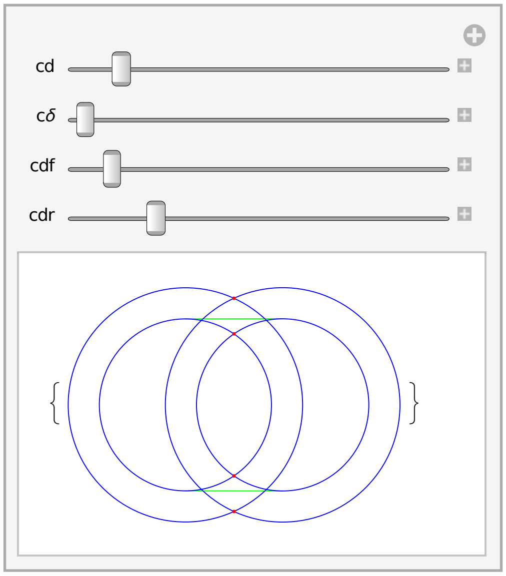
```
(*Author: HKUST,Robotics Institute, Wenchao DING, William Wu*)
(* Key idea of solving the max-min problem:
   First,
we solve the inner problem by deriving the closed-form solution of Δr(dr)
  (dr in the script, i.e., the inflation we need)
  such that the two inflated balls
   exactly cover the convex hull of the two original balls.
   This step can be done by expressing
  the geometry relationship (we can use the $Solve$
      function in mathematica).

   Afterwards, we solve the outer problem using the aforementioned
   closed- form expression.
   It then becomes a pure maximization problem which can
   be solved using $Maximize$ function in mathematica.
   Alternatively, we can check the second-order derivative
   and first-order derivative to verify that the Δr is
   monotonically decreasing with respect to δ(i.e., difference
    between the radius of the two balls, corresponds to α in the paper)
  for given dmax(d in the script) and df.
 *)

(* A visualization tool for problem description*)
Manipulate[
 r1 = d/2 - δ + df;
 r2 = d/2 + δ + df;
 c0 = Circle[{0, 0}, r1];
 c1 = Circle[{d, 0}, r2];
 rc0 = Circle[{0, 0}, r1 + dr];
 rc1 = Circle[{d, 0}, r2 + dr];
 θ = ArcSin[(r2 - r1)/d];
 p0 = {-r1 Sin[θ], r1 Cos[θ]};
 p1 = {d - r2 Sin[θ], r2 Cos[θ]};
 p2 = {-r1 Sin[θ], -r1 Cos[θ]};
 p3 = {d - r2 Sin[θ], -r2 Cos[θ]};
 {ans = Solve[{x, y} ∈ c1 && {x, y} ∈ c0, {x, y}];
  ans2 = Solve[{x, y} ∈ rc1 && {x, y} ∈ rc0, {x, y}];
  (*Solve[{x,y}∈rc1&&{x,y}∈rc0&&({x,y}∈Line[{p0,p1}]||{x,y}∈Line[{p2,p3}]),dr];*)
  Graphics[{{Green, Line[{p0, p1}], Line[{p2, p3}]}, {Blue, c0, c1, rc0, rc1},
     {Red, PointSize[0.01], Point[{x, y}] /. ans, Point[{x, y}] /. ans2}}] /.
    {d → cd, δ → cδ, df → cdf, dr → cdr}},
 {cd, 1, 10}, {cδ, 0.001, cd/2}, {cdf, 0.00, 10}, {cdr, 0.01, 3}
]
```




```
(* Main program starts here.
  It may need a long time to output results. Just wait for some time.
  pls run this block first if any of
 the block following appears to be undefined. *)

r1 = d/2 - δ + df;
r2 = d/2 + δ + df;
c0 = Circle[{0, 0}, r1];
c1 = Circle[{d, 0}, r2];
rc0 = Circle[{0, 0}, r1 + dr];
rc1 = Circle[{d, 0}, r2 + dr];
θ = ArcSin[(r2 - r1)/d];
p0 = {-r1 Sin[θ], r1 Cos[θ]};
p1 = {d - r2 Sin[θ], r2 Cos[θ]};
p2 = {-r1 Sin[θ], -r1 Cos[θ]};
p3 = {d - r2 Sin[θ], -r2 Cos[θ]};
ans = Solve[d > 0 && δ > 0 && df > 0 && {x, y} ∈ c1 && {x, y} ∈ c0, {x, y}];
ans2 = Solve[d > 0 && δ > 0 && df > 0 && {x, y} ∈ rc1 && {x, y} ∈ rc0, {x, y}];
(* The following function is to set up the geometry relationship. *)
inner = Solve[d > 0 && δ > 0 && df > 0 && {x, y} ∈ rc1 && {x, y} ∈ rc0 &&
   ({x, y} ∈ Line[{p0, p1}] || {x, y} ∈ Line[{p2, p3}]), {x, y, dr}]

(* Note that the closed-form solution of Δr(dr) contains four cases
   after solving the equation. Two of them can be grouped together depending
   on the valid interval of delta. And the two groups are actually symmetric
   and identical. The resulting solution of Δr is
   continue (differentiable) everywhere as verified later. *)

(*You can either run inner =
 Solve or use the results below directly for convenience.*)

    inner = {{x →
```

$$\text{ConditionalExpression}\Big[\frac{d^4 - 8\,d^2\,\delta^2 + 8\,d\,\delta^3 + 16\,df\,\delta^3}{2\,d\,(d^2 - 8\,\delta^2)} -$$
$$\sqrt{2}\,\sqrt{\Big(\Big(d^6\,\delta^2 + 2\,d^5\,df\,\delta^2 + 2\,d^4\,df^2\,\delta^2 - 12\,d^4\,\delta^4 -}$$
$$16\,d^3\,df\,\delta^4 - 16\,d^2\,df^2\,\delta^4 + 48\,d^2\,\delta^6 + 32\,d\,df\,\delta^6 +$$
$$32\,df^2\,\delta^6 - 64\,\delta^8\Big) \Big/ \Big(d^2\,(d^2 - 8\,\delta^2)^2\Big)\Big),$$
$$d > 2\,\sqrt{2}\,\sqrt{\delta^2}\ \&\&\ df > 0\ \&\&\ \delta > 0\Big],\ y \to$$
$$\text{ConditionalExpression}\Big[$$
$$-\frac{1}{4}\sqrt{\Big(\frac{1}{\delta^2}\Big(d^4 - 8\,d^2\,\delta^2 + 16\,\delta^4 - 4\,d^3}$$
$$\Big(\frac{d^4 - 8\,d^2\,\delta^2 + 8\,d\,\delta^3 + 16\,df\,\delta^3}{2\,d\,(d^2 - 8\,\delta^2)} - \sqrt{2}\,\sqrt{\Big(\Big(d^6\,\delta^2 +}$$
$$2\,d^5\,df\,\delta^2 + 2\,d^4\,df^2\,\delta^2 - 12\,d^4\,\delta^4 -$$
$$16\,d^3\,df\,\delta^4 - 16\,d^2\,df^2\,\delta^4 + 48\,d^2\,\delta^6 +$$
$$32\,d\,df\,\delta^6 + 32\,df^2\,\delta^6 - 64\,\delta^8\Big) \Big/$$
$$\Big(d^2\,(d^2 - 8\,\delta^2)^2\Big)\Big)\Big) + 16\,d\,\delta^2$$
$$\Big(\frac{d^4 - 8\,d^2\,\delta^2 + 8\,d\,\delta^3 + 16\,df\,\delta^3}{2\,d\,(d^2 - 8\,\delta^2)} - \sqrt{2}$$
$$\sqrt{\Big(\Big(d^6\,\delta^2 + 2\,d^5\,df\,\delta^2 + 2\,d^4\,df^2\,\delta^2 -}$$
$$12\,d^4\,\delta^4 - 16\,d^3\,df\,\delta^4 - 16\,d^2\,df^2\,\delta^4 +$$
$$48\,d^2\,\delta^6 + 32\,d\,df\,\delta^6 + 32\,df^2\,\delta^6 -$$
$$64\,\delta^8\Big) \Big/ \Big(d^2\,(d^2 - 8\,\delta^2)^2\Big)\Big)\Big) +$$
$$4\,d^2\,\Big(\frac{d^4 - 8\,d^2\,\delta^2 + 8\,d\,\delta^3 + 16\,df\,\delta^3}{2\,d\,(d^2 - 8\,\delta^2)} -$$
$$\sqrt{2}\,\sqrt{\Big(\Big(d^6\,\delta^2 + 2\,d^5\,df\,\delta^2 + 2\,d^4\,df^2\,\delta^2 -}$$
$$12\,d^4\,\delta^4 - 16\,d^3\,df\,\delta^4 - 16\,d^2\,df^2\,\delta^4 +$$
$$48\,d^2\,\delta^6 + 32\,d\,df\,\delta^6 + 32\,df^2\,\delta^6 -$$
$$64\,\delta^8\Big) \Big/ \Big(d^2\,(d^2 - 8\,\delta^2)^2\Big)\Big)\Big)^2 -$$
$$16\,\delta^2\,\Big(\frac{d^4 - 8\,d^2\,\delta^2 + 8\,d\,\delta^3 + 16\,df\,\delta^3}{2\,d\,(d^2 - 8\,\delta^2)} -$$

$$\sqrt{2}\,\sqrt{\left(\left(\left(d^6\,\delta^2 + 2\,d^5\,df\,\delta^2 + 2\,d^4\,df^2\,\delta^2 - 12\,d^4\,\delta^4 - 16\,d^3\,df\,\delta^4 - 16\,d^2\,df^2\,\delta^4 + 48\,d^2\,\delta^6 + 32\,d\,df\,\delta^6 + 32\,df^2\,\delta^6 - 64\,\delta^8\right)\Big/\left(d^2\,(d^2 - 8\,\delta^2)^2\right)\right)\right)^2}\Bigg)\Bigg),$$

$$d > 2\,\sqrt{2}\,\sqrt{\delta^2}\ \&\&\ df > 0\ \&\&\ \delta > 0\Big],\ dr \to$$

ConditionalExpression[

$$\frac{1}{2}\Bigg((-d - 2\,df + 2\,\delta) + \sqrt{\Bigg(\Bigg(\frac{d^4 - 8\,d^2\,\delta^2 + 8\,d\,\delta^3 + 16\,df\,\delta^3}{2\,d\,(d^2 - 8\,\delta^2)} - \sqrt{2}\,\sqrt{\left(\left(d^6\,\delta^2 + 2\,d^5\,df\,\delta^2 + 2\,d^4\,df^2\,\delta^2 - 12\,d^4\,\delta^4 - 16\,d^3\,df\,\delta^4 - 16\,d^2\,df^2\,\delta^4 + 48\,d^2\,\delta^6 + 32\,d\,df\,\delta^6 + 32\,df^2\,\delta^6 - 64\,\delta^8\right)\Big/\left(d^2\,(d^2 - 8\,\delta^2)^2\right)\right)}\Bigg)^2 + \frac{1}{16\,\delta^2}\Bigg(d^4 - 8\,d^2\,\delta^2 +}$$

$$16\,\delta^4 - 4\,d^3\left(\frac{d^4 - 8\,d^2\,\delta^2 + 8\,d\,\delta^3 + 16\,df\,\delta^3}{2\,d\,(d^2 - 8\,\delta^2)} - \sqrt{2}\,\sqrt{\left(\left(d^6\,\delta^2 + 2\,d^5\,df\,\delta^2 + 2\,d^4\,df^2\,\delta^2 - 12\,d^4\,\delta^4 - 16\,d^3\,df\,\delta^4 - 16\,d^2\,df^2\,\delta^4 + 48\,d^2\,\delta^6 + 32\,d\,df\,\delta^6 + 32\,df^2\,\delta^6 - 64\,\delta^8\right)\Big/\left(d^2\,(d^2 - 8\,\delta^2)^2\right)\right)}\right) +$$

$$16\,d\,\delta^2\left(\frac{d^4 - 8\,d^2\,\delta^2 + 8\,d\,\delta^3 + 16\,df\,\delta^3}{2\,d\,(d^2 - 8\,\delta^2)} - \sqrt{2}\,\sqrt{\left(\left(d^6\,\delta^2 + 2\,d^5\,df\,\delta^2 + 2\,d^4\,df^2\,\delta^2 - 12\,d^4\,\delta^4 - 16\,d^3\,df\,\delta^4 - 16\,d^2\,df^2\,\delta^4 + 48\,d^2\,\delta^6 + 32\,d\,df\,\delta^6 + 32\,df^2\,\delta^6 - 64\,\delta^8\right)\Big/\left(d^2\,(d^2 - 8\,\delta^2)^2\right)\right)}\right) +$$

$$4\,d^2\left(\frac{d^4-8\,d^2\,\delta^2+8\,d\,\delta^3+16\,df\,\delta^3}{2\,d\,(d^2-8\,\delta^2)}-\sqrt{2}\sqrt{\left(\left(d^6\,\delta^2+2\,d^5\,df\,\delta^2+2\,d^4\,df^2\,\delta^2-12\,d^4\,\delta^4-16\,d^3\,df\,\delta^4-16\,d^2\,df^2\,\delta^4+48\,d^2\,\delta^6+32\,d\,df\,\delta^6+32\,df^2\,\delta^6-64\,\delta^8\right)/\left(d^2\,(d^2-8\,\delta^2)^2\right)\right)}\right)^2-$$

$$16\,\delta^2\left(\frac{d^4-8\,d^2\,\delta^2+8\,d\,\delta^3+16\,df\,\delta^3}{2\,d\,(d^2-8\,\delta^2)}-\sqrt{2}\sqrt{\left(\left(d^6\,\delta^2+2\,d^5\,df\,\delta^2+2\,d^4\,df^2\,\delta^2-12\,d^4\,\delta^4-16\,d^3\,df\,\delta^4-16\,d^2\,df^2\,\delta^4+48\,d^2\,\delta^6+32\,d\,df\,\delta^6+32\,df^2\,\delta^6-64\,\delta^8\right)/\left(d^2\,(d^2-8\,\delta^2)^2\right)\right)}\right)^2\Bigg)\Bigg),$$

$$d>2\sqrt{2}\sqrt{\delta^2}\ \&\&\ df>0\ \&\&\ \delta>0\Big]\Big\},\ \Big\{x\to$$

$$\text{ConditionalExpression}\Big[\frac{d^4-8\,d^2\,\delta^2+8\,d\,\delta^3+16\,df\,\delta^3}{2\,d\,(d^2-8\,\delta^2)}-\sqrt{2}\sqrt{\left(\left(d^6\,\delta^2+2\,d^5\,df\,\delta^2+2\,d^4\,df^2\,\delta^2-12\,d^4\,\delta^4-16\,d^3\,df\,\delta^4-16\,d^2\,df^2\,\delta^4+48\,d^2\,\delta^6+32\,d\,df\,\delta^6+32\,df^2\,\delta^6-64\,\delta^8\right)/\left(d^2\,(d^2-8\,\delta^2)^2\right)\right)},$$

$$d>2\sqrt{2}\sqrt{\delta^2}\ \&\&\ df>0\ \&\&\ \delta>0\Big],\ y\to$$

$$\text{ConditionalExpression}\Big[\frac{1}{4}\sqrt{\Bigg(\frac{1}{\delta^2}\Bigg(d^4-8\,d^2\,\delta^2+16\,\delta^4-4\,d^3\left(\frac{d^4-8\,d^2\,\delta^2+8\,d\,\delta^3+16\,df\,\delta^3}{2\,d\,(d^2-8\,\delta^2)}-\sqrt{2}\sqrt{\left(\left(d^6\,\delta^2+2\,d^5\,df\,\delta^2+2\,d^4\,df^2\,\delta^2-\right.\right.}\right.}$$

$$12\,d^4\,\delta^4 - 16\,d^3\,df\,\delta^4 - 16\,d^2\,df^2\,\delta^4 +$$
$$48\,d^2\,\delta^6 + 32\,d\,df\,\delta^6 + 32\,df^2\,\delta^6 -$$
$$64\,\delta^8\Big) \Big/ \Big(d^2\,(d^2 - 8\,\delta^2)^2\Big)\Big)\Bigg) +$$

$$16\,d\,\delta^2\,\left(\frac{d^4 - 8\,d^2\,\delta^2 + 8\,d\,\delta^3 + 16\,df\,\delta^3}{2\,d\,(d^2 - 8\,\delta^2)} -\right.$$
$$\sqrt{2}\,\sqrt{}\Big(\Big(d^6\,\delta^2 + 2\,d^5\,df\,\delta^2 + 2\,d^4\,df^2\,\delta^2 -$$
$$12\,d^4\,\delta^4 - 16\,d^3\,df\,\delta^4 - 16\,d^2\,df^2\,\delta^4 +$$
$$48\,d^2\,\delta^6 + 32\,d\,df\,\delta^6 + 32\,df^2\,\delta^6 -$$
$$64\,\delta^8\Big) \Big/ \Big(d^2\,(d^2 - 8\,\delta^2)^2\Big)\Big)\Bigg) +$$

$$4\,d^2\,\left(\frac{d^4 - 8\,d^2\,\delta^2 + 8\,d\,\delta^3 + 16\,df\,\delta^3}{2\,d\,(d^2 - 8\,\delta^2)} -\right.$$
$$\sqrt{2}\,\sqrt{}\Big(\Big(d^6\,\delta^2 + 2\,d^5\,df\,\delta^2 + 2\,d^4\,df^2\,\delta^2 -$$
$$12\,d^4\,\delta^4 - 16\,d^3\,df\,\delta^4 - 16\,d^2\,df^2\,\delta^4 +$$
$$48\,d^2\,\delta^6 + 32\,d\,df\,\delta^6 + 32\,df^2\,\delta^6 -$$
$$64\,\delta^8\Big) \Big/ \Big(d^2\,(d^2 - 8\,\delta^2)^2\Big)\Big)\Bigg)^2 -$$

$$16\,\delta^2\,\left(\frac{d^4 - 8\,d^2\,\delta^2 + 8\,d\,\delta^3 + 16\,df\,\delta^3}{2\,d\,(d^2 - 8\,\delta^2)} -\right.$$
$$\sqrt{2}\,\sqrt{}\Big(\Big(d^6\,\delta^2 + 2\,d^5\,df\,\delta^2 + 2\,d^4\,df^2\,\delta^2 -$$
$$12\,d^4\,\delta^4 - 16\,d^3\,df\,\delta^4 - 16\,d^2\,df^2\,\delta^4 +$$
$$48\,d^2\,\delta^6 + 32\,d\,df\,\delta^6 + 32\,df^2\,\delta^6 -$$
$$64\,\delta^8\Big) \Big/ \Big(d^2\,(d^2 - 8\,\delta^2)^2\Big)\Big)\Bigg)^2\Bigg)\Bigg),$$

$$d > 2\,\sqrt{2}\,\sqrt{\delta^2}\ \&\&\ df > 0\ \&\&\ \delta > 0\Big],\ dr \to$$

ConditionalExpression$\Big[$

$$\frac{1}{2}$$
$$(-d - 2\,df + 2\,\delta) +$$
$$\sqrt{}\left(\left(\frac{d^4 - 8\,d^2\,\delta^2 + 8\,d\,\delta^3 + 16\,df\,\delta^3}{2\,d\,(d^2 - 8\,\delta^2)} -\right.\right.$$

$$\sqrt{2}\ \sqrt{\Big(\Big(d^6\,\delta^2+2\,d^5\,df\,\delta^2+2\,d^4\,df^2\,\delta^2-}$$

$$12\,d^4\,\delta^4-16\,d^3\,df\,\delta^4-16\,d^2\,df^2\,\delta^4+$$

$$48\,d^2\,\delta^6+32\,d\,df\,\delta^6+32\,df^2\,\delta^6-$$

$$64\,\delta^8\Big)\Big/\Big(d^2\,(d^2-8\,\delta^2)^2\Big)\Big)\Big)^2+$$

$$\frac{1}{16\,\delta^2}\Bigg(d^4-8\,d^2\,\delta^2+16\,\delta^4-4\,d^3$$

$$\Bigg(\frac{d^4-8\,d^2\,\delta^2+8\,d\,\delta^3+16\,df\,\delta^3}{2\,d\,(d^2-8\,\delta^2)}-$$

$$\sqrt{2}\ \sqrt{\Big(\Big(d^6\,\delta^2+2\,d^5\,df\,\delta^2+2\,d^4\,df^2\,\delta^2-}$$

$$12\,d^4\,\delta^4-16\,d^3\,df\,\delta^4-16\,d^2\,df^2\,\delta^4+$$

$$48\,d^2\,\delta^6+32\,d\,df\,\delta^6+32\,df^2\,\delta^6-$$

$$64\,\delta^8\Big)\Big/\Big(d^2\,(d^2-8\,\delta^2)^2\Big)\Big)\Bigg)+$$

$$16\,d\,\delta^2\Bigg(\frac{d^4-8\,d^2\,\delta^2+8\,d\,\delta^3+16\,df\,\delta^3}{2\,d\,(d^2-8\,\delta^2)}-$$

$$\sqrt{2}\ \sqrt{\Big(\Big(d^6\,\delta^2+2\,d^5\,df\,\delta^2+2\,d^4\,df^2\,\delta^2-}$$

$$12\,d^4\,\delta^4-16\,d^3\,df\,\delta^4-16\,d^2\,df^2\,\delta^4+$$

$$48\,d^2\,\delta^6+32\,d\,df\,\delta^6+32\,df^2\,\delta^6-$$

$$64\,\delta^8\Big)\Big/\Big(d^2\,(d^2-8\,\delta^2)^2\Big)\Big)\Bigg)+$$

$$4\,d^2\Bigg(\frac{d^4-8\,d^2\,\delta^2+8\,d\,\delta^3+16\,df\,\delta^3}{2\,d\,(d^2-8\,\delta^2)}-\sqrt{2}$$

$$\sqrt{\Big(\Big(d^6\,\delta^2+2\,d^5\,df\,\delta^2+2\,d^4\,df^2\,\delta^2-12}$$

$$d^4\,\delta^4-16\,d^3\,df\,\delta^4-16\,d^2\,df^2\,\delta^4+$$

$$48\,d^2\,\delta^6+32\,d\,df\,\delta^6+32\,df^2\,\delta^6-$$

$$64\,\delta^8\Big)\Big/\Big(d^2\,(d^2-8\,\delta^2)^2\Big)\Big)\Bigg)^2-$$

$$16\,\delta^2\Bigg(\frac{d^4-8\,d^2\,\delta^2+8\,d\,\delta^3+16\,df\,\delta^3}{2\,d\,(d^2-8\,\delta^2)}-\sqrt{2}$$

$$\sqrt{\Big(\Big(d^6\,\delta^2+2\,d^5\,df\,\delta^2+2\,d^4\,df^2\,\delta^2-12}$$

$$d^4\,\delta^4-16\,d^3\,df\,\delta^4-16\,d^2\,df^2\,\delta^4+$$

$$48\,d^2\,\delta^6+32\,d\,df\,\delta^6+32\,df^2\,\delta^6-$$

$$64\,\delta^8\Big)\Big/\Big(d^2\,(d^2-8\,\delta^2)^2\Big)\Big)\Big)^2\Big)\Big),$$

$$d>2\sqrt{2}\,\sqrt{\delta^2}\ \&\&\ df>0\ \&\&\ \delta>0\Big]\Big\},\ \Big\{x\to$$

$$\text{ConditionalExpression}\Big[$$

$$\frac{d^4-8\,d^2\,\delta^2+8\,d\,\delta^3+16\,df\,\delta^3}{2\,d\,(d^2-8\,\delta^2)}+$$

$$\sqrt{2}$$

$$\sqrt{}\Big(\Big(d^6\,\delta^2+2\,d^5\,df\,\delta^2+2\,d^4\,df^2\,\delta^2-12\,d^4\,\delta^4-$$

$$16\,d^3\,df\,\delta^4-16\,d^2\,df^2\,\delta^4+48\,d^2\,\delta^6+32\,d\,df\,\delta^6+$$

$$32\,df^2\,\delta^6-64\,\delta^8\Big)\Big/\Big(d^2\,(d^2-8\,\delta^2)^2\Big)\Big),$$

$$2\,\delta<d<2\sqrt{2}\,\sqrt{\delta^2}\ \&\&\ df>0\ \&\&$$

$$\delta>$$

$$0\Big],$$

$$y\to\text{ConditionalExpression}\Big[$$

$$-\frac{1}{4}$$

$$\sqrt{}\Big(\frac{1}{\delta^2}\Big(d^4-8\,d^2\,\delta^2+16\,\delta^4-$$

$$4\,d^3\Big(\frac{d^4-8\,d^2\,\delta^2+8\,d\,\delta^3+16\,df\,\delta^3}{2\,d\,(d^2-8\,\delta^2)}+\sqrt{2}$$

$$\sqrt{}\Big(\Big(d^6\,\delta^2+2\,d^5\,df\,\delta^2+2\,d^4\,df^2\,\delta^2-$$

$$12\,d^4\,\delta^4-16\,d^3\,df\,\delta^4-16\,d^2\,df^2\,\delta^4+$$

$$48\,d^2\,\delta^6+32\,d\,df\,\delta^6+32\,df^2\,\delta^6-$$

$$64\,\delta^8\Big)\Big/\Big(d^2\,(d^2-8\,\delta^2)^2\Big)\Big)\Big)+$$

$$16\,d\,\delta^2\Big(\frac{d^4-8\,d^2\,\delta^2+8\,d\,\delta^3+16\,df\,\delta^3}{2\,d\,(d^2-8\,\delta^2)}+$$

$$\sqrt{2}\,\sqrt{}\Big(\Big(d^6\,\delta^2+2\,d^5\,df\,\delta^2+2\,d^4\,df^2\,\delta^2-$$

$$12\,d^4\,\delta^4-16\,d^3\,df\,\delta^4-16\,d^2\,df^2\,\delta^4+$$

$$48\,d^2\,\delta^6+32\,d\,df\,\delta^6+32\,df^2\,\delta^6-$$

$$64\,\delta^8\Big)\Big/\Big(d^2\,(d^2-8\,\delta^2)^2\Big)\Big)\Big)+$$

$$4\,d^2\left(\frac{d^4-8\,d^2\,\delta^2+8\,d\,\delta^3+16\,df\,\delta^3}{2\,d\,(d^2-8\,\delta^2)}+\sqrt{2}\,\sqrt{\left(\left(d^6\,\delta^2+2\,d^5\,df\,\delta^2+2\,d^4\,df^2\,\delta^2-12\,d^4\,\delta^4-16\,d^3\,df\,\delta^4-16\,d^2\,df^2\,\delta^4+48\,d^2\,\delta^6+32\,d\,df\,\delta^6+32\,df^2\,\delta^6-64\,\delta^8\right)\Big/\left(d^2\,(d^2-8\,\delta^2)^2\right)\right)}\right)^2-$$

$$16\,\delta^2\left(\frac{d^4-8\,d^2\,\delta^2+8\,d\,\delta^3+16\,df\,\delta^3}{2\,d\,(d^2-8\,\delta^2)}+\sqrt{2}\,\sqrt{\left(\left(d^6\,\delta^2+2\,d^5\,df\,\delta^2+2\,d^4\,df^2\,\delta^2-12\,d^4\,\delta^4-16\,d^3\,df\,\delta^4-16\,d^2\,df^2\,\delta^4+48\,d^2\,\delta^6+32\,d\,df\,\delta^6+32\,df^2\,\delta^6-64\,\delta^8\right)\Big/\left(d^2\,(d^2-8\,\delta^2)^2\right)\right)}\right)^2\Bigg)\Bigg),$$

$$2\,\delta<d<2\sqrt{2}\,\sqrt{\delta^2}\ \&\&\ df>0\ \&\&\ \delta>0\Big],\ dr\to$$

ConditionalExpression[

$$\frac{1}{2}\left(-d-2\,df+2\,\delta\right)+\sqrt{\left(\left(\frac{d^4-8\,d^2\,\delta^2+8\,d\,\delta^3+16\,df\,\delta^3}{2\,d\,(d^2-8\,\delta^2)}+\sqrt{2}\,\sqrt{\left(\left(d^6\,\delta^2+2\,d^5\,df\,\delta^2+2\,d^4\,df^2\,\delta^2-12\,d^4\,\delta^4-16\,d^3\,df\,\delta^4-16\,d^2\,df^2\,\delta^4+48\,d^2\,\delta^6+32\,d\,df\,\delta^6+32\,df^2\,\delta^6-64\,\delta^8\right)\Big/\left(d^2\,(d^2-8\,\delta^2)^2\right)\right)}\right)^2+\right.}$$

$$\frac{1}{16\,\delta^2}\left(d^4-8\,d^2\,\delta^2+16\,\delta^4-4\,d^3\left(\frac{d^4-8\,d^2\,\delta^2+8\,d\,\delta^3+16\,df\,\delta^3}{2\,d\,(d^2-8\,\delta^2)}+\sqrt{2}\,\sqrt{\left(\left(d^6\,\delta^2+2\,d^5\,df\,\delta^2+2\,d^4\,df^2\,\delta^2-12\,d^4\,\delta^4-16\,d^3\,df\,\delta^4-16\,d^2\,df^2\,\delta^4+\right.\right.}\right.\right.$$

$48\, d^2\, \delta^6 + 32\, d\, df\, \delta^6 + 32\, df^2\, \delta^6 -$

$64\, \delta^8) / (d^2\, (d^2 - 8\, \delta^2)^2)))) +$

$16\, d\, \delta^2\, \Big(\frac{d^4 - 8\, d^2\, \delta^2 + 8\, d\, \delta^3 + 16\, df\, \delta^3}{2\, d\, (d^2 - 8\, \delta^2)} +$

$\sqrt{2}\, \sqrt{}\, ((d^6\, \delta^2 + 2\, d^5\, df\, \delta^2 + 2\, d^4\, df^2\, \delta^2 -$

$12\, d^4\, \delta^4 - 16\, d^3\, df\, \delta^4 - 16\, d^2\, df^2\, \delta^4 +$

$48\, d^2\, \delta^6 + 32\, d\, df\, \delta^6 + 32\, df^2\, \delta^6 -$

$64\, \delta^8) / (d^2\, (d^2 - 8\, \delta^2)^2)))\Big) +$

$4\, d^2\, \Big(\frac{d^4 - 8\, d^2\, \delta^2 + 8\, d\, \delta^3 + 16\, df\, \delta^3}{2\, d\, (d^2 - 8\, \delta^2)} + \sqrt{2}$

$\sqrt{}\, ((d^6\, \delta^2 + 2\, d^5\, df\, \delta^2 + 2\, d^4\, df^2\, \delta^2 - 12$

$d^4\, \delta^4 - 16\, d^3\, df\, \delta^4 - 16\, d^2\, df^2\, \delta^4 +$

$48\, d^2\, \delta^6 + 32\, d\, df\, \delta^6 + 32\, df^2\, \delta^6 -$

$64\, \delta^8) / (d^2\, (d^2 - 8\, \delta^2)^2)))\Big)^2 -$

$16\, \delta^2\, \Big(\frac{d^4 - 8\, d^2\, \delta^2 + 8\, d\, \delta^3 + 16\, df\, \delta^3}{2\, d\, (d^2 - 8\, \delta^2)} + \sqrt{2}$

$\sqrt{}\, ((d^6\, \delta^2 + 2\, d^5\, df\, \delta^2 + 2\, d^4\, df^2\, \delta^2 - 12$

$d^4\, \delta^4 - 16\, d^3\, df\, \delta^4 - 16\, d^2\, df^2\, \delta^4 +$

$48\, d^2\, \delta^6 + 32\, d\, df\, \delta^6 + 32\, df^2\, \delta^6 -$

$64\, \delta^8) / (d^2\, (d^2 - 8\, \delta^2)^2)))\Big)^2)),$

$2\, \delta < d < 2\, \sqrt{2}\, \sqrt{\delta^2}\ \&\&\ df > 0\ \&\&\ \delta > 0]\},$

$\{x \to \mathtt{ConditionalExpression}[$

$\frac{d^4 - 8\, d^2\, \delta^2 + 8\, d\, \delta^3 + 16\, df\, \delta^3}{2\, d\, (d^2 - 8\, \delta^2)} +$

$\sqrt{2}$

$\sqrt{}\, ((d^6\, \delta^2 + 2\, d^5\, df\, \delta^2 + 2\, d^4\, df^2\, \delta^2 - 12\, d^4\, \delta^4 -$

$16\, d^3\, df\, \delta^4 - 16\, d^2\, df^2\, \delta^4 + 48\, d^2\, \delta^6 + 32\, d\, df\, \delta^6 +$

$32\, df^2\, \delta^6 - 64\, \delta^8) / (d^2\, (d^2 - 8\, \delta^2)^2)),$

$$2\,\delta < d < 2\sqrt{2}\,\sqrt{\delta^2}\ \&\&\ df > 0\ \&\&$$
$$\delta >$$
$$0\Big],$$

$$y \to \text{ConditionalExpression}\Big[$$

$$\frac{1}{4}$$
$$\sqrt{\left(\frac{1}{\delta^2}\left(d^4 - 8\,d^2\,\delta^2 + 16\,\delta^4 - \right.\right.}$$
$$4\,d^3\left(\frac{d^4 - 8\,d^2\,\delta^2 + 8\,d\,\delta^3 + 16\,df\,\delta^3}{2\,d\,(d^2 - 8\,\delta^2)} + \sqrt{2}\right.$$
$$\sqrt{}\left(\left(d^6\,\delta^2 + 2\,d^5\,df\,\delta^2 + 2\,d^4\,df^2\,\delta^2 - \right.\right.$$
$$12\,d^4\,\delta^4 - 16\,d^3\,df\,\delta^4 - 16\,d^2\,df^2\,\delta^4 +$$
$$48\,d^2\,\delta^6 + 32\,d\,df\,\delta^6 + 32\,df^2\,\delta^6 -$$
$$\left.\left.\left.64\,\delta^8\right)\Big/\left(d^2\,(d^2 - 8\,\delta^2)^2\right)\right)\right) +$$
$$16\,d\,\delta^2\left(\frac{d^4 - 8\,d^2\,\delta^2 + 8\,d\,\delta^3 + 16\,df\,\delta^3}{2\,d\,(d^2 - 8\,\delta^2)} + \right.$$
$$\sqrt{2}\,\sqrt{}\left(\left(d^6\,\delta^2 + 2\,d^5\,df\,\delta^2 + 2\,d^4\,df^2\,\delta^2 - \right.\right.$$
$$12\,d^4\,\delta^4 - 16\,d^3\,df\,\delta^4 - 16\,d^2\,df^2\,\delta^4 +$$
$$48\,d^2\,\delta^6 + 32\,d\,df\,\delta^6 + 32\,df^2\,\delta^6 -$$
$$\left.\left.\left.64\,\delta^8\right)\Big/\left(d^2\,(d^2 - 8\,\delta^2)^2\right)\right)\right) +$$
$$4\,d^2\left(\frac{d^4 - 8\,d^2\,\delta^2 + 8\,d\,\delta^3 + 16\,df\,\delta^3}{2\,d\,(d^2 - 8\,\delta^2)} + \right.$$
$$\sqrt{2}\,\sqrt{}\left(\left(d^6\,\delta^2 + 2\,d^5\,df\,\delta^2 + 2\,d^4\,df^2\,\delta^2 - \right.\right.$$
$$12\,d^4\,\delta^4 - 16\,d^3\,df\,\delta^4 - 16\,d^2\,df^2\,\delta^4 +$$
$$48\,d^2\,\delta^6 + 32\,d\,df\,\delta^6 + 32\,df^2\,\delta^6 -$$
$$\left.\left.\left.64\,\delta^8\right)\Big/\left(d^2\,(d^2 - 8\,\delta^2)^2\right)\right)\right)^2 -$$
$$16\,\delta^2\left(\frac{d^4 - 8\,d^2\,\delta^2 + 8\,d\,\delta^3 + 16\,df\,\delta^3}{2\,d\,(d^2 - 8\,\delta^2)} + \right.$$
$$\sqrt{2}\,\sqrt{}\left(\left(d^6\,\delta^2 + 2\,d^5\,df\,\delta^2 + 2\,d^4\,df^2\,\delta^2 - \right.\right.$$
$$12\,d^4\,\delta^4 - 16\,d^3\,df\,\delta^4 - 16\,d^2\,df^2\,\delta^4 +$$

$$48\, d^2\, \delta^6 + 32\, d\, df\, \delta^6 + 32\, df^2\, \delta^6 -$$

$$64\, \delta^8\Big) \Big/ \Big(d^2\, (d^2 - 8\, \delta^2)^2\Big)\Big)\Big)^2\Big)\Big),$$

$$2\, \delta < d < 2\sqrt{2}\, \sqrt{\delta^2}\ \&\&\ df > 0\ \&\&\ \delta > 0\Big],\ dr \to$$

$$\text{ConditionalExpression}\Big[$$

$$\frac{1}{2}$$

$$(-d - 2\, df + 2\, \delta) +$$

$$\sqrt{\Big(\Big(\frac{d^4 - 8\, d^2\, \delta^2 + 8\, d\, \delta^3 + 16\, df\, \delta^3}{2\, d\, (d^2 - 8\, \delta^2)} +}$$

$$\sqrt{2}\ \sqrt{\Big(\Big(d^6\, \delta^2 + 2\, d^5\, df\, \delta^2 + 2\, d^4\, df^2\, \delta^2 -}$$

$$12\, d^4\, \delta^4 - 16\, d^3\, df\, \delta^4 - 16\, d^2\, df^2\, \delta^4 +$$

$$48\, d^2\, \delta^6 + 32\, d\, df\, \delta^6 + 32\, df^2\, \delta^6 -$$

$$64\, \delta^8\Big) \Big/ \Big(d^2\, (d^2 - 8\, \delta^2)^2\Big)\Big)\Big)^2 +$$

$$\frac{1}{16\, \delta^2}\ \Big(d^4 - 8\, d^2\, \delta^2 + 16\, \delta^4 - 4\, d^3$$

$$\Big(\frac{d^4 - 8\, d^2\, \delta^2 + 8\, d\, \delta^3 + 16\, df\, \delta^3}{2\, d\, (d^2 - 8\, \delta^2)} +$$

$$\sqrt{2}\ \sqrt{\Big(\Big(d^6\, \delta^2 + 2\, d^5\, df\, \delta^2 + 2\, d^4\, df^2\, \delta^2 -}$$

$$12\, d^4\, \delta^4 - 16\, d^3\, df\, \delta^4 - 16\, d^2\, df^2\, \delta^4 +$$

$$48\, d^2\, \delta^6 + 32\, d\, df\, \delta^6 + 32\, df^2\, \delta^6 -$$

$$64\, \delta^8\Big) \Big/ \Big(d^2\, (d^2 - 8\, \delta^2)^2\Big)\Big)\Big) +$$

$$16\, d\, \delta^2\ \Big(\frac{d^4 - 8\, d^2\, \delta^2 + 8\, d\, \delta^3 + 16\, df\, \delta^3}{2\, d\, (d^2 - 8\, \delta^2)} +$$

$$\sqrt{2}\ \sqrt{\Big(\Big(d^6\, \delta^2 + 2\, d^5\, df\, \delta^2 + 2\, d^4\, df^2\, \delta^2 -}$$

$$12\, d^4\, \delta^4 - 16\, d^3\, df\, \delta^4 - 16\, d^2\, df^2\, \delta^4 +$$

$$48\, d^2\, \delta^6 + 32\, d\, df\, \delta^6 + 32\, df^2\, \delta^6 -$$

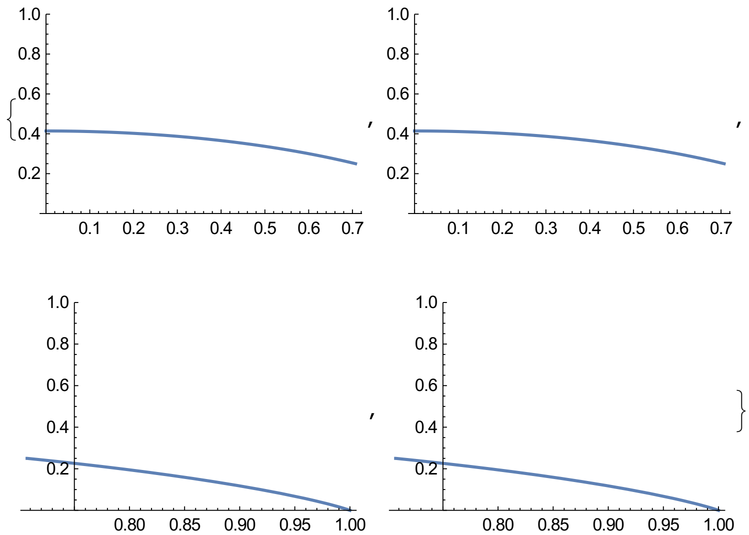
$$64\, \delta^8\Big) \Big/ \Big(d^2\, (d^2 - 8\, \delta^2)^2\Big)\Big)\Big) +$$

$$4\, d^2\ \Big(\frac{d^4 - 8\, d^2\, \delta^2 + 8\, d\, \delta^3 + 16\, df\, \delta^3}{2\, d\, (d^2 - 8\, \delta^2)} + \sqrt{2}$$

```
√ ((d^6 δ^2 + 2 d^5 df δ^2 + 2 d^4 df^2 δ^2 - 12
      d^4 δ^4 - 16 d^3 df δ^4 - 16 d^2 df^2 δ^4 +
     48 d^2 δ^6 + 32 d df δ^6 + 32 df^2 δ^6 -
     64 δ^8) / (d^2 (d^2 - 8 δ^2)^2)))^2 -
16 δ^2 ((d^4 - 8 d^2 δ^2 + 8 d δ^3 + 16 df δ^3) / (2 d (d^2 - 8 δ^2)) + √2
   √ ((d^6 δ^2 + 2 d^5 df δ^2 + 2 d^4 df^2 δ^2 - 12
      d^4 δ^4 - 16 d^3 df δ^4 - 16 d^2 df^2 δ^4 +
     48 d^2 δ^6 + 32 d df δ^6 + 32 df^2 δ^6 -
     64 δ^8) / (d^2 (d^2 - 8 δ^2)^2)))^2)),
2 δ < d < 2 √2 √(δ^2) && df > 0 && δ > 0]}};
```

```
inner[[1]](* the first case of the closed
 form solution of [x,y] position and Δr (dr) *)

inner[[2]](* the second case of the closed
 form solution of [x,y] position and Δr (dr) *)

inner[[3]](* the third case of the closed
 form solution of  [x,y] position and Δr (dr) *)

inner[[4]](* the fourth case of the closed
 form solution of [x,y] position and Δr (dr) *)

(* we represent Δr (dr) w.r.t d, δ and df. We make use of the observation that
  the extrema only occurs at df = 0. As such we limit df
  to a very small value in the visualizaton. *)

pdr = dr /. inner;
```

```
(* We plot  Δr with respect to δ for visualization.*)
Plot[#, {δ, 0.00001, 1}, PlotRange → {0, 1}] & /@ (pdr /. {d → 2, df → 0.00000001})
```




```
(*The function of  Δr = f(δ) can be obtained via grouping
  plot 1 and plot 3 (plot 2 and plot 4 are the other symmetric case)
  In the plot, we can find that as δ →0 the maximum  Δr can be found. *)
(*Here we solve the outer maximization problem using built-
  in function in Mathematica (numerical method). Note that for δ→0,
  we may encounter some numerical issue, since the two intersections
 points used in the formulation are reducing to one.
    The results of numerical method are just for your reference,
and  later on we will show this result analytically *)

Maximize[{#, δ < df, δ < 0.7} /. d → 2, {δ, df} ∈ Rectangle[{0.0001, 0.0001}, {1, 1}]] & /@
 Take[pdr, 2]

{{0.414184, {δ → 0.0001, df → 0.0001}}, {0.414184, {δ → 0.0001, df → 0.0001}}}
```

```
(*   To obtain the maximum Δr theoretically,
we calculate the first order and second order derivative for δ in (0, d/2).
     We have the following results:
     1. function Δr = f(δ) is continue and differentiable for δ in (0, d/2).
     2. first order derivative < 0 for δ in (0, d/2)
     3. As  δ→0, the limit of first order derivative approaches 0
     4. second order derivative < 0 for δ in (0, d/2)
     5. function is continue at  δ = 0.
   Given 1-5,
we can conclude that Δr is monotonically decreasing for δ in (0, d/2). And at δ=
 0,  Δr takes its maximum value.
*)
(*1.1 check the continuity at d = 2√2 δ. Although the Δr derived
   by Mathematica is undefined at d =
  2√2 δ (due to the Solve function in Mathematica
      has some limitation when dealing with geometry),
   we can check the (left and right) limit of Δr as d →  2√2 δ.
   On the other hand, the function evaluation at d = 2√2 δ is actually well definec
    By applying the geometry replationship and using some off-the-shelf
       geometry solver, we can obtain when d = 2√2  δ,
Δr=(-4+2√2+√(33-20√2))/4≈0.25.
       which is consistent with the
    left limit and right limit after plugging in df = 0. *)
Limit[Take[pdr, {1}] /. {d → 2}, δ →  √2 / 2 , Direction → 1]
TrueQ[Limit[Take[pdr, {1}] /. {d → 2}, δ →  √2 / 2 , Direction → 1] ==
  Limit[Take[pdr, {3}] /. {d → 2}, δ →  √2 / 2 , Direction → -1]]
```

$$\left\{\text{ConditionalExpression}\left[\frac{1}{4\,(1+\text{df})}\left(-4+2\sqrt{2}+2\left(-4+\sqrt{2}\right)\text{df}-4\,\text{df}^2+\sqrt{\left(33-20\sqrt{2}+\left(96-52\sqrt{2}\right)\text{df}-16\left(-7+3\sqrt{2}\right)\text{df}^2-16\left(-4+\sqrt{2}\right)\text{df}^3+16\,\text{df}^4\right)}\right),\ \text{df}>0\right]\right\}$$

True

```
(*1.2 check the continuity of first order derivative at d = 2√2 δ *)
dri11 = Limit[D[pdr[[1]], δ], df → 0];
dri31 = Limit[D[pdr[[3]], δ], df → 0];
TrueQ[Limit[dri11 /. {d → 2 }, δ →  √2 / 2 , Direction → 1] ==
  Limit[dri31 /. {d → 2}, δ →  √2 / 2 , Direction → -1]]
```

True

```
dri12 = D[dri11, δ];
dri32 = D[dri31, δ];
```

```
(*2. first order derivative <0 for delta in (0, d/2) *)
test[{ConditionalExpression[x_, cond_]}, var_] := Reduce[x < 0, var];
test[{Limit[dri11 /. d → 2, df → 0]}, δ]
test[{Limit[dri31 /. d → 2, df → 0]}, δ]
```

$0 < \delta < \frac{1}{\sqrt{2}}$

$\frac{1}{\sqrt{2}} < \delta < 1$

```
(*3. As δ →0, the limit of first order derivative approaches 0*)
Limit[dri11 /. {d → 2 }, δ → 0 , Direction → -1]
```

0

```
(*4. second order derivative < 0 delta in (0, d/2) *)

test[{dri12 /. d → 2}, δ]
test[{dri32 /. d → 2}, δ]
```

$-\frac{1}{\sqrt{2}} < \delta < \frac{1}{\sqrt{2}}$

$-1 < \delta < -\frac{1}{\sqrt{2}} \; || \; \frac{1}{\sqrt{2}} < \delta < 1$

```
(*5. check the continuity at δ = 0 to make sure there is no invalid extrema.
     Specifically, for δ = 0, we can easility get Δ_R = (√2 - 1)/2 ×d.
     On the other hand, for δ → 0⁺,
we can use the following to calculate the limit
     (plug in df = 0).  Note that δ → 0⁺ and δ → 0⁻ are two symmetric cases,
    and the limits are the same. Therefore,
function δ_R = f(δ)is continue at δ = 0 *)
Limit[Take[pdr, {1}], δ → 0 , Direction → -1]
```

$\left\{\text{ConditionalExpression}\left[-\frac{d}{2} - df + \sqrt{\frac{d^2}{2} + d\,df + df^2}\;,\; df > 0 \,\&\&\, d > 0\right]\right\}$