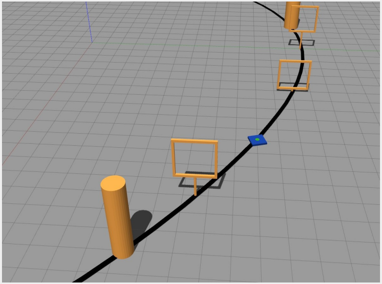
Autonomous Mobile Robotics

Robot Localization

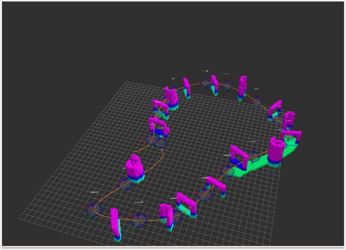
Geesara Kulathunga

March 13, 2023



Robot Localization

- A Taxonomy of Localization Problems
- Markov localization
  - Environment Sensing
  - Motion in the Environment
  - Localization in the Environment
- EKF localization with known correspondence
- Particle filter localization with known correspondence



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- Passive Versus Active Approaches
  - In passive, robot is controlled through some other means, robot motion is not aiming at facilitating localization

Algorithm Markov_localization($bel(x_{t-1}), u_t, z_t, m$)**:**

for all x_t do

$\overline{bel}(x_t) = \int p(x_t \mid u_t, x_{t-1}, m)\ bel(x_{t-1})\ dx$

$bel(x_t) = \eta\ p(z_t \mid x_t, m)\ \overline{bel}(x_t)$

endfor

return $bel(x_t)$

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- **Markov localization** is derived from the algorithm **Bayes filter**
- However, it requires information about the **map** to **estimate the measurement model** $p(z_t|x_t, m)$
- Markov localization addresses the **global localization**, the position tracking, and the kidnapped robot problem in **static environment**

Markov localization

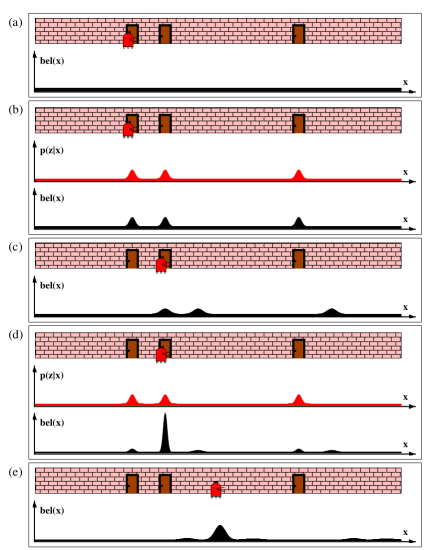
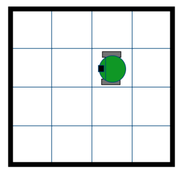


Illustration of the Markov localization algorithm, Thrun, Sebastian. "Probabilistic robotics." Communications of the ACM 45.3 (2002): 52-57.

Grid-based localization



.02	.05	.05	.05
.02	.05	.18	.05
.05	.05	.18	.05
.05	.05	.05	.05

robot initial belief

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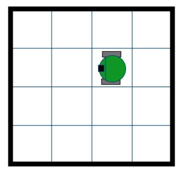

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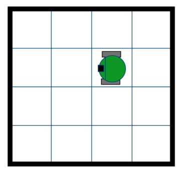

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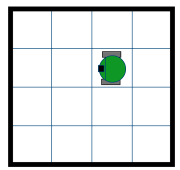

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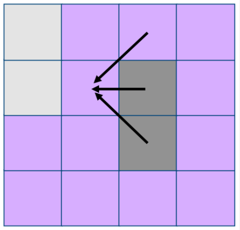
- Consider the initial belief of the robot position is given
- If **the control command** to the robot is given by $\delta x, \delta y$ = -1.0 cells, 0.0 cells, what is the probability that robot be in the position (2,3)
- The following outcomes are possible when **the control command** is being applied

.00	.00	.00
.00	.00	1.0
.00	.00	.00

$\xrightarrow{(\Delta x, \Delta y)}$

.00	.20	.00
.00	.50	.10
.00	.20	.00

- How many possible ways to get to (2,3)?



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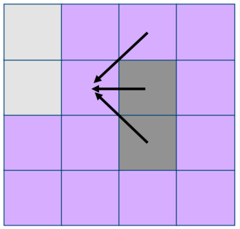

  - Prediction step

$$p(x_k|z_{1:k-1}, u_{1:k-1}) = \Sigma_{x_{k-1}\in X}\ p(x_k|x_{k-1}, u_{k-1})p(x_{k-1}|z_{1:k-1}, u_{0:k-1}) \quad (1)$$

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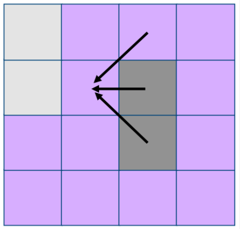

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  - Correction step

$$p(x_k|z_{1:k}, u_{0:k-1}) = \frac{p(z_k|x_k) p(x_k|z_{1:k-1}, u_{0:k-1})}{p(z_k|z_{1:k-1}, u_{0:k-1})} \tag{2}$$

, where

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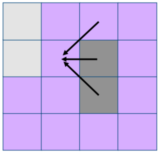

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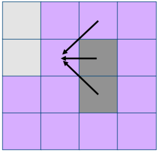
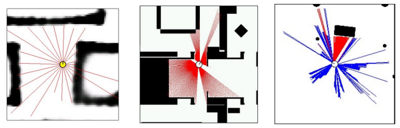


  - Prediction step

$$
\begin{aligned}
p(x_{i,t}|u_t) &= \Sigma_{j=1}^{n} p\left(x_{i,t}|x_{j,t-1}, u_t\right) p\left(x_{j,t-1}\right) \\
&= p\left(x_{i,t} = (2,3)|x_{j,t-1} = (3,3), u_t = (-1,0)\right) p\left(x_{j,t-1} = (3,3)\right) \\
&+ p\left(x_{i,t} = (2,3)|x_{j,t-1} = (2,3), u_t = (-1,0)\right) p\left(x_{j,t-1} = (2,3)\right) \\
&+ p\left(x_{i,t} = (2,3)|x_{j,t-1} = (3,2), u_t = (-1,0)\right) p\left(x_{j,t-1} = (3,2)\right) \\
&+ p\left(x_{i,t} = (2,3)|x_{j,t-1} = (3,4), u_t = (-1,0)\right) p\left(x_{j,t-1} = (3,4)\right) \\
&= 0.5 \cdot 0.18 + 0.1 \cdot 0.05 + 0.18 \cdot 0.2 + 0.05 \cdot 0.2 \\
&= 0.141
\end{aligned}
$$

Correction step

- How can we estimate the $p(z_t|x_{i,t})$?



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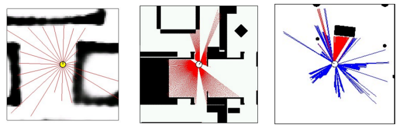

- If each sensor reading consists of N measurements, i.e., $z = \{z_1, ..., z_n\}$, assuming each such **measurement is independent** given the robot pose,

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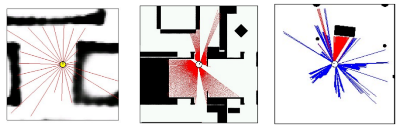

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- Such measurements can be caused by known obstacles, dynamic obstacles, reflections, etc.

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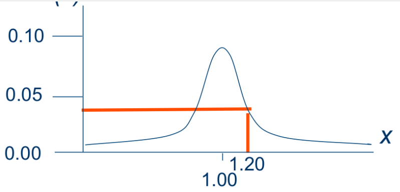
- $p(z_t|x_{i,t})$ getting measurement z_t from state $x_{i,t}$
- Let z_t be 1.2m and range sensor has the following distribution



- $p(z_t)$ probability of the sensor measurement z_t. Calculated so that the sum over all states $x_{i,t}$ equals 1

$$1 = \Sigma_{i=1}^{n} p(x_{i,t}|z_t = 1.2)$$

$$1 = \frac{\Sigma_{i=1}^{n} p(z_t = 1.2|x_{i,t}) p(x_{i,t}|u_{i,t})}{p(z_t = 1.2)}$$

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- Can we calculate this?

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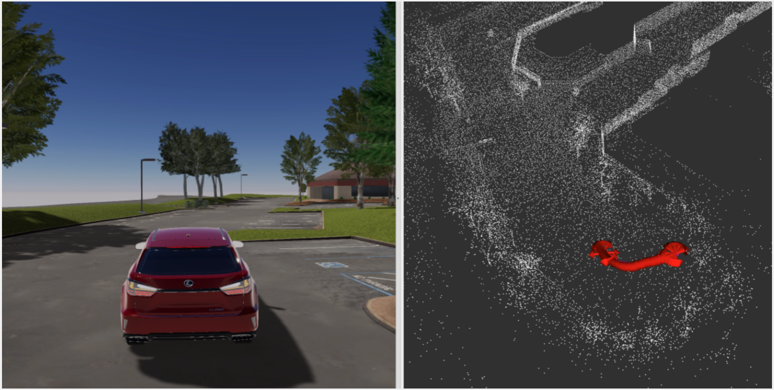
EKF Localization



https://autowarefoundation.gitlab.io/autoware.auto/AutowareAuto/ekf-localization-howto.html

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- Output is a new, revised estimation: μ_t and Σ_t

KF	EKF
	$\boxed{\Phi_k = \left.\frac{\partial f(\mathbf{x}_k, t)}{\partial \mathbf{x}}\right\vert_{\mathbf{x}_k}}$
$\hat{\mathbf{x}}_k^- = \Phi_k \mathbf{x}_k$	$\boxed{\hat{\mathbf{x}}_k^- = f(\mathbf{x}_k, \mathbf{t})}$
$\mathbf{P}_k^- = \Phi_k \mathbf{P}_k \Phi_k^T + \mathbf{Q}_k$	$\mathbf{P}_k^- = \Phi_k \mathbf{P}_k \Phi_k^T + \mathbf{Q}_k$
	$\boxed{\mathbf{H} = \left.\frac{\partial h(\hat{\mathbf{x}}_k^-)}{\partial \hat{\mathbf{x}}}\right\vert_{\hat{\mathbf{x}}_k^-}}$
$\mathbf{y} = \mathbf{z}_k - \mathbf{H}_k \hat{\mathbf{x}}_k^-$	$\mathbf{y} = \mathbf{z}_k - \boxed{h(\hat{x}_k^-)}$
$\mathbf{K_k} = \mathbf{P_k^-} \mathbf{H_k}^T (\mathbf{H_k} \mathbf{P_k^-} \mathbf{H_k}^T + \mathbf{R}_k)^{-1}$	$\mathbf{K_k} = \mathbf{P_k^-} \mathbf{H_k}^T (\mathbf{H_k} \mathbf{P_k^-} \mathbf{H_k}^T + \mathbf{R}_k)^{-1}$
$\hat{\mathbf{x}}_k = \hat{\mathbf{x}}_k^- + \mathbf{K_k} \mathbf{y}$	$\hat{\mathbf{x}}_k = \hat{\mathbf{x}}_k^- + \mathbf{K_k} \mathbf{y}$
$\mathbf{P}_k = (\mathbf{I} - \mathbf{K_k} \mathbf{H_k}) \mathbf{P_k^-}$	$\mathbf{P}_k = (\mathbf{I} - \mathbf{K_k} \mathbf{H_k}) \mathbf{P_k^-}$

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Probabilistic Motion Model

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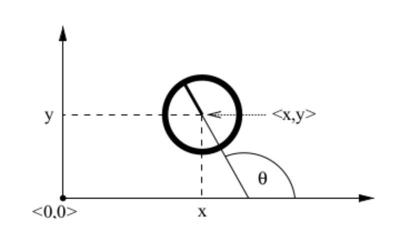
- Robot pose $[x \quad y \quad \theta]^\top$, shown in a global coordinate system



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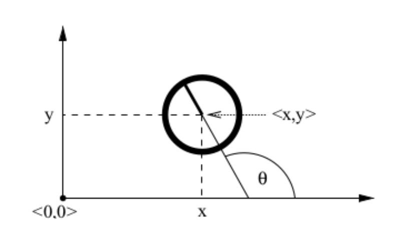

- Probabilistic kinematic model, or motion model (velocity motion model or odometry motion model), describes the posterior distribution over kinematic states that a robot assumes when executing the motion command $\mathbf{u}_t$ at $\mathbf{x}_t$

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Velocity Motion Model (Noise-free)

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- Differential drives, Ackerman drives, and synchro-drives can be controlled in this way
- Let $\mathbf{x}_{t-1} = [x_{t-1} \quad y_{t-1} \quad \theta_{t-1}]^\top, \mathbf{x}_t = [x_t \quad y_t \quad \theta_t]^\top$ be pose and time $t-1$ and successor pose, respectively, after applying applying control u_{t-1} for δt duration

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- A robot can be control through linear and angular velocities $\mathbf{u}_t = [v_t \quad \omega_t]^\top$
- Differential drives, Ackerman drives, and synchro-drives can be controlled in this way
- Let $\mathbf{x}_{t-1} = [x_{t-1} \quad y_{t-1} \quad \theta_{t-1}]^\top, \mathbf{x}_t = [x_t \quad y_t \quad \theta_t]^\top$ be pose and time $t-1$ and successor pose, respectively, after applying applying control u_{t-1} for δt duration
- If **both velocities are kept at a fixed value** for the entire time interval, [t-1, t], **robot moves on a circle with radius** $r = |\frac{v}{u}|$

Velocity Motion Model (Noise-free)

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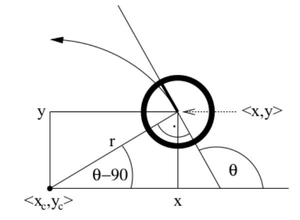
- If **both velocities are kept at a fixed value** for the entire time interval, [t-1, t], **robot moves on a circle with radius** $r = |\frac{v}{u}|$
- For linear motion, r becomes infinite
- After δt units of time, the noise-free robot has progressed $v\delta t$ along the circle, which caused its heading direction to turn by $\omega\delta t$

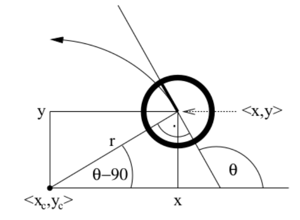
Velocity Motion Model (Noise-free)



- The center of the circle is at, assuming v and ω, denoted linear and angular velocities relative to $< x, y >$,

$$\begin{bmatrix} x_c \\ y_c \end{bmatrix} = \begin{bmatrix} x - \frac{v}{\omega} sin(\theta) \\ y + \frac{v}{\omega} cos(\theta) \end{bmatrix}$$

Velocity Motion Model (Noise-free)



- The center of the circle is at, assuming v and ω, denoted linear and angular velocities relative to $< x, y >$,

$$\begin{bmatrix} x_c \\ y_c \end{bmatrix} = \begin{bmatrix} x - \frac{v}{\omega} sin(\theta) \\ y + \frac{v}{\omega} cos(\theta) \end{bmatrix}$$

- After δt time, ideal robot will be at $\mathbf{x}_{t+1} = \begin{bmatrix} x_{t+1} & y_{t+1} & \theta_{t+1} \end{bmatrix}$

$$= \begin{bmatrix} x_c + \frac{v}{\omega} sin(\theta_t + \omega\delta t) \\ y_c - \frac{v}{\omega} cos(\theta_t + \omega\delta t) \\ \theta_t + \omega\delta t \end{bmatrix} = \begin{bmatrix} x_t \\ y_t \\ \theta_t \end{bmatrix} + \begin{bmatrix} -\frac{v}{\omega} sin(\theta_t) + \frac{v}{\omega} sin(\theta_t + \omega\delta t) \\ \frac{v}{\omega} cos(\theta_t) - \frac{v}{\omega} cos(\theta_t + \omega\delta t) \\ \omega\delta t \end{bmatrix}$$

- In reality, robot motion is subject to noise, to model such noise, which is formed a zero-centered random variable with finite variance, we can do the following approach

$$\begin{pmatrix} \hat{v}_t \\ \hat{\omega}_t \end{pmatrix} = \begin{pmatrix} v_t \\ \omega_t \end{pmatrix} + \begin{pmatrix} \varepsilon_{\alpha_1 v_t^2 + \alpha_2 \omega_t^2} \\ \varepsilon_{\alpha_3 v_t^2 + \alpha_4 \omega_t^2} \end{pmatrix} = \begin{pmatrix} v_t \\ \omega_t \end{pmatrix} + N(0, M_t) \qquad (3)$$

where $M_t = \begin{pmatrix} \varepsilon_{\alpha_1 v_t^2 + \alpha_2 \omega_t^2} & 0 \\ 0 & \varepsilon_{\alpha_3 v_t^2 + \alpha_4 \omega_t^2} \end{pmatrix}$

- Real motion model

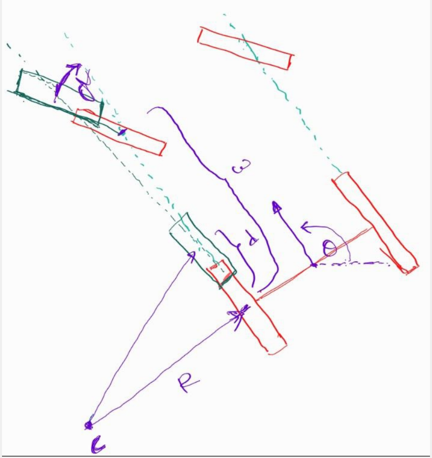
$$\underbrace{\begin{pmatrix} x_{t+1} \\ y_{t+1} \\ \theta_{t+1} \end{pmatrix}}_{\mathbf{x}_{t+1}} = \underbrace{\begin{pmatrix} x_t \\ y_t \\ \theta_t \end{pmatrix} + \begin{pmatrix} -\frac{\hat{v}_t}{\hat{\omega}_t} sin(\theta) + \frac{\hat{v}_t}{\hat{\omega}_t} sin(\theta + \hat{\omega}_t \delta t) \\ \frac{\hat{v}_t}{\hat{\omega}_t} cos(\theta) - \frac{\hat{v}_t}{\hat{\omega}_t} cos(\theta + \hat{\omega}_t \delta t) \\ \hat{\omega}_t \delta t + \hat{\gamma} \delta t \end{pmatrix}}_{f(u_t, \mathbf{x}_t)} \quad (4)$$

, where $\hat{\gamma} \sim \varepsilon_{\alpha_5 v_t^2 + \alpha_6 \omega_t^2}$

- Approximated motion model, i.e., replacing true motion $\hat{v}_t$ and $\hat{\omega}_t$ by executed control (v_t, ω_t)

$$\underbrace{\begin{pmatrix} x_{t+1} \\ y_{t+1} \\ \theta_{t+1} \end{pmatrix}}_{\mathbf{x}_{t+1}} = \underbrace{\begin{pmatrix} x_t \\ y_t \\ \theta_t \end{pmatrix} + \begin{pmatrix} -\frac{v_t}{\omega_t} sin(\theta) + \frac{v_t}{\omega_t} sin(\theta + \omega_t \delta t) \\ \frac{v_t}{\omega_t} cos(\theta) - \frac{v_t}{\omega_t} cos(\theta + \omega_t \delta t) \\ \omega_t \delta t \end{pmatrix}}_{f(u_t, \mathbf{x}_t)} + N(0, Q_t) \quad (5)$$

Consider the following robot model



The front tire is pointing in direction α relative to the wheelbase. Over a short time period the car moves forward and the rear wheel ends up further ahead and slightly turned inward, as depicted with the green dotted tire. Over such a short time frame we can approximate this as a turn around a C .

Prove that

- $\beta = \frac{d}{w} tan(\alpha)$
- $R = \frac{d}{\beta}$, where $d = \delta t v$, if robot robot move with v forward velocity for δt time
- The position of the C is given by
  $Cx = x - Rsin(\theta), \quad Cy = y + Rcos(\theta)$
- If robot move forward for time δt, the new pose is given by
  $$\begin{bmatrix} x_{new} \\ y_{new} \\ \theta_{new} \end{bmatrix} = \begin{bmatrix} x - Rsin(\theta) + Rsin(\theta + \beta) \\ y + Rcos(\theta) - Rcos(\theta + \beta) \\ \theta + \beta \end{bmatrix}$$

Remark: $sin(-\theta) = -sin(\theta), cos(-\theta) = cos(\theta)$

How can we define the state variables, control inputs, and the system model?

How can we define the measurement model?

EKF Localization with known correspondence

State variables, control inputs, and the system model

- state variables: $\mathbf{x} = [x\ , y\ , \theta]$
- control inputs: $\mathbf{u} = [v,\ \alpha]$
- $\bar{\mathbf{x}} = \mathbf{x} + f(\mathbf{x}, \mathbf{u}) + N(0, Q)$, where Q is the white noise

Measurement model

If the installed sensor gives a noisy bearing and range to multiple known landmarks, bearing and range can be estimated in the following way, e.g., let p_x, p_y be a landmark location,

$$r = \sqrt{(p_x - x)^2 + (p_y - y)^2}, \quad \phi = arctan(\frac{p_y - y}{p_x - x}) - \theta \qquad (6)$$
$$\mathbf{z} = h(\mathbf{x}, P) + N(0, R),$$

R is the white noise

- Approximated motion model, i.e., replacing true motion $\hat{v}_t$ and $\hat{\omega}_t$ by executed control (v_t, ω_t)

$$\underbrace{\begin{pmatrix} x_{t+1} \\ y_{t+1} \\ \theta_{t+1} \end{pmatrix}}_{\mathbf{x}_{t+1}} = \underbrace{\begin{pmatrix} x_t \\ y_t \\ \theta_t \end{pmatrix} + \begin{pmatrix} -\frac{v_t}{\omega_t} sin(\theta) + \frac{v_t}{\omega_t} sin(\theta + \omega_t \delta t) \\ \frac{v_t}{\omega_t} cos(\theta) - \frac{v_t}{\omega_t} cos(\theta + \omega_t \delta t) \\ \omega_t \delta t \end{pmatrix}}_{f(u_t, \mathbf{x}_t)} + N(0, Q_t) \tag{7}$$

- Let μ_{t-1}, Σ_{t-1} be the previous optimal state ($\hat{\mathbf{x}}_{t-1}^{-}$) as a Gaussian distribution

- Taylor expansion is used to linearize the function $f(u_t, \mathbf{x}_{t-1})$

$$f(u_t, \mathbf{x}_{t-1}) \approx f(u_t, \hat{\mathbf{x}}^-_{t-1}) + \Phi_t(\mathbf{x}_{t-1} - \hat{\mathbf{x}}^-_{t-1}) \tag{8}$$

$$\Phi_t = \frac{\partial f(u_t, \hat{\mathbf{x}}^-_{t-1})}{\partial \hat{\mathbf{x}}^-_{t-1}} = \begin{pmatrix} \frac{\partial x'}{\partial \hat{\mathbf{x}}^-_{t-1,x}} & \frac{\partial x'}{\partial \hat{\mathbf{x}}^-_{t-1,y}} & \frac{\partial x'}{\partial \hat{\mathbf{x}}^-_{t-1,\theta}} \\ \frac{\partial y'}{\partial \hat{\mathbf{x}}^-_{t-1,x}} & \frac{\partial y'}{\partial \hat{\mathbf{x}}^-_{t-1,y}} & \frac{\partial y'}{\partial \hat{\mathbf{x}}^-_{t-1,\theta}} \\ \frac{\partial \theta'}{\partial \hat{\mathbf{x}}^-_{t-1,x}} & \frac{\partial \theta'}{\partial \hat{\mathbf{x}}^-_{t-1,y}} & \frac{\partial \theta'}{\partial \hat{\mathbf{x}}^-_{t-1,\theta}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & \frac{v_t}{\omega_t}(-cos(\hat{\mathbf{x}}^-_{t-1,\theta}) + cos(\hat{\mathbf{x}}^-_{t-1,\theta} + \omega_t \delta t)) \\ 0 & 1 & \frac{v_t}{\omega_t}(-sin(\hat{\mathbf{x}}^-_{t-1,\theta}) + sin(\hat{\mathbf{x}}^-_{t-1,\theta} + \omega_t \delta t)) \\ 0 & 0 & 1 \end{pmatrix}$$

where $\hat{\mathbf{x}}^-_{t-1} = \hat{\mathbf{x}}^-_{t-1,x}, \hat{\mathbf{x}}^-_{t-1,y}, \hat{\mathbf{x}}^-_{t-1,\theta}$ denotes the mean estimate factored into its individual three values

EKF Localization with known correspondence

- Motion model with respect to control

$$V_t = \frac{\partial f(u_t, \hat{\mathbf{x}}_{t-1}^{-})}{\partial u_t} = \begin{pmatrix} \frac{\partial x'}{\partial v_t} & \frac{\partial x'}{\partial \omega_t} \\ \frac{\partial y'}{\partial v_t} & \frac{\partial y'}{\partial \omega_t} \\ \frac{\partial \theta'}{\partial v_t} & \frac{\partial \theta'}{\partial \omega_t} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{-sin(\theta)+sin(\theta+\omega_t\delta t)}{\omega_t} & \frac{v_t(sin(\theta)-sin(\theta+\omega_t\delta t))}{\omega_t^2} + \frac{v_t(cos(\theta+\omega_t\delta t)\delta t}{\omega_t} \\ \frac{cos(\theta)-cos(\theta+\omega_t\delta t)}{\omega_t} & -\frac{v_t(cos(\theta)-cos(\theta+\omega_t\delta t))}{\omega_t^2} + \frac{v_t(sin(\theta+\omega_t\delta t)\delta t}{\omega_t} \\ 0 & \delta t \end{pmatrix} \tag{9}$$

Let's calculate using sympy
`https://colab.research.google.com/drive/1Zd3ymJ0Cq83X_G1eTQXpJxPFiLBtS8Bh?usp=sharing`

EKF Localization with known correspondence

- Correction step: sensor reading

$$\underbrace{\begin{bmatrix} r_t^i \\ \theta_t^i \end{bmatrix}}_{z_t^i} = \underbrace{\begin{pmatrix} \sqrt{(m_{j,x} - x)^2 + (m_{j,y} - y)^2} \\ atan2(m_{j,y} - y, m_{j,x} - x) - \theta \end{pmatrix}}_{h(x_{t,j,m})} + N(0, R) \tag{10}$$

  , where $m_{j,x}, m_{j,y}$ denotes the coordinates of jth landmark detection at time t, $R = \begin{bmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_r^2 \end{bmatrix}$, and $\mathbf{x}_{t,x}^- = x, \mathbf{x}_{t,y}^- = y$

- The Taylor approximation of the measurement model
  $h(x_t, j, m) \approx h(\hat{\mathbf{x}}_t^-, j, m) + H_t^i(x_t - \hat{\mathbf{x}}_t^-)$

$$
\begin{aligned}
H_t^i = \frac{\partial h(\hat{\mathbf{x}}_t^-, j, m)}{\partial \hat{\mathbf{x}}_t^-} &= \begin{pmatrix} \frac{\partial r_t^i}{\partial \hat{\mathbf{x}}_{t,x}^-} & \frac{\partial r_t^i}{\partial \hat{\mathbf{x}}_{t,y}^-} & \frac{\partial r_t^i}{\partial \hat{\mathbf{x}}_{t,\theta}^-} \\ \frac{\partial \Phi_t^i}{\partial \hat{\mathbf{x}}_{t,x}^-} & \frac{\partial \Phi_t^i}{\partial \hat{\mathbf{x}}_{t,y}^-} & \frac{\partial \Phi_t^i}{\partial \hat{\mathbf{x}}_{t,\theta}^-} \end{pmatrix} \\
&= \begin{pmatrix} -\frac{m_{j,x} - \hat{\mathbf{x}}_{t,x}^-}{\sqrt{q}} & -\frac{m_{j,y} - \hat{\mathbf{x}}_{t,y}^-}{\sqrt{q}} & 0 \\ \frac{m_{j,y} - \hat{\mathbf{x}}_{t,y}^-}{q} & -\frac{m_{j,x} - \hat{\mathbf{x}}_{t,x}^-}{q} & -1 \end{pmatrix}
\end{aligned} \tag{11}
$$

where $q = (m_{j,x} - \hat{\mathbf{x}}_{t,x}^-)^2 + (m_{j,y} - \hat{\mathbf{x}}_{t,y}^-)^2$

We can formulate the location problem with EKF with known correspondence

EKF

$$\boxed{\Phi_t = \left.\frac{\partial f(\mathbf{x}_t, t)}{\partial \mathbf{x}}\right|_{\mathbf{x}_t}}$$

$$\boxed{\hat{\mathbf{x}}_k^- = f(\mathbf{x}_t, \mathbf{t})}$$

$$\mathbf{P}_t^- = \Phi_t \mathbf{P}_t \Phi_t^\mathsf{T} + \mathbf{Q}_t$$

$$\boxed{\mathbf{H} = \left.\frac{\partial h(\hat{\mathbf{x}}_t^-)}{\partial \hat{\mathbf{x}}}\right|_{\hat{\mathbf{x}}_t^-}}$$

$$\mathbf{y} = \mathbf{z}_t - \boxed{h(\hat{x}_t^-)}$$

$$\mathbf{K_t} = \mathbf{P_t^-} \mathbf{H_t}^\mathsf{T} (\mathbf{H_t} \mathbf{P_t^-} \mathbf{H_t}^\mathsf{T} + \mathbf{R}_t)^{-1}$$

$$\hat{\mathbf{x}}_k = \hat{\mathbf{x}}_t^- + \mathbf{K_t} \mathbf{y}$$

$$\mathbf{P}_t = (\mathbf{I} - \mathbf{K_t} \mathbf{H_t}) \mathbf{P_t^-}$$

PF Localization with known correspondence

- Monte Carlo localization (MCL), also known as particle filter localization
- Whenever the robot moves, it shifts the particles to predict its new state after the movement. Whenever the robot senses something, the particles are resampled. Ultimately, the particles should converge towards the actual position of the robot

PF Localization with known correspondence

- Initialize a set of N particles x_k^i random or some prior distribution $p(x_0)$
- Prediction
  - Apply control input u_{k-1} on the state of each particle $\hat{x}_{k-1|k-1}^i$, to which add random noise
  - The obtained predicted a set of particles $\hat{x}_{k|k-1}^i$
- Correction
  - Estimate the measurement for each particles $\hat{x}_{k|k-1}^i$
  - Evaluate the particle importance: difference between obtained measurement z_k and estimated particle measurements $\hat{z}_k^i$, i.e., $innov_k^i = z_k - \bar{z}_k^i$ (innovation or measurement residual)
  - Importance sampling: way to select importance particles $w_k^i = det(2\pi R)^{-1/2} e^{1/2 (innov_k^i)^\top R^{-1} (innov_k^i)}$
  - Estimate $\hat{x}_{k|k}^i$ as the average value of the all the particles

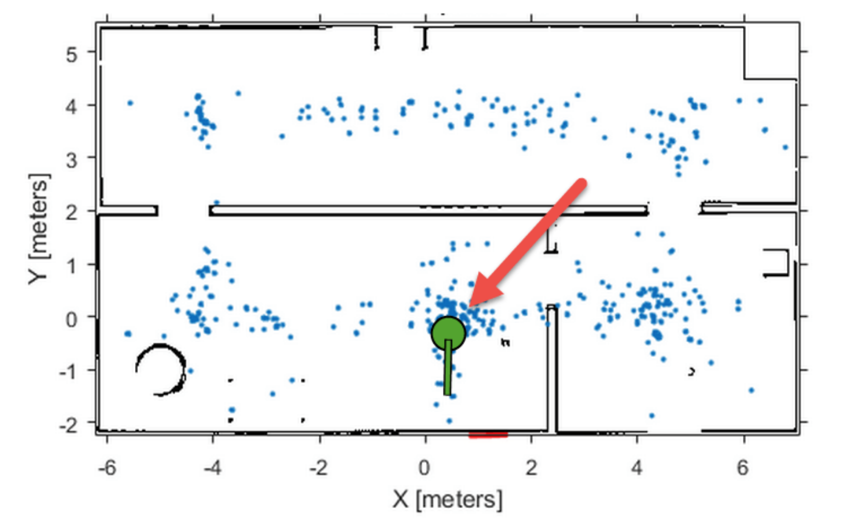
- Sampling motion model

Algorithm 1: Sample motion model velocity

Input: $\mathbf{u}_k, \mathbf{x}_k$
Result: $\mathbf{x}_{k+1} = (x', y', \theta')^\top$
$v = N(\mathbf{u}_k^v, \alpha_v)$
$\omega = N(\mathbf{u}_k^\omega, \alpha_\omega)$
$x' = \mathbf{x}_k^x + \delta_k \cdot v \cdot cos(x_k^\theta)$
$y' = \mathbf{x}_k^y + \delta_k \cdot v \cdot sin(x_k^\theta)$
$\theta' = \mathbf{x}_k^\theta + \delta_k \cdot \omega$

Algorithm requires as input the current robot pose $\mathbf{x}_k$, and the desired control $\mathbf{u}_k$ that is expressed as normal distributions separately for velocity and angular velocity with α_v, α_ω variances, respectively.

PF Localization with known correspondence



https://nl.mathworks.com/help/nav/ug/monte-carlo-localization-algorithm.html

- Measurement update

Algorithm 2: Measurement update

Input: $f_k^i, \mathbf{x}_k, m$

Result: w

$\hat{r} = \sqrt{(m_{j,x} - \mathbf{x}_k^x)^2 + (m_{j,y} - \mathbf{x}_k^y)^2}$

$\hat{\Phi} = atan2(m_{j,y} - \mathbf{x}_k^y, m_{j,x} - \mathbf{x}_k^x)$

$w = prob(r_k^i - \hat{r}, \sigma_r) \cdot prob(\Phi_k^i - \hat{\Phi}, \sigma_\Phi)$

Algorithm requires as input an observed feature $f_k^i = (r_k^i, \Phi_k^i)$, current robot pose $\mathbf{x}_k$, and the map m

- Monte Carlo Localization based on particle filter

Algorithm 3: Monte Carlo Localization based on particle filter

Input: $\mathbf{x}_{k-1}, \mathbf{u}_k, m, z_k$
Result: $\mathbf{x}_k = (x', y', \theta')^\top$
for *m=1 to to M do* **do**
  for *p=1 to to N do* **do**
    $\mathbf{x}_k^{[p]} = sample\_motion\_model(\mathbf{u}_k, \mathbf{x}_{k-1}^{[m]})$
    $\mathbf{w}_k^{[p]} = measurement\_model(z_t, \mathbf{x}_k^{[m]}, m)$
  end
  $importance\_sampling(\mathbf{w}_k)$ Resample according to importance weights
  $\mathbf{w}_k = \frac{\mathbf{w}_k}{\Sigma_{p=0}^{N}\mathbf{w}_k^{[p]}}$
  $\mathbf{x}_k = \frac{1}{N}\Sigma_{p=0}^{N}\mathbf{x}_k^{[p]}$
end

Algorithm requires as input an observed feature z_k, robot pose $\mathbf{x}_{k-1}$, the map m. The pth particle is denoted by $\mathbf{x}_k^{[p]}$ and corresponding weight, denoted $\mathbf{w}_k^{[p]}$

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