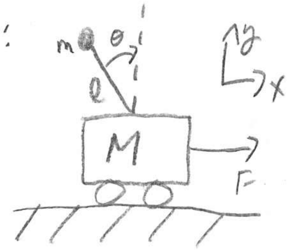
Clouse
ASEN5014
HW2

Problem 1:



$$(M+m)\ddot{x} - ml\ddot{\theta}\cos\theta + ml\dot{\theta}^2\sin\theta = F$$

$$l\ddot{\theta} - g\sin\theta = \ddot{x}\cos\theta$$

$$y = x - l\sin\theta = h_1$$

a) Linearize EOMs

$$(M+m)\ddot{x} - m(g\sin\theta + \ddot{x}\cos\theta)\cos\theta + ml\dot{\theta}^2\sin\theta - F = 0$$

$$\vec{u} = F$$

$$\vec{x} = \begin{pmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{pmatrix}, \quad \dot{\vec{x}} = \begin{pmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta} \\ \ddot{\theta} \end{pmatrix} = \begin{bmatrix} \dot{x} \\ \dfrac{-mg\sin\theta\cos\theta + ml\dot{\theta}^2\sin\theta - F}{M+m-m\cos\theta} \\ \dot{\theta} \\ \ddot{\theta} = \dfrac{\ddot{x}\cos\theta + g\sin\theta}{l} \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix}$$

$$\frac{\partial f_2}{\partial \theta} = \frac{(-mg\cos^2\theta + mg\sin^2\theta + ml\dot{\theta}^2\cos\theta)(M+m(1-\cos\theta)) - (g\cos\theta + l\dot{\theta}^2)m\sin\theta(m\sin\theta)}{(M+m(1-\cos\theta))^2}$$

$$\left.\frac{\partial f_2}{\partial \theta}\right|_{\substack{\theta=0 \\ \dot{\theta}=0}} = \frac{(-mg)M}{M^2} = -\frac{m}{M}g$$

$$\frac{\partial f_2}{\partial \dot{\theta}} = \frac{2ml\dot{\theta}\sin\theta}{M+m(1-\cos\theta)} \Rightarrow \left.\frac{\partial f_2}{\partial \dot{\theta}}\right|_{\substack{\theta=0 \\ \dot{\theta}=0}} = 0$$

$$\frac{\partial f_3}{\partial \dot{\theta}} = 1$$

$$\frac{\partial f_4}{\partial \theta} = \frac{\partial}{\partial \theta}\left[\frac{f_2\cos\theta + \sin\theta}{l}\right] = \frac{\partial f_2}{\partial \theta}\frac{\cos\theta}{l} - f_2\frac{\sin\theta}{l}$$

$$\left.\frac{\partial f_4}{\partial \theta}\right|_{\substack{\theta=0 \\ \dot{\theta}=0}} = -\frac{m}{M}\frac{g}{l}$$

$$\frac{\partial f_4}{\partial \dot{\theta}} = \frac{\partial f_2}{\partial \dot{\theta}}\frac{\cos\theta}{l} \Rightarrow \left.\frac{\partial f_4}{\partial \dot{\theta}}\right|_{\substack{\theta=0 \\ \dot{\theta}=0}} = 0$$

$$\frac{\partial h_1}{\partial x} = 1$$

$$\left.\frac{\partial h_1}{\partial \theta}\right|_{\theta=0} = -l\cos\theta\Big|_{\theta=0} = -l$$

$$\left.\frac{\partial f_2}{\partial u}\right|_{\theta=0} = -\frac{1}{M} \ ; \quad \left.\frac{\partial f_4}{\partial u}\right|_{\theta=0} = \left[\frac{\partial f_2}{\partial \theta}\frac{\cos\theta}{l} - f_2\frac{\sin\theta}{l}\right]_{\theta=0} = -\frac{1}{Ml}$$

Problem 1 cont.

$$\delta\dot{\vec{x}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{m}{M}g & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{m}{M}\frac{g}{\ell} & 0 \end{pmatrix} \delta\vec{x} + \begin{pmatrix} 0 \\ -\frac{1}{M} \\ 0 \\ -\frac{1}{M\ell} \end{pmatrix} \partial\vec{u}$$

$$\delta y = [1\ 0\ -\ell\ 0]\,\delta\vec{x} + 0\cdot\delta\vec{u}$$

b) $\delta\ddot{\theta} = \delta\ddot{x}/\ell \Rightarrow \delta\ddot{x} = \ell\delta\ddot{\theta} = \frac{d}{dt}\dot{x} = \frac{d^2}{dt^2}x \Rightarrow x = \ell\iint \frac{d^2}{dt^2}\theta \Rightarrow \theta = \frac{x}{\ell} + c$

rows 2 & 4 are not independent

let $\vec{x}_L = \begin{pmatrix}\theta \\ \dot{\theta}\end{pmatrix}$, $u = F$, y

about $\vec{x}_L = \begin{pmatrix}0 \\ 0\end{pmatrix}$

$$\dot{\vec{x}}_L = \begin{bmatrix} 0 & 1 \\ -\frac{m}{M}\frac{g}{\ell} & 0 \end{bmatrix}\vec{x} + \begin{bmatrix} 0 \\ \frac{-1}{M\ell} \end{bmatrix}\vec{u}$$

$$y = [0\ \ 0]\,\vec{x} + 0\,\vec{u}$$

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Problem 2: Given SS model

$$\dot{\vec{x}} = \begin{bmatrix} 0 & 1 \\ -25 & -4 \end{bmatrix} \vec{x} + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \vec{u} \quad , \quad \vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\vec{y} = I \, \vec{x} + [0] \, \vec{u}$$

a) Find TF matrix

$$\frac{\underline{Y}(s)}{\underline{U}(s)} = C(sI-A)^{-1}B + D$$

$$sI - A = \begin{pmatrix} s & -1 \\ 25 & s+4 \end{pmatrix} \Rightarrow \det(sI-A) = s^2 + 4s + 25 = A(s)$$

$$\text{cofactor} = \begin{pmatrix} s+4 & -25 \\ 1 & s \end{pmatrix}$$

$$\text{cofactor}^T = \begin{matrix} s+4 & 1 \\ -25 & s \end{matrix}$$

$$\begin{pmatrix} s+4 & 1 \\ -25 & s \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{matrix} s+4 & s+4+1 \\ -25 & s-25 \end{matrix}$$

$$\boxed{\frac{\underline{Y}(s)}{\underline{U}(s)} = \begin{pmatrix} B_{11}(s) & B_{12}(s) \\ B_{21}(s) & B_{22}(s) \end{pmatrix} \cdot \frac{1}{A(s)} = \begin{pmatrix} s+4 & s+5 \\ -25 & s-25 \end{pmatrix} \cdot \frac{1}{s^2+4s+25}}$$

b) on next page

c) graph on next page

The $T(s)_{21}$ term is the input 1, output 2 channel. It starts at zero because the B matrix shows that the first input has no direct impact on the second output. Looking at the Laplace transform, it is in the form of a decaying sinewave:

$$\mathcal{L}^{-1}\left[\frac{25}{s^2+4s+25}\right] = \mathcal{L}^{-1}\left[\frac{-25 \, \frac{\sqrt{21}}{\sqrt{21}}}{(s+2)+21}\right] = -\frac{25}{\sqrt{21}} \cdot e^{-2at} \sin\omega t \cdot u(t)$$

The other terms are superpositions of decaying sine + cosine waves, The outputs directly affect their outputs, which lead to the cos terms, The impulse with zero I.C.'s cause this response.