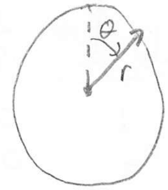
1. Given: $\ddot{r} = r\dot{\theta}^2 - \frac{k}{r^2} + u_1(t)$ $\qquad r(0)=r_0$

$\qquad \dot{r}(0)=0$

$\ddot{\theta} = \frac{-2\dot{\theta}\dot{r}}{r} + \frac{1}{r}u_2(t)$ $\qquad \theta(0)=0$

$\qquad \dot{\theta}(0)=\omega_0$



a)

Let $\vec{x} = \begin{pmatrix} r \\ \dot{r} \\ \theta \\ \dot{\theta} \end{pmatrix}$, $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

$$\boxed{\dot{\vec{x}} = \begin{pmatrix} \dot{r} \\ \ddot{r} \\ \dot{\theta} \\ \ddot{\theta} \end{pmatrix} = \begin{pmatrix} \dot{r} \\ r\dot{\theta}^2 - \frac{k}{r^2} \\ \dot{\theta} \\ \frac{-2\dot{\theta}\dot{r}}{r} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & \frac{1}{r} \end{pmatrix} \vec{u}}$$

b)

$\delta\dot{\vec{x}} = \frac{\delta \vec{f}}{\delta \vec{x}}\delta\vec{x} + \cancel{\frac{\delta \vec{f}}{\delta \vec{u}}\delta\vec{u}}$ 0, $\quad x_n = \begin{pmatrix} r_0 \\ 0 \\ \omega_0 t + c \\ \omega_0 \end{pmatrix}$

$\frac{\delta f_1}{\delta \vec{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$

$\frac{\delta f_2}{\delta \vec{x}} = \begin{bmatrix} \dot{\theta}^2 + 2\frac{k}{r^3} & 0 & 0 & 2r\dot{\theta} \end{bmatrix}_n$

$= \begin{bmatrix} \omega_0^2 + 2\frac{k}{r_0^3} & 0 & 0 & 2r_0\omega_0 \end{bmatrix}$

$\frac{\delta f_3}{\delta \vec{x}} = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$

$\frac{\delta f_4}{\delta \vec{x}} = \begin{bmatrix} \frac{2\dot{\theta}\dot{r}}{r^2} - \frac{u_2}{r^2} & -\frac{2\dot{\theta}}{r} & 0 & -\frac{2\dot{r}}{r} \end{bmatrix}_n$

$= \begin{bmatrix} 0 & -\frac{2\omega_0}{r_0} & 0 & 0 \end{bmatrix}$

$$\Rightarrow \boxed{\delta\dot{\vec{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \omega_0^2 + 2\frac{k}{r_0^3} & 0 & 0 & 2r_0\omega_0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{2\omega_0}{r_0} & 0 & 0 \end{bmatrix} \delta\vec{x}}$$

1. cont

c) $$\left.\frac{\delta \vec{f}}{\delta \vec{u}}\right|_{\begin{subarray}{l}u_n\\x_n\end{subarray}} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & \frac{1}{r} \end{bmatrix}_n = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & \frac{1}{r_0} \end{bmatrix}$$

$$\boxed{\begin{aligned} \dot{\vec{X}} &= \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ \omega_0^2 + 2\frac{K}{r_0^3} & 0 & 0 & 2r_0\omega_0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{2\omega_0}{r_0} & 0 & 0 \end{bmatrix}}_{A} \vec{X} + \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & \frac{1}{r_0} \end{bmatrix}}_{B} \vec{u} \\ \vec{y} &= \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{C} \vec{X} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{D} \vec{u} \end{aligned}}$$

2. Using the axioms found on pp 159-160 of Brogan

a) for even fcns, $f(-t)=f(t)$, so symmetric about the y-axis

1. $\vec{x}\in G$, $\vec{y}\in G$, $\vec{z}\in G \Rightarrow \vec{x}+\vec{y}=\vec{z}$

   True, because even functions add to be other even functions
2. $\vec{x}+\vec{y}=\vec{y}+\vec{x} \Rightarrow$ True
3. Associative Addition $\Rightarrow$ True
4. $\exists$ zero-vector $\Rightarrow$ True, 0 is an even fcn
5. $-\vec{x}$ exist for all $\vec{x} \Rightarrow$ True, all $-f(t)$ is even for even $f(t)$
6. scalar Mult $\Rightarrow$ True
7. Associative Mult. of scalars $\Rightarrow$ True
8. Distributive Mult. of scalars $\Rightarrow$ True

$\boxed{\text{Is a vector space}}$

b) $G=\{\text{all real pairs}\}$, $g_1+g_2=(a_1,a_2)\#(b_1,b_2)=(a_1+2b_1, a_2+3b_2)$

1. True, see addition definition ↑
2. $g_1+g_2=(a_1+2b_1, a_2+3b_2)$

   $g_2+g_1=(b_1+2a_1, b_2+3a_2)$ $\Rightarrow a_1+2b_1=b_1+2a_1 \Rightarrow b_1=a_1$

   False, only true if $a_1=b_1$, $\therefore$ cannot add any two vectors commutatively

$\boxed{\therefore \text{Not a vector space}}$

c) $G=\{\text{all real pairs}\}$, $c(a_1,a_2)=\begin{cases}(0,0) & \text{if } c=0\\ (ca_1, \frac{a_2}{c}) & \text{if } c\neq 0\end{cases}$

1. True
2. Commutative Addition: True
3. Associative Addition: True
4. $\exists$ zero vector: True
5. $-\vec{x}$ exist for all $\vec{x}$: True
6. Scalar mult, & unit scalar Mult: True, $1\cdot\vec{x}=1\cdot(a_1,a_2)=(1a_1,\frac{a_2}{1})$
7. Assoc. Mult: $b(c\vec{x})=b\cdot(c\cdot a_1,\frac{a_2}{c})=(b\cdot c\cdot a_1,\frac{a_2}{c\cdot b})$

   $(bc)\vec{x}=(b\cdot c\cdot a_1,\frac{a_2}{bc})$ $\Rightarrow$ True
8. Dist. Mult: $(c+b)\vec{x}=((c+b)a_1,\frac{a_2}{c+b})$

   $c\vec{x}+b\vec{x}=(ca_1,\frac{a_2}{c})+(ba_1,\frac{a_2}{b})=((c+b)a_1, a_2\cdot(\frac{1}{b}+\frac{1}{c}))$ $\neq$ False

$\boxed{\therefore \text{Not a vector space}}$

3. Find basis for $N(A)$, $R(A)$, & dim for both

a) Square matrix, so if $\det(A) \neq 0$ it is L.I. and full rank.

Used Matlab for determinant calculation, see Attached result

$\det(A) = 8.456 \times 10^3$

$\Rightarrow \dim(N(A)) = 0 \Rightarrow N(A) = \vec{0}$

$\dim(R(A)) = 5$

$$R(A) = \text{columns of } A = \left\{ \begin{bmatrix} 1.0000 \\ -10.5092 \\ 4.5354 \\ 0.9451 \\ 1.0000 \end{bmatrix}, \begin{bmatrix} -10.5092 \\ 1.0000 \\ -0.1899 \\ 0.2517 \\ 1.0000 \end{bmatrix}, \begin{bmatrix} 4.5354 \\ -0.1899 \\ 1.0000 \\ -5.3037 \\ 1.0000 \end{bmatrix}, \begin{bmatrix} 0.9451 \\ 0.2517 \\ -5.3037 \\ 1.0000 \\ 1.0000 \end{bmatrix}, \begin{bmatrix} 1.0000 \\ 1.0000 \\ 1.0000 \\ 1.0000 \\ 2.0000 \end{bmatrix} \right\}$$

b) $\dim(R(A))$ = number of pivots in R matrix, derived from QR composition.

$\dim(N(A)) = n - r$ if A is $m \times n$

$A = QR$

See attached Matlab for Q & R matrices

$\text{rank}(A) = 5$, $\therefore$ $\boxed{\dim(R(A)) = 5, \ \dim(N(A)) = 2}$

basis of $R(A)$ = columns of Q. See Attached Matlab result.

basis of $N(A)$: if $A^T = Q_{A^T} R_{A^T}$, Last two columns of Q_{A^T} are the basis of $N(A)$.

See attached Matlab result.