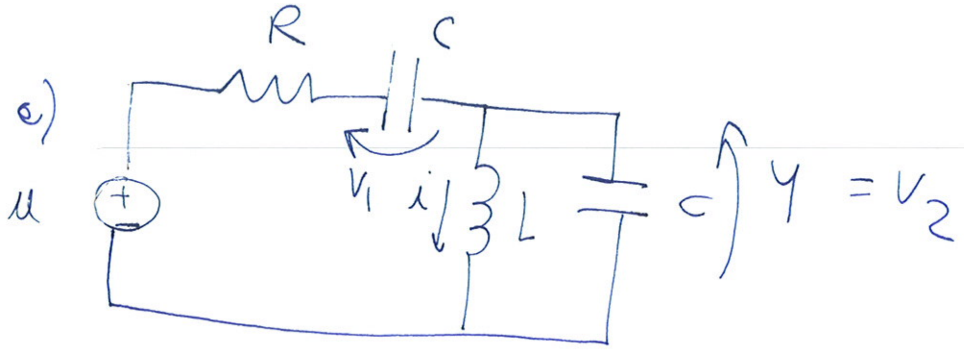
1) a)



$$\begin{cases} \dot{v}_1 = C^{-1} \dfrac{(u - v_2 - v_1)}{R} \\[2ex] \dot{v}_2 = \left(\dfrac{u - v_2 - v_1}{R} - i \right) C^{-1} \\[2ex] \dot{i} = \dfrac{v_2}{L} \end{cases}$$

$$x = \begin{bmatrix} v_1 \\ v_2 \\ i \end{bmatrix}, \quad \dot{x} = \overbrace{\begin{bmatrix} -\frac{1}{RC} & -\frac{1}{RC} & 0 \\ -\frac{1}{RC} & -\frac{1}{RC} & -\frac{1}{C} \\ 0 & \frac{1}{L} & 0 \end{bmatrix}}^{A} x + \overbrace{\begin{bmatrix} \frac{1}{RC} \\ \frac{1}{RC} \\ 0 \end{bmatrix}}^{B} u$$

$$y = \underbrace{\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}}_{C} x$$

b) $$H(s) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} s + \frac{1}{RC} & \frac{1}{RC} & 0 \\ \frac{1}{RC} & s + \frac{1}{RC} & \frac{1}{C} \\ 0 & -\frac{1}{L} & s \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{RC} \\ \frac{1}{RC} \\ 0 \end{bmatrix}$$

$$\det(sI-A) = \frac{1}{L}\left(s+\frac{1}{RC}\right)\frac{1}{C} + s\left[\left(s+\frac{1}{RC}\right)^2-\left(\frac{1}{RC}\right)^2\right]$$

$$= \frac{1}{LC}\left(s+\frac{1}{RC}\right) + s\left[s^2+\frac{2s}{RC}\right]$$

$$= s^3+\frac{2s^2}{RC}+\frac{s}{LC}+\frac{1}{LRC^2}$$

$$H(s) = \frac{\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} \ast & \ast & \ast \\ -\frac{s}{RC} & s\left(s+\frac{1}{RC}\right) & \ast \\ \ast & \ast & \ast \end{bmatrix}\begin{bmatrix} \frac{1}{RC} \\ \frac{1}{RC} \\ 0 \end{bmatrix}}{\det(sI-A)}$$

$$= \frac{\frac{1}{RC}s^2}{s^3+\frac{2s^2}{RC}+\frac{s}{LC}+\frac{1}{LRC^2}} = \frac{s^2}{RCs^3+2s^2+\frac{R}{L}s+\frac{1}{LC}}$$

2) a) $\chi_A(\lambda) = (\lambda-2)^2(\lambda+1)$

$$\lambda = 2 \quad \operatorname{Ker}[A-2I] = \operatorname{Ker}\begin{bmatrix} -3 & 0 & -3 \\ 3 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix} = \operatorname{Im}\underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{v_1}$$

$$\operatorname{Ker}[A-2I]^2 = \operatorname{Ker}\begin{bmatrix} 9 & 0 & 9 \\ -9 & 0 & -9 \\ 0 & 0 & 0 \end{bmatrix} = \operatorname{Im}\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & -1 \end{bmatrix}$$

v_2

$\lambda = -1$

$$\operatorname{Ker}[A+I] = \operatorname{Ker}\begin{bmatrix} 0 & 0 & -3 \\ 3 & 3 & 4 \\ 0 & 0 & 3 \end{bmatrix} = \operatorname{Im}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$A^k v_2 = \lambda^k v_2 + k\lambda^{k-1}(A-2I)v_2$$

$$= \lambda^k \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + k\lambda^{k-1}\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = 2^k \begin{bmatrix} 1 \\ -k/2 \\ -1 \end{bmatrix}$$

$$A^k = \begin{bmatrix} 0 & 2^k & (-1)^k \\ 2^k & -k2^{k-1} & -(-1)^k \\ 0 & -2^k & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}^{-1} \longrightarrow = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (-1)^k & 0 & (-1)^k - 2^k \\ 2^k - (-1)^k & 2^k & 2^k - (-1)^k + k2^{k-1} \\ 0 & 0 & 2^k \end{bmatrix}$$

b) $x(k) = A^k x_0 = \begin{bmatrix} 0 \\ 2^k \\ 0 \end{bmatrix}$

3) $\mathcal{X}_R(1) = \operatorname{Im} B = \operatorname{Im} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$\mathcal{X}_R(2) = \operatorname{Im} [B, AB] = \operatorname{Im} \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} = \operatorname{Im} \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$

$\mathcal{X}_R(3) = \operatorname{Im} [B, AB, A^2B] = \operatorname{Im} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \mathcal{X}_R(2)$

$\mathcal{X}_R = \mathcal{X}_R(2) = \operatorname{Im} \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$

$\mathcal{X}_C(1) = A^{-1}\mathcal{X}_R(1) \qquad \mathcal{X}_R(1)^\perp = \operatorname{Ker} [1 \ 0 \ 1]$

$= \operatorname{Im} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{bmatrix}$

$\mathcal{X}_C(1) = \operatorname{Ker}\left(\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} A \right) = \operatorname{Ker} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$= \operatorname{Im} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\mathcal{X}_C(2) = A^{-2}\mathcal{X}_R(2), \quad \mathcal{X}_2(2)^\perp = \operatorname{Ker} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \operatorname{Im} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$= \operatorname{Ker}\left([0 \ 1 \ 0] A \right) = \operatorname{Ker} [0 \ 0 \ 0] = \mathbb{R}^3$

$\mathcal{X}_C = \mathcal{X}_C(2) = \mathbb{R}^3$

b) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = A^k \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \in \mathcal{X}_R(k)$

Per $k=1$ $\quad \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} -2 \\ 0 \\ -2 \end{bmatrix} \notin \operatorname{Im} B$

Per $k=2$ $\quad \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \underbrace{\begin{bmatrix} -4 \\ \\ -2 \end{bmatrix}}_{A^2 x_0} = \begin{bmatrix} 5 \\ 0 \\ 2 \end{bmatrix} \in \operatorname{Im} [B, AB]$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - A^2 x_0 = R_2 \begin{bmatrix} u(1) \\ u(0) \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 5 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u(1) \\ u(0) \end{bmatrix}$$

$$\boxed{u(1) = -1 \,, \quad u(0) = 3}$$

4) $X_c(1) = \operatorname{Im} B = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$$X_c(2) = \operatorname{Im}\left[B, AB\right] = \operatorname{Im}\begin{bmatrix} -1 & -1 \\ 0 & 0 \\ 0 & 3 \\ 1 & 1 \end{bmatrix}$$

$$X_c(3) = \operatorname{Im}\left[B, AB, A^2B\right] = \operatorname{Im}\begin{bmatrix} -1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 3 & 6 \\ 1 & 1 & 1 \end{bmatrix} = X_c(2)$$

$$X_c = \operatorname{Im}\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad T = \begin{bmatrix} \underbrace{\begin{matrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ -1 & 0 \end{matrix}}_{T_1} & \underbrace{\begin{matrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{matrix}}_{T_2} \end{bmatrix}, \quad T^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\hat{A} = T^{-1}AT = T^{-1}\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ -3 & 1 & 0 & 3 \\ -1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \overbrace{\boxed{\begin{matrix} 1 & 0 \\ -3 & 1 \end{matrix}}}^{A_R} & \begin{matrix} 1 & 1 \\ 0 & 3 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \underbrace{\boxed{\begin{matrix} 2 & 1 \\ 1 & 2 \end{matrix}}}_{A_{NR}} \end{bmatrix}$$

$$\hat{B} = T^{-1}B = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{matrix} \}\, B_R \\ \\ \}\, B_{NR} \\ \end{matrix}$$

$$\hat{C} = CT = \begin{bmatrix} \underbrace{0 \quad 1}_{C_R} & \underbrace{-1 \quad 0}_{C_{NR}} \end{bmatrix}$$

$\sigma(A_R) = \{ 1 \}$ (AUT. RAGG.)

$\sigma(A_{NR}) \to \chi_{A_{NR}}(\lambda) = \lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3)$

$\sigma(A_{NR}) = \{1, 3\}$

$$H(s) = C_R (sI - A_R)^{-1} B_R$$

$$= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s-1 & 0 \\ 3 & s-1 \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \frac{3}{(s-1)^2}$$

$$h(t) = \mathcal{L}^{-1}\left\{ \frac{3}{(s-1)^2} \right\} = \sum \mathrm{Res} \left\{ \frac{3e^{st}}{(s-1)^2} \right\}$$

$$= \frac{d}{ds} 3e^{st} \Big|_{s=1} = 3te^{st} \Big|_{s=1} = 3te^{t} \cdot 1(t)$$

5) se v è un autovettore di A

$$Av = \lambda v \rightarrow BAv = B\lambda v$$

essendo $AB = BA$ $\quad ABv = \lambda Bv$

quindi Bv è un autovettore di A associato all'autovalore λ. Essendo l'autospazio associato a λ di dimensione 1, Bv e v devono essere multipli tra loro, cioè $Bv \in \text{lin}\, v$ e quindi $\exists \mu$ tale che $Bv = \mu v$.