Challenge 1

$M_{m\times n}, m>n$, $y \in \{\mathbb{R}^m = R(M) \oplus N(M^T)\}$

$x \in \{\mathbb{R}^n = R(M^T) \oplus N(M)\}$ $\dim(N(M))=0$

$w \in N(M^T)$

w is a member of a subspace that is not contained by $\mathbb{R}^n$, which x is a member of. The only component of y that can affect x is contained within R(M)

Challenge 2

a) Let $\vec{x} = \begin{bmatrix}2\\3\end{bmatrix}$, $A = I_{2\times 2} \Rightarrow b = \begin{bmatrix}2\\3\end{bmatrix} \Rightarrow [A \mid b] = \left[\begin{array}{cc|c}1&0&2\\0&1&3\end{array}\right]$

let $W' = \left[\begin{array}{cc|c}1&2&0\\0&3&1\end{array}\right] \Rightarrow \begin{bmatrix}1&2\\0&3\end{bmatrix} \cdot \begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}8\\9\end{bmatrix} \neq \begin{bmatrix}0\\1\end{bmatrix}$ $\boxed{\text{False}}$

b) $\boxed{\text{False}}$ the system of eqns. produces the same solution if any rows are scaled or added, but swapping rows swaps elements of $\vec{x}$

c) $\boxed{\text{True}}$, $\text{Rank}(A_{n\times n}) = n$ iff all columns are L.I., Reduced-REF requires that only the pivot 1 be in the pivot column, and there can only be one pivot per row. A pivot represents an LI column.

d) $\boxed{\text{True}}$, elementary row operations are used to create RREF. A finite # of operations are used.

e) consistent → at least one soln. According to Rouche-Capelli Thm, inconsistent if $\text{Rank}([A \mid b]) > \text{Rank}(A)$. if $A = \begin{bmatrix}1&0&2\\0&1&0\\0&0&0\end{bmatrix}$ and $b = \begin{bmatrix}0\\0\\1\end{bmatrix}$, $[A \mid b]$ is RREF but $\text{Rank}(A) = 2$ and $\text{Rank}([A \mid b]) = 3$, $\therefore \boxed{\text{False}}$

f) $A_{m\times n}$, $x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $r = \text{rank}(A) \to Ax = 0$ solns are in $N(A)$. $\underline{\dim}(N(A)) = n-r$ $\boxed{\text{True}}$

g) $\boxed{\text{True}}$, pivots only exist in non-zero rows, and # pivots = Rank