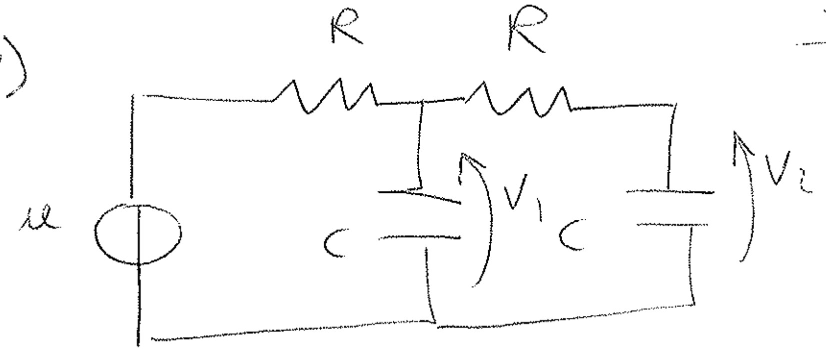
Soluzione

1)



$$C\dot{V}_1 = \frac{u - V_1}{R} - \frac{V_1 - V_2}{R}$$

$$C\dot{V}_2 = \frac{V_1 - V_2}{R}, \quad y = V_2$$

$$\dot{x} = \begin{bmatrix} \dot{V}_1 \\ \dot{V}_2 \end{bmatrix} = \overbrace{\begin{bmatrix} -\frac{2}{RC} & \frac{1}{RC} \\ \frac{1}{RC} & -\frac{1}{RC} \end{bmatrix}}^{A} x + \overbrace{\begin{bmatrix} \frac{1}{RC} \\ 0 \end{bmatrix}}^{B} u$$

$$y = \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{C} x \qquad D = 0$$

$$\chi_A(\lambda) = \det \begin{bmatrix} -\lambda - \frac{2}{RC} & \frac{1}{RC} \\ \frac{1}{RC} & -\frac{1}{RC} - \lambda \end{bmatrix} = \lambda^2 + \lambda \frac{3}{RC} + \frac{1}{(RC)^2}$$

$$= (\lambda - \lambda_1)(\lambda - \lambda_2)$$

con $\lambda_1 = \frac{-3 + \sqrt{5}}{2RC}$ $\quad \lambda_2 = \frac{-3 - \sqrt{5}}{2RC}$

modi $= \left\{ e^{-\frac{3+\sqrt{5}}{2RC}t}, \; e^{-\frac{3-\sqrt{5}}{2RC}t} \right\}$

2)

$$A = \begin{bmatrix} 3 & 2 & 0 \\ -2 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad \chi_A(\lambda) = \det \begin{bmatrix} 3-\lambda & 2 & 0 \\ -2 & -1-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{bmatrix}$$

$$= (2-\lambda)\Big[(3-\lambda)(-1-\lambda)+4\Big] = (2-\lambda)(\lambda-1)^2$$

$$\lambda = 1 \rightarrow \operatorname{Ker}\,[A - I] = \operatorname{Ker} \begin{bmatrix} 2 & 2 & 0 \\ -2 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \operatorname{Im} \begin{array}{c} v_1 \\ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \end{array}$$

$$\operatorname{Ker}\,[A - I]^2 = \operatorname{Ker} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \operatorname{Im} \begin{array}{cc} v_1 & v_2 \\ \multicolumn{2}{c}{\begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & 0 \end{bmatrix}} \end{array}$$

$$\operatorname{Ker}\,[A - 2I] = \operatorname{Ker} \begin{bmatrix} 1 & 2 & 0 \\ -2 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \operatorname{Im} \begin{array}{c} v_3 \\ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{array}$$

$$e^{At} v_1 = e^{t} v_1 = \psi_1(t)$$

$$e^{At} v_2 = e^{t} \Big[I - (A - I)t\Big] v_2 = e^{t} \begin{bmatrix} 2t \\ 1-2t \\ 0 \end{bmatrix} = \psi_2(t)$$

$$e^{At} v_3 = e^{2t} v_3$$

$$\Psi(t) = \begin{bmatrix} e^{t} & 2te^{t} & 0 \\ -e^{t} & (1-2t)e^{t} & 0 \\ 0 & 0 & e^{2t} \end{bmatrix}$$

$$e^{At} = \Psi(t)\,\Psi(0)^{-1} = \begin{bmatrix} e^t & 2te^t & 0 \\ -e^t & (1-2t)e^t & 0 \\ 0 & 0 & e^{2t} \end{bmatrix} \cdot$$

$$\cdot \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \Psi(t) \cdot \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} e^t(1+2t) & 2te^t & 0 \\ e^t(-2t) & (1-2t)e^t & 0 \\ 0 & 0 & e^{2t} \end{bmatrix}$$

$$e^{At} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} e^t(1+2t) \\ -2te^t \\ 0 \end{bmatrix}$$

3) a) $X_R(1) = \operatorname{Im} B = \operatorname{Im} \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$

$$X_R(2) = \operatorname{Im} \begin{bmatrix} B & AB \end{bmatrix} = \operatorname{Im} \begin{bmatrix} -1 & 1 \\ 1 & -1 \\ -1 & 0 \end{bmatrix}$$

$$X_R(3) = \operatorname{Im} \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \operatorname{Im} \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & -1 \\ -1 & 0 & -1 \end{bmatrix} \subset X_R(2)$$

$$X_R(k) = X_R(2) = \operatorname{Im} \begin{bmatrix} -1 & 1 \\ 1 & -1 \\ -1 & 0 \end{bmatrix}, \quad \forall k \geq 2$$

b)

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = R_2 V_2 = \begin{bmatrix} B & AB \end{bmatrix} \begin{bmatrix} u(1) \\ u(0) \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} u(1) \\ u(0) \end{bmatrix}$$

$\Rightarrow \quad u(1) = -1 \quad , \quad u(0) = 1$

4) $AB = \begin{bmatrix} -1 \\ -2 \\ 2 \\ 0 \end{bmatrix} \notin \mathrm{Im}\, B$

$$A^2 B = \begin{bmatrix} -1 \\ -3 \\ 3 \\ 0 \end{bmatrix} \in \mathrm{Im}\,[B, AB]$$

INFATTI $\begin{bmatrix} -1 & -1 & -1 \\ -1 & -2 & -3 \\ 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ HA RANGO 2 (LA TERZA RIGA È UGUALE ALLA SECONDA CAMBIATA DI SEGNO)

$$\mathrm{Im}\, R = \mathrm{Im}\, T_1 = \mathrm{Im} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}$$

$$T = [T_1, T_2] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad T^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\hat{A} = T^{-1} A T = T^{-1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} \boxed{\begin{matrix} 1 & 0 \\ 1 & 1 \end{matrix}}_{A_R} & \begin{matrix} 0 & 0 \\ 1 & -1 \end{matrix} \\ \underbrace{\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}}_{\vec{0}} & \boxed{\begin{matrix} 2 & -1 \\ 0 & 3 \end{matrix}}_{A_{NR}} \end{bmatrix}$$

$$\hat{B} = T^{-1} B = \begin{bmatrix} \boxed{\begin{matrix} 1 \\ 1 \end{matrix}}_{B_R} \\ \underbrace{\begin{matrix} 0 \\ 0 \end{matrix}}_{0} \end{bmatrix}$$

$$\sigma(A_R) = \{1\}$$

$$\sigma(B_R) = \{2, 3\}$$

$$\hat{C} = CT = \begin{array}{cc} \begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \end{bmatrix} \\ C_R \quad\quad C_{NR} \end{array}$$

$$H(s) = C_R (sI - A_R)^{-1} B_R$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \left(\begin{bmatrix} s-1 & 0 \\ -1 & s-1 \end{bmatrix} \right)^{-1} \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$= \frac{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} s-1 & 0 \\ 1 & s-1 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix}}{(s-1)^2} = \frac{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -(s-1) \\ -s \end{bmatrix}}{(s-1)^2}$$

$$= \frac{\begin{bmatrix} -(s-1) \\ s \end{bmatrix}}{(s-1)^2} = \begin{bmatrix} -\frac{1}{s-1} \\ \frac{s}{(s-1)^2} \end{bmatrix}$$

$$h(t) = \mathcal{L}^{-1}\{H(s)\} = \begin{bmatrix} -e^t \\ te^t + e^t \end{bmatrix}$$

5) $$\chi_A(\lambda) = \det \begin{bmatrix} -\lambda & 1 & 0 & 0 & 0 \\ 0 & -\lambda & 1 & 0 & 0 \\ 0 & 0 & -\lambda & 1 & 0 \\ 0 & 0 & 0 & -\lambda & 1 \\ 1 & 0 & 0 & 0 & -\lambda \end{bmatrix}$$

$$= 1 \cdot \underbrace{\det \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\lambda & 1 & 0 & 0 \\ 0 & -\lambda & 1 & 0 \\ 0 & 0 & 0 & -\lambda \end{bmatrix}}_{=1} - \lambda \underbrace{\det \begin{bmatrix} -\lambda & 1 & 0 & 0 \\ 0 & -\lambda & 1 & 0 \\ 0 & 0 & -\lambda & 1 \\ 0 & 0 & 0 & -\lambda \end{bmatrix}}_{=\lambda^4}$$

$$= 1 - \lambda^5$$

Per il teorema di Hamilton-Cailey

$A^5 = I$, quindi

SCRIVIAMO $K = K \bmod 5 + \lfloor K/5 \rfloor 5$

dove $\lfloor K/5 \rfloor$ INDICA LA PARTE INTERA DI $\frac{K}{5}$ e

$K \bmod 5$ è IL RESTO DELLA DIVISIONE DI K in 5

$$A^K = A^{K \bmod 5} \cdot \underbrace{A^{\lfloor K/5 \rfloor 5}}_{= I^{\lfloor K/5 \rfloor} = I} = A^{K \bmod 5}$$

quindi $A^K x_0 = A^{K \bmod 5} x_0$ è periodico

di periodo PARI A 5 PASSI.