

No.

Date.

1. $\because 10 \equiv -1 \pmod{11}$

若 $n = a_k a_{k-1} \dots a_1 a_0$ (十进制)

则 $\sum_{i=0}^k a_i \cdot 10^i \equiv \sum_{i=0}^k a_i \cdot (-1)^i \pmod{11}$

因此 $n \equiv 0 \pmod{11} \Leftrightarrow$ 各位交替和能被11整除

2. $\because a_i$ 和 b_i 是完全剩余系

$\therefore \sum_{i=1}^n a_i \equiv \sum_{i=1}^n b_i \equiv \frac{n(n-1)}{2} \pmod{n}$

$\because n \equiv 0 \pmod{2} \quad \therefore \frac{n(n-1)}{2} \equiv \frac{n}{2} \pmod{n}$

$\therefore \sum_{i=1}^n (a_i + b_i) \equiv n \equiv 0 \pmod{n}$

但若 $a_i + b_i$ 是完全剩余系, 则应有 $\sum_{i=1}^n (a_i + b_i) \equiv \frac{n}{2} \pmod{n}$, 矛盾

$\therefore a_i + b_i$ 不是模 n 的完全剩余系

3. ① 证: $a=1$ 时 $h_1^p \equiv h_1^p$, 显然成立

$a=2$ 时 对 $\forall h_1, h_2 \in \mathbb{Z}$

$(h_1 + h_2)^p = \sum_{k=0}^p h_1^{p-k} h_2^k \quad \because 0 < k < p \text{ 时, } p \mid C_p^k = \frac{p!}{k!(p-k)!}$

$\therefore (h_1 + h_2)^p \equiv h_1^p + h_2^p \pmod{p}$

若 $a=m$ 时成立且 $m \in \mathbb{N}_+$

则当 $a=m+1$ 时, 使 $n = h_1 + \dots + h_m \quad \therefore \text{左} = (n + h_{m+1})^p \equiv n^p + h_{m+1}^p \pmod{p}$

$\therefore n^p \equiv h_1^p + h_2^p + \dots + h_m^p \quad \therefore a=m+1 \text{ 时亦成立}$

\therefore 综上, 若 p 为素数, 则对 $\forall h_1, h_2, \dots, h_a \in \mathbb{Z}, (h_1 + \dots + h_a)^p \equiv h_1^p + \dots + h_a^p \pmod{p}$

② 若 $h_1 = h_2 = \dots = h_a = 1$, 则 $a^p \equiv a \pmod{p}$ \therefore 费马小定理成立

③ $\because (a, m) = 1 \quad \therefore \prod_{i=1}^{\phi(m)} (a r_i) = \prod_{i=1}^{\phi(m)} r_i \pmod{m} \quad (r_1, \dots, r_{\phi(m)} \text{ 是所有与 } m \text{ 互质的剩余类})$

即 $a^{\phi(m)} \prod_{i=1}^{\phi(m)} r_i = \prod_{i=1}^{\phi(m)} r_i \pmod{m}$

$\because \prod r_i$ 与 m 互质

$\therefore a^{\phi(m)} \equiv 1 \pmod{m}$

No.

Date.

4. 当 $k=1$ 时, 显然 x_1 通过 m_1 的完全剩余系

当 $k=2$ 时若成立, 则当 $k=2$ 时

若使 $y = x_1 + m_1 x_2 + \dots + m_1 m_2 \dots m_{k-2} x_{k-1}$

则 y 通过 $m_1 m_2 \dots m_{k-1}$ 的完全剩余系, 使 $M = m_1 m_2 \dots m_{k-1}$

$\because (m_1 m_2 \dots m_{k-1}, m_k) = 1$ 且 x_k 通过 m_k 的完全剩余系

① 若 $k=2$ 且 $x_1 + m_1 x_2 \equiv x_1' + m_1 x_2' \pmod{m_1 m_2}$

$$\Rightarrow x_1 + m_1 x_2 \equiv x_1' + m_1 x_2' \pmod{m_1}$$

$$\Rightarrow x_1 \equiv x_1' \pmod{m_1} \Rightarrow x_1 = x_1'$$

$$\therefore x_2 = x_2' \quad \therefore x_1 + m_1 x_2 \text{ 通过 } m_1 m_2 \text{ 的完全剩余系}$$

② 若 $k > 2$, 则令 $y = x_1$, $M = m_1$, $x_k = x_2$

则 $y + M x_k$ 通过 $M m_k$ 的完全剩余系

\therefore 综上, 成立

5.

$\because a_1, \dots, a_{\varphi(m)}$ 为 m 的一个缩系

$\therefore (m-a_1), (m-a_2), \dots, (m-a_{\varphi(m)})$ 也是 m 的一个缩系

$$\therefore \sum_{i=1}^{\varphi(m)} a_i \equiv \sum_{i=1}^{\varphi(m)} (m-a_i) \pmod{m}$$

$$\therefore 2 \sum_{i=1}^{\varphi(m)} a_i \equiv m \varphi(m) \pmod{m}$$

$$\therefore \sum_{i=1}^{\varphi(m)} a_i \equiv m \frac{\varphi(m)}{2} \pmod{m} \equiv 0 \pmod{m}$$

6. (1) $\because p-i \equiv -i \pmod{p}$ 且 p 是奇素数

$$\text{又 } (p-1)! \equiv -1 \pmod{p} \quad \therefore 1^2 \cdot 3^2 \cdot \dots \cdot (p-2)^2 \equiv (-1)^{\frac{p-1}{2}} (p-1)! \equiv (-1)^{\frac{p-1}{2}} \pmod{p}$$

$$(2) \text{ 同理 } 2^2 \cdot 4^2 \cdot \dots \cdot (p-1)^2 \equiv (-1)^{\frac{p-1}{2}} (p-1)! \equiv (-1)^{\frac{p-1}{2}} \pmod{p}$$

No.

Date.

7. 若证 $\forall x \in \mathbb{Z}, (\frac{1}{5}x^5 + \frac{1}{3}x^3 + \frac{7}{15}x) \in \mathbb{Z}$

$$\Leftarrow \frac{1}{15}(3x^5 + 5x^3 + 7x)$$

$$\Leftarrow 3x^5 + 5x^3 + 7x \equiv 0 \pmod{15}, \text{ 设 } f(x) = 3x^5 + 5x^3 + 7x$$

$$\Leftarrow f(x) \equiv 0 \pmod{15} \quad \text{则解此同余式等价于}$$

$$\begin{cases} f(x) \equiv 0 \pmod{3} \Rightarrow x \equiv 0, 1, 2 \pmod{3} \\ f(x) \equiv 0 \pmod{5} \Rightarrow x \equiv 0, 1, 2, 3, 4 \pmod{5} \end{cases}$$

$$\text{由中国剩余定理 } \begin{cases} x \equiv b_1 \pmod{5} \\ x \equiv b_2 \pmod{3} \end{cases} \Leftrightarrow x \equiv 5b_1 + 3b_2 \pmod{15}$$

$\therefore b_1$ 取 $0, 1, 2, 3, 4$; b_2 取 $0, 1, 2$ 时, 可得解集为 $\mathbb{Z} \therefore$ 成立

$$\therefore \forall x \in \mathbb{Z}, (\frac{1}{5}x^5 + \frac{1}{3}x^3 + \frac{7}{15}x) \in \mathbb{Z}$$

8. $111x \equiv 75 \pmod{321}$

$$\textcircled{1} \text{ 求 } d = (111, 321) \quad 321 = 2 \times 111 + 99 \Rightarrow 111 = 1 \times 99 + 12 \Rightarrow 99 = 8 \times 12 + 3 \Rightarrow 12 = 4 \times 3 + 0$$

$\therefore d = 3$ 且 $3 \mid 75 \therefore$ 有解且恰有 3 个

$$\textcircled{2} \text{ 经化简, 原式等价于 } 37x \equiv 25 \pmod{107}$$

$$\textcircled{3} \text{ 求 } 37^{-1} \pmod{107}: 107 = 2 \times 37 + 33 \Rightarrow 37 = 1 \times 33 + 4 \Rightarrow 33 = 8 \times 4 + 1 \Rightarrow 4 = 1 \times 4 + 0$$

$$\therefore \text{回代 } 1 = 33 - 8 \times 4 = 33 - 8 \times (37 - 1 \times 33) = 9 \times 33 - 8 \times 37 = 9 \times (107 - 2 \times 37) - 8 \times 37 = 9 \times 107 - 26 \times 37$$

$$\therefore 37^{-1} \equiv -26 \equiv 81 \pmod{107}$$

$$\textcircled{4} \therefore \text{原式等价于 } x \equiv 25 \times 81 \equiv 102 \pmod{107}$$

$$\therefore \text{原方程的解为 } x \equiv 102, 102 + 107, 102 + 214 \pmod{321}$$

$$\therefore x \equiv 102, 209, 316 \pmod{321}$$

No.

Date.

/ /

$$9. \begin{cases} 3x \equiv 5 \pmod{4} \\ 5x \equiv 2 \pmod{7} \end{cases} \Rightarrow \begin{cases} x \equiv 15 \pmod{4} \\ x \equiv 6 \pmod{7} \end{cases} \Rightarrow \begin{cases} x \equiv 3 \pmod{4} \\ x \equiv 6 \pmod{7} \end{cases}$$

$$\therefore x = 4k + 3 \Rightarrow 4k + 3 \equiv 6 \pmod{7} \Rightarrow 4k \equiv 3 \pmod{7} \Rightarrow k \equiv 6 \pmod{7}$$

$$\therefore x \equiv 4 \times 6 + 3 \equiv 27 \pmod{28}$$

$$\therefore x \equiv 27 \pmod{28}$$