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Geometry and measure

1 Introduction

Here I will write some observations about geometric measure theory. Every fact written here is a necessary theorem for achieving each sub goal. I aim to calculate the dimension of generalized Cantor sets and to explore some key theorems.

1.1 Acknowledged results from measure theory

1.2 Hausdorff measure

The Hausdorff measure generalises the notion of measure for lower dimensional objects in higher-dimensional space. The idea is essentially similar to the construction of Lebesgue's measure except that we take a lower limit instead of an infinum. We define a cover of E by sets of diameter less then δ as a δ-cover of E. And we conceder only countable covers. We note that

$$\mathcal{H}^s_\delta(E) = \inf_C \sum_{I \in C} \omega_s \left(\frac{\operatorname{diam}(I)}{2}\right)^s$$

where $s \in \mathbb{R}_{\geq 0}$ is a dimension, $\omega_s \in \mathbb{R}$ is a coefficient, preferably continuous or smooth as a function of s, and C is a δ-cover of E. We may assume that

$$\omega_s = \frac{\pi^{s/2}}{\Gamma(1 + s/2)}$$

We define the Hausdorff measure as a limit of the previous value. It exists because $\mathcal{H}^s_\delta(E)$ is increasing function of δ. We note

$$\mathcal{H}^s(E) = \lim_{\delta \to 0^+} \mathcal{H}^s_\delta(E)$$

I shall introduce the notion of s-variation of a cover S as

$$\operatorname{Var}^s(S) = \sum_{I \in S} \omega_s \left(\frac{\operatorname{diam}(I)}{2}\right)^s$$

Proposition: For a natural $n \geq 0$, ω_n is a volume of a unit n-dimensional ball.

1.3 Properties of Hausdorff measure

Proposition: Hausdorff measure is a Borel measure for regular topology.

Proposition: In the definition of Hausdorff measure we can consider only closed or open sets.

Proposition: Hausdorff measure of dimension $m \in N$ coincide on m-dimensional affine subspaces with their Lebesgue measure.

Proposition: For a Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}^m$ we have the following inequality

$$\mathcal{H}^s(f[E]) \leq \operatorname{Lip}(f)^s \mathcal{H}^s(E)$$

for every $s > 0$ and $E \subseteq \mathbb{R}^n$. And $\dim(E) < \dim(f[E])$.

Proposition: The n-dimetional Hausdorff measure traced to a n-dimentional C^1-submanifold of R^m induces the area measure on this submanifold and coincides with the integral measure via parametrisation on it.

Remark: Proofs to those proposition can be found in the book "Geometric measure theory" be Francesco Maggi.

1.4 Hausdorff dimension

To a set S we can associate a number $s = \inf\{a \geq 0 \mid \mathcal{H}^a(S) = 0\}$. It's called its Hausdorff dimension.

Proposition:

2 Dimension of cantor sets

Here we calculate the dimension of generalized set. Let $n \in \mathbb{N}$ and $m \in \mathbb{N}^*$ so that $2m < n$. Then we can define C_k ($k \in \mathbb{N}$) define recursively by agreeing that $C_0 = \{[0,1]\}$ and we obtain C_{k+1} from C_k by cutting out the open middle part from each segment of C_k and living side parts of length m/n of original interval. We will note $C = \lim C_k = \bigcap C_k$.

image

Obviously C_k is a $(m/n)^k$-cover of C, so

$$\mathcal{H}^s_{(m/n)^k} \leq \sum_{I \in C_k} \omega_s (\frac{\text{diam}(I)}{2})^s = \omega_s 2^k ((m/n)^k)/2)^s = \omega_s/2^s (2(m/n)^s)^k$$

And if $s > \log_{n/m}(2)$ we have right side approaching 0 as k tends to infinity. That means that $\dim(C) \leq \log_{n/m}(2)$.

Now we need to prove the inequality in the other direction. Let $s = \log_{n/m}(2)$. And let S be a $(m/n)^k$-cover of C. In fact by the construction C is an intersection of compacts on a real line, so is compact. And by one of the previous propositions we can conceder only open covers. Then by compactness we can leave only a finite number of sets in S and this way we reduce its Hausdorff variance and we can extend the resting elements to closed intervals of the same diameter. This does not change the variance. The new cover is noted by S'. Now in every interval of S' we can find 2 maximal intervals from some C_i and C_j, so the they are disjoint. If we can't do that, then there are no points of C in this interval and we can throw away that set also. So now we have 2 maximal intervals J and J' in I. They are ordered. Between them we have an interval K and as they are maximal $I \setminus J \setminus K \setminus J'$ does not contain any points from C and we can through those parts away from the covering. By the construction

$$|J|, |J'| \leq \frac{m}{n} \cdot \frac{n}{n-2m} |K| = \frac{m}{n-2m} |K|$$

Now we have $1/2(|J| + |J'|) \leq \frac{m}{n-2m} |K|$

$$|I|^s = (|J|+|J'|+|K|)^s \geq ((1+\frac{n-2m}{2m}))(|J|+|J'|))^s = (\frac{n}{m} 1/2(|J|+|J'|))^s = 2(1/2(|J|+|J'|))^s \geq |J|^s+|J'|^s$$

Where the last step is done by concavity of function $x \mapsto x^s$. That means that we can reduce this any cover to a C_k cover which has a smaller s-variation. That means that for dimension $s = \log_{n/m}(2)$ the $\mathcal{H}^s(C)$ is finite as the s-variation of C_k is always $\omega_s/2^s$.

Remark: This is a variation on the proof given in the book "The geometry of fractal sets" by K. J. Flaconer, generelised to the case of arbitrary m and n. In this book the proof is done for the case $m = 1, n = 3$.

Proposition: There is a subset of $[0,1]$ with a Hausdorff dimension 1, but Lebesgue measure 0.

To show that we shall use Cantor's sets. Let $C_{m/n}$ be a set discussed in a previous paragraph. Then $S = \bigcap C_{m/(2m+1)}$ is a set of dimension 1. As for every $0 \leq s < 1$ there is such m, that $\log_{n/m}(2) = \log_{(2m+1)/m}(2) > s$, as $\log_{(2m+1)/m}(2) \to 1$. And thus $\mathcal{H}^s(S) > \mathcal{H}^s(C_{m/(2m+1)}) = \infty$.

3 Weak* topology and compactness

As to a positive measure we can associate an integral, we need to utilise some results from functional analysis.

For topological spaces Y_i and set of functions $f_i : X \to Y_i$. We can define the smallest, coarsest topology on Y that makes those functions continuous. By the definition such topology is defined by $\tau(\{f_i\}) = \bigcap\{\tau \mid \tau \text{ is a topology of } X \text{ and } f_i \text{ are continuous}\}$. As an example a topology of product is exactly $\tau(\{\pi_i\})$, where π_i are canonical projections.

Proposition: Let τ be a topology on X. Then $\tau = \tau(\{f_i\})$ iff every function $g : W \to X$ such that $f_i \circ g$ are continuous is continuous.

Remark: This is a well known property of caorsest topology, but I checked that it's also an alternative characterisation of such topology.

If $\tau = \tau(\{f_i\})$ and $g : W \to X$ is such function that $f_i \circ g$ are continuous. It's sufficient to check that for all elements of prebase of $\tau(\{f_i\})$ the inverse image is open, but the prebase consists of elements of the form $f_i^{-1}[U]$ and its inverse image is $(f_i \circ g)^{-1}[U]$ which is open by hypotheses.

If τ is a such topology, that for every function $g : W \to X$ it's continuous iff $f_i \circ g$ are continuous. The in particular we have $\text{id} : (X,\tau) \to (X,\tau)$ continuous and that means that $f_i = f_i \circ \text{id}$ are continuous and we have $\tau(\{f_i\}) \subseteq \tau$. On the other hand we have $\mathrm{id}' : (X, \tau(\{f_i\})) \to (X, \tau)$ continuous because $f_i = f_i \circ \mathrm{id}' : (X, \tau(\{f_i\})) \to Y_i$ are continuous by the definition of coarsest topology. Thus we have id' continuous and that means that $\tau \subseteq \tau(\{f_i\})$. And finally $\tau = \tau(\{f_i\})$.

Theorem (Tichonoff): Product of compact spaces is compact.

General structure: Let I be a set of indices and E_i for $i \in I$ be a topological space with a topology τ_i. The prebase of the product topology on $\prod_{i \in I} E_i$ is $\{\pi_i^{-1}[U] \mid i \in I, U \in \tau_i\}$. a set of products of open subspaces of one spaces on others. All the finite intersections form a base of product topology. Its elements are products of open sets where almost all factors are E_i.

Maximal covers: Let's note that a set of covers that does not contain finite sub-covers for a partially ordered set with the relation of inclusion. For every chain we have its union which does not contain a finite sub-cover, which otherwise would have been in some element of chain. Thus each chain has an upper bound. By the Zorn's lemma we find a maximal element M.

Let X be a topological space and $M \subseteq \tau$ a maximal cover that does not contain a finite sub-cover. **Then if $V \in M^c$, we have $U_1, \ldots, U_n \in M$ such that $V \cup U_1 \cup \ldots \cup U_n = X$.** Because otherwise we could have added V to M and M would not be maximum. **If $U, V \in M^c$ then $\overline{UV} \in M^c$.** In other words M^c is a multiplicative system, which is similar to the statement that $\mathfrak{p}^c$ is a multiplicative for a prime ideal $\mathfrak{p}$. This is true due to the fact that we have $U_1, \ldots, U_k \in M$ and $V_1, \ldots, V_l \in M$ such that $U \cup U_1 \cup \ldots \cup U_n = X = V \cup V_1 \cup \ldots \cup V_l$ and thus $(U \cap V) \cup U_1 \cup \ldots \cup U_k \cup V_1 \cup \ldots \cup V_l = X$, which implies that $U \cap V \in X$.

Alexander's lemma about prebase: Let B be a prebase of a topological space X. Then if in every cover of X by elements of B exists a finite subcover, then the space X is compact. If X is not compact, then we have a M maximal cover that does not contain a finite sub-cover. Then to every $x \in X$ we can associate its neighborhood $V_x \in M$. Then we find some element of a basis $U_x = U_{1,x} \cup \ldots \cup U_{n_x,x} \subseteq V_x$ where $U_{i,x} \in B$ are elements of prebase. Thus by maximality $U_x \in M$ as $U_x \subseteq V_x$. But as $U_x = U_{1,x} \cup \ldots \cup U_{n_x,x}$ and as M^c is a multiplicative system, for some i we have $U_{i,x} \in M$. It means that in M we have a sub-cover of X by elements of a prebase B. And by hypotheses we can chose a finite sub-cover which gives a contradiction.

Tichonoff theorem's proof: Let $\mathcal{S} = (U_i)_{i \in I}$ be a cover of a product $E = \prod_{j \in J} E_j$ of compact space by elements of canonical prebase. Let's suppose that it does not contain a finite sub-cover. For every $j \in J$ we shall pose $S_j = \{\pi_j^{-1}[V_{i,j}] = U_i \mid V_{i,j} \in \tau_j, i \in I_j\}$. Then $(V_{i,j})_{i \in I}$ cannot be a cover of E_j, because otherwise we can extract a finite sub-cover of E_j and hence of E. So we can chose $x_j \in E_j$ such that $x_j \notin \bigcup_{i \in I_j} V_{i,j}$. Let $x = (x_j)_{j \in J}$ and it does not lie in every set of $\mathcal{S}$, thus it's not a cover and we get a contradiction.

Remark: This is the most non trivial part of the proof of Banach-Alaglaua theorem and as I had this proof noted I've decided to also put it here.

3.1 Topologies on spaces E and E^*

In this section E is a normed vector space and E^* is it's dual space of continuous 1-forms on E. Then on the space E apart from it's metric topology we have the weak topology $\sigma(E, E^*) = \tau(\{f\}_{f \in E^*})$. As $f \in E^*$ are continuous for the regular topology, the topology $\sigma(E, E^*)$ is finer then the regular topology, which we call strong.

On the space E^* we also have strong topology with the norm for linear operators. And we have a weak* topology $\sigma(E^*, E) = \tau(\{v\}_{v \in E})$.

Proposition: Weak* topology is a trace topology from the space $\mathbb{R}^E$ with a topology of product.

Proof: Let $\tau(\{\pi_v\}_{v \in E})$ be the trace topology. Then it is easy to see that $\pi_v = v$ as both function are evoluations at v and thus $\tau(\{\pi_v\}_{v \in E}) = \tau(\{v\}_{v \in E}) = \sigma(E^*, E)$ is a weak* topology.

Remark: In the book "Functional Analysis" by Haim Brezis the part above is done by setting an hom-memorpism and the verification of its bicontinuity. As you have seen there is actually nothing to prove as it's just 2 notions of the same thing projection and dual-space.

Theorem (Banach-Alaoglu): The closed unit ball $B = \{f \in E^* \mid |f| \leq 1\}$ is compact in the weak* topology $\sigma(E^*, E)$.

Proof:

$$B = \left\{ f \in \mathbb{R}^E \mid \begin{cases} |f(x)| < |x|, \ \forall x \in E \\ f(\lambda x) = \lambda f(x), \ \forall \lambda \in \mathbb{R}, x \in E \\ f(x + y) = f(x) + f(y) \ \forall x, y \in E \end{cases} \right\}$$

Hence it is intesection of following sets $B = K \cap \bigcap_{x,y \in E} A_{x,y} \cap \bigcap_{x \in E, \lambda \in \mathbb{R}} B_{\lambda,x}$, where $K = \{f \in \mathbb{R}^E \mid |f(x)| \leq |x|\} = \prod_{x \in E} [-|x|, |x|]$ is compact by Tichonoff theorem, where for $x, y \in E$ we define $A_{x,y} = \{f \in \mathbb{R}^E \mid f(x + y) - f(x) - f(y) = 0\}$ is colsed since evoluations and addition are continious and thus $f \mapsto$

$f(x+y) - f(x) - f(y)$ is continous and $A_{x,y}$. For a similar reason $B_{\lambda,x} = \{f \in \mathbb{R}^E \mid f(\lambda x) - \lambda f(x) = 0\}$ is closed which proves that B is compact.

4 Measures and convergence

Let μ be a Radon measure

5 Analysis results

For a ball $B = B(x,r)$ of center x and radius r we shall note ${}^{\epsilon}B = B(x,(1+\epsilon)r)$ for every $\epsilon > 0$.

Vitali's covering theorem: Let $\epsilon > 0$ and $\mathcal{F}$ be any collection of any collection of nondegenearted closed balls in $\mathbb{R}^n$ with

$$\sup\{\operatorname{diam} B \mid B \in \mathcal{F}\} < \infty$$

Then for every $\epsilon > 1$ there exist a countable family $\mathcal{G}$ of disjoint balls in $\mathcal{F}$ such that

$$\bigcup_{B\in\mathcal{F}} B \subseteq \bigcup_{B\in\mathcal{G}} B^{\epsilon}$$

Proof: Set $D = \sup\{\operatorname{diam} B \mid B \in \mathcal{F}\}$. Set

$$\mathcal{F}_j = \left\{ B \in \mathcal{F} \mid \frac{D}{\epsilon^j} < \operatorname{diam} B \leq \frac{D}{\epsilon^{j-1}} \right\}, \quad j = 1, 2, \ldots$$

We define $\mathcal{G}_j \subseteq \mathcal{F}_j$ as follows

- Let $\mathcal{G}_1$ be any maximal disjoint collection of balls in $\mathcal{F}_1$.
- Assuming $\mathcal{G}_1, \ldots, \mathcal{G}_{k-1}$ have been selected, we chose $\mathcal{G}_k$ to be any maximal disjoint subcollection of

$$\{B \in \mathcal{F}_k \mid B \cap B' = \emptyset \text{ for all } B' \in \bigcup_{j=1}^{k-1} \mathcal{G}_j\}$$

They exist by Zorn's Lemma. Finally, define $\mathcal{G} = \bigcup_{j\in\mathbb{N}^*} \mathcal{G}_j$ a collection of disjoint balls and $\mathcal{G} \subseteq \mathcal{F}$.

Prooving that for each ball $B \in \mathcal{F}$, there exists a ball $B' \in \mathcal{G}$ such that $B \cap B' \neq \emptyset$ and $B \subseteq {}^{\epsilon}B'$. Fix $B \in \mathcal{F}$, there exists and index j such that $B \in \mathcal{F}_j$ and by maximality of $\mathcal{G}_k$ there exists a ball $B' \in \bigcup_{k=1}^{j} \mathcal{G}_k$ with $B \cap B' \neq \emptyset$. But $\operatorname{diam} B' > \frac{D}{\epsilon^j}$ and $\operatorname{diam} B \leq \frac{D}{\epsilon^{j-1}}$; so that

$$\operatorname{diam} B \leq \frac{D}{\epsilon^{j-1}} < \epsilon \operatorname{diam} B'$$

Thus $B \subseteq {}^{\epsilon}B'$.

Remark: This is a generalised version of the proof from the book "Measure theory and fine properties of functions" where it is done for an obscure case of $\epsilon = 5$.

Whitney covering theorem: Let $C \subseteq \mathbb{R}^n$ be a closed set and $f : C \to \mathbb{R}$, $d : C \to \mathbb{R}^{n*}$ be continuous functions. We shall use notions

$$R(y,x) = \frac{f(y) - f(x) - d(x)(y-x)}{|x-y|}, \quad \forall x, y \in C, x \neq y$$
$$\rho_k(\delta) = \sup\{|R(x,y)| \mid 0 < |x-y| \leq \delta, x, y \in K\}$$

if we suppose that for every compact $K \subseteq C$

$$\rho_K(\delta) \to 0 \text{ as } \delta \to 0 \tag{1}$$

Then there exists a fuction $\overline{f} \in C^1(\mathbb{R}^n, \mathbb{R})$ and $D\overline{f}|_C = d$.

Proof: Let $U = C^c$ be an open set. Let $r(x) = \frac{1}{4}\min(1, \operatorname{dist}(x,C))$. By Vitali's covering theorem there exist a countable set $\{x_j\}_{j\in\mathbb{N}}$ and a countable set of disjoint closed balls $\{B_j = B(x_j, r(x_j))\}_{j\in\mathbb{N}}$ such that $\bigcup_{j\in\mathbb{N}} {}^2B_j = U$. We need $\frac{1}{2}$ in the definition of $r(x)$ to make sure that ${}^2B_j \subseteq U$. Then for every $x \in U$ we shall define $S_x = \{x_j \mid B(x, 2r(x)) \cap B(x_j, 2r(x)) \neq \emptyset\}$.

Now we chech that S_x is bounded for each dimention. Let $x_j \in S_x$ then $|r(x) - r(x_j)| \leq 1/4|x - x_j|$ because $|r(x) - r(x_j)| = 1/4|\min(1, \operatorname{dist}(x,C)) - \min(1, \operatorname{dist}(x_j, C))|$ and without loss of generality we can consider 3 cases:

1. $\mathrm{dist}(x, C), \mathrm{dist}(x_j, C) > 1$ then $|\min(1, \mathrm{dist}(x, C)) - \min(1, \mathrm{dist}(x_j, C))| = 0 \leq |x - x_j|$.
2. $\mathrm{dist}(x, C) \leq 1, \mathrm{dist}(x_j, C) > 1$, then $|\min(1, \mathrm{dist}(x, C)) - \min(1, \mathrm{dist}(x_j, C))| = 1 - \mathrm{dist}(x, C) < \mathrm{dist}(x_j, C) - \mathrm{dist}(x, C) = |x_j - s| - |x - s| \leq |x_j - x|$, where s is a projection of x on C.
3. $\mathrm{dist}(x, C) \leq \mathrm{dist}(x_j, C) \leq 1$, then $|\min(1, \mathrm{dist}(x, C)) - \min(1, \mathrm{dist}(x_j, C))| = \mathrm{dist}(x_j, C) - \mathrm{dist}(x, C) \leq |x_j - x|$.

So we have $|r(x) - r(x_j)| \leq 1/4|x - x_j| \leq 1/4|2r(x) - 2r(x_j)| = 1/2|r(x) + r(x_j)$ as $x_j \in S_x$. And hence

$$r(x) - r(x_j) \leq 1/2(r(x) + r(x_j)) \;\Rightarrow\; r(x) \leq 3r(x_j)$$
$$r(x_j) - r(x) \leq 1/2(r(x) + r(x_j)) \;\Rightarrow\; r(x_j) \leq 3r(x)$$

In addition we have $|x - x_j| + r(x_j) \leq 4(r(x) + r(x_j)) + r(x_j) \leq 4r(x) + 12r(x) + 3r(x) = 19r(x)$. Which means that $B(x_j, r(x_j)) \subseteq B(x, 19r(x))$ and since $B(x_j, r(x_j))$ are disjoint we have an inquality on volumes:

$$\#S_x \omega_n (r(x)/3)^n \leq \#S_x \omega_n (r(x_j))^n = \sum_{x_j \in S_x} \mathrm{Vol}\, B_j \leq \mathrm{Vol}(B(x, 19r(x)) = \omega_n (19r(x))^n$$

Therefor $\#S_x \leq (3 \cdot 19)^n = 57^n$ is bounded by a fixed constant in each dimention.

The goal of that part is to construct the function $\overline{f}$.

6 Countably n-rectifiable sets

7 Grassmannian

In this section we introduce the topological space $G(m, n)$.

Similarly to projective spaces $P\mathbb{R}^n$ we can generalise this notion to smaller subspaces than hyperplanes. The set of m dimensional subspaces of a vector space $\mathbb{R}^n$ is called grassmannian and noted by $G(m, n)$. It has a topology identified from a topology of orthogonal projection on m-dimensional subspaces.

8 Varifold

An m-dimensional varifold V is a Radon measure over $\mathbb{R}^n \times G(n, m)$ endowed with a product topology. We say $\|V\|$ is a measure in $\mathbb{R}^n$ which is reciprocally projection of a varifold V by π_1^{-1}.
Proposition: For varifolds we concider weak* topology. Then we have a convergence criteria that $V_i \to V$ if and only if

$$\int f dV_i \to \int f dV$$

for every continuous function $f : \mathbb{R}^n \times G(m, n) \to R$ with a compact support.