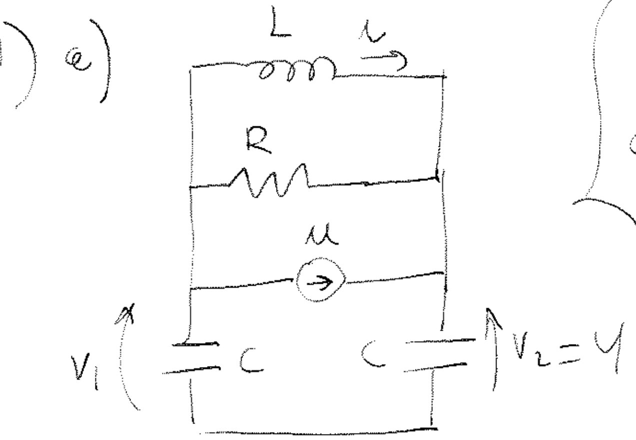
1) a)



$$\begin{cases} C\dot{v}_1 = -u - i - \dfrac{v_1 - v_2}{R} \\ C\dot{v}_2 = u + i + \dfrac{v_1 - v_2}{R} \\ L\dot{i} = v_1 - v_2 \end{cases}$$

$$x = \begin{bmatrix} v_1 \\ v_2 \\ i \end{bmatrix} \qquad \dot{x} = \underbrace{\begin{bmatrix} -\frac{\cancel{R}}{CR} & \frac{\cancel{R}}{CR} & -\frac{1}{C} \\ \frac{1}{RC} & -\frac{1}{RC} & \frac{1}{C} \\ \frac{1}{L} & -\frac{1}{L} & 0 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} -\frac{1}{C} \\ \frac{1}{C} \\ 0 \end{bmatrix}}_{B} u$$

$$y = \underbrace{[0 \;\; 1 \;\; 0]}_{C} x + \underbrace{0}_{D} u$$

b) $X_R(1) = \operatorname{Im} B = \operatorname{Im} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

$$X_R(2) = X_R(1) + \operatorname{Im} A \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = X_e(1) + \operatorname{Im} \begin{bmatrix} \frac{2}{RC} \\ -\frac{2}{RC} \\ -\frac{2}{L} \end{bmatrix} = \operatorname{Im} \begin{bmatrix} -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$X_R(3) = X_R(2) + \operatorname{Im} A \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = X_R(2) + \operatorname{Im} \begin{bmatrix} -\frac{1}{C} \\ \frac{1}{C} \\ 0 \end{bmatrix} = X_R(2)$$

$$X_R = X_R(2)$$

c) $$T = \begin{bmatrix} +1 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = [\underbrace{\,T_1\,}\ \ \underbrace{\,T_2\,}], \quad T^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\hat{A} = T^{-1} A T = T^{-1} \begin{bmatrix} -\frac{2}{RC} & -\frac{1}{C} & \frac{1}{RC} \\ \frac{2}{RC} & +\frac{1}{C} & -\frac{1}{RC} \\ \frac{2}{L} & 0 & -\frac{1}{L} \end{bmatrix} = \left[\begin{array}{cc|c} -\frac{2}{RC} & -\frac{1}{C} & \frac{1}{RC} \\ \frac{2}{L} & 0 & -\frac{1}{L} \\ \hline 0 & 0 & 0 \end{array}\right]$$

A_R

A_{NR}

0

$$\hat{B} = T^{-1} B = \begin{bmatrix} -\frac{1}{C} \\ 0 \\ 0 \end{bmatrix} \begin{matrix} \}B_R \\ \\ \}B_{NR} \end{matrix}, \quad \hat{C} = CT = [\underbrace{-1 \quad 0}_{C_R} \quad \underbrace{1}_{C_{NR}}], \quad \hat{D} = 0$$

d) $$H(s) = C_R (sI - A_R)^{-1} B_R = [-1, 0] \begin{bmatrix} s + \frac{2}{RC} & \frac{1}{C} \\ -\frac{2}{L} & s \end{bmatrix}^{-1} \begin{bmatrix} -\frac{1}{C} \\ 0 \end{bmatrix}$$

$$= \frac{\frac{s}{C}}{s^2 + \frac{2}{RC} s + \frac{2}{LC}} = \frac{s}{Cs^2 + \frac{2}{R} s + \frac{2}{L}}$$

2) $\chi_A(\lambda) = (\lambda+2)(\lambda-3)(\lambda-2)$

$\lambda = -2$

$$V_{-2} = \operatorname{Ker}\left[A+2I\right] = \operatorname{Ker}\begin{bmatrix} 0 & 0 & 0 \\ -4 & 5 & 1 \\ 4 & 0 & 4 \end{bmatrix} = \operatorname{Im}\overset{v_1}{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}}$$

$\lambda = 3$

$$V_3 = \operatorname{Ker}\left[A-3I\right] = \operatorname{Ker}\begin{bmatrix} -5 & 0 & 0 \\ -4 & 0 & 1 \\ 4 & 0 & -1 \end{bmatrix} = \operatorname{Im}\overset{v_2}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}$$

$\lambda = 2$

$$V_2 = \operatorname{Ker}\left[A-2I\right] = \operatorname{Ker}\begin{bmatrix} -4 & 0 & 0 \\ -4 & 1 & 1 \\ 4 & 0 & 0 \end{bmatrix} = \operatorname{Im}\overset{v_3}{\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}}$$

$$A^k = \left[(-2)^k v_1,\ 3^k v_2,\ 2^k v_3\right]\left[v_1, v_2, v_3\right]^{-1}$$

$$= \begin{bmatrix} (-2)^k & 0 & 0 \\ (-2)^k & 3^k & 2^k \\ -(-2)^k & 0 & -2^k \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix}^{-1}}_{= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix}} = \begin{bmatrix} (-2)^k & 0 & 0 \\ (-2)^k - 2^k & 3^k & 3^k - 2^k \\ -(-2)^k + 2^k & 0 & 2^k \end{bmatrix}$$

3) a) $X_R(0) = \cancel{\mathrm{Im}}\ \{0\}$

$$X_R(1) = \mathrm{Im}\ B = \mathrm{Im}\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$X_R(2) = \mathrm{Im}\ [B, AB] = \mathrm{Im}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \mathrm{Im}\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$X_R(3) = X_R(2) + \mathrm{Im}\ A\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = X_R(2) + \mathrm{Im}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = X_R(2) = X_R$$

b) $$T = [T_1, T_2] = \left[\underbrace{\begin{matrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{T_1} \ \underbrace{\begin{matrix} 1 \\ 0 \\ 0 \\ 0 \end{matrix}}_{T_2}\right], \quad T^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\hat{A} = T^{-1}AT = T^{-1}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \left[\begin{array}{ccc|c} \multicolumn{3}{c|}{A_R} & \\ 0 & 1 & 1 & 1 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \multicolumn{3}{c|}{0} & A_{NR} \end{array}\right]$$

$$\hat{B} = T^{-1}B = \begin{array}{c} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \end{array} \begin{array}{l} \big] B_R \\ \big] 0 \end{array}$$

4) $R = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ -1 & -2 & 1 \end{bmatrix}$ $\qquad \varphi = [0 \ 0 \ 1] R^{-1} = [1 \ 0 \ 1]$

$$P = \begin{bmatrix} \varphi \\ \varphi A \\ \varphi A^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} \qquad P^{-1} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & -1 \end{bmatrix}$$

$$A_c = PAP^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ 2 & -1 & 1 \end{bmatrix} P^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$B_c = PB = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$f = -\varphi\, d(A) \qquad d(\lambda) = (\lambda+1)^3 = \lambda^3 + 3\lambda^2 + 3\lambda + 1$

$= -\varphi(A^3 + 3A^2 + 3A + I) = -\varphi A^3 - 3\varphi A^2 - 3\varphi A - \varphi$

$\varphi A^3 = [2 \ -1 \ 1]$

$f = -\left([2 \ -1 \ 1] + [3 \ -3 \ 0] + [0 \ 3 \ 0] + [1 \ 0 \ 1]\right)$

$= -[6, -1, 2] = [-6, 1, -2]$

5) $$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix}}_{\bar{A}} \begin{bmatrix} x \\ z \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ B \end{bmatrix}}_{\bar{B}} u$$

$$\bar{R} = \left[\bar{B}, \bar{A}\bar{B}, \ldots, \bar{A}^{2n-1}\bar{B} \right]$$

$$= \begin{bmatrix} 0 & B & 0 & AB & & \cancel{A^{n-1}B} & 0 & A^{n-1} \\ B, & 0, & AB, & 0, & \ldots, & \cancel{A^{n-1}B}, & A^{n-1}B, & B \end{bmatrix}$$

$$\operatorname{rank} \bar{R} = \operatorname{rank} \begin{bmatrix} B, AB, \ldots, A^{n-1}B, & 0, & 0, \ldots, 0 \\ 0 & 0, \ldots, 0 & , B, AB, \ldots, A^{n-1}B \end{bmatrix}$$

$$= 2 \operatorname{rank} \left[B, AB, \ldots, A^{n-1}B \right]$$

$$= 2 \operatorname{rank} R$$

quindi $\operatorname{rank} \bar{R} = 2n \iff \operatorname{rank} R = n$

$\iff (A, B)$ è raggiungibile