Critical Mach Numbers

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November 17, 2009

The behavior of collisionless shocks is governed largely by the Mach number of the shock. Above (or below) certain 'critical' values of the Mach number, features such as shock overshoots, substantial increases in ion reflection, or upstream whistler waves appear or disappear. This document describes in detail the calculation several different types of critical Mach numbers for a shock with given upstream plasma parameters. The theory describing the significance of the critical Mach numbers calculated here is contained in a review article by Kennel *et al.* in the monograph "Collisionless Shocks in the Heliosphere: A Tutorial Review" [KEH85], and the method to solve for the first critical Mach number is described in some detail in a paper by Edmiston and Kennel [EK84]. In fact, this document is mostly a record of my attempts to duplicate the results shown in these papers.

1 First, Second, and Foreshock Critical Mach Numbers

1.1 RH Equations

These first, second, and foreshock Mach numbers are defined by the downstream value of the shock normal velocity, so solving for them is a matter of solving the Rankine-Hugoniot equations for a defined u_{2x}.

Upstream/downstream values are denoted with a subscript of 1/2. The only angles we are concerned with are the upstream and downstream θ_{bn} angles, so for simplicity these are denoted as θ_1 and θ_2.

Useful definitions:

$$V^2 = kT/m \qquad \beta = 8\pi NkT/B^2 \qquad NV^2 = \beta B^2/8\pi M$$

$$C_s = (\gamma kT/M)^{1/2} \qquad C_A = B/(4\pi NM)^{1/2} \qquad C_f^2 = (C_A^2 + C_s^2)/2 + (((C_A^2 + C_s^2)/2)^2 - C_A^2 C_s^2 \cos^2\theta_1)^{1/2}$$

From geometry, we have:

$$\begin{aligned} B_{1x} &= B_1 \cos\theta_1 & B_{2x} &= B_2 \cos\theta_2 \\ B_{1z} &= B_1 \sin\theta_1 & B_{2x} &= B_2 \sin\theta_2 \end{aligned}$$

From Maxwell's equations:

$$B_{1x} = B_{2x} \equiv B_x \tag{M1}$$

$$U_{1x}B_{1z} = U_{2x}B_{2z} - U_{2z}B_x \tag{M2}$$

From the Rankine-Hugoniot relations:

$$N_1 U_{1x} = N_2 U_{2x} \tag{RH1}$$

$$N_1(U_{1x}^2 + V_1^2) + B_{1z}^2/8\pi M = N_2(U_{2x}^2 + V_2^2) + B_{2z}^2/8\pi M \tag{RH2}$$

$$B_{1z}B_x/4\pi M = B_{2z}B_x/4\pi M - N_2 U_{2x} U_{2z} \tag{RH3}$$

$$\begin{aligned} &N_1 U_{1x}(\gamma V_1^2/(\gamma-1) + U_{1x}^2/2) + U_{1x}B_{1z}^2/4\pi M = \\ &\qquad N_2 U_{2x}(\gamma V_2^2/(\gamma-1) + U_{2x}^2/2 + U_{2z}^2/2) + B_{2z}/4\pi M(B_{2z}U_{2x} - B_x U_{2z})) \end{aligned} \tag{RH4}$$

(Note: there is a typo in the original paper's energy flux equation. U_{1z} on the LHS should be U_{1x}. The coplanarity theorem also makes the equation involving the y components superfluous.)

We seek to make the equations dimensionless. From M1 and RH1, we have the density and magnetic compression:

$$B_2/B_1 = \cos\theta_1/\cos\theta_2 \qquad N_2/N_1 = U_{1x}/U_{2x}$$

We define dimensionless velocities in terms of the bulk velocities relative to the thermal velocities.

$$u_{1x} \equiv U_{1x}/V_1 \qquad u_{2x} \equiv U_{2x}/V_2 \qquad u_{2z} \equiv U_{2z}/V_2 \qquad v \equiv V_2/V_1$$

From M2, we find:

$$u_{1x} \tan\theta_1 = v(u_{2x}\tan\theta_2 - u_{2z}) \tag{1}$$

From RH2:

$$\beta_1(u_{1x}^2+1) + \sin^2\theta_1 = v\beta_1(u_{2x}^2+1)u_{1x}/u_{2x} + \cos^2\theta_1\tan^2\theta_2 \tag{2}$$

From RH3:

$$\sin\theta_1\cos\theta_1 = \tan\theta_2\cos^2\theta_1 - v\beta_1 u_{1x}u_{2z}/2 \tag{3}$$

From RH4:

$$\begin{split}\beta_1 u_{1x}(\gamma/(\gamma-1) + u_{1x}^2/2) + 2u_{1x}\sin^2\theta_1 = \\ v\left(\beta_1 u_{1x} v(\gamma/(\gamma-1) + u_{2x}^2/2 + u_{2z}^2/2) + 2\cos^2\theta_1\left(u_{2x}\tan^2\theta_2 - u_{2z}\tan\theta_2\right)\right)\end{split} \tag{4}$$

We solve Equation 1 for v, yielding

$$v = u_{1x}\tan\theta_1/(u_{2x}\tan\theta_2 - u_{2z}) \tag{5}$$

Plugging in Equation 5 to Equations 2, 3, and 4 yields:

$$\beta_1(u_{1x}^2+1) + \sin^2\theta_1 = \beta_1(u_{2x}^2+1)\frac{u_{1x}^2\tan\theta_1}{u_{2x}^2\tan\theta_2 - u_{2x}u_{2z}} + \cos^2\theta_1\tan^2\theta_2 \tag{6}$$

$$\sin\theta_1\cos\theta_1 = \tan\theta_2\cos^2\theta_1 - \frac{\beta_1 u_{1x}^2\tan\theta_1}{2(u_{2x}\tan\theta_2/u_{2z} - 1)} \tag{7}$$

$$\beta_1\left(\frac{\gamma}{\gamma-1} + \frac{u_{1x}^2}{2}\right) + 2\sin^2\theta_1 = \beta_1\left(\frac{u_{1x}\tan\theta_1}{u_{2x}\tan\theta_2 - u_{2z}}\right)^2\left(\frac{\gamma}{\gamma-1} + \frac{u_{2x}^2}{2} + \frac{u_{2z}^2}{2}\right) + 2\cos\theta_1\sin\theta_1\tan\theta_2 \tag{8}$$

Solving Equation 7 for u_{1x} yields

$$u_{1x} = \left(\frac{2(\tan\theta_2\cos^2\theta_1 - \sin\theta_1\cos\theta_1)(u_{2x}\tan\theta_2/u_{2z} - 1)}{\beta_1\tan\theta_1}\right)^{1/2} \tag{9}$$

Plugging 9 into 6 and 8 yields two equations:

$$\frac{2(\tan\theta_2\cos^2\theta_1 - \sin\theta_1\cos\theta_1)}{\tan\theta_1}\left(\frac{u_{2x}}{u_{2z}}\tan\theta_2 - 1 - \frac{(u_{2x}^2+1)\tan\theta_1}{u_{2x}u_{2z}}\right) + \beta_1 + \sin{\theta_1}^2 - \cos^2\theta_1\tan^2\theta_2 = 0 \tag{10}$$

$$\begin{split}\beta_1\frac{\gamma}{\gamma-1} + \frac{(\tan\theta_2\cos^2\theta_1 - \sin\theta_1\cos\theta_1)(u_{2x}\tan\theta_2/u_{2z} - 1)}{\tan\theta_1} + 2\sin^2\theta_1 \\ - \frac{2(\tan\theta_2\cos^2\theta_1 - \sin\theta_1\cos\theta_1)\tan\theta_1}{u_{2z}(u_{2x}\tan\theta_2 - u_{2z})}\left(\frac{\gamma}{\gamma-1} + \frac{u_{2x}^2}{2} + \frac{u_{2z}^2}{2}\right) - 2\cos\theta_1\sin\theta_1\tan\theta_2 = 0\end{split} \tag{11}$$

We can then solve Equation 10 for u_{2z}:

$$u_{2z} = \frac{u_{2x}\tan\theta_2 - (u_{2x}^2+1)\tan\theta_1/u_{2x}}{1 + \tan\theta_1(\cos^2\theta_1\tan^2\theta_2 - \beta_1 - \sin^2\theta_1)/(2(\tan\theta_2\cos^2\theta_1 - \sin\theta_1\cos\theta_1))} \tag{12}$$

I won't write out Equation 11 with Equation 12 inserted. However, the insertion of 12 makes 11 a single variable equation, which is much easier to solve than a set of multivariable equations. The only trick to solving 11 is to avoid the two places where the equation can blow up, which are at

$$\tan\theta_2 = \frac{u_{2x}^2+1}{u_{2x}^2}\tan\theta_1 \tag{13}$$

and

$$u_{2x}\tan\theta_2 - u_{2z} = 0 \tag{14}$$

13 can be directly computed, and 14 can be computed by using 12 to find a polynomial in $\tan\theta_2$:

$$\tan^3\theta_2 + \left(2\frac{u_{2x}^2+1}{u_{2x}^2} - \frac{\beta_1 + \sin^2\theta_1}{\cos^2\theta_1}\right)\tan\theta_2 - \left(2\frac{u_{2x}^2+1}{u_{2x}^2}\right)\tan\theta_1 = 0 \tag{15}$$

1.2 Values of u_{2x} for Different Critical Mach Numbers

Type	U_{2x}	u_{2x}	Notes
First	C_{s2}	$\gamma^{1/2}$	U_{2x} is downstream sound speed
Second	C_{i2}	$\sqrt{1/(T_{e2}/T_{i2}+1)}$	U_{2x} is downstream ion thermal speed, ion reflection present
Second (K85)*	C_{i2}	$\sqrt{\gamma/(T_{e2}/T_{i2}+1)}$	"
Foreshock**	$3C_{i2}\cos\theta_2$	$3\cos\theta_2\sqrt{1/(T_{e2}/T_{i2}+1)}$	Significant escape upstream of downstream ions
Foreshock (K85)*	$3C_{i2}\cos\theta_2$	$3\cos\theta_2\sqrt{\gamma/(T_{e2}/T_{i2}+1)}$	"

* The different second and foreshock Mach numbers boils down to a factor of $\sqrt{\gamma}$. The K85 value is what is used in the review paper, but the definition of C_{i2} as $\sqrt{kT/M}$. The plots in this document use the starred versions which include the factor of $\sqrt{\gamma}$.

** The foreshock Mach number includes a factor of $\cos\theta_2$ in U_{2x} (and therefore in u_{2x}.) The modified value of u_{2x} must be inserted back into the polynomial of Equation 15 and the polynomial must be resolved to find different coefficients for $\tan\theta_2$.

1.3 Numerical RH Solution and Calculation of M_f^*

The value of u_{2x} in the above table can be plugged into Equations 13 and 15 to find the singular values of Equation 11. We can then use a numerical procedure like the ZBRENT procedure in the IDLASTRO library to numerically solve for θ_2, being careful to avoid the spurious zeros. From there u_{1x} may be obtained by plugging in to Equations 10 and 9. Using the definition of C_f it is possible to show that M_f is a function of u_{1x} and the given upstream parameters:

$$M_f = u_{1x}\left(\frac{2+\gamma\beta_1}{2\beta_1} + \left(\left(\frac{2+\gamma\beta_1}{2\beta_1}\right)^2 - \frac{2\gamma}{\beta_1}\cos^2\theta_1\right)^{1/2}\right)^{1/2} \tag{16}$$

The critical Mach number M_f^* is then given by calculating M_f for the value of u_{1x} from 9. Using this method, we are able to reproduce the results of [EK84] and [KEH85]. Some example plots, which match those from these two papers, are reproduced in Section 3 as Figures 1, 2, 3, and 4.

2 Whistler Critical Mach Numbers

The whistler critical Mach number is the Mach number below which a whistler wave can stand in the upstream plasma. Solving for the whistler critical Mach number does not involve solving the RH conditions for the shock. According to [KEH85], the whistler critical Mach number M_w^* is given by:

$$M_w^* = \frac{\cos\theta_1}{2\mu}\frac{C_A}{C_f} \tag{17}$$

where $\mu \approx 1/42.85$ is the square root of the proton/electron mass ratio. Other critical whistler Mach numbers, including the nonlinear and the group velocity critical Mach number, are defined in Krasnoselskikh *et al.* [KLSL02]. The [KEH85] whistler critical Mach number is reproduced in Figure 5 in Section 3.

3 Plots

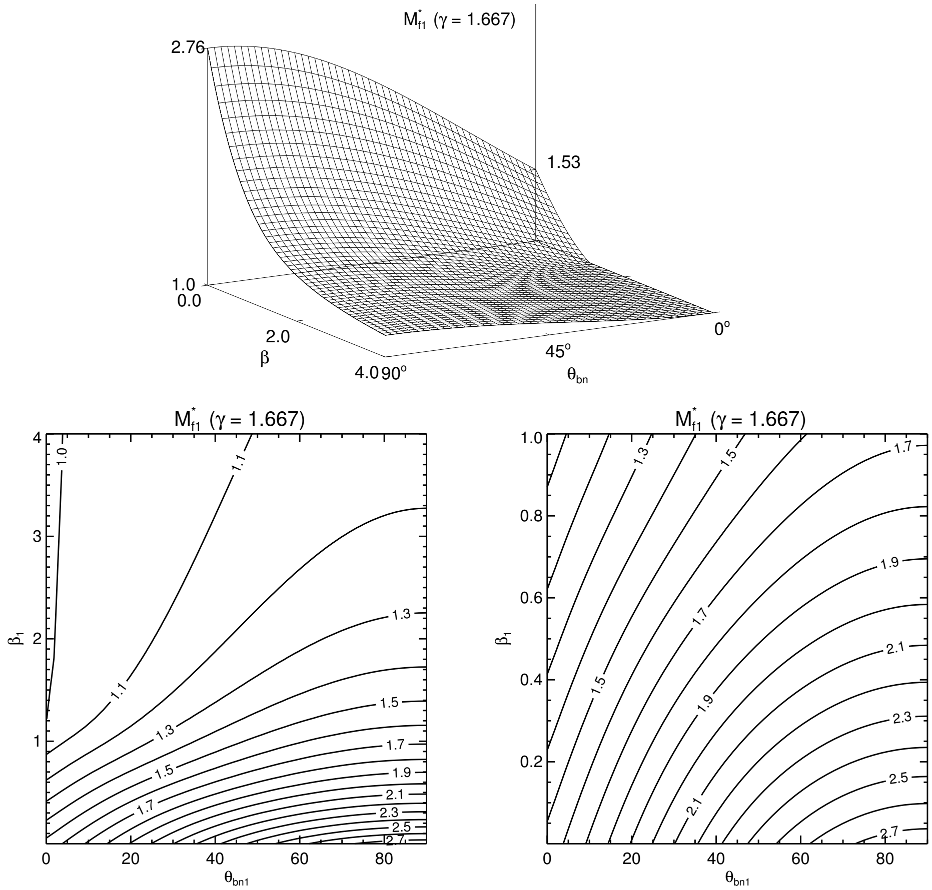

Figure 1: EK Figure 4: First critical Mach number

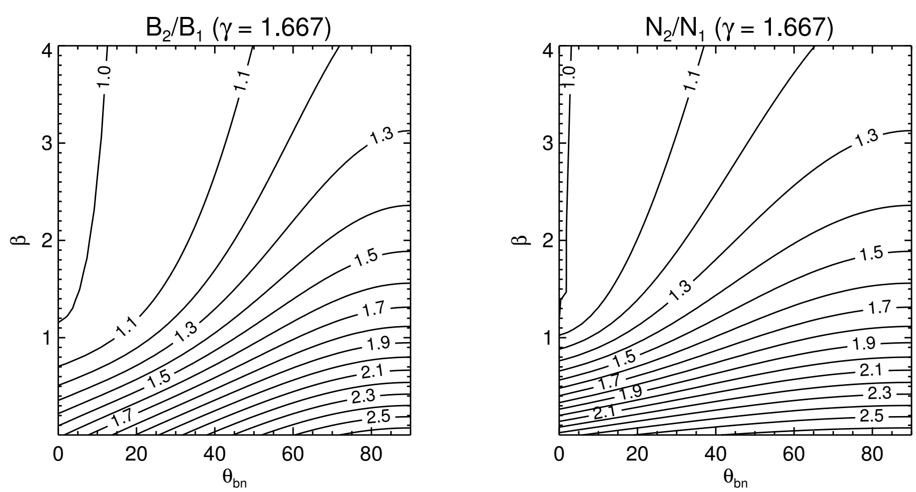

Figure 2: EK Figure 10: Density and magnetic field compression ratios at the first critical Mach number (Note: I think that the figures in the original paper are mislabeled!)

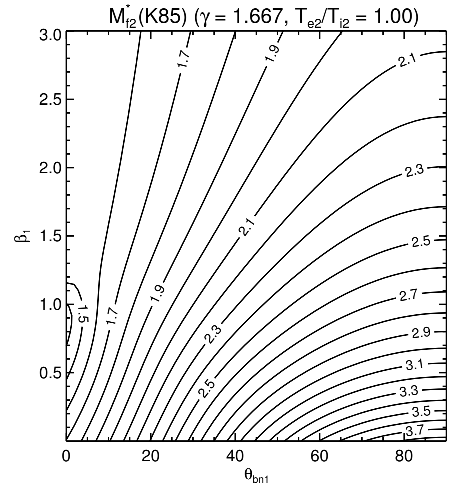

Figure 3: KEH Figure 13c: Second critical Mach number, for the case of $T_{e2}/T_{i2} = 1.0$

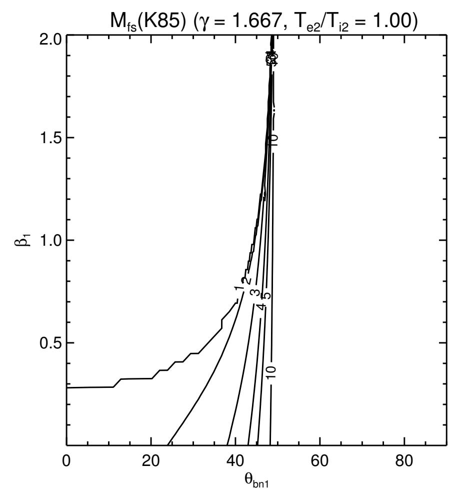

Figure 4: KEH Figure 14b: Foreshock critical Mach number, for the case of $T_{e2}/T_{i2} = 1.0$

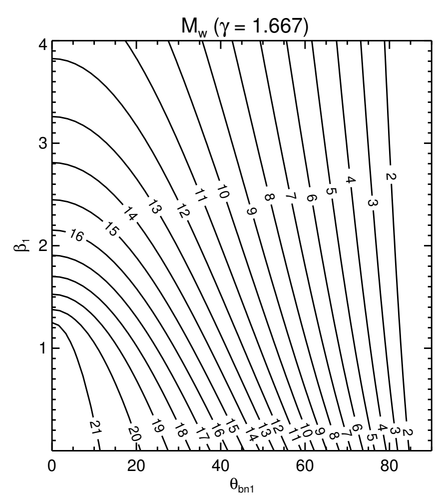

Figure 5: KEH Figure 9b: Whistler critical Mach number

References

[EK84] J. P. Edmiston and C. F. Kennel. A parametric survey of the first critical Mach number for a fast MHD shock. *Journal of Plasma Physics*, 32:429–441, 1984.

[KEH85] C. F. Kennel, J. P. Edmiston, and T. Hada. A quarter century of collisionless shock research. *Washington DC American Geophysical Union Geophysical Monograph Series*, 34:1–36, 1985.

[KLSL02] V. V. Krasnoselskikh, B. Lembège, P. Savoini, and V. V. Lobzin. Nonstationarity of strong collisionless quasiperpendicular shocks: Theory and full particle numerical simulations. *Physics of Plasmas*, 9:1192–1209, April 2002.