}\right]_0^E a_s
\end{aligned} \tag{35}
$$

Further simplifying:

$$
\begin{aligned}
\Delta e &= \frac{\sqrt{1-e^2}\text{sin}i}{n^2 a}\left[-2e\text{cos}\omega\text{sin}E + \text{cos}\omega\left[\frac{3E}{2}+\frac{\text{sin}2E}{4}\right]+\sqrt{1-e^2}\text{sin}\omega\frac{\text{cos}^2E-1}{2}\right]a_s \\
&= \frac{\sqrt{1-e^2}\text{sin}i}{n^2 a}\left[\frac{3}{2}\text{cos}\omega\, E - 2e\text{cos}\omega\text{sin}E + \frac{1}{4}\text{cos}\omega\text{sin}2E + \sqrt{1-e^2}\text{sin}\omega\frac{\frac{1}{2}\left[1+\text{cos}2E\right]-1}{2}\right]a_s \\
&= \frac{\sqrt{1-e^2}\text{sin}i}{n^2 a}\left[\frac{3}{2}\text{cos}\omega\, E - 2e\text{cos}\omega\text{sin}E + \frac{1}{4}\text{cos}\omega\text{sin}2E - \frac{\sqrt{1-e^2}}{4}\text{sin}\omega(1-\text{cos}2E)\right]a_s
\end{aligned} \tag{36}
$$

D.

$\frac{\partial \bar{R}}{\partial \sigma}$ is easy to find, since no part of it is dependent on $\sigma/f/E/M$:

$$\frac{\partial \bar{R}}{\partial \sigma} = \frac{\partial}{\partial \sigma}\left(-\frac{3}{2}ae\text{sin}\omega\text{sin}i\, a_s\right) = 0 \tag{37}$$

$\frac{\partial \bar{R}}{\partial \omega}$ is non-zero:

$$\frac{\partial \bar{R}}{\partial \omega} = \frac{\partial}{\partial \omega}\left(-\frac{3}{2}ae\text{sin}\omega\text{sin}i\, a_s\right) = -\frac{3}{2}ae\text{cos}\omega\text{sin}i\, a_s \tag{38}$$

The secular change of a is found by

$$\frac{n}{2\pi}\Delta a(2\pi) = \frac{n}{2\pi}\frac{2\text{sin}i}{n^2}\left[\sqrt{1-e^2}\text{cos}\omega(0) - \text{sin}\omega(1-1)\right]a_s = 0 \tag{39}$$

This result matches Equation 13 when $\bar{R}$ is used:

$$\frac{2}{na}\frac{\partial \bar{R}}{\partial \sigma} = 0 \tag{40}$$

The secular change of e is found by

$$
\begin{aligned}
\frac{n}{2\pi}\Delta e(2\pi) &= \frac{n}{2\pi}\frac{\sqrt{1-e^2}\text{sin}i}{n^2 a}\left[\frac{3}{2}\text{cos}\omega\,(2\pi) - 2e\text{cos}\omega(0) + \frac{1}{4}\text{cos}\omega(0) - \frac{\sqrt{1-e^2}}{4}\text{sin}\omega(1-1)\right]a_s \\
&= \frac{3}{2}\frac{\sqrt{1-e^2}\text{sin}i}{na}\text{cos}\omega
\end{aligned} \tag{41}
$$

which matches Equation 14 when $\bar{R}$ is used

$$
\begin{aligned}
\frac{1}{na^2 e}\left[(1-e^2)\frac{\partial \bar{R}}{\partial \sigma} - \sqrt{1-e^2}\frac{\partial \bar{R}}{\partial \omega}\right] &= \frac{1}{na^2 e}\left[(1-e^2)(0) - \sqrt{1-e^2}\left(-\frac{3}{2}ae\text{cos}\omega\text{sin}i\, a_s\right)\right] \\
&= \frac{3}{2}\frac{\sqrt{1-e^2}\text{sin}i}{na}\text{cos}\omega
\end{aligned} \tag{42}
$$

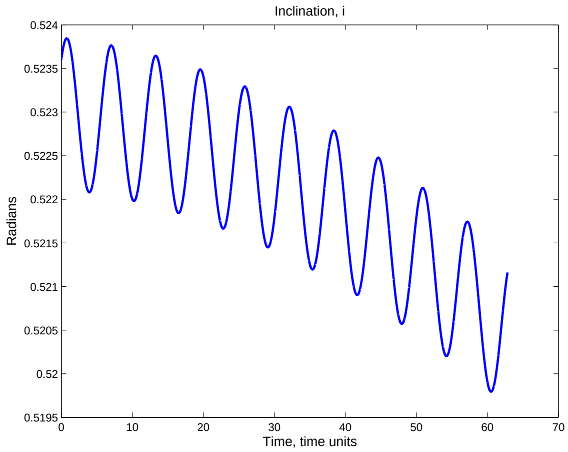
E.

The simulation initial conditions are as follows:

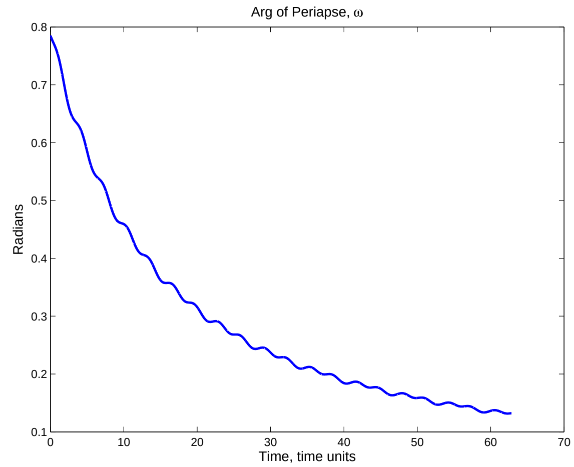
Table 1. My caption

Parameter	Value
μ	1
a	1
e	0.01
i	30°
Ω	0°
ω	45°

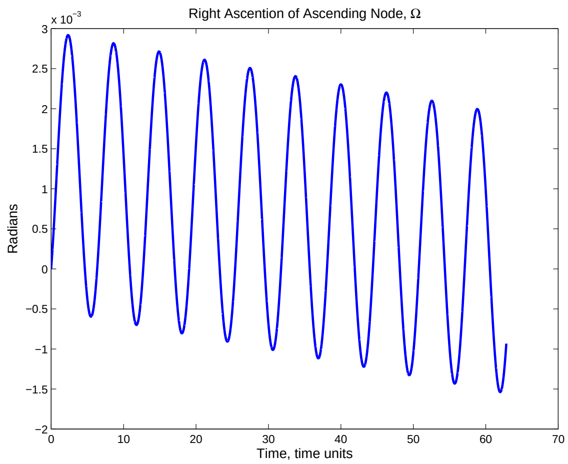
The results of this simulation are shown in Figures 3 and 4 below:



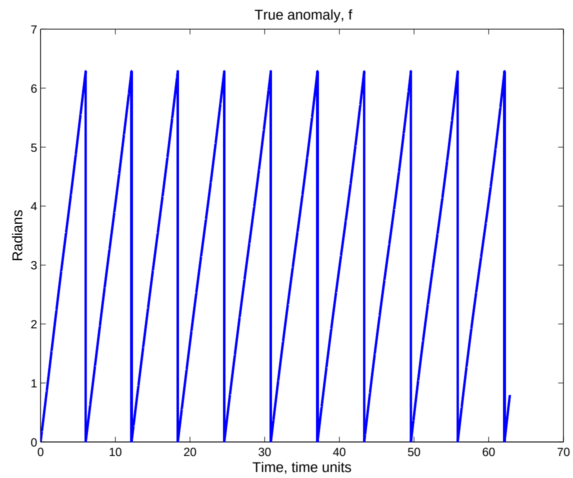
(a) Inclination



(b) Argument of periapsis

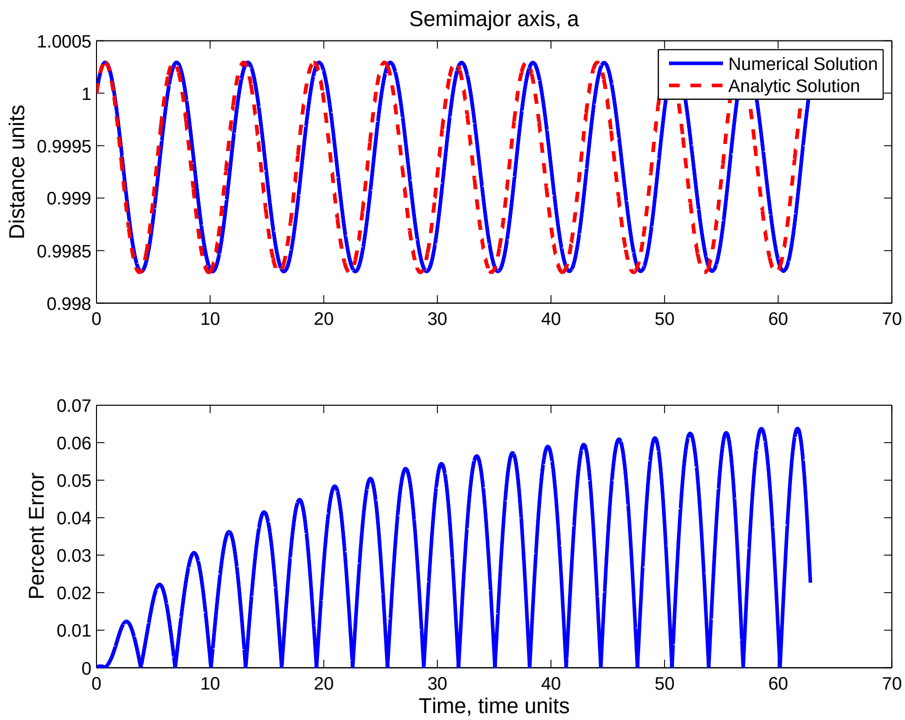


(c) RAAN

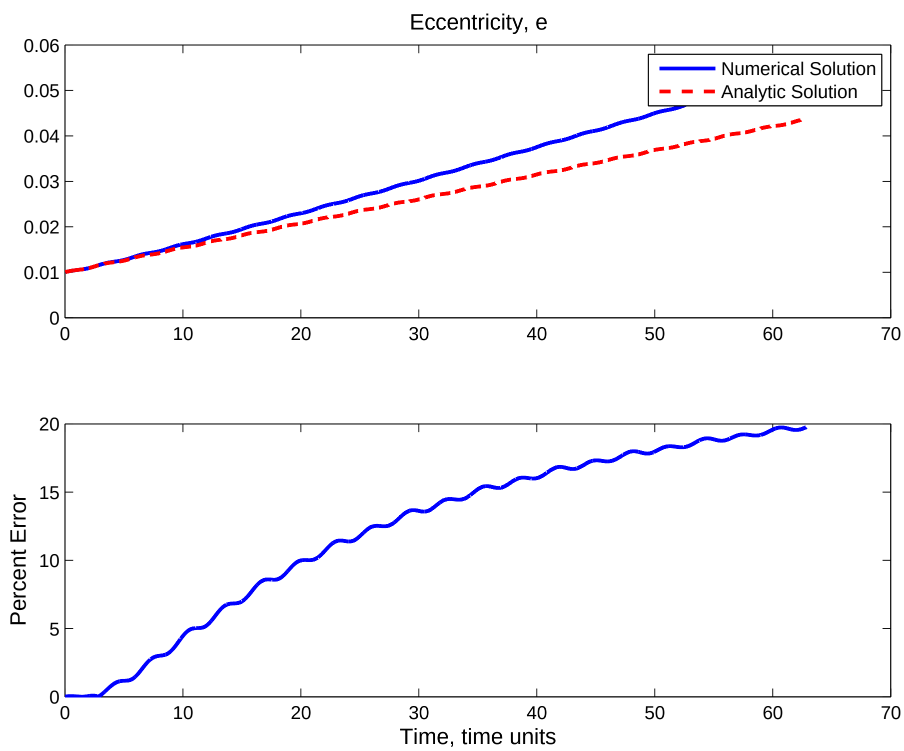


(d) True anomaly

Figure 3. Simulation results, orbital elements



(a) Euler angles wrt LVLH



(b) Body Rates

Figure 4. Simulation results, a and e

The argument of periapsis shows the most change of the orbit-orientation angles. While inclination and RAAN have secular change as well, they are quite small compared to ω. Although a does not experience secular change, the numerical solution diverges a small amount from the analitical solution. The divergence starts small and grows, likely due to the cosω and sinω terms changing. The eccentricity experiences the largest error between the two solutions. Since there is secular change in e, the change predicted by Δe becomes less accurate as time goes on and e grows over each orbit.

F.

The short period term $\bar{\Delta} a$ is as follows:

$$\begin{aligned}\bar{\Delta} a &= \frac{1}{2\pi}\int_0^{2\pi}\Delta a(E)dM = \frac{1}{2\pi}\int_0^{2\pi}\frac{2\sqrt{1-e^2}\text{sin}i\,a_s}{n^2}\left[\sqrt{1-e^2}\text{cos}\omega\text{sin}E - \text{sin}\omega\left(1-\text{cos}E\right)\right]\left(1-e\text{cos}E\right)dE \\ &= \frac{\sqrt{1-e^2}\text{sin}i\,a_s}{\pi\, n^2}\int_0^{2\pi}\left[\sqrt{1-e^2}\text{cos}\omega\text{sin}E - \sqrt{1-e^2}e\text{cos}\omega\text{cos}E\text{sin}E - \text{sin}\omega + e\text{sin}\omega\text{cos}E + \text{sin}\omega\text{cos}E - e\text{sin}\omega\text{cos}^2E\right]dE \\ &= -\frac{2\sqrt{1-e^2}\text{sin}i\text{sin}\omega\, a_s}{n^2}\end{aligned} \tag{43}$$

The short period term $\bar{\Delta} e$ is as follows:

$$\bar{\Delta} e = \frac{1}{2\pi}\int_0^{2\pi}\left(\Delta e(E) - \frac{1}{n}\dot{\bar{e}}M\right)dM \tag{44}$$

Breaking up the integral into two parts, we can solve for the first:

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\Delta e(E)dM = \frac{1}{2\pi}\int_0^{2\pi}&\frac{\sqrt{1-e^2}\text{sin}i\,a_s}{n^2a} \\ &\left[\frac{3}{2}\text{cos}\omega E - 2e\text{cos}\omega\text{sin}E + \frac{1}{4}\text{cos}\omega\text{sin}2E - \frac{\sqrt{1-e^2}}{4}\text{sin}\omega + \frac{\sqrt{1-e^2}}{4}\text{sin}\omega\text{cos}2E\right]\left(1-e\text{cos}E\right)dE\end{aligned} \tag{45}$$

Using the double-angle identities, distributing, and using Eqns 1, 2, 3, 4, and 5:

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\Delta e(E)dM = \frac{1}{2\pi}\int_0^{2\pi}&\frac{\sqrt{1-e^2}\text{sin}i\,a_s}{n^2a} \\ &\left[\frac{3}{2}\text{cos}\omega E - 2e\text{cos}\omega\text{sin}E + \frac{2}{4}\text{cos}\omega\text{sin}E\text{cos}E - \frac{\sqrt{1-e^2}}{4}\text{sin}\omega + \frac{\sqrt{1-e^2}}{4}\text{sin}\omega\left(1-2\text{sin}^2E\right)\right](1-e\end{aligned} \tag{46}$$

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\Delta e(E)dM = \frac{1}{2\pi}\int_0^{2\pi}&\frac{\sqrt{1-e^2}\text{sin}i\,a_s}{n^2a} \\ &\left[\frac{3}{2}\text{cos}\omega E - 2e\text{cos}\omega\text{sin}E + \frac{1}{2}\text{cos}\omega\text{sin}E\text{cos}E + \frac{\sqrt{1-e^2}}{2}\text{sin}\omega\text{sin}^2E\right]\left(1-e\text{cos}E\right)dE\end{aligned} \tag{47}$$

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\Delta e(E)dM &= \frac{1}{2\pi}\frac{\sqrt{1-e^2}\text{sin}i\,a_s}{n^2a}\int_0^{2\pi}\left[\frac{3}{2}\text{cos}\omega E - \frac{\sqrt{1-e^2}}{2}\text{sin}\omega\text{sin}^2E\right]dE \\ &= \frac{\sqrt{1-e^2}\text{sin}i\,a_s}{n^2a}\left[\frac{3\pi}{2}\text{cos}\omega - \frac{\sqrt{1-e^2}}{4}\text{sin}\omega\right]\end{aligned} \tag{48}$$

Now for the second part:

$$\begin{aligned}\frac{1}{2\pi n}\int_0^{2\pi}\dot{e}MdM &= \frac{1}{2\pi}\frac{\sqrt{1-e^2}\text{sin}i\,a_s}{n^2a}\int_0^{2\pi}\left[\text{cos}\omega + \text{cos}(\omega+f)\frac{(e+\text{cos}f)}{1+e\text{cos}f}\right]MdM \\ &= \frac{1}{2\pi}\frac{\sqrt{1-e^2}\text{sin}i\,a_s}{n^2a}\left[2\pi^2\text{cos}\omega + \int_0^{2\pi}\left(\text{cos}\omega\text{cos}E - e\text{cos}\omega\text{cos}E - \sqrt{1-e^2}\text{sin}\omega\text{sin}E\right)\text{cos}E\left(E - e\text{sin}E\right)dE\right]\end{aligned} \tag{49}$$

The remaining integral becomes:

$$\begin{aligned}
&\int_0^{2\pi} \left(\cos\omega\cos E - e\cos\omega\cos E - \sqrt{1-e^2}\sin\omega\sin E\right)\cos E\,(E - e\sin E)\,dE \\
&= \int_0^{2\pi} \left(\cos\omega\cos^2 E - e\cos\omega\cos^2 E - \sqrt{1-e^2}\sin\omega\sin E\cos E\right)(E - e\sin E)\,dE \\
&= \pi^2\cos\omega + \frac{\pi\sqrt{1-e^2}}{2}\sin\omega
\end{aligned} \tag{50}$$

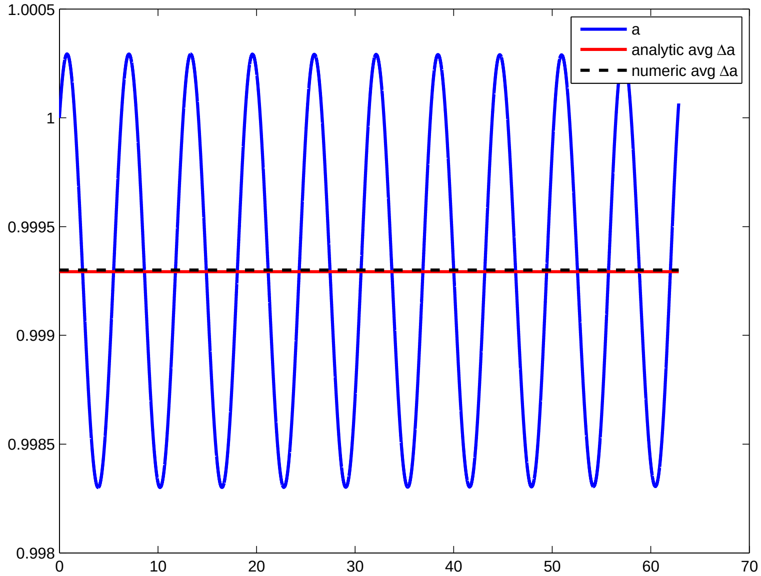
Combining Eqns 49 and 50:

$$\frac{1}{2\pi n}\int_0^{2\pi} \dot{e}M\,dM = \frac{1}{2\pi}\frac{\sqrt{1-e^2}\sin i\, a_s}{n^2 a}\left[3\pi^2\cos\omega + \frac{\pi\sqrt{1-e^2}}{2}\sin\omega\right] \tag{51}$$

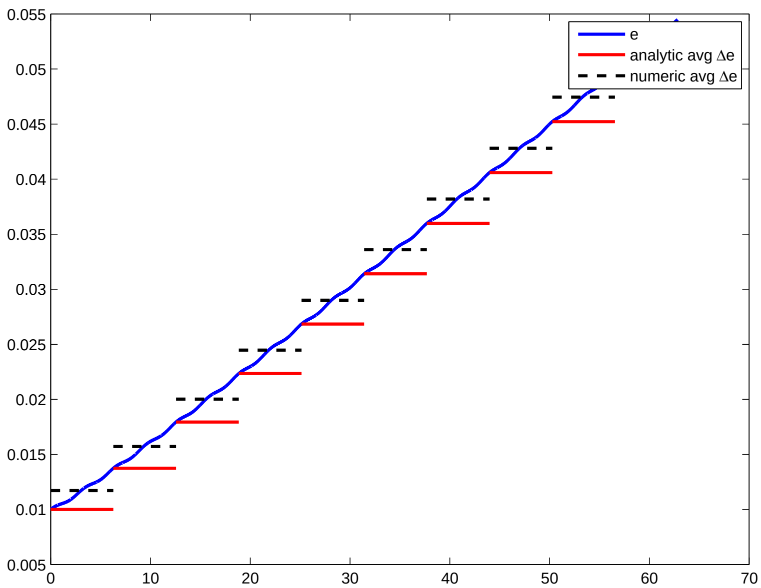
And combining Eqns 48 and 51 into 52 yield:

$$\bar{\Delta e} = \frac{1}{2\pi}\int_0^{2\pi}\left(\Delta e(E) - \frac{1}{n}\bar{\dot{e}}M\right)dM = 0 \tag{52}$$

The results of the short-period averaging are shown in Figure 5 below.



(a) Semimajor axis short-period averages



(b) Eccentricity short-period averages

Figure 5. Average short-period changes

As expected, the $\bar{\Delta a}$ in the numerical and analytical solutions are very close since there is no secular change in a. However, the $\bar{\Delta e}$ solutions, taken at the beginning of each orbit, differ quite a bit. This means that e *only* changes secularly.