The Concept of Varifold

Ulrich Menne

ABSTRACT. We survey—by means of 20 examples—the concept of varifold, as generalised submanifold, with emphasis on regularity of integral varifolds with mean curvature, while keeping prerequisites to a minimum. Integral varifolds are the natural language for studying the variational theory of the area integrand if one considers, for instance, existence or regularity of stationary (or stable) surfaces of dimension at least three or the limiting behaviour of sequences of smooth submanifolds under area and mean curvature bounds.

Introduction

Motivation

Apart from generalisation, there are two main reasons to include nonsmooth surfaces in geometric variational problems in Euclidean space or Riemannian manifolds: firstly, the separation of existence proofs from regularity considerations and, secondly, the modelling of nonsmooth physical objects. Following the first principle, varifolds were introduced by F. Almgren in 1965 to prove, for every intermediate dimension, the existence of a generalised *minimal surface* (i.e., a surface with vanishing first variation of area) in a given compact smooth Riemannian manifold. Then, in 1972, an important partial regularity result for such varifolds was established by W. Allard in his foundational paper "On the first variation of a varifold" [2]. These pioneering works still have a strong influence in geometric analysis as well as related fields. Particularly prominent examples are the proof of the Willmore conjecture by F. Marques and A. Neves and the development of a regularity theory for stable generalised minimal surfaces in one codimension by N. Wickramasekera, both published in 2014. Following the second principle, varifolds were employed by Almgren's PhD student K. Brakke in 1978 to create a mathematical model of the motion of grain boundaries in an annealing pure metal. This was the starting point of the rapid development of mean curvature flow, even for smooth surfaces.

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What is a Varifold?

Constructing nonparametric models of nonsmooth surfaces usually requires entering the realm of geometric measure theory. In the simplest case, we associate to each smooth surface M the measure *over the ambient space* whose value at a set A equals the surface measure of the intersection $A \cap M$. For varifolds, it is in fact expedient to similarly record information on the tangent planes of the surfaces. This yields weak continuity of area, basic compactness theorems, and the possibility to retain tangential information in the limit.

Main Topics Covered

We will introduce the notational infrastructure together with a variety of examples. This will then allow us to formulate the compactness theorem for integral varifolds, Theorem 1, and two key regularity theorems for integral varifolds with mean curvature, Theorems 2 and 3. The accompanying drawings generally stress certain aspects deemed important at the expense of accuracy. A version of this article with additional references is available from `https://arxiv.org/abs/1705.05253`.

> *the attempt to comprise all objects that should be considered m-dimensional surfaces*

Notation

Suppose throughout this note that m and n are integers and $1 \leq m \leq n$. We will make use of the m-dimensional Hausdorff measure $\mathcal{H}^m$ over $\mathbf{R}^n$. This measure is one of several natural outer measures over $\mathbf{R}^n$ which associate the usual values with subsets of m-dimensional continuously differentiable submanifolds of $\mathbf{R}^n$.

First Order Quantities

Amongst the initial reasons for developing the notion of *m-dimensional varifold* in $\mathbf{R}^n$ was the attempt to comprise all objects that should be considered m-dimensional surfaces of locally finite area in $\mathbf{R}^n$. Its definition is tailored for compactness.

Definition (Varifold and weight)**.** If $\mathbf{G}(m,n)$ is the space of unoriented m-dimensional subspaces of $\mathbf{R}^n$, endowed with its natural topology, then by an *m-dimensional varifold V in $\mathbf{R}^n$*, we mean a Radon measure V over $\mathbf{R}^n \times \mathbf{G}(n,m)$, and we denote by $\|V\|$ the *weight* of V, that is, its canonical projection into $\mathbf{R}^n$.

The theory of Radon measures yields a metrisable topology on the space of m-dimensional varifolds in $\mathbf{R}^n$ such that $V_i \to V$ as $i \to \infty$ if and only if

$$\int k \, \mathrm{d}V_i \to \int k \, \mathrm{d}V \quad \text{as } i \to \infty$$

whenever $k : \mathbf{R}^n \times \mathbf{G}(m,n) \to \mathbf{R}$ is a continuous function with compact support.

Proposition (Basic compactness)**.** *Whenever κ is a real-valued function on the bounded open subsets of $\mathbf{R}^n$, the set of m-dimensional varifolds V in $\mathbf{R}^n$ satisfying $\|V\|(Z) \le \kappa(Z)$ for each bounded open subset Z of $\mathbf{R}^n$ is compact.*

In general, neither does the weight $\|V\|$ of V need to be m-dimensional, nor does the Grassmannian information of V need to be related to the geometry of $\|V\|$:

Example 1 (Region with associated plane)**.** For each plane $T \in \mathbf{G}(n,m)$ and each open subset Z of $\mathbf{R}^n$, the product of the Lebesgue measure restricted to Z, with the Dirac measure concentrated at T, is an m-dimensional varifold in $\mathbf{R}^n$.

Example 2 (Point with associated plane)**.** For $a \in \mathbf{R}^n$ and $T \in \mathbf{G}(n,m)$, the Dirac measure concentrated at (a,T) is an m-dimensional varifold in $\mathbf{R}^n$.

This generality is in fact expedient to describe higher- (or lower-) dimensional approximations of more regular m-dimensional surfaces, so as to include information on the m-dimensional tangent planes of the limit. Such situations may be realised by elaborating on Examples 1 and 2. However, we will mainly consider *integral* varifolds, that is, varifolds in the spirit of the next basic example, which avoid the peculiarities of the preceding two examples.

Example 3 (Part of a submanifold)**.** If B is a Borel subset of a closed m-dimensional continuously differentiable submanifold M of $\mathbf{R}^n$, then, by Riesz's representation theorem, the associated varifold $\mathbf{v}_m(B)$ in $\mathbf{R}^n$ may be defined by

$$\int k \, \mathrm{d}\mathbf{v}_m(B) = \int_B k(x, \operatorname{Tan}(M,x)) \, \mathrm{d}\mathcal{H}^m x$$

whenever $k : \mathbf{R}^n \times \mathbf{G}(m,n) \to \mathbf{R}$ is a continuous function with compact support, where $\operatorname{Tan}(M,x)$ is the tangent space of M at x. Then, the weight $\|\mathbf{v}_m(B)\|$ over $\mathbf{R}^n$ equals the restriction $\mathcal{H}^m \llcorner B$ of m-dimensional Hausdorff measure to B.

Definition (Integral varifold)**.** An m-dimensional varifold V in $\mathbf{R}^n$ is called *integral* if and only if there exists a sequence of Borel subsets B_i of closed m-dimensional continuously differentiable submanifolds M_i of $\mathbf{R}^n$ with $V = \sum_{i=1}^{\infty} \mathbf{v}_m(B_i)$.

An integral varifold V is determined by its weight $\|V\|$; in fact, associated to $\|V\|$ there are a multiplicity function Θ and a tangent plane function τ with values in the nonnegative integers and $\mathbf{G}(n,m)$, respectively, such that

$$V(k) = \int k(x, \tau(x)) \Theta(x) \, \mathrm{d}\mathcal{H}^m x$$

whenever $k : \mathbf{R}^n \times \mathbf{G}(m,n) \to \mathbf{R}$ is a continuous function with compact support.

Example 4 (Sum of two submanifolds)**.** Suppose M_i are closed m-dimensional continuously differentiable submanifolds M_i of $\mathbf{R}^n$, for $i \in \{1,2\}$, and $V = \mathbf{v}_m(M_1) + \mathbf{v}_m(M_2)$. Then, for $\mathcal{H}^m$ almost all $z \in M_1 \cap M_2$, we have

$$\Theta(z) = 2 \quad \text{and} \quad \tau(z) = \operatorname{Tan}(M_1, z) = \operatorname{Tan}(M_2, z).$$

The theory of varifolds is particularly successful in the variational study of the area integrand. To describe some aspects of this, diffeomorphic deformations are needed in order to define and compute its first variation.

Definition (Area of a varifold)**.** If B is a Borel subset of $\mathbf{R}^n$ and V is called an m-dimensional varifold V in $\mathbf{R}^n$, then *the area of V in B* equals $\|V\|(B)$.

Example 5 (Area via multiplicity)**.** If V is integral, then $\|V\|(B) = \int_B \Theta \, \mathrm{d}\mathcal{H}^m$.

Definition (Push-forward)**.** Suppose V is an m-dimensional varifold in $\mathbf{R}^n$ and ϕ is a continuously differentiable diffeomorphism of $\mathbf{R}^n$ onto $\mathbf{R}^n$. Then, the *push-forward of V by ϕ* is the m-dimensional varifold $\phi_\# V$ defined by

$$\int k \, \mathrm{d}\phi_\# V = \int k(\phi(z), \operatorname{image}(\mathrm{D}\,\phi(z)|S)) \, j_m\phi(z,S) \, \mathrm{d}V(z,S)$$

whenever $k : \mathbf{R}^n \times \mathbf{G}(m,n) \to \mathbf{R}$ is a continuous function with compact support. Here, $j_m\phi(z,S)$ denotes the m-dimensional Jacobian of the restriction to S of the differential $\mathrm{D}\,\phi(z) : \mathbf{R}^n \to \mathbf{R}^n$.

Example 6 (Push-forward)**.** If B and M are as in Example 3, then, by the transformation formula (or area formula), $\phi_\#(\mathbf{v}_m(B)) = \mathbf{v}_m(\operatorname{image}(\phi|B))$.

Definition (First variation)**.** The *first variation* of a varifold V in $\mathbf{R}^n$ is defined by

$$(\delta V)(\theta) = \int \operatorname{trace}(\mathrm{D}\,\theta(z) \circ S_\natural) \, \mathrm{d}V(z,S)$$

whenever $\theta : \mathbf{R}^n \to \mathbf{R}^n$ is a smooth vector field with compact support, where $S_\natural$ denotes the canonical orthogonal projection of $\mathbf{R}^n$ onto S.

Example 7 (Meaning of the first variation)**.** If the maps $\phi_t : \mathbf{R}^n \to \mathbf{R}^n$ satisfy $\phi_t(z) = z + t\theta(z)$ whenever $t \in \mathbf{R}$, $|t| \operatorname{Lip} \theta < 1$, and $z \in \mathbf{R}^n$, then one may verify that, for such t, ϕ_t is a smooth diffeomorphism of $\mathbf{R}^n$ onto $\mathbf{R}^n$ and $\|(\phi_t)_\# V\|(\operatorname{spt} \theta)$ is a smooth function of t whose differential at 0 equals $(\delta V)(\theta)$.

Example 8 (First variation of a smooth submanifold I)**.** Suppose that M is a properly embedded, twice continuously differentiable m-dimensional submanifold-with-boundary in $\mathbf{R}^n$ and $V = \mathbf{v}_m(M)$. Then, integrating by parts, one obtains

$$(\delta V)(\theta) = -\int_M \mathbf{h}(M,z) \bullet \theta(z) \, d\mathcal{H}^m z + \int_{\partial M} \nu(M,z) \bullet \theta(z) \, d\mathcal{H}^{m-1} z,$$

where $\mathbf{h}(M,z) \in \mathbf{R}^n$ is the mean curvature vector of M at $z \in M$, $\nu(M,z)$ is the outer normal of M at $z \in \partial M$, and $\bullet$ denotes the inner product in $\mathbf{R}^n$. In fact, $\mathbf{h}(M,z)$ belongs to the normal space, $\operatorname{Nor}(M,z)$, of M at z.

Second Order Quantities

In general, even integral varifolds—which already are significantly more regular than general varifolds—give meaning only to *first* order properties such as tangent planes but do not possess any *second* order properties such as curvatures. Yet, for varifolds of locally bounded first variation, one may define mean curvature.

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Example 9 (A particular continuously differentiable submanifold without—even approximate—curvatures; see R. Kohn [3])**.** There exists a nonempty closed continuously differentiable m-dimensional submanifold M of $\mathbf{R}^{m+1}$ such that for any *twice* continuously differentiable m-dimensional submanifold N of $\mathbf{R}^{m+1}$, there holds $\mathcal{H}^m(M \cap N) = 0$.

Definition (Locally bounded first variation)**.** A varifold V in $\mathbf{R}^n$ is said to be *of locally bounded first variation* if and only if $\|\delta V\|(Z)$, defined as

$$\sup\{(\delta V)(\theta) : \theta \text{ a smooth vector field compactly supported in } Z \text{ with } |\theta| \le 1\}$$

whenever Z is an open subset of $\mathbf{R}^n$, is finite on bounded sets. Then $\|\delta V\|$ may be uniquely extended to a Radon measure over $\mathbf{R}^n$, also denoted by $\|\delta V\|$.

Clearly, for any open subset Z of $\mathbf{R}^n$, $\|\delta V\|(Z)$ is lower semicontinuous in V.

Example 10 (Functions of locally bounded variation)**.** If $m = n$, then, m-dimensional varifolds in $\mathbf{R}^m$ of locally bounded first variation are in natural correspondence to real-valued functions of locally bounded first variation on $\mathbf{R}^m$.

Example 11 (First variation of a smooth submanifold II)**.** If M and V are as in Example 8, then we have $\|\delta V\| = |\mathbf{h}(M,\cdot)| \mathcal{H}^m \llcorner M + \mathcal{H}^{m-1} \llcorner \partial M$.

Theorem 1 (Compactness of integral varifolds; see Allard [2, 6.4])**.** *If κ is a real-valued function on the bounded open subsets of $\mathbf{R}^n$, then the set of integral varifolds with $\|V\|(Z) + \|\delta V\|(Z) \le \kappa(Z)$ for each bounded open subset Z of $\mathbf{R}^n$ is compact.*

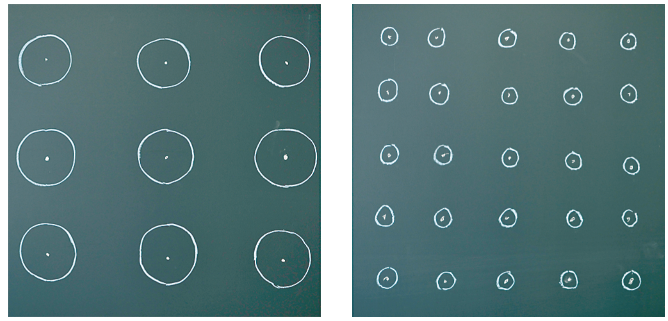

Figure 1. One-dimensional submanifolds (by the spheres drawn) of $\mathbf{R}^2$ converging to a limit nonzero, nonintegral varifold with all one-dimensional tangent planes equally weighted at every point of $\mathbf{R}^2$.

The preceding theorem is fundamental both for varifolds and the consideration of limits of submanifolds. The summand $\|\delta V\|(Z)$ therein may not be omitted:

Example 12 (Lattices of spheres; see Figure 1)**.** If V_i is the varifold associated to

$$M_i = \mathbf{R}^{m+1} \cap \{z : |iz - a| = 3^{-1} i^{-1/m} \text{ for some } a \in \mathbf{Z}^{m+1}\},$$

for every positive integer i, then the nonzero limit of this sequence is the product of the Lebesgue measure over $\mathbf{R}^{m+1}$ with an invariant Radon over $\mathbf{G}(m+1,m)$.

Definition (Mean curvature)**.** Suppose V is a varifold in $\mathbf{R}^n$ of locally bounded first variation and σ is the singular part of $\|\delta V\|$ with respect to $\|V\|$. Then, there exist a σ almost unique, σ measurable function $\boldsymbol{\eta}(V,\cdot)$ with values in the unit sphere $\mathbf{S}^{n-1}$ as well as a $\|V\|$ almost unique, $\|V\|$ measurable, locally $\|V\|$ summable, $\mathbf{R}^n$-valued function $\mathbf{h}(V,\cdot)$—called the *mean curvature of* V—satisfying

$$(\delta V)(\theta) = -\int \mathbf{h}(V,z) \bullet \theta(z) \, d\|V\| \, z + \int \boldsymbol{\eta}(V,z) \bullet \theta(z) \, d\sigma \, z$$

whenever $\theta : \mathbf{R}^n \to \mathbf{R}^n$ is a smooth vector field with compact support.

Example 13 (Varifold mean curvature of a smooth submanifold)**.** If M, V, and ν are as in Example 8, then $\sigma = \mathcal{H}^{m-1} \llcorner \partial M$, $\boldsymbol{\eta}(V,z) = \nu(M,z)$ for σ almost all z, and $\mathbf{h}(V,z) = \mathbf{h}(M,z)$ for $\|V\|$ almost all z.

Yet, in general, the mean curvature vector may have tangential parts, and the behaviour of σ and $\boldsymbol{\eta}(V,\cdot)$ may differ from that of boundary and outer normal:

Example 14 (Weighted plane)**.** If $m = n$, Θ is a positive, continuously differentiable function on $\mathbf{R}^m$, and $V(k) = \int k(z, \mathbf{R}^m) \Theta(z) \, d\mathcal{H}^m z$ whenever $k : \mathbf{R}^m \times \mathbf{G}(m,m) \to \mathbf{R}$ is a continuous function with compact support, then

$$\mathbf{h}(V,z) = \operatorname{grad}(\log \circ \Theta)(z) \quad \text{for } \|V\| \text{ almost all } z.$$

Example 15 (Primitive of the Cantor function)**.** If C is the Cantor set in $\mathbf{R}$, $f : \mathbf{R} \to \mathbf{R}$ is the associated function (i.e., $f(x) = \mathcal{H}^d(C \cap \{\chi : \chi \le x\})$ for $x \in \mathbf{R}$, where $d = \log 2 / \log 3$), and V is the varifold in $\mathbf{R}^2 \simeq \mathbf{R} \times \mathbf{R}$ associated to the graph of a primitive function of f, then V

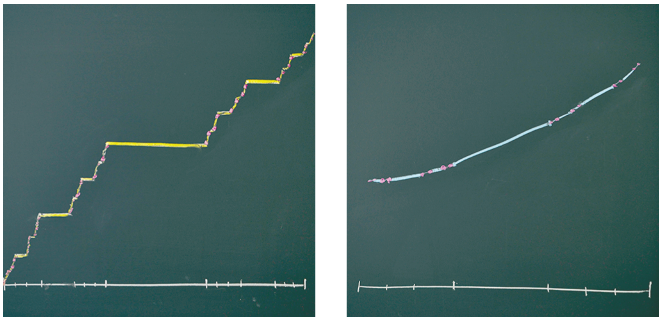

Figure 2. Varifold (blue, on the right) with fractal "boundary." Left: Cantor function. Right: A primitive of the Cantor function.

is an integral varifold of locally bounded first variation, $\mathbf{h}(V,z) = 0$ for $\|V\|$ almost all z, and $\operatorname{spt}\sigma$ corresponds to C via the orthogonal projection onto the domain of f, as in Figure 2.

Definition (Classes $H(p)$ of summability of mean curvature)**.** For $1 \le p \le \infty$, the class $H(p)$ is defined to consist of all m-dimensional varifolds V in $\mathbf{R}^n$ of locally bounded first variation satisfying the following three conditions: if $p > 1$, then $\|\delta V\|$ is absolutely continuous with respect to $\|V\|$; if $1 < p < \infty$, then their mean curvature is locally p-th power $\|V\|$ summable; and if $p = \infty$, it is locally $\|V\|$ essentially bounded.

Observe that $(\int_Z |\mathbf{h}(V,z)|^p \, \mathrm{d}\|V\| z)^{1/p}$ is lower semicontinuous in $V \in H(p)$ if Z is an open subset of $\mathbf{R}^n$ and $1 < p < \infty$; a similar statement holds for $p = \infty$. By Theorem 1, this entails further compactness theorems for integral varifolds.

Example 16 (Critical scaling for $p = m$)**.** If $V \in H(m)$, $0 < r < \infty$, and $\phi : \mathbf{R}^n \to \mathbf{R}^n$ satisfies $\phi(z) = rz$ for $z \in \mathbf{R}^n$, then $\phi_\# V \in H(m)$ and we have

$$\int_{\operatorname{image}(\phi|B)} |\mathbf{h}(\phi_\# V, z)|^m \, \mathrm{d}\|\phi_\# V\| z = \int_B |\mathbf{h}(V,z)|^m \, \mathrm{d}\|V\| z,$$

and, in case $m = 1$, additionally $\|\delta(\phi_\# V)\|(\operatorname{image}(\phi|B)) = \|\delta V\|(B)$ whenever B is a Borel subset of $\mathbf{R}^n$.

Regularity by First Variation Bounds

For twice continuously differentiable submanifolds M, the mean curvature vector $\mathbf{h}(M,z)$ at $z \in M$ is defined as trace of the second fundamental form

$$\mathbf{b}(M,z) : \operatorname{Tan}(M,z) \times \operatorname{Tan}(M,z) \to \operatorname{Nor}(M,z).$$

For varifolds, bounds on the mean curvature (or, more generally, on the first variation) are defined without reference to a second order structure and entail more regularity, similar to the case of weak solutions of the Poisson equation. The key additional challenges are nongraphical behaviour and higher multiplicity:

Example 17 (Cloud of spheres)**.** If $1 \le p < m < n$ and Z is an open subset of $\mathbf{R}^n$, then there exists a countable collection C of m-dimensional spheres in $\mathbf{R}^n$ such that $V = \sum_{M \in C} \mathbf{v}_m(C) \in H(p)$ and $\operatorname{spt}\|V\| = \operatorname{Closure} Z$.

In contrast, if $p \ge m$, then $\mathscr{H}^m \llcorner \operatorname{spt}\|V\| \le \|V\|$ whenever V is an integral varifold in $H(p)$; in particular, $\operatorname{spt}\|V\|$ has locally finite m-dimensional measure.

Definition (Singular)**.** An m-dimensional varifold V in $\mathbf{R}^n$ is called *singular* at z in $\operatorname{spt}\|V\|$ if and only if there is no neighbourhood of z in which V corresponds to a positive multiple of an m-dimensional continuously differentiable submanifold.

Example 18 (Zero sets of smooth functions)**.** If A is a closed subset of $\mathbf{R}^m$, then there exists a nonnegative smooth function $f : \mathbf{R}^m \to \mathbf{R}$ with $A = \{x : f(x) = 0\}$.

Example 19 (Touching; see Figure 3)**.** Suppose A is a closed subset of $\mathbf{R}^m$ without interior points such that $\mathscr{H}^m(A) > 0$, f is related to A as in Example 18, $n = m+1$, $\mathbf{R}^n \simeq \mathbf{R}^m \times \mathbf{R}$, and $M_i \subset \mathbf{R}^n$, for $i \in \{1,2\}$, correspond to the graph of f, and $\mathbf{R}^m \times \{0\}$, respectively. Then, $V = \mathbf{v}_m(M_1) + \mathbf{v}_m(M_2)$ (see Examples 4 and 13) is an integral varifold in $H(\infty)$ which is singular at each point of $M_1 \cap M_2 \simeq A \times \{0\}$.

Proposition (Structure of one-dimensional integral varifolds in $H(1)$; see [4, 12.5])**.** *If $m = 1$ and $V \in H(1)$ is an integral varifold, then, near $\|V\|$ almost all points, V may be locally represented by finitely many Lipschitzian graphs.*

Proposition (Regularity of one-dimensional integral varifolds with vanishing first variation; see Allard and Almgren [1, §3])**.** *If $m = 1$ and V is an integral varifold with $\delta V = 0$, then the set of singular points of V has $\mathscr{H}^m$ measure zero.*

It is a key open problem to determine whether the hypothesis $m = 1$ is essential; the difficulty of the case $m \ge 2$ arises from the possible presence of holes:

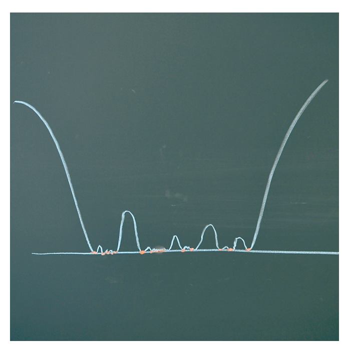

Figure 3. A simple integral varifold with bounded mean curvature whose set of singular points has positive weight measure.

Example 20 (Swiss cheese; see Brakke or [4, 10.8])**.** If $2 \le m < n$, then there exist $V \in H(\infty)$ integral, $T \in \mathbf{G}(n,m)$, and a Borel subset B of $T \cap \operatorname{spt}\|V\|$ with $\mathscr{H}^m(B) > 0$ such that no $b \in B$ belongs to the interior of the orthogonal projection of $\operatorname{spt}\|V\|$ onto T relative to T. So, for $\mathscr{H}^m$ almost all $b \in B$, $\tau(b) = T$ and V is singular at b (cf. Example 4); see Figure 4.

Yet, the preceding varifold needs to have nonsingular points in $\operatorname{spt}\|V\|$. To treat $\|V\|$ almost all points, however, weaker concepts of regularity are needed.

Theorem 2 (Allard's regularity theorem—qualitative version; see Allard [2, 8.1 (1)])**.** *If $p > m \ge 2$ and $V \in H(p)$ is integral, then the set of singular points of V contains no interior points relative to* $\operatorname{spt}\|V\|$.

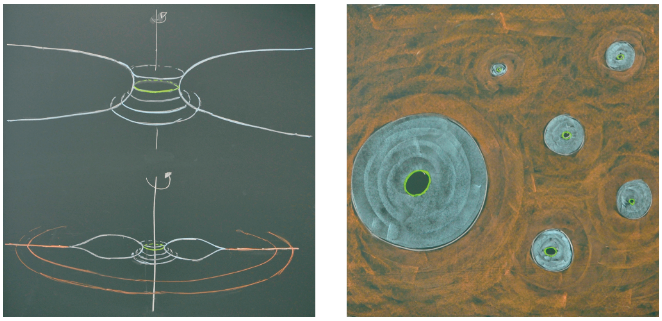

Figure 4. Constructing an integral varifold with bounded mean curvature with holes near a set of positive weight measure. Left: Building blocks (from side). Right: Result (from top).

Theorem 3 (Existence of a second order structure; see [5, 4.8])**.** *If V is an integral varifold in $H(1)$, then there exists a countable collection C of twice continuously differentiable m-dimensional submanifolds such that C covers $\|V\|$ almost all of $\mathbf{R}^n$. Moreover, for $M \in C$, there holds $\mathbf{h}(V,z) = \mathbf{h}(M,z)$ for $\|V\|$ almost all z.*

Hence, one may define a concept of second fundamental form of V such that, for $\|V\|$ almost all z, its trace is $\mathbf{h}(V,z)$. Moreover, Theorem 3 is the base for studying the fine behaviour of integral varifolds in $H(p)$; see [5] and [4].

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Ulrich Menne's research concerns geometric measure theory, differential and convex geometry, and nonlinear elliptic partial differential equations.