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Geometry and measure

1 Introduction

Here are my observations about geometric measure theory.

1.1 Acknowledged results and definition from analysis and measure theory

Definition: An *outer measure* on X is a set function on X with values in $[0, \infty]$ with

- $\mu(\emptyset) = 0$
- $E \subseteq \bigcup_{h \in \mathbb{N}} E_h \quad \Rightarrow \quad \mu(E) \leq \sum_{h \in \mathbb{N}} \mu(E_h)$

Carathéodory's theorem: if μ is an outer measure on X and $\mathcal{M}(\mu)$ is the family of those $E \subseteq X$ such that

$$\mu(F) = \mu(E \cap F) + \mu(F \setminus E), \qquad \forall F \subseteq X$$

then $\mathcal{M}(\mu)$ is a σ-algebra and μ is a measure on $\mathcal{M}(\mu)$.

Definition: μ is a *Borel measure* on a topological space X if it is an outer measure on X such that $\mathcal{B}(X) \subseteq \mathcal{M}(\mu)$.

Definition: A measure μ is said to be *absolutely continuous with respect to* measure λ if for any set A, $\lambda(A) = 0$ implies $\mu(A) = 0$ and we write it $\mu << \lambda$

Definition: We say that a Borel measure μ is *regular* if for every

Definition: An outer measure μ is a *Radon measure* on a topological space if it is locally finite (finite on all compact spaces).

Definition: For a function $f : X \to Y$ between metric spaces we can define its *Lipschitz constant* $\mathbf{Lip}(f) = \inf\{L \in \mathbb{R} \mid d(f(x), f(y)) \leq d(x, y) \forall x, y \in X\}$

1.2 Hausdorff measure

The Hausdorff measure generalises the notion of measure for lower dimensional objects in higher-dimensional space. The idea is essentially similar to the construction of Lebesgue's measure except that we take a lower limit instead of an infinum. We define a cover of E by sets of diameter less then δ as a δ-cover of E. And we conceder only countable covers. We note that

$$\mathcal{H}^s_\delta(E) = \inf_C \sum_{I \in C} \omega_s \left(\frac{\operatorname{diam}(I)}{2}\right)^s$$

where $s \in \mathbb{R}_{\geq 0}$ is a dimension, $\omega_s \in \mathbb{R}$ is a coefficient, preferably continuous or smooth as a function of s, and C is a δ-cover of E. We may assume that

$$\omega_s = \frac{\pi^{s/2}}{\Gamma(1 + s/2)}$$

We define the Hausdorff measure as a limit of the previous value. It exists because $\mathcal{H}^s_\delta(E)$ is increasing function of δ. We note

$$\mathcal{H}^s(E) = \lim_{\delta \to 0^+} \mathcal{H}^s_\delta(E)$$

I shall introduce the notion of s-variation of a cover S as

$$\operatorname{Var}^s(S) = \sum_{I \in S} \omega_s \left(\frac{\operatorname{diam}(I)}{2}\right)^s$$

Proposition: For a natural $n \geq 0$, ω_n is a volume of a unit n-dimensional ball.

1.3 Properties of Hausdorff measure

Proposition: Hausdorff measure is a Borel measure for regular topology.

Proposition: In the definition of Hausdorff measure we can consider only closed or open sets.

Proposition: Hausdorff measure of dimension $m \in N$ coincide on m-dimensional affine subspaces with their Lebesgue measure.

Proposition: For a Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}^m$ we have the following inequality

$$\mathcal{H}^s(f[E]) \leq \mathrm{Lip}(f)^s \mathcal{H}^s(E)$$

for every $s > 0$ and $E \subseteq \mathbb{R}^n$. And $\dim(E) < \dim(f[E])$.

Proposition: The n-dimetional Hausdorff measure traced to a n-dimentional C^1-submanifold of R^m induces the area measure on this submanifold and coincides with the integral measure via parametrisation on it.

Remark: Proofs to those proposition can be found in the book "Geometric measure theory" be Francesco Maggi.

1.4 Hausdorff dimension

To a set S we can associate a number $s = \inf\{a \geq 0 \mid \mathcal{H}^a(S) = 0\}$. It's called its Hausdorff dimension.

Proposition:

2 Dimension of cantor sets

Here we calculate the dimension of generalized set. Let $n \in \mathbb{N}$ and $m \in \mathbb{N}^*$ so that $2m < n$. Then we can define C_k ($k \in \mathbb{N}$) define recursively by agreeing that $C_0 = \{[0, 1]\}$ and we obtain C_{k+1} from C_k by cutting out the open middle part from each segment of C_k and living side parts of length m/n of original interval. We will note $C = \lim C_k = \bigcap C_k$.

image

Obviously C_k is a $(m/n)^k$-cover of C, so

$$\mathcal{H}^s_{(m/n)^k} \leq \sum_{I \in C_k} \omega_s (\frac{\mathrm{diam}(I)}{2})^s = \omega_s 2^k ((m/n)^k)/2)^s = \omega_s/2^s (2(m/n)^s)^k$$

And if $s > \log_{n/m}(2)$ we have right side approaching 0 as k tends to infinity. That means that $\dim(C) \leq \log_{n/m}(2)$.

Now we need to prove the inequality in the other direction. Let $s = \log_{n/m}(2)$. And let S be a $(m/n)^k$-cover of C. In fact by the construction C is an intersection of compacts on a real line, so is compact. And by one of the previous propositions we can conceder only open covers. Then by compactness we can leave only a finite number of sets in S and this way we reduce its Hausdorff variance and we can extend the resting elements to closed intervals of the same diameter. This does not change the variance. The new cover is noted by S'. Now in every interval of S' we can find 2 maximal intervals from some C_i and C_j, so the they are disjoint. If we can't do that, then there are no points of C in this interval and we can throw away that set also. So now we have 2 maximal intervals J and J' in I. They are ordered. Between them we have an interval K and as they are maximal $I \setminus J \setminus K \setminus J'$ does not contain any points from C and we can through those parts away from the covering. By the construction

$$|J|, |J'| \leq \frac{m}{n} \cdot \frac{n}{n-2m}|K| = \frac{m}{n-2m}|K|$$

Now we have $1/2(|J| + |J'|) \leq \frac{m}{n-2m}|K|$

$$|I|^s = (|J|+|J'|+|K|)^s \geq ((1+\frac{n-2m}{2m}))(|J|+|J'|))^s = (\frac{n}{m}1/2(|J|+|J'|))^s = 2(1/2(|J|+|J'|))^s \geq |J|^s+|J'|^s$$

Where the last step is done by concavity of function $x \mapsto x^s$. That means that we can reduce this any cover to a C_k cover which has a smaller s-variation. That means that for dimension $s = \log_{n/m}(2)$ the $\mathcal{H}^s(C)$ is finite as the s-variation of C_k is always $\omega_s/2^s$.

Remark: This is a variation on the proof given in the book "The geometry of fractal sets" by K. J. Flaconer, generelised to the case of arbitrary m and n. In this book the proof is done for the case $m = 1, n = 3$.

Proposition: There is a subset of $[0, 1]$ with a Hausdorff dimension 1, but Lebesgue measure 0.

To show that we shall use Cantor's sets. Let $C_{m/n}$ be a set discussed in a previous paragraph. Then $S = \bigcap C_{m/(2m+1)}$ is a set of dimension 1. As for every $0 \leq s < 1$ there is such m, that $\log_{n/m}(2) = \log_{(2m+1)/m}(2) > s$, as $\log_{(2m+1)/m}(2) \to 1$. And thus $\mathcal{H}^s(S) > \mathcal{H}^s(C_{m/(2m+1)}) = \infty$.

3 Weak* topology and compactness

As to a positive measure we can associate an integral, we need to utilise some results from functional analysis.

For topological spaces Y_i and a set of functions $f_i : X \to Y_i$, we can define the smallest, coarsest topology on X that makes those functions continuous. By definition such topology is $\tau(\{f_i\}) = \bigcap\{\tau \mid \tau \text{ is a topology on } X \text{ and } f_i \text{ are continuous}\}$. As an example, the product topology is exactly $\tau(\{\pi_i\})$, where π_i are canonical projections.

Proposition: Let τ be a topology on X. Then $\tau = \tau(\{f_i\})$ if and only if every function $g : W \to X$ such that $f_i \circ g$ are continuous is continuous.

Remark: This is a well-known property of caorsest topology, but I checked that it is also an alternative characterisation of such topology.
If $\tau = \tau(\{f_i\})$ and $g : W \to X$ is such function that $f_i \circ g$ are continuous. It's sufficient to check that for all elements of prebase of $\tau(\{f_i\})$ the inverse image is open, but the prebase consists of elements of the form $f_i^{-1}[U]$ and its inverse image is $(f_i \circ g)^{-1}[U]$ which is open by hypotheses.

If τ is a such topology, that for every function $g : W \to X$ it is continuous if and only if $f_i \circ g$ are continuous, then in particular we have $\mathrm{id} : (X, \tau) \to (X, \tau)$ continuous and that means that $f_i = f_i \circ \mathrm{id}$ are continuous and we have $\tau(\{f_i\}) \subseteq \tau$. On the other hand we have $\mathrm{id}' : (X, \tau(\{f_i\})) \to (X, \tau)$ continuous because $f_i = f_i \circ \mathrm{id}' : (X, \tau(\{f_i\})) \to Y_i$ are continuous by the definition of coarsest topology. Thus we have id' continuous and that means that $\tau \subseteq \tau(\{f_i\})$. And finally $\tau = \tau(\{f_i\})$.

Theorem (Tychonoff): Product of compact spaces is compact.

General structure: Let I be a set of indices and E_i for $i \in I$ be a topological space with a topology τ_i. The prebase of the product topology on $\prod_{i \in I} E_i$ is $\{\pi_i^{-1}[U] \mid i \in I, U \in \tau_i\}$. a set of products of open subspaces of one spaces on others. All the finite intersections form a base of product topology. Its elements are products of open sets where almost all factors are E_i.

Maximal covers: Let's note that a set of covers that does not contain finite sub-covers for a partially ordered set with the relation of inclusion. For every chain we have its union which does not contain a finite sub-cover, which otherwise would have been in some element of chain. Thus each chain has an upper bound. By the Zorn's lemma we find a maximal element M.

Let X be a topological space and $M \subseteq \tau$ a maximal cover that does not contain a finite sub-cover. **Then if $V \in M^c$, we have $U_1, \ldots, U_n \in M$ such that $V \cup U_1 \cup \ldots \cup U_n = X$.** Because otherwise we could have added V to M and M would not be maximum. **If $U, V \in M^c$ then $U \cap V \in M^c$.** In other words M^c is a multiplicative system, which is similar to the statement that $\mathfrak{p}^c$ is a multiplicative for a prime ideal $\mathfrak{p}$. This is true due to the fact that we have $U_1, \ldots, U_k \in M$ and $V_1, \ldots, V_l \in M$ such that $U \cup U_1 \cup \ldots \cup U_n = X = V \cup V_1 \cup \ldots \cup V_l$ and thus $(U \cap V) \cup U_1 \cup \ldots \cup U_k \cup V_1 \cup \ldots \cup V_l = X$, which implies that $U \cap V \in M^c$.

Alexander's lemma about prebase: Let B be a prebase of a topological space X. Then if in every cover of X by elements of B there exists a finite subcover, then the space X is compact. If X is not compact, then we have a M maximal cover that does not contain a finite sub-cover. Then to every $x \in X$ we can associate its neighborhood $V_x \in M$. Then we find some element of a basis $U_x = U_{1,x} \cup \ldots \cup U_{n_x,x} \subseteq V_x$ where $U_{i,x} \in B$ are elements of prebase. Thus by maximality $U_x \in M$ as $U_x \subseteq V_x$. But as $U_x = U_{1,x} \cup \ldots \cup U_{n_x,x}$ and as M^c is a multiplicative system, for some i we have $U_{i,x} \in M$. It means that in M we have a sub-cover of X by elements of a prebase B. And by hypotheses we can chose a finite sub-cover which gives a contradiction.

Tichonoff theorem's proof: Let $\mathcal{S} = (U_i)_{i \in I}$ be a cover of a product $E = \prod_{j \in J} E_j$ of compact space by elements of canonical prebase. Let's suppose that it does not contain a finite sub-cover. For every $j \in J$ we shall pose $S_j = \{\pi_j^{-1}[V_{i,j}] = U_i \mid V_{i,j} \in \tau_j, i \in I_j\}$. Then $(V_{i,j})_{i \in I}$ cannot be a cover of E_j, because otherwise we can extract a finite sub-cover of E_j and hence of E. So we can chose $x_j \in E_j$ such that $x_j \notin \bigcup_{i \in I_j} V_{i,j}$. Let $x = (x_j)_{j \in J}$ and it does not lie in every set of $\mathcal{S}$, thus it is not a cover and we get a contradiction.

Remark: This is the most non-trivial part of the proof of Banach-Alaoglu theorem and as I had this proof noted I have decided to also put it here.

3.1 Topologies on spaces E and E^*

In this section, E is a normed vector space and E^* is its dual space of continuous 1-forms on E. On the space E, apart from its metric topology, we have the weak topology $\sigma(E, E^*) = \tau(\{f\}_{f \in E^*})$. As $f \in E^*$ is continuous with respect to the regular topology, the topology $\sigma(E, E^*)$ is coarser then the regular topology, which we call strong.

On the space E^*, we also have strong topology with the operator norm. Additionally, we have the weak* topology $\sigma(E^*, E) = \tau(\{v\}_{v \in E})$.

Proposition: The weak* topology is a trace topology from the space $\mathbb{R}^E$ with the product topology.

Proof: Let $\tau(\{\pi_v\}_{v \in E})$ be the trace topology. Then it is easy to see that $\pi_v = v$ as both function are evaluations at v and thus $\tau(\{\pi_v\}_{v \in E}) = \tau(\{v\}_{v \in E}) = \sigma(E^*, E)$ is a weak* topology.

Remark: In the book "Functional Analysis" by Haim Brezis, the part above is done by establishing an homeomorpism and the verification of its bicontinuity. As you have seen, there is actually nothing substantial to prove since these are just two notions of the same concept – projection and evaluation in the dual-space.

Theorem (Banach-Alaoglu): The closed unit ball $B = \{f \in E^* \mid |f| \leq 1\}$ is compact in the weak* topology $\sigma(E^*, E)$.

Proof:

$$B = \left\{ f \in \mathbb{R}^E \mid \begin{cases} |f(x)| < |x|, \ \forall x \in E \\ f(\lambda x) = \lambda f(x), \ \forall \lambda \in \mathbb{R}, x \in E \\ f(x+y) = f(x) + f(y) \ \forall x, y \in E \end{cases} \right\}$$

Hence it is intersection of the following sets $B = K \cap \bigcap_{x,y \in E} A_{x,y} \cap \bigcap_{x \in E, \lambda \in \mathbb{R}} B_{\lambda,x}$, where $K = \{f \in \mathbb{R}^E \mid |f(x)| \leq |x|\} = \prod_{x \in E} [-|x|, |x|]$ is compact by Tichonoff theorem, where for $x, y \in E$, we define $A_{x,y} = \{f \in \mathbb{R}^E \mid f(x+y) - f(x) - f(y) = 0\}$, which is closed since evaluations and addition are continuous, and thus $f \mapsto f(x+y) - f(x) - f(y)$ is continuous and $A_{x,y}$. For similar reasons $B_{\lambda,x} = \{f \in \mathbb{R}^E \mid f(\lambda x) - \lambda f(x) = 0\}$ is closed. This proves that B is compact.

4 Measures and convergence

4.1 Vector valued measure

Let X be a topological space and V a Banach space, then $\mu : \mathcal{B}(X) \to Y$ is a V-valued Borel measure if

$$\sum_n \mu(E_n) = \mu(\bigcup_n E_n)$$

for any disjoint countable family $\{E_n\}$ of Borel sets. From that definition we have $\mu(A) + \mu(\emptyset) = \mu(A \cup \emptyset) = \mu(A)$ and thus $\mu(\emptyset) = 0$. This is a quite a strong property as the convergence of the sum does not depend on the order, which in finite dimenetions is equivalent to the absolute convergence of that series.

Let μ be a vector valued measure. Then the *total variation* $|\mu|$ of a Borel set A by measure μ is defined by:

$$|\mu|(A) = \sup\{\sum_n |\mu(A_n)| \mid \{A_n\} \text{ countable partition of } A\}$$

Proposition: Total variation is a positive bounded measure.

It is easy to see that $|\mu|(\emptyset) = 0$ since all partitions of an empty set consist of empty sets which measure is zero. The image of $|\mu|$ by the definition consists of positive numbers. Lastly we shall verify σ-additivity. Let $\{S_n\}$ be a disjoint countable collection of Borel sets. Then

$$\sum_n |\mu|(S_n) = \sum_n \sup\{\sum_m |\mu(S_{n,m})| \mid (S_{n,m})_m \text{ is a countable Borel partition of } S_n\}$$

Then we remark that for each choice of $\{S_{n,m}\}$, it is a countable Borel partition of $S = \bigcup_n S_n$, and thus $|\mu|(S) > \sum_n |\mu|(S_n)$. On the other hand if $\{A_k\}$ is a countable Borel partition of S then we have partitions of S_n defined as $\{S_{n,k} = A_k \cap S_n\}_k$ and we have the following inequality:

$$\sum_k |\mu(A_k)| = \sum_k |\sum_n \mu(S_{n,k})| \leq \sum_n \sum_k |\mu(S_{n,k})|$$

which implies $|\mu|(S) \leq \sum_n |\mu|(S_n)$ and we conclude that $|\mu|$ is a positive measure.

Let's verify that total variation is bounded. That is a tricker question and we shall follow the proof from "...". The measure can be partitioned into projection measures $\mu = (\mu_i)_{i=1}^n$. As all the norms are equivalent we can concider $|\cdot| = \|\cdot\|_1$. Then as we have the following inequality:

$$\sup\{\sum_i |\mu(X_i)| \mid X_i \text{ is a borel partition of } X\} \leq \sum_j \sup\{\sum_i |\mu_j(X_i)| \mid X_i \text{ is a borel partition of } X\}$$

It is sufficient to prove that for real valued measures its total variation is bound. If we suppose it is not, then we have a real valued measure μ, countable Borel partition of X $\{X_m\}_m$ and $n \in \mathbb{N}$ such that

$$\sum_{m=0}^{n} |\mu(X_m)| > 2(|\mu(X)| + 1)$$

Let $P = \{X_i|\mu(X_i) > 0\}$ and $N = \{X_i|\mu(X_i) < 0\}$. Then we have $|\mu(\bigcup P)| > |\mu(X)| + 1$ or $|\mu(\bigcup N)| > |\mu(X)| + 1$, thus we have a set E such that $|\mu(E)| > |\mu(X)| + 1$. Then we have $|\mu(E^c)| = |\mu(X) - \mu(E)| \geq |\mu(E)| - |\mu(X)| > 1$. Then by additivity of $|\mu|$ we have $|\mu|(E) = \infty$ or $|\mu|(E^c) = \infty$; supposing the latter we pose $E_1 = E$ (or $= F$) we always have $\mu(E_1) > 1$ and if we continue the same procedure for $X = E^c$ we construct by the choice axiom the following sequence of disjoint sets $(E_i)_i$ and $|\mu|(E_i) > 1$ and thus $\sum \mu(E_i)$ does not converge and we have a contracdiction to the definition of vector valued measure. Thus μ is bound.

By setting

$$\mu_+ = \frac{|\mu| + \mu}{2} \qquad \mu_- = \frac{|\mu| - \mu}{2}$$

we have μ_+ and μ_- positive bounded measures and $\mu = \mu_+ - \mu_-$ which ports a name a *Jordan decomposition.*

The *mass* of μ is set to be $\|\mu\| = |\mu|(X)$. Does it coincide with the operator norm?

4.2 Differentiation of Radon measure and Radon-Nikodym Theorem

Remark: That section is an adopted version of paragraph 1.6 from the book "Measure theory and fine properties of functions" for vector measures.

Definition: Let μ and ν be Radon measures on $\mathbb{R}^n$. Then we can define upper and lower derivatives of ν by μ by

$$\overline{D}_\mu \nu(x) = \begin{cases} \limsup_{r\to 0} \frac{\nu(B(x,r))}{\mu(B(x,r))} & \text{if } \mu(B(x,r)) > 0 \text{ for all } r > 0 \\ +\infty & \text{if } \mu(B(x,r)) = 0 \text{ for some } r > 0 \end{cases}$$

$$\underline{D}_\mu \nu(x) = \begin{cases} \liminf_{r\to 0} \frac{\nu(B(x,r))}{\mu(B(x,r))} & \text{if } \mu(B(x,r)) > 0 \text{ for all } r > 0 \\ +\infty & \text{if } \mu(B(x,r)) = 0 \text{ for some } r > 0 \end{cases}$$

If $\overline{D}_\mu \nu(x) = \underline{D}_\mu \nu(x) < +\infty$ then we say that ν is *differentiable* with respect to μ at x and we write

$$D_\mu \nu(x) = \overline{D}_\mu \nu(x) = \underline{D}_\mu \nu(x)$$

Definition: Let μ be Radon measure and ν be a vector measure on $\mathbb{R}^n$. Then we define a derivatives as

$$D_\mu \nu(x) = \begin{cases} \lim_{r\to 0} \frac{\nu(B(x,r))}{\mu(B(x,r))} & \text{if } \mu(B(x,r)) > 0 \text{ for all } r > 0 \\ +\infty & \text{if } \mu(B(x,r)) = 0 \text{ for some } r > 0 \end{cases}$$

Upper and lower derivatives lemmas: For $\alpha \in \mathbb{R}_{>0}$ we have

- $A \subseteq \{x \in \mathbb{R}^n \mid \underline{D}_\mu \nu(x) \leq \alpha\}$ **implies** $\nu(A) \leq \alpha\mu(A)$.
- $A \subseteq \{x \in \mathbb{R}^n \mid \overline{D}_\mu \nu(x) \geq \alpha\}$ **implies** $\nu(A) \geq \alpha\mu(A)$.

Proposition: Let μ be a Radon measure and ν be a Radon or Vector measure. Then

- $D_\mu \nu$ **exists and is finite** μ**-a.e.**
- $D_\mu \nu$ **is** μ**-measurable.**

We start by proving statements for μ-Radon measure and then I will pass to Vector space. We may assume $\mu(\mathbb{R}^n), \nu(\mathbb{R}^n) < \infty$. Otherwise we can take restriction on bounded neiboouds of each point. Let's prove the first assertion.

Let $I = \{x \mid D_\mu \nu(x) = \infty\}$. Observe that for each $\alpha > 0$, $I \subseteq \{x \mid D_\mu \nu(x) \geq \alpha\}$. Thus by Lemma

Radon-Nikodym Theorem: Let ν be a vector measure and μ a Radon measure on $\mathbb{R}^n$. Then

$$\nu(A) = \int_A D_\mu \nu \, d\mu$$

4.3 Representation of vector valued measures

In the context of geometric measure theory we are interested in the vector space $E = C_c^0(\mathbb{R}^n, \mathbb{R}^m)$ with the supremum norm. Then its dual space is $E^* = \{L : E \to \mathbb{R} \mid L \text{ is linear and continious}\}$ is a vector space of bound measure. Then on the E^* from now and on we will consider the weak* star topology. To make the connection with measure we shall state the result for Reisz's representation of E^*. In fact every functional $L \in E^*$ can be represented by an $\mathbb{R}^m$-valued Radon measure μ, such that

$$\langle L, \phi \rangle = \int \phi d\mu$$

We shall denote $\mathcal{M}(X, \mathbb{R}^n)$ the space of $\mathbb{R}^n$ measures on X endowed with weak* topology.

4.4 Riesz representation theoremes for vector valued measure

The map

$$\begin{aligned} \Lambda : \mathcal{M}(X, \mathbb{R}^n) &\to C_0(X, \mathbb{R}^n)^* \\ \mu \quad &\mapsto \quad \Lambda_\mu \end{aligned}$$

is an isometry

4.5 Interpretation of Banach-Alaoglu theorem for vector valued measures

5 Analysis results

For a ball $B = B(x, r)$ of center x and radius r we shall note ${}^\epsilon B = B(x, (1+\epsilon)r)$ for every $\epsilon > 0$.

Vitali's covering theorem: Let $\mathcal{F}$ be any collection of nondegenearted closed balls in $\mathbb{R}^n$ with

$$\sup\{\operatorname{diam} B \mid B \in \mathcal{F}\} < \infty$$

Then for every $\epsilon > 1$ there exist a countable family $\mathcal{G}$ of disjoint balls in $\mathcal{F}$ such that

$$\bigcup_{B \in \mathcal{F}} B \subseteq \bigcup_{B \in \mathcal{G}} {}^{2\epsilon} B$$

Proof: Set $D = \sup\{\operatorname{diam} B \mid B \in \mathcal{F}\}$. Set

$$\mathcal{F}_j = \left\{ B \in \mathcal{F} \mid \frac{D}{\epsilon^j} < \operatorname{diam} B \leq \frac{D}{\epsilon^{j-1}} \right\}, \quad j = 1, 2, \ldots$$

We define $\mathcal{G}_j \subseteq \mathcal{F}_j$ as follows

- Let $\mathcal{G}_1$ be any maximal disjoint collection of balls in $\mathcal{F}_1$.
- Assuming $\mathcal{G}_1, \ldots, \mathcal{G}_{k-1}$ have been selected, we chose $\mathcal{G}_k$ to be any maximal disjoint subcollection of

$$\{B \in \mathcal{F}_k \mid B \cap B' = \emptyset \text{ for all } B' \in \bigcup_{j=1}^{k-1} \mathcal{G}_j\}$$

They exist by Zorn's Lemma. Finally, define $\mathcal{G} = \bigcup_{j \in \mathbb{N}^*} \mathcal{G}_j$ a collection of disjoint balls and $\mathcal{G} \subseteq \mathcal{F}$.

Proving that for each ball $B \in \mathcal{F}$, there exists a ball $B' \in \mathcal{G}$ such that $B \cap B' \neq \emptyset$ and $B \subseteq {}^\epsilon B'$. Fix $B \in \mathcal{F}$, there exists and index j such that $B \in \mathcal{F}_j$ and by maximality of $\mathcal{G}_k$ there exists a ball $B' \in \bigcup_{k=1}^j \mathcal{G}_k$ with $B \cap B' \neq \emptyset$. But $\operatorname{diam} B' > \frac{D}{\epsilon^j}$ and $\operatorname{diam} B \leq \frac{D}{\epsilon^{j-1}}$; so that

$$\operatorname{diam} B \leq \frac{D}{\epsilon^{j-1}} < \epsilon \operatorname{diam} B'$$

Thus $B \subseteq {}^{2\epsilon}B'$.

Remark: This is a generalised version of the proof from the book "Measure theory and fine properties of functions" where it is done for the smallest integral case $\epsilon = 2$. The generalised proof shows the reason why the final dilatation is $5 = 1 + 2\epsilon$, but actually it is true for dilatation ≥ 3 and the smallest such integer is 4.

Whitney covering theorem: Let $C \subseteq \mathbb{R}^n$ be a closed set and $f : C \to \mathbb{R}$, $d : C \to \mathbb{R}^{n*}$ be continuous functions. We shall use notions

$$R(y,x) = \frac{f(y) - f(x) - d(x)(y-x)}{|x-y|}, \quad \forall x, y \in C, x \neq y$$
$$\rho_k(\delta) = \sup\{|R(x,y)| \mid 0 < |x-y| \leq \delta, x, y \in K\}$$

if we suppose that for every compact $K \subseteq C$

$$\rho_K(\delta) \to 0 \text{ as } \delta \to 0 \tag{1}$$

Then there exists a fuction $\overline{f} \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$ and $D\overline{f}|_C = d$.

Remark: I seek to give a more explicit version of the proof given in the book "Measure theory and fine properties of functions". In books that looked at about geometric measure theory this proof usually is not stated and pointed to the book of Federer wheres at least in version of that book the theorem is proved in much more general context and the theorem statement differs from the one we want.

Proof: The main challenge is to find a suitable extension of f. To construct this extension we will select regularly enough points in the complementary set and make a such function so that on those points it is an extension via averaged linear extrapolation and in between we interpolate by some close enough points. Let $U = C^c$ be a complementary open set. Let $r(x) = \frac{1}{4}\min(1, \operatorname{dist}(x, C))$. By Vitali's covering theorem there exist a countable set $\{x_j\}_{j\in\mathbb{N}}$ and a countable set of disjoint closed balls $\{B_j = B(x_j, r(x_j))\}_{j\in\mathbb{N}}$ such that $\bigcup_{j\in\mathbb{N}} {}^2B_j = U$. We need $\frac{1}{2}$ in the definition of $r(x)$ to make sure that ${}^2B_j \subseteq U$. Then for every $x \in U$ we shall define $S_x = \{x_j \mid B(x, 2r(x)) \cap B(x_j, 2r(x_j)) \neq \emptyset\}$.

Now we chech that S_x is bounded for each dimention. Let $x_j \in S_x$ then $|r(x) - r(x_j)| \leq 1/4|x - x_j|$ because $|r(x) - r(x_j)| = 1/4|\min(1, \operatorname{dist}(x, C)) - \min(1, \operatorname{dist}(x_j, C))|$ and without loss of generality we can consider 3 cases:

1. $\operatorname{dist}(x, C), \operatorname{dist}(x_j, C) > 1$ then $|\min(1, \operatorname{dist}(x, C)) - \min(1, \operatorname{dist}(x_j, C))| = 0 \leq |x - x_j|$.
2. $\operatorname{dist}(x, C) \leq 1, \operatorname{dist}(x_j, C) > 1$, then $|\min(1, \operatorname{dist}(x, C)) - \min(1, \operatorname{dist}(x_j, C))| = 1 - \operatorname{dist}(x, C) < \operatorname{dist}(x_j, C) - \operatorname{dist}(x, C) = |x_j - s| - |x - s| \leq |x_j - x|$, where s is a projection of x on C.
3. $\operatorname{dist}(x, C) \leq \operatorname{dist}(x_j, C) \leq 1$, then $|\min(1, \operatorname{dist}(x, C)) - \min(1, \operatorname{dist}(x_j, C))| = \operatorname{dist}(x_j, C) - \operatorname{dist}(x, C) \leq |x_j - x|$.

So we have $|r(x) - r(x_j)| \leq 1/4|x - x_j| \leq 1/4|2r(x) - 2r(x_j)| = 1/2(r(x) + r(x_j))$ as $x_j \in S_x$. And hence

$$r(x) - r(x_j) \leq 1/2(r(x) + r(x_j)) \;\Rightarrow\; r(x) \leq 3r(x_j)$$
$$r(x_j) - r(x) \leq 1/2(r(x) + r(x_j)) \;\Rightarrow\; r(x_j) \leq 3r(x)$$

In addition we have $|x - x_j| + r(x_j) \leq 2(r(x) + r(x_j)) + r(x_j) \leq 2r(x) + 6r(x) + 3r(x) = 11r(x)$. Which means that $B(x_j, r(x_j)) \subseteq B(x, 11r(x))$ and since $B(x_j, r(x_j))$ are disjoint we have an inquality on volumes:

$$\#S_x\omega_n(r(x)/3)^n \leq \#S_x\omega_n(r(x_j))^n = \sum_{x_j\in S_x} \operatorname{Vol} B_j \leq \operatorname{Vol}(B(x, 11r(x)) = \omega_n(11r(x))^n$$

Therefor $\#S_x \leq (3 \cdot 11)^n = 33^n$ is bounded by a fixed constant in each dimention.

The goal of that part is to construct the function $\overline{f}$. Let $\mu : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^\infty$ function such that $0 \leq \mu \leq 1$, $\mu(t) = 1$ if $t \leq 1$ and $\mu(t) = 0$ if $t \geq 2$. Then for each $j = 1, ...$ we set $u_j(x) = \mu\left(\frac{|x-x_j|}{2r(x_j)}\right)$ for $x \in \mathbb{R}^n$. Then $u_j \in \mathcal{C}^\infty$, $0 \leq u_j \leq 1$ and $u_j \equiv 1$ on $B(x_j, 2r(x_j))$ and $u_j \equiv 0$ on $B(x_j, 4r(x_j))$.

Lipschitz function extension theorem: Let X **be a metric space,** $A \subseteq X$ **and** $f : A \to \mathbb{R}$**. Then there exists a Lipschitz function** $\overline{f} : X \mapsto \mathbb{R}$ **such that** $\operatorname{Lip}(f) = \operatorname{Lip}(\overline{f})$ and $\overline{f}|_A = f$.

This is a proof from "Simons Lectures on geometric measure theory". Let's set $L = \operatorname{Lip}(f)$. Then we define

$$\overline{f}(x) = \inf_{y\in A}(f(y) + Ld(x,y))$$

By the definition, for all $x \in A$, $\overline{f}(x) \leq f(x)$ as in particular we can chose $y = x$. Furthermore, for all $a, b \in A$ and $x \in X$, we have an inequality for a Lipschitz function $f(b) - f(a) \leq Ld(b, a) \leq Ld(b, x) + Ld(a, x)$ and thus we have

$$f(a) + Ld(a, x) \geq f(b) - Ld(b, x)$$

and if we apply an infinum over a, we have $\overline{f}(x) \geq f(b) - L(b, x)$ and if $x \in A$ we can chose $b = x$ and we have an inequality in the other direction and thus the equality $\overline{f}(x) = f(x)$.

Now we check the Lipschitz constant

Consequence: Let X be a metric space, $A \subseteq X$ and $f : A \to \mathbb{R}^n$. Then there exists a Lipschitz function $\overline{f} : X \mapsto \mathbb{R}^{\mathbb{m}}$ such that $\overline{f}|_A = f$

Let's set $\overline{f} = (\overline{f}_i)_i$ extension by coordinate functions.

Remark: I was thinking about extending the theorem to the case where function take vector values, but I can only prove it for the maximum norm.

6 Countably n-rectifiable sets

Let $M \subseteq \mathbb{R}^{n+k}$ is called an *n-rectifiable* if

$$M \subseteq M_0 \cup \bigcup_{i \in \mathbb{N}^*} F_i[\mathbb{R}^n]$$

where $\mathcal{H}^n(M_0) = 0$ and F_i are Lipschitz functions.
Proposition: Alternatively we can define an n-rectifiable set by inclusion $M \subseteq M_0 \cup \bigcup_{i \in \mathbb{N}^*} F_i[A_i]$, where the only thing we change is $F_i : A_i \to \mathbb{R}^{n+m}$ the domain of Lipschitz function.
Clearly the definition in the proposition covers a wider variety of sets. To prove that they are equal we shall recall an extension theorem.

7 Grassmannian

In this section we introduce the topological space $G(m, n)$.

Similarly to projective spaces $P\mathbb{R}^n$ one can generalise this notion to smaller subspaces than hyperplanes. The set of m dimensional subspaces of a vector space $\mathbb{R}^n$ is called grassmannian and noted by $G(m, n)$. It has a topology identified from a topology of orthogonal projection on m-dimensional subspaces.

8 Varifold

An m-dimensional varifold V is a Radon measure over $\mathbb{R}^n \times G(n, m)$ endowed with a product topology. We say $\|V\|$ is a measure in $\mathbb{R}^n$ which is reciprocally projection of a varifold V by π_1^{-1}.
Proposition: For varifolds we concider weak* topology. Then we have a convergence criteria that $V_i \to V$ if and only if

$$\int f dV_i \to \int f dV$$

for every continuous function $f : \mathbb{R}^n \times G(m, n) \to R$ with a compact support.