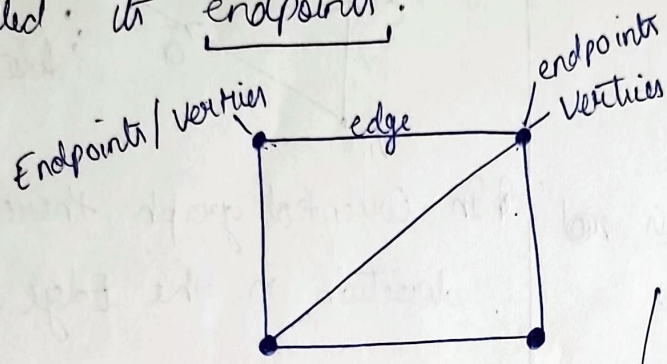
Graph Theory

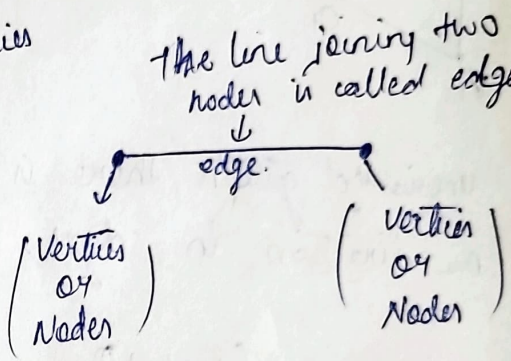
Introduction:-

→ Introduced by Euler.

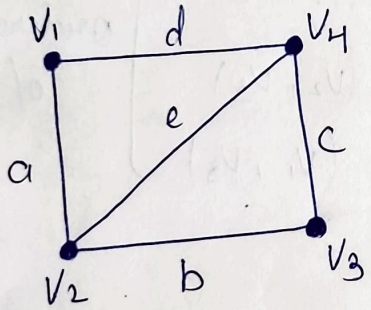
→ A Graph is a set of vertices (or nodes) and Edges (E).
denoted by (V) [set of vertices]
(E) [set of edges]

→ Each edge has one or two vertices associated with it, called its endpoints.





ex:-



So, V_1, V_2, V_3, V_4 are vertices (Nodes) & a, b, c, d, e are edges. ~~of this~~

Q in given graph find the set of ~~given~~ vertices & edges?

Sol formulae:- Vertices (V) = $\{V_1, V_2, V_3, V_4\}$

$\boxed{\text{Edges (E)} = \{a, b, c, d, e\}}$

$\boxed{\text{Edge (E)} = \{(a,d), (a,b), (b,c) (c,d)\}}$

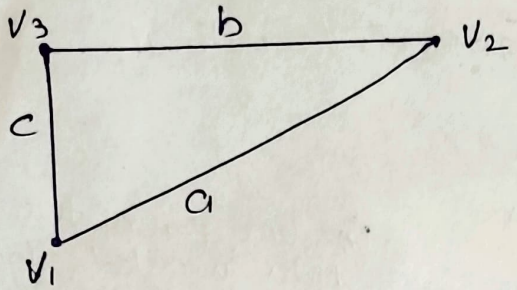
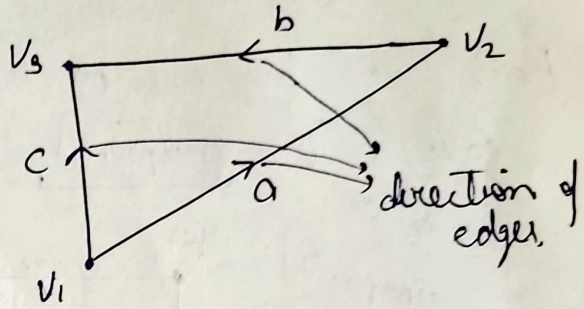
Graph (G) = (V, E)

Questions / Problems

① Utilities problem, ② Travelling Salesman Problem

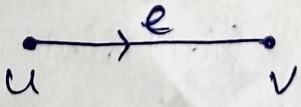
③ Round Robin Tournaments ④ Networking Problem.

Difference b/w Unoriented / Undirected Graph & Oriented / directed Graph :-

Unoriented ~~Undirected~~ / Undirected Graph	Oriented / directed Graph
	
In unoriented graph, there is no direction in edges.	In Oriented graph, there is direction in the Edge.
$a \to (v_1, v_2)$, $a \to (v_2, v_1)$; $b \to (v_3, v_2)$, $b \to (v_2, v_3)$; $c \to (v_3, v_1)$, $c \to (v_1, v_3)$ } unordered pairs of edges.	$a = (v_1, v_2)$, $b = (v_2, v_3)$, $c = (v_1, v_3)$] ordered pairs of edges

Suppose $e = (u, v)$ is a directed edge in graph.

then.



(i) u is called initial vertex of e & v is terminal vertex of e.

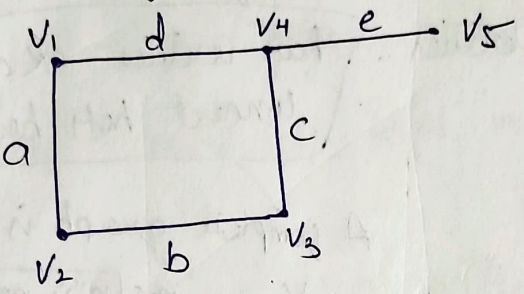
(ii) e is said to be incident from u & to be incident on v.

Types of Graph :-

① **Simple Graph :-** It is a graph in which each edge connects two different vertices and where no two edges connects the same pair of vertices.

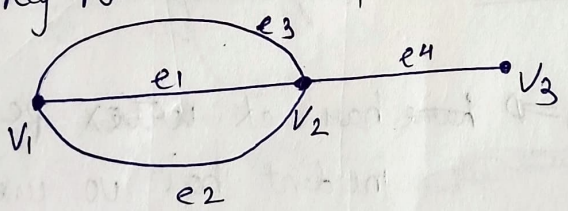
→ Simple Graph does not contain any self loop and parallel edges.

ex:-



② **Multi Graph :-** These graphs may have multiple edges connecting same vertices. or we can say that they have some parallel edges.

ex:-

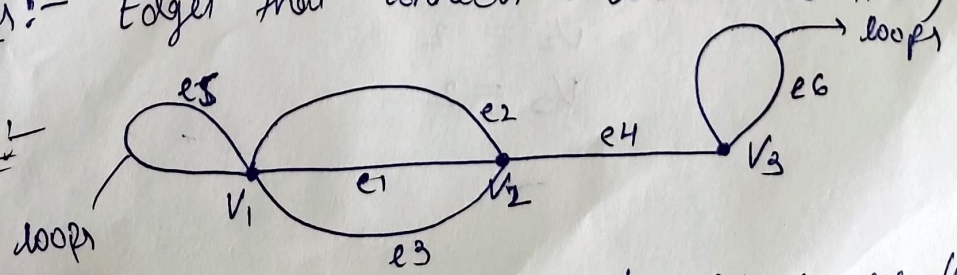


e_1, e_2, e_3 are parallel edges

③ **Pseudo Graph :-** These are the graph that include Loops, and possibly multiple edges connecting same pair of vertices.

Loops :- Edges that connects a vertices to itself.

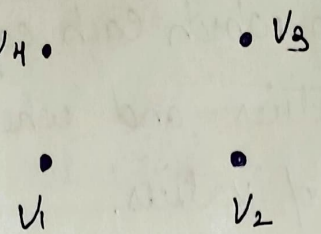
ex:-



edges e_5, e_6 are loops

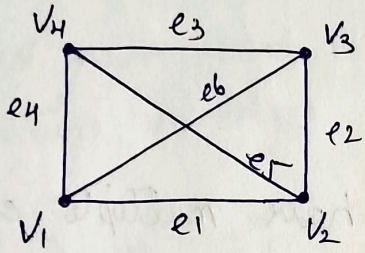
④ **Null Graph** :- it is a graph with no edges.

ex:-



⑤ **Complete Graph** :- it has an edge b/w every pair of vertices. (has vertices ek dusre se connect hoti hai)

ex:-



A complete graph is denoted by K_n, n is no. of vertices

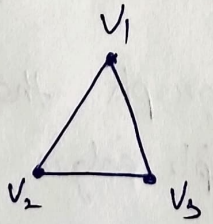
$$K_n = \frac{n(n-1)}{2} \text{ edges}$$

$\rightarrow K_n = \frac{4(4-1)}{2} \Rightarrow \frac{4\times3}{2} = \frac{12}{2}$ 6 edges

⑥ **Regular Graph** :- degree of each vertex is same.

degree of vertex ⇒ has ek vertex pe kitni edge incident hai vo use vertex ki degree hoti hai.

ex:-



V_1 vertex pe 2 edge incident ho rahi hai toh V_1 ki degree 2 hai.

degree $V_1 = 2$
$V_2 = 2$
$V_3 = 2$

Degree of vertex in Linear Graph :-

It is the number of edges including self loops that incident at vertex V is called degree of vertex.

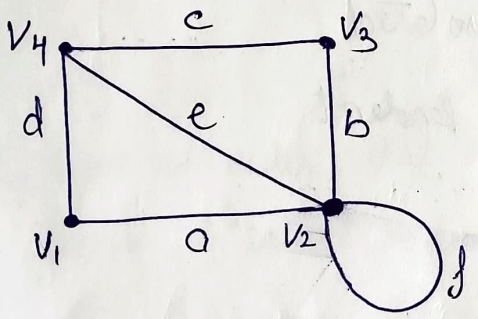
$$\boxed{\deg(V) = n_e + 2n_L}$$

$n_e \rightarrow$ no. of edges incident at vertex V.

$n_L \rightarrow$ no. of self loops incident at vertex V.

Q1 find the degree of each vertex in given graph.

(a)



Sol $\boxed{\deg(V_1) = n_e + 2n_L}$

$\deg(V_1) = (a, d) = 2$

~~$\deg(V_2) = (a, b, e, f)$~~

$\deg(V_2) \rightarrow$ no. of edges (a, b, e) , self loops (f)

$\deg(V_2) = 3 + 2\times1 \Rightarrow 3+2 = 5$

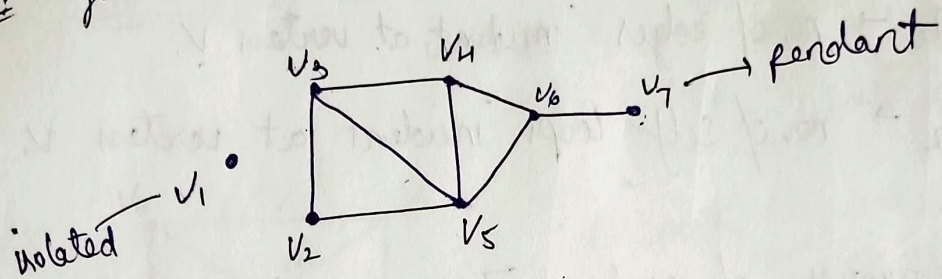
$\deg(V_3) = (c, b) = 2$

$\deg(V_4) = (c, d, e) = 3$

Imp. Points :- (i) vertex of degree zero is called as isolated vertex.

(ii) vertex of degree one is called as Pendant vertex.

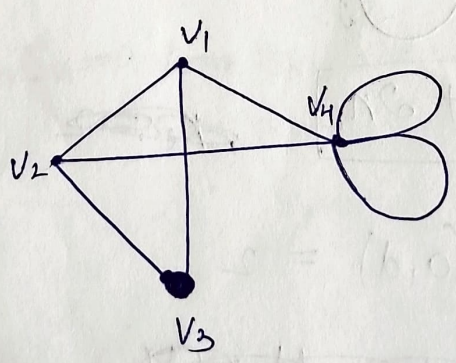
ex:- find which vertex is isolated or which is Pendant vertex.



$\deg(v_1) = 0 \rightarrow$ isolated

$\deg(v_7) = 1 \rightarrow$ pendant

Q2



Sol

$\deg(v_1) = 3 + 2 \times 0 = 3$

$\deg(v_2) = 3$

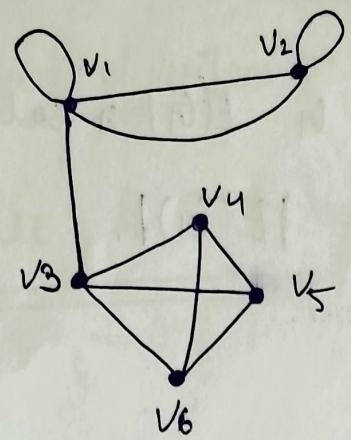
$\deg(v_3) = 2$

$\deg(v_4) = 2 + 2 \times 2$

$= 2 + 4$

$= 6.$

Q3



Sol

$\deg(v_1) = 3 + 2\times1 \Rightarrow 5$

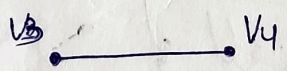
$\deg(v_2) = 2 + 2\times1 \Rightarrow 2+2 \Rightarrow 4$

$\deg(v_3) = 4$, $\deg(v_4) = 3$, $\deg(v_5) = 3$

$\deg(v_6) = 3$.

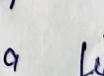
Note :-

(i) Any pair of nodes that is connected by an edge in a graph is called adjacent nodes.



(v_3, v_4) pair of nodes connected by an edge.

(ii) A node that is not adjacent to another node is called isolated node. ex:-



(iii) A graph $G(V, E)$ is said to finite if it has a finite no. of vertices and finite no. of edges, otherwise it is an infinite graph.

(iv) if G is finite the $V(G)$ denotes the no. of vertices in Graph (G) & is called the order of G. denoted by $|V(G)|$

(v) Similarly the no. of edges in G $E(G)$ is called the size of G. denoted by $|E(G)|$.

Note :-

max. no. of edges in simple graph with n vertices

$$= \boxed{\frac{n(n-1)}{2}}$$

$$\boxed{\text{deg} = \text{sum of all degree of vertices}}$$

$$\boxed{\text{deg} = 2 \times \text{no. of edges}}$$

max. degree of any vertex in simple graph with n vertices is $\boxed{n-1}$

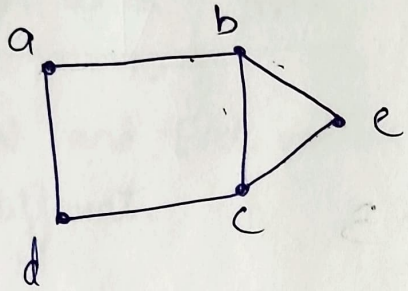
$$\boxed{\text{Length of a walk of a graph} = \text{no. of edges in a walk.}}$$

degree of vertex → No. of edges incident on a vertex.

Indegree & Out degree :- $O(v)$

$I(v)$

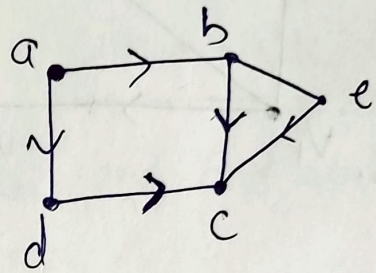
① Undirected graph



$\deg(a) = 2$

$\deg(b) = 3$

② Directed graph.



~~Indegree~~ :-

Outdegree → edges going away from a vertex.

Indegree → edges ~~coming~~ ~~comming~~ coming towards a vertex.

A vertex with zero indegree is called a source

A vertex with zero outdegree is called a sink

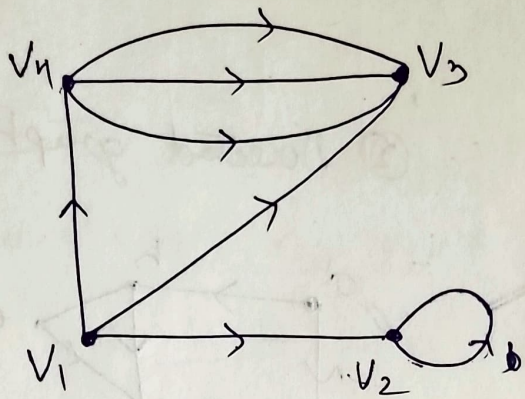
ex:- a se 2 edges bhar ki tarf ja rahi hai to vo outdegree me ayega.

$O(a) = 2$

or a se koi edge andar nhi a rhi hai to indegree 0 hogi

$I(a) = 0$.

Ex:- find in deg, & out deg



→ in deg $(V_1) = 0$, Out deg $(V_1) = 3$

Total degree $(V_1) = 3$.

→ in deg $(V_2) = 2$, out deg $(V_2) = 1$

when there is self loop it contributes 2 degrees 1 for incoming & 1 for outgoing.

→ in deg $(V_4) = 1$ out deg $(V_4) = 3$

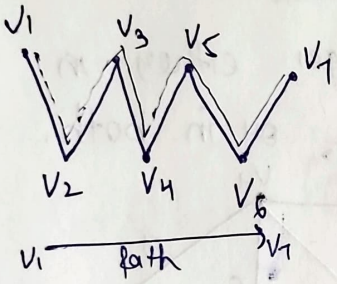
→ in deg $(V_3) = 4$, out deg $(V_3) = 0$

Walk on Graph :- (a particular sequence in which you travers a graph.)

A walk is nothing but sequence of edges / vertices of a given graph.

Open walk / Path

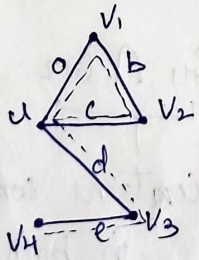
Initial and End vertex are different.



→ $(V_1, V_2, V_3, V_4, V_5, V_6, V_7)$

Initial vertex (V_1) } different
end vertex (V_7)

ex:-



Path (a, b, c, d, e)

$E(a) = (u, V_1)$ } means
$E(c) = (V_2, u)$ } 'u' appears twice.

So it is not an open path.

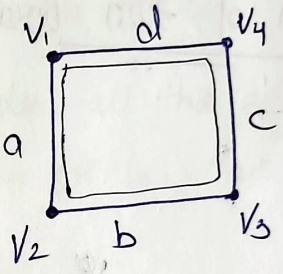
Path (bcde) → it is open path

→ No vertex should appear more than once.

Closed walk / Circuit

Initial and End vertex are Same.

ex:-



Path (a, b, c, d)

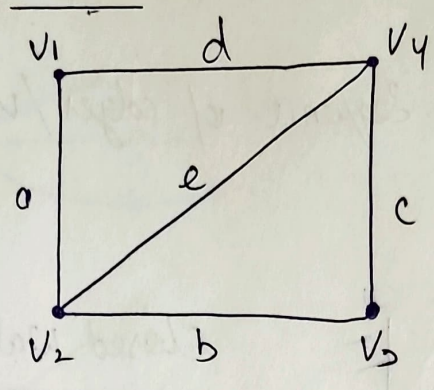
$V_1 \to V_2 \to V_3 \to V_4$

so initial & endpoints are same.

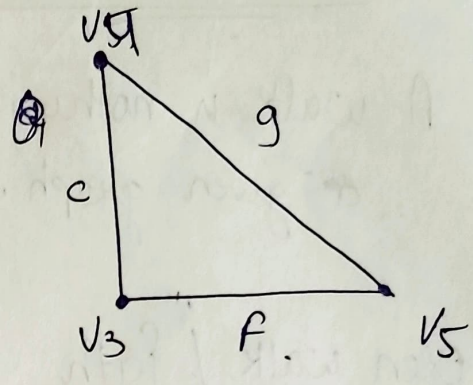
Also called as Circular Path, cycle.

Operation on Graph :-

ex:-



G_1

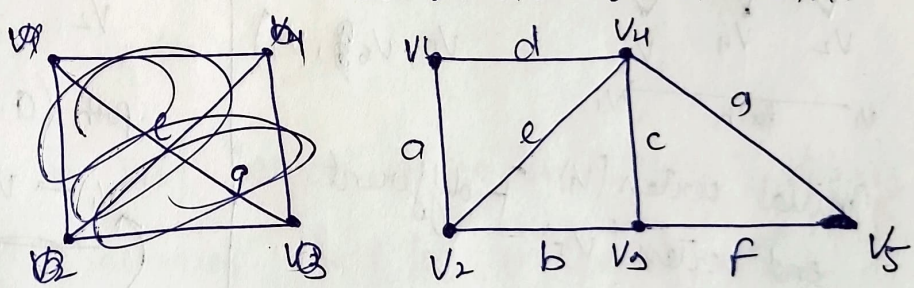


G_2

① **Union of two Graphs :-** $G = G_1 \cup G_2$

contains edges either in G_1 or G_2 or in both.

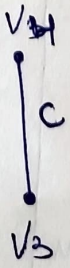
$G_1 \cup G_2 \Rightarrow$



② **Intersection of two graphs :-** $G = G_1 \cap G_2$

Contains common edges in both G_1 & G_2.

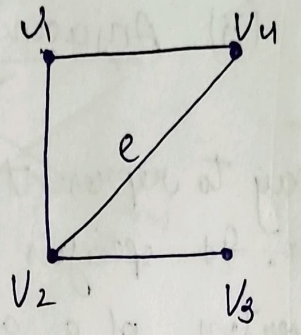
$G_1 \cap G_2 =$



③ <u>difference of two graphs</u> :- $G = \underbrace{G_1 - G_2}$

contains all the edges in G_1 but not in G_2

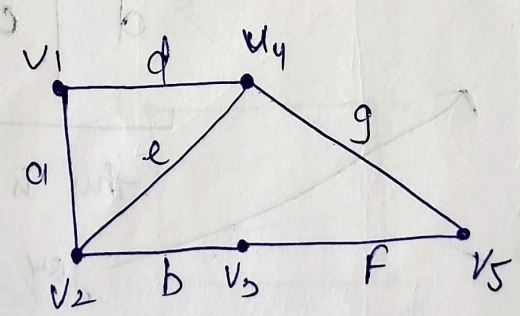
$G_1 - G_2 =$



④ <u>Addition of two graphs</u> :- $G = \underbrace{G_1 \oplus G_2}$

includes all the edges either in G_1 or G_2 but not in both

$G_1 \oplus G_2 = (G_1 \cup G_2) - (G_1 \cap G_2)$



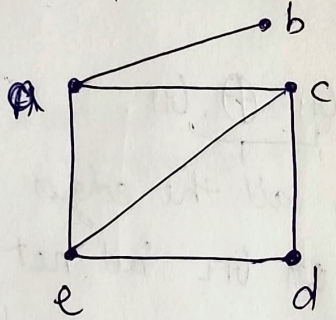
Graph Representation :-

We represents graphs using two methods

(i) Adjacency lists (ii) Adjacency Matrices.

(i) **Adjacency list** :- It is a way to represent a graph without multiple edges. It specifies all the vertices that are adjacent to each vertex of a graph.

ex:-



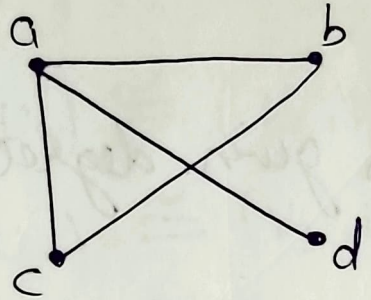
Vertex	Adjacent vertices
a	b, c, e
b	a
c	d, a, e
e	a, d, c
d	c, e

this is Adjacency list for this graph.

(ii) **Adjacency Matrices** :- $A(A_G)$ is a $n \times n$ zero-one matrix with 1 as its (i,j) entry when i & j are adjacent & 0 otherwise.

$$A = [a_{ij}] = \begin{cases} 1, & \text{if } (i,j) \text{ is edge of } G \\ 0, & \text{otherwise} \end{cases}$$

Q Represent following graph with use of adjacency matrix.

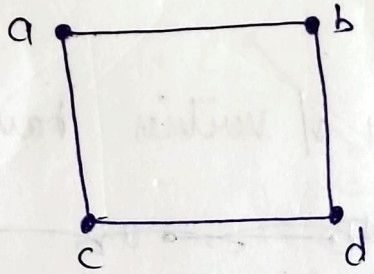


$$\Rightarrow \underbrace{\begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 1 & 1 & 1 \\ b & 1 & 0 & 1 & 0 \\ c & 1 & 1 & 0 & 0 \\ d & 1 & 0 & 0 & 0 \end{array}}_{\text{adjacency matrix.}}$$

Q draw a graph with adjacency Matrix.

$$\begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 1 & 1 & 0 \\ b & 1 & 0 & 0 & 1 \\ c & 1 & 0 & 0 & 1 \\ d & 0 & 1 & 1 & 0 \end{array}$$

Sol



graph.