3 Velocity Distribution Functions

Let us define a generalized Gaussian probability density function as:

$$f_s(x) = \frac{A_o}{\sqrt{2\pi\sigma^2}} \, e^{-\dfrac{(x - x_o)^2}{2\sigma^2}} \tag{3.1}$$

where x_o is the displacement of the peak from x = 0, A_o is a normalization amplitude, s denotes the specific set (later used for particle species) of data the distribution describes, and σ^2 is the variance (e.g., see Section 2.3.2). For this distribution, one can find the Full Width at Half Maximum (FWHM) as:

$$FWHM = 2\sqrt{2\ln 2}\,\sigma \tag{3.2}$$

which is an expression for the width of the distribution at half its peak value. If we change x → v, where v is a velocity, then the distribution in Equation 3.1 is now referred to as a Maxwell-Boltzmann distribution, or Maxwellian. A one dimensional Maxwellian is given by:

$$f_s(v) = \frac{n_o}{\sqrt{\pi}\, V_{T_s}} \, e^{-\left(\dfrac{v - v_o}{V_{T_s}}\right)^2} \tag{3.3}$$

where v_o is the drift speed of the peak relative to zero, n_o is the particle number density, and we have replaced $2\sigma^2$ with ${V_{T_s}}^2$, the thermal speed[7], which is given by:

$$V_{T_s} = \sqrt{\frac{2k_B T_s}{m_s}} \tag{3.4}$$

where k_B is Boltzmann's constant, T_s is the temperature, m_s is the mass, and s is the particle species. Therefore, one can show that FWHM $= 2\sqrt{\ln 2}\, V_{T_s}$.

The general representation of a two dimensional multivariate distribution is given by the following:

$$f(x, y) = \alpha \, e^{-\left(\dfrac{\beta}{\sqrt{2(1-\rho^2)}}\right)^2} \tag{3.5a}$$

$$\alpha = \frac{A_o}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \tag{3.5b}$$

$$\beta^2 = \left[\left(\frac{x - x_o}{\sigma_x}\right)^2 + \left(\frac{y - y_o}{\sigma_y}\right)^2 - \left(\frac{2\rho(x - x_o)(y - y_o)}{\sigma_x\sigma_y}\right)\right] \tag{3.5c}$$

[7]Note that this version is referred to as the *most probable speed* for a 1D Gaussian.

where we define ρ and σ_j in the following manner:

$$\rho = \frac{cov\left(x, y\right)}{\sigma_x \sigma_y} \tag{3.6a}$$

$$cov\left(x, y\right) = E\left[\left(x - \mu_x\right)\left(y - \mu_y\right)\right] \tag{3.6b}$$

where ρ is the correlation between x and y, μ_x = E[x] is the expected value of the aggregate data set X = $\cup_i\ x_i$, and μ_{x_i} (e.g., see Section 2.3). In the limit $\rho \to 0$ (i.e., x and y are uncorrelated), Equation 3.5b reduces to:

$$f\left(x, y\right) = \frac{A_x A_y}{2\pi\sigma_x\sigma_y}\ e^{-\frac{1}{2}\left[\left(\frac{x - \mu_x}{\sigma_x}\right)^2 + \left(\frac{y - \mu_y}{\sigma_y}\right)^2\right]} \ . \tag{3.7}$$

In general, for uncorrelated variables, we can write:

$$f\left(x, y, z\right) = f\left(x\right) f\left(y\right) f\left(z\right) \tag{3.8}$$

but if we let $V_x \to V_\perp \cos\phi$, $V_y \to V_\perp \sin\phi$, and $V_z \to V_\parallel$, where ϕ is the phase angle of the velocity and $\partial f / \partial\phi = 0$, the distribtuion is gyrotropic.

3.1 Bi-Maxwellian Distributions

One can show that a gyrotropic distribution satisfies $V_{T\perp,x} = V_{T\perp,y} \equiv V_{T\perp}$. If we substitute into Equation 3.7 x $\to V_\parallel$, y $\to V_\perp$, $\mu_j \to V_{o,j}$, and $\sigma_j \to V_{T,j}/\sqrt{2}$ we arrive at a bi-Maxwellian distribution given by:

$$f\left(V_\parallel, V_\perp\right) = \frac{n_o}{\pi^{3/2} {V_{T\perp}}^2 V_{T\parallel}}\ e^{-\left[\left(\frac{V_\parallel - v_{o\parallel}}{V_{T\parallel}}\right)^2 + \left(\frac{V_\perp - v_{o\perp}}{V_{T\perp}}\right)^2\right]} \tag{3.9}$$

3.1.1 Derivatives of Parameters: Bi-Maxwellian Distributions

In the use of numerical methods like the Levenberg-Marquardt algorithm [e.g., *Markwardt*, 2009], it is useful to define the derivatives of a function with respect to the free parameters. In the case of velocity distributions, these are the density, thermal speeds, drift speeds, and exponent (for self-similar and kappa distributions). First, we define some simplifying terms for brevity. Let us define the following:

$$u_j = V_j - v_{oj} \tag{3.10a}$$

$$w_j = \frac{u_j}{V_{Tj}} \tag{3.10b}$$

$$\psi\left(z\right) = \frac{\Gamma'\left(z\right)}{\Gamma\left(z\right)} \equiv \text{digamma function} \tag{3.10c}$$

Now we can proceed and define the partial derivatives of Equation 3.9 which we will denote as $f^{(m)}$ for brevity:

$$\frac{\partial f^{(m)}}{\partial n_o} = \frac{f^{(m)}}{n_o} \tag{3.11a}$$
$$\frac{\partial f^{(m)}}{\partial V_{T\parallel}} = \left[\frac{2\ \left({w_\parallel}^2 - 1\right)}{V_{T\parallel}}\right]\ f^{(m)} \tag{3.11b}$$
$$\frac{\partial f^{(m)}}{\partial V_{T\perp}} = \left[\frac{2\ \left({w_\perp}^2 - 1\right)}{V_{T\perp}}\right]\ f^{(m)} \tag{3.11c}$$
$$\frac{\partial f^{(m)}}{\partial v_{o\parallel}} = \left(\frac{2\ w_\parallel}{V_{T\parallel}}\right)\ f^{(m)} \tag{3.11d}$$
$$\frac{\partial f^{(m)}}{\partial v_{o\perp}} = \left(\frac{2\ w_\perp}{V_{T\perp}}\right)\ f^{(m)} \tag{3.11e}$$

3.2 Bi-Kappa Distributions

A generalized power-law particle distribution is given by a bi-kappa distribution [e.g., *Livadiotis*, 2015; *Mace and Sydora*, 2010], for electrons here as:

$$f\left(V_\perp, V_\parallel\right) = \left[\frac{1}{\pi\left(\kappa - \frac{3}{2}\right)}\right]^{3/2} \frac{n_o\ \Gamma\left(\kappa + 1\right)}{{V_{T\perp}}^2\ V_{T\parallel}\ \Gamma\left(\kappa - \frac{1}{2}\right)} \left\{1 + \frac{1}{\left(\kappa - \frac{3}{2}\right)}\left[\left(\frac{V_\parallel - v_{o\parallel}}{V_{T\parallel}}\right)^2 + \left(\frac{V_\perp - v_{o\perp}}{V_{T\perp}}\right)^2\right]\right\}^{-(\kappa+1)} \tag{3.12}$$

Note that we have again defined V_{Tj} as the most probable speed of a 1D Gaussian for consistency, i.e., it does not depend upon κ.

3.2.1 Derivatives of Parameters: Bi-Kappa Distributions

In the use of numerical methods like the Levenberg-Marquardt algorithm [e.g., *Markwardt*, 2009], it is useful to define the derivatives of a function with respect to the free parameters. In the case of velocity distributions, these are the density, thermal speeds, drift speeds, and exponent (for self-similar and kappa distributions). First, we define some simplifying terms for brevity. Let us define the following:

$$u_j = V_j - v_{oj} \tag{3.13a}$$
$$w_j = \frac{u_j}{V_{Tj}} \tag{3.13b}$$
$$\psi\left(z\right) = \frac{\Gamma'\left(z\right)}{\Gamma\left(z\right)} \equiv \text{digamma function} \tag{3.13c}$$
$$D\left(w_\parallel, w_\perp, \kappa\right) = {w_\parallel}^2 + {w_\perp}^2 + \left(\kappa - \tfrac{3}{2}\right) \tag{3.13d}$$

Now we can proceed and define the partial derivatives of Equation 3.12 which we will denote as $f^{(\kappa)}$

for brevity:

$$\frac{\partial f^{(\kappa)}}{\partial n_o} = \frac{f^{(\kappa)}}{n_o} \tag{3.14a}$$

$$\frac{\partial f^{(\kappa)}}{\partial V_{T\parallel}} = \left[\frac{2\ {w_{\parallel}}^2\ \left(\kappa + \frac{1}{2}\right) - {w_{\perp}}^2 - \left(\kappa - \frac{3}{2}\right)}{V_{T\parallel}\ D\left(w_{\parallel}, w_{\perp}, \kappa\right)} \right]\ f^{(\kappa)} \tag{3.14b}$$

$$\frac{\partial f^{(\kappa)}}{\partial V_{T\perp}} = \left\{ \frac{2\ \left[\kappa\ {w_{\perp}}^2 - {w_{\parallel}}^2 - \left(\kappa - \frac{3}{2}\right)\right]}{V_{T\perp}\ D\left(w_{\parallel}, w_{\perp}, \kappa\right)} \right\}\ f^{(\kappa)} \tag{3.14c}$$

$$\frac{\partial f^{(\kappa)}}{\partial V_{o\parallel}} = \left[\frac{2\ w_{\parallel}\ \left(\kappa + 1\right)}{V_{T\parallel}\ D\left(w_{\parallel}, w_{\perp}, \kappa\right)} \right]\ f^{(\kappa)} \tag{3.14d}$$

$$\frac{\partial f^{(\kappa)}}{\partial V_{o\perp}} = \left[\frac{2\ w_{\perp}\ \left(\kappa + 1\right)}{V_{T\perp}\ D\left(w_{\parallel}, w_{\perp}, \kappa\right)} \right]\ f^{(\kappa)} \tag{3.14e}$$

$$\frac{\partial f^{(\kappa)}}{\partial \kappa} = \left\{ \frac{\left({w_{\parallel}}^2 + {w_{\perp}}^2\right)\left(\kappa - \frac{1}{2}\right) - \frac{3}{2}\ \left(\kappa - \frac{3}{2}\right)}{\left(\kappa - \frac{3}{2}\right)\ D\left(w_{\parallel}, w_{\perp}, \kappa\right)} - \ln\left|1 + \frac{{w_{\parallel}}^2 + {w_{\perp}}^2}{\left(\kappa - \frac{3}{2}\right)}\right| + \psi\left(\kappa + 1\right) - \psi\left(\kappa - \frac{1}{2}\right) \right\}\ f^{(\kappa)} \tag{3.14f}$$

3.3 Self-Similar Distributions

When a distribution evolves under the action of inelastic scattering, the result is a *self-similar distribution* [*Dum et al.*, 1974; *Dum*, 1975; *Goldman*, 1984; *Horton et al.*, 1976; *Horton and Choi*, 1979; *Jain and Sharma*, 1979], which in one dimension is given by:

$$f_s\left(x, t\right) = C_o\ e^{-\left(\frac{x}{x_o}\right)^p} \tag{3.15}$$

where we define C_o by defining:

$$n_o = \int_{-\infty}^{\infty} dv\ f_s\left(v, t\right) \tag{3.16a}$$

$$= 2 \int_{0}^{\infty} dv\ f_s\left(v, t\right)\ \text{(if symmetric)} \tag{3.16b}$$

where the general solution to Equation 3.16b is given by:

$$\int_{0}^{\infty} dx\ x^n e^{-\alpha x^p} = \frac{\Gamma\left(k\right)}{p \alpha^k} \tag{3.17}$$

for n > -1, $p > 0$, $\alpha > 0$, and $k = (n + 1)/p$[8]. For $n = 0$, we can show:

$$C_o = \frac{n_o \; p \; \alpha^{1/p}}{2 \; \Gamma\left(1/p\right)} \tag{3.18}$$

Physically we can see that $\alpha \rightarrow {V_{T_s}}^{-p}$, therefore the one dimensional form of the self-similar distribution can be given by:

$$f_s\left(v, t\right) = \frac{n_o \; p}{2 \; V_{T_s} \; \Gamma\left(1/p\right)} \; e^{-\left(\frac{v}{V_{T_s}}\right)^p} \tag{3.19}$$

which in the limit as $p \rightarrow 2$ reduces to:

$$f_s\left(v, t\right) = \frac{n_o}{\sqrt{\pi} \; V_{T_s}} \; e^{-\left(\frac{v}{V_{T_s}}\right)^2} \tag{3.20}$$

which matches Equation 3.3 for $v_o \rightarrow 0$.

For the 3D case, the self-similar solution reduces to (for even p):

$$f\left(V_x, V_y, V_z\right) = \left[\frac{p}{2\Gamma\left(\frac{n+1}{p}\right)}\right]^3 \frac{n_o}{\left(V_{T_x} \; V_{T_y} \; V_{T_z}\right)^{n+1}} \; e^{-\left[\left(\frac{V_x}{V_{T_x}}\right)^2 + \left(\frac{V_y}{V_{T_y}}\right)^2 + \left(\frac{V_z}{V_{T_z}}\right)^2\right]} \tag{3.21}$$

where we can follow the same lines of reasoning that lead to Equation 3.9 to find (for $n \rightarrow 0$):

$$f\left(V_\parallel, V_\perp\right) = \left[\frac{p}{2\Gamma\left(\frac{1}{p}\right)}\right]^3 \frac{n_o}{{V_{T\perp}}^2 V_{T\parallel}} \; e^{-\left[\left(\frac{V_\parallel}{V_{T\parallel}}\right)^p + \left(\frac{V_\perp}{V_{T\perp}}\right)^p\right]} \tag{3.22}$$

After some manipulation and letting $V_j \rightarrow V_j$ - v_{oj} we find:

$$f\left(V_\parallel, V_\perp\right) = \left[2\Gamma\left(\frac{1+p}{p}\right)\right]^{-3} \frac{n_o}{{V_{T\perp}}^2 V_{T\parallel}} \; e^{-\left[\left(\frac{V_\parallel - v_{o\parallel}}{V_{T\parallel}}\right)^p + \left(\frac{V_\perp - v_{o\perp}}{V_{T\perp}}\right)^p\right]} \tag{3.23}$$

Note that we have again defined V_{Tj} as the most probable speed of a 1D Gaussian for consistency, i.e.,

[8]Note that $\Gamma\left(1/p\right)/p = \Gamma\left(1 + 1/p\right)$

it does not depend upon p.

3.3.1 Derivatives of Parameters: Self-Similar Distributions

In the use of numerical methods like the Levenberg-Marquardt algorithm [e.g., *Markwardt*, 2009], it is useful to define the derivatives of a function with respect to the free parameters. In the case of velocity distributions, these are the density, thermal speeds, drift speeds, and exponent (for self-similar and kappa distributions). First, we define some simplifying terms for brevity. Let us define the following:

$$u_j = V_j - v_{oj} \tag{3.24a}$$
$$w_j = \frac{u_j}{V_{Tj}} \tag{3.24b}$$
$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} \equiv \text{digamma function} \tag{3.24c}$$

Now we can proceed and define the partial derivatives of Equation 3.23 which we will denote as $f^{(s)}$ for brevity:

$$\frac{\partial f^{(s)}}{\partial n_o} = \frac{f^{(s)}}{n_o} \tag{3.25a}$$
$$\frac{\partial f^{(s)}}{\partial V_{T\parallel}} = \left(\frac{p\ w_\parallel - 1}{V_{T\parallel}} \right)\ f^{(s)} \tag{3.25b}$$
$$\frac{\partial f^{(s)}}{\partial V_{T\perp}} = \left(\frac{p\ w_\perp - 2}{V_{T\perp}} \right)\ f^{(s)} \tag{3.25c}$$
$$\frac{\partial f^{(s)}}{\partial v_{o\parallel}} = \left(\frac{p\ {w_\parallel}^{p-1}}{V_{T\parallel}} \right)\ f^{(s)} \tag{3.25d}$$
$$\frac{\partial f^{(s)}}{\partial v_{o\perp}} = \left(\frac{p\ {w_\perp}^{p-1}}{V_{T\perp}} \right)\ f^{(s)} \tag{3.25e}$$
$$\frac{\partial f^{(s)}}{\partial p} = \left[\frac{3\ \psi\left(\frac{1+p}{p}\right)}{p^2} - {w_\parallel}^p\ \ln w_\parallel - {w_\perp}^p\ \ln w_\perp \right]\ f^{(s)} \tag{3.25f}$$