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Ch 4 Prob 21

$x(t) = x_0 + \dot{x}_0 t + \frac{1}{2}\ddot{x}_0 t^2$
$\dot{x}(t) = \dot{x}_0 + \ddot{x}_0 t$
$\ddot{x}(t) = \ddot{x}_0$

$\mathbf{X}(t) = \begin{pmatrix} x \\ \dot{x} \\ \ddot{x} \end{pmatrix}$; $\bar{P}_0 = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; $\bar{\mathbf{X}}(t_0) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$R = \sigma_Y^2 = 1$, $Y(t_1) = 2$

a) $\hat{\mathbf{X}}(t_0)$ using batch

$\hat{\mathbf{X}}(t_0) = (H^T R^{-1} H + \bar{P}_0^{-1})^{-1}(H^T R^{-1} Y + \bar{P}_0^{-1} \bar{\mathbf{X}}(t_0))$

$\bar{P}_0^{-1} = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad R^{-1} = 1$

$\dot{x} = \dot{x}_0 + \ddot{x} t$
$x = x_0 + \dot{x}_0 t + \frac{1}{2}\ddot{x} t^2$
$= x_0 + \dot{x} t - \frac{1}{2}\ddot{x} t^2$

$G(\mathbf{X}) = x \Rightarrow \tilde{H} = \frac{\partial G}{\partial \mathbf{X}} = [1 \quad t \quad +\frac{1}{2}t^2]$

$H(t_1) = [1 \quad 1 \quad +\frac{1}{2}]$

$\hat{\mathbf{X}}(t_0) = \left(\begin{bmatrix} 1 & 1 & +\frac{1}{2} \\ 1 & 1 & +\frac{1}{2} \\ +\frac{1}{2} & +\frac{1}{2} & +\frac{1}{4} \end{bmatrix} + \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)^{-1} \left(\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{pmatrix} \right)$

$= \frac{1}{29}\begin{pmatrix} 52 & -32 & -8 \\ -32 & 42 & -4 \\ -8 & -4 & 28 \end{pmatrix} \cdot \begin{pmatrix} 9/4 \\ 5/2 \\ 2 \end{pmatrix} = \boxed{\begin{pmatrix} 21/29 \\ 25/29 \\ 28/29 \end{pmatrix}}$

cont →

b) $\underline{X}(t_0)$ using CKF

$\Phi: \underline{X}(t_i) = \Phi(t_i, t_0)\, \underline{X}(t_0)$

$$= \begin{bmatrix} 1 & t & \frac{1}{2}t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} X_0 \\ \dot{X}_0 \\ \ddot{X}_0 \end{pmatrix}$$

$$\tilde{H}_0 = \frac{\partial G_i}{\partial \underline{X}_i} \cdot \frac{\partial \underline{X}_i}{\partial \underline{X}_0} = \tilde{H}_i\, \Phi(t, t_0)$$

$$K = \bar{P}_0 \cdot \tilde{H}_0^T (\tilde{H}_0 \bar{P}_0 \tilde{H}_0^T + R) = \frac{1}{29} \begin{pmatrix} 16 \\ 8 \\ 2 \end{pmatrix}$$

$$\hat{\underline{X}}_0 = \bar{\underline{X}}_0 + K(\underline{Y}_1 - \tilde{H}_0 \bar{x}_0)$$

$$\boxed{\hat{\underline{X}}_0 = \begin{pmatrix} 21/29 \\ 25/29 \\ 28/29 \end{pmatrix}}$$

c) $\mathbf{X}(t_1)$ using CKF, then $\mathbf{X}(t_0)$

Φ by inspection: $\mathbf{X} = \Phi(t, t_0)\mathbf{X}(t_0)$

$$\Rightarrow \begin{matrix} x_0 & \dot{x}_0 t & \frac{1}{2}\ddot{x}t^2 \\ 0 & \dot{x}_0 & \ddot{x}_0 t \\ 0 & 0 & \ddot{x}_0 \end{matrix} = \underline{\begin{bmatrix} 1 & t & \frac{1}{2}t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}} \begin{pmatrix} x_0 \\ \dot{x}_0 \\ \ddot{x}_0 \end{pmatrix}$$

$$\tilde{H}\Big|_{t_0} = [1\ 0\ 0]$$

$$\hat{\mathbf{X}}(t_1) = \bar{\mathbf{X}}(t_1) + K_1(\mathbf{Y} - \tilde{H}\bar{\mathbf{X}}(t_1))$$

$$K_1 = \bar{P}_1 \tilde{H}^T(\tilde{H}\bar{P}_1\tilde{H}^T + R)^{-1}$$

$$\bar{X}_1 = \Phi(t_1, t_0)\bar{X}_0 = \begin{pmatrix} 2.5 \\ 2 \\ 1 \end{pmatrix}$$

$$\bar{P}_1 = \Phi(t_1, t_0)\bar{P}_0\Phi^T(t_1, t_0) = \begin{pmatrix} 6.25 & 2.5 & 0.5 \\ 2.5 & 3 & 1 \\ 0.5 & 1 & 1 \end{pmatrix}$$

$$K_1 = \begin{pmatrix} 25/4 \\ 5/2 \\ 1/2 \end{pmatrix}\left(\frac{25}{4} + 1\right)^{-1} = \begin{pmatrix} 25/29 \\ 10/29 \\ 2/29 \end{pmatrix}$$

$$\hat{\mathbf{X}}(t_1) = \begin{pmatrix} 5/2 \\ 2 \\ 1 \end{pmatrix} + \frac{1}{29}\begin{pmatrix} 25 \\ 10 \\ 2 \end{pmatrix}(2 - 2.5) = \boxed{\frac{1}{29}\begin{pmatrix} 66 \\ 53 \\ 28 \end{pmatrix}}$$

$$\hat{\mathbf{X}}(t_0) = \Phi^{-1}(t, t_0)\hat{\mathbf{X}}(t_1)$$

$$= \begin{pmatrix} 1 & -1 & 0.5 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 66 \\ 53 \\ 28 \end{pmatrix}\frac{1}{29}$$

$$= \underline{\begin{matrix} 21/29 \\ 25/29 \\ 28/29 \end{matrix}}$$

Ch 4 Problem 30

$\vec{x} \Rightarrow n \times 1$

$= N(\bar{x}, P)$

$\vec{y} \Rightarrow m \times 1$

$= H\vec{x} + \vec{\varepsilon}$

$H \Rightarrow m \times n$

$\vec{\varepsilon} \Rightarrow m \times 1$

$\vec{\varepsilon} = N(0, R)$

a) $E[\vec{y}] = ?$

$E[\vec{y}] = E[H\vec{x} + \vec{\varepsilon}]$

$\boxed{= H\bar{x} + 0}$

b) Show $P_y = HPH^T + R$

$$P_y = E[(H\vec{x} + \vec{\varepsilon} - H\bar{x})(H\vec{x} + \vec{\varepsilon} - H\bar{x})^T]$$

$$= E[H\vec{x}\vec{x}^T H^T + \vec{\varepsilon}\vec{x}^T H^T - H\bar{x}\vec{x}^T H^T + H\vec{x}\vec{\varepsilon}^T + \vec{\varepsilon}\vec{\varepsilon}^T - H\bar{x}\vec{\varepsilon}^T$$

$$- H\vec{x}\bar{x}^T H^T - \vec{\varepsilon}\bar{x}^T H^T + H\bar{x}\bar{x}^T H^T]$$

$$= E[\underbrace{H\vec{x}\vec{x}^T H^T - H\bar{x}\vec{x}^T H^T}_{P} + \underbrace{\vec{\varepsilon}\vec{\varepsilon}^T}_{R} + H(\underbrace{\vec{x} - \bar{x}}_{0})\vec{\varepsilon}^T + \vec{\varepsilon}(\underbrace{\vec{x}^T - \bar{x}^T}_{0})H^T]$$

$$\underbrace{- H\bar{x}\cancel{\bar{x}^T} H^T + H\bar{x}\cancel{\bar{x}^T} H^T}$$

$$P = E[(x - \bar{x})(\ )^T] = E[xx^T - \bar{x}x^T - x\bar{x}^T + \bar{x}\bar{x}^T]$$

$$= E[xx^T] - \bar{x}\bar{x}^T - \cancel{\bar{x}\bar{x}^T} + \cancel{\bar{x}\bar{x}^T}$$

$$\Rightarrow \boxed{P_y = HPH^T + R}$$

c) Show $P_{xy} = PH^T$

$$P_{xy} = E[(\vec{x} - \bar{x})(H\vec{x} + \varepsilon - H\bar{x})^T]$$

$$= E[\vec{x}\vec{x}^T H^T - \bar{x}\vec{x}^T H^T + \vec{x}\vec{\varepsilon}^T - \bar{x}\vec{\varepsilon}^T - \vec{x}\bar{x}^T H^T + \bar{x}\bar{x}^T H^T]$$

$$= E[\vec{x}\vec{x}^T - \bar{x}\vec{x}^T]H^T + E[(\vec{x} - \bar{x})\vec{\varepsilon}] - \bar{x}\bar{x}^T H^T + \bar{x}\bar{x}^T H^T$$

$$\boxed{= PH^T}$$

Ch 4 Problem 31

$$\vec{y} = H\vec{x} + \vec{\varepsilon}$$

$$\vec{\varepsilon} = N(\bar{\varepsilon}, R)$$

$$\bar{x} = \vec{x} + \vec{e}$$

$$\vec{e} = N(0, \bar{P})$$

$$E[\vec{e}\vec{\varepsilon}^T] = 0$$

$$\hat{x} = \bar{x} + K(\vec{y} - H\bar{x})$$

a) bias?

$$E[\hat{x}] = E[\vec{x} + \vec{e} + K(H\vec{x} + \vec{\varepsilon} - H\vec{x} - H\vec{e})]$$

$$= \vec{x} + \cancel{KH\vec{x}} + K\bar{\varepsilon} - \cancel{KH\vec{x}}$$

$$= \vec{x} + K\bar{\varepsilon} \Rightarrow \boxed{\text{bias} = K\bar{\varepsilon}}$$

b) $P = ?$

$$P = E[(\hat{x} - x - K\bar{\varepsilon})(\;)^T]$$

$$= E[(\cancel{\vec{x}} + \vec{e} + \cancel{KH\vec{x}} + K\vec{\varepsilon} - \cancel{KH\vec{x}} - KH\vec{e} - \cancel{\vec{x}} - K\bar{\varepsilon})(\;)^T]$$

$$= E[\underbrace{\vec{e}\vec{e}^T}_{\bar{P}} + \cancel{K\vec{\varepsilon}\vec{e}^T}^{0} - KH\vec{e}\vec{e}^T - \cancel{K\bar{\varepsilon}\vec{e}^T}^{0}$$
$$+ \cancel{\vec{e}\vec{\varepsilon}^T K^T} + \underbrace{K\vec{\varepsilon}\vec{\varepsilon}^T K^T}_{R} - \cancel{KH\vec{e}\vec{\varepsilon}^T K^T} - K\bar{\varepsilon}\vec{\varepsilon}^T K^T$$
$$- \vec{e}\vec{e}^T H^T K^T - \cancel{K\vec{\varepsilon}\vec{e}^T H^T K^T} + KH\vec{e}\vec{e}^T H^T K^T + \cancel{K\bar{\varepsilon}\vec{e}^T H^T K^T}$$
$$- \cancel{\vec{e}\bar{\varepsilon}^T K^T} - K\vec{\varepsilon}\bar{\varepsilon}^T K^T + \cancel{KH\vec{e}\bar{\varepsilon}^T K^T} + K\bar{\varepsilon}\bar{\varepsilon}^T K^T]$$

$$= E[\vec{e}\vec{e}^T - KH\vec{e}\vec{e}^T + K\vec{\varepsilon}\vec{\varepsilon}^T K^T - K\bar{\varepsilon}\vec{\varepsilon}^T K^T - \vec{e}\vec{e}^T H^T K^T + KH\vec{e}\vec{e}^T H^T K^T$$
$$- K\vec{\varepsilon}\bar{\varepsilon}^T K^T + K\bar{\varepsilon}\bar{\varepsilon}^T K^T$$

$$= E[((I - KH)\vec{e})((I - KH)\vec{e})^T + K(\vec{\varepsilon} - \bar{\varepsilon})(\vec{\varepsilon} - \bar{\varepsilon})^T K^T]$$

$$= E[(I - KH)\vec{e}\vec{e}^T (I - KH)^T] + KRK^T$$

$$\boxed{= (I - KH)\bar{P}(I - KH)^T + KRK^T}$$

cont →

c) $\vec{\epsilon} = \bar{\epsilon} + \vec{\epsilon}' \Rightarrow E(\vec{\epsilon}) = E(\bar{\epsilon} + \epsilon') = E(\epsilon') + \bar{\epsilon} = \bar{\epsilon} \Rightarrow E(\epsilon') = 0$

$$\hat{x} = \bar{x} + K\left([H\ I]\begin{pmatrix} x \\ \bar{\epsilon} \end{pmatrix} + \epsilon' - [H\ I]\begin{pmatrix} x+e \\ \bar{\epsilon} \end{pmatrix}\right)$$

$$= \begin{pmatrix} x+e \\ \bar{\epsilon} \end{pmatrix} + K\epsilon' - K[H\ I]\begin{pmatrix} e \\ 0 \end{pmatrix}$$

$$E[\hat{x}] = E\left[\begin{pmatrix} x+e \\ \bar{\epsilon} \end{pmatrix} + K\epsilon' - K[H\ I]\begin{pmatrix} e \\ 0 \end{pmatrix}\right]$$

$$\underline{\underline{E[\hat{x}] = \begin{pmatrix} x \\ \bar{\epsilon} \end{pmatrix}}}$$