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ASEN5070
HW3

Problem 1: Dynamical System $\dot{\vec{X}}(t) = A\vec{X}(t)$

Given STM $\Phi = \begin{bmatrix} e^{-2at} & 0 \\ 0 & e^{bt} \end{bmatrix}$

Determine A.

$$\dot{\Phi} = A\Phi \Rightarrow \frac{d\Phi}{dt} = \begin{bmatrix} -2ae^{-2at} & 0 \\ 0 & be^{bt} \end{bmatrix} = \dot{\Phi}$$

$$A\Phi\Phi^{-1} = \dot{\Phi}\Phi^{-1} \Rightarrow A = \dot{\Phi}\Phi^{-1}$$

$$|\Phi| = e^{(-2a+b)t}$$

$$\Phi^{-1} = \frac{1}{e^{(-2a+b)t}} \begin{bmatrix} e^{bt} & 0 \\ 0 & e^{-2at} \end{bmatrix} = \begin{bmatrix} \frac{e^{bt}}{e^{(-2a+b)t}} & 0 \\ 0 & \frac{e^{-2at}}{e^{(-2a+b)t}} \end{bmatrix}$$

$$= \begin{bmatrix} e^{bt+2at-bt} & \\ 0 & e^{-2at+2at-bt} \end{bmatrix} = \begin{bmatrix} e^{2at} & 0 \\ 0 & e^{-bt} \end{bmatrix}$$

$$A = \begin{bmatrix} -2ae^{-2at} & 0 \\ 0 & be^{bt} \end{bmatrix} \begin{bmatrix} e^{2at} & 0 \\ 0 & e^{-bt} \end{bmatrix}$$

$$\begin{bmatrix} -2ae^{(-2at+2at)} & 0 \\ 0 & be^{(bt-bt)} \end{bmatrix}$$

$$\boxed{A = \begin{bmatrix} -2a & 0 \\ 0 & b \end{bmatrix}}$$

Problem 2: Given the solution of

$$\dot{\vec{x}}(t_i) = A(t_i)\vec{x}(t_i)$$

is

$$\vec{x}(t_i) = \Phi(t_i, t_k)\vec{x}(t_k)$$

where

$$\Phi(t_k, t_k) = [I]$$

a) show that $\dot{\Phi}(t_i, t_k) = A(t_i)\Phi(t_i, t_k)$

$$\frac{d}{dt}(\vec{x}(t_i)) = \frac{d}{dt}(\Phi(t_i, t_k)\underbrace{\vec{x}(t_k)}_{\text{const}}))$$

$$\dot{\vec{x}}(t_i) = \dot{\Phi}(t_i, t_k)\vec{x}(t_k)$$

$$= A(t_i)\vec{x}(t_i) = A(t_i)\Phi(t_i, t_k)x(t_k)$$

$$\Rightarrow \dot{\Phi}(t_i, t_k)\vec{x}(t_k) = A(t_i)\Phi(t_i, t_k)\vec{x}(t_k)$$

Mult both sides by $\frac{x_k^T}{x_k x_k^T}$

$$\Rightarrow \dot{\Phi}(t_i, t_k) = A(t_i)\Phi(t_i, t_k)$$

b) show that $\Phi(t_i, t_j) = \Phi(t_i, t_k)\Phi(t_k, t_j)$

$$x(t_i) = \Phi(t_i, t_k)\vec{x}(t_k)$$

$$= \Phi(t_i, t_j)\vec{x}(t_j)$$

$$\vec{x}(t_k) = \Phi(t_k, t_j)\vec{x}(t_j)$$

$$\vec{x}(t_i) = \Phi(t_i, t_j)\vec{x}(t_j) = \Phi(t_i, t_k)\Phi(t_k, t_j)\vec{x}(t_j)$$

$$\Phi(t_i, t_j) = \Phi(t_i, t_k)\Phi(t_k, t_j)$$

c) show that $\Phi^{-1}(t_i, t_k) = \Phi(t_k, t_i)$

$$\vec{x}(t_i) = \Phi(t_i, t_k)\vec{x}(t_k)$$

$$\vec{x}(t_k) = \Phi(t_k, t_i)\vec{x}(t_i) = \Phi(t_k, t_i)\Phi(t_i, t_k)\vec{x}(t_k)$$

$$[I] = \Phi(t_k, t_i)\Phi(t_i, t_k)$$

$$\Phi^{-1}(t_i, t_k) = \Phi(t_k, t_i)$$

Cont →

Problem 2 (cont)

d) Show that $\dot{\Phi}^{-1}(t_i,t_k) = -\Phi^{-1}(t_i,t_k)A(t_i)$

$$\frac{d}{dt}\left(\vec{x}(t_k) = \Phi^{-1}(t_i,t_k)\,\vec{x}(t_i)\right)$$

$$\Rightarrow \vec{0} = \dot{\Phi}^{-1}(t_i,t_k)\,\vec{x}(t_i) + \Phi^{-1}(t_i,t_k)\,\underbrace{\dot{\vec{x}}(t_i)}_{A(t_i)\vec{x}(t_i)}$$

$$\Rightarrow \dot{\Phi}^{-1}(t_i,t_k)\,\vec{x}(t_i) = -\Phi^{-1}(t_i,t_k)A(t_i)\,\vec{x}(t_i)$$

$$\Rightarrow \dot{\Phi}^{-1}(t_i,t_k) = -\Phi^{-1}(t_i,t_k)A(t_i)$$

Problem 3: Determine least squares estimate for $\hat{x}$

$$J(\vec{x}) = \frac{1}{2}\vec{\epsilon}^T W \vec{\epsilon} + \frac{1}{2}\vec{\eta}^T \bar{W} \vec{\eta}$$

where $\vec{\eta} = (\bar{\vec{x}} - \vec{x})$, $\vec{\epsilon} = \vec{y} - H\vec{x}$

W being symmetric: ← App. B.7.2

$$\frac{\partial}{\partial \vec{x}}\left(\frac{1}{2}\vec{\epsilon} W \vec{\epsilon}\right) = (W^{\frac{1}{2}}\vec{\epsilon})^T W^{\frac{1}{2}} \frac{\partial \vec{\epsilon}}{\partial \vec{x}} = \epsilon^T \underbrace{W^{\frac{1}{2}T} W^{\frac{1}{2}}}_{=W} \cdot -H$$

$$= -H^T W \vec{\epsilon} = -H^T W \vec{y} + H^T W H \vec{x}$$

$\bar{W}$ symmetry:

$$\frac{\partial}{\partial \vec{x}}\left(\frac{1}{2}\vec{\eta} \bar{W} \vec{\eta}\right) = (\bar{W}^{\frac{1}{2}}\vec{\eta})^T \bar{W}^{\frac{1}{2}} \frac{\partial \vec{\eta}}{\partial \vec{x}} = \vec{\eta}^T \bar{W} \cdot -I$$

$$= -\bar{W}\vec{\eta} = -\bar{W}\bar{\vec{x}} + \bar{W}\vec{x}$$

$$\frac{\partial J(\vec{x})}{\partial \vec{x}} = 0 = -H^T W \vec{y} + H^T W H \vec{x} - \bar{W}\bar{\vec{x}} + \bar{W}\vec{x}$$

$$\underline{\hat{x} = (H^T W H + \bar{W})^{-1}(+H^T W \vec{y} + \bar{W}\bar{\vec{x}})}$$

Problem 4: Determine Φ for

$$A = \begin{bmatrix} a & 0 \\ b & g \end{bmatrix}, \quad a \neq g$$

Need a different, equivalent A

$$\dot{x} = ax$$

$$\ddot{x} = bx + g\dot{x} = bx + agx = (b+ag)x \Rightarrow \begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \underbrace{\begin{bmatrix} a & 0 \\ b+ag & 0 \end{bmatrix}}_{A} \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$$

$$\dot{\Phi} = \begin{bmatrix} a & 0 \\ b+ag & 0 \end{bmatrix} \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} = \begin{pmatrix} a\phi_{11} & a\phi_{12} \\ (b+ag)\phi_{11} & (b+ag)\phi_{12} \end{pmatrix} = \begin{pmatrix} \dot{\phi}_{11} & \dot{\phi}_{12} \\ \dot{\phi}_{21} & \dot{\phi}_{22} \end{pmatrix}$$

$$\frac{d\phi_{11}}{dt} = a\phi_{11} \Rightarrow \frac{d\phi_{11}}{\phi_{11}} = a\,dt \Rightarrow \phi_{11} = c_{11}e^{at}$$

$$t_0: \; 1 = c_{11}e^{at_0} \Rightarrow c_{11} = e^{-at_0}$$

$$\underline{\phi_{11} = e^{a\Delta t}}$$

$$\phi_{12} = c_{12}e^{at}$$

$$t_0: \; 0 = c_{12}e^{at} \Rightarrow c_{12} = 0 \Rightarrow \underline{\phi_{12} = 0}$$

$$\frac{d\phi_{21}}{dt} = (b+ag)e^{a\Delta t} \Rightarrow d\phi_{21} = (b+ag)e^{a\Delta t}dt$$

$$\phi_{21} = a(b+ag)e^{a\Delta t} + c_{21}$$

$$t_0: \; 0 = a(b+ag)\cancel{e}^{1} + c_{21} \Rightarrow c_{21} = -a(b+ag)$$

$$\phi_{21} = a(b+ag)(e^{a\Delta t} - 1)$$

$$\dot{\phi}_{22} = 0, \quad \phi_{22}(t_0) = 1 = \phi_{22}(t)$$

$$\boxed{\Phi = \begin{bmatrix} e^{a\Delta t} & 0 \\ a(b+ag)(e^{a\Delta t} - 1) & 0 \end{bmatrix}}$$

Problem 5: Find Φ for $\ddot{x} = -abx$

$\vec{x} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$, $\dot{\vec{x}} = \begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} \dot{x} \\ -abx \end{bmatrix}$

A: $\begin{bmatrix} \dot{x} \\ -abx \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -ab & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$

$$\Phi = \mathcal{L}^{-1}\left[(s[I]-A)^{-1}\right] = \mathcal{L}^{-1}\left[\begin{bmatrix} s & -1 \\ +ab & s \end{bmatrix}^{-1}\right] \; ; \quad \begin{vmatrix} s & -1 \\ ab & s \end{vmatrix} = s^2+ab = s^2+(\sqrt{ab})^2$$

$$= \mathcal{L}^{-1}\begin{bmatrix} \frac{s}{s^2+ab} & +\frac{1}{s^2+ab} \\ -\frac{ab}{s^2+ab} & \frac{s}{s^2+ab} \end{bmatrix}$$

$$\boxed{\Phi = \begin{bmatrix} \cos(\sqrt{ab}\,\Delta t) & \frac{1}{\sqrt{ab}}\sin(\sqrt{ab}\,\Delta t) \\ -\sqrt{ab}\sin(\sqrt{ab}\,\Delta t) & \cos(\sqrt{ab}\,\Delta t) \end{bmatrix}}$$

where $\Delta t = t - t_0$

Problem 6: Show that

$$\Phi(t,t_0) = \begin{bmatrix} 3e^{at} & 0 \\ 0 & 2e^{-bt} \end{bmatrix}$$

Satisfies $\dot{\Phi} = A\Phi$, but that Φ is not a STM $(t_0 = 0)$

$$\dot{\Phi} = \begin{pmatrix} 3ae^{at} & 0 \\ 0 & -2be^{-bt} \end{pmatrix}$$

$$\left.\begin{aligned} 3ae^{at} &= A_{11} \cdot 3e^{at} + A_{12} \cdot 0 \Rightarrow A_{11} = a \\ 0 &= A_{11} \cdot 0 + A_{12} \cdot 2e^{-bt} \Rightarrow A_{12} = 0 \\ 0 &= A_{21} \cdot 3ae^{at} + A_{22} \cdot 0 \Rightarrow A_{21} = 0 \\ -2be^{-bt} &= A_{21} \cdot 0 + A_{22} \cdot 2e^{-bt} \Rightarrow A_{22} = -b \end{aligned}\right\}$$

A is a valid matrix, so $\dot{\Phi} = A\Phi$ is satisfied

<u>However,</u>

STMs: $\Phi(t_0, t_0) = [I]$

$$\begin{bmatrix} 3e^{a(0)} & 0 \\ 0 & 2e^{-b(0)} \end{bmatrix} = \begin{bmatrix} 3(1) & 0 \\ 0 & 2(1) \end{bmatrix} \neq [I]$$

Problem 7: Show that when

$$\Phi^{-1}(t,t_K) = -(J\Phi(t,t_K)J)^T = \left[\begin{array}{c|c} \Phi_4^T & -\Phi_2^T \\ \hline -\Phi_3^T & \Phi_1^T \end{array}\right]$$

that $\Phi(t,t_K)$ is symplectic : $\Phi(t,t_K)^T J\,\Phi(t,t_K) = J$

$J = \left[\begin{array}{c|c} 0 & I \\ \hline -I & 0 \end{array}\right]$, J is orthoganal: $J^{-1} = J^T$

also, $J^T = -J$

$$\Phi(t,t_K) = \left[\begin{array}{c|c} \Phi_1 & \Phi_2 \\ \hline \Phi_3 & \Phi_4 \end{array}\right]$$

$$\Phi^{-1} = -J^T\Phi^T J^T$$

$\Rightarrow$

$$[I] = -J^T\Phi^T J^T \Phi$$

$$\Rightarrow (J^T)^{-1} = -\Phi^T J^T \Phi$$

$$\Rightarrow \underline{J = \Phi^T J \Phi}$$