Tracce delle soluzioni

1.
Vedi dispense del corso.

2.

$$\begin{cases} m D^2 x_1 = f - k x_1 - b D x_1 + k(x_2 - x_1) + b(D x_2 - D x_1) \\ m D^2 x_2 = -k(x_2 - x_1) - b(D x_2 - D x_1) \end{cases}$$

$$\begin{cases} (bD + k) x_2 = m D^2 x_1 + 2 b D x_1 + 2 k x_1 - f \\ (m D^2 + b D + k) x_2 = b D x_1 + k x_1 \end{cases}$$

$$(m D^2 + b D + k)(m D^2 x_1 + 2 b D x_1 + 2 k x_1 - f) =$$
$$= (bD + k)(b D x_1 + k x_1)$$

$$(m D^2 + b D + k)(m D^2 + 2 b D + 2 k) x_1 - (m D^2 + b D + k) f =$$
$$= (bD + k)^2 x_1$$

$$(m^2 D^4 + 3 b m D^3 + (3 k m + 2 b^2) D^2 + 4 b k D + 2 k^2) x_1$$
$$- (b^2 D^2 + 2 b k D + k^2) x_1 = (m D^2 + b D + k) f$$

1

$$\boxed{\begin{aligned} & m^2 D^4 x_1 + 3 b m D^3 x_1 + (3 k m + b^2) D^2 x_1 + 2 b k D x_1 + k^2 x_1 = \\ & \quad = m D^2 f + b D f + k f \end{aligned}}$$

2

$$G(s) = \frac{m s^2 + b s + k}{m^2 s^4 + 3 b m s^3 + (3 k m + b^2) s^2 + 2 b k s + k^2}$$

3.

$$U(s) = \frac{1}{s^2} \qquad Y(s) = G(s)\,U(s) = \frac{32}{s^2(s+2)^3(s+4)}$$

$$= \frac{K_{11}}{s^2} + \frac{K_{12}}{s} + \frac{K_{21}}{(s+2)^3} + \frac{K_{22}}{(s+2)^2} + \frac{K_{23}}{s+2} + \frac{K_3}{s+4}$$

RESIDUI

$$K_{11} = \left.\frac{32}{(s+2)^3(s+4)}\right|_{s=0} = \frac{32}{32} = 1$$

$$K_{21} = \left.\frac{32}{s^2(s+4)}\right|_{s=-2} = \frac{32}{4\cdot 2} = 4$$

$$K_3 = \left.\frac{32}{s^2(s+2)^3}\right|_{s=-4} = \frac{32}{4\cdot 4\cdot(-8)} = -\frac{1}{4}$$

$$K_{12} + K_{23} + K_3 = 0 \qquad \boxed{K_{12} + K_{23} = \frac{1}{4}}$$

calcolo di K_{12} : $K_{12} = D\left[\frac{32}{(s+2)^3(s+4)}\right]_{s=0} = -\frac{7}{4}$

$$= -(32)\left.\frac{3(s+2)^2(s+4)+(s+2)^3}{(s+2)^6(s+4)^2}\right|_{s=0} = -32\cdot\frac{3\cdot 4\cdot 4 + 4\cdot 2}{2^6\cdot 4^2} = -8\,\frac{3\cdot 4+2}{2^3\cdot 2\cdot 4} =$$

$$-\frac{7}{4} + K_{23} = \frac{1}{4}$$

$$K_{23} = \frac{1}{4} + \frac{7}{4} = \frac{8}{4} = 2$$

$$K_{23} = 2$$

$$y(t) = \left(t - \frac{7}{4} + 4 \cdot \frac{1}{2!} t^2 e^{-2t} + K_{22}\, t e^{-2t} + 2 e^{-2t} - \frac{1}{4} e^{-4t} \right) 1(t)$$

$$K_{22} = D\left[\frac{32}{s^2(s+4)} \right]\Bigg|_{s=-2} =$$

$$= -32 \cdot \frac{2s(s+4) + s^2}{s^4(s+4)^2}\Bigg|_{s=-2} = -32 \frac{-\cancel{4}\cdot 2 + \cancel{4}}{2^4 \cdot \cancel{4}} =$$

$$= -\cancel{16}\cdot 2 \frac{-1}{\cancel{16}} = 2$$

$$y(t) = \left(-\frac{7}{4} + t + 2t^2 e^{-2t} + 2t e^{-2t} + 2e^{-2t} - \frac{1}{4} e^{-4t} \right) \cdot 1(t)$$

$$Y(s) = \frac{1}{s^2} - \frac{7}{4} \cdot \frac{1}{s} + \frac{4}{(s+2)^3} + \frac{2}{(s+2)^2} + \frac{2}{s+2} - \frac{1}{4} \cdot \frac{1}{s+4}$$

ok!

Il grado relativo del sistema è uguale a 4. Considerato che il segnale di ingresso ha grado massimo di continuità pari a 0 segue dalla nota proprietà che il grado massimo del segnale di uscita è uguale a 4.

4. Vedi dispense del corso.

5.

a) Funzione di Trasferimento:

$$P(s) = 100\,\frac{(1-s)^2}{s\,(s+2)^3} \;\Rightarrow\; P(j\omega) = 12.5\,\frac{(1-j\omega)^2}{j\omega\,(1+0.5\,j\omega)^3}$$

Ascissa dell'asintoto: $\nabla_a = 12.5\,(-1-1-0.5-0.5-0.5) = -43.75$
Argomento della funzione di trasferimento:

$$\arg P(j\omega) = -\frac{\pi}{2} - 3\,\arctan(0.5\,\omega) - 2\,\arctan\omega$$

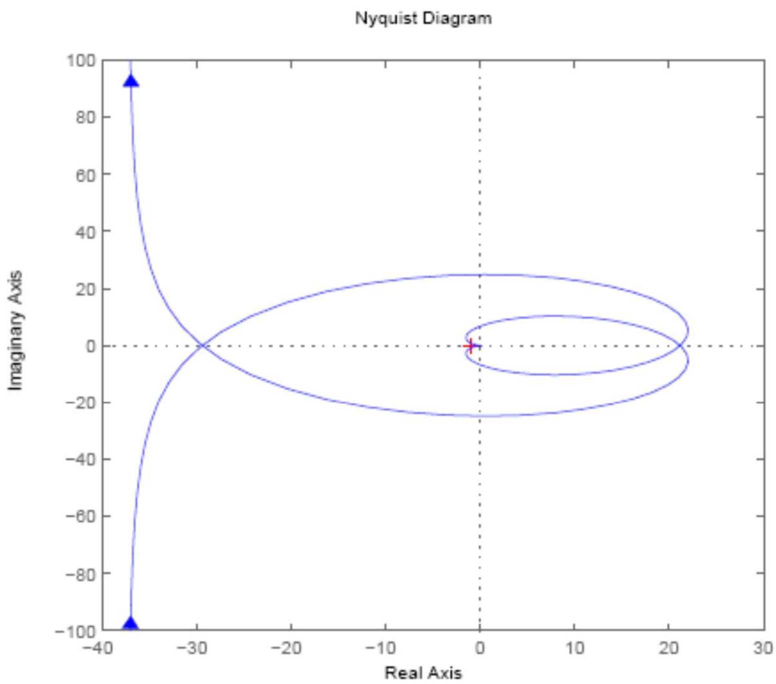
$$\text{per}\,\omega \to 0 \;\Rightarrow\; \arg P(j\omega) \to -\frac{\pi}{2} \qquad \text{per}\,\omega \to +\inf \;\Rightarrow\; \arg P(j\omega) \to -3\,\pi$$

Intersezione con l'asse reale negativo:

$$\arg P(j\omega_p) = -\pi \Rightarrow 3\,\arctan\frac{\omega_p}{2} + 2\,\arctan\omega_p = \frac{\pi}{2}$$

Attraverso una stima numerica si ottiene: $\omega_p \simeq 0,47$ [rad/s]
Intesezione:

$$|P(j\omega_p)| = 12.5\,\frac{(1+\omega_p^2)}{\omega_p\,\left(1+(\frac{\omega_p}{2})^2\right)^{\frac{3}{2}}} \simeq 29.95$$



b) Il diagramma polare completo circonda due volte in senso orario il punto -1 e il guadagno di anello non ha poli a parte reale positiva, quindi le radici di $1 + P(s)$ sono:

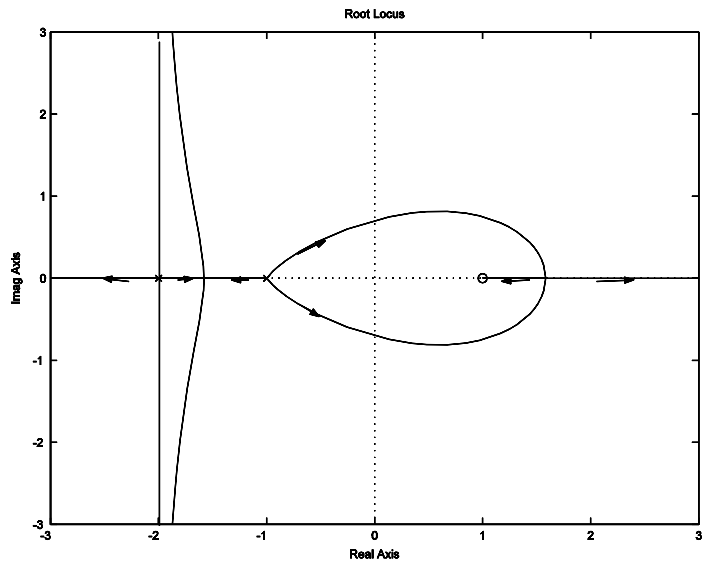
$$\begin{aligned} n \in \mathbb{C}_+ :&\quad 2 \\ n \in \mathbb{C}_- :&\quad 2\;\;(4-2) \\ n \in j\,\mathbb{R} :&\quad 0 \end{aligned}$$

6.
L'equazione caratteristica è riscrivibile come

$$1+K_1\frac{s-1}{(s+1)^3(s+2)^2}=0 \quad \text{con } K_1=-K\in(-\infty,0]$$

e quindi si tratta di un luogo inverso. Presenta 4 asintoti il cui centro ha ascissa

$$\sigma_a=\frac{-1-1-1-2-2-(+1)}{5-1}=-2$$



Le radici doppie sono determinabili risolvendo l'equazione:

$$\frac{1}{s+1}+\frac{1}{s+1}+\frac{1}{s+1}+\frac{1}{s+2}+\frac{1}{s+2}-\frac{1}{s-1}=0$$

$$\frac{3}{s+1}+\frac{2}{s+2}-\frac{1}{s-1}=0$$

$$4s^2-10=0$$

$$s_{1,2}=\pm\sqrt{5/2}=\pm1,5811$$

7.

$$P(j\omega) = \frac{10}{(1+j\omega)(1+0{,}4j\omega)^2}$$

ω_0? $\Rightarrow$ $\arg P(j\omega_0) = -\pi + \frac{45}{180}\cdot\pi = -2{,}3562$

$\arg P(j\omega_0) = -\operatorname{arctg}\omega_0 - 2\operatorname{arctg} 0{,}4\cdot\omega_0$

ω_0	$\arg P(j\omega_0)$
1	$-1{,}5464$
2	$-2{,}4566$
1,8	$-2{,}3117$
1,86	$-2{,}3568$

Impongo

$\omega_n = \omega_0 = 1{,}86$ rad/sec

attenuazione in centro banda $= \frac{\tau_1+\tau_2}{\tau_1+\tau_2+\tau_{12}} = \frac{1}{|P(j\omega_0)|} =$

$\frac{1}{\sqrt{\tau_1\tau_2}} = \omega_n$ $\quad \frac{1}{\tau_1\tau_2} = \omega_n^2$ $\quad \frac{1}{10\tau_2^2} = \omega_n^2$

$\frac{\tau_1}{\tau_2} = 10$ $\quad \tau_1 = 10\tau_2$

$\tau_2^2 = \frac{1}{10\cdot\omega_n^2} \Rightarrow \tau_2 = \frac{1}{\sqrt{10}\cdot\omega_n} = 0{,}170$ sec

$\tau_1 = 10\tau_2 = 1{,}70$ sec

$$\left| P(j\omega_0) \right| = \frac{10}{\sqrt{1+\omega_0^2}\cdot\left(1+(0{,}4\cdot\omega_0)^2\right)} =$$

$$= \frac{10}{\#} = 3{,}0481$$

$$\frac{\tau_1+\tau_2}{\tau_1+\tau_2+\tau_{12}} = \frac{1}{\left| P(j\omega_0) \right|}$$

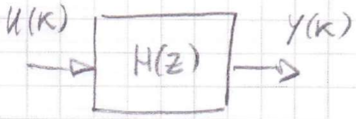
$$(\tau_1+\tau_2)\cdot\left| P(j\omega_0) \right| = (\tau_1+\tau_2)+\tau_{12}$$

$$\tau_{12} = (\tau_1+\tau_2)\left[\,\left| P(j\omega_0) \right| - 1\,\right]$$

$$= (1{,}70+0{,}170)\left[\,2{,}0481\,\right]$$

$$= 1{,}87\cdot 2{,}0481 \cong 3{,}83 \text{ sec}$$

8.



$$H(z) = \frac{z^2+z+1}{(z-1)\left(z+\frac{1}{2}\right)} \qquad u(k) = 1(k)$$

$$U(z) = \frac{z}{z-1} \qquad Y(z) = H(z)U(z) = \frac{\left(z^2+z+1\right)z}{(z-1)^2\left(z+\frac{1}{2}\right)}$$

$$Y(z)\cdot z^{-1} = \frac{z^2+z+1}{(z-1)^2\left(z+\frac{1}{2}\right)} = \frac{C_{11}}{(z-1)^2} + \frac{C_{12}}{z-1} + \frac{C_2}{z+\frac{1}{2}}$$

$$C_{11} = \left.\frac{z^2+z+1}{z+\frac{1}{2}}\right|_{z=1} = \frac{3}{\frac{3}{2}} = 2$$

$$C_2 = \left.\frac{z^2+z+1}{(z-1)^2}\right|_{z=-\frac{1}{2}} = \frac{\frac{1}{4}-\frac{1}{2}+1}{\frac{9}{4}} = \frac{\frac{1-2+4}{4}}{\frac{9}{4}} = \frac{\cancel{3}^{1}}{\cancel{4}}\cdot\frac{\cancel{4}}{\cancel{9}_{3}} = \frac{1}{3}$$

$$C_{12} + C_2 = 1 \qquad C_{12} = 1 - \frac{1}{3} = \frac{2}{3}$$

$$Y(z) = 2\frac{z}{(z-1)^2} + \frac{2}{3}\cdot\frac{z}{z-1} + \frac{1}{3}\cdot\frac{z}{z+\frac{1}{2}}$$

$$y(k) = \mathcal{Z}^{-1}\left[Y(z)\right] = 2k\cdot 1(k) + \frac{2}{3}1(k) + \frac{1}{3}\cdot\left(-\frac{1}{2}\right)^k 1(k)$$