1) $\sigma(A) = \{3, 2, -1\}$

$$X_R(1) = \text{Im}\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \quad X_R(2) = \text{Im}\begin{bmatrix} 1 & 3 \\ -1 & -2 \\ -1 & -3 \end{bmatrix}$$

$$= \text{Im}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$X_R(2) = \text{Im}\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 2 \\ -1 & 0 & 0 \end{bmatrix} = X_R(1) = X_R$$

$$T_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \quad T^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\hat{A} = T^{-1}AT = T^{-1}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ -3 & 0 & -1 \end{bmatrix} = \left[\begin{array}{cc|c} 3 & 0 & 0 \\ 0 & 2 & 0 \\ \hline 0 & 0 & -1 \end{array}\right] \quad A_R, \; A_{NR}$$

$$\hat{B} = T^{-1}B = \left[\begin{array}{c} 1 \\ -1 \\ \hline 0 \end{array}\right] B_R$$

$\lambda = -1$ è NON raggiungibile $\Rightarrow$ a) NON è possibile
b) è possibile

b) $$F_R = -\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -1 & -2 \end{bmatrix}^{-1}(A_R + I)^2$$

$$f_R = - [1, 1] \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}^2$$

$$= - [1, 1] \begin{bmatrix} 16 & 0 \\ 0 & 9 \end{bmatrix} = - [16, 9]$$

$$\hat{F} = [-16, -9, 0]$$

$$u = \hat{F} z = \hat{F} T^{-1} x$$

$$F = \hat{F} T^{-1} = [-16, -9, 0]$$

2) $X_{no}(0) = \operatorname{Ker} C = \operatorname{Im} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}$

$$X_{no}(1) = \operatorname{Ker} \begin{bmatrix} C \\ CA \end{bmatrix} = \operatorname{Ker} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

$$= \operatorname{Im} \begin{bmatrix} 0 & 1 \\ 0 & -1 \\ 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$X_{no}(2) = \operatorname{Ker} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix} = \operatorname{Ker} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \operatorname{Im} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$X_{no}(3) = \operatorname{Ker} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 1 & 0 & 2 \end{bmatrix} = X_{no}(2)$$

$X_{no}(k) = X_{no}(2), \quad \forall k \geq 2.$

3) e) $\mathcal{X}_R(1) = \operatorname{Im} B = \operatorname{Im}\begin{bmatrix}0\\1\\0\\0\end{bmatrix}$

$\mathcal{X}_R(2) = \operatorname{Im}\begin{bmatrix}B, AB\end{bmatrix} = \operatorname{Im}\begin{bmatrix}0 & 1\\1 & -1\\0 & 0\\0 & 0\end{bmatrix} = \operatorname{Im}\begin{bmatrix}0 & 1\\1 & 0\\0 & 0\\0 & 0\end{bmatrix}$

$\mathcal{X}_R(3) = \operatorname{Im}\begin{bmatrix}0 & 1 & 1\\1 & 0 & -1\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix} = \mathcal{X}_R(2) = \mathcal{X}_R$

$\mathcal{X}_{NO}(0) = \operatorname{Ker} C$

$\mathcal{X}_{NO}(1) = \operatorname{Ker}\begin{bmatrix}C\\CA\end{bmatrix} = \operatorname{Ker}\begin{bmatrix}1 & 1 & 0 & 1\\0 & 0 & 0 & 2\end{bmatrix} = \operatorname{Ker}\begin{bmatrix}1 & 1 & 0 & 1\\0 & 0 & 0 & 1\end{bmatrix}$

$\mathcal{X}_{NO}(2) = \operatorname{Ker}\begin{bmatrix}1 & 1 & 0 & 1\\0 & 0 & 0 & 1\\0 & 0 & 0 & 1\end{bmatrix} = \mathcal{X}_{NO}(1)$

$\mathcal{X}_{NO} = \operatorname{Im}\begin{bmatrix}1 & 0\\-1 & 0\\0 & 1\\0 & 0\end{bmatrix}$

$\operatorname{Im} T_2 = \mathcal{X}_{NO} \cap \mathcal{X}_R = \operatorname{Im}\begin{bmatrix}1\\-1\\0\\0\end{bmatrix}$

$T_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \quad T_4 = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \quad T_3 = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$

$T = \begin{bmatrix}1 & 1 & 0 & 0\\0 & -1 & 0 & 0\\0 & 0 & 0 & 1\\0 & 0 & 1 & 0\end{bmatrix}, \quad T^{-1} = \begin{bmatrix}1 & 1 & 0 & 0\\0 & -1 & 0 & 0\\0 & 0 & 0 & 1\\0 & 0 & 1 & 0\end{bmatrix}$

$$\hat{A} = T^{-1} A T = T^{-1} \begin{bmatrix} 1 & 0 & -1 & 0 \\ -1 & 0 & 2 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \left[\begin{array}{cc|c|c} 0 & 0 & 1 & 0 \\ 1 & 0 & -2 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

$A_{R,O}$ $\quad 0 \quad$ $A_{R,NO}$ $\quad$ $A_{NR,O}$ $\quad$ $A_{NR,NO}$ $\quad 0$

$$\hat{B} = T^{-1} B = \begin{bmatrix} 1 \\ -1 \\ \hline 0 \\ 0 \end{bmatrix} \begin{array}{l} \to B_{R,O} \\ \to B_{R,NO} \\ \left.\begin{array}{c} \\ \\ \end{array}\right\} 0 \end{array}$$

$$\hat{C} = CT = \left[\begin{array}{c|c|c|c} 1 & 0 & 1 & 0 \end{array}\right]$$

$C_{R,O}$ $\quad 0 \quad$ $C_{NR,O}$ $\quad 0$

b) $H(s) = C_{R,O}\,(sI - A_{R,O})^{-1}\, B_{R,O} = 1 \cdot s^{-1} \cdot 1 = \dfrac{1}{s}$

4) $P = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$, $A^T P + PA + Q - PBR^{-1}B^T P = 0$

$$\begin{bmatrix} 0 & 0 \\ a+b & b+c \end{bmatrix} + \begin{bmatrix} 0 & a+b \\ 0 & b+c \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} b \\ c \end{bmatrix} \begin{bmatrix} b & c \end{bmatrix} = 0$$

$$\begin{cases} 1 - b^2 = 0 \\ a + b - bc = 0 \\ 2b + 2c + 1 - c^2 = 0 \end{cases}$$

$b = \pm 1$

se $b = -1 \rightarrow -2 + 2c + 1 - c^2 = 0 \rightarrow c^2 - 2c + 1 = 0$

$\rightarrow c = 1 \rightarrow a - 1 + 1 = 0 \rightarrow a = 0$ non accettabile

se $b = 1 \rightarrow 2 + 2c + 1 - c^2 = 0 \rightarrow c^2 - 2c - 3 = 0$

$c = 1 \pm \sqrt{4} = 3, -1 \rightarrow$ non accettabile.

$c = 3$, $a + 1 - 3 = 0 \rightarrow a = 2$

$P = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$, $u^*(t) = -R^{-1} B^T P x(t)$

$= \begin{bmatrix} -1, & -3 \end{bmatrix} x(t)$

5) $\Rightarrow$ Facendo la scomposizione di Kalman si ha che $X_R = X_R \cap X_{NO}$

quindi $T_1 = [\ ]$ da cui $H(s) = 0$

$\Leftarrow$ Per assurdo, sia $v \neq 0$, $v \in X_R$, $v \notin X_{NO}$, essendo $v \in X_R$

esiste un controllo $\bar{u}$ definito sull'intervallo temporale $[0,1]$ tale che la sol. del sistema con $x(0) = 0$ soddisfa $x(1) = v$

essendo $x(1) = v \notin X_{NO}$

$y(1) = C x(1) \neq 0$, quindi $H(s) \neq 0$.