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(\* Key idea of solving the max-min problem:

First,

we solve the inner problem by deriving the closed-form solution of  $\Delta_r(dr)$

( $dr$  in the script, i.e., the inflation we need)

such that the two inflated balls

exactly cover the convex hull of the two original balls.

This step can be done by expressing

the geometry relationship (we can use the `$Solve$` function in mathematica).

Afterwards, we solve the outer problem using the aforementioned closed- form expression.

It then becomes a pure maximization problem which can

be solved using `$Maximize$` function in mathematica.

Alternatively, we can check the second-order derivative

and first-order derivative to verify that the  $\Delta_r$  is

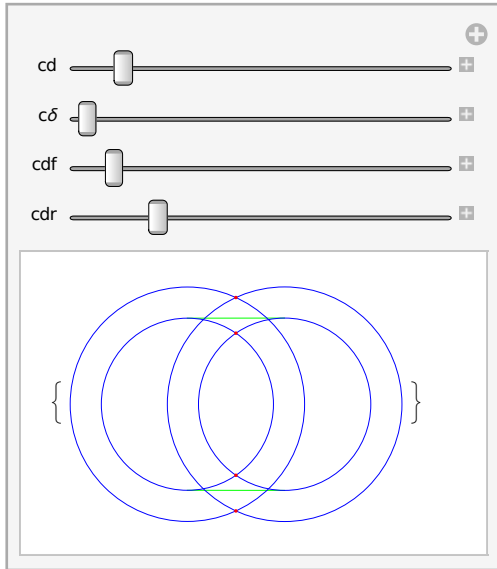
monotonically decreasing with respect to  $\delta$  (i.e., difference

between the radius of the two balls, corresponds to  $\alpha$  in the paper) for given  $d_{\max}$  ( $d$  in the script) and  $d_f$ .

\*)

(\* A visualization tool for problem description\*)

```
Manipulate[
  r1 = d/2 -  $\delta$  + df;
  r2 = d/2 +  $\delta$  + df;
  c0 = Circle[{0, 0}, r1];
  c1 = Circle[{d, 0}, r2];
  rc0 = Circle[{0, 0}, r1 + dr];
  rc1 = Circle[{d, 0}, r2 + dr];
   $\theta$  = ArcSin[ $\frac{r2 - r1}{d}$ ];
  p0 = {-r1 Sin[ $\theta$ ], r1 Cos[ $\theta$ ]};
  p1 = {d - r2 Sin[ $\theta$ ], r2 Cos[ $\theta$ ]};
  p2 = {-r1 Sin[ $\theta$ ], -r1 Cos[ $\theta$ ]};
  p3 = {d - r2 Sin[ $\theta$ ], -r2 Cos[ $\theta$ ]};
  {ans = Solve[{x, y}  $\in$  c1 && {x, y}  $\in$  c0, {x, y}];
  ans2 = Solve[{x, y}  $\in$  rc1 && {x, y}  $\in$  rc0, {x, y}];
  (*Solve[{x,y} $\in$ rc1&&{x,y} $\in$ rc0&&({x,y} $\in$ Line[{p0,p1}]||{x,y} $\in$ Line[{p2,p3}]},dr];*)
  Graphics[{
    {Green, Line[{p0, p1}], Line[{p2, p3}]},
    {Blue, c0, c1, rc0, rc1},
    {Red, PointSize[0.01], Point[{x, y}] /. ans, Point[{x, y}] /. ans2}}] /.
    {d  $\rightarrow$  cd,  $\delta$   $\rightarrow$  cd $\delta$ , df  $\rightarrow$  cdf, dr  $\rightarrow$  cdr}},
  {cd, 1, 10}, {cd $\delta$ , 0.001, cd/2}, {cdf, 0.00, 10}, {cdr, 0.01, 3}
]
```



(\* Main program starts here.

It may need a long time to output results. Just wait for some time.

pls run this block first if any of

the block following appears to be undefined. \*)

```

r1 = d/2 - δ + df;
r2 = d/2 + δ + df;
c0 = Circle[{0, 0}, r1];
c1 = Circle[{d, 0}, r2];
rc0 = Circle[{0, 0}, r1 + dr];
rc1 = Circle[{d, 0}, r2 + dr];
θ = ArcSin[ $\frac{r_2 - r_1}{d}$ ];
p0 = {-r1 Sin[θ], r1 Cos[θ]};
p1 = {d - r2 Sin[θ], r2 Cos[θ]};
p2 = {-r1 Sin[θ], -r1 Cos[θ]};
p3 = {d - r2 Sin[θ], -r2 Cos[θ]};
ans = Solve[d > 0 && δ > 0 && df > 0 && {x, y} ∈ c1 && {x, y} ∈ c0, {x, y}];
ans2 = Solve[d > 0 && δ > 0 && df > 0 && {x, y} ∈ rc1 && {x, y} ∈ rc0, {x, y}];
(* The following function is to set up the geometry relationship. *)
inner = Solve[d > 0 && δ > 0 && df > 0 && {x, y} ∈ rc1 && {x, y} ∈ rc0 &&
  ({x, y} ∈ Line[{p0, p1}] || {x, y} ∈ Line[{p2, p3}]), {x, y, dr}]

(* Note that the closed-form solution of Δr(dr) contains four cases
after solving the equation. Two of them can be grouped together depending
on the valid interval of delta. And the two groups are actually symmetric
and identical. The resulting solution of Δr is
continue (differentiable) everywhere as verified later. *)

(*You can either run inner =
Solve or use the results below directly for convenience.*)

```

inner = { {x →

$$\begin{aligned}
& \text{ConditionalExpression} \left[ \frac{d^4 - 8 d^2 \delta^2 + 8 d \delta^3 + 16 d f \delta^3}{2 d (d^2 - 8 \delta^2)} - \right. \\
& \quad \sqrt{2} \sqrt{\left( (d^6 \delta^2 + 2 d^5 d f \delta^2 + 2 d^4 d f^2 \delta^2 - 12 d^4 \delta^4 - \right. \\
& \quad \quad 16 d^3 d f \delta^4 - 16 d^2 d f^2 \delta^4 + 48 d^2 \delta^6 + 32 d d f \delta^6 + \\
& \quad \quad \left. 32 d f^2 \delta^6 - 64 \delta^8) / (d^2 (d^2 - 8 \delta^2)^2) \right)}, \\
& \quad \left. d > 2 \sqrt{2} \sqrt{\delta^2} \ \&\& \ d f > 0 \ \&\& \ \delta > 0 \right], \ y \rightarrow \\
& \text{ConditionalExpression} \left[ \right. \\
& \quad - \frac{1}{4} \sqrt{\left( \frac{1}{\delta^2} \left( d^4 - 8 d^2 \delta^2 + 16 \delta^4 - 4 d^3 \right. \right. \\
& \quad \quad \left( \frac{d^4 - 8 d^2 \delta^2 + 8 d \delta^3 + 16 d f \delta^3}{2 d (d^2 - 8 \delta^2)} - \sqrt{2} \sqrt{\left( (d^6 \delta^2 + \right. \right.} \\
& \quad \quad \quad 2 d^5 d f \delta^2 + 2 d^4 d f^2 \delta^2 - 12 d^4 \delta^4 - \\
& \quad \quad \quad 16 d^3 d f \delta^4 - 16 d^2 d f^2 \delta^4 + 48 d^2 \delta^6 + \\
& \quad \quad \quad \left. 32 d d f \delta^6 + 32 d f^2 \delta^6 - 64 \delta^8) / \right.} \\
& \quad \quad \quad \left. \left. (d^2 (d^2 - 8 \delta^2)^2) \right) \right) + 16 d \delta^2} \\
& \quad \quad \left( \frac{d^4 - 8 d^2 \delta^2 + 8 d \delta^3 + 16 d f \delta^3}{2 d (d^2 - 8 \delta^2)} - \sqrt{2} \right. \\
& \quad \quad \quad \left. \sqrt{\left( (d^6 \delta^2 + 2 d^5 d f \delta^2 + 2 d^4 d f^2 \delta^2 - \right. \right.} \\
& \quad \quad \quad \quad 12 d^4 \delta^4 - 16 d^3 d f \delta^4 - 16 d^2 d f^2 \delta^4 + \\
& \quad \quad \quad \quad 48 d^2 \delta^6 + 32 d d f \delta^6 + 32 d f^2 \delta^6 - \\
& \quad \quad \quad \quad \left. \left. 64 \delta^8) / (d^2 (d^2 - 8 \delta^2)^2) \right) \right) + \\
& \quad \quad 4 d^2 \left( \frac{d^4 - 8 d^2 \delta^2 + 8 d \delta^3 + 16 d f \delta^3}{2 d (d^2 - 8 \delta^2)} - \right. \\
& \quad \quad \quad \left. \sqrt{2} \sqrt{\left( (d^6 \delta^2 + 2 d^5 d f \delta^2 + 2 d^4 d f^2 \delta^2 - \right. \right.} \\
& \quad \quad \quad \quad 12 d^4 \delta^4 - 16 d^3 d f \delta^4 - 16 d^2 d f^2 \delta^4 + \\
& \quad \quad \quad \quad 48 d^2 \delta^6 + 32 d d f \delta^6 + 32 d f^2 \delta^6 - \\
& \quad \quad \quad \quad \left. \left. 64 \delta^8) / (d^2 (d^2 - 8 \delta^2)^2) \right) \right)^2 - \\
& \quad \quad \left. 16 \delta^2 \left( \frac{d^4 - 8 d^2 \delta^2 + 8 d \delta^3 + 16 d f \delta^3}{2 d (d^2 - 8 \delta^2)} - \right. \right.
\end{aligned}$$

$$\sqrt{2} \sqrt{\left( (d^6 \delta^2 + 2 d^5 df \delta^2 + 2 d^4 df^2 \delta^2 - 12 d^4 \delta^4 - 16 d^3 df \delta^4 - 16 d^2 df^2 \delta^4 + 48 d^2 \delta^6 + 32 d df \delta^6 + 32 df^2 \delta^6 - 64 \delta^8) / (d^2 (d^2 - 8 \delta^2)^2) \right)^2 \right)},$$

$$d > 2 \sqrt{2} \sqrt{\delta^2} \ \&\& \ df > 0 \ \&\& \ \delta > 0 \Big], \ dr \rightarrow$$

ConditionalExpression[

$$\frac{1}{2}$$

$$(-d - 2 df + 2 \delta) +$$

$$\sqrt{\left( \left( \frac{d^4 - 8 d^2 \delta^2 + 8 d \delta^3 + 16 df \delta^3}{2 d (d^2 - 8 \delta^2)} - \right. \right.$$

$$\left. \sqrt{2} \sqrt{\left( (d^6 \delta^2 + 2 d^5 df \delta^2 + 2 d^4 df^2 \delta^2 - 12 d^4 \delta^4 - 16 d^3 df \delta^4 - 16 d^2 df^2 \delta^4 + 48 d^2 \delta^6 + 32 d df \delta^6 + 32 df^2 \delta^6 - 64 \delta^8) / (d^2 (d^2 - 8 \delta^2)^2) \right)^2} + \frac{1}{16 \delta^2} \left( d^4 - 8 d^2 \delta^2 + \right.$$

$$\left. 16 \delta^4 - 4 d^3 \left( \frac{d^4 - 8 d^2 \delta^2 + 8 d \delta^3 + 16 df \delta^3}{2 d (d^2 - 8 \delta^2)} - \right. \right.$$

$$\left. \sqrt{2} \sqrt{\left( (d^6 \delta^2 + 2 d^5 df \delta^2 + 2 d^4 df^2 \delta^2 - 12 d^4 \delta^4 - 16 d^3 df \delta^4 - 16 d^2 df^2 \delta^4 + 48 d^2 \delta^6 + 32 d df \delta^6 + 32 df^2 \delta^6 - 64 \delta^8) / (d^2 (d^2 - 8 \delta^2)^2) \right)^2} + \right.$$

$$\left. 16 d \delta^2 \left( \frac{d^4 - 8 d^2 \delta^2 + 8 d \delta^3 + 16 df \delta^3}{2 d (d^2 - 8 \delta^2)} - \right. \right.$$

$$\left. \sqrt{2} \sqrt{\left( (d^6 \delta^2 + 2 d^5 df \delta^2 + 2 d^4 df^2 \delta^2 - 12 d^4 \delta^4 - 16 d^3 df \delta^4 - 16 d^2 df^2 \delta^4 + 48 d^2 \delta^6 + 32 d df \delta^6 + 32 df^2 \delta^6 - 64 \delta^8) / (d^2 (d^2 - 8 \delta^2)^2) \right)^2} + \right.$$

$$\begin{aligned}
& 4 d^2 \left( \frac{d^4 - 8 d^2 \delta^2 + 8 d \delta^3 + 16 d f \delta^3}{2 d (d^2 - 8 \delta^2)} - \sqrt{2} \right. \\
& \quad \sqrt{\left( (d^6 \delta^2 + 2 d^5 d f \delta^2 + 2 d^4 d f^2 \delta^2 - 12 d^4 \delta^4 - 16 d^3 d f \delta^4 - 16 d^2 d f^2 \delta^4 + \right. \\
& \quad \left. 48 d^2 \delta^6 + 32 d d f \delta^6 + 32 d f^2 \delta^6 - 64 \delta^8) / (d^2 (d^2 - 8 \delta^2)^2) \right)} \Big)^2 - \\
& 16 \delta^2 \left( \frac{d^4 - 8 d^2 \delta^2 + 8 d \delta^3 + 16 d f \delta^3}{2 d (d^2 - 8 \delta^2)} - \sqrt{2} \right. \\
& \quad \sqrt{\left( (d^6 \delta^2 + 2 d^5 d f \delta^2 + 2 d^4 d f^2 \delta^2 - 12 d^4 \delta^4 - 16 d^3 d f \delta^4 - 16 d^2 d f^2 \delta^4 + \right. \\
& \quad \left. 48 d^2 \delta^6 + 32 d d f \delta^6 + 32 d f^2 \delta^6 - 64 \delta^8) / (d^2 (d^2 - 8 \delta^2)^2) \right)} \Big)^2 \Big), \\
& d > 2 \sqrt{2} \sqrt{\delta^2} \ \&\& d f > 0 \ \&\& \delta > 0 \Big], \{x \rightarrow \\
& \text{ConditionalExpression}\left[ \right. \\
& \quad \frac{d^4 - 8 d^2 \delta^2 + 8 d \delta^3 + 16 d f \delta^3}{2 d (d^2 - 8 \delta^2)} - \\
& \quad \sqrt{2} \\
& \quad \sqrt{\left( (d^6 \delta^2 + 2 d^5 d f \delta^2 + 2 d^4 d f^2 \delta^2 - 12 d^4 \delta^4 - \right. \\
& \quad \left. 16 d^3 d f \delta^4 - 16 d^2 d f^2 \delta^4 + 48 d^2 \delta^6 + 32 d d f \delta^6 + \right. \\
& \quad \left. 32 d f^2 \delta^6 - 64 \delta^8) / (d^2 (d^2 - 8 \delta^2)^2) \right)}, \\
& d > 2 \sqrt{2} \sqrt{\delta^2} \ \&\& d f > 0 \ \&\& \delta > 0 \Big], y \rightarrow \\
& \text{ConditionalExpression}\left[ \right. \\
& \quad \frac{1}{4} \\
& \quad \sqrt{\left( \frac{1}{\delta^2} \left( d^4 - 8 d^2 \delta^2 + 16 \delta^4 - \right. \right. \\
& \quad \left. \left. 4 d^3 \left( \frac{d^4 - 8 d^2 \delta^2 + 8 d \delta^3 + 16 d f \delta^3}{2 d (d^2 - 8 \delta^2)} - \sqrt{2} \right. \right. \right. \\
& \quad \left. \left. \left. \sqrt{\left( (d^6 \delta^2 + 2 d^5 d f \delta^2 + 2 d^4 d f^2 \delta^2 - \right. \right. \right. \right.
\end{aligned}$$

$$\begin{aligned}
& \left( \frac{12 d^4 \delta^4 - 16 d^3 df \delta^4 - 16 d^2 df^2 \delta^4 + 48 d^2 \delta^6 + 32 d df \delta^6 + 32 df^2 \delta^6 - 64 \delta^8}{d^2 (d^2 - 8 \delta^2)^2} \right) + \\
& 16 d \delta^2 \left( \frac{d^4 - 8 d^2 \delta^2 + 8 d \delta^3 + 16 df \delta^3}{2 d (d^2 - 8 \delta^2)} - \right. \\
& \quad \sqrt{2} \sqrt{\left( (d^6 \delta^2 + 2 d^5 df \delta^2 + 2 d^4 df^2 \delta^2 - 12 d^4 \delta^4 - 16 d^3 df \delta^4 - 16 d^2 df^2 \delta^4 + 48 d^2 \delta^6 + 32 d df \delta^6 + 32 df^2 \delta^6 - 64 \delta^8) / (d^2 (d^2 - 8 \delta^2)^2) \right)} + \\
& 4 d^2 \left( \frac{d^4 - 8 d^2 \delta^2 + 8 d \delta^3 + 16 df \delta^3}{2 d (d^2 - 8 \delta^2)} - \right. \\
& \quad \sqrt{2} \sqrt{\left( (d^6 \delta^2 + 2 d^5 df \delta^2 + 2 d^4 df^2 \delta^2 - 12 d^4 \delta^4 - 16 d^3 df \delta^4 - 16 d^2 df^2 \delta^4 + 48 d^2 \delta^6 + 32 d df \delta^6 + 32 df^2 \delta^6 - 64 \delta^8) / (d^2 (d^2 - 8 \delta^2)^2) \right)^2} - \\
& 16 \delta^2 \left( \frac{d^4 - 8 d^2 \delta^2 + 8 d \delta^3 + 16 df \delta^3}{2 d (d^2 - 8 \delta^2)} - \right. \\
& \quad \sqrt{2} \sqrt{\left( (d^6 \delta^2 + 2 d^5 df \delta^2 + 2 d^4 df^2 \delta^2 - 12 d^4 \delta^4 - 16 d^3 df \delta^4 - 16 d^2 df^2 \delta^4 + 48 d^2 \delta^6 + 32 d df \delta^6 + 32 df^2 \delta^6 - 64 \delta^8) / (d^2 (d^2 - 8 \delta^2)^2) \right)^2} \Bigg),
\end{aligned}$$

$$d > 2 \sqrt{2} \sqrt{\delta^2} \ \&\& df > 0 \ \&\& \delta > 0 \Big], \ dr \rightarrow$$

ConditionalExpression[

$$\frac{1}{2}$$

$$(-d - 2 df + 2 \delta) +$$

$$\sqrt{\left( \left( \frac{d^4 - 8 d^2 \delta^2 + 8 d \delta^3 + 16 df \delta^3}{2 d (d^2 - 8 \delta^2)} - \right. \right.$$

$$\sqrt{2} \sqrt{\left( (d^6 \delta^2 + 2 d^5 df \delta^2 + 2 d^4 df^2 \delta^2 - 12 d^4 \delta^4 - 16 d^3 df \delta^4 - 16 d^2 df^2 \delta^4 + 48 d^2 \delta^6 + 32 d df \delta^6 + 32 df^2 \delta^6 - 64 \delta^8) / (d^2 (d^2 - 8 \delta^2)^2) \right)^2 +$$

$$\frac{1}{16 \delta^2} \left( d^4 - 8 d^2 \delta^2 + 16 \delta^4 - 4 d^3 \right. \\ \left. \left( \frac{d^4 - 8 d^2 \delta^2 + 8 d \delta^3 + 16 df \delta^3}{2 d (d^2 - 8 \delta^2)} - \right. \right.$$

$$\sqrt{2} \sqrt{\left( (d^6 \delta^2 + 2 d^5 df \delta^2 + 2 d^4 df^2 \delta^2 - 12 d^4 \delta^4 - 16 d^3 df \delta^4 - 16 d^2 df^2 \delta^4 + 48 d^2 \delta^6 + 32 d df \delta^6 + 32 df^2 \delta^6 - 64 \delta^8) / (d^2 (d^2 - 8 \delta^2)^2) \right)^2 +$$

$$16 d \delta^2 \left( \frac{d^4 - 8 d^2 \delta^2 + 8 d \delta^3 + 16 df \delta^3}{2 d (d^2 - 8 \delta^2)} - \right.$$

$$\sqrt{2} \sqrt{\left( (d^6 \delta^2 + 2 d^5 df \delta^2 + 2 d^4 df^2 \delta^2 - 12 d^4 \delta^4 - 16 d^3 df \delta^4 - 16 d^2 df^2 \delta^4 + 48 d^2 \delta^6 + 32 d df \delta^6 + 32 df^2 \delta^6 - 64 \delta^8) / (d^2 (d^2 - 8 \delta^2)^2) \right)^2 +$$

$$4 d^2 \left( \frac{d^4 - 8 d^2 \delta^2 + 8 d \delta^3 + 16 df \delta^3}{2 d (d^2 - 8 \delta^2)} - \sqrt{2} \right. \\ \sqrt{\left( (d^6 \delta^2 + 2 d^5 df \delta^2 + 2 d^4 df^2 \delta^2 - 12 d^4 \delta^4 - 16 d^3 df \delta^4 - 16 d^2 df^2 \delta^4 + 48 d^2 \delta^6 + 32 d df \delta^6 + 32 df^2 \delta^6 - 64 \delta^8) / (d^2 (d^2 - 8 \delta^2)^2) \right)^2 -$$

$$16 \delta^2 \left( \frac{d^4 - 8 d^2 \delta^2 + 8 d \delta^3 + 16 df \delta^3}{2 d (d^2 - 8 \delta^2)} - \sqrt{2} \right. \\ \sqrt{\left( (d^6 \delta^2 + 2 d^5 df \delta^2 + 2 d^4 df^2 \delta^2 - 12 d^4 \delta^4 - 16 d^3 df \delta^4 - 16 d^2 df^2 \delta^4 + 48 d^2 \delta^6 + 32 d df \delta^6 + 32 df^2 \delta^6 -$$

$$\begin{aligned}
& 64 \delta^8) / (d^2 (d^2 - 8 \delta^2)^2)) \Big)^2 \Big) \Big), \\
& d > 2 \sqrt{2} \sqrt{\delta^2} \ \&\& df > 0 \ \&\& \delta > 0 \Big] \Big\}, \{x \rightarrow \\
& \text{ConditionalExpression} \Big[ \\
& \quad \frac{d^4 - 8 d^2 \delta^2 + 8 d \delta^3 + 16 df \delta^3}{2 d (d^2 - 8 \delta^2)} + \\
& \quad \sqrt{2} \\
& \quad \sqrt{\Big( (d^6 \delta^2 + 2 d^5 df \delta^2 + 2 d^4 df^2 \delta^2 - 12 d^4 \delta^4 - \\
& \quad 16 d^3 df \delta^4 - 16 d^2 df^2 \delta^4 + 48 d^2 \delta^6 + 32 d df \delta^6 + \\
& \quad 32 df^2 \delta^6 - 64 \delta^8) / (d^2 (d^2 - 8 \delta^2)^2) \Big)}, \\
& \quad 2 \delta < d < 2 \sqrt{2} \sqrt{\delta^2} \ \&\& df > 0 \ \&\& \\
& \quad \delta > \\
& \quad 0 \Big], \\
& y \rightarrow \text{ConditionalExpression} \Big[ \\
& \quad -\frac{1}{4} \\
& \quad \sqrt{\Big( \frac{1}{\delta^2} \Big( d^4 - 8 d^2 \delta^2 + 16 \delta^4 - \\
& \quad 4 d^3 \Big( \frac{d^4 - 8 d^2 \delta^2 + 8 d \delta^3 + 16 df \delta^3}{2 d (d^2 - 8 \delta^2)} + \sqrt{2} \\
& \quad \sqrt{\Big( (d^6 \delta^2 + 2 d^5 df \delta^2 + 2 d^4 df^2 \delta^2 - \\
& \quad 12 d^4 \delta^4 - 16 d^3 df \delta^4 - 16 d^2 df^2 \delta^4 + \\
& \quad 48 d^2 \delta^6 + 32 d df \delta^6 + 32 df^2 \delta^6 - \\
& \quad 64 \delta^8) / (d^2 (d^2 - 8 \delta^2)^2) \Big) \Big) + \\
& \quad 16 d \delta^2 \Big( \frac{d^4 - 8 d^2 \delta^2 + 8 d \delta^3 + 16 df \delta^3}{2 d (d^2 - 8 \delta^2)} + \\
& \quad \sqrt{2} \sqrt{\Big( (d^6 \delta^2 + 2 d^5 df \delta^2 + 2 d^4 df^2 \delta^2 - \\
& \quad 12 d^4 \delta^4 - 16 d^3 df \delta^4 - 16 d^2 df^2 \delta^4 + \\
& \quad 48 d^2 \delta^6 + 32 d df \delta^6 + 32 df^2 \delta^6 - \\
& \quad 64 \delta^8) / (d^2 (d^2 - 8 \delta^2)^2) \Big) \Big) +
\end{aligned}$$



$$\begin{aligned}
& 4 d^2 \left( \frac{d^4 - 8 d^2 \delta^2 + 8 d \delta^3 + 16 d f \delta^3}{2 d (d^2 - 8 \delta^2)} + \right. \\
& \quad \left. \sqrt{2} \sqrt{\left( (d^6 \delta^2 + 2 d^5 d f \delta^2 + 2 d^4 d f^2 \delta^2 - \right. \right. \\
& \quad \quad 12 d^4 \delta^4 - 16 d^3 d f \delta^4 - 16 d^2 d f^2 \delta^4 + \\
& \quad \quad 48 d^2 \delta^6 + 32 d d f \delta^6 + 32 d f^2 \delta^6 - \\
& \quad \quad \left. \left. 64 \delta^8) / (d^2 (d^2 - 8 \delta^2)^2) \right) \right)^2 - \\
& 16 \delta^2 \left( \frac{d^4 - 8 d^2 \delta^2 + 8 d \delta^3 + 16 d f \delta^3}{2 d (d^2 - 8 \delta^2)} + \right. \\
& \quad \left. \sqrt{2} \sqrt{\left( (d^6 \delta^2 + 2 d^5 d f \delta^2 + 2 d^4 d f^2 \delta^2 - \right. \right. \\
& \quad \quad 12 d^4 \delta^4 - 16 d^3 d f \delta^4 - 16 d^2 d f^2 \delta^4 + \\
& \quad \quad 48 d^2 \delta^6 + 32 d d f \delta^6 + 32 d f^2 \delta^6 - \\
& \quad \quad \left. \left. 64 \delta^8) / (d^2 (d^2 - 8 \delta^2)^2) \right) \right)^2 \Bigg),
\end{aligned}$$

$$2 \delta < d < 2 \sqrt{2} \sqrt{\delta^2} \ \&\& \ d f > 0 \ \&\& \ \delta > 0 \Big], \ dr \rightarrow$$

ConditionalExpression[

$$\frac{1}{2}$$

$$\begin{aligned}
& (-d - 2 d f + 2 \delta) + \\
& \sqrt{\left( \left( \frac{d^4 - 8 d^2 \delta^2 + 8 d \delta^3 + 16 d f \delta^3}{2 d (d^2 - 8 \delta^2)} + \right. \right. \\
& \quad \left. \left. \sqrt{2} \sqrt{\left( (d^6 \delta^2 + 2 d^5 d f \delta^2 + 2 d^4 d f^2 \delta^2 - \right. \right. \right. \\
& \quad \quad 12 d^4 \delta^4 - 16 d^3 d f \delta^4 - 16 d^2 d f^2 \delta^4 + \\
& \quad \quad 48 d^2 \delta^6 + 32 d d f \delta^6 + 32 d f^2 \delta^6 - \\
& \quad \quad \left. \left. \left. 64 \delta^8) / (d^2 (d^2 - 8 \delta^2)^2) \right) \right)^2 + \right.
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{16 \delta^2} \left( d^4 - 8 d^2 \delta^2 + 16 \delta^4 - 4 d^3 \right. \\
& \quad \left( \frac{d^4 - 8 d^2 \delta^2 + 8 d \delta^3 + 16 d f \delta^3}{2 d (d^2 - 8 \delta^2)} + \right. \\
& \quad \left. \sqrt{2} \sqrt{\left( (d^6 \delta^2 + 2 d^5 d f \delta^2 + 2 d^4 d f^2 \delta^2 - \right. \right. \\
& \quad \quad 12 d^4 \delta^4 - 16 d^3 d f \delta^4 - 16 d^2 d f^2 \delta^4 +
\end{aligned}$$

$$\begin{aligned}
& \left( \frac{48 d^2 \delta^6 + 32 d df \delta^6 + 32 df^2 \delta^6 - 64 \delta^8}{d^2 (d^2 - 8 \delta^2)^2} \right) + \\
& 16 d \delta^2 \left( \frac{d^4 - 8 d^2 \delta^2 + 8 d \delta^3 + 16 df \delta^3}{2 d (d^2 - 8 \delta^2)} + \right. \\
& \quad \sqrt{2} \sqrt{\left( (d^6 \delta^2 + 2 d^5 df \delta^2 + 2 d^4 df^2 \delta^2 - 12 d^4 \delta^4 - 16 d^3 df \delta^4 - 16 d^2 df^2 \delta^4 + \right.} \\
& \quad \left. 48 d^2 \delta^6 + 32 d df \delta^6 + 32 df^2 \delta^6 - 64 \delta^8) / (d^2 (d^2 - 8 \delta^2)^2) \right) + \\
& 4 d^2 \left( \frac{d^4 - 8 d^2 \delta^2 + 8 d \delta^3 + 16 df \delta^3}{2 d (d^2 - 8 \delta^2)} + \sqrt{2} \right. \\
& \quad \sqrt{\left( (d^6 \delta^2 + 2 d^5 df \delta^2 + 2 d^4 df^2 \delta^2 - 12 d^4 \delta^4 - 16 d^3 df \delta^4 - 16 d^2 df^2 \delta^4 + \right.} \\
& \quad \left. 48 d^2 \delta^6 + 32 d df \delta^6 + 32 df^2 \delta^6 - 64 \delta^8) / (d^2 (d^2 - 8 \delta^2)^2) \right)^2 - \\
& 16 \delta^2 \left( \frac{d^4 - 8 d^2 \delta^2 + 8 d \delta^3 + 16 df \delta^3}{2 d (d^2 - 8 \delta^2)} + \sqrt{2} \right. \\
& \quad \sqrt{\left( (d^6 \delta^2 + 2 d^5 df \delta^2 + 2 d^4 df^2 \delta^2 - 12 d^4 \delta^4 - 16 d^3 df \delta^4 - 16 d^2 df^2 \delta^4 + \right.} \\
& \quad \left. 48 d^2 \delta^6 + 32 d df \delta^6 + 32 df^2 \delta^6 - 64 \delta^8) / (d^2 (d^2 - 8 \delta^2)^2) \right)^2 \Bigg), \\
& \left. 2 \delta < d < 2 \sqrt{2} \sqrt{\delta^2} \ \&\& df > 0 \ \&\& \delta > 0 \right\}, \\
\{x \rightarrow \text{ConditionalExpression}\Big[ & \\
& \frac{d^4 - 8 d^2 \delta^2 + 8 d \delta^3 + 16 df \delta^3}{2 d (d^2 - 8 \delta^2)} + \\
& \sqrt{2} \\
& \sqrt{\left( (d^6 \delta^2 + 2 d^5 df \delta^2 + 2 d^4 df^2 \delta^2 - 12 d^4 \delta^4 - \right.} \\
& \quad \left. 16 d^3 df \delta^4 - 16 d^2 df^2 \delta^4 + 48 d^2 \delta^6 + 32 d df \delta^6 + \right. \\
& \quad \left. 32 df^2 \delta^6 - 64 \delta^8) / (d^2 (d^2 - 8 \delta^2)^2) \right), & \\
\end{aligned}$$

$$\begin{aligned}
& 2 \, \delta < d < 2 \, \sqrt{2} \, \sqrt{\delta^2} \, \&\& df > 0 \, \&\& \\
& \delta > \\
& 0], \\
y \rightarrow \text{ConditionalExpression}\Big[ & \\
& \frac{1}{4} \\
& \sqrt{\left(\frac{1}{\delta^2} \left(d^4 - 8 \, d^2 \, \delta^2 + 16 \, \delta^4 - \right. \right.} \\
& \quad 4 \, d^3 \left( \frac{d^4 - 8 \, d^2 \, \delta^2 + 8 \, d \, \delta^3 + 16 \, df \, \delta^3}{2 \, d \, (d^2 - 8 \, \delta^2)} + \sqrt{2} \right. \\
& \quad \left. \sqrt{\left( (d^6 \, \delta^2 + 2 \, d^5 \, df \, \delta^2 + 2 \, d^4 \, df^2 \, \delta^2 - \right. \right.} \\
& \quad \quad 12 \, d^4 \, \delta^4 - 16 \, d^3 \, df \, \delta^4 - 16 \, d^2 \, df^2 \, \delta^4 + \\
& \quad \quad 48 \, d^2 \, \delta^6 + 32 \, d \, df \, \delta^6 + 32 \, df^2 \, \delta^6 - \\
& \quad \quad \left. \left. 64 \, \delta^8 \right) / (d^2 \, (d^2 - 8 \, \delta^2)^2) \right) \Big) + \\
& \quad 16 \, d \, \delta^2 \left( \frac{d^4 - 8 \, d^2 \, \delta^2 + 8 \, d \, \delta^3 + 16 \, df \, \delta^3}{2 \, d \, (d^2 - 8 \, \delta^2)} + \right. \\
& \quad \left. \sqrt{2} \, \sqrt{\left( (d^6 \, \delta^2 + 2 \, d^5 \, df \, \delta^2 + 2 \, d^4 \, df^2 \, \delta^2 - \right. \right.} \\
& \quad \quad 12 \, d^4 \, \delta^4 - 16 \, d^3 \, df \, \delta^4 - 16 \, d^2 \, df^2 \, \delta^4 + \\
& \quad \quad 48 \, d^2 \, \delta^6 + 32 \, d \, df \, \delta^6 + 32 \, df^2 \, \delta^6 - \\
& \quad \quad \left. \left. 64 \, \delta^8 \right) / (d^2 \, (d^2 - 8 \, \delta^2)^2) \right) \Big) + \\
& \quad 4 \, d^2 \left( \frac{d^4 - 8 \, d^2 \, \delta^2 + 8 \, d \, \delta^3 + 16 \, df \, \delta^3}{2 \, d \, (d^2 - 8 \, \delta^2)} + \right. \\
& \quad \left. \sqrt{2} \, \sqrt{\left( (d^6 \, \delta^2 + 2 \, d^5 \, df \, \delta^2 + 2 \, d^4 \, df^2 \, \delta^2 - \right. \right.} \\
& \quad \quad 12 \, d^4 \, \delta^4 - 16 \, d^3 \, df \, \delta^4 - 16 \, d^2 \, df^2 \, \delta^4 + \\
& \quad \quad 48 \, d^2 \, \delta^6 + 32 \, d \, df \, \delta^6 + 32 \, df^2 \, \delta^6 - \\
& \quad \quad \left. \left. 64 \, \delta^8 \right) / (d^2 \, (d^2 - 8 \, \delta^2)^2) \right) \Big)^2 - \\
& \quad 16 \, \delta^2 \left( \frac{d^4 - 8 \, d^2 \, \delta^2 + 8 \, d \, \delta^3 + 16 \, df \, \delta^3}{2 \, d \, (d^2 - 8 \, \delta^2)} + \right. \\
& \quad \left. \sqrt{2} \, \sqrt{\left( (d^6 \, \delta^2 + 2 \, d^5 \, df \, \delta^2 + 2 \, d^4 \, df^2 \, \delta^2 - \right. \right.} \\
& \quad \quad 12 \, d^4 \, \delta^4 - 16 \, d^3 \, df \, \delta^4 - 16 \, d^2 \, df^2 \, \delta^4 +
\end{aligned}$$

$$\begin{aligned}
& \left. \left( \frac{48 d^2 \delta^6 + 32 d df \delta^6 + 32 df^2 \delta^6 - 64 \delta^8}{(d^2 (d^2 - 8 \delta^2)^2)} \right)^2 \right), \\
& 2 \delta < d < 2 \sqrt{2} \sqrt{\delta^2} \ \&\& df > 0 \ \&\& \delta > 0 \Big], \ dr \rightarrow \\
& \text{ConditionalExpression}\Big[ \\
& \quad \frac{1}{2} \\
& \quad \quad (-d - 2 df + 2 \delta) + \\
& \quad \quad \sqrt{\left( \left( \frac{d^4 - 8 d^2 \delta^2 + 8 d \delta^3 + 16 df \delta^3}{2 d (d^2 - 8 \delta^2)} + \right. \right.} \\
& \quad \quad \quad \left. \left. \sqrt{2} \sqrt{\left( (d^6 \delta^2 + 2 d^5 df \delta^2 + 2 d^4 df^2 \delta^2 - \right. \right. \right. \\
& \quad \quad \quad \left. \left. \left. 12 d^4 \delta^4 - 16 d^3 df \delta^4 - 16 d^2 df^2 \delta^4 + \right. \right. \right. \\
& \quad \quad \quad \left. \left. \left. 48 d^2 \delta^6 + 32 d df \delta^6 + 32 df^2 \delta^6 - \right. \right. \right. \\
& \quad \quad \quad \left. \left. \left. 64 \delta^8 \right) / (d^2 (d^2 - 8 \delta^2)^2) \right)^2} + \right. \\
& \quad \quad \frac{1}{16 \delta^2} \left( d^4 - 8 d^2 \delta^2 + 16 \delta^4 - 4 d^3 \right. \\
& \quad \quad \quad \left( \frac{d^4 - 8 d^2 \delta^2 + 8 d \delta^3 + 16 df \delta^3}{2 d (d^2 - 8 \delta^2)} + \right. \\
& \quad \quad \quad \left. \sqrt{2} \sqrt{\left( (d^6 \delta^2 + 2 d^5 df \delta^2 + 2 d^4 df^2 \delta^2 - \right. \right. \right. \\
& \quad \quad \quad \left. \left. \left. 12 d^4 \delta^4 - 16 d^3 df \delta^4 - 16 d^2 df^2 \delta^4 + \right. \right. \right. \\
& \quad \quad \quad \left. \left. \left. 48 d^2 \delta^6 + 32 d df \delta^6 + 32 df^2 \delta^6 - \right. \right. \right. \\
& \quad \quad \quad \left. \left. \left. 64 \delta^8 \right) / (d^2 (d^2 - 8 \delta^2)^2) \right)^2} + \right. \\
& \quad \quad 16 d \delta^2 \left( \frac{d^4 - 8 d^2 \delta^2 + 8 d \delta^3 + 16 df \delta^3}{2 d (d^2 - 8 \delta^2)} + \right. \\
& \quad \quad \quad \left. \sqrt{2} \sqrt{\left( (d^6 \delta^2 + 2 d^5 df \delta^2 + 2 d^4 df^2 \delta^2 - \right. \right. \right. \\
& \quad \quad \quad \left. \left. \left. 12 d^4 \delta^4 - 16 d^3 df \delta^4 - 16 d^2 df^2 \delta^4 + \right. \right. \right. \\
& \quad \quad \quad \left. \left. \left. 48 d^2 \delta^6 + 32 d df \delta^6 + 32 df^2 \delta^6 - \right. \right. \right. \\
& \quad \quad \quad \left. \left. \left. 64 \delta^8 \right) / (d^2 (d^2 - 8 \delta^2)^2) \right)^2} + \right. \\
& \quad \quad 4 d^2 \left( \frac{d^4 - 8 d^2 \delta^2 + 8 d \delta^3 + 16 df \delta^3}{2 d (d^2 - 8 \delta^2)} + \sqrt{2} \right.
\end{aligned}$$

$$\begin{aligned}
& \sqrt{\left( (d^6 \delta^2 + 2 d^5 df \delta^2 + 2 d^4 df^2 \delta^2 - 12 d^4 \delta^4 - 16 d^3 df \delta^4 - 16 d^2 df^2 \delta^4 + \right. \\
& \quad \left. 48 d^2 \delta^6 + 32 d df \delta^6 + 32 df^2 \delta^6 - 64 \delta^8) / (d^2 (d^2 - 8 \delta^2)^2) \right)^2 - \\
& \quad 16 \delta^2 \left( \frac{d^4 - 8 d^2 \delta^2 + 8 d \delta^3 + 16 df \delta^3}{2 d (d^2 - 8 \delta^2)} + \sqrt{2} \right. \\
& \quad \left. \sqrt{\left( (d^6 \delta^2 + 2 d^5 df \delta^2 + 2 d^4 df^2 \delta^2 - 12 d^4 \delta^4 - 16 d^3 df \delta^4 - 16 d^2 df^2 \delta^4 + \right. \right. \\
& \quad \left. \left. 48 d^2 \delta^6 + 32 d df \delta^6 + 32 df^2 \delta^6 - 64 \delta^8) / (d^2 (d^2 - 8 \delta^2)^2) \right)^2} \right), \\
& \quad \left. 2 \delta < d < 2 \sqrt{2} \sqrt{\delta^2} \ \&\& df > 0 \ \&\& \delta > 0 \right] \} \} ;
\end{aligned}$$

```

inner[[1]](* the first case of the closed
form solution of [x,y] position and Δr (dr) *)

inner[[2]](* the second case of the closed
form solution of [x,y] position and Δr (dr) *)

inner[[3]](* the third case of the closed
form solution of [x,y] position and Δr (dr) *)

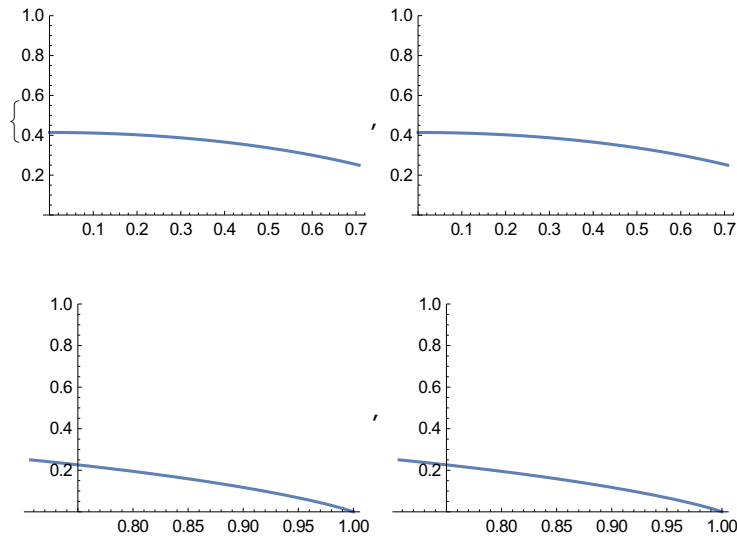
inner[[4]](* the fourth case of the closed
form solution of [x,y] position and Δr (dr) *)

(* we represent Δr (dr) w.r.t d, δ and df. We make use of the observation that
the extrema only occurs at df = 0. As such we limit df
to a very small value in the visualizaton. *)

pdr = dr /. inner;

```

```
(* We plot  $\Delta_r$  with respect to  $\delta$  for visualization.*)
Plot[#, { $\delta$ , 0.00001, 1}, PlotRange -> {0, 1}] & /@ {pdr /. {d -> 2, df -> 0.00000001}}
```



(\*The function of  $\Delta_r = f(\delta)$  can be obtained via grouping plot 1 and plot 3 (plot 2 and plot 4 are the other symmetric case)  
 In the plot, we can find that as  $\delta \rightarrow 0$  the maximum  $\Delta_r$  can be found. \*)  
 (\*Here we solve the outer maximization problem using built-in function in Mathematica (numerical method). Note that for  $\delta \rightarrow 0$ , we may encounter some numerical issue, since the two intersections points used in the formulation are reducing to one.  
 The results of numerical method are just for your reference, and later on we will show this result analytically \*)

```
Maximize[{#,  $\delta < df$ ,  $\delta < 0.7$ ] /. d -> 2, { $\delta$ , df} ∈ Rectangle[{0.0001, 0.0001}, {1, 1}]] & /@
Take[pdr, 2]
{{0.414184, { $\delta \rightarrow 0.0001$ , df -> 0.0001}}, {0.414184, { $\delta \rightarrow 0.0001$ , df -> 0.0001}}}
```

(\* To obtain the maximum  $\Delta_r$  theoretically,  
we calculate the first order and second order derivative for  $\delta$  in  $(0, d/2)$ .

We have the following results:

1. function  $\Delta_r = f(\delta)$  is continue and differentiable for  $\delta$  in  $(0, d/2)$ .
2. first order derivative  $< 0$  for  $\delta$  in  $(0, d/2)$
3. As  $\delta \rightarrow 0$ , the limit of first order derivative approaches 0
4. second order derivative  $< 0$  for  $\delta$  in  $(0, d/2)$
5. function is continue at  $\delta = 0$ .

Given 1-5,

we can conclude that  $\Delta_r$  is monotonically decreasing for  $\delta$  in  $(0, d/2)$ . And at  $\delta = 0$ ,  $\Delta_r$  takes its maximum value.

\*)

(\*1.1 check the continuity at  $d = 2\sqrt{2}\delta$ . Although the  $\Delta_r$  derived  
by Mathematica is undefined at  $d =$

$2\sqrt{2}\delta$  (due to the Solve function in Mathematica  
has some limitation when dealing with geometry),

we can check the (left and right) limit of  $\Delta_r$  as  $d \rightarrow 2\sqrt{2}\delta$ .

On the other hand, the function evaluation at  $d = 2\sqrt{2}\delta$  is actually well defined

By applying the geometry replationship and using some off-the-shelf  
geometry solver, we can obtain when  $d = 2\sqrt{2}\delta$ ,

$$\Delta_r = \left( -4 + 2\sqrt{2} + \sqrt{33 - 20\sqrt{2}} \right) / 4 \approx 0.25.$$

which is consistent with the left limit and right limit. \*)

```
Limit[Take[pdr, {1}] /. {d -> 2}, δ -> √2 / 2, Direction -> 1]
TrueQ[Limit[Take[pdr, {1}] /. {d -> 2}, δ -> √2 / 2, Direction -> 1] ==
  Limit[Take[pdr, {3}] /. {d -> 2}, δ -> √2 / 2, Direction -> -1]]
{ConditionalExpression[ $\frac{1}{4(1+df)}$ ,
  (-4 + 2√2 + 2(-4 + √2)df - 4df² + √(33 - 20√2 + (96 - 52√2)df -
    16(-7 + 3√2)df² - 16(-4 + √2)df³ + 16df⁴)), df > 0]}
```

True

(\*1.2 check the continuity of first order derivative at  $d = 2\sqrt{2}\delta$  \*)

```
dri11 = Limit[D[pdr[[1]], δ], df -> 0];
```

```
dri31 = Limit[D[pdr[[3]], δ], df -> 0];
```

```
TrueQ[Limit[dri11 /. {d -> 2}, δ -> √2 / 2, Direction -> 1] ==
  Limit[dri31 /. {d -> 2}, δ -> √2 / 2, Direction -> -1]]
```

True

```
dri12 = D[dri11, δ];
```

```
dri32 = D[dri31, δ];
```

```
(*2. first order derivative <0 for delta in (0, d/2) *)
test[{ConditionalExpression[x_, cond_]}, var_] := Reduce[x < 0, var];
test[{Limit[dri11 /. d -> 2, df -> 0]}, δ]
test[{Limit[dri31 /. d -> 2, df -> 0]}, δ]
```

$$0 < \delta < \frac{1}{\sqrt{2}}$$

$$\frac{1}{\sqrt{2}} < \delta < 1$$

```
(*3. As δ → 0, the limit of first order derivative approaches 0*)
```

```
Limit[dri11 /. {d -> 2}, δ -> 0, Direction -> -1]
```

```
0
```

```
(*4. second order derivative < 0 delta in (0, d/2) *)
```

```
test[{dri12 /. d -> 2}, δ]
```

```
test[{dri32 /. d -> 2}, δ]
```

$$-\frac{1}{\sqrt{2}} < \delta < \frac{1}{\sqrt{2}}$$

$$-1 < \delta < -\frac{1}{\sqrt{2}} \quad || \quad \frac{1}{\sqrt{2}} < \delta < 1$$

```
(*5. check the continuity at δ = 0 to make sure there is no invalid extrema.
```

Specifically, for  $\delta = 0$ , we can easility get  $\Delta_R = \frac{\sqrt{2}-1}{2} \times d$ .

On the other hand, for  $\delta \rightarrow 0^+$ ,

we can use the following to calculate the limit

(plug in  $df = 0$ ). Note that  $\delta \rightarrow 0^+$  and  $\delta \rightarrow 0^-$  are two symmetric cases,

and the limits are the same. Therefore,

```
function δR = f(δ) is continue at δ = 0 *)
```

```
Limit[Take[pdr, {1}], δ -> 0, Direction -> -1]
```

$$\left\{ \text{ConditionalExpression}\left[-\frac{d}{2} - df + \sqrt{\frac{d^2}{2} + d df + df^2}, df > 0 \text{ \&\& } d > 0\right] \right\}$$