

1. (1) 对于仿射密码, $P = C = \mathbb{Z}_{26}$, $K = \{(a, b) \in \mathbb{Z}_{26} \times \mathbb{Z}_{26}; \gcd(a, 26) = 1\}$

$$e_K(x) = (ax + b) \bmod 26$$

$$\begin{aligned} \because x, y \in \mathbb{Z}_{26} \quad \therefore \Pr[Y=y] &= \sum_{k \in \dots} \Pr[K=k] \Pr[x = d_k(y)] \\ &= \sum_{k \in \dots} \frac{1}{312} \Pr[x = a^{-1}(y+b)] \\ &= \frac{1}{312} \sum_{k \in \dots} \Pr[x = a^{-1}(y-b)] \end{aligned}$$

易知, $a \in A$, $|A|=12$, $b \in \mathbb{Z}_{26}$, $K = (a, b) \in A \times \mathbb{Z}_{26}$

\therefore 固定 y, a , 则 $a^{-1}(y-b)$ 构成 \mathbb{Z}_{26} 的一个置换

固定 y, b , 则 $a^{-1}(y-b)$ 构成 A 的一个置换

$$\therefore \sum_{k \in A \times \mathbb{Z}_{26}} \Pr[x = a^{-1}(y-b)] = \sum_{x \in \mathbb{Z}_{26}} \Pr[x = x] \times 12 = 12$$

$$\therefore \forall y \in \mathbb{Z}_{26}, \text{有 } \Pr[Y=y] = \frac{1}{26}$$

$$\begin{aligned} \therefore \forall x, y \in \mathbb{Z}_{26}, \Pr[Y=y|x] &= \Pr[K = (y-x) \bmod 26] \times 12 \\ &= \frac{1}{312} \times 12 = \frac{1}{26} \end{aligned}$$

$$\begin{aligned} \therefore \Pr[X=y] &= \frac{\Pr[X] \Pr[Y|x]}{\Pr[Y]} \\ &= \frac{\Pr[X] \frac{1}{26}}{\frac{1}{26}} = \Pr[X] \end{aligned}$$

\therefore 仿射密码是完善保密的

(2) $\forall x \in P, y \in C$

$$\Pr[Y=y] = \sum_{(a,b) \in K} \Pr[(a,b)] = \sum_{a \in A} \Pr[a] \cdot \frac{1}{26} = \frac{1}{26} \sum_{a \in A} \Pr[a] = \frac{1}{26}$$

$$\frac{1}{2} \Pr[Y] = \sum_{x \in P} \Pr[X=x] \Pr[Y|x] = \frac{1}{26}$$

$$\Pr[X=x|y] = \frac{\Pr[Y=y] \Pr[X=x]}{\Pr[Y]} = \Pr[X=x]$$

\therefore 成立