SSV control

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1 Introduction

2 Kinematic model

Consider a skid-steering vehicle (SSV) moving on a plane, the vehicle configuration is defined by

$$\mathbf{q} = \begin{bmatrix} x & y & \theta \end{bmatrix}^T$$

where x, y represent the position of the vehicle center of gravity with respect to an inertial frame, and θ the vehicle heading, i.e., the orientation of the x axis of the local vehicle reference frame with respect to the corresponding axis of the inertial frame. The velocities of the vehicle frame with respect to the inertial frame are given by

$$\dot{\mathbf{q}} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ \omega \end{bmatrix} \tag{1}$$

where v_x and v_y denote the vehicle linear velocity in the local frame, and ω the angular velocity of the vehicle around its yaw axis.

To derive the vehicle kinematic model, three Instantaneous Center of Rotations (ICRs) are introduced, one for the vehicle, and one for each couple (left/right) of wheels and/or for each track. From Kennedy's theorem [1], one knows that the three ICRs lie on a line parallel to the y axis of the local frame. Defining the coordinates of the three ICRs as $(x_{\mathrm{ICR}}, y_{\mathrm{ICR}})$ for the vehicle ICR, and $(x_{\mathrm{ICR_L}}, y_{\mathrm{ICR_L}})$, $(x_{\mathrm{ICR_R}}, y_{\mathrm{ICR_R}})$ for the left and right couples of wheels/tracks (Figure 1). Their coordinates are expressed by

$$\begin{aligned} x_{\mathrm{ICR}} &= x_{\mathrm{ICR_L}} = x_{\mathrm{ICR_R}} = -\frac{v_y}{\omega} \\ y_{\mathrm{ICR}} &= \frac{v_x}{\omega} \\ y_{\mathrm{ICR_L}} &= \frac{v_L - v_y}{\omega} \\ y_{\mathrm{ICR_R}} &= \frac{v_R - v_y}{\omega} \end{aligned} \tag{2}$$

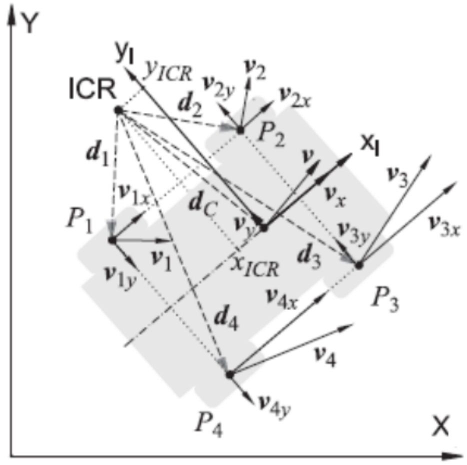

Figure 1: Skid-steering kinematic quantities.

where v_L and v_R are the linear velocities of the left and right wheels, respectively, assuming that the two wheels on the left/right side moves at the same velocity, i.e., $v_{1x} = v_{2x} = v_L$ and $v_{3x} = v_{4x} = v_R$.
The value of y_{ICR} ranges within $\pm\infty$, depending on vehicle trajectory curvature. In particular, it goes to infinite for rectilinear trajectories. The same does not happen for y_{ICR_L} and y_{ICR_R} as numerator and denominator becomes infinitesimal of the same order leading to a finite ratio.

Using equation (2), the following kinematic relation can be derived

$$\begin{bmatrix} v_x \\ v_y \\ \omega \end{bmatrix} = \mathbf{J}_v \begin{bmatrix} v_L \\ v_R \end{bmatrix} \tag{3}$$

where

$$\mathbf{J}_v = \frac{1}{y_{\text{ICR}_\text{L}} - y_{\text{ICR}_\text{R}}} \begin{bmatrix} -y_{\text{ICR}_\text{R}} & y_{\text{ICR}_\text{L}} \\ x_{\text{ICR}} & -x_{\text{ICR}} \\ -1 & 1 \end{bmatrix}$$

Note that, in case of a symmetric vehicle $x_{\text{ICR}} = 0$, and y_{ICR_L}, y_{ICR_R} are symmetric with respect to the x axis of the local reference frame. Consequently, relation (3) can be rewritten as

$$v_x = \frac{v_L + v_R}{2} \qquad \omega = \frac{v_R - v_L}{2d} \tag{4}$$

where $2d$ is the robot track width, i.e., the distance between the left and right wheel/track contact point, and $v_y = 0$. From now on, the vehicle is assumed to

be symmetric.
Finally, defining the slip coefficient s of a wheel as

$$s = \frac{v - r\omega}{v}$$

where r is the wheel radius, and v, ω are the wheel linear velocity and the corresponding angular velocity, relation (4) can be rewritten as

$$\begin{aligned} v_x &= \frac{v_L\left(1 - s_L\right) + v_R\left(1 - s_R\right)}{2} \\ \omega &= \frac{v_R\left(1 - s_R\right) - v_L\left(1 - s_L\right)}{2d} \end{aligned} \tag{5}$$

Substituting now (5) into (1) (remember that for a symmetric vehicle $v_y = 0$), the kinematic model of a SSV can be rewritten as

$$\begin{aligned} \dot{x} &= \frac{(1 - s_R)\cos\theta}{2} v_R + \frac{(1 - s_L)\cos\theta}{2} v_L \\ \dot{y} &= \frac{(1 - s_R)\sin\theta}{2} v_R + \frac{(1 - s_L)\sin\theta}{2} v_L \\ \dot{\theta} &= \frac{1 - s_R}{2d} v_R - \frac{1 - s_L}{2d} v_L \end{aligned} \tag{6}$$

3 Feedback linerization

Let P be a point along the x axis of the local reference frame attached to the robot, located at a distance ε from the origin of this frame.
The coordinates of point P with respect to the inertial frame are given by

$$\begin{aligned} x_P &= x + \varepsilon\cos\theta \\ y_P &= y + \varepsilon\sin\theta \end{aligned}$$

and its velocity by

$$\begin{aligned} \dot{x}_P &= \dot{x} - \varepsilon\dot{\theta}\sin\theta \\ \dot{y}_P &= \dot{y} + \varepsilon\dot{\theta}\cos\theta \end{aligned} \tag{7}$$

Substituting now in (7) the equations of the kinematic model (6), gives rise to

$$\begin{aligned} \dot{x}_P &= \frac{1 - s_R}{2d}\left(d\cos\theta - \varepsilon\sin\theta\right) v_R + \frac{1 - s_L}{2d}\left(d\cos\theta + \varepsilon\sin\theta\right) v_L \\ \dot{y}_P &= \frac{1 - s_R}{2d}\left(d\sin\theta + \varepsilon\cos\theta\right) v_R + \frac{1 - s_L}{2d}\left(d\sin\theta - \varepsilon\cos\theta\right) v_L \end{aligned} \tag{8}$$

From (8), it follows that the SSV kinematic model can be transformed into the model of a particle P moving in the plane

$$\begin{aligned} \dot{x}_P &= v_{x_P} \\ \dot{y}_P &= v_{y_P} \end{aligned}$$

using the following relations

$$\begin{bmatrix} v_{x_P} \\ v_{y_P} \end{bmatrix} = \frac{1}{2d} \begin{bmatrix} \eta_R \left(d\cos\theta - \varepsilon \sin\theta \right) & \eta_L \left(d\cos\theta + \varepsilon \sin\theta \right) \\ \eta_R \left(d\sin\theta + \varepsilon \cos\theta \right) & \eta_L \left(d\sin\theta - \varepsilon \cos\theta \right) \end{bmatrix} \begin{bmatrix} v_R \\ v_L \end{bmatrix}$$

where

$$\eta_R = 1 - s_R \qquad \eta_L = 1 - s_L$$

or, equivalently

$$\begin{bmatrix} v_R \\ v_L \end{bmatrix} = \frac{1}{\varepsilon \eta_R \eta_L} \begin{bmatrix} \eta_L \left(\varepsilon \cos\theta - d\sin\theta \right) & \eta_L \left(\varepsilon \sin\theta + d\cos\theta \right) \\ \eta_R \left(\varepsilon \cos\theta + d\sin\theta \right) & \eta_R \left(\varepsilon \sin\theta - d\cos\theta \right) \end{bmatrix} \begin{bmatrix} v_{x_P} \\ v_{y_P} \end{bmatrix} \tag{9}$$

Finally, considering that the standard inputs of a SSV are the linear velocity v and the angular velocity ω, the mapping can be rewritten as

$$\begin{bmatrix} v \\ \omega \end{bmatrix} = \frac{1}{\varepsilon \eta_R \eta_L} \begin{bmatrix} \varepsilon \gamma_{11} \cos\theta - d\gamma_{12} \sin\theta & \varepsilon \gamma_{11} \sin\theta + d\gamma_{12} \cos\theta \\ \varepsilon \gamma_{22} \cos\theta - d\gamma_{21} \sin\theta & \varepsilon \gamma_{22} \sin\theta + d\gamma_{21} \cos\theta \end{bmatrix} \begin{bmatrix} v_{x_P} \\ v_{y_P} \end{bmatrix} \tag{10}$$

where

$$\gamma_{11} = \frac{\eta_L + \eta_R}{2} \qquad \gamma_{12} = \frac{\eta_L - \eta_R}{2} \qquad \gamma_{21} = \frac{\eta_L + \eta_R}{d} \qquad \gamma_{22} = \frac{\eta_L - \eta_R}{d}$$

Substituting (9) in (6), one obtains the following closed-loop system

$$\begin{aligned} \dot{x} &= \left(v_{x_P} \cos\theta + v_{y_P} \sin\theta \right) \cos\theta \\ \dot{y} &= \left(v_{x_P} \cos\theta + v_{y_P} \sin\theta \right) \sin\theta \\ \dot{\theta} &= \frac{v_{y_P} \cos\theta - v_{x_P} \sin\theta}{\varepsilon} \end{aligned}$$

3.1 Robustness analysis

Consider now the presence of uncertainty in the slip estimate, the linearizing law (9) can be rewritten as

$$\begin{bmatrix} v_R \\ v_L \end{bmatrix} = \frac{1}{\varepsilon \eta_R^{\text{est}} \eta_L^{\text{est}}} \begin{bmatrix} \eta_L^{\text{est}} \left(\varepsilon \cos\theta - d\sin\theta \right) & \eta_L^{\text{est}} \left(\varepsilon \sin\theta + d\cos\theta \right) \\ \eta_R^{\text{est}} \left(\varepsilon \cos\theta + d\sin\theta \right) & \eta_R^{\text{est}} \left(\varepsilon \sin\theta - d\cos\theta \right) \end{bmatrix} \begin{bmatrix} v_{x_P} \\ v_{y_P} \end{bmatrix} \tag{11}$$

where

$$\eta_R^{\text{est}} = 1 - s_R^{\text{est}} \qquad \eta_L^{\text{est}} = 1 - s_L^{\text{est}}$$

and s_R^{est} and s_L^{est} are the slip estimates.

The dynamics of the closed-loop system, accounting also for the uncertainty on the slip, is governed by (6) and (11), i.e.,

$$
\begin{aligned}
\dot{x} &= \frac{(1-s_R)\cos\theta}{2}v_R + \frac{(1-s_L)\cos\theta}{2}v_L \\
\dot{y} &= \frac{(1-s_R)\sin\theta}{2}v_R + \frac{(1-s_L)\sin\theta}{2}v_L \\
\dot{\theta} &= \frac{1-s_R}{2d}v_R - \frac{1-s_L}{2d}v_L \\
v_R &= \frac{\varepsilon\cos\theta - d\sin\theta}{\varepsilon\eta_R^{\mathrm{est}}}v_{x_P} + \frac{\varepsilon\sin\theta + d\cos\theta}{\varepsilon\eta_R^{\mathrm{est}}}v_{y_P} \\
v_L &= \frac{\varepsilon\cos\theta + d\sin\theta}{\varepsilon\eta_L^{\mathrm{est}}}v_{x_P} + \frac{\varepsilon\sin\theta - d\cos\theta}{\varepsilon\eta_L^{\mathrm{est}}}v_{y_P}
\end{aligned}
\tag{12}
$$

To perform the stability analysis, constant input values for the feedback linearized system (12) are defined, i.e.,

$$
v_{x_P} = \bar{v}\cos\bar{\theta} \qquad v_{y_P} = \bar{v}\sin\bar{\theta}
$$

and the dynamics of the hidden state is considered

$$
\dot{\Delta\theta} = \frac{\bar{v}}{2}\left[\left(\frac{\eta_R}{d\eta_R^{\mathrm{est}}} - \frac{\eta_L}{d\eta_L^{\mathrm{est}}}\right)\cos\Delta\theta - \left(\frac{\eta_R}{e\eta_R^{\mathrm{est}}} + \frac{\eta_L}{e\eta_L^{\mathrm{est}}}\right)\sin\Delta\theta\right] \tag{13}
$$

where $\Delta\theta = \theta - \bar{\theta}$.
Equation (13) is characterised by two different equilibria:

- the equilibria

$$
\theta_s = \bar{\theta} + \arctan\left(\frac{\varepsilon}{d}\frac{\eta_R\eta_L^{\mathrm{est}} - \eta_L\eta_R^{\mathrm{est}}}{\eta_R\eta_L^{\mathrm{est}} + \eta_L\eta_R^{\mathrm{est}}}\right) + 2k\pi \qquad k\in\mathbb{Z}
$$

  are asymptotically stable, with a basin of attraction corresponding to the open interval $\left(\bar{\theta} + (2k-1)\pi, \bar{\theta} + (2k+1)\pi\right)$;

- the equilibria

$$
\theta_u = \bar{\theta} + \arctan\left(\frac{\varepsilon}{d}\frac{\eta_R\eta_L^{\mathrm{est}} - \eta_L\eta_R^{\mathrm{est}}}{\eta_R\eta_L^{\mathrm{est}} + \eta_L\eta_R^{\mathrm{est}}}\right) + (2k+1)\pi \qquad k\in\mathbb{Z}
$$

  are unstable, practically excluding backward motion.

In fact, the partial derivative of the right hand side of (13), evaluated at the two equilibria, is equal to

$$
\pm\frac{\bar{v}}{2}\frac{\sqrt{\varepsilon^2\left(\eta_R\eta_L^{\mathrm{est}} - \eta_L\eta_R^{\mathrm{est}}\right)^2 + d^2\left(\eta_R\eta_L^{\mathrm{est}} + \eta_L\eta_R^{\mathrm{est}}\right)^2}}{\varepsilon d\eta_L^{\mathrm{est}}\eta_R^{\mathrm{est}}}
$$

where the minus sign holds for θ_s, and the plus sign for θ_u. As a consequence, assuming that the slip estimator ensures $\eta_*^{\mathrm{est}} \in (0,2)$, this allows to conclude on the stability of the equilibria.

4 Notes

Concerning slip estimation:

- a solution exploiting nonlinear control techniques to derive an adaptive observer ensuring exponential convergence to the real slip coefficients [2];

Concerning trajectory tracking control:

- a MPC approach exploiting a representation of the kinematic model (and slip estimation) based on Gaussian processes [3];
- a learning-based MPC control approach that exploits a relatively simple nominal vehicle model, which is improved based on measurement data and tools from machine learning [4];

References

[1] J. Uicker, J. Pennock, and J. Shigley, *Theory of machines and mechanism*, 4th ed. Oxford University Press, 2010.

[2] R. Tazzari, I. A. Azzollini, and L. Marconi, "An adaptive observer approach to slip estimation for agricultural tracked vehicles," in *2021 European Control Conference (ECC)*, 2021, pp. 1591–1596.

[3] L. Gentilini, D. Mengoli, S. Rossi, and L. Marconi, "Data-driven model predictive control for skid-steering unmanned ground vehicles," in *2022 IEEE Workshop on Metrology for Agriculture and Forestry (MetroAgriFor)*, 2022, pp. 80–85.

[4] J. Kabzan, L. Hewing, A. Liniger, and M. N. Zeilinger, "Learning-based model predictive control for autonomous racing," *IEEE Robotics and Automation Letters*, vol. 4, no. 4, pp. 3363–3370, 2019.