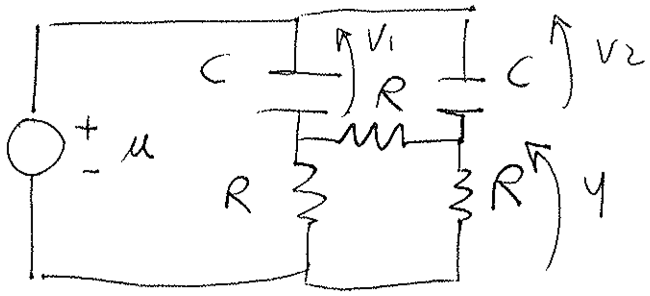
ES. 1



a)

$$C\dot{V}_1 = \frac{u - V_1}{R} + \frac{V_2 - V_1}{R}$$

$$C\dot{V}_2 = \frac{u - V_2}{R} + \frac{V_1 - V_2}{R}$$

$$y = u - V_2$$

$$x = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \quad \begin{cases} \dot{x} = \frac{1}{RC}\overbrace{\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}}^{A} x + \frac{1}{RC}\overbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}^{B} \\ y = \overbrace{\begin{bmatrix} 0 & -1 \end{bmatrix}}^{C} x + \overset{D}{\begin{bmatrix} 1 \end{bmatrix}} u \end{cases}$$

b) $$\chi_A(\lambda) = \lambda^2 + \frac{4}{RC}\lambda + \frac{3}{(RC)^2}$$

$$\lambda = -\frac{2}{RC} \pm \sqrt{\frac{4-3}{(RC)^2}} = -\frac{2}{RC} \pm \frac{1}{RC}$$

$$= -\frac{3}{RC}, -\frac{1}{RC}$$

$$\sigma(A) = \left\{ -\frac{3}{RC}, -\frac{1}{RC} \right\}$$

$$\text{modi} = \left\{ e^{-\frac{3}{RC}t}, e^{-\frac{t}{RC}} \right\}$$

c) $$R = \begin{bmatrix} \frac{1}{RC} & -\frac{1}{(RC)^2} \\ \frac{1}{RC} & -\frac{1}{(RC)^2} \end{bmatrix}$$

$$X_R = \operatorname{Im} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Es 2

$$\chi_A(\lambda) = (\lambda-2)(\lambda^2+1) = (\lambda-2)(\lambda+j)(\lambda-j)$$

$$\sigma(A) = \{2, j, -j\}$$

$$\text{Ker}\,[2I-A] = \text{Ker}\begin{bmatrix} 0 & 6 & 6 \\ 0 & 1 & -1 \\ 0 & 2 & 3 \end{bmatrix} = \text{Im}\overset{v_1}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}$$

$$\text{Ker}\,[j-A] = \text{Ker}\begin{bmatrix} j-2 & 0 & 0 \\ 0 & j-1 & -1 \\ 0 & 2 & j+1 \end{bmatrix} = \text{Im}\overset{v_2}{\begin{bmatrix} 0 \\ 1 \\ j-1 \end{bmatrix}}$$

$$e^{At}v_1 = \begin{bmatrix} e^{2t} \\ 0 \\ 0 \end{bmatrix} \qquad e^{At}v_2 = e^{jt}\begin{bmatrix} 0 \\ 1 \\ j-1 \end{bmatrix} = \begin{bmatrix} 0 \\ \cos t \\ -\cos t - \text{sen}\, t \end{bmatrix} + j\begin{bmatrix} 0 \\ \text{sen}\, t \\ \cos t - \text{sen}\, t \end{bmatrix}$$

a)

$$\Psi(t) = \begin{bmatrix} e^{2t} & 0 & 6 \\ 0 & \cos t & \text{sen}\, t \\ 0 & -\cos t - \text{sen}\, t & \cos t - \text{sen}\, t \end{bmatrix}$$

$$\Psi(0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \qquad \Psi(0)^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

b)

$$e^{At} = \Psi(t)\Psi(0)^{-1} = \begin{bmatrix} e^{2t} & 0 & 0 \\ 0 & \cos t + \text{sen}\, t & \text{sen}\, t \\ 0 & -2\,\text{sen}\, t & \cos t - \text{sen}\, t \end{bmatrix}$$

c)

$$e^{At}x_0 = \begin{bmatrix} 0 \\ \cos t + \text{sen}\, t \\ -2\,\text{sen}\, t \end{bmatrix}$$

ES. 3

a) $$X_R(\phi) = \operatorname{Im} B = \operatorname{Im}\begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$X_R(2) = \operatorname{Im}[B, AB] = \operatorname{Im}\begin{bmatrix}1 & 1\\0 & 0\\0 & -1\end{bmatrix}$$

$$X_R(3) = \operatorname{Im}[B, AB, A^2B] = \operatorname{Im}\begin{bmatrix}1 & 1 & 1\\0 & 0 & 0\\0 & -1 & -2\end{bmatrix} = \operatorname{Im}\begin{bmatrix}1 & 1\\0 & 0\\0 & -1\end{bmatrix} = X_R(2)$$

$X_R(k) = X_R(2), \quad \forall k \geq 2.$

$$X_c(1) = [I \quad 0] \operatorname{Ker}[A \; B] = [I \quad 0] \operatorname{Ker}\left[\overbrace{\begin{matrix}1 & -1 & 0\\0 & 0 & 0\\-1 & 2 & 1\end{matrix}}^{A}\ \begin{matrix}1\\0\\0\end{matrix}\right]$$

$$= [I \quad 0] \operatorname{Im}\begin{bmatrix}1 & 1\\1 & 0\\-1 & 1\\0 & -1\end{bmatrix} = \operatorname{Im}\begin{bmatrix}1 & 1\\1 & 0\\-1 & 1\end{bmatrix}$$

$$X_c(2) = [I \quad 0] \operatorname{Ker}\left[\underbrace{\begin{matrix}1 & -1 & 0\\0 & 0 & 0\\-2 & 3 & 1\end{matrix}}_{A^2}\ \begin{matrix}1 & 1\\0 & 0\\0 & -1\end{matrix}\right] \overset{[I\ 0]}{\overset{\downarrow}{=}} \operatorname{Im}\begin{bmatrix}1 & 1 & 0\\1 & 0 & 0\\-1 & 2 & 1\\0 & -1 & -1\\0 & 0 & 1\end{bmatrix} = \mathbb{R}^3$$

$X_c(k) = \mathbb{R}^3, \; \forall k \geq 2$

b) 1 PASSO → $0 = Ax_0 + Bu(0)$

$$0 = \begin{bmatrix}0\\0\\1\end{bmatrix} + \begin{bmatrix}1\\0\\0\end{bmatrix}u(0) \Rightarrow \text{non possibile}$$

2 PASSI

$$0 = A^2x_0 + ABu(0) + Bu(1)$$

$$\begin{bmatrix}0\\0\\1\end{bmatrix} + \begin{bmatrix}1 & 1\\0 & 0\\-1 & 0\end{bmatrix}\begin{bmatrix}u(0)\\u(1)\end{bmatrix} = 0 \Rightarrow \begin{matrix}u(0) = 1\\u(1) = -1\end{matrix}$$

ES 4   a)

$$B = \begin{array}{c}\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \\ \begin{matrix} b_1 & b_2 \end{matrix}\end{array}, \quad Ab_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \quad Ab_2 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} \in \mathrm{Im}\,[b_2]$$

$$A^2 b_1 = \cancel{\begin{bmatrix} 1 \\ -1 \\ -2 \\ 0 \end{bmatrix}} \begin{bmatrix} 0 \\ 2 \\ 0 \\ 4 \end{bmatrix} \in \mathrm{Im}\,[Ab_1]$$

$$X_R = \mathrm{Im} \begin{array}{c}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} \\ \begin{matrix} b_1 & b_2 & Ab_1 \end{matrix}\end{array} = \mathrm{Im} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad T^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\hat{A} = T^{-1} A T = T^{-1} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} = \left[\begin{array}{ccc|c} 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \begin{matrix} \nearrow A_R \\ \to A_{R,NR} \\ \downarrow 0 \\ \searrow A_{NR} \end{matrix}$$

$$\hat{B} = T^{-1} B = \left[\begin{array}{cc} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ \hline 0 & 0 \end{array}\right] \begin{matrix} B_R \\ \\ \\ B_{NR} \end{matrix} \qquad \hat{C} = CT = [\overbrace{1 \ \ 1 \ \cancel{|} \ 0}^{C_R} \,|\, \overset{C_{NR}}{0}]$$

b) $\sigma(A_R) = \{0, -1, 2\}, \quad \sigma(A_{NR}) = \{1\}$

c) $$H(s) = [1 \ 1 \ 0] \begin{bmatrix} s & 0 & -1 \\ 0 & s+1 & 0 \\ 0 & 0 & s-2 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \frac{[1 \ 1 \ 0] \left[\begin{array}{c|c|c} * & 0 & s+1 \\ \hline * & s(s-2) & 0 \\ \hline * & * & * \end{array}\right] \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}}{s(s+1)(s-2)} = \frac{[1 \ 1 \ 0] \begin{bmatrix} s+1 & 0 \\ 0 & s(s-2) \\ * & * \end{bmatrix}}{s(s+1)(s-2)}$$

$$= \left[\frac{1}{s(s-2)} \, , \, \frac{1}{s+1} \right]$$

ES 5

DIMOSTRIAMO che OGNI SOTTOSPAZIO INVARIANTE RISPETTO AD A CONTENENTE $\mathrm{Im}\, B$ CONTIENE X_R.

$\mathrm{Im}\, B \subset V$ per ipotesi

$A\, \mathrm{Im}\, B \subset AV \subset V$ perché V è INVAR. RISP. AD A

$$A^2\, \mathrm{Im}\, B = A \underbrace{A\, \mathrm{Im}\, B}_{\subset V} \subset AV \subset V$$

Proseguendo si dimostra che $A^k\, \mathrm{Im}\, B \subset V, \forall k \in \mathbb{N}$ quindi

$$\mathrm{Im}\, R = \underset{\subset V}{\mathrm{Im}\, B} + \underset{\subset V}{\mathrm{Im}\, AB} + \dots + \underset{\subset V}{\mathrm{Im}\, A^{n-1}B} \subset V$$