Splitting Problems for Vector Bundles. Part III

Splitting algebraic vector bundles

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Let T be a paracompact topological space (e.g., a CW complex). Then there are natural bijections

$$\mathcal{V}_r^{\mathbb{R}}(T) \cong [T, Gr_r(\mathbb{R})]$$

and

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Corollary

Let T be a paracompact topological space (e.g., a CW complex). Then the maps

$$\mathcal{V}_r^{\mathbb{K}}(T) \to \mathcal{V}_r^{\mathbb{K}}(T \times [0,1])$$

are bijections for all $r \geq 0$ and $\mathbb{K} = \mathbb{R}, \mathbb{C}$.

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Theorem (Quillen, Suslin, 1976)

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Bass-Quillen conjecture (1972/1976)

Let X be a regular Noetherian affine scheme of finite Krull dimension. Then the maps

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The category obtained in this form is called

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Caveat: This map is not bijective in general!

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Let S be either $Spec(k)$ for some field k or $Spec(\mathbb{Z})$ and $r \geq 0$. Then there are natural bijections

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- Asok-Hoyois-Wendt (2015): as above (in essence, theorem is equivalent to the Bass-Quillen conjecture for all **affine** $X \in Sm_S$)

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When does an algebraic vector bundle $\mathcal{E}$ over X of rank r admit an isomorphism $\mathcal{E} \cong \mathcal{E}' \oplus \underline{1}$ for some vector bundle $\mathcal{E}'$?

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In other words: When does $[\mathcal{E}]$ lie in the image of the map $s_{r-1} : \mathcal{V}_{r-1}(X) \to \mathcal{V}_r(X), [\mathcal{E}'] \mapsto [\mathcal{E}' \oplus \underline{1}]$?

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The inclusion $\xi_{r-1} : Gr_{r-1} \hookrightarrow Gr_r$ induces a commutative diagram

$$\begin{tikzcd}
\mathcal{V}_{r-1}(X) \arrow[r, "s_{r-1}"] \arrow[d, "\cong"] & \mathcal{V}_r(X) \arrow[d, "\cong"] \\
{[X, Gr_{r-1}]} \arrow[r, "(\xi_{r-1})_*"] & {[X, Gr_r]}.
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Each of these simpler lifting problems can be reduced to studying cohomology.

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The morphism ξ_{r-1} fits into a fiber sequence

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Using the motivic Moore-Postnikov factorization, one obtains for $i \geq 0$ cohomological obstructions

$$o_i \in H^{i+1}_{Nis}(X, \pi_i^{\mathbb{A}^1}(\mathbb{A}^r_k \setminus 0, *)(\det \theta))$$

for a lift of $\theta : X \to Gr_r$ along ξ_{r-1}.

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Theorem (Morel, 2006)

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- for $i < d$ by Morel's computations above.

$\square$

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Motivic Euler classes

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For $r \geq 2$, one has $\pi_i^{\mathbb{A}^1}(\mathbb{A}_k^r \setminus 0, *) \cong \begin{cases} 0 & \text{if } i \leq r-2 \\ \mathbf{K}_r^{MW} & \text{if } i = r-1. \end{cases}$

Theorem (Morel, 2006)

Let X be a smooth affine k-scheme of dimension $d \geq 2$. A vector bundle $\mathcal{E}$ over X of rank d admits a decomposition $\mathcal{E} \cong \mathcal{E}' \oplus \underline{1}$ if and only if the **Euler class** $o_{d-1} \in H^d_{Nis}(X, \mathbf{K}_d^{MW}(\det\theta))$ is zero.

Proof.

$H^{i+1}_{Nis}(X, \pi_i^{\mathbb{A}^1}(\mathbb{A}_k^d \setminus 0, *)(\det\theta)) = 0$

- for $i \geq d$ by dimension reasons

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Proof.

$H_{Nis}^{i+1}(X, \pi_i^{\mathbb{A}^1}(\mathbb{A}_k^d \setminus 0, *)(\det\theta)) = 0$

- for $i \geq d$ by dimension reasons
- for $i < d-1$ by Morel's computations.

$\square$

Thank you!