Method to get DKL coordinates from LMS values (scaled to have equal luminance units such that luminance = L+M). The idea is explained in Stephen Westland's book and is based on David Brainard's Appendix (1996).

See the code lms2dkl available here: https://in.mathworks.com/matlabcentral/fileexchange/40640-computational-colour-science-using-matlab-2e. The same logic is also used in Psychtoolbox's ComputeDKL_M code, with the only exception is that LMS values are expressed in normal units (by integrating with the cone fundamentals that are scaled to unity). This code also takes in the cone fundamentals and CMF ybar and computes the scale that is required to convert LMS values in luminance units.

Suppose the LMS coordinates of the background are lms_b = [b1 b2 b3]'.

The logic used is as follows:

1. Start with Matrix B defined as:

$$B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -b1/b2 & 0 \\ -1 & -1 & (b1+b2)/b3 \end{bmatrix}$$

2. Invert this matrix to get B_inv.
3. Find the three isolating stimuli as columns of B_inv.
4. Normalize each isolating stimulus by its pooled contrast.
5. Get normalizing constants that are used to rescale B (using steps described below). Eventually we get a diagonal matrix D which is pre-multiplied to B.

Below, we show that the diagonal matrix D in step 5 can be computed directly as:

$$\boldsymbol{\frac{1}{(b1+b2)}\begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{1+(b2/b1)^2} & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

Proof: First, note that it is easy to invert B directly (i.e., by computing the Adjoint and Determinant), since it only has two entries which are different from 0 or 1.
B_inv is simply:

$$\frac{1}{(b1+b2)}\begin{bmatrix} b1 & b2 & 0 \\ b2 & -b2 & 0 \\ b3 & 0 & b3 \end{bmatrix}$$

Proof: $B*B^{-1}$ =

$$\frac{1}{b1+b2}\begin{bmatrix} 1 & 1 & 0 \\ 1 & -\frac{b1}{b2} & 0 \\ -1 & -1 & \frac{b1+b2}{b3} \end{bmatrix} X \begin{bmatrix} b1 & b2 & 0 \\ b2 & -b2 & 0 \\ b3 & 0 & b3 \end{bmatrix}$$

$$=\frac{1}{b1+b2}\begin{bmatrix} b1+b2 & b2-b2 & 0 \\ b1+b2(-\frac{b1}{b2}) & b2+(-\frac{b1}{b2})(-b2) & 0 \\ -b1-b2+\frac{b1+b2}{b3}b3 & -b2+b2 & \frac{b1+b2}{b3}b3 \end{bmatrix}$$

$$=\frac{1}{b1+b2}\begin{bmatrix} b1+b2 & 0 & 0 \\ 0 & b1+b2 & 0 \\ 0 & 0 & b1+b2 \end{bmatrix}$$ which is the identity matrix

Therefore, columns of B_inv are simply (using the terminology used in lms2dkl matlab code and assigning L=b1+b2):
Lum = (1/L) * [b1 b2 b3]'
Chro_LM = (1/L) * [b2 -b2 0]'
Chro_S = (1/L)* [0 0 b3]';

Pooled cone contrast:
Lum_pooled = (1/L) * norm ([b1/b1 b2/b2 b3/b3]) = (1/L)*√3
Chro_LM_pooled = (1/L) * norm([b2/b1 -b2/b2 0/b3]) = (1/L)*$\sqrt{1+(b2/b1)^2}$
Chro_S_pooled = (1/L) * norm([0/b1 0/b2 b3/b3] = 1/L

Therefore,
Lum_unit = Lum/lum_pooled = [b1 b2 b3]'/√3
ChroLM_unit = Chro_LM/chro_LM_pooled = [b2 -b2 0]/($\sqrt{1+(b2/b1)^2}$)
Chro_S_unit = Chro_S/chro_S_pooled = [0 0 b3]

Lum_norm = B*Lum_unit is

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & -\frac{b1}{b2} & 0 \\ -1 & -1 & \frac{b1+b2}{b3} \end{bmatrix}(\frac{1}{\sqrt{3}})[b1\ b2\ b3]^T = \left(\frac{1}{\sqrt{3}}\right) * [(b1+b2)\ 0\ 0]^T = \left[\frac{L}{\sqrt{3}}\ 0\ 0\right]^T.$$

The normalizing constant is the inverse of the first element, which is $\frac{\sqrt{3}}{L}$

Similarly,
chroLM_norm = B*chroLM_unit. Let c = $\sqrt{1 + (b2/b1)^2})$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & -\dfrac{b1}{b2} & 0 \\ -1 & -1 & \dfrac{b1+b2}{b3} \end{bmatrix} (\frac{1}{c})[b2 \ -b2 \ 0]^T = \left(\frac{1}{c}\right)[\ 0 \ (b1+b2) \ 0]^T = \left(\frac{1}{c}\right)[0 \ L \ 0]^T.$$

The normalizing constant is the inverse of the second element: c/L = $\sqrt{1 + (b2/b1)^2})/\mathrm{L}$

Finally,
chroS_norm = B*chroS_unit is simply

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & -\dfrac{b1}{b2} & 0 \\ -1 & -1 & \dfrac{b1+b2}{b3} \end{bmatrix} [0 \ 0 \ b3]^T = [\ 0 \ 0 \ (b1+b2)]^T = [0 \ 0 \ L]^T.$$

The normalizing constant is the inverse of the third element: $1/\mathrm{L}$

Hence, the full normalizing constant D is

$$\frac{1}{L}\begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{1+(b2/b1)^2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$