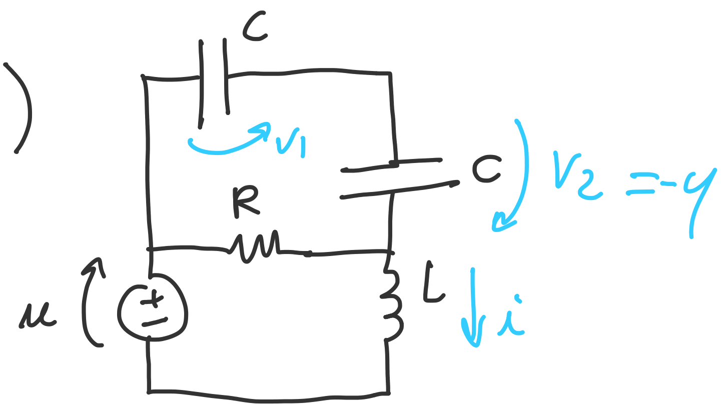
mercoledì 24 novembre 2021 10:51

1)



$$C\dot{V}_1 = -i - \frac{\cancel{u} + V_1 + V_2 - \cancel{u}}{R} = -i - \left(\frac{V_1 + V_2}{R}\right)$$

$$C\dot{V}_2 = -i - \left(\frac{V_1 + V_2}{R}\right)$$

$$L\dot{i} = u + V_1 + V_2$$

$$x = \begin{bmatrix} V_1 \\ V_2 \\ i \end{bmatrix}$$

$$\dot{x} = \overbrace{\begin{bmatrix} -\frac{1}{RC} & -\frac{1}{RC} & -\frac{1}{C} \\ -\frac{1}{RC} & -\frac{1}{RC} & -\frac{1}{C} \\ \frac{1}{L} & \frac{1}{L} & 0 \end{bmatrix}}^{A} x + \overbrace{\begin{bmatrix} 0 \\ 0 \\ \frac{1}{L} \end{bmatrix}}^{B} u$$

$$y = \underbrace{\begin{bmatrix} 0 & -1 & 0 \end{bmatrix}}_{C} x$$

$$X_R(1) = \mathrm{Im}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \mathrm{Im}\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1/L \end{bmatrix} \quad \underbrace{\begin{bmatrix} 1 \end{bmatrix}}_{M}$$

$$X_R(2) = X_R(1) + \mathrm{Im}\, AM = X_R(1) + \mathrm{Im}\begin{bmatrix} -1/C \\ -1/C \\ 0 \end{bmatrix}$$

$$= \mathrm{Im}\begin{array}{c} \begin{array}{cc} & M \end{array} \\ \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array}$$

$$X_R(3) = X_R(2) + \mathrm{Im}\, AM = X_R(2) + \mathrm{Im}\begin{bmatrix} -2/RC \\ -2/RC \\ 2/L \end{bmatrix}$$

$$= X_R(2) = X_R$$

2) $$\chi_A(\lambda) = (\lambda^2 - 4\lambda + 4)\lambda$$
$$= (\lambda - 2)^2 \lambda$$

$\lambda = 2$ $$\ker(A - 2I) = \ker\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 4 & 2 & -2 \end{bmatrix}$$

$$= \mathrm{Im}\begin{array}{c} v_1 \\ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \end{array}, \quad \dim \ker(A - 2I) = 1$$

$$\Rightarrow \mu_A(\lambda) = \chi_A(\lambda) \quad @$$

$$\ker(A - 2I)^2 = \ker\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -8 & 0 & 4 \end{bmatrix} = \mathrm{Im}\begin{array}{c} \begin{array}{cc} & v_2 \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 0 \end{bmatrix} \end{array}$$

$\lambda = 0$

$$\ker A = \mathrm{Im}\begin{array}{c} v_3 \\ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{array}$$

$$A^k v_2 = \lambda^{k} v_2 + \lambda^{k-1} k (A-2I) v_2$$

$$= 2^k \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 2^{k-1} k \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} k\, 2^{k-1} \\ 2^k \\ 2k \cdot 2^{k-1} \end{bmatrix}$$

$$A^k = \left[A^k v_1, A^k v_2, A^k v_3 \right] \left[v_1, v_2, v_3 \right]^{-1}$$

$$= \begin{bmatrix} 2^k & k \cdot 2^{k-1} & 0 \\ 0 & 2^k & 0 \\ 2 \cdot 2^k & 2 \cdot k \cdot 2^{k-1} & \delta(k) \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}^{-1}}_{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}}$$

$$= \begin{bmatrix} 2^k & k \cdot 2^{k-1} & 0 \\ 0 & 2^k & 0 \\ 2(2^k - \delta(k)) & 2k \cdot 2^{k-1} & \delta(k) \end{bmatrix}$$ (b)

c) $\sigma(A) = \{2, 0\} \rightarrow$ IL SISTEMA È INSTABILE

3) $$X_R(1) = \mathrm{Im}\, B = \mathrm{Im} \overbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}^{M}$$

$$X_R(2) = X_R(1) + \mathrm{Im}\, AM = X_R(1) + \mathrm{Im} \begin{bmatrix} 1 & 0 \\ -2 & 1 \\ -1 & 2 \\ -1 & 0 \end{bmatrix}$$

$$= \mathrm{Im} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & -2 & 1 \\ \dots \end{bmatrix} = \mathrm{Im} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ \dots \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & -1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & \underbrace{-1}_{M} \end{bmatrix}$$

$$X_R(3) = X_R(2) + \operatorname{Im} AM = X_R(2) + \operatorname{Im}\begin{bmatrix} 0 \\ -2 \\ -4 \\ 0 \end{bmatrix}$$

$$= X_R(2) = X_R$$

4) e) $X_R(1) = \operatorname{Im} B$

$$X_R(2) = \operatorname{Im}[B, AB] = \operatorname{Im}\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & -1 \\ 1 & -1 \end{bmatrix}$$

$$X_R(3) = \operatorname{Im}[B, AB, A^2B] = \operatorname{Im}\begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 1 \\ 1 & -1 & 1 \end{bmatrix} = X_R(2) = X_R$$

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{bmatrix} \qquad T^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{bmatrix}$$

$$\underbrace{\text{cols } 1,2}_{T1} \quad \underbrace{\text{cols } 3,4}_{T2}$$

$$\hat{A} = T^{-1}AT = T^{-1}\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 1 & 0 \\ 1 & -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

A_R 0 A_{NR}

$$\hat{B} = T^{-1}B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$\to B_R$ $\to 0$

$$\hat{C} = CT = [\,\underbrace{1\ \ 0}_{C_R}\ \ \underbrace{0\ \ 0}_{C_{NR}}\,]$$

$$H(s) = C_R\,(sI - A_R)^{-1} B_R$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s+1 & -1 \\ 0 & s \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{1}{s(s+1)} \begin{bmatrix} s & +1 \\ 0 & s+1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{s(s+1)}$$

5) $$X_R(2) = \operatorname{Im}[B, AB] = \operatorname{Im}\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -2 & e+1 \\ 1 & -1 \end{bmatrix} = \operatorname{Im}\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -2 & e-1 \\ 1 & 0 \end{bmatrix}$$

se $e = 1$ $\quad X_R(2) = X_R(1) = X_R$

se $e \neq 1$

$$X_R(2) = \operatorname{Im}\begin{array}{c} \begin{array}{cc} & M \end{array} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -2 & 1 \\ 1 & 0 \end{bmatrix} \end{array}$$

$$X_R(3) = X_R(2) + \operatorname{Im} AM = X_R(2) + \operatorname{Im}\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \operatorname{Im}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 1 \end{bmatrix} = \operatorname{Im}\begin{array}{c} \begin{array}{ccc} & & M \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

$\lfloor 1 \;\; 0 \;\; 0 \rfloor \qquad \lfloor 1 \;\; 0 \;\; 0 \rfloor$

$$X_R(4) = X_R(3) + \mathrm{Im}\, AM = X_R(3) + \mathrm{Im}\begin{bmatrix} a \\ 1 \\ 0 \\ a \end{bmatrix}$$

$$= \mathrm{Im}\begin{bmatrix} 0 & 0 & 0 & a \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & a \end{bmatrix} = \begin{cases} X_R(3) = X_R & \text{se } a = 0 \\ \mathbb{R}^4 & \text{se } a \neq 0 \end{cases}$$

quindi $$X_R = \begin{cases} \mathrm{Im}\begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix} & \text{se } a = 1 \\ \mathrm{Im}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} & \text{se } a = 0 \\ \mathbb{R}^4 & \text{altrimenti} \end{cases}$$

6) a) Sia $x_0 \in V$, $x(t) = e^{At} x_0$

$$e^{At} A x_0 = A e^{At} x_0 = A x(t)$$

quindi $e^{At}(A x_0)$ è periodica, essendo $x(t)$ periodica e $A x_0 \in V$

essendo x_0 generico $A(V) \subset V$

b) Per assurdo, sia λ autovalore

b) PER ASSURDO, SIA λ AUTOVALORE

DI $a|_V$, con $\mathrm{Re}\{\lambda\} \neq 0$ e

SIA $x \in \mathbb{C}^n$ UN AUTOVETTORE ASSOCIATO,

ALLORA $a|_V x = \lambda x \Rightarrow Ax = \lambda x$

$$\Rightarrow e^{At} x = e^{\lambda t} x \qquad \lambda = a + bj$$

$$\|e^{At} x\| = |e^{\lambda t}| \, \|x\|$$

$$= |e^{at}| \underbrace{\cancel{|e^{bjt}|}}_{=1} \|x\|$$

$$= e^{at} \|x\|$$

se $a > 0$ $\|e^{At} x\|$ è MONOTONA STRETTAMENTE CRESCENTE

se $a < 0$ $\|e^{At} x\|$ è MONOTONA STRET. DECRESCENTE,

IN ENTRAMBI I CASI $e^{At} x$ NON

è PERIODICA.