**Coordinate Transformations for Unsteady Frames**

Johnathan Clouse[[1]](#footnote-1)

University of Colorado, Boulder, CO, 80309

A Frenet frame is a useful coordinate system to describe motion with respect to the motion of a particle. A system utilizing a ground station, mothership, and Micro Aerial Vehicle (MAV) is explored. MAV motion is determined by coordinate transformation to provide position, velocity, and acceleration with respect to the ground station. Error between direct numerical differentiation of the position and rotational kinematics is investigated.

# Nomenclature

= tangent vector of Frenet frame

= binormal vector of Frenet frame

= normal vector of Frenet frame

= position vector of Micro Aerial Vehicle (MAV) expressed in the ground station (GS) Cartesian frame

= velocity vector of MAV expressed in the GS Cartesian frame

= acceleration vector of MAV expressed in the GS Cartesian frame

t = time of measurement

G = fundamental metric tensor between two frames

**=** rotation rate between two frames

= angular acceleration between two frames

= MAV quantities expressed in the mothership (MS) Frenet frame

= MS quantities expressed in the GS Cartesian frame

# Introduction

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HE Frenet frame is a useful set of coordinates centered on a particle. The basis vectors describe the motion of the particle, showing the instantaneous direction of motion and the instantaneous radius of curvature of the path1. This project focuses on the use of the Frenet frame on the paths of an unmanned aerial vehicle (UAV) mothership (MS) and Micro Aerial Vehicles (MAVs) deployed by the MS to act as sensory equipment1. The MAV motion with respect to the MS are a known quantity. Sensory data is then relayed from the MS to a ground station (GS), so coordinate transformations must be applied to obtain the MAV motion with respect to a Cartesian frame centered on the GS.

The Frenet frame for a particle on a path is obtained by defining the tangent vector as

One can see that the tangent vector is solely in the direction of motion. The binormal vector is defined as

Thus, the binormal vector is normal to both the direction of motion and the particle acceleration. It is important to note that this vector cannot be defined when the velocity and acceleration are collinear. Finally, the normal vector is found to complete the three-dimensional coordinate frame by

This normal vector is normal to the curve of the path, and so it is useful to find the normal component of a particle’s acceleration.

To find the Frenet basis vectors for a given instant, the first and second derivatives of the position vector must be known. An analytic expression for the position vector will easily yield an analytic expression for these derivatives, but it is not the case when only time-stamped position data are provided. In the latter case, one can forward-differentiate the *n*th data point by

This method will reduce the amount of data points by one for each derivative taken. For example, given five position vectors in time, one will end up with four velocity vectors and three acceleration vectors.

# Simulation of Known Analytical Path

Consider the path of the MS, with respect to the GS and expressed in a Cartesian frame1:

And the path of an MAV in the MS Frenet frame1:

Units are not given, so generic distance units (DU) and time units (TU) are used in the plots.

The derivatives required to obtain the Frenet bases are obtained by numerical forward differentiation, rather than analytically, to allow the script to handle discrete data from another source. For this analytical case, 1000 time-points are used. Figure 1 and Figure 2 show the positions of the mothership and MAV, with their Frenet basis vectors.

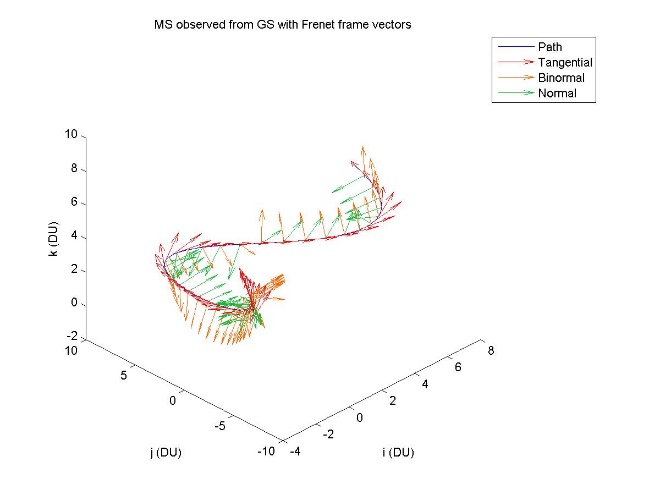


Figure :Mothership position wrt GS Cartesian, with its Frenet basis vectors

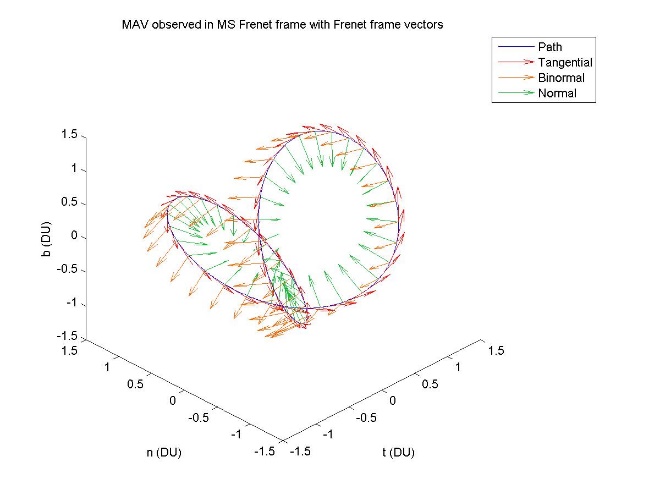


Figure : MAV position wrt MS Frenet frame, with its Frenet basis vectors

The mothership’s motion experiences an inflection in its path curvature, causing the normal and binormal Frenet vectors to change substantially. The MAV experiences a path with no inflections, so the change in the Frenet vectors is continuous.

The mothership’s speed and accelerations with respect to the GS Cartesian frame is shown in Figure 3 and Figure 4 below:

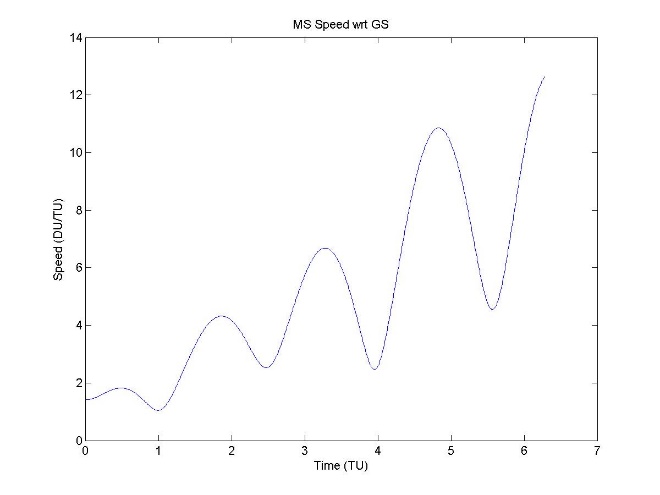


Figure : MS speed wrt GS

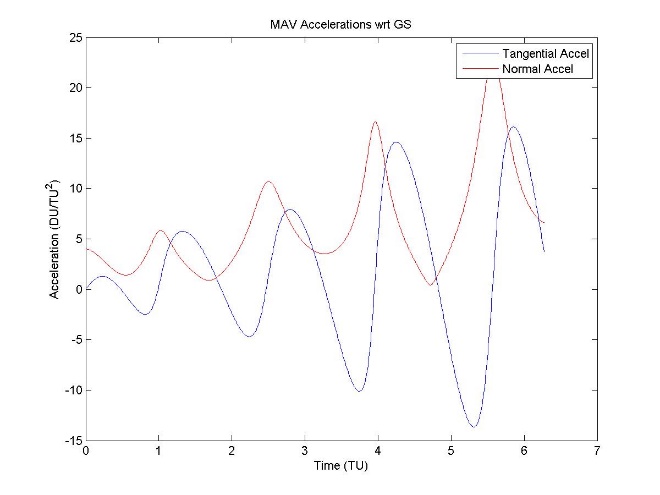


Figure : MS normal and tangential accelerations wrt GS

The MAV speed and accelerations with respect to the MS Frenet frame is shown in Figure 5 and Figure 6.

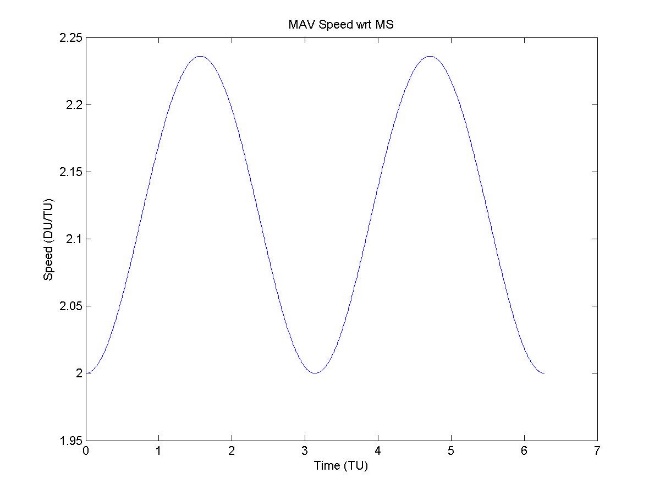


Figure : MAV speed wrt MS

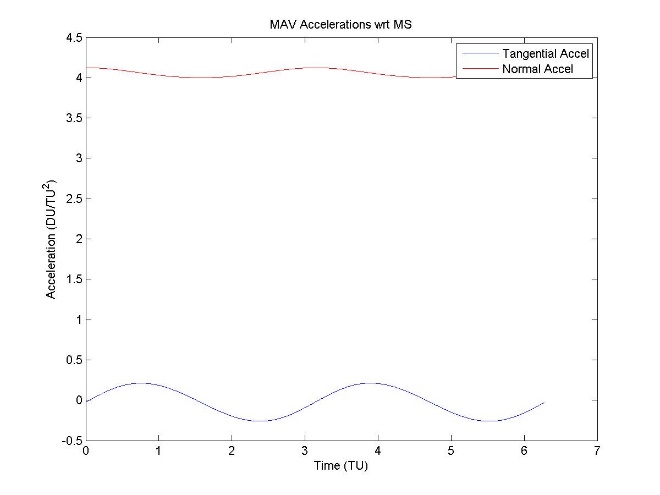


Figure : MAV accelerations wrt MS

The fundamental metric tensor between the GS Cartesian and MS Frenet frames is

(7)

The position of the MAV in the GS coordinate frame is then represented by

The resultant MAV positions are shown in Figure 7. There is a large change in MAV position in the vicinity of the MS inflection point.

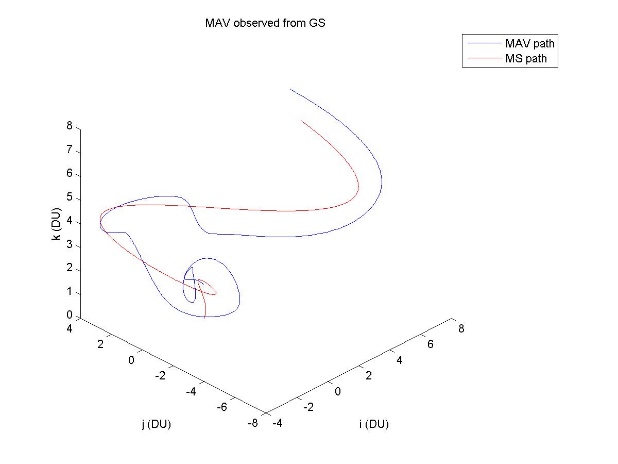
The MAV speed and acceleration with respect to the ground station frame are found by finding the rotational rate of the MS Frenet frame as seen by the GS and using relative motion equations. The rotational rate of one frame with respect to another can be found by2 

Figure : MAV position in GS Cartesian frame

where the fundamental metric tensor between the frames, G, is

between the GS Cartesian and MS Frenet frames, and the Euler angles are found by2

The velocity in the GS Cartesian frame is calculated by

The acceleration in the GS Cartesian frame is

The angular velocity is the time-derivative of . The application of this kinematic method can be seen in Figure 8 and Figure 9 for the speed calculation in both the rotating coordinate method and direct numerical differentiation of ***r***. Figure 10 shows the normal and tangential accelerations of the MAV in the GS Cartesian frame as calculated from the above kinematic equations; and Figure 11 shows the error between the kinematic solution and direct numerical differentiation of ***r***.

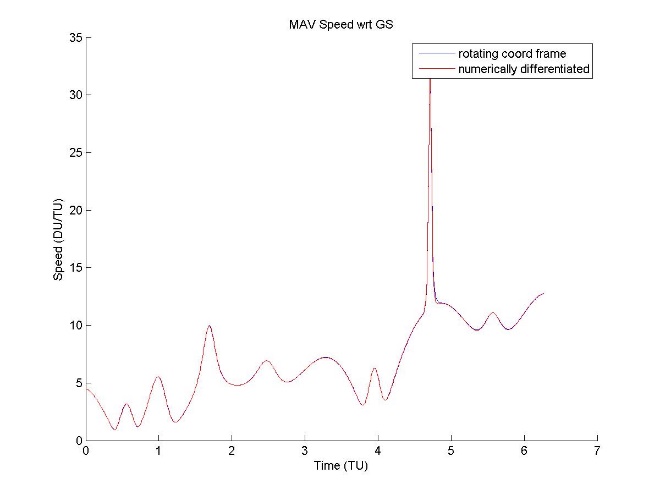


Figure :MAV speed in GS Cartesian frame

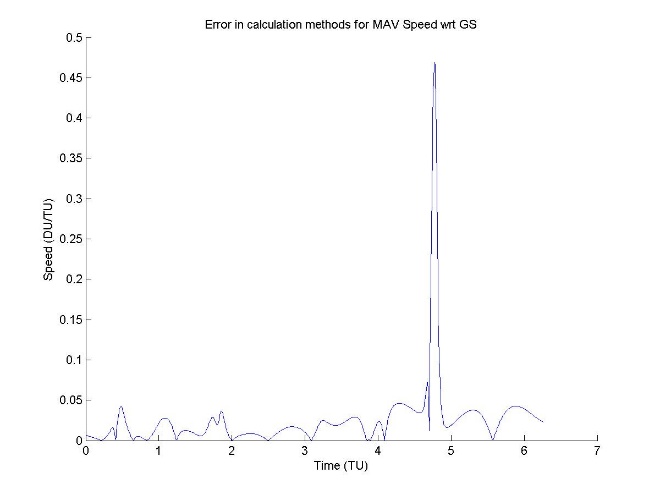


Figure : Error between methods for MAV speed wrt GS

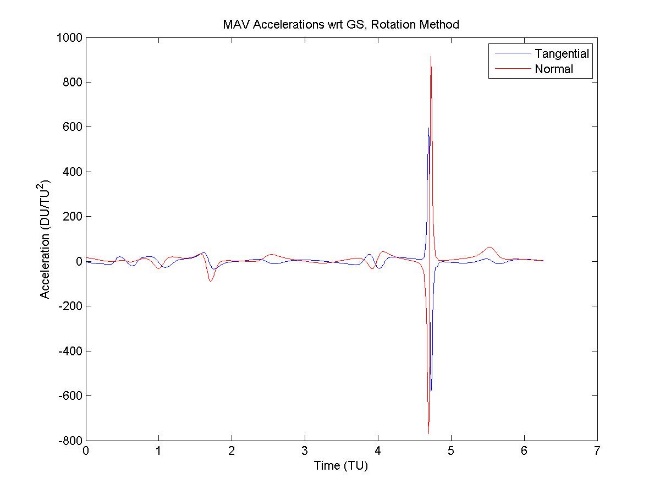


Figure :MAV accelerations in the GS Cartesian frame

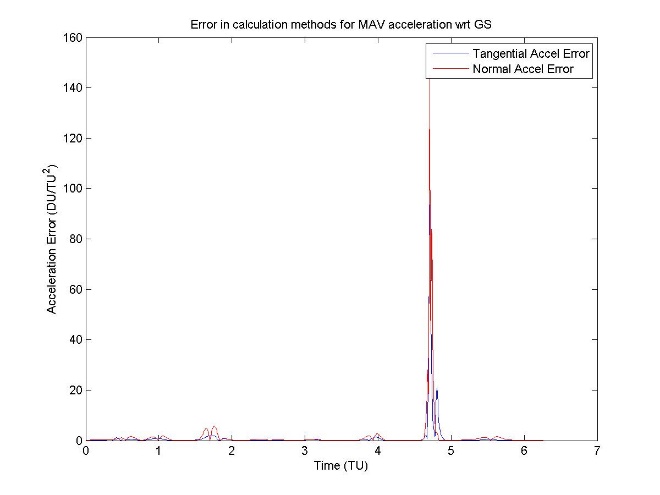


Figure : Error between methods for MAV accelerations.

From the above acceleration plots, it is clear that the inflection point of the MS affects the acceleration of the MAV viewed by the GS. The fast change in the mothership’s normal and binormal Frenet vectors require such a change in the MAV’s position as well, causing the spike in acceleration. The error between the two methods in acceleration calculation are also affected by this discontinuity.

# Discrete Data for MAV Position

Now, a data file with MAV positions in the MS Frenet frame is used to provide MAV position data; MS position is the same as the previous section. The data file lists 100 evenly-spaced data points for the same timescale as the last section. Figure 12 shows the MAV position in the MS Frenet frame, with its own Frenet vectors. Figure 13 shows the MAV position in the GS Cartesian frame. The speed and accelerations of the MAV can be seen in Figure 14 and Figure 15.

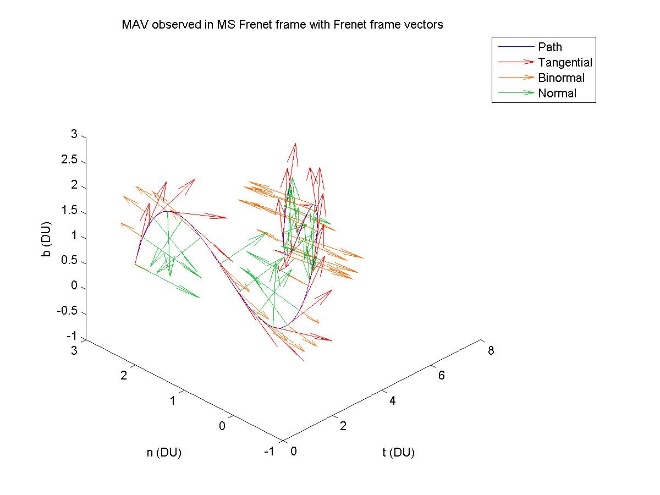


Figure : MAV position in MS Frenet frame, with Frenet vectors

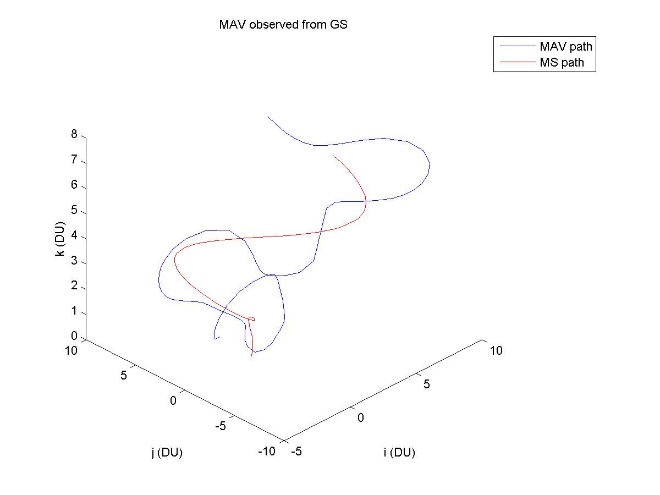


Figure : MAV position in GS Cartesian frame



Figure : MAV speed wrt MS

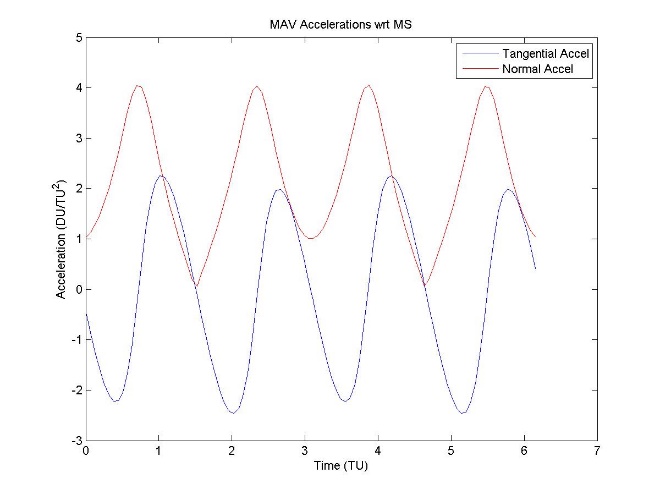


Figure : MAV accelerations wrt MS

The curvature of the MAV’s path in the MS Frenet frame experiences inflections in its own path this time. The MAV position in the GS Cartesian frame again experiences a large position change in the vicinity of the MS inflection.

The speed of the MAV with respect to the GS is found kinematically and numerically, and the results are displayed in Figure 16. The error between the two methods’ results are shown in Figure 17. The kinematically-computed accelerations of the MAV with respect to the GS are seen in Figure 18, and the numerically-differentiated result is shown in Figure 19. Figure 20 shows the error between the acceleration computations.

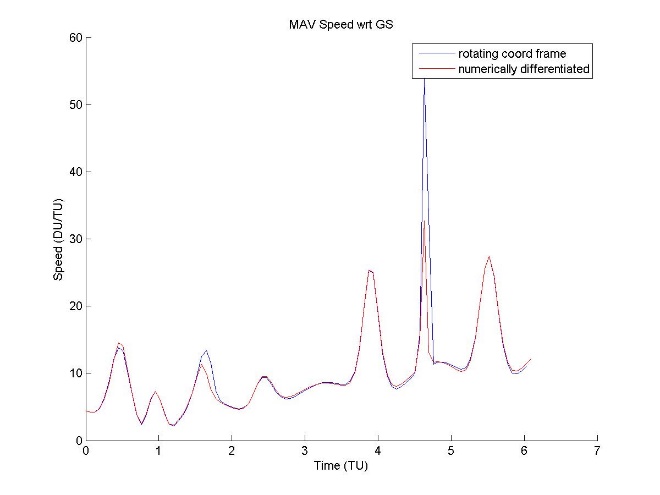


Figure : MAV speed wrt GS

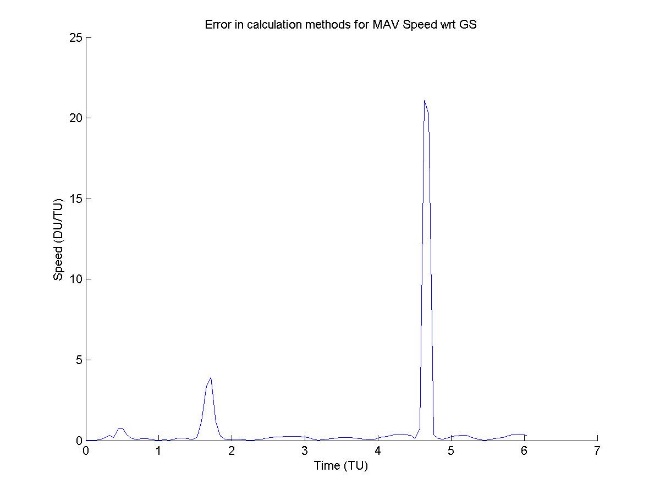


Figure : Error between methods for MAV speed wrt GS

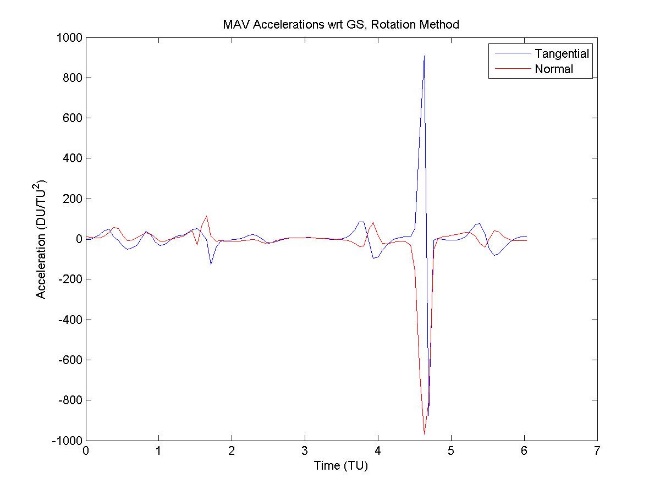


Figure : MAV accelerations wrt GS Cartesian frame, rotation method

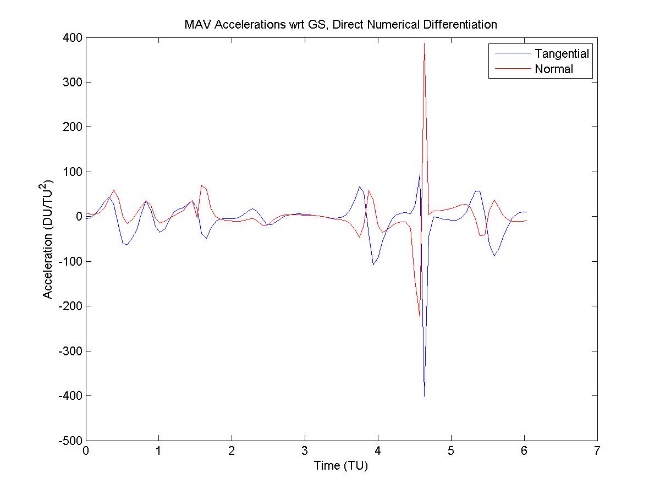


Figure : MAV accelerations wrt GS Cartesian frame, direct numerical differentiation

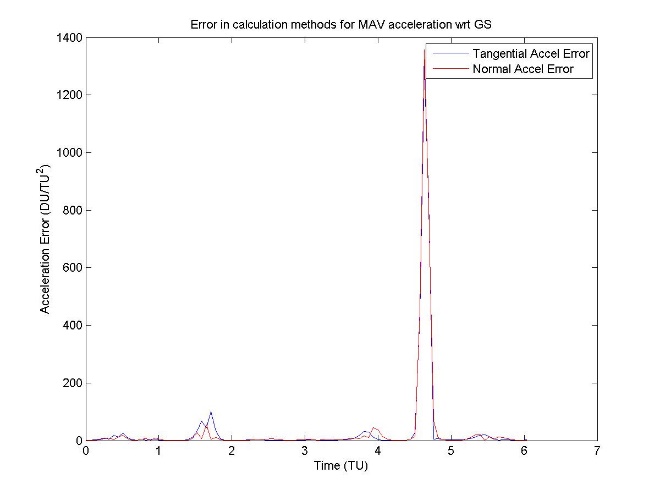


Figure : Error between methods for MAV accelerations.

The error between the kinematic and numerical differentiation methods does not track as well as before. This is due in part to the larger time step between the provided positions of the MAV, giving the numerical differentiation less granularity. The MS inflection point exacerbates the spike in the accelerations and the error calculation in the region. In fact, the accelerations are shown to increase in opposite directions between the two methods.

# Conclusion

The Frenet frame is a useful way to describe things with respect to a particle in motions, but there are caveats to its usage. First, the frame is undefined if the velocity and acceleration are collinear. One may work around this by fixing the basis vectors when the velocity and acceleration are nearly collinear, but care should be take that this does not cause different problems. Second, motion of a vehicle maintaining some position in the Frenet of another particle is at the mercy of the second particle’s motion. Inflections in the path could cause unsafe accelerations in the vehicle, or cause it to diverge from the desired path as it corrects its motion.

While a MAV may experience the aforementioned issues, an orbital mothership may be a better-suited use for the Frenet frame. Since two-body motion results in a continuously curved path, no inflections would be experienced and a secondary vehicle’s motion could easily be described.

# References

1Hussein, M., “Project: Coordinate Transformations for Unsteady Frames,” Fall 2016.

2Hussein, M., “Project Supplement: Rotating Coordinate Frames: Euler Angles,” Fall 2016.

# Appendix

MATLAB code for the project is on the following pages.

1. Graduate Student, Aerospace Engineering Sciences, 1111 Engineering Drive, Boulder, CO, 80309-0429 [↑](#footnote-ref-1)