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ROUND III

Questions

Organized by

GRAMOLY

Instructions

1. There would be 4 subjective questions worth 10 points each.
2. Time duration is 36 hours. Beginning from 12:00 of 4th March (GMT+5.5) and ending on 23:59 of 5th March (GMT+5.5).
3. You have to submit each problem as a single pdf which could be either handwritten or LaTeX-ed.
4. There is a different upload option for the problem involving programming, so kindly it upload there separately.
5. Since it is a subjective exam, your thought process and elegance of method will be used for grading.
6. For the same reason as above, you need to show every step of your work, and state any approximations used, etc.
7. Discussion and plagiarism are prohibited. It will result in immediate disqualification.
8. You can only use desmos, Wikipedia but reference them.
9. Any use of other resources not limited to books or advanced calculators like Wolfram alpha is not allowed or needed.
10. Any sort of suspected plagiarisation will result in disqualification.
11. For the problem that involves programming, you may use any local IDE, you must share the code that you used.
12. Incase you want to request for a clarification in a problem. Submit your clarification **here**.
13. If there is a legitimate error in the problem, we will update it **here**. Please keep checking it regularly.
14. After completing the paper submit it at **https://r3-fizika.gramoly.org/submit**

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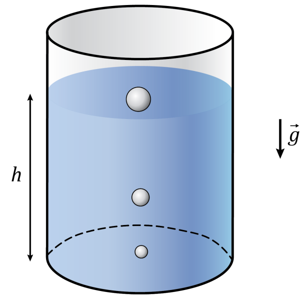
Problem 1

A. Physics of a carbonated drink

Carbonated bottled drinks contain dissolved carbon dioxide CO_2, which tends to be in the gaseous phase under normal condition. When you pour a carbonated drink into a glass, you can observe numerous small bubbles rising from the bottom. Those bubbles grow in size as they ascend and their speed changes as they travel upward.

A1. **0.5pts.**

Consider a bubble as a perfect sphere with initial radius $r_0 = 2.0 \cdot 10^{-4} m$ located at the bottom of a glass filled with fluid of density $\rho_f = 1.0 \cdot 10^3 kg/m^3$ up to the height $h = 0.2\ m$. When the bubble reaches the surface between fluid and air, its radius increases two times from initial size near bottom of the glass.



Estimate amount of energy ΔE dissipated due to viscous effects during ascending of that bubble. For calculations use acceleration due to gravity as $g = 9.8\ m/s^2$. Assume that surface tension effects are negligibly small

A2. **0.2pts.**

Estimate velocity v_h of the bubble at the distance $h = 0.2\ m$ from bottom of the glass
For calculations use dynamic viscosity of carbonated drink as $\eta_f = 1.0 \cdot 10^{-3} Pa \cdot s$

A3. **1.3pts.**

Estimate acceleration a_h of the bubble at the distance $h = 0.2\ m$ from bottom of the glass. Assume that surface tension effects are negligibly small.

B. Sonoluminescence

Sonoluminescence is a wonderful phenomena where an external acoustic wave causes disturbances in the fluid, leading to the growth and rapid collapse of bubbles in the liquid. During that bubble collapse, content of the bubble become heated to extremely high temperatures, which causes excitation of the gas with emission of light.

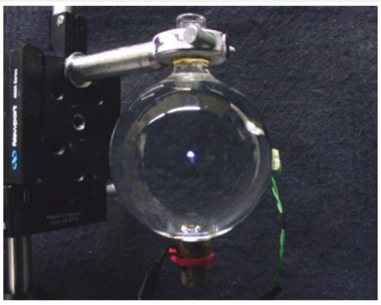
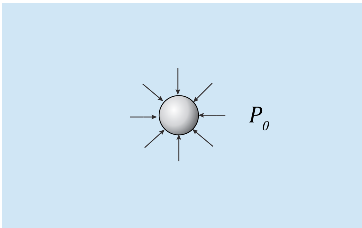


Photo of single bubble sonoluminescence (light blue spot in the center of glass sphere filled with fluid). Credit to Suslick K.

In this section of the problem you will have an opportunity to describe some key aspects of the sonoluminescence process

B1. **1.2pts.**

Consider a large volume of incompressible fluid of density ρ, which is initially at rest. That container with fluid has a small spherical cavity (an empty space without gas or fluid) of initial radius R_0, which starts to collapse due to an ambient fluid pressure P_0.

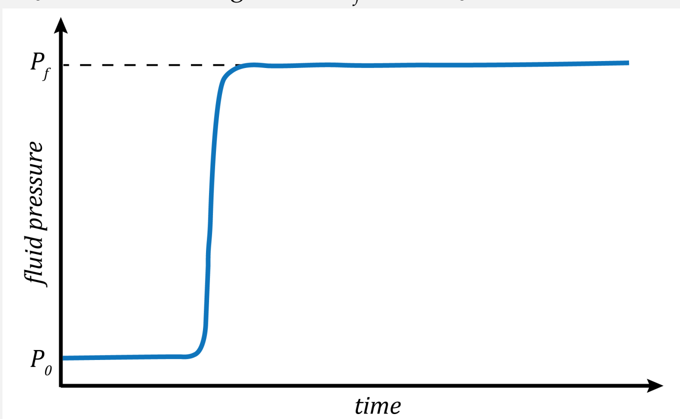


Evaluate velocity v_1 of the boundary of cavity when radius of the bubble would decrease to some value R_1. Assume that for such small size of the cavity surface tension and viscous effects can be neglected

As was shown in the previous section velocity of the boundary of an empty cavity would tend to increase to infinity during collapse of the cavity at small radius $R \to 0$. Thus, modeling of an empty cavity is unrealistic, which means that bubble in fluid has to be filled with some gas or vapor. This would create a counterbalance pressure inside the bubble, allowing it to bounce back at some small non-zero radius

* * *

Sonoluminescence effect occurs when changes in ambient pressure of fluid cause a bubble with noble gas to shrink, which increases temperature of the gas significantly. Usually in experiments with sonoluminescence, fluid pressure is varied as a harmonic function of time. However, here for simplicity, will consider only a rough approximation of the process, with one step of such variations in pressure, when fluid pressure is increased very fast from initial value P_0 to some new high value $P_f = 100P_0$



For the next few questions consider a bubble, which initially has a stationary radius $R_0 = 1.0 \cdot 10^{-4}\ m$, being filled with a noble gas xenon at temperature $T_0 = 300.0\ K$ and pressure P_0. Shrinkage of bubble due to increased ambient fluid pressure can be accurately modeled with adiabatic process, which can be described with adiabatic constant γ:

$$\gamma = \frac{C_p}{C_v} = \frac{5}{3}$$

where C_p and C_v are molar heat capacities of xenon at constant pressure and constant volume respectively.

B2. **1.2pts.**

Determine minimal radius of the bubble R_{min} due to increase in ambient fluid pressure from P_0 to $P_f = 100P_0$. Assume that surface tension and viscous friction effects can be neglected. Also for simplicity assume that presence of water vapor inside the bubble is negligibly small
Note: For your calculations, you are not expected to use manual iterative approach for solving transcendental equations. Instead it could be easier to use calculations in some spreadsheet such as MS Excel, or write a simple script for plotting a complicated function

B3. **0.3pts.**

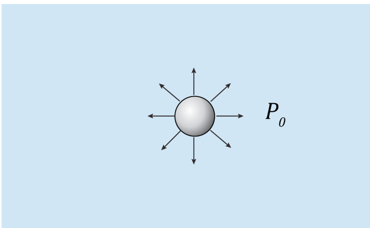
Evaluate maximum temperature T_{max} of the noble gas inside cavity during its shrinkage.

Stability

It is interesting that a lot of processes for bubbles differ based on the initial equilibrium size of the bubble. If radius of the bubble is larger than certain critical value R_c , then after small disturbances in ambient fluid pressure, the bubble will start growing uncontrollably. Otherwise if initial equilibrium radius of cavity is smaller than certain threshold, it will be stable to small variations of pressure

B4. **1.2pts.**

Consider a small bubble with radius $R_0 = 5.0 \cdot 10^{-6} m$, which is held at pressure $P_0 = 10^5\ Pa$ in equilibrium with distilled water. Then fluid pressure was disturbed so that bubble expanded to radius R.



Calculate critical radius R_c for this system, if entire process can be treated as isothermal, with vapor pressure at that temperature equal to $P_v = 0.5 \cdot 10^5\ Pa$. Coefficient of surface tension of water is $\sigma = 7.2 \cdot 10^{-2} N/m$

Small oscillations

In the next questions, assume that radius of the bubble is less than critical value R_c unless otherwise stated. For those smaller bubbles, minor disturbances in ambient fluid pressure would cause small harmonic oscillations with frequency ω. That frequency can be decomposed at three components as

$$\omega^2 = \omega_p^2 + \omega_\sigma^2 - \omega_\eta^2$$

where ω_p is natural frequency of the bubble inside liquid, neglecting effects of surface tension and attenuation due to viscosity. While ω_σ and ω_η are additional correction terms associated with surface tension effects and viscosity respectively.

B6. **1.2pts.**

Evaluate frequency ω_p of small harmonic oscillations of the bubble of radius $R_0 = 1.0 \cdot 10^{-4} m$ assuming that surface tension and viscosity effects can be neglected. Consider that oscillating bubble is filled with xenon, which has adiabatic constant $\gamma = 5/3$. The bubble is floating inside distilled water of density $\rho = 1.0 \cdot 10^3 kg/m^3$. Assume that during those oscillations fluid pressure P_f is always close to initial pressure P_0 inside bubble:

$$P_f \approx P_0 = 1.0 \cdot 10^5 Pa$$

For small $z \ll 1$ can be used following approximation

$$(1-z)^n \approx 1 - nz + \frac{n(n-1)z^2}{2}$$

B7. **0.2pts.**

It is known that for larger bubbles with initial radius $R_0 = 1.0 \cdot 10^{-4} m$ ratio between components of frequencies related to surface tension ω_σ to the frequency ω_p calculated with neglecting surface tension and viscous effects is

$$z_{\sigma 0} = \frac{\omega_{\sigma 0}}{\omega_{p0}} = 0.1$$

Determine ratio $z_{\sigma 1} = \omega_{\sigma 1}/\omega_{p1}$ for a smaller bubble with radius $R_1 = R_0/10$

B8. **0.3pts.**

It is known that for bubble with radius $R_0 = 1.0 \cdot 10^{-4} m$ ratio between components of frequency related to viscous forces ω_η to frequency ω_p calculated by neglecting surface tension and viscous effects is

$$z_{\eta 0} = \frac{\omega_{\eta 0}}{\omega_{p0}} = 4.0 \cdot 10^{-4}$$

Estimate similar ratio $z_{\eta 1} = \frac{\omega_{\eta 1}}{\omega_{p1}}$ for bubble with smaller radius $R_1 = R_0/10$

Forces due to variable pressure

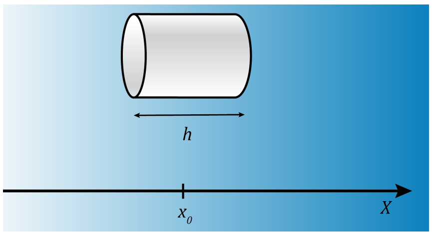
In case when pressure is spatially varied, effective force applied on the object inside such pressure field would be non-zero. For example, hydro static pressure varies with depth, creating buoyancy force in the vertical direction. In this section we will describe horizontal forces exerted on bubbles in fluid in the presence of variable pressure

B9. **0.2pts.**

Consider a small cylinder with a height h and cross-sectional area is S placed inside a fluid, which has spatially varied pressure P characterized with coordinate x as

$$P = P_0(1 + c_1 x^2)$$

where c_1 is some known constant



Center of the cylinder is located at the coordinate $x_0 \gg h$, while main axis of the cylinder is parallel to the X axis. Evaluate effective horizontal force F_x exerted on this cylinder from fluid

B10. **0.5pts.**

A rigid sphere of radius R is placed inside a fluid, with variable pressure, which changes with horizontal coordinate x as

$$P = P_0(1 + c_1x^2)$$

where c_1 is some known constant
Center of the sphere is located at coordinate $x_0 \ll R$. Determine effective horizontal force F_x applied to the sphere from the surrounding fluid

B11. **0.2pts.**

Now consider a similar case with a rigid sphere of radius R placed inside fluid, where pressure P_f changes with horizontal coordinate x and time t as

$$P_f = P_0 + c_2 \sin kx \cos \omega t$$

where P_0, c_2, k and ω are known constants
Evaluate average effective horizontal force $\langle F_x \rangle$ exerted at the sphere, if known that radius of the sphere is much smaller than parameter $1/k$

B12. **0.3pts.**

Now consider an oscillating bubble in a standing wave, with fluid pressure described as a function of horizontal coordinate x and time t as

$$P = P_0 + c_2 \sin kx \cos \omega t$$

where P_0, c_2, k, ω and ϕ are some positive constants
It can be shown that in such variable pressure, radius of the bubble R would change as a function of time as

$$R = R_0 - \delta \sin kx \cos(\omega t + \phi)$$

where R_0 and δ are some positive constants
Let's call bubbles with size larger than critical radius R_c as "Large" and bubbles with radii smaller than R_c as "Small". It can be shown that parameter ϕ is equal 0 or π
Let ϕ_L and ϕ_S are coefficients ϕ for "Large" and "Small" bubbles respectively. Briefly justify in 1-2 sentences or in a few short equations, which values 0 or π should we use for parameters ϕ_L and ϕ_S

B13. **0.7pts.**

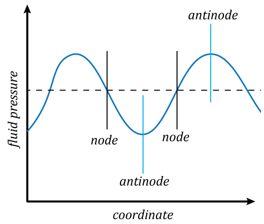
Consider bubble described in the question $B12$ with small oscillations such that $\delta \ll R_0$. Determine average horizontal force $\langle F_x \rangle$ exerted at this oscillating bubble

B14. **0.5pts.**

Consider that fluid pressure changes with time t and coordinate x as

$$P = P_0 + c_3 \sin kx \cos \omega t$$

This is an equation of a standing wave, which can be characterized with "nodes" and "antinodes"

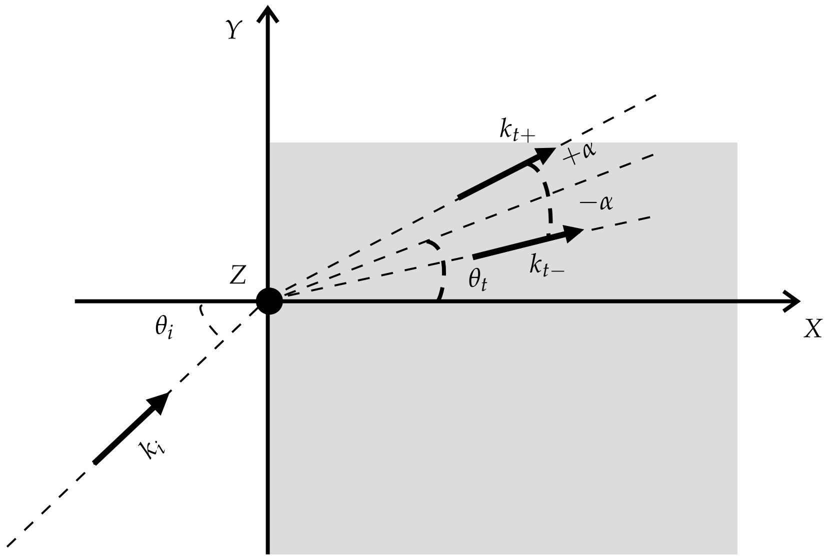


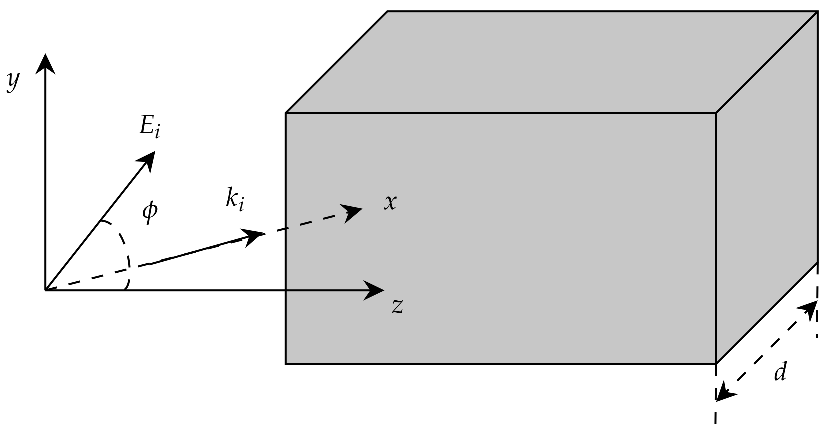
Describe briefly in $1-2$ sentences where would gather "Large" and "Small" bubble in that standing wave (either close to nodes or in the vicinity of antinodes)

Problem 2

A.

We have a hypothetical crystal, whose refractive index depends on both the polarisation and direction of propagation of an incoming EM wave.
Let us choose a Cartesian reference frame, where the boundary between the medium we are investigating and vacuum is $x = 0$ plane. The incident wave, lies in the xy plane, having a wave vector $\vec{k}_i$, making an angle θ_i with the $x-$ axis (As shown in the figure). Let us define two polarisations, S and P. Where S polarisation is when the wave is polarised perpendicular to the incident xy plane and P is the polarisation when the electric field of the wave lies in the xy plane. Let us assume two real and positive refractive indices for S and P waves η_S and η_P repectively. Without loss of generality assume $\eta_P > \eta_S$.
Say the incoming wave is linearly polarised confined in the xy plane, with its electric field making an angle $\frac{\pi}{4}$ with the z axis. This wave is a mixture of both S and P polarisation. It splits into two at different angles $\theta_{t\pm} = \theta_t \pm \alpha$, where $\vec{k}_{t+}$ corresponds to S, and $\vec{k}_{t-}$ corresponds to P.





A1. **1.5 pts.**

Find n_S and n_P in terms of θ_t, θ_i, α. Assume that $n_S = n_0 + \Delta n$ and $n_P = n_0 - \Delta n$, where $\Delta n / n_0 << 1$ and we only look for first-order terms in $\Delta n / n_0$.

Now assume the incidence to be normal, and the electric field still makes an angle $\pi/4$ with the z-axis (As can be seen in the figure). The crystal has a thickness $t >> \lambda$.

A2. **1 pts.**

Find the values of t for which the light exiting crystal is either circularly polarised or it is linearly polarised, but rotated by and angle $\pi/2$ with respect to the incident polarisation. You may neglect the difference between Reflection coefficients for S and P.

B.

Let us investigate a plane EM wave, of frequency ω travelling in a static uniform magnetic field $B_0\hat{z}$, where $\hat{z}$ is the unit vector z of the chosen Cartesian frame of reference. B_0 is much stronger than the field of the wave and the wave propagation is also along $\hat{z}$.
This medium contains N_e bound electrons per unit volume, each obeying the classical equation of motion.

$$e(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}) + m_e\omega^2 r = -m_e\ddot{\vec{r}}, \vec{v} = \frac{d\vec{r}}{dt} \tag{1}$$

Where m_e and $-e$ are the mass and charge of electron respectively.

B1. **4.5 pts.**

Prove that the wave propagation depends on polarisation, by calculating the refractive index for circular polarisation (either left handed or right).

Now consider the propagation of a linearly polarised in the same wave. Asuume the field at $z = 0$ to be given by $\vec{E}_i(z = 0, t) = E_i e^{-i\omega t}\hat{x}$, and a relatively weak magnetic field such that $\omega >> \omega_c$ and higher order terms of ω_c/ω can be neglected.

B2. **3 pts.**

Calculate the electric field at $z = l$, and find the angle by which the polarization has rotated.

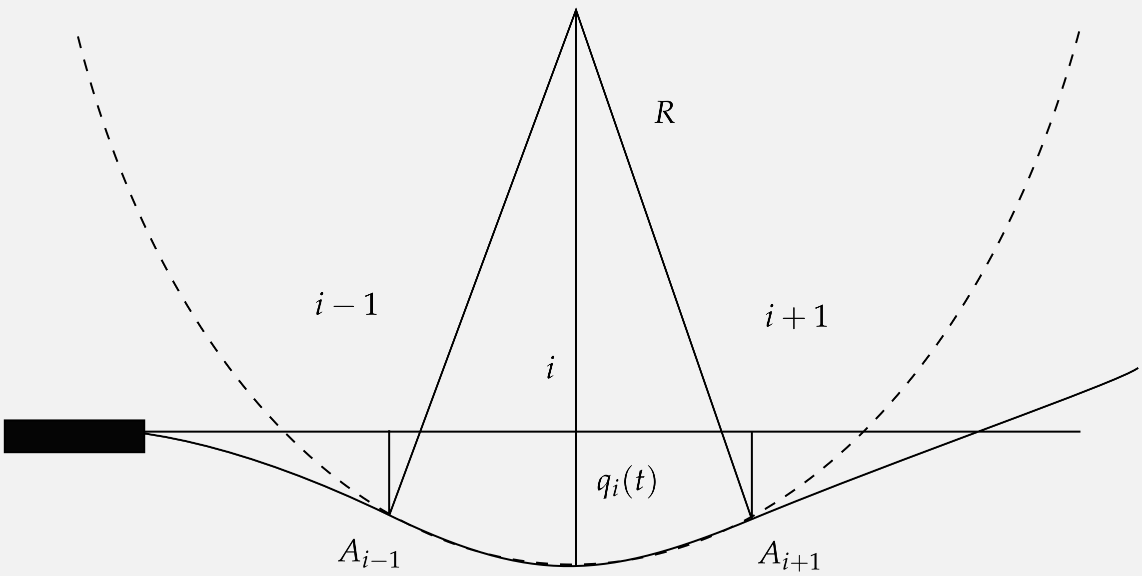
Problem 3

We have a thin, homogeneous knife of mass density μ, total mass M and length L, which is at rest in a horizontal position. This knife is deformed and a vibration ensues.

A. Analysis using a Discontinuous model

Say the knife consists of N segments each of length δ and mass $\mu\delta$. Let us denote the center of a segment i by A_i and p_i the deviation of A_i from the equilibrium position along the vertical. Also, assume that this deformation is "weak" so we can safely take the angle made by each segment with horizontal to be small. Under these imposed conditions, we can hypothesise that each segment has a fixed length (δ), equal to horizontal projection of deformation.

You may find the following information useful. The elastic energy per unit length of a knife which is bent can be expressed as $YI/2R_0^2$. Where Y is the Young's modulus, R_0 is the radius of curvature at the point. I is the so-called "elastic moment of inertia" equal to $st^2/12$, where s is the cross-sectional area and t is the thickness. Note that, the knife oscillates so fast the oscillations can be assumed adiabatic. Also, for simplicity we ignore the effects of gravity.



A1. **0.2 pts.**

Find the expression of the knife's kinetic energy.

A2. **0.3 pts.**

Find an approximate expression of R_i, the radius of circle passing through A_{i-1}, A_i, A_{i+1} in terms of the coordinates p_{i-1}, p_i, p_{i+1}

A3. **0.3 pts.**

Calculate the potential energy of the knife, neglecting any border effects.

A4. **0.2 pts.**

Write the Lagrangian of the knife and write the Euler Lagrange equations.

Extending this analysis to continuous Limits

Let $\Phi(x,t)$ be an interpolation function of the knife satisfying the condition $\Phi(i\delta, t) = p_i(t)\ \forall\ A_i$. Solve the remainder of the problem in the limit that $N \to \infty$, $\delta \to 0$ and $N\delta = L$

B1. **2 pts.**

Find a partial differential equation that is satisfied by $\Phi(x,t)$.

B2. **3 pts.**

What are the angular frequencies of the knife's oscillations.
Find solutions of the form $\Phi(x,t) = e^{-i\omega t}\Psi(x)$. Assume the following boundary conditions, the knife is fixed at one end, the other end is free, infinite radius of curvature i.e 0 second and third derivative.

B3. **0.2 pts.**

Give a reasonable estimate of the fundamental frequency of oscillations of a 1m long stainless steel knife of 3.5 mm thickness. Density of Steel 7850kg/m^3. Youngs modulus of steel $2.1 * 10^{11} Pa$

B4. **2.4 pts.**

Calculate the Lagrangian density for the system.

B5. **1.4 pts.**

Using the Euler Lagrange equations recover the wave equation we got before.

Problem 4

The following problem has two parts, writing the algorithm and then programming it.

10 pts.

Using the expression for the force between a charge and a metal sphere. First, write how one can compute the electrostatic force between two metal balls of radius r_1 and r_2, charged with charges q_1 and q_2 respectively, in free space, with the distance between their centres being d. (The actual computation may be tedious.)

Say this force is F Consider the following dimensionless quantities:

$$f = \frac{4\pi\epsilon_0 d^2}{{q_1}^2} \cdot F$$

$$u = \frac{r_1}{d}$$

$$v = \frac{r_2}{d}$$

$$p = \frac{q_1}{q_2}$$

Express all equations in the above procedure in terms of these dimensionless quantities.

Write a program in any language (Python, C, etc.) to compute f given u, v, p. The maximum uncertainty in f from the calculation must not be more than $\approx 1\%$ for $p \approx 1$ $u \approx v \approx 0.5$.