The Virial Theorem

In

Stellar Astrophysics

by

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To the kindness, wisdom, humanity, and memory of

D. Nelson Limber

and

Uco van Wijk

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Preface to the Pachart Edition

As Fred Hoyle has observed, most readers assume a preface is written first and thus contains the author's hopes and aspirations. In reality most prefaces are written after the fact and contain the authors' views of his accomplishments. So it is in this case and I am forced to observe that my own perception of the subject has deepened and sharpened the considerable respect I have always had for the virial theorem. A corollary aspect of this expanded perspective is an awareness of how much remains to be done. Thus by no means can I claim to have prepared here a complete and exhaustive discussion of the virial theorem; rather this effort should be viewed as a guided introduction, punctuated by a few examples. I can only hope that the reader will proceed with the attitude that this constitutes not an end in itself, but an establishment of a point of view that is useful in comprehending some of the aspects of the universe.

A second traditional role of a preface is to provide a vehicle for acknowledging the help and assistance the author received in the preparation of his work. In addition to the customary accolades for proof reading which in this instance go to George Sonneborn and Dr. John Faulkner, and manuscript preparation by Mrs. Delores Chambers, I feel happily compelled to heap praise upon the publisher.

It is not generally appreciated that there are only a few thousand astronomers in the United States and perhaps twice that number in the entire world. Only a small fraction of these could be expected to have an interest in such an apparently specialized subject. Thus the market for such a work compared to a similar effort in another domain of physical sciences such as Physics, Chemistry or Geology is miniscule. This situation has thereby forced virtually all contemporary thought in astrophysics into the various journals, which for economic reasons similar to those facing the would-be book publisher; find little room for contemplative or reflective thought. So it is a considerable surprise and great pleasure to find a publisher willing to put up with such problems and produce works of this type for the small but important audience that has need of them.

Lastly I would like to thank my family for trying to understand why anyone would write a book that won't make any money.

George W. Collins, II
The Ohio State University
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Preface to the Internet Edition

Not only might one comfortably ask "why one would write a book on this subject?", but one might further wonder why anyone would resurrect it from the past. My reasons revolve around the original reasons for writing the monograph in the first place. I have always regarded the virial theorem as extremely powerful in understanding problems of stellar astrophysics, but I have also found it to be poorly understood by many who study the subject. While it is obvious that the theorem has not changed in the quarter-century that has passed since I first wrote the monograph, pressures on curricula have reduced the exposure of students to the theorem even below that of the mid 20th century. So it does not seem unreasonable that I make it available to any who might learn from it. I would only ask that should readers find it helpful in their research, that they make the proper attribution should they employ its contents.

The original monograph was published by Pachart Press and had its origin in a time before modern word processors and so lacked many of the cosmetic niceties that can currently be generated. The equations were more difficult to read and sections difficult to emphasize. The format I chose then may seem a little archaic by today's standards and the referencing methods rather different from contemporary journals. However, I have elected to stay close to the original style simply as a matter of choice. Because some of the derivations were complicated and tedious, I elected to defer them to a "notes" section at the end of each chapter. I have kept those notes in this edition, but enlarged the type font so that they may be more easily followed. However, confusion arose in the main text between superscripts referring to references and entries in the notes sections. I have attempted to reduce that confusion by using italicized superscripts for referrals to the notes section. I have also added some references that appeared after the manuscript was originally prepared. These additions are in no-way meant to be exhaustive or complete. It is hoped that they are helpful. I have also corrected numerous typographical errors that survived in the original monograph, but again, the job is likely to be incomplete. Finally, the index was converted from the Pachart Edition by means of a page comparison table. Since such a table has an inherent one page error, the entries in the index could be off by a page. However, that should be close enough for the reader to find the appropriate reference.

I have elected to keep the original notation even though the Einstein summation convention has become common place and the vector-dyadic representation is slipping from common use. The reason is partly sentimental and largely not wishing to invest the time required to convert the equations. For similar reasons I have decided not to re-write the text even though I suspect it could be more clearly rendered. To the extent corrections have failed to be made or confusing text remains the fault is solely mine

Lastly, I would like to thank John Martin and Charlie Knox who helped me through the vagaries of the soft- and hardware necessary to reclaim the work from the original. Continuing thanks is due A.G. Pacholczyk for permitting the use of the old Copyright to allow the work to appear on the Internet.

George W. Collins, II
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Introduction

1. A Brief Historical Review

Although most students of physics will recognize the name of the viria1 theorem, few can state it correcet1y and even fewer appreciate its power. This is largely the result of its diverse development and somewhat obscure origin, for the viria1 theorem did not spring full blown in its present form but rather evolved from the studies of the kinetic theory of gases. One of the lasting achievements of 19th century physics was the development of a comprehensive theory of the behavior of confined gases which resulted in what is now known as thermodynamics and statistical mechanics. A brief, but impressive, account of this historical development can be found in "The Dynamical Theory of Gases" by Sir James Jeans[1] and in order to place the viria1 theorem in its proper prospective, it is worth recounting some of that history.

Largely inspired by the work of Carnot on heat engines, R. J. E. C1aussius began a long study of the mechanical nature of heat in 1851[2]. This study led him through twenty years to the formulation of what we can now see to be the earliest clear presentation of the viria1 theorem. On June 13, 1870, Claussius delivered a lecture before the Association for Natural and Medical Sciences of the Lower Rhine "On a Mechanical Theorem Applicable to Heat."[3] In giving this lecture, C1aussius stated the theorem as "*The mean vis viva of the system is equal to its viria1*."[4] In the 19th century, it was commonplace to assign a Latin name to any special characteristic of a system. Thus, as is known to all students of celestial mechanics the *vis viva* integral is in reality the total kinetic energy of the system. C1aussius also turned to the Latin word *virias* (the plural of vis) meaning forces to obtain his 'name' for the term involved in the second half of his theorem. This scalar quantity which he called the *viria1* can be represented in terms of the forces $\mathbf{F}_i$ acting on the system as $\frac{1}{2}\left\langle \sum_i \mathbf{F_i} \bullet \mathbf{r}_i \right\rangle$ and can be shown to be 1/2 the average potential energy of the system. So, in the more contemporary language of energy, C1aussius would have stated that the average kinetic energy is equal to 1/2 the average potential energy. Although the characteristic of the system C1aussius called the viria1 is no longer given much significance as a physical concept, the name has become attached to the theorem and its evolved forms.

Even though C1aussius' lecture was translated and published in Great Britain in a scant six weeks, the power of the theorem was slow in being recognized. This lack of recognition prompted James Clerk Maxwell four years later to observe that *"as in this country the importance of this theorem seems hardly to be appreciated, it may be as well to explain it a little* *more fully*."[5] Maxwell's observation is still appropriate over a century later and indeed serves as the "*raison d'etre*" for this book.

After the turn of the century the applications of the theorem became more varied and widespread. Lord Rayleigh formulated a generalization of the theorem in 1903[6] in which one can see the beginnings of the tensor viria1 theorem revived by Parker[7] and later so extensively developed by Chandrasekhar during the 1960's.[8] Poincare used a form of the viria1 theorem in 1911[9] to investigate the stability of structures in different cosmological theories. During the 1940's Paul Ledoux developed a variational form of the virial theorem to obtain pulsational periods for stars and investigate their stability.[10] Chandrasekhar and Fermi extended the virial theorem in 1953 to include the presence of magnetic fields[11]

At this point astute students of celestial mechanics will observe that the virial theorem can be obtained directly from Lagrange's Identity by simply averaging it over time and making a few statements concerning the stability of the system. Indeed, it is this derivation which is most often used to establish the virial theorem. Since Lagrange predates Claussius by a century, some comment is in order as to who has the better claim to the theorem.

In 1772 the Royal Academy of Sciences of Paris published J. L. Lagrange's "Essay on the Problem of Three Bodies."[12] In this essay he developed what can be interpreted as Lagrange's identity for three bodies. Of course terms such as "moment of inertia", "potential" and "kinetic energy" do not appear, but the basic mathematical formulation is present. It does appear that this remained a special case germane to the three-body problem until the winter of 1842-43 when Karl Jacobi generalized Lagrange's result to n-bodies. Jacobi's formulation closely parallels the present representation of Lagrange's identity including the relating of what will later be known as the virial of Claussius to the potential.[13] He continues on in the same chapter to develop the stability criterion for n-body systems which bears his name. It is indeed a very short step from this point to what is known as the Classical Virial Theorem. It is difficult to imagine that the contemporary Claussius was unaware of this work. However, there are some notable and important differences between the virial theorem of Claussius and that which can be deduced from Jacobi's formulation of Lagrange's identity. These differences are amplified by considering the state of physics during the last half of the 19th century. The passion for unification which pervaded 20th century physics was not extant in the time of Jacobi and Claussius. The study of heat and classical dynamics of gravitating systems were regarded as two very distinct disciplines. The formulation of statistical mechanics which now provides some measure of unity between the two had not been accomplished. The characterization of the properties of a gas in terms of its internal and kinetic energy had not yet been developed. The very fact that Claussius required a new term, the virial, for the theorem makes it clear that its relationship to the internal energy of the gas was not clear. In addition, although he makes use of time averages in deriving the theory, it is clear from the development that he expected these averages to be interpreted as phase or ensemble averages. It is this last point which provides a major distinction between the virial theorem of Claussius and that obtainable from Lagrange's Identity. The point is subtle and often overlooked today. Only if the system is ergodic (in the sense of obeying the ergodic theorem) are phase and time averages the same. We will return to this point later in some detail. Thus it is fair to say that although the dynamical foundation for the virial theorem existed well before Claussius' pronouncement, by demonstrating its applicability to thermodynamics he made a new and fundamental contribution to physics.

2. The Nature of the Theorem

By now the reader may have gotten some feeling for the wide ranging applicability of the virial theorem. Not only is it applicable to dynamical and thermodynamical systems, but we shall see that it can also be formulated to deal with relativistic (in the sense of special relativity) systems, systems with velocity dependent forces, viscous systems, systems exhibiting macroscopic motions such as rotation, systems with magnetic fields and even some systems which require general relativity for their description. Since the theorem represents a basic structural relationship that the system must obey, applying the Calculus of Variations to the theorem can be expected to provide information regarding its dynamical behavior and the way in which the presence of additional phenomena (e.g., rotation, magnetic fields, etc.) affect that behavior.

Let us then prepare to examine why this theorem can provide information concerning systems whose complete analysis may defy description. Within the framework of classical mechanics, most of the systems I mentioned above can be described by solving the force equations representing the system. These equations can usually be obtained from the beautiful formalisms of Lagrange and Hamilton or from the Boltzmann transport equation. Unfortunately, those equations will, in general, be non-linear, second-order, vector differential equations which, exhibit closed form solutions only in special cases. Although additional cases may be solved numerically, insight into the behavior of systems in general is very difficult to obtain in this manner. However, the virial theorem generally deals in scalar quantities and usually is applied on a global scale. It is indeed this reduction in complexity from a vector description to a scalar one which enables us to solve the resulting equations. This reduction results in a concomitant loss of information and we cannot expect to obtain as complete a description of a physical system as would be possible from the solution of the force equations.

There are two ways of looking at the reason for this inability to ascertain the complete physical structure of a system from energy considerations alone. First, the number of separate scalar equations one has at his disposal is fewer in the energy approach than in the force approach. That is, the energy considerations yield equations involving only energies or 'energy-like' scalars while the force equations, being vector equations, yield at least three separate 'component' equations which in turn will behave as coupled scalar equations. One might sum up this argument by simply saying that there is more information contained in a vector than in a scalar.

The second method of looking at the problem is to note that energies are normally first integrals of forces. Thus the equations we shall be primarily concerned with are related to the first integral of the defining differential force equations. The integration of a function leads to a loss of 'information' about that function. That is, the detailed structure of the function over a discrete range is lumped into a single quantity known as integral of the function, and in doing so any knowledge of that detailed structure is lost. Therefore, since the process of integration results in a loss of information, we cannot expect the energy equation (representing the first integral of the force equation) to yield as complete a picture of the system as would the solution of the force equations themselves. However, this loss of detailed structure is somewhat compensated for, firstly by being able to solve the resulting equations due to their greater simplicity, and secondly, by being able to consider more difficult problems whose formulation in detail is at present beyond the scope of contemporary physics.

3. The Scope and Structure of the Book

Any introduction to a book would be incomplete if it failed to delimit its scope. Initially one might wonder at such an extensive discussion of a single theorem. In reality it is not possible to cover in a single text all of the diverse applications and implications of this theorem. All areas of physical science in which the concepts of force and energy are important are touched by the virial theorem. Even within the more restricted study of astronomy, the virial theorem finds applications in the dust and gas of interstellar space as well as cosmological considerations of the universe as a whole. Restriction of this investigation to stars and stellar systems would admit discussions concerning the stability of clusters, galaxies and clusters of galaxies which could in themselves fill many separate volumes. Thus, we shall primarily concern ourselves with the application of the virial theorem to the astrophysics of stars and star-like objects. Indeed, since research into these objects is still an open and aggressively pursued subject, I shall not even be able to guarantee that this treatment is complete and comprehensive. Since, as I have already noted, the virial theorem does not by its very nature provide a complete description of a physical system but rather extensive insight into its behavior, let me hope that this same spirit of incisive investigation will pervade the rest of this work.

With regard to the organization and structure of what follows, let me emphasize that this is a book for students - young and old. To that end, I have endeavored to avoid such phrases as "it can easily be shown that.....", or others designed to extol the intellect of the author at the expense of the reader. Thus, in an attempt to clarify the development I have included most of the algebraic steps of the development. The active professional or well prepared student may skip many of these steps without losing content or continuity. The skeptic will wish to read them all. However, in order not to burden the casual reader, the more tedious algebra has been relegated to notes at the end of each chapter. Each chapter of the book has been subdivided into sections (as has the introduction), which represent a particular logically cohesive unit. At the end of each chapter, I have chosen to provide a brief summary of what I feel constitutes the major thread of that chapter. A comfortable rapport with the content of these summaries may encourage the reader in the belief that he is understanding what the author intended.

References

1 Jeans, J. H. 1925 *Dynamical Theory of Gases*, Cambridge University Press, London, p. 11.

2 Claussius, R. J. E. 1851. Phil. Mag. S. 4 Vol. 2, pp. 1-21, 102-119.
Claussius, R. J. E. 1856. Phil. Mag. S. 4, Vol. 12, p. 81.
Claussius, R. J. E. 1862. Phil. Mag. S. 4, Vol. 24, pp. 81-97, 201-213.

3 Claussius, R. J. E. 1870. Phil. Mag. S. 4, Vol. 40, p. 122.

4 Claussius, R. J. E. 1870. Phil. Mag. S. 4, Vol. 40, p. 124.

5 Maxwell, J. C. 1874. *Scientific Papers. Vol. 2*, p. 410, Dover Publications, Inc., N. Y.

6 Rayleigh, L. 1903. *Scientific Papers. Vol. 4*, p. 491, Cambridge, England.

7 Parker, E. N. 1954. Phys. Rev. 96, pp. 1686-9.

8 Chandrasekhar, S., and Lebovitz, N.R., (1962) Ap.J. 136, pp. 1037-1047 and references therein.

9 Poincare, H. 1911. *Lectures on Cosmological Theories*, Hermann, Paris.

10 Ledoux, P. 1945, Ap. J. 102, pp. 134-153.

11 Chandrasekhar, S. and Fermi, E. 1953, Ap. J. l18, p.116.

12 Lagrange, J. L. 1873. *Oeuvres de Lagrange* ed: M. J.-A. Serret Gauthier-Villars, Paris, pp. 240, 241.

13 Jacobi, C. G. J. 1889. *Varlesungen uber Dynamik* ed: A. Clebsch G. Reimer, Berlin, pp. 18-22.