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Problem 7: S&J 3.15

$$[C] = e^{-[\tilde{\gamma}]} = e^{-\phi[\tilde{e}]} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(-\phi[\tilde{e}]\right)^n$$

$$= [I]_{3\times3} - \frac{1}{1}\cdot\phi[\tilde{e}] + \frac{1}{2!}\phi^2[\tilde{e}]^2 - \frac{1}{3!}\phi^3[\tilde{e}]^3 + \frac{1}{4!}\phi^4[\tilde{e}]^4 \ldots$$

n = odds : (see Mathematica Notebook)

$$[\tilde{e}]^n = \begin{pmatrix} 0 & -\hat{e}_3(\hat{e}_1^2+\hat{e}_2^2+\hat{e}_3^2)^{\frac{n-1}{2}} & \hat{e}_2(\hat{e}_1^2+\hat{e}_2^2+\hat{e}_3^2)^{\frac{n-1}{2}} \\ \hat{e}_3(\hat{e}_1^2+\hat{e}_2^2+\hat{e}_3^2)^{\frac{n-1}{2}} & 0 & -\hat{e}_1(\hat{e}_1^2+\hat{e}_2^2+\hat{e}_3^2)^{\frac{n-1}{2}} \\ -\hat{e}_2(\hat{e}_1^2+\hat{e}_2^2+\hat{e}_3^2)^{\frac{n-1}{2}} & \hat{e}_1(\hat{e}_1^2+\hat{e}_2^2+\hat{e}_3^2)^{\frac{n-1}{2}} & 0 \end{pmatrix}(-1)^{\left(\frac{n-1}{2}\right)}$$

$$= \begin{pmatrix} 0 & -\hat{e}_3 & \hat{e}_2 \\ \hat{e}_3 & 0 & -\hat{e}_1 \\ -\hat{e}_2 & \hat{e}_1 & 0 \end{pmatrix}(-1)^{\frac{n-1}{2}} = \underline{[\tilde{e}](-1)^{\frac{n-1}{2}}}$$

$$\hat{e}\hat{e}^T = \begin{pmatrix} \hat{e}_1^2 & \hat{e}_1\hat{e}_2 & \hat{e}_1\hat{e}_3 \\ \hat{e}_1\hat{e}_2 & \hat{e}_2^2 & \hat{e}_2\hat{e}_3 \\ \hat{e}_1\hat{e}_3 & \hat{e}_2\hat{e}_3 & \hat{e}_3^2 \end{pmatrix}$$

n = evens : (see Mathematica Notebook)

$$[\tilde{e}]^n = \begin{pmatrix} -\hat{e}_2^2-\hat{e}_3^2 & \hat{e}_1\hat{e}_2 & \hat{e}_1\hat{e}_3 \\ \hat{e}_1\hat{e}_2 & -\hat{e}_1^2-\hat{e}_3^2 & \hat{e}_2\hat{e}_3 \\ \hat{e}_1\hat{e}_3 & \hat{e}_2\hat{e}_3 & -\hat{e}_1^2-\hat{e}_2^2 \end{pmatrix}(-1)^{\frac{n}{2}-1}\cancelto{1}{(\hat{e}_1^2+\hat{e}_2^2+\hat{e}_3^2)}^{\frac{n}{2}-1}$$

$$= \begin{pmatrix} \hat{e}_1^2-1 & \hat{e}_1\hat{e}_2 & \hat{e}_1\hat{e}_3 \\ \hat{e}_1\hat{e}_2 & \hat{e}_2^2-1 & \hat{e}_1\hat{e}_2 \\ \hat{e}_1\hat{e}_3 & \hat{e}_1\hat{e}_2 & \hat{e}_3^2-1 \end{pmatrix}(-1)^{\frac{n}{2}-1} = \underline{(\hat{e}\hat{e}^T - [I]_{3\times3})(-1)^{\frac{n-1}{2}}}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \ldots$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \ldots$$

$$[C] = [I]_{3\times3} - \phi[\tilde{e}] + \frac{1}{2!}\phi^2\cdot(\hat{e}\hat{e}^T - [I]_{3\times3}) + \frac{1}{3!}\phi^3[\tilde{e}] - \frac{1}{4!}\phi^4(\hat{e}\hat{e}^T - [I]_{3\times3}) \ldots$$

$$= [I]_{3\times3}\cos\phi - \sin\phi[\tilde{e}] + [I]_{3\times3}\hat{e}\hat{e}^T - [I]_{3\times3}\hat{e}\hat{e}^T + \frac{1}{2!}\phi^2\hat{e}\hat{e}^T - \frac{1}{4!}\phi^4\hat{e}\hat{e}^T$$

$$= [I]_{3\times3}\cos\phi - \sin\phi[\tilde{e}] + [I]_{3\times3}\hat{e}\hat{e}^T - \cos\phi\,\hat{e}\hat{e}^T$$

$$\underline{[C] = [I]_{3\times3}\cos\phi - \sin\phi[\tilde{e}] + (1-\cos\phi)\hat{e}\hat{e}^T}$$