Controlling chaos in a network of oscillators

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We have extended the technique of stabilization of unstable periodic orbits to the case of spatially distributed networks of oscillators in a chaotic regime. The control is achieved via minute kicks to the variables of oscillators. These systems of many degrees of freedom exhibit high-dimensional chaos. We discuss the relevance of such control for cognitive processes.

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I. INTRODUCTION

Spatiotemporal coherent systems with many degrees of freedom appear in such diverse fields as, for example, hydrodynamics, chemistry, and model neural networks describing biological or artificial neurons. The role of such networks in information processing by a human brain has been recently stressed by Destexhe and Babloyantz [1,2]. They showed that a network of excitatory and inhibitory neurons under the influence of thalamic inputs exhibits behaviors akin to human sleep and wake cycles [3].

Thus it seems natural to relate the mechanisms leading to cognition and information processing to the dynamical properties of chaotic networks. The latter may be considered as an infinite reservoir of unstable periodic orbits. If some of these orbits could be stabilized as a result of the action of external or internal small perturbations, they could be thought of as coding devices for information processing. Thus a theory of cognition based on chaotic dynamics could be proposed.

A method for stabilization of unstable periodic orbits within a chaotic attractor was proposed by Ott, Grebogi, and Yorke (OGY) [4]. In this method the stabilization is achieved by submitting parameters of the system to *small* external perturbations.

The method of OGY has been applied to many theoretical as well as experimental systems [5,6]. However, these model systems always have few degrees of freedom. For example, Romeiras *et al.* [7] applied the OGY method to control chaos in a mechanical rotor with four degrees of freedom. Peng, Petrov, and Showalter [8] used the OGY method for stabilizing periodic oscillations in a chemical model exhibiting chaos. Other studies deal with the control of transient chaos [9] or chaos in a nonlinear oscillator model [10].

To our best knowledge the OGY method has only been used for stabilization of periodic orbits in systems with few degrees of freedom. Chaotic networks of biological or computational interest are usually extremely large. In this paper we have used the OGY for the stabilization of unstable periodic orbits in a network of moderate size.

Section II describes the spatially extended network. Section III summarizes the salient features of the OGY method. In Sec. IV, four unstable periodic orbits of the network are identified. Section V is devoted to the stabilization of these orbits by the OGY method. In Sec. VI, the information processing ability of the network is outlined.

II. THE MODEL

Let us consider a square network of $N \times N$ oscillating units. The equations describing the dynamics of the system are

$$\frac{dW_j}{dt} = W_j - (1+i\beta)|W_j|^2 W_j + (1+i\alpha) D \sum_k C_{jk} W_k \quad (j = 1, \ldots, N^2). \tag{1}$$

The constant parameters α, β, and D are real. Each oscillator of the network is described by a complex variable W_j and is labeled by a number j. In order to simplify the notations, the index j does not represent the Cartesian coordinates in the network, but designates a single oscillator. The matrix C_{jk} is defined as the *connectivity matrix* of the network. If only first-neighbors diffusive interaction is considered, the connectivity in a two-dimensional square network is expressed as

$$\sum_k C_{jk} W_k = W_{j_1} + W_{j_2} + W_{j_3} + W_{j_4} - 4 W_j,$$

where the indices j_1, j_2, j_3, and j_4 denote the four nearest neighbors of the unit j. The boundary conditions of the network are of the zero-flux type.

Equations (1) are generic as they constitute the normal form of an oscillatory network near a supercritical Hopf bifurcation [11]. These equations are analogous to the complex Ginzburg Landau equation derived for oscillating reaction-diffusion systems [11].

The parameters and the connectivity of the network are chosen such that, in the absence of control, the system exhibits chaotic activity. A way to achieve chaotic dynamics is to consider parameters for which the uniform oscillations of the network are unstable. The uniform oscillations of the network may be expressed analytically by

$$W_j(t) = \exp(-i\beta t + \phi_0) \;\; \forall\, j, \tag{2}$$

where $0 \leq \phi_0 < 2\pi$ is an arbitrary phase. One can show that the solution (2) of Eqs. (1) loses its stability if the following condition is fulfilled:

$$\frac{1+\alpha\beta}{1+\alpha^2} < D\left(\cos\frac{\pi}{N-1} - 1\right). \tag{3}$$

Note that condition (3) is analogous to the Benjamin-Feir instability condition, derived for continuous media [12].

Chaotic activity is generated in a network of 9×9 oscillators by the following procedure. For the values of the parameters $\alpha = -10$ and $\beta = 2$ in Eqs. (1), D is decreased such that Eq. (3) is satisfied. By decreasing D from the critical value $D = 2.5$, the uniform oscillations lose their stability and various nonuniform dynamical regimes appear. We first observe various periodic regimes, then quasiperiodic dynamics arise. Finally the onset of chaos is seen for $D = 1.9$. In the following, we fix the value of $D = 1.3$. As shown in Fig. 1, the broadband power spectrum of the variable $\mathrm{Re}W_{(3,3)}(t)$ indicates the presence of chaotic activity. The latter may also be observed with the help of a Poincaré section. As such a Poincaré section is essential in the following, let us define it more precisely.

In order to simplify the notations, let us rewrite Eqs. (1) in a general form,

$$\dot{\mathbf{x}} = \mathbf{f}_p(\mathbf{x}), \tag{4}$$

where $\mathbf{x} = (W_1, W_2, \ldots, W_{N^2})$ is the state vector and p denotes a control parameter, e.g., one of the parameters α, β, or D of Eqs. (1).

We denote by $\mathbf{\Phi}(\mathbf{x}_0, t)$ the dynamical flow, i.e., the general solution of Eq. (4) with initial condition $\mathbf{x}(0) = \mathbf{x}_0$. A Poincaré section is obtained by the intersection of the flow $\mathbf{\Phi}$ with a hyperplane satisfying the equation $\Pi \equiv \mathbf{a}\cdot\boldsymbol{\xi} = b$, where $\mathbf{a}$ is a vector and b a real number. On the hyperplane Π the Poincaré map is then defined as

$$\boldsymbol{\mathcal{P}} : \Pi \longrightarrow \Pi : \boldsymbol{\xi} \longmapsto \mathbf{\Phi}(\boldsymbol{\xi}, t_\Pi), \tag{5}$$

where t_Π is the shortest time interval such that $\mathbf{\Phi}(\boldsymbol{\xi}, t_\Pi) \in \Pi$ and $\dot{\mathbf{\Phi}}(\boldsymbol{\xi}, 0)\cdot\dot{\mathbf{\Phi}}(\boldsymbol{\xi}, t_\Pi) > 0$.

An example of such a Poincaré section is illustrated in Fig. 2. Due to the absence of a well-defined structure in this Poincaré section, we can guess that the chaotic attractor is not of low dimension.

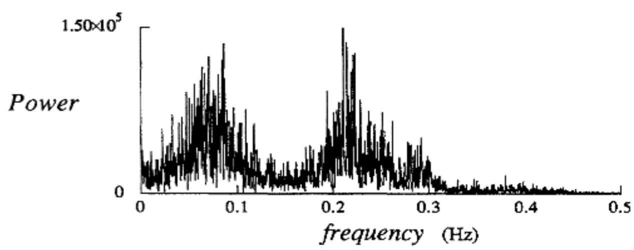

FIG. 1. Power spectrum of the variable $\mathrm{Re}W_{(3,3)}(t)$ of the oscillator at coordinate $(3,3)$ in the 9×9 network. Parameters of Eqs. (1) are $\alpha = -10, \beta = 2$, and $D = 1.3$.

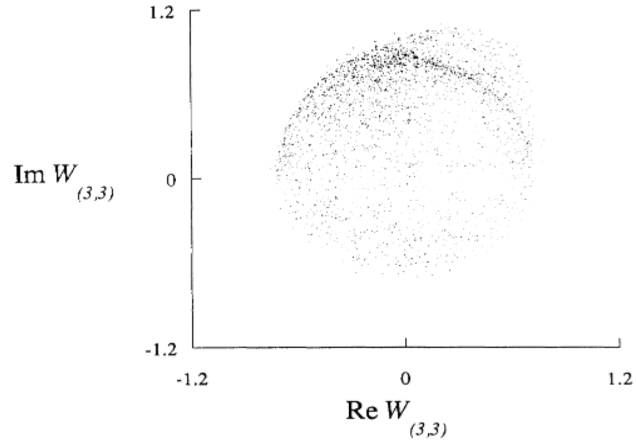

FIG. 2. Projection of the Poincaré section of the flow on the subspace $(\mathrm{Re}W_{(3,3)}, \mathrm{Im}W_{(3,3)})$. Parameters of Eq. (5) are $b = 0$ and $\mathbf{a} = (1, 1, \ldots, 1)$. Other parameter values are as in Fig. 1.

This guess is further confirmed when the fractal dimension of the chaotic attractor is estimated. We used the well-known algorithm of Grassberger-Procaccia [13] in order to estimate the correlation dimension D_2. However, the procedure did not converge for embedding dimensions up to 14. We conclude that the underlying chaotic attractor is not of low dimension.

In the next section we introduce a summary of the procedure developed by Ott, Grebogi, and Yorke [4] for the stabilization of unstable periodic orbits of a chaotic system.

III. THE OGY METHOD

Let us consider the Poincaré map $\boldsymbol{\mathcal{P}}(\boldsymbol{\xi}^{(n)})$ defined in Eq. (5), $\boldsymbol{\xi}^{(n)}$ representing the nth intersection in the phase space of the flow with a surface of section.

In a Poincaré map, a periodic orbit $\mathcal{C}(t) = \mathcal{C}(t+T)$, the solution of Eq. (4), is represented by a fixed point $\boldsymbol{\mathcal{P}}^k(\boldsymbol{\xi}_F) = \boldsymbol{\xi}_F$, for a given $k \geq 1$. Here we consider the simplest case, $k = 1$.

The Poincaré map may be linearized in the neighborhood of the fixed point, and we may write

$$\boldsymbol{\mathcal{P}}(\boldsymbol{\xi}) \approx \boldsymbol{\xi}_F + \mathbf{M}(\boldsymbol{\xi} - \boldsymbol{\xi}_F) \tag{6}$$

with the matrix $\mathbf{M} = \frac{\partial \boldsymbol{\mathcal{P}}(\boldsymbol{\xi}_F)}{\partial \boldsymbol{\xi}}$. The eigenvalues $\{\lambda_k\}$ of $\mathbf{M}$ determine the stability of the periodic orbit $\mathcal{C}(t)$. If there exists one eigenvalue with a modulus $|\lambda_u| > 1$, the periodic solution $\mathcal{C}(t)$ of Eq. (4) is unstable.

The OGY method considers a local control feedback as, in phase space, the Eq. (6) holds only locally. Assume there is only one unstable direction around the fixed point $\boldsymbol{\xi}_F$, corresponding to the eigenvector $\mathbf{u}$ and the eigenvalue λ_u of the transposed matrix $\mathbf{M}^T$. Then, the prescription of the OGY method [4] is to perform small perturbations δp around p according to the control feedback law,

$$\delta p^{(n)} = -\frac{\lambda_u}{1-\lambda_u}\frac{\mathbf{u}\cdot(\boldsymbol{\xi}^{(n)}-\boldsymbol{\xi}_F)}{\mathbf{u}\cdot\mathbf{g}}. \tag{7}$$

In this expression $\mathbf{g} = \frac{d\boldsymbol{\xi}_F}{dp}$ and the dot designates the scalar product. The feedback law (7) is applied only when $|\boldsymbol{\xi}^{(n)}-\boldsymbol{\xi}_F| < \delta\boldsymbol{\xi}^*$, where $\delta\boldsymbol{\xi}^*$ is a function of the maximum allowed variation $\delta p_{\max}$ of p (see Ref. [4]). The effect of the OGY procedure is to pull the state $\boldsymbol{\xi}^{(n)}$ towards the stable manifold of the fixed point $\boldsymbol{\xi}_F$.

In summary, the basic ingredients of the OGY method are (i) the localization of unstable fixed points of a Poincaré map associated with the system's dynamics and (ii) the computation of the unstable eigenvectors of the matrix $\mathbf{M}^T$ associated with the linearized Poincaré map around the fixed points.

As seen in this section, the stabilization of unstable periodic orbits by the OGY method necessitates their previous identification. In the next section we give two methods for identifying unstable periodic orbits of our network.

IV. UNSTABLE PERIODIC ORBITS OF THE NETWORK

In the present section we look for unstable fixed points of the Poincaré map $\boldsymbol{\mathcal{P}}$, corresponding to unstable periodic orbits of the flow. First, we could wonder whether there exist analytical solutions of Eqs. (1) which correspond to unstable periodic orbits. Indeed, as we saw, the only simple analytical solution of Eqs. (1) consists in uniform oscillations of the entire network, where each oscillator follows the periodic dynamics which would exist in the absence of coupling. The analytical expression of such an unstable periodic orbit $\mathcal{C}_0(t)$ is given by Eq. (2).

Other unstable periodic orbits could be found by numerical methods. We can proceed in two different ways. In both cases, the first step of the procedure is the same, and consists in analyzing the recurrence of the flow on the Poincaré section Π. Starting from a random initial condition $\boldsymbol{\xi}^{(0)}$, the distances $d^{(n)} = |\boldsymbol{\xi}^{(n)} - \boldsymbol{\mathcal{P}}(\boldsymbol{\xi}^{(n)})|$ between two successive iterates of the Poincaré map are scanned. When $d^{(n)}$ is lower than a fixed value ϵ, the vector $\boldsymbol{\xi}^{(n)}$ is recorded as an approximate value of a fixed point of the Poincaré map. After the scanning of a large number of iterates, a great number of approximate fixed points may be found. Then statistical methods may be invoked to elicit the precise position of several fixed points, as well as estimates of the corresponding matrix $\mathbf{M}$ defined in Eq. (6). Such an algorithm has been proposed by Auberbach *et al.* [14].

In our network, the statistical methods did not seem to furnish satisfactory results for the determination of fixed points and linearization of the Poincaré section, for a reasonable integration time. However, as in our case the differential equations describing the system are available, a method developed by Sparrow [15] seems more appropriate for the determination of the unstable periodic orbits.

The Sparrow method. Sparrow has proposed a method which yields the precise position of fixed points [15]. This method is based on the algorithm of Newton for finding roots of nonlinear equations. Given an approximate value $\boldsymbol{\xi}$ of the fixed point $\boldsymbol{\xi}_F$ and an approximate value τ of the period T, the method consists in linearizing the flow Φ associated with the dynamics of the network described by Eqs. (1). Therefore the following fixed-point equation:

$$\boldsymbol{\Phi}(\boldsymbol{\xi}_F, T) = \boldsymbol{\xi}_F \tag{8}$$

after linearization gives

$$\begin{gathered}[\mathbf{1}-\mathbf{A}(\tau)]\delta\boldsymbol{\xi} - \mathbf{f}_p(\boldsymbol{\xi}_1)\delta\tau = \boldsymbol{\xi}-\boldsymbol{\xi}_1,\\ \mathbf{a}\cdot\delta\boldsymbol{\xi} = 0,\end{gathered} \tag{9}$$

where we denote $\boldsymbol{\xi}_1 = \boldsymbol{\Phi}(\boldsymbol{\xi},\tau)$. The matrix $\mathbf{A}(\tau)$ is obtained by the simultaneous integration, in the time interval τ, of the following equations:

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{f}_p(\mathbf{x}),\\ \dot{\mathbf{A}} &= \tfrac{\partial \mathbf{f}_p(\mathbf{x}(t))}{\partial \mathbf{x}}\,\mathbf{A},\end{aligned}$$

with the initial condition $\mathbf{x}(0) = \boldsymbol{\xi}$ and $\mathbf{A}(0) = \mathbf{1}$.

Once Eq. (9) has been solved, the variations $\delta\boldsymbol{\xi}$ and $\delta\tau$ are added to the initial values $\boldsymbol{\xi}$ and τ, and a new iteration is performed. After a few iterations, the variables $\boldsymbol{\xi}$ and τ converge rapidly towards good approximations of $\boldsymbol{\xi}_F$ and T, even if the periodic orbit is unstable [15].

The eigenvalues $\{\lambda_k\}$ of the matrix $\mathbf{A}(T)$ are the Floquet multipliers of the periodic orbit $\mathcal{C}(t)$. The Floquet exponents $\{\mu_k\}$ of the orbit are related to the Floquet multipliers by the equation

$$\lambda_k = e^{\mu_k T}. \tag{10}$$

Finally, one can show that the Lyapunov exponents of the orbit $\mathcal{C}(t)$ coincide with the real part of the Floquet exponents

$$\ell_k = \mathrm{Re}\,\mu_k.$$

In the sequel $\mathbf{A}(T)$ will be denoted simply by $\mathbf{A}$ and $\mathbf{f}_p(\boldsymbol{\xi}_F)$ by $\mathbf{f}$.

Unstable periodic orbits. With the help of the Sparrow method, we identify four unstable periodic orbits in the following manner. Starting with random initial conditions for the network, and applying the procedures described above, a first unstable periodic orbit $\mathcal{C}_1$ can be identified. We did not find other unstable periodic orbits with random initial conditions.

However, other orbits could be found by choosing particular initial conditions which could be constructed in the following manner. We consider the eigenvectors of the connectivity matrix C_{jk}, i.e., the states $\mathbf{e}$ of the network such that the condition $\sum_k C_{jk}\, e_k = \sigma e_j$ is satisfied. Here σ is an eigenvalue of the connectivity matrix. In the case considered here, namely the diffusive connectivity with zero-flux boundary conditions, the expressions of the eigenvalues $\sigma_{(m,n)}$ and the associated eigenvectors $e_j^{(m,n)}$ are given by

$$e_j^{(m,n)} = \cos\left(\frac{m\pi}{N-1}k_1\right)\cos\left(\frac{n\pi}{N-1}k_2\right),$$
$$\sigma_{(m,n)} = 2\left(\cos\frac{m\pi}{N-1} - 1\right) + 2\left(\cos\frac{n\pi}{N-1} - 1\right),$$
$$m, n = 0, \ldots, N-1$$

where k_1 and k_2 represent the Cartesian coordinates of the oscillator j in the network. By taking initial conditions corresponding to specific values of (m,n), the Sparrow method furnishes an unstable periodic orbit. For example, the orbit $\mathcal{C}_2$ is obtained with $\mathrm{Re}W_j(0) = e_j^{(1,0)}, \mathrm{Im}W_j(0) = 0$, and another orbit $\mathcal{C}_3$ with $(m,n) = (1,1)$. For all values of (m,n) with m or n greater than 2, we could not find orbits topologically different from $\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2$, and $\mathcal{C}_3$.

Thus it seems that there are no other periodic orbits satisfying $\mathcal{P}(\boldsymbol{\xi}_F) = \boldsymbol{\xi}_F$. This does not preclude the presence of other orbits satisfying other conditions, e.g., $\boldsymbol{\mathcal{P}}^k(\boldsymbol{\xi}_F) = \boldsymbol{\xi}_F$, with $k > 1$.

We checked the validity of our computer program by retrieving the orbit $\mathcal{C}_0$ mentioned above [see Eq. (2)]. In this case, we could compare the values of the Lyapunov exponents computed numerically with their analytical expressions. A very satisfactory agreement was found.

The orbits $\mathcal{C}_1, \mathcal{C}_2$, and $\mathcal{C}_3$ exhibit spatial structures, represented in Fig. 3 at a given instant t. The orbit $\mathcal{C}_1$ corresponds in fact to a *rotating wave* of oscillatory activity around the central unit of the network. This dynamical feature is not represented here. Such a wave is characterized by the fact that the amplitude of the oscillator in the center of the network vanishes. This property is completely analogous to the presence of a "phase defect" generating a spiral wave in continuous oscillatory medium [16].

The orbits $\mathcal{C}_2$ and $\mathcal{C}_3$ correspond to *standing waves* of the network, possessing various symmetries. The orbit $\mathcal{C}_2$ is antisymmetric with respect to the reflection around a median—one segment joining the middle of two opposite sides of the square—and constant in the other perpendicular direction. The orbit $\mathcal{C}_3$ is invariant under reflections around the two diagonal axis of the square.

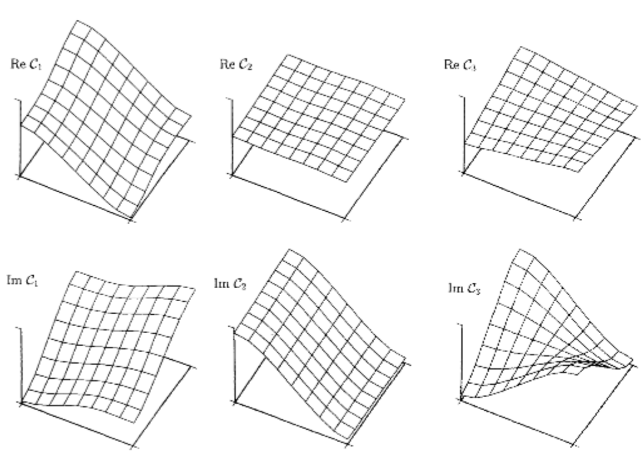

FIG. 3. Snapshots of network activity for the unstable periodic orbits $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$. The vertical axes are scaled by the same constant interval $[-1.2, 1.2]$. Parameter values are as in Fig. 1.

We note also that all the three orbits $\mathcal{C}_1, \mathcal{C}_2$, and $\mathcal{C}_3$ are associated with two unstable directions in the phase space. For $\mathcal{C}_2$, these directions correspond to two distinct real eigenvalues, whereas for $\mathcal{C}_1$ and $\mathcal{C}_3$ the instability is associated with complex conjugate eigenvalues.

In Table I we summarize the salient features of all unstable periodic orbits found in our network. These are characterized by the oscillatory period and the Floquet exponents with positive real parts. From Table I we note that the orbit $\mathcal{C}_1$ is characterized by the lowest Lyapunov exponents. Therefore it is not surprising that $\mathcal{C}_1$ is also the more recurrent unstable periodic orbit. This fact is visible in the power spectrum of Fig. 1, where the frequency of one peak coincides with the frequency of $\mathcal{C}_1$. Such an association does not seem to be present for the other unstable periodic orbits. In the next section we apply the OGY procedure to stabilize the unstable periodic orbits of Table I.

V. CONTROLLING CHAOS IN A NETWORK

After identifying a number of unstable periodic orbits of the network of oscillators described by Eqs. (1), we use the OGY method to stabilize these orbits. In particular, we will discuss the case of large Floquet multipliers associated with some of the periodic orbits.

In order to apply the OGY method we need the eigenvectors and eigenvalues of the matrix $\mathbf{M}$, introduced in Eq. (6). By linearizing Eq. (5) we can easily show the relation between the matrix $\mathbf{A}$ [Eq. (9)] computed with the Sparrow method and the matrix $\mathbf{M}$,

$$\mathbf{M} = \left(\mathbf{1} - \frac{\mathbf{f}\,\mathbf{a}\cdot}{\mathbf{a}\cdot\mathbf{f}}\right)\mathbf{A}.$$

Moreover, we can verify that if $\tilde{\mathbf{u}}$ is an eigenvector of $\mathbf{A}$, the vector $\mathbf{u} = (\mathbf{1} - \frac{\mathbf{f}\mathbf{a}\cdot}{\mathbf{a}\cdot\mathbf{f}})\,\tilde{\mathbf{u}}$ is an eigenvector of $\mathbf{M}$ with the same eigenvalue. Note that a convenient choice for $\mathbf{a}$ is to take $\mathbf{a} = \mathbf{f}$. As the matrix $\mathbf{A}$, and hence the eigenvectors $\tilde{\mathbf{u}}$, are determined by the Sparrow method, we are ready to apply the OGY procedure.

TABLE I. Characteristics of four unstable periodic orbits of Eqs. (1). T is the period of each orbit. The third column shows the positive Lyapunov exponents of these unstable periodic orbits. The fourth column gives the imaginary part of the corresponding Floquet exponents.

Orbits	T	$\ell_k = \mathrm{Re}\,\mu_k$	$\mathrm{Im}\,\mu_k$
$\mathcal{C}_0$	π	1.038	
$\mathcal{C}_1$	13.66	0.123	0.348
		0.123	- 0.348
$\mathcal{C}_2$	15.4	0.754	
		0.720	
$\mathcal{C}_3$	2.25	0.651	1.0
		0.651	- 1.0

Equation (7) assumes that there exists only one unstable direction. If this is not the case, it can be easily shown that m unstable directions will require m independent variations of different parameters and a straightforward generalization of Eq. (7) is possible [17]. In this way the unique Eq. (7) is replaced by a linear system of m equations.

We implemented the OGY method by perturbing the *variables* rather than the parameters of the system. This idea was suggested in [6] and, as it is shown below, this choice does not affect the basic principles of the method. Indeed, let us assume that there exist m independent vectors $\mathbf{v}_j$ along which we may perturb the state of the dynamical system. Then we construct a new Poincaré map $\boldsymbol{\mathcal{P}}_\epsilon$ by introducing m new parameters ϵ_j, such as

$$\boldsymbol{\mathcal{P}}_\epsilon(\boldsymbol{\xi}) = \boldsymbol{\mathcal{P}}(\boldsymbol{\xi} + \epsilon_1 \mathbf{v}_1 + \epsilon_2 \mathbf{v}_2 + \cdots + \epsilon_m \mathbf{v}_m).$$

In the case where the new parameters $\epsilon_j = 0$, we recover the original Poincaré section. Therefore, in the same spirit as in the OGY method, we can write the conditions which determine small variations of ϵ_j around zero, such that the system state is pulled onto the stable manifold of the fixed point $\boldsymbol{\xi}_F$. These conditions are expressed by the linear system

$$\left(\sum_{j=1}^{m} \mathbf{u}_k \cdot \mathbf{v}_j \right) \epsilon_j = \mathbf{u}_k \cdot (\boldsymbol{\xi}_F - \boldsymbol{\xi}), \tag{11}$$

where the vectors $\{\mathbf{u}_k\}$ represent the unstable directions of the matrix $\mathbf{M}^T$. The solvability condition of Eq. (11) may be stated as follows. The system (11) possesses a solution only if the orthogonal of the subspace generated by the vectors $\{\mathbf{v}_k\}$ contains no vectors $\{\mathbf{u}_k\}$. A simple choice corresponds to $\mathbf{v}_k = \mathbf{u}_k$.

We applied this procedure to the network of oscillators as described above and the periodic orbits $\mathcal{C}_1$ and $\mathcal{C}_3$ of Table I were stabilized, with the choice $\mathbf{v}_k = \mathbf{u}_k$ ($k = 1, 2$). Figure 4 shows the transition in time of the variable $\mathrm{Re}W_{(3,3)}(t)$ during the stablilization of $\mathcal{C}_1$. The control is performed under the form of microkicks applied periodically to the network, whenever the dynamical flow intersects the Poincaré section.

Due to the large amplitude of the greatest Floquet multiplier, $|\lambda_u| = e^{\ell_1 T} \sim 10^5$, associated with the largest Lyapunov exponent ℓ_1 of $\mathcal{C}_2$, the stabilization of this orbit is not easy to achieve. Indeed, a minute deviation from the unstable periodic orbit $\mathcal{C}_2$ is dramatically amplified over a period of time T. This difficulty can be avoided following different strategies.

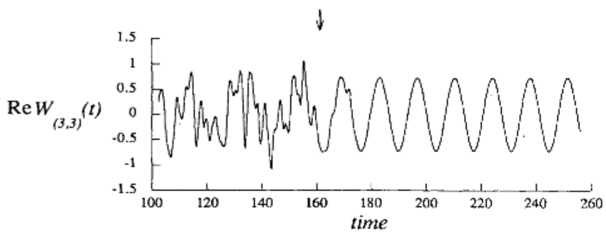

FIG. 4. $\mathrm{Re}W_{(3,3)}(t)$ before and after stabilization of periodic orbit $\mathcal{C}_1$. The arrow indicates the time at which the feedback control is applied for the first time.

A first method for reducing the effect of large Floquet multipliers is to consider several hyperplanes of section in the phase space, such that the Poincaré map $\boldsymbol{\mathcal{P}}$ is decomposed in several factors. For example, we may consider two hyperplanes, let $\Pi_1 = \Pi$ and $\Pi_2 \neq \Pi$, and two maps $\boldsymbol{\mathcal{P}}_1$ and $\boldsymbol{\mathcal{P}}_2$ defined by

$$\boldsymbol{\mathcal{P}}_1 : \Pi_1 \longrightarrow \Pi_2 : \boldsymbol{\xi}_1 \longmapsto \Phi(\boldsymbol{\xi}_1, t_{\Pi_2}),$$

$$\boldsymbol{\mathcal{P}}_2 : \Pi_2 \longrightarrow \Pi_1 : \boldsymbol{\xi}_2 \longmapsto \Phi(\boldsymbol{\xi}_2, t_{\Pi_1}),$$

such that $\boldsymbol{\mathcal{P}}(\boldsymbol{\xi}) = \boldsymbol{\mathcal{P}}_2(\boldsymbol{\mathcal{P}}_1(\boldsymbol{\xi}))$. In these equations the time intervals t_{Π_1} and t_{Π_2} are defined in a similar way as for Eq. (5). The generalization to more than two hyperplanes is obvious. Hence, the matrix $\mathbf{M}$ defined in Eq. (6) decomposes into $\mathbf{M} = \mathbf{M}_2\mathbf{M}_1$. We can define also $\boldsymbol{\xi}_{F1} = \boldsymbol{\xi}_F$, $\boldsymbol{\mathcal{P}}_1(\boldsymbol{\xi}_{F1}) = \boldsymbol{\xi}_{F2}$, and $\boldsymbol{\mathcal{P}}_2(\boldsymbol{\xi}_{F2}) = \boldsymbol{\xi}_{F1}$, and assume that there is only one unstable direction $\mathbf{u}_1$ around $\boldsymbol{\xi}_{F1}$ and $\mathbf{u}_2$ around $\boldsymbol{\xi}_{F2}$, such that $\mathbf{M}_1^T\mathbf{u}_1 = \lambda_1\mathbf{u}_2$, and $\mathbf{M}_2^T\mathbf{u}_2 = \lambda_2\mathbf{u}_1$, with $\lambda_u = \lambda_1\lambda_2$. With these notations, it is not difficult to derive a feedback law, analogous to Eq. (7), which proceeds in two steps

$$\begin{aligned} \delta p_1^{(n)} &= -\lambda_1 \frac{\mathbf{u}_1 \cdot (\boldsymbol{\xi}^{(n)} - \boldsymbol{\xi}_{F1})}{\mathbf{u}_2 \cdot \mathbf{g}_2 - \lambda_1 \mathbf{u}_1 \cdot \mathbf{g}_1}, \\ \delta p_2^{(n)} &= -\lambda_2 \frac{\mathbf{u}_2 \cdot (\boldsymbol{\xi}^{(n)} - \boldsymbol{\xi}_{F2})}{\mathbf{u}_1 \cdot \mathbf{g}_1 - \lambda_2 \mathbf{u}_2 \cdot \mathbf{g}_2}, \end{aligned} \tag{12}$$

where $\mathbf{g}_k = \frac{d\boldsymbol{\xi}_{Fk}}{dp}$ as previously. The advantage of considering the feedback in several steps, as in Eqs. (12), is that the value of the partial Floquet multiplier λ_k, associated to step k may be arbitrarily reduced when the number of steps increase. Indeed, if the total number of intermediate steps is l, we can evaluate $|\lambda_k| \sim |\lambda_u|^{1/l}$, which decreases with l.

Thus the OGY method may be applied in a situation where $|\lambda_u| \gg 1$, provided that the Poincaré section is decomposed in several factors.

In this paper we used another method for reducing the greatest Floquet multiplier associated with $\mathcal{C}_2$. The general idea of the method is as follows. Let us note T_1, the period of an unstable periodic orbit $\mathcal{C}(t)$. Suppose that there exists a matrix $H(t)$ such that $H(t)\mathcal{C}(t)$ is still a periodic function of period T_2. Then we may apply the OGY method, not directly on the variable $\mathbf{x}(t)$, but on the transformed variable $\mathbf{y}(t) = H(t)\mathbf{x}(t)$, in order to stabilize the unstable periodic orbit $H(t)\mathcal{C}(t)$. As the greatest Lyapunov exponents related to the variables $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are identical, Eq. (10) shows that the greatest Floquet multiplier of the periodic orbit, expressed in terms of the new variables, is proportional to $|\lambda_u|^{T_2/T_1}$. Therefore, if $T_2 < T_1$, the greatest Floquet multiplier associated with the periodic orbit $H(t)\mathcal{C}(t)$ can be reduced.

This method is most appropriate for the study of Eqs. (1). Indeed, in this case it can be shown that all periodic orbits of period T may be expressed under the form $W_j(t) = e^{i(2\pi/T)t}Z_j$, where Z_j is independent of

t [18]. Therefore, if we choose the function $H(t) = e^{i\omega t}$, the periodic orbit $\mathcal{C}(t)$ of period T_1 is transformed into a periodic orbit of period

$$T_2 = \frac{T_1}{1 + \frac{\omega T_1}{2\pi}}.$$

For example, in the case of the unstable periodic orbit $\mathcal{C}_2$, $T_1 = 15.4$. By choosing $\omega = \beta = 2$, we get $T_2 = 2.61$ and the greatest Floquet multiplier of the orbit $e^{i\beta t}\mathcal{C}_2$ is of order 1. Thus, with this method, we could stabilize the unstable periodic orbit $\mathcal{C}_2$.

VI. CONCLUSION

In this paper we have shown that the technique developed by Ott, Grebogi, and Yorke could be applied to spatially distributed networks of moderate size, exhibiting chaos which is not of low dimension. In systems with few degrees of freedom the OGY method necessitates the perturbation of a single parameter if the unstable periodic orbit possesses only one positive Lyapunov exponent. In our system, the existence of two positive Lyapunov exponents implies that two independent parameters must be perturbed. We implemented the OGY control procedure by submitting the network to two independent periodic kicks of very small amplitude. The kicks were applied to the variables instead of the usual procedure where the parameters are subject to fluctuations. We discussed also how to stabilize a periodic orbit with large Floquet multipliers.

The possibility of identification and stabilization of periodic orbits out of a spatiotemporal chaotic reservoir opens the way for a possible explanation of cognitive processes. Indeed, if the cortical activity generates spatiotemporal chaos, it is a reservoir of unstable periodic orbits. One could think of this infinite number of periodic orbits as infinite ways of coding information. The interest of the technique is that the stabilization is achieved via very small internal or external perturbations. The information transfer does not necessitate appreciable change in the network. This point is of importance in the case of cortical networks, as one needs not to invoke costly metabolic changes of long duration.

The learning and retrieval of information from cortical networks could be understood in the following manner. Let us assume that a perturbation of a given nature and magnitude is able to stabilize an orbit of well-defined order and frequency. Moreover, such an orbit is absent in the reservoir of a nonlearned network. The learning process changes the network connectivity according to a given learning rule, for example Hebbian learning rules. Only in this "learned" network, orbits appear which could be stabilized by a small perturbation which triggers the process of recognition and memory.

The stabilization of unstable orbits in a chaotic neural network could also be of computational value. Moreover, as in principle the number of periodic orbits even in a small network is infinite, a coding device which uses such orbits will have a very large capacity. In a separate paper [19] we have shown how a chaotic categorizer may be constructed with the help of the results achieved in this paper.

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