

Warp Drives and Closed Timelike Curves

Barak Shoshany*

✉ bshoshany@brocku.ca ID 0000-0003-2222-127X 🌐 <https://baraksh.com/>
 🏢 Department of Physics, Brock University
 📍 1812 Sir Isaac Brock Way, St. Catharines, Ontario, L2S 3A1, Canada

Ben Snodgrass

✉ benedict.snodgrass@math.uni-freiburg.de ID 0009-0002-0546-5045
 🏢 Department of Pure Mathematics, University of Freiburg
 📍 Ernst-Zermelo-Straße 1, 79104 Freiburg im Breisgau, Germany

Abstract

It is commonly accepted that superluminal travel may be used to facilitate time travel. This is a purely special-relativistic argument, using the fact that for observers in two frames of reference, separated by a spacelike interval, the non-causal (spacelike) future of one observer includes part of the causal past of the other. In this paper we provide a concrete realization of this argument in a curved general-relativistic spacetime, using warp drives as the means of faster-than-light travel. By generalizing the usual warp drive metric to allow for a non-unit lapse function, we allow the warp drive to switch between reference frames in a purely geometric way. With an additional modification allowing the warp drive to have compact support, this permits us to glue two warp drives together to construct a closed timelike geodesic, such that a test particle following the geodesics of the two warp drives travels back to its own past. This provides a precise mathematical model for the connection between faster-than-light travel and time travel in general relativity, and the first such model to be explicitly formulated using two warp drives. We also give a detailed discussion of weak energy condition violations in the non-unit-lapse warp drive.

Contents

1	Introduction	2
1.1	Warp drives	2
1.2	Superluminal travel and time travel	3
1.3	Warp drives and closed timelike curves	3
2	Geodesic rest frame transitions	5
2.1	Warp drives with unit lapse	5
2.2	Warp drives with non-unit lapse	7
3	Creating a closed timelike geodesic	10
3.1	The double-warp-drive metric	10
3.2	Requirements for time travel	11
3.3	The curvature tensors	13
3.4	A specific example	15

*Author to whom any correspondence should be addressed.

4 The weak energy condition with a non-unit lapse	19
4.1 Violations of the weak energy condition	19
4.2 Discussion and literature review	20
4.3 The case $\rho = 0 = \partial_{[i}\beta_{j]}$	22
5 Conclusions	24
A In-depth analysis of rest frame transitions	25
A.1 The setting	25
A.2 Overview and intuition	27
A.3 Formalism and proofs	28
B Tables of symbols	33
References	35

1 Introduction

1.1 Warp drives

Since Alcubierre's seminal 1994 paper [1], the warp drive has developed from a vague, science-fictional concept to a subject of genuine scientific interest [2, 3, 4]. Whilst not generally considered to be a realistic option for human interstellar travel in the near future, it serves as an interesting theoretical model, opening up new insights into the surprising possibilities that arise from the curved spacetime geometry of general relativity. Studying warp drives may also expose pathologies in the theory itself.

The Alcubierre warp drive is defined by the metric

$$ds^2 = -dt^2 + dx^2 + dy^2 + (dz - fv dt)^2, \quad (1.1)$$

where $f \equiv f(x, y, z - \zeta(t))$ with $f(0, 0, 0) = 1$, $f \rightarrow 0$ at spatial infinity, and $v \equiv v(t) \equiv \partial_t \zeta(t)$. Note that this metric is flat on a hypersurface $t = \text{constant}$ if and only if $v(t) = 0$. It can easily be shown that the path

$$\alpha(\tau) = (\tau, 0, 0, \zeta(\tau)), \quad \tau \in (-\infty, \infty), \quad (1.2)$$

is a geodesic, parameterised by proper time τ .

The idea behind the warp drive is to take advantage of a loophole in general relativity: that while all massive objects are constrained to move along timelike paths, space itself has no such restriction. Roughly speaking, a warp drive exploits this by having a shell of curved spacetime (the ‘warp bubble’) embedded in a flat background spacetime, which can accelerate its flat interior to arbitrarily high speeds, without the passengers inside feeling any acceleration whatsoever. An observer inside the warp bubble will follow a timelike geodesic, but its speed is unbounded – and in particular, it is not limited by the speed of light, so the observer can effectively travel between two spacelike-separated points.

There are numerous problems with warp drives as they have so far been conceived, even setting aside the extreme engineering difficulties. The most notable are their seemingly-inevitable violation of the energy conditions [5, 6], the typically vast amounts of (negative) energy required to sustain them [1, 7], and the formation of event horizons [1, 8].

Despite those issues, warp drives are an important theoretical tool in general relativity, most importantly in the study of causality and its violations. In this paper we will focus on the question of how warp drives can be used to create closed timelike curves. For this purpose we will propose a generalised warp drive model, and we will spend some time studying one of the above issues, the violation of the weak energy condition, in the context of this new model.

1.2 Superluminal travel and time travel

In this paper, we shall implement the following well-known method of using superluminal travel to facilitate time travel. Consider two reference frames, S and S' , with some relative velocity u along the x axis. Suppose one can travel a large distance at a superluminal speed¹ $v > 1$ in a given reference frame. Then, if u is large enough for a given v (as discussed in detail in Section 3.2), one can travel back in time by travelling superluminally between two points in the S frame, transitioning to the S' frame, and returning superluminally in the S' frame to the starting point.

This is proven mathematically in Section 3.2, but it is visually clear from the spacetime diagram² in Figure 1. Here, the traveller uses two warp drives to follow the path of the blue arrows, which both represent superluminal paths in S and S' respectively. First, they travel from $x = 0$ out along the upper arrow, finishing at a speed u and perhaps landing on a space station at rest in S' . Then, they travel back along the lower arrow³. The traveller arrives back at $x = 0$ *earlier* than when they left, so they have travelled backwards in time.

The light cones of S and S' are highlighted in pink; of course, the blue paths are outside the light cones, as they are spacelike. Note that along the upper arrow, moving in the positive x direction, the traveller is moving forwards in time according to S , but backwards in time according to S' . Similarly, along the lower arrow, moving in the negative x direction, the traveller is moving forwards in time according to S' , but backwards in time according to S . This is how the ‘trick’ works: for any object moving superluminally in one frame of reference, there is another frame of reference in which they are moving backwards in time.

1.3 Warp drives and closed timelike curves

To our knowledge, [9] was the first to argue that warp drives can be used to construct closed timelike curves in the fashion described above; see also a summary in [10]. While this work serves as a clear and useful proof of principle, it has several shortcomings, which we would like to highlight – and resolve – in this paper.

First, as the traveller moves from S to S' , they must also change their reference frame. The reason is that if they stayed in the reference frame of S , they must still move towards the future of S (i.e. towards increasing values of t), so time travel would be impossible.

There is something seemingly ‘magical’ about the frame transition from S to S' ; before the transition, even if the traveller moved at arbitrarily high speeds, $v \rightarrow \infty$, they still could never move back in time. However, once they land on the space station at rest in S' , and a Lorentz transformation is performed, travel to the past suddenly becomes possible.

In this paper, we will provide a concrete warp drive metric that takes this ‘landing’ into account, and therefore makes this ‘magical’ part precise. One issue that we will have to overcome is that warp drives, as previously formulated, reside fully within a single reference frame from beginning to end, and thus do not allow such rest frame transitions to occur. Of course, this is not an issue if one is travelling within the warp bubble in a spaceship; they can simply accelerate using the spaceship’s rockets to facilitate the landing on the moving target. However, this does introduce a significant conceptual hurdle, namely that it is impossible to create a closed timelike *geodesic* using a warp drive alone.

A closed timelike *curve* can be constructed, for example with rocket propulsion or perhaps some sort of conveyor belt, but it will not be a *geodesic* – meaning that it is not a feature of the spacetime itself, but requires external forces to work. Furthermore, including external acceleration will considerably

¹Throughout this paper we will be using units where $c = 1$.

²The reader may reproduce this spacetime diagram interactively using the Mathematica notebook available at <https://github.com/bshoshany/spacetime-diagrams>.

³In this diagram, changes in velocity happen instantaneously. This could of course be smoothed out to make the acceleration finite everywhere.

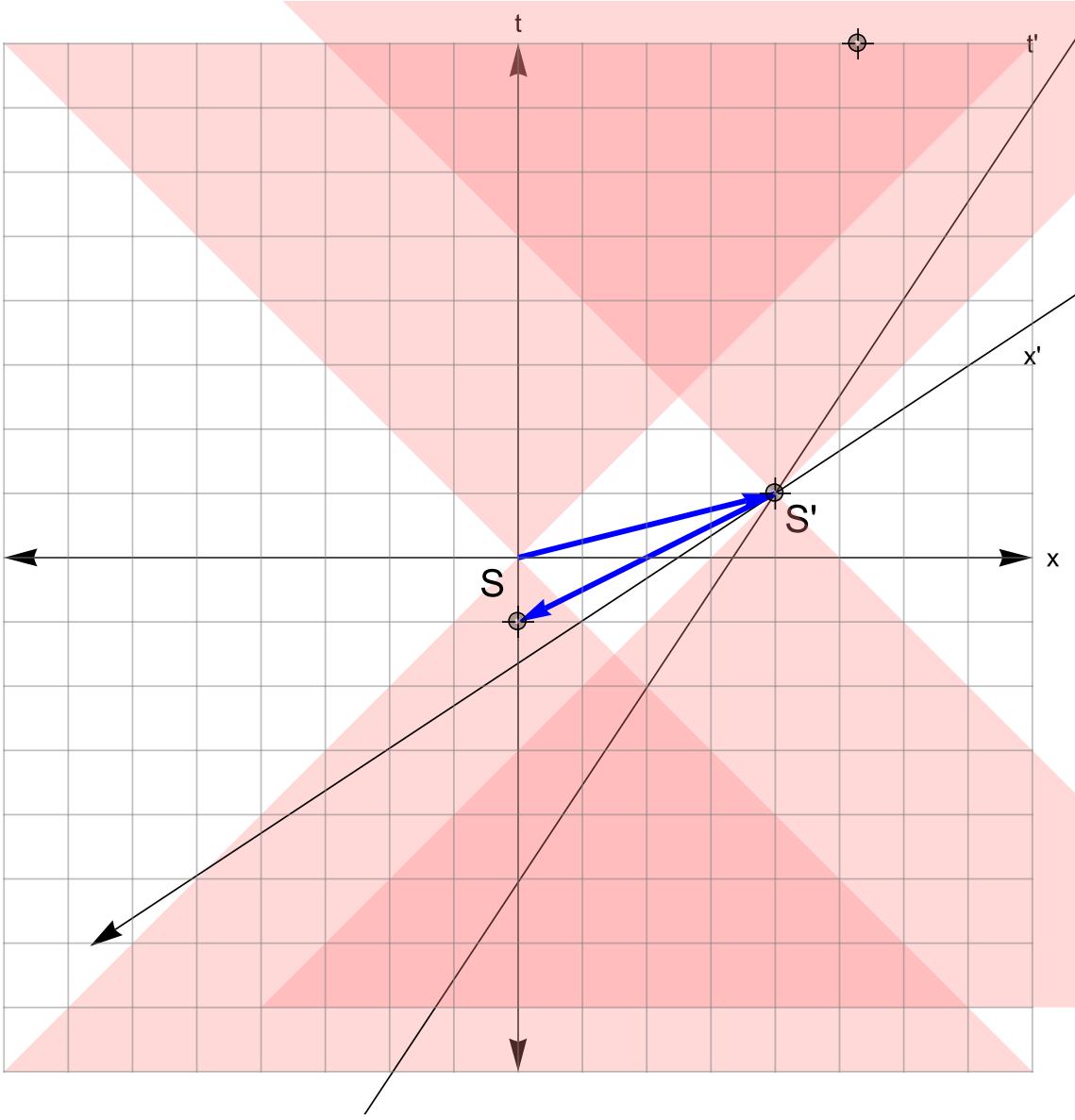


Figure 1: Spacetime diagram showing how superluminal travel may allow for time travel. In the unprimed coordinates (representing S), the path moving in the positive x direction is moving forwards in t , and in the primed coordinates (representing S'), the path moving along the negative x direction is moving forwards in t' . However, each path viewed in the other frame is travelling backwards in their respective time coordinates. Here the relative velocity between the frames is $u \approx 0.66$ (Lorentz factor $\gamma \approx 1.33$), and the speed of the warp drive in both directions is $v \approx 4$.

complicate the theoretical and mathematical analysis; we cannot simply write down a metric that will contain a closed timelike curve, we would have to consider the external acceleration separately. To resolve this issue, we will generalise the warp drive metric such that it is inherently capable of changing frames as needed.

The second issue is that the usual warp drives are non-compact, meaning that the two warp drives –

one from S to S' , another from S' back to S – overlap. In previous work, it was simply assumed that this overlapping was negligible, leaving the geodesics inside the two warp drives unaffected. In this paper we will make this construction more precise and explicit by introducing *compact* warp drives, which do not overlap with each other.

The third and final issue is that, in previous work, the metric of one warp drive was considered, but a complete metric containing both the outgoing and incoming warp drives, as well as an *explicit* closed timelike geodesic, was not derived. Our main result in this paper will be a complete and explicit spacetime metric, (3.5), containing two warp drives of a new, generalised kind, which:

- Begin and end in different rest frames,
- Do not overlap with each other, and
- Create an explicit closed timelike geodesic.

This construction will provide a much more solid foundation to the widely-held belief that faster-than-light travel allows for time travel. Furthermore, the explicit metric we will construct will provide a foundation for future investigations of various consequences of time travel, such as time travel paradoxes [2, 11, 12, 13], in a concrete general-relativistic setting.

Our paper relies on the definition of a new generalised warp drive with non-unit lapse function. Therefore, it would be interesting to investigate whether this resolves what is often considered to be the most problematic feature of warp drives, namely the violation of the pointwise weak energy condition (WEC). We do this by extending the arguments presented in [6] to allow for a non-unit lapse.

Unfortunately, we find that this does not help to avoid WEC violations, subject to some reasonable conditions. We also give an analysis of the case of zero Eulerian energy density and a curl-free shift vector field, as this was not accounted for in [6]. Interestingly, we conclude that the WEC violations are likely nothing more than an artefact of the particular class of metrics, and have nothing to do with the metrics' potential to describe warp drives, or superluminal travel in general.

This paper introduces many different mathematical symbols. To prevent confusion and help make the mathematical formulas presented in the paper more clear, we provide a convenient table of symbols in Appendix B. We denote 4-dimensional spacetime indices using Greek letters, and 3-dimensional spatial indices using lowercase English letters.

Throughout the paper, we have used both the Mathematica package OGRe (An Object-Oriented General Relativity Package) [14] and a beta version of the Python package OGRePy [15] to facilitate calculations of curvature tensors, geodesics, and other relevant geometrical quantities.

2 Geodesic rest frame transitions

2.1 Warp drives with unit lapse

First, we give a preliminary definition of a *geodesic rest frame transition*, which will suffice for our purposes. We give a more general definition and thorough analysis in Appendix A, from a coordinate-independent starting point.

We take a general metric $g_{\mu\nu}$ expressed in the ADM formalism and coordinates (t, x, y, z) as⁴

$$ds^2 = -N^2 dt^2 + \gamma_{ij}(dx^i - \beta^i dt)(dx^j - \beta^j dt). \quad (2.1)$$

The metric is assumed to be asymptotically flat in these coordinates, that is, as $r \equiv \sqrt{x^2 + y^2 + z^2} \rightarrow \infty$,

$$N \rightarrow 1, \quad \beta \rightarrow \mathbf{0}, \quad \gamma_{ij} \rightarrow \delta_{ij}.$$

⁴In this paper, as in most warp drive papers, we define the shift vector β as the negative of what is usually used in other areas. With this convention, passengers in the warp drive (usually) move with a 3-velocity of β .

We say that (2.1) allows for a *geodesic rest frame transition* if:

1. $\exists T_{flat} > 0$ such that if $t \in (-\infty, 0] \cup [T_{flat}, \infty)$, $g_{\mu\nu} = \eta_{\mu\nu}$, where $\eta_{\mu\nu}$ is the Minkowski metric.⁵
2. There exists a timelike geodesic Γ_{RFT} such that for $t \leq 0$, Γ_{RFT} has tangent vector $v^\mu = (1, \mathbf{0})$, and that for $t \geq T_{flat}$, Γ_{RFT} has tangent vector $v^\mu \neq (1, \mathbf{0})$.⁶

For $t \notin (0, T_{flat})$, the coordinates (t, x, y, z) describe an inertial reference frame, since $g_{\mu\nu} = \eta_{\mu\nu}$. This encapsulates the idea of an observer starting at rest in this reference frame and finishing at rest in a different reference frame.

We shall now demonstrate that warp drives do not allow for rest frame transitions. We will focus specifically on warp drives of the Natário class [16]⁷, which is the set of asymptotically-flat spacetimes of the form (2.1) with $N = 1$ and $\gamma_{ij} = \delta_{ij}$:

$$ds^2 = -dt^2 + \delta_{ij}(dx^i - \beta^i dt)(dx^j - \beta^j dt). \quad (2.2)$$

Define $\Sigma_{t^*} \equiv t^{-1}(t^*)$ for $t^* \in \mathbb{R}$. The normal vector field to Σ_t is given by

$$\begin{aligned} \mathbf{n} &\equiv -dt \\ \implies n_\mu &= (-1, \mathbf{0}), \quad n^\mu = (1, \boldsymbol{\beta}). \end{aligned} \quad (2.3)$$

Consider an observer starting at rest in the rest frame described by these coordinates at a time $t < 0$, with 4-velocity $(1, 0, 0, 0)$ (as they would along Γ_{RFT} , if it exists). This means that the observer is initially Eulerian, by which we mean that their 4-velocity coincides with the normal vector field. Consider the following general formula, valid in any ADM spacetime:

$$n^\nu \nabla_\nu n_\mu = \frac{1}{N} (\nabla_\mu N + n_\mu \mathcal{L}_\mathbf{n} N), \quad (2.4)$$

where \mathcal{L} denotes the Lie derivative. From this formula, one can easily show that the integral curves of the vector field n^μ (the paths of Eulerian observers) are geodesics if and only if $\partial_i N = 0 \iff N = N(t)$. This is clearly the case for (2.2), so the observer, wherever they go, must have 4-velocity n^μ .

From the known expression for an inverse metric expressed in the ADM formalism, we can see that

$$n^\mu = -Ng^{0\mu} = -g^{0\mu}, \quad (2.5)$$

since, as $t \rightarrow T_{flat}$, $g^{0\mu} \rightarrow \eta^{0\mu} = (-1, 0, 0, 0)$ and $n^\mu \rightarrow (1, \mathbf{0})$. The observer must have this same 4-velocity, and we see that they have returned to being at rest in the original frame of reference. Therefore, (2.2) does *not* allow for rest frame transitions, so it cannot be used to facilitate time travel using the method described in the introduction without complicating the model with an external, non-geodesic source of acceleration.

The above argument can easily be extended to the case where $N = N(t)$, $\delta_{ij} \rightarrow \gamma_{ij}$ for arbitrary $N(t) > 0$ and positive-definite γ_{ij} , since Eulerian observers still follow geodesics and we must still have $n^\mu \rightarrow (1, \mathbf{0})$ as $t \rightarrow T_{flat}$. Therefore, in order to find a metric allowing for a rest frame transition, we have no choice but to introduce spatial dependence into the lapse. This will be the subject of the next section.

⁵This may seem unnecessarily restrictive. Could a free-falling observer inside a warp drive not be cast out of the warp drive with a non-zero velocity, without having the warp drive completely vanish? Yes, in principle, but we argue that once they have left the warp drive, for their 4-velocity to be well-defined with respect to the original frame, they must be in a simply connected region of flat spacetime which stretches to future timelike infinity. If this is the case, then there will be no problem in ‘flattening’ the warp drive, since this cannot affect the 4-velocity of the observer any more. Therefore, any metric describing a rest-frame-transitioning warp drive could be easily modified to fit with this definition, without changing the environment of Γ_{RFT} at all. For a detailed discussion, see Appendix A.

⁶Since the sections of Γ_{RFT} with $t \notin (0, T_{flat})$ are in flat spacetime, these tangent vectors have constant components in these regions.

⁷This class includes the original Alcubierre drive, and should not be confused with the zero-expansion warp drives, described in the second half of Natário’s paper.

2.2 Warp drives with non-unit lapse

In this section, we introduce a generalised warp drive that starts from rest in some reference frame and travels along an arbitrary path $\mathbf{r}(t)$ for $0 < t < T_1$ where $T_1 > 0$, finishing at a constant 3-velocity $(0, 0, u)$ for $t \geq T_1$. We then show how we can manipulate the metric so that the warp curvature flattens whilst keeping the free-falling passengers moving at a speed u along the z axis. This means that we have found a warp drive capable of ‘landing’ on a target moving at a speed u .

First, we introduce a generic warp drive following an arbitrary path $\mathbf{r}(t) = (X(t), Y(t), Z(t))$, with a shift vector $\beta = a(t)\mathbf{v}(t, \mathbf{x})$. We set $\partial_t \mathbf{r}(t) = \mathbf{v}(t, \mathbf{r}(t))$ so that $(t, \mathbf{r}(t))$ is an integral curve of $(1, \mathbf{v}(t, \mathbf{x}))$ ⁸. Our new metric is given by

$$ds^2 = -N^2 dt^2 + \delta_{ij} (dx^i - \beta^i dt)(dx^j - \beta^j dt). \quad (2.6)$$

The introduction of the function a and a non-unit lapse N are our modifications to the Natário form of the metric, and their significance shall soon become clear. We also set \mathbf{v} such that

$$(\partial_i v^j)(t, \mathbf{r}(t)) = 0, \quad (2.7)$$

that is, \mathbf{v} is flat at the warp drive centre $(t, \mathbf{r}(t))$, and as usual, we also require $\mathbf{v} \rightarrow \mathbf{0}$ at spatial infinity. All functions describing the metric are assumed to be at least C^2 , such that the Riemann tensor is continuous.

The geodesic Lagrangian is

$$\mathcal{L}_g = \frac{1}{2} \left(-N^2 \dot{t}^2 + \delta_{ij} (\dot{x}^i - \beta^i \dot{t})(\dot{x}^j - \beta^j \dot{t}) \right), \quad (2.8)$$

where we shall use a dot to denote differentiation with respect to proper time τ . The Euler-Lagrange equations then imply

$$\begin{aligned} 0 &= \frac{d}{d\tau} (-N^2 \dot{t} - \delta_{jk} \beta^j (\dot{x}^k - \beta^k \dot{t})) + \dot{t}^2 N \partial_t N + \dot{t} \cdot \delta_{jk} (\partial_t \beta^j) (\dot{x}^k - \beta^k \dot{t}), \\ 0 &= \frac{d}{d\tau} (\dot{x}^i - \beta^i \dot{t}) + \dot{t}^2 N \partial_{x^i} N + \dot{t} \cdot \delta_{jk} (\partial_{x^i} \beta^j) (\dot{x}^k - \beta^k \dot{t}). \end{aligned} \quad (2.9)$$

The unit normal vector field to hypersurfaces of constant t , Σ_t , is given by

$$\mathbf{n} \equiv -N dt \implies n_\mu = (-N, \mathbf{0}), \quad n^\mu = \frac{1}{N} (1, \beta). \quad (2.10)$$

Again, an integral curve of this vector field will satisfy the geodesic equations if and only if $\partial_i N = \mathbf{0}$, as can be seen from (2.9). The path $(\tau, \mathbf{r}(\tau))$ is thus a geodesic wherever $N = 1$ and $a = 1$, as it is an integral curve of n^μ . It is also parameterised by proper time.

Now take $0 < T_1 < T_2$ and consider the case that

$$\begin{cases} \mathbf{v}(t, \mathbf{x}) = \mathbf{0} & t \leq 0, \mathbf{x} \in \mathbb{R}^3, \\ \mathbf{v}(t, \mathbf{r}(t)) = (0, 0, u) & t \geq T_1, \end{cases} \quad (2.11)$$

for some fixed $0 < u < 1$, and \mathbf{v} is unconstrained for $0 < t < T_1$. We take $a(t)$ as follows:

$$\begin{cases} a(t) = 1 & t \leq T_1, \\ 0 < a(t) < 1 & T_1 < t < T_2, \\ a(t) = 0 & t \geq T_2, \end{cases} \quad (2.12)$$

⁸In the original Alcubierre drive (1.1), this condition is a generalisation of setting f as a function of $z - \zeta(t)$ and then setting the metric coefficient as $v(t) = \partial_t \zeta(t)$ where $v(t) = \partial_t \zeta(t)$ and $f = 1$ at the warp drive centre.

and set the lapse as

$$N = 1 - b \cdot (z - Z)s. \quad (2.13)$$

where $s \rightarrow 0$ at spatial infinity with $s(t, \mathbf{r}(t)) = 1$, $(\partial_i s)(t, \mathbf{r}(t)) = 0$ and

$$\begin{aligned} b(t) &\equiv -u\lambda^2 \cdot \partial_t a, \\ \lambda &\equiv (1 - u^2(1 - a)^2)^{-\frac{1}{2}}. \end{aligned} \quad (2.14)$$

We call a and b the *transition functions* for the shift vector and lapse, respectively. s and a must also be chosen such that $N > 0$ everywhere.

Note that for $t \geq T_2$, a and b vanish, so $g_{\mu\nu} = \eta_{\mu\nu}$, and the metric becomes Minkowskian (so T_2 corresponds to T_{flat} in the previous section). $b(t)$ vanishes for $t \notin (T_1, T_2)$, so $N = 1$ for $t \notin (T_1, T_2)$, and Eulerian observers follow geodesics here, so taking

$$\chi_1(\tau) \equiv (\tau, \mathbf{r}(\tau)), \quad 0 \leq \tau \leq \tau_1 \equiv T_1, \quad (2.15)$$

we see $\chi_1([0, \tau_1])$ is a geodesic. However, in the region with $t \in (T_1, T_2)$, Eulerian observers do not follow geodesics. For $T_1 \leq t \leq T_2$, consider the path χ_2 satisfying

$$\begin{aligned} \dot{\chi}_2(\tau) &= (\lambda, 0, 0, \lambda u), \\ \chi_2(\tau_1) &= (T_1, \mathbf{r}(T_1)), \end{aligned} \quad (2.16)$$

so that at $\tau = \tau_1 > 0$, a may start to change. Since $0 \leq a \leq 1$ and $|u| < 1$, we have

$$1 \leq \lambda \leq \gamma_u, \quad (2.17)$$

where

$$\gamma_u \equiv (1 - u^2)^{-\frac{1}{2}}$$

is the Lorentz factor, and this ordinary differential equation has a unique, smooth solution for all $\tau \geq \tau_1$ ⁹. We now demonstrate that the solution to (2.16) is a geodesic.

For χ_2 ,

$$\frac{dz}{dt} = \frac{dz}{d\tau} \cdot \frac{d\tau}{dt} = \lambda u \cdot \frac{1}{\lambda} = u, \quad (2.18)$$

so parameterised by coordinate time, this is the path

$$(\chi_2 \circ \tau)(t) = (t, X(T_1), Y(T_1), Z(T_1) + u(t - T_1)) = (t, \mathbf{r}(t)), \quad (2.19)$$

and looking at (2.13), $N = 1$ along this path (but crucially, $\partial_z N \neq 0$). As can be checked from (2.6), this has the consequence that $\dot{\chi}_2$ is normalised, i.e. $g(\dot{\chi}_2, \dot{\chi}_2) = -1$, and we see that τ is the proper time of the path, with

$$\tau(t) = T_1 + \int_{T_1}^t \sqrt{-g(\partial_t \chi_2, \partial_t \chi_2)} dt', \quad T_1 < t \leq T_2. \quad (2.20)$$

⁹It can also be seen that since $\frac{dt}{d\tau} = \lambda \geq 1$ is bounded between two finite values, the duration of the transition as measured by proper time is bounded and strictly less than the duration as measured by coordinate time.

Taking the first geodesic equation, along χ_2 , we have

$$\begin{aligned}
0 &= \frac{d}{d\tau}(-1^2 \cdot \lambda - \lambda \cdot (u - au) \cdot au) + \lambda^2 \cdot 1 \cdot ub + u\lambda\partial_t a(\lambda u(1-a)) \\
&= -\frac{d}{d\tau}(\lambda + \lambda u^2 a(1-a)) + \lambda^2 ub + \lambda u^2 \frac{da}{d\tau}(1-a) \\
&= -\dot{\lambda} + \lambda^2 ub - au^2 \frac{d}{d\tau}(\lambda(1-a)) \\
&= -\dot{\lambda} - u^2 a(1-a)\dot{\lambda} + u^2 \lambda a \dot{a} + \lambda^2 ub \\
&= u^2 \lambda \dot{a} (\lambda^2(1-a) + au^2 \lambda^2(1-a)^2 + au^2) + \lambda^2 ub \\
&= u^2 \lambda^3 \dot{a} ((1-a) + au^2(1-a)^2 + a(1-u^2(1-a)^2)) + \lambda^2 ub \\
&= u^2 \lambda^3 \dot{a} + \lambda^2 ub \\
\implies 0 &= u\lambda^2 \partial_t a + b,
\end{aligned} \tag{2.21}$$

where in the fifth line we have used the identity $\dot{\lambda} = -\lambda^3 u^2 (1-a)\dot{a}$, and in the sixth the definition of λ . Therefore, we see that we recover (2.14) as a necessary condition for χ_2 to be geodesic.

The geodesic equations for $x(\tau)$ and $y(\tau)$ are trivially satisfied since, along χ_2 ,

$$\dot{x} = \beta^x = \partial_x N = \dot{y} = \beta^y = \partial_y N = 0.$$

So finally, considering the equation for $z(\tau)$:

$$\begin{aligned}
0 &= \frac{d}{d\tau}(\lambda u(1-a)) - \lambda^2 \cdot 1 \cdot b + 0 \\
&= \frac{d}{d\tau}(\lambda u(1-a)) - \lambda^2 b,
\end{aligned} \tag{2.22}$$

where we have used that $(\partial_i \beta^j)(t, \mathbf{r}(t)) = 0 \implies \partial_z \beta^z = 0$ and $(\partial_i s)(t, \mathbf{r}(t)) = 0$ along χ_2 . Now looking at the third line in (2.21), we see this can be written as

$$\begin{aligned}
0 &= \frac{1}{au}(-\dot{\lambda} + \lambda^2 ub) - \lambda^2 b \\
\implies 0 &= u\lambda^2 \partial_t a + b.
\end{aligned} \tag{2.23}$$

Therefore, (2.14) is a necessary *and* sufficient condition for χ_2 to be geodesic.

Finally, we note that the combination of the two above paths

$$\chi(\tau) \equiv \begin{cases} \chi_1(\tau) & 0 \leq \tau < \tau_1, \\ \chi_2(\tau) & \tau_1 \leq \tau \leq \tau_2, \end{cases} \tag{2.24}$$

where $\tau_2 \equiv \tau(T_2)$, is also a geodesic. This can be seen by noting that both the position and tangent vector of χ_1 at $\tau = \tau_1$ match those of χ_2 at $\tau = \tau_1$, so extending the geodesic χ_1 is equivalent to solving the geodesic equation with initial conditions

$$\begin{aligned}
\chi(\tau_1) &= (T_1, \mathbf{r}(T_1)), \\
\dot{\chi}(\tau_1) &= (1, 0, 0, u),
\end{aligned} \tag{2.25}$$

which is exactly χ_2 . Written another way, the entire curve

$$\Gamma_{RFT} \equiv \{p \in M : \exists t \in [0, T_2] : p = (t, \mathbf{r}(t))\} \tag{2.26}$$

is a geodesic, and it satisfies the requirements described in the previous section. This means that a free-falling observer starting at rest in the centre of the bubble does not fall out of it, and their final 4-velocity at time $t = T_2$ is $(\gamma_u, 0, 0, \gamma_u u)$ – not the same as their initial 4-velocity of $(1, 0, 0, 0)$. Since for $t \geq T_2$ we have $g_{\mu\nu} = \eta_{\mu\nu}$, we see that an observer following this geodesic has transitioned between rest reference frames.

3 Creating a closed timelike geodesic

3.1 The double-warp-drive metric

In this section, we demonstrate how the above method of performing a rest frame transition may be employed to write down a metric describing a spacetime containing a *closed timelike geodesic*. To the authors' knowledge, this is the first complete and well-defined example of a spacetime metric which *explicitly* uses warp drives to create a closed timelike curve, and furthermore, the closed timelike curve we find is a geodesic, requiring no acceleration due to external forces, which would complicate the theoretical and mathematical analysis.

The idea, as described in the introduction, is to have an observer start at rest in a reference frame S , travel in a superluminal warp drive of the form (2.6), transition to a different reference frame¹⁰ \hat{S} , and then travel back to the starting point in a similar warp drive. It will turn out that if the two warp drives move fast enough, this will result in the observer finishing the journey at a point in the causal past of their departure.

First, we introduce a generic spacetime, which we suggestively call M_{CTC} , containing two warp drives that take an observer on a journey to and from rest at a given spatial point in S , which we shall take to be $(x, y, z) = (0, 0, 0)$. For the return journey, we shall need to define a warp drive in the Lorentz-boosted frame \hat{S} , which moves at speed u along the z axis with respect to S . We thus take the metric

$$ds^2 = -\hat{N}^2 d\hat{t}^2 + \delta_{ij}^{\hat{x}}(d\hat{x}^i - \hat{\beta}^i d\hat{t})(d\hat{x}^j - \hat{\beta}^j d\hat{t}), \quad (3.1)$$

where $(\hat{t}, \hat{x}, \hat{y}, \hat{z})$ is the Lorentz-boosted coordinate system, using the convention that indices with hats denote \hat{S} coordinates and those without hats denote S coordinates. The functions \hat{N} , $\hat{\beta}$, \hat{v} , \hat{a} , \hat{b} , \hat{s} , \hat{r} , \hat{X} , \hat{Y} , and \hat{Z} are defined exactly analogously to their S counterparts, but with \hat{S} coordinates as their arguments. There are only two other changes:

1. The critical times, $t = 0, T_1, T_2$, have now become $\hat{t} = \hat{T}_0, \hat{T}_1, \hat{T}_2$ respectively.
2. \hat{v} is set such that the warp drive finishes at rest in S :

$$\begin{cases} \hat{v}(\hat{t}, \hat{x}) = 0 & \hat{t} \leq \hat{T}_0, \hat{x} \in \mathbb{R}^3, \\ \hat{v}(\hat{t}, \hat{r}(\hat{t})) = (0, 0, -u) & \hat{t} \geq \hat{T}_1. \end{cases} \quad (3.2)$$

Applying the Lorentz transformation

$$\begin{aligned} \hat{t} &= \gamma_u(t - uz), \\ \hat{x} &= x, \\ \hat{y} &= y, \\ \hat{z} &= \gamma_u(z - ut), \end{aligned} \quad (3.3)$$

gives the metric (3.1) in S coordinates as

$$\begin{aligned} ds^2 &= -\hat{N}^2 \gamma_u^2 (dt - u dz)^2 + (dx - \hat{\beta}^x \gamma_u (dt - u dz))^2 \\ &\quad + (dy - \hat{\beta}^y \gamma_u (dt - u dz))^2 + \gamma_u^2 ((dz - u dt) - \hat{\beta}^z (dt - u dz))^2 \end{aligned} \quad (3.4)$$

where all the functions have their arguments in terms of S coordinates.

Now we put these two warp drive spacetimes together. Defining $h_{\mu\nu}$ as the perturbation¹¹ to the Minkowski metric $\eta_{\mu\nu}$ that gives the metric (2.6) and the corresponding perturbation for (3.4) as $\hat{h}_{\mu\nu}$, we now consider the spacetime with metric

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + \hat{h}_{\mu\nu}. \quad (3.5)$$

¹⁰We use \hat{S} here, instead of S' as in the introduction, to facilitate more concise notation of related parameters.

¹¹If ${}^{(1)}g_{\mu\nu}$ is the metric (2.6), then $h_{\mu\nu} \equiv {}^{(1)}g_{\mu\nu} - \eta_{\mu\nu}$. Similarly, $\hat{h}_{\mu\nu} \equiv {}^{(2)}g_{\mu\nu} - \eta_{\mu\nu} \implies \hat{h}_{\mu\nu} \equiv {}^{(2)}g_{\mu\nu} - \eta_{\mu\nu}$, as $\eta_{\mu\nu}$ is invariant under Lorentz transformations.

This spacetime contains the two warp drives described by $h_{\mu\nu}$ and $\hat{h}_{\mu\nu}$. To avoid collision of the warp drives, we make the further assumption that, viewed as functions of (t, x, y, z) , their supports do not overlap:

$$\text{supp } \beta \cap \text{supp } \hat{\beta} = \emptyset = \text{supp}(N - 1) \cap \text{supp}(\hat{N} - 1).$$

An object travelling inside the first warp drive must be deposited in the right place in \hat{S} to be picked up by the returning warp drive. Therefore, the point $(T_2, \mathbf{r}(T_2))$ must correspond to $(\hat{T}_0, \hat{\mathbf{r}}(\hat{T}_0))$. Using (3.3), this means

$$\begin{aligned}\hat{T}_0 &= \gamma_u(T_2 - uZ(T_2)), \\ \hat{X}(\hat{T}_0) &= X(T_2), \\ \hat{Y}(\hat{T}_0) &= Y(T_2), \\ \hat{Z}(\hat{T}_0) &= \gamma_u(Z(T_2) - uT_2).\end{aligned}\tag{3.6}$$

Note that the second warp drive does not have to leave immediately – we are free to set $\hat{\mathbf{v}} = \mathbf{0}$ for some interval of time after $\hat{t} = \hat{T}_0$.

We may set similar conditions for the return journey to ensure that the returning observer is deposited by the second warp drive at exactly the same spatial coordinates $(x, y, z) = (0, 0, 0)$ in S . Therefore

$$(T_{\text{finish}}, \mathbf{0}) \leftrightarrow (\hat{T}_2, \hat{\mathbf{r}}(\hat{T}_2)),$$

where T_{finish} is the return time of the observer as measured in S . Thus, we have

$$\begin{aligned}\hat{T}_2 &= \gamma_u T_{\text{finish}}, \\ \hat{X}(\hat{T}_2) &= 0, \\ \hat{Y}(\hat{T}_2) &= 0, \\ \hat{Z}(\hat{T}_2) &= -\gamma_u u T_{\text{finish}}.\end{aligned}\tag{3.7}$$

3.2 Requirements for time travel

Now we define the outgoing and returning average speeds in the z and $-\hat{z}$ directions, as measured in the warp drives' respective frames:

$$\bar{v}^z \equiv \frac{Z(T_2)}{T_2},\tag{3.8}$$

$$\hat{v}^z \equiv -\frac{\hat{Z}(\hat{T}_2) - \hat{Z}(\hat{T}_0)}{\hat{T}_2 - \hat{T}_0}.\tag{3.9}$$

Using (3.6) and (3.7), it is straightforward to find

$$T_{\text{finish}} = T_2 \frac{\bar{v}^z + \hat{v}^z - u(1 + \bar{v}^z \hat{v}^z)}{\hat{v}^z - u}.\tag{3.10}$$

Clearly, if either \bar{v}^z or \hat{v}^z are less than 1, we cannot have $T_{\text{finish}} < 0$. So assuming $\bar{v}^z, \hat{v}^z > 1$, we get the *time-travel condition*:

$$u > \frac{\bar{v}^z + \hat{v}^z}{1 + \bar{v}^z \hat{v}^z}.\tag{3.11}$$

Let us now assume $T_{\text{finish}} < 0$. We define the connection between these two paths at the origin in S

$$C \equiv \{(t, \mathbf{0}) : t \in [T_{\text{finish}}, 0)\},\tag{3.12}$$

and note that the entire path given by

$$\Gamma_{CTC} \equiv \Gamma_{RFT} \cup \hat{\Gamma}_{RFT} \cup C\tag{3.13}$$

is a geodesic, where Γ_{RFT} and $\hat{\Gamma}_{RFT}$ are defined analogously to (2.26).

Since the supports of the relevant functions do not overlap, the analysis from Section 2.2 still holds. Therefore, we already know that Γ_{RFT} and $\hat{\Gamma}_{RFT}$ are geodesics, and that this path is continuous at the intersections by (3.6), (3.7), (3.12). In Section 2.2, we proved that

$$\Gamma_{RFT} = \chi_1([0, \tau_1]) \cup \chi_2([\tau_1, \tau_2])$$

is a geodesic, and a similar proof applies here; the only other thing we have to check is that the tangent vectors at the start and end points also match, which indeed turns out to be the case. In S , the observer finishes the outgoing journey with 4-velocity $(\gamma_u, 0, 0, u\gamma_u)$ which, in \hat{S} , is $(1, 0, 0, 0)$, agreeing with (3.2). Similarly, the observer finishes the return journey with 4-velocity $(\gamma_u, 0, 0, -u\gamma_u)$ in \hat{S} , which in S , is $(1, 0, 0, 0)$, agreeing with (3.12) and (2.11). Thus Γ_{CTC} is a geodesic.

For definiteness, we now also consider the proper time τ and its relation to coordinate time t . This will give insight into the points at which the observer is travelling backwards in time, according to S . Within Γ_{RFT} , we can use the same $t(\tau)$ as we did in Section 2.2, that is,

$$t(\tau) = \begin{cases} \tau & \tau \in [0, \tau_1], \\ \tau_1 + \epsilon^{-1}(\tau) & \tau \in [\tau_1, \tau_2], \end{cases} \quad (3.14)$$

where¹²

$$\begin{aligned} \epsilon(t) &\equiv \int_{T_1}^t \sqrt{-g(\partial_t \chi, \partial_t \chi)} dt', \quad t \in [T_1, T_2], \\ \tau_2 &\equiv \tau_1 + \epsilon(T_2). \end{aligned} \quad (3.15)$$

We can then similarly define $\hat{e}(\hat{t})$ along with

$$\begin{aligned} \tau_3 &= \tau_2 + \hat{T}_1 - \hat{T}_0, \\ \tau_4 &= \tau_3 + \hat{e}(\hat{T}_2). \end{aligned} \quad (3.16)$$

τ_4 is the total proper time experienced by the observer in one round trip along Γ_{CTC} . This gives us the full description of the coordinate times t and \hat{t} in terms of the proper time τ along Γ_{CTC} :

$$\begin{aligned} t(\tau) &= \begin{cases} \tau & \tau \in [0, \tau_1], \\ T_1 + \epsilon^{-1}(\tau) & \tau \in [\tau_1, \tau_2], \end{cases} \\ \hat{t}(\tau) &= \begin{cases} \hat{T}_0 + (\tau - \tau_2) & \tau \in [\tau_2, \tau_3], \\ \hat{T}_1 + \hat{e}^{-1}(\tau) & \tau \in [\tau_3, \tau_4]. \end{cases} \end{aligned} \quad (3.17)$$

Then, using the reverse Lorentz transformation, the point $(\hat{t}, \hat{\mathbf{r}}(\hat{t}))$ in \hat{S} has time coordinate

$$t = \gamma_u(\hat{t} + u\hat{Z}(\hat{t})), \quad (3.18)$$

and we find the full expression for $t(\tau)$:

$$t(\tau) = \begin{cases} \tau & \tau \in [0, \tau_1], \\ T_1 + \epsilon^{-1}(\tau) & \tau \in [\tau_1, \tau_2], \\ \gamma_u(\hat{T}_0 + (\tau - \tau_2) + u\hat{Z}(\hat{T}_0 + (\tau - \tau_2))) & \tau \in [\tau_2, \tau_3], \\ \gamma_u(\hat{T}_1 + \hat{e}^{-1}(\tau) + u\hat{Z}(\hat{T}_1 + \hat{e}^{-1}(\tau))) & \tau \in [\tau_3, \tau_4]. \end{cases} \quad (3.19)$$

As expected, one can show that $\dot{t}(\tau) > 0$ for $\tau \notin (\tau_2, \tau_3)$, and that for $\tau \in (\tau_2, \tau_3)$

$$\dot{t}(\tau) < 0 \iff \partial_{\hat{t}} \hat{Z}(\hat{T}_0 + (\tau - \tau_2)) < -\frac{1}{u} < -1, \quad (3.20)$$

so in order to be travelling backwards in time in S , the observer must be moving faster than $\frac{1}{u}$ in the negative \hat{z} direction.

¹² $\chi(\tau)$ is again the parametrisation of the outgoing geodesic, with $(\chi \circ \tau)(t) = (t, \mathbf{r}(t))$.

3.3 The curvature tensors

In this section, we give the curvature tensors associated with the metric (3.5). Since the metric is defined piecewise on disjoint regions of M_{CTC} , we can safely write

$${}^{(CTC)}R_{\rho\sigma\mu\nu} = R_{\rho\sigma\mu\nu} + \hat{R}_{\rho\sigma\mu\nu}, \quad (3.21)$$

where $R_{\rho\sigma\mu\nu}$ and $\hat{R}_{\rho\sigma\mu\nu}$ are the Riemann tensors associated to (2.6) and (3.4).

For this, we shall need the extrinsic curvatures of the spacelike hypersurfaces defined by constant time Σ_t and $\hat{\Sigma}_{\hat{t}}$, and it will be simplest to define them each in their respective frames as follows:

$$\begin{aligned} n_\mu &= (-N, \mathbf{0}), \\ \gamma_{\mu\nu} &\equiv \eta_{\mu\nu} + h_{\mu\nu} + n_\mu n_\nu, \\ K_{\mu\nu} &\equiv \frac{1}{2} \mathcal{L}_n \gamma_{\mu\nu}, \end{aligned} \quad (3.22)$$

for the outgoing warp drive in S coordinates and

$$\begin{aligned} \hat{n}_{\hat{\mu}} &= (-\hat{N}, \mathbf{0}), \\ \hat{\gamma}_{\hat{\mu}\hat{\nu}} &\equiv \eta_{\hat{\mu}\hat{\nu}} + \hat{h}_{\hat{\mu}\hat{\nu}} + \hat{n}_{\hat{\mu}} \hat{n}_{\hat{\nu}}, \\ \hat{K}_{\hat{\mu}\hat{\nu}} &\equiv \frac{1}{2} \mathcal{L}_{\hat{n}} \hat{\gamma}_{\hat{\mu}\hat{\nu}}, \end{aligned} \quad (3.23)$$

for the returning warp drive in \hat{S} coordinates.

We shall make use of the Gauss, Codazzi, and Ricci equations¹³, given here in general:

$$\gamma_\rho^\alpha \gamma_\sigma^\beta \gamma_\mu^\gamma \gamma_\nu^\delta {}^{(4)}R_{\alpha\beta\gamma\delta} = {}^{(3)}R_{\rho\sigma\mu\nu} + K_{\rho\mu} K_{\sigma\nu} - K_{\rho\nu} K_{\sigma\mu}, \quad (3.24)$$

$$\gamma_\rho^\alpha n^\beta \gamma_\mu^\gamma \gamma_\nu^\delta {}^{(4)}R_{\alpha\beta\gamma\delta} = D_\mu K_{\rho\nu} - D_\nu K_{\rho\mu}, \quad (3.25)$$

$$\gamma_\alpha^\rho n^\sigma \gamma_\beta^\mu n^\nu {}^{(4)}R_{\rho\sigma\mu\nu} = -\mathcal{L}_n K_{\alpha\beta} + \frac{1}{N} D_\alpha D_\beta N + K_{\alpha\rho} K_\beta^\rho, \quad (3.26)$$

where D is the induced covariant derivative on the hypersurface in question, and ${}^{(4)}R_{\rho\sigma\mu\nu}$, ${}^{(3)}R_{\rho\sigma\mu\nu}$ are the full and induced Riemann tensors respectively.

In the following, as in [6], instead of using the coordinate basis

$$\{\partial_t, \partial_x, \partial_y, \partial_z\}$$

to specify the components of tensors, we shall use the non-coordinate, orthonormal basis

$$\{\mathbf{n}, \partial_x, \partial_y, \partial_z\} \quad (3.27)$$

as this will allow us to use Equations (3.24)-(3.26) directly. Our notation will be such that a tensor with an index n means that that index has been contracted with n^μ , for example $R_{\mu n} \equiv R_{\mu\rho} n^\rho$.

It turns out that K_{ij} takes a pleasingly simple form:

$$K_{ij} = \frac{1}{N} \partial_{(i} \beta_{j)}, \quad (3.28)$$

¹³See, for example, [17].

and with this in mind, we evaluate $R_{\rho\sigma\mu\nu}$. Since the induced metric is flat, $\gamma_{ij} = \delta_{ij}$, we have $D_i = \partial_i$ and ${}^{(3)}R_{\rho\sigma\mu\nu} = 0$. Also noting that $\gamma_i^\rho = \delta_i^\rho$, Equations (3.24)-(3.26) give us:

$$\begin{aligned} R_{ijkl} &= K_{ik}K_{jl} - K_{il}K_{jk}, \\ R_{inkl} &= \gamma_i^\alpha n^\beta \gamma_k^\gamma \gamma_l^\delta R_{\alpha\beta\gamma\delta} = \partial_k K_{il} - \partial_l K_{ik}, \\ R_{injn} &= \gamma_i^\rho n^\sigma \gamma_j^\mu n^\nu R_{\rho\sigma\mu\nu} = -\mathcal{L}_n K_{ij} + K_{ik}K_j^k + \frac{1}{N}\partial_i\partial_j N. \end{aligned} \quad (3.29)$$

In contracting these expressions, we shall use repeatedly that our basis is orthonormal, and thus for any tensor $Q_{\mu\nu}$ we may write

$$g^{\mu\nu}Q_{\mu\nu} = -Q_{nn} + \delta^{ij}Q_{ij}.$$

We now find the Ricci tensor:

$$\begin{aligned} R_{nn} &= g^{\mu\nu}R_{\mu\nu nn} = -R_{nnnn} + \delta^{ij}R_{ijnj} \\ &= -\mathcal{L}_n K - 2K_{jk}K^{jk} + K_{jk}K^{jk} + \frac{1}{N}\Delta N \\ &= -\mathcal{L}_n K - K_{jk}K^{jk} + \frac{1}{N}\Delta N, \\ R_{ni} &= g^{\mu\nu}R_{\mu\nu ni} = -R_{nnni} + \delta^{jk}R_{jnki} \\ &= \partial_j K_i^j - \partial_i K, \\ R_{ij} &= g^{\mu\nu}R_{\mu\nu ij} = -R_{ninj} + \delta^{kl}R_{kilj} \\ &= -\left(-\mathcal{L}_n K_{ij} + K_{ik}K_j^k + \frac{1}{N}\partial_i\partial_j N\right) + KK_{ij} - K_{ik}K_j^k \\ &= \mathcal{L}_n K_{ij} + KK_{ij} - 2K_{ik}K_j^k - \frac{1}{N}\partial_i\partial_j N, \end{aligned} \quad (3.30)$$

where $K \equiv K_\rho^\rho = K_i^i$ and we have used the useful identity

$$\delta^{ij}\mathcal{L}_n K_{ij} = \mathcal{L}_n K + 2K_{ij}K^{ij} \quad (3.31)$$

to simplify R_{nn} . From this we get the Ricci scalar:

$$\begin{aligned} R &= -R_{nn} + \delta^{ij}R_{ij} \\ &= -(-\mathcal{L}_n K - K_{jk}K^{jk} + \frac{1}{N}\Delta N) + \delta^{ij}(\mathcal{L}_n K_{ij} + KK_{ij} - 2K_{ik}K_j^k - \frac{1}{N}\partial_i\partial_j N) \\ &= 2\mathcal{L}_n K + K^2 + K_{kl}K^{kl} - \frac{2}{N}\Delta N. \end{aligned} \quad (3.32)$$

This allows us to evaluate the Einstein tensor $G_{\mu\nu}$, which is of course related to the energy-momentum tensor by the Einstein equation

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}.$$

We shall give the decomposition of the energy-momentum tensor here:

$$\begin{aligned} \rho &\equiv \frac{1}{8\pi G}G_{nn} = \frac{1}{16\pi G}(K^2 - K_{ij}K^{ij}), \\ \phi_i &\equiv \frac{1}{8\pi G}G_{ni} = \frac{1}{8\pi G}(\partial_j K_i^j - \partial_i K), \\ T_{ij} &\equiv \frac{1}{8\pi G}G_{ij} = \frac{1}{8\pi G}(\mathcal{L}_n K_{ij} + KK_{ij} - 2K_{ik}K_j^k - \frac{1}{N}\partial_i\partial_j N \\ &\quad - \frac{1}{2}\delta_{ij}(2\mathcal{L}_n K + K^2 + K_{kl}K^{kl} - \frac{2}{N}\Delta N)). \end{aligned} \quad (3.33)$$

Since Equations (3.29) - (3.33) are covariant expressions (if one resets $\partial_i \rightarrow D_i$), and $\hat{\gamma}_{\hat{i}\hat{j}} = \delta_{\hat{i}\hat{j}}$ is too, analogous equations hold for $\hat{R}_{\rho\sigma\mu\nu}$, with $\beta \rightarrow \hat{\beta}$, $\mathbf{n} \rightarrow \hat{\mathbf{n}}$, $K_{\mu\nu} \rightarrow \hat{K}_{\mu\nu}$ and $D_i \rightarrow \hat{D}_i$.

3.4 A specific example

Let us now give a specific choice of the functions that describe our metric such that the curve Γ_{CTC} is indeed a closed timelike curve, to prove that such a choice is possible and make the result even more concrete. We first define the bump function

$$q : \mathbb{R} \rightarrow \mathbb{R}, \quad q(x) = \begin{cases} 140x^3(1-x)^3 & 0 \leq x \leq 1, \\ 0 & \text{otherwise,} \end{cases} \quad (3.34)$$

which can be seen in Figure 2, and its primitive

$$q^{(-1)} : \mathbb{R} \rightarrow \mathbb{R}, \quad q^{(-1)}(x) = \int_0^x q(y) dy. \quad (3.35)$$

$q^{(-1)}$ is a (non-strictly) monotone-increasing $C^3(\mathbb{R})$ function with $q^{(-1)}(x) = 0$ for $x \leq 0$ and $q^{(-1)}(x) = 1$ for $x \geq 1$, a ‘smooth step function’. We shall use q and its primitives repeatedly for our construction of functions such as β , s , a , and b . Note that there are many possible choices of q ; the following discussion would work just as well with any $q \in C^2(\mathbb{R})$ with

$$q(x) \in [0, 1], \quad \text{supp } q \subset [0, 1], \quad \text{and } \int_0^1 q(x) dx = 1.$$

First, we give the functions describing the outgoing warp drive in S . We start with \mathbf{v} , the velocity vector field of the warp drive. We shall take it to be compactly supported, and take

$$r(t, x, y, z) \equiv ((x - X(t))^2 + (y - Y(t))^2 + (z - Z(t))^2)^{\frac{1}{2}}, \quad (3.36)$$

from which we define, similarly to [1],

$$\begin{aligned} s(r) &= 1 - q^{(-1)}\left(\frac{r - r_1}{r_2 - r_1}\right), \\ \mathbf{v}(t, \mathbf{x}) &= \partial_t \mathbf{r}(t) \cdot s(r), \end{aligned} \quad (3.37)$$

for some $r_2 > r_1$. We see that s satisfies $s(0) = 1$ and $s'(0) = 0$. The function s appears in the definition of the lapse (2.13), but here it also serves to define \mathbf{v} . r_1 is the radius of the region of flat space inside the warp bubble as measured by its passengers, and r_2 is the radius of the warp bubble, as measured by external observers in S . The direction of \mathbf{v} does not vary in (x, y, z) .

We wish for the outgoing warp drive to start at rest and finish with a constant velocity $(0, 0, u)$ in S . Therefore, the acceleration along the z axis, $\partial_t^2 Z$, must be zero outside $0 < t < T_2$, and it must take both signs at different times to allow for the necessary acceleration and deceleration. Here, we take it to be comprised of two bump functions next to each other with opposite signs, as follows:

$$\begin{aligned} \partial_t^2 Z(t) &= k_1 q\left(\frac{t}{t_1}\right) - k_2 q\left(\frac{t - t_1}{T_1 - t_1}\right), \\ \partial_t Z(t)|_{t=0} &= 0, \\ Z(0) &= 0, \end{aligned} \quad (3.38)$$

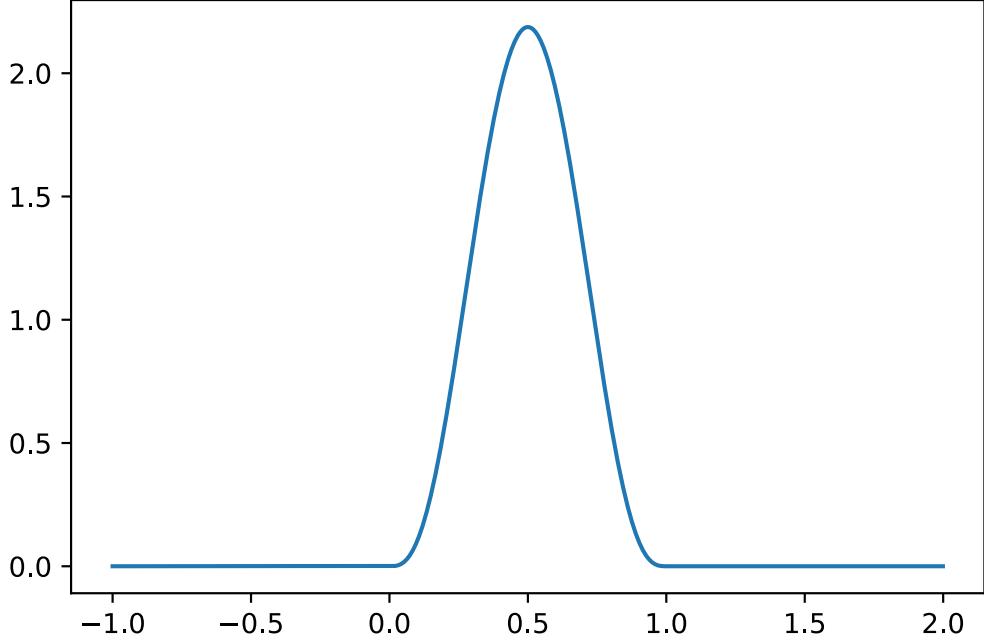


Figure 2: The bump function $q(x) = 140x^3(1-x)^3$, $0 < x < 1$.

for $k_1, k_2 > 0$ and $0 < t_1 < T_1$ where we require

$$\begin{aligned} \int_0^{T_1} \partial_t^2 Z(t) dt &= t_1 k_1 - (T_1 - t_1) k_2 = u, \\ \implies t_1 &= \frac{u + T_1 k_2}{k_1 + k_2}. \end{aligned} \tag{3.39}$$

The warp drive accelerates until $t = t_1$, and decelerates from $t = t_1$ to $t = T_1$. We get the full solution

$$Z(t) = k_1 t_1^2 q^{(-2)}\left(\frac{t}{t_1}\right) - k_2 (T_1 - t_1)^2 q^{(-2)}\left(\frac{t - t_1}{T_1 - t_1}\right), \tag{3.40}$$

where

$$q^{(-2)}(x) \equiv \int_0^x q^{(-1)}(y) dy. \tag{3.41}$$

We also define

$$\kappa \equiv q^{(-2)}(1) = \int_0^1 q^{(-1)}(y) dy.$$

One can calculate that with the above choice of q , we have $\kappa = \frac{1}{2}$. Note that for $x \geq 1$,

$$q^{(-2)}(x) = \kappa + (x - 1).$$

This gives us

$$Z(T_2) = k_1 t_1^2 \left(\kappa + \frac{T_2}{t_1} - 1 \right) - k_2 (T_1 - t_1)^2 \left(\kappa + \frac{T_2 - t_1}{T_1 - t_1} - 1 \right). \tag{3.42}$$

Next, we choose a such that

$$a(t) = 1 - q^{(-1)} \left(\frac{t - T_1}{t_2 - T_1} \right), \quad (3.43)$$

where $t_2 = T_2 - u\gamma_u r_2$, and we assume $t_2 > T_1$. The outgoing warp drive vanishes at $t = t_2 < T_2$. This is to avoid special relativistic issues of simultaneity, where if the outgoing warp drive were to disappear at T_2 in S with the returning warp drive appearing at \hat{T}_0 in \hat{S} , there would be an overlap (a collision) due to the finite extension of the warp drives.

Using (2.14), this gives us¹⁴

$$b(t) = \frac{u\lambda^2}{t_2 - T_1} q \left(\frac{t - T_1}{t_2 - T_1} \right), \quad (3.44)$$

where λ is defined as before. Note that b is C^2 , which is important as b forms part of a metric component. The fact that q is C^2 implies that the Riemann tensor is at least C^0 everywhere.

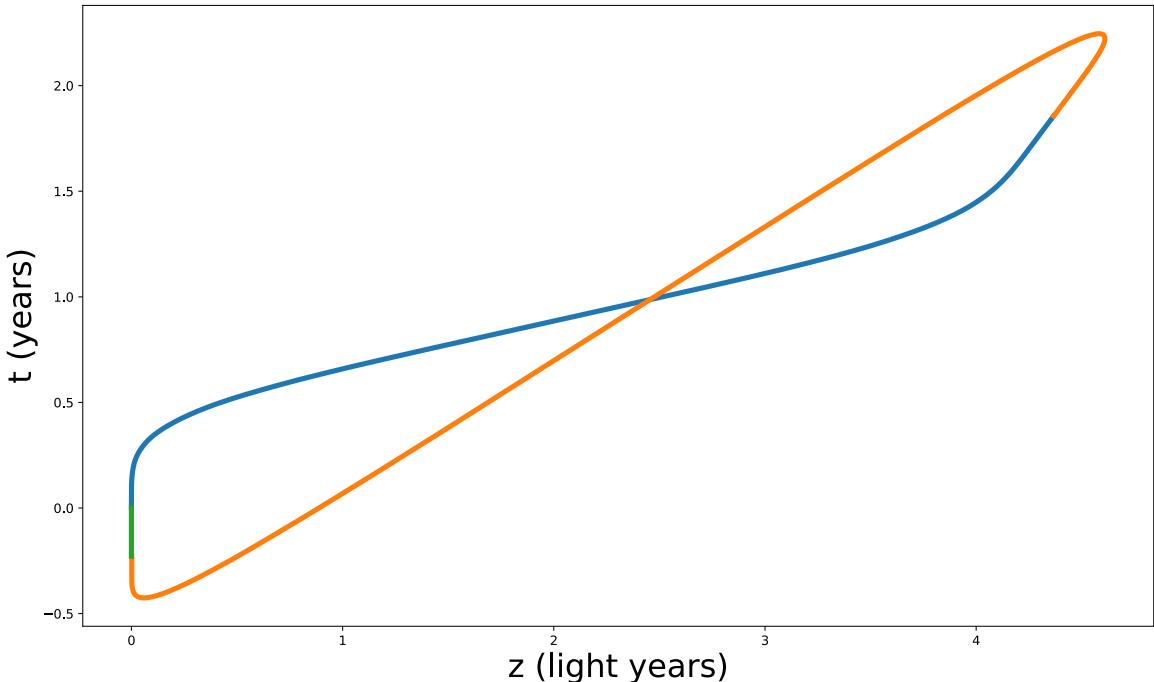


Figure 3: The path in our example projected onto the $t - z$ plane. The blue line is the outgoing path, the orange line is the returning path, and the green line is the section corresponding to C . For this plot we have chosen $u = \frac{3}{4}$, $\alpha = 10$, $\omega = 0.95$, and $T = \frac{1}{v} \cdot 4.4$ light-years. Note that the two warp drives do not collide in the middle, due to their displacement in x , which is not shown in the plot.

Next, we set X and Y . It is important that these are not both simply set to be zero; since the returning warp drive is travelling backwards in time in S (that is, $\dot{t}(\tau) < 0$), if X, Y, \hat{X}, \hat{Y} were simply zero, there would be a collision between the future- and past-directed warp drives. Therefore, we set

$$\begin{aligned} X(t) &= 2r_2 q^{(-1)} \left(\frac{t}{t_x} \right), \\ Y(t) &= 0, \end{aligned} \quad (3.45)$$

¹⁴We do not directly prove that the resulting lapse function is positive everywhere. However, it is clear from (2.13) that for a given b , we can choose r_2 small enough to ensure $N > 0$ everywhere.

which allows the outgoing warp drive to ‘sidestep’ the returning one. We assume $t_x \ll T_2$, so that the outgoing warp drive is well out of the way before the returning one gets near.

Finally, we define the corresponding functions for the returning warp drive in exactly the same way, but with hats added and various appearances of \hat{T}_0 . Once again, there are a couple of changes:

1. $\partial_{\hat{t}}^2 \hat{Z}$ is set with an extra minus sign:

$$\partial_{\hat{t}}^2 \hat{Z}(\hat{t}) = -\hat{k}_1 q \left(\frac{\hat{t} - \hat{T}_0}{\hat{t}_1 - \hat{T}_0} \right) + \hat{k}_2 q \left(\frac{\hat{t} - \hat{t}_1}{\hat{T}_1 - \hat{t}_1} \right),$$

with $\hat{k}_1, \hat{k}_2 > 0$. This gives the same condition $(\hat{t}_1 - \hat{T}_0)\hat{k}_1 - (\hat{T}_1 - \hat{t}_1)\hat{k}_2 = u$.

2. $\hat{X}(\hat{t}) = 2r_2 \left(1 - q^{(-1)} \left(\frac{\hat{t} - \hat{T}_0}{\hat{t}_x - \hat{T}_0} \right) \right)$ (we may leave r_1 and r_2 the same for the returning warp drive).

To construct an example, we are free to choose these parameters as we like. We take

$$\begin{aligned} t_1 &= \frac{1}{2}T_1, \\ k_2 &= \alpha \frac{u}{T_1} \implies k_1 = (\alpha + 2) \frac{u}{T_1}, \end{aligned} \tag{3.46}$$

where we leave $\alpha > 0$ as a free parameter. Plugging this into (3.42), and using $\kappa = \frac{1}{2}$, we find

$$\bar{v}^z = \frac{Z(T_2)}{T_2} = u \left(1 + \frac{\omega}{4}(\alpha - 1) \right), \tag{3.47}$$

where $\omega \equiv \frac{T_1}{T_2}$.

We also choose

$$\begin{aligned} \hat{t}_1 - \hat{T}_0 &= \frac{1}{2}(\hat{T}_1 - \hat{T}_0), \\ \hat{k}_1 &= (\alpha + 2) \frac{u}{\hat{T}_1 - \hat{T}_0}, \\ \hat{k}_2 &= \alpha \frac{u}{\hat{T}_1 - \hat{T}_0}, \end{aligned} \tag{3.48}$$

and set

$$\frac{\hat{T}_1 - \hat{T}_0}{\hat{T}_2 - \hat{T}_0} = \frac{T_1}{T_2} = \omega, \tag{3.49}$$

meaning

$$\bar{v}^z = \hat{v}^z = v. \tag{3.50}$$

Intuitively, these conditions mean that the outgoing path looks the same in S as the returning path does in \hat{S} . This is in that the profile of the velocity along the z axis has the same shape, albeit rescaled by a factor $\frac{T_2}{T_2 - \hat{T}_0}$ to allow the warp drives to travel different distances in their respective frames. Both warp drives have the same average speed v .

Renaming $T_2 \rightarrow T$ and using (3.10), we get

$$T_{\text{finish}} = T \left(\frac{2v - u(1 + v^2)}{v - u} \right). \tag{3.51}$$

This result can be stated in words as follows: If an observer travels in a warp drive at an average speed of v in a frame S for a time T , transitions to a frame \hat{S} which is moving at speed u with respect to S , and then returns in another warp drive moving at speed $-v$ in \hat{S} to the starting point in S , they will arrive at a time T_{finish} . If

$$u > \frac{2v}{1 + v^2}, \tag{3.52}$$

then $T_{\text{finish}} < 0$, and the observer has travelled back in time.

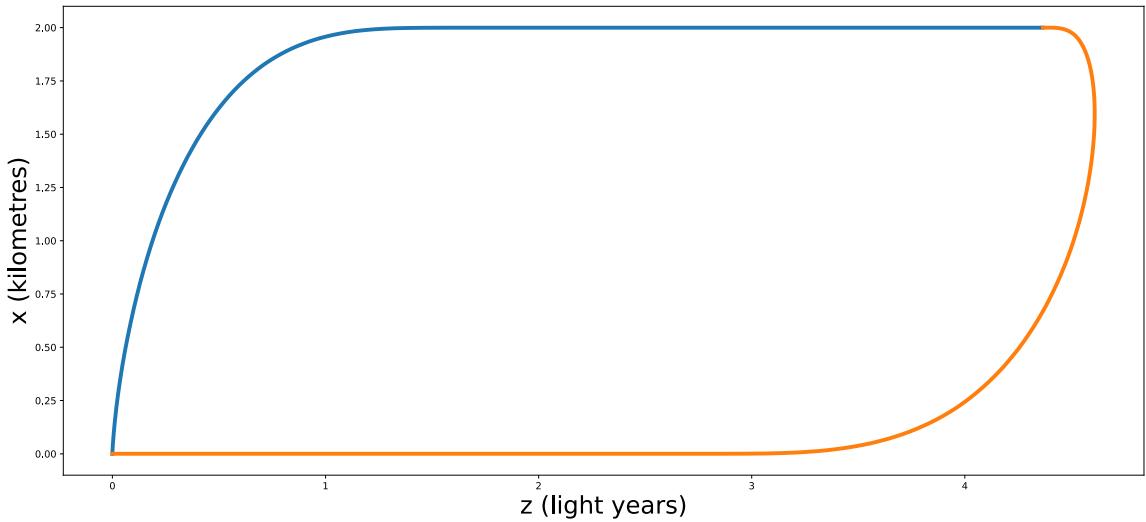


Figure 4: The path in our example projected onto the $x - z$ plane. The blue line is the outgoing path, and the orange line is the returning path. We have made the same choice of variables as in the $t - z$ diagram in Figure 3, and in addition took $t_x = \hat{t}_x - \hat{T}_0 = 0.8$ years and $r_2 = 1$ km. Note that the vertex at $x = z = 0$ does not imply the velocity is discontinuous, since the traveller is stationary in S there.

4 The weak energy condition with a non-unit lapse

4.1 Violations of the weak energy condition

The study of the energy conditions in relation to warp drive spacetimes is key, as it gives us a way to determine whether the matter needed to sustain a given spacetime geometry is ‘exotic’ or not. All warp drive spacetimes considered in the literature so far violate the weak energy condition¹⁵. Whilst it is still unclear if energy condition violations automatically mean that the corresponding metric is unphysical, it is usually considered problematic. This is because known sources of negative energy, for example the Casimir effect, are believed only to be able to produce tiny amounts of negative energy, in contrast to the typically vast negative energy requirements of warp drives.

We shall study the pointwise weak energy condition (WEC) in this section. In [6], it was shown that a metric in the Natário class always entails violation of the WEC, subject to some reasonable conditions. Here we shall follow a modified version of an argument in [6] applied to metrics of the form

$$ds^2 = -N^2 dt^2 + \delta_{ij}(dx^i - \beta^i dt)(dx^j - \beta^j dt), \quad (4.1)$$

and show that the introduction of a non-unit lapse N does not help to avoid WEC violation. We also consider the non-trivial case where $\rho = 0$ everywhere, which was not considered in [6]. Importantly, the arguments in this section hold for an *arbitrary* positive-definite lapse, not just our particular modification (2.13) necessary for creating a closed timelike geodesic.

Our goal will be to study what exactly it is about metrics of the form (4.1) that gives rise to violation of the weak energy condition, without consideration for the shape of N and β (or whether they describe a warp drive in particular). We do not prove violation of the pointwise null energy condition (NEC) as was done in [6], as the arguments therein do not readily extend to the case with $N \neq 1$.

¹⁵Recently, [18], [19] and [20] claimed to bypass this limitation. However, these claims were refuted in [6], so we will not consider them here.

The pointwise weak energy condition (WEC) is the requirement that

$$T_{\mu\nu}t^\mu t^\nu \geq 0 \text{ for all timelike vectors } t^\mu \in T_p M. \quad (4.2)$$

The physical justification behind this is that the energy density at a given point as measured by any timelike observer should be non-negative. Clearly then, in order to prove WEC violation, only one timelike vector t^μ such that $T_{\mu\nu}t^\mu t^\nu < 0$ is needed. Here, we consider the so-called Eulerian energy density $\rho = T_{\mu\nu}n^\mu n^\nu$. Using (3.28) and (3.33), we find

$$\rho = \frac{1}{16\pi G} \frac{1}{N^2} ((\partial_i \beta_i)^2 - \partial_{(i} \beta_j) \partial_{(i} \beta_j), \quad (4.3)$$

where we are free to write β with a lowered index since it is a spatial vector and therefore $\beta_i = \delta_{ij}\beta^j$. It is a simple task to rewrite this as

$$\rho = \frac{1}{16\pi G} \frac{1}{N^2} \left(\partial_i (\beta_i \partial_j \beta_j - \beta_j \partial_j \beta_i) - \partial_{[i} \beta_{j]} \partial_{[i} \beta_{j]} \right), \quad (4.4)$$

and we see that the inside of the brackets is a 3-divergence plus a negative-semi-definite term. Multiplying by N^2 and integrating over a sphere of radius r , $B_t(r) \subset \Sigma_t$, we find

$$\int_{B_t(r)} N^2 \rho d^3x = \frac{1}{16\pi G} \left(\int_{\partial B_t(r)} (\beta(\nabla \cdot \beta) - (\beta \cdot \nabla)\beta) \cdot \vec{n} dS - \int_{B_t(r)} \partial_{[i} \beta_{j]} \partial_{[i} \beta_{j]} d^3x \right), \quad (4.5)$$

where we have used the divergence theorem. The last term on the right-hand side is clearly non-positive, and the first vanishes as $r \rightarrow \infty$ if we assume that β decays fast enough. If we take $\beta = O(r^{-\frac{1}{2}})$ for large r , then

$$\beta(\nabla \cdot \beta) - (\beta \cdot \nabla)\beta = O(r^{-2}). \quad (4.6)$$

Therefore, we have

$$(\beta(\nabla \cdot \beta) - (\beta \cdot \nabla)\beta) \cdot \vec{n} = O(r^{-2}), \quad (4.7)$$

and since $dS \propto r^2$, the integral vanishes as $r \rightarrow \infty$. If this is the case, we then find by taking $r \rightarrow \infty$ in (4.5) that

$$\int_{\Sigma_t} N^2 \rho d^3x \leq 0, \quad (4.8)$$

with a strict inequality if $\partial_{[i} \beta_{j]} \neq 0$ anywhere, that is, β is not curl-free. Then we see that if $\rho > 0$ somewhere, $\rho < 0$ somewhere else, and the weak energy condition must be violated somewhere on Σ_t .

4.2 Discussion and literature review

The argument in the previous section holds if

1. The integrand $\beta(\nabla \cdot \beta) - (\beta \cdot \nabla)\beta$ is C^1 (continuously differentiable) everywhere such that the divergence theorem can be applied. This means that β must be C^2 , giving a Riemann tensor that is just C^0 (continuous).
2. $\beta = O(r^{-\frac{1}{2}})$ for large r . This can even be weakened further if, for example, β is asymptotically orthogonal to the sphere's normal vector \vec{n} such that $(\beta(\nabla \cdot \beta) - (\beta \cdot \nabla)\beta) \cdot \vec{n} = O(r^{-2})$.
3. $\rho \neq 0$ somewhere or $\partial_{[i} \beta_{j]} \neq 0$ for some i, j , somewhere. However, if we assume that $\rho = 0$ and $\partial_{[i} \beta_{j]} = 0$ everywhere, this does not imply that the spacetime is flat or even that $T_{\mu\nu} = 0$. We shall discuss this case further below.

An interesting feature of the above proof is that it never references the shape of the spacetime, and certainly does not require that the spacetime is in any way ‘warp-drive-like’, with a shell of curved spacetime enclosing a flat interior. Instead, it shows that there is something fundamental about metrics of this form, subject to these conditions, that requires the WEC to be violated somewhere. Put another way, Σ_t having zero *intrinsic* curvature coupled with it having non-zero *extrinsic* curvature leads to negative energy somewhere.

Although this seems disappointing in terms of the search for a positive-energy warp drive, this could actually be good news. It tells us that the negative energy is nothing but a symptom of the choice of metric, and does *not* have anything to do with *superluminal travel*. Therefore the WEC violations associated with known warp drives do not, by themselves, give us any reason to believe that superluminal travel always requires negative energy.

An often-cited study [21] claims to prove that superluminal travel always leads to WEC violation. However, this is done using a definition of superluminal travel based on the idea that a superluminal path should be in some sense ‘faster’ than all neighbouring paths. It also relies on the so-called generic condition, which essentially means the curvature does not vanish entirely along any causal geodesic.¹⁶

However, even just the original Alcubierre drive (1.1) with $f(x, y, z - \zeta(t))$ chosen such that it is 1 on an open region containing $(x, y, z - \zeta(t)) = (0, 0, 0)$ satisfies neither of these conditions. Indeed, most warp drives considered in the literature (e.g. [1], [7], [16]) either are or can be easily modified such that a passenger is in a Riemann-flat environment for the entire journey, so the generic condition cannot be satisfied.

There will also always be a path just next to the first superluminal path which is just as fast, so the path is not faster than all its neighbours. In addition, using the definition of a superluminal path presented in [21] has the consequence that all superluminal paths are null, which is clearly not the case with even the Alcubierre drive. The study presented in [21] is therefore only applicable to a limited set of cases which satisfy its particular definition of superluminal travel.

It is worth noting that if the boundary condition (condition 2 above) is not satisfied, one can find an asymptotically-flat β that goes to zero at spatial infinity, with $\rho \geq 0$ everywhere. It can be shown that, for a scalar $\psi \equiv \psi(t, r)$, where $r = \sqrt{x^2 + y^2 + z^2}$ and $\beta = \nabla\psi$, the Eulerian energy density is given by

$$\rho = \frac{1}{8\pi G} \frac{1}{r^2} \partial_r \left(r(\partial_r \psi(t, r))^2 \right). \quad (4.11)$$

So for example,

$$\psi(r) = \int_0^r \frac{s^{\frac{7}{4}}}{(1+s)^2} ds \quad (4.12)$$

has $\rho \geq 0$ everywhere. It is unclear if it is possible to make β ‘warp-drive-like’ whilst having $\rho \geq 0$ everywhere, though, and even if it were, one would still need to prove that $T_{\mu\nu} t^\mu t^\nu > 0$ for *all* timelike vectors t^μ . This does demonstrate, however, that negative *Eulerian* energy density is only unavoidable if the decay condition $\beta = O(r^{-\frac{1}{2}})$ is enforced, and metrics of this form that are only asymptotically flat, for example, do not *necessarily* violate the WEC.

In any case, if we restrict ourselves to a sufficiently-smooth and quickly-decaying vector field β , we can see that the addition of a non-unit lapse to the Natário class of warp drives does not change the

¹⁶The generic condition states that along any null geodesic with tangent vector ξ^μ there is at least one point where

$$\xi_{[\alpha} R_{\beta]\rho\sigma[\gamma} \xi_{\delta]} \xi^\rho \xi^\sigma \neq 0, \quad (4.9)$$

and along any timelike geodesic with tangent vector t^μ there is a point where

$$R_{\alpha\beta\gamma\delta} t^\alpha t^\delta \neq 0. \quad (4.10)$$

WEC violation, as was suggested as a possibility in [6]. This is ignoring the case that $\rho = 0 = \partial_{[i}\beta_{j]}$ everywhere, discussed below.

The integrated Eulerian energy density,

$$E_{tot} = \int_{\Sigma_t} \rho d^3x, \quad (4.13)$$

can however be made arbitrarily small simply by making N large where $\nabla\beta$ is large. If we imagine that β takes a typical ‘warp-drive-like’ shape with $\nabla\beta$ only large inside a thin shell surrounding the passengers, then we see that ρ is only large inside the shell. Then making N large here can reduce E_{tot} to arbitrarily small values, without affecting the inside passengers or even the duration of the journey as measured by them. The trade-off is that this would make the duration of the journey as experienced by the matter *inside* the shell larger by a factor of order N , as already noted in [3] (chapter 11, footnote 1).

4.3 The case $\rho = 0 = \partial_{[i}\beta_{j]}$

As mentioned in condition 3 above, the case where $\rho = 0$ and $\partial_{[i}\beta_{j]} = 0$ everywhere does not *necessarily* violate the WEC. After all, Eulerian observers are just one class of observers, and even if they see $\rho = 0$, this does not mean that the spacetime is flat or even a vacuum solution. While it is unclear if this case could ever correspond to anything resembling a warp drive, it is interesting to see if WEC violations are unavoidable in all non-trivial metrics of the form (4.1), subject only to conditions 1 and 2.

First, we note that

$$\partial_{[i}\beta_{j]} = \frac{1}{2} {}^{(3)}d\beta \equiv 0 \iff \exists \psi \text{ such that } {}^{(3)}d\psi = \beta, \quad (4.14)$$

where ${}^{(3)}d$ denotes the induced exterior derivative on Σ_t . Again following arguments in [6], we know that

$$\text{WEC} \implies \rho + \bar{p} \geq 0, \quad (4.15)$$

where

$$\bar{p} \equiv \frac{1}{3} \delta^{ij} T_{ij}. \quad (4.16)$$

We can prove this by considering the six vectors in the orthonormal basis $(\mathbf{n}, \partial_x, \partial_y, \partial_z)$:

$$\begin{aligned} (\xi_1)^\mu_\pm &= (1, \pm a_1, 0, 0), \\ (\xi_2)^\mu_\pm &= (1, 0, \pm a_2, 0), \\ (\xi_3)^\mu_\pm &= (1, 0, 0, \pm a_3), \end{aligned} \quad (4.17)$$

with $|a_1|, |a_2|, |a_3| < 1$ such that $(\xi_i)_\pm$ is timelike. Since these are all timelike vectors, we have for each i :

$$\text{WEC} \implies \begin{aligned} (\xi_i)_+^\mu (\xi_i)_+^\nu T_{\mu\nu} &\geq 0, \\ (\xi_i)_-^\mu (\xi_i)_-^\nu T_{\mu\nu} &\geq 0. \end{aligned} \quad (4.18)$$

Expanding these inequalities out and summing them together for each i , the cross terms cancel and we find, since $\rho = T_{\mu\nu} n^\mu n^\nu$,

$$\text{WEC} \implies 2\rho + 2a_i^2 T_{ii} \geq 0, \quad (4.19)$$

where no summation is implied by the repeated index i . Taking the limit $a_i \rightarrow 1$, we find

$$\text{WEC} \implies \rho + T_{ii} \geq 0. \quad (4.20)$$

Summing over i and dividing by 3, we finally see that

$$\text{WEC} \implies \rho + \frac{1}{3}\delta^{ij}T_{ij} = \rho + \bar{p} \geq 0. \quad (4.21)$$

Here we are assuming $\rho = 0$, so we have

$$\text{WEC} \implies \bar{p} \geq 0. \quad (4.22)$$

If the 3-momentum tensor T_{ij} is even slightly anisotropic, meaning its eigenvalues are not all the same, this can be strengthened to

$$\text{WEC} + \text{anisotropic } T_{ij} \implies \bar{p} > 0. \quad (4.23)$$

These implications are strictly one-way.

Let us assume that the WEC holds. Looking at the equation for ρ in (3.33), we see

$$\rho \equiv 0 \iff K^2 = K_{ij}K^{ij}. \quad (4.24)$$

Using this, $\beta_i = \partial_i\psi$, and the equation for T_{ij} in (3.33), we see that

$$\begin{aligned} \bar{p} &= -K^2 - \mathcal{L}_n K + \frac{1}{N}\Delta N \\ &= -\frac{1}{N^2}(\Delta\psi)^2 - \frac{1}{N}\left(\partial_t\left(\frac{1}{N}\Delta\psi\right) + \partial_i\psi\partial_i\left(\frac{1}{N}\Delta\psi\right)\right) + \frac{1}{N}\Delta N, \end{aligned} \quad (4.25)$$

where we have used $K = \frac{1}{N}\Delta\psi$, and that

$$\mathcal{L}_n K = n^\rho \partial_\rho K = \frac{1}{N}(\partial_t K + \beta^i \partial_i K). \quad (4.26)$$

We now multiply by N and integrate over Σ_t . We get

$$\int_{\Sigma_t} N\bar{p} d^3x = \int_{\Sigma_t} \left(-\frac{1}{N}(\Delta\psi)^2 - \partial_t\left(\frac{1}{N}\Delta\psi\right) - \nabla\psi \cdot \nabla\left(\frac{1}{N}\Delta\psi\right) + \Delta N \right) d^3x. \quad (4.27)$$

The last term vanishes since $\Delta N = \nabla \cdot \nabla N$ is a divergence, if we make the additional assumption that $N = 1 + O(r^{-1})$ for large r . The condition that $\nabla\psi = \beta = O(r^{-\frac{1}{2}})$ is then sufficient to use integration by parts on the penultimate term. Using this, we find

$$\begin{aligned} \int_{\Sigma_t} N\bar{p} d^3x &= \int_{\Sigma_t} \left(-\frac{1}{N}(\Delta\psi)^2 - \partial_t\left(\frac{1}{N}\Delta\psi\right) - \left(0 - \frac{1}{N}(\Delta\psi)^2\right) + 0 \right) d^3x \\ &= \int_{\Sigma_t} \left(-\partial_t\left(\frac{1}{N}\Delta\psi\right) \right) d^3x = - \int_{\Sigma_t} \partial_t K d^3x. \end{aligned} \quad (4.28)$$

Now, since $N > 0$ and the WEC means $\bar{p} \geq 0$, we must have

$$\text{WEC} \implies \partial_t \int_{\Sigma_t} K d^3x \leq 0. \quad (4.29)$$

If either $\bar{p} > 0$ somewhere or T_{ij} is anisotropic somewhere,

$$\implies \partial_t \int_{\Sigma_t} K d^3x < 0. \quad (4.30)$$

It is not immediately obvious that this conflicts with anything else we know about warp drives, but there is one thing we can take from this. If we start in flat space at $t = 0$, we have $K \equiv 0$. Therefore, any warp drive of the form (2.6) arising from flat space *must* have, for all $t \geq 0$,

$$\int_{\Sigma_t} K d^3x \leq 0, \quad (4.31)$$

and if $\exists t > 0$ such that $\int_{\Sigma_t} K d^3x < 0$, the spacetime cannot re-flatten without violating the WEC. If $\bar{p} > 0$ anywhere or T_{ij} is anisotropic anywhere, this will be the case. Thus, the only possibility is that $T_{ij} = 0$. Then our energy-momentum tensor becomes

$$\begin{aligned} \rho &= 0, \\ \phi_i &= \frac{1}{8\pi G} \left(\partial_j \left(\frac{1}{N} (\partial_i \partial_j \psi) \right) - \partial_i \left(\frac{1}{N} \Delta \psi \right) \right), \\ T_{ij} &= 0. \end{aligned} \quad (4.32)$$

Now we can simply take $\phi^i = \delta^{ij} \phi_j$ and use it construct the timelike vector

$$t^\mu = \left(1, -\frac{1}{2} (\phi_j \phi^j)^{-\frac{1}{2}} \phi^i \right).$$

We then calculate

$$T_{\mu\nu} t^\mu t^\nu = -(\phi_j \phi^j)^{\frac{1}{2}} \leq 0, \quad (4.33)$$

with equality if and only if $\phi^i = 0$. So the WEC means also that $\phi_i = 0$, which altogether shows that $T_{\mu\nu} = 0$, so there is no energy-momentum source. If we have $g_{\mu\nu} = \eta_{\mu\nu}$, $\partial_t g_{\mu\nu} = 0$ on Σ_0 (starting from Minkowski space) and we have $T_{\mu\nu} = 0$ for all $t > 0$, the unique solution to Einstein's equations is of course that $g_{\mu\nu} = \eta_{\mu\nu}$ for $t > 0$.

In this paper, we are considering warp drives that appear from and vanish into flat space, which allows a passenger to exit the warp bubble. Such warp drives are the most natural ones that could hypothetically be used for interstellar travel. In this case, there is no chance this could occur without violating the WEC. The only possible way for a passenger of such a warp drive to be able to get out is if the extrinsic curvature scalar K is non-zero only in regions far from the passenger.

In any case, this certainly forbids simple warp drives with a velocity parameter $v(t)$ like (1.1), since changes in the spacetime are irreversible¹⁷ by (4.29). This result indicates that although it may be possible to find a non-trivial metric of the form (4.1) subject to conditions 1 and 2 that does not violate the WEC, it is unlikely that it would be possible to utilise this to make a useful warp drive metric.

In conclusion, in this section we have discussed and extended the results of [6], and found that the addition of an arbitrary lapse N does not make a difference with regards to the WEC violation, although it can be used to reduce the *total* energy requirements of the warp drive. If we relax condition 3 of Section 4.2, it is unclear if WEC violations are unavoidable as a *fundamental* property of metrics of this form, but we have argued that it is unlikely that these could actually describe useful positive energy warp drive metrics.

5 Conclusions

In this paper we provided a concrete, precise, and complete example of a spacetime geometry which proves that a warp drive can be used to facilitate not only faster-than-light travel, but also time travel. While previous authors have already given good arguments for why this should be possible, the

¹⁷This does not violate time-reversibility, since K also changes sign under $t \rightarrow -t$.

construction we have presented here makes those arguments precise and gives, for the first time, a complete double-warp-drive metric that explicitly contains a closed timelike geodesic.

In order to accomplish this, we generalised the notion of a warp drive to allow for a non-unit lapse function, a generalisation that has previously been alluded to in the literature, but never studied in depth. This was necessary in order to allow the warp drive itself to transition between two reference frames, without requiring acceleration from any external forces, which would make the theoretical and mathematical analysis significantly more complicated.

Using this generalised warp drive, we were able to glue two warp drives together, and use them to create a closed timelike curve which is a *geodesic*, a result of the pure spacetime geometry itself. Our new double-warp-drive spacetime thus joins the existing list of exotic solutions to Einstein's field equations which contain explicit closed timelike curves, and it may be used to provide new perspectives in the study of the nature of time and causality.

We also demonstrated that weak energy condition violations arise in metrics of the general form

$$ds^2 = -N^2 dt^2 + \delta_{ij}(dx^i - \beta^i dt)(dx^j - \beta^j dt), \quad (5.1)$$

subject to some reasonable conditions, *whether or not* we choose N and β such that the spacetime is 'warp-drive-like', that is, contains a warp bubble. Furthermore, these violations occur whether or not the warp drive is superluminal. This suggests that the WEC violations arise not from any supposed deep connection between superluminal travel and negative energy, but rather from the particular form of the metric. Positive-energy warp drives may still be allowed within classical general relativity, but only if they could be constructed using a different type of metric.

We hope that this paper will provide a useful tool for future investigations of warp drives and their potential use for faster-than-light travel and/or time travel. In particular, we hope to explore the following avenues of research in future work:

- Use the closed timelike geodesics resulting from double-warp-drive geometry to study various consequences of time travel, such as time travel paradoxes, in a concrete general-relativistic setting.
- Check whether our generalised class of non-unit-lapse warp drives necessarily violates other energy conditions, such as the strong and null energy conditions.
- Generalise our warp drive metric even further, by allowing the hypersurfaces Σ_t to be intrinsically curved, which may (or may not) allow us to prevent the WEC violations.

Acknowledgements

The authors would like to thank Mitacs for funding this research as part of the Globalink Research Internship program. Ben Snodgrass would also like to thank Katharina Schramm, Zipora Stober, Alicia Savelli, and Jared Wogan for stimulating discussions and support, with special thanks to Jared Wogan for helping with the OGRePy package.

A In-depth analysis of rest frame transitions

A.1 The setting

In this section we will consider an arbitrary spacetime M subject to the following conditions, as illustrated in Figure 5:

1. M is diffeomorphic to \mathbb{R}^4 and geodesically complete.

2. M is Riemann-flat outside a compact region $K \subset M$, where K is diffeomorphic to D^4 , the solid 4D ball. We also define $F \equiv M \setminus K$.
3. There exists a region $\mathcal{R} \subset M$ diffeomorphic to a 4D Riemann-flat cylinder of infinite length $(\mathbb{R} \times \text{Int } D^3)$, such that $\mathcal{R} \cap K \neq \emptyset$ and the intersection $\partial K \cap \mathcal{R}$ is composed of two disjoint 3-surfaces.
4. \mathcal{R} contains an inextendible timelike geodesic Γ .

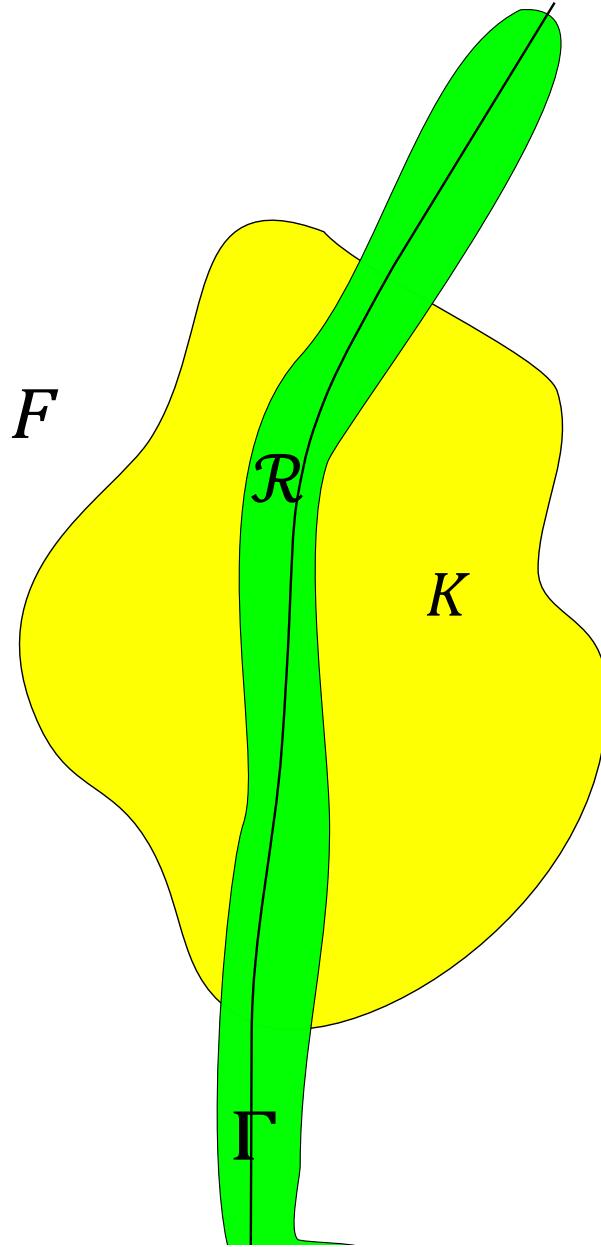


Figure 5: Schematic diagram showing Γ , \mathcal{R} , K and F .

This setting describes a warp drive that appears from flat space and subsequently vanishes, returning to flat space. The interior of the warp drive is \mathcal{R} , and the openness of \mathcal{R} ensures that the flat interior of the

warp drive has finite size, and is not just a single point¹⁸. \mathcal{R} must contain a timelike geodesic in order to allow a free-falling observer to travel inside the flat interior. This is a very general conception of a warp drive¹⁹, and in particular makes no assumptions about the existence of a special time coordinate where, for example, the lapse function is unity or spacelike hypersurfaces Σ_t are intrinsically flat.

The compactness of the curved region K may seem like an unnecessary and artificial restriction, given that many warp drives studied in the literature do not obey this constraint (for example, [1], [16], and [6]). However, if we imagine we are starting in a Riemann-flat universe and creating this warp drive, information of the warp drive's existence should not be able to travel arbitrarily fast. The existence of curvature at arbitrary distances would imply that this information had travelled at arbitrarily high speeds.

A.2 Overview and intuition

We first give an overview of the arguments and results presented in this section, before giving their formal proofs. First of all, we argue that there exists an open sub-region $\tilde{\mathcal{R}} \subset \mathcal{R}$ that is isometric to a ‘cylinder’ of flat spacetime, that is, one can choose a coordinate system (τ, x, y, z) in $\tilde{\mathcal{R}}$ such that $g_{\mu\nu} = \eta_{\mu\nu}$, with the boundary of a hypersurface of constant τ being a 2-sphere with constant spatial coordinates. The path $(\tau, \mathbf{0}), -\infty < \tau < \infty$ is Γ , so $\Gamma \subset \tilde{\mathcal{R}}$. These would be the local coordinates used by an observer on the geodesic Γ .

We then show that a time coordinate t (which induces a normal vector field $\mathbf{n} = -dt$) can be chosen on F such that t defines a reference frame. An observer following Γ is at rest in this reference frame before entering K . Since F extends to the section of Γ beyond K , this gives us a well-defined and natural way to determine whether the observer exits K with a different 4-velocity from the one they entered it with.

In Minkowski space, a spacelike hypersurface of zero extrinsic curvature gives rise to a constant normal vector \mathbf{n} , which is the 4-velocity of observers at rest in a frame that has this hypersurface as the set of points with $t = \text{constant}$. We call a time coordinate ‘locally Minkowskian’ at a point p if the neighbourhood of p is Riemann-flat and the hypersurface $t = \text{constant}$ is extrinsically flat in a neighbourhood of p .

We next show that there exists an extension of t inside $\tilde{\mathcal{R}}$ such that t is locally Minkowskian everywhere in $\tilde{\mathcal{R}}$ too. Such a choice of t gives a local frame of reference inside $\tilde{\mathcal{R}}$ and thus has useful intuitive value, as well as significance in relation to the freedom of choice of the lapse N , as we shall see. It also gives rise to a partial foliation of M , $\Sigma_{t^*} \equiv t^{-1}(t^*)$ for $t^* \in \mathbb{R}$.

After choosing such a t , we show that at any point $p \in \tilde{\mathcal{R}}$, some $\tilde{x}, \tilde{y}, \tilde{z}$ can be chosen on an open set containing p where the metric takes the following form:

$$ds^2 = -N^2 dt^2 + d\tilde{x}^2 + d\tilde{y}^2 + d\tilde{z}^2.$$

From this, we deduce that since $\tilde{\mathcal{R}}$ is flat, N cannot be of a more general form than

$$N(t, \tilde{\mathbf{x}}) = p(t) + \mathbf{q}(t) \cdot \tilde{\mathbf{x}},$$

for some p and \mathbf{q} depending only on time. This was in fact the original motivation for the ansatz (2.13). Lapses taking this form can be viewed as a uniform ‘acceleration of the time coordinate’, that is, the integral curve of n^μ would correspond to an accelerating observer. This means that the

¹⁸A point-like interior would only be able to transport a point-like particle without experiencing tidal forces, so this is a reasonable assumption.

¹⁹Technically, the CTC spacetimes described in Section 3 would not satisfy these requirements, as the curve Γ_{CTC} clearly does not exist inside a region like \mathcal{R} , despite containing two rest frame transitions. This is a mere technicality though, and to weaken the restrictions on M to allow for this would increase their complexity significantly, without adding extra physical insight. Clearly, if one has two compact warp drives subject to the above conditions, they could be embedded in the same flat spacetime without issue.

non-accelerating observers in the warp drive accelerate relative to the Eulerian observer following n^μ . Since the vector n^μ is what we measure the net change in velocity of observers traversing $\tilde{\mathcal{R}}$ against, this corresponds to a rest frame transition. See Figure 7 for a visual depiction of this.

A.3 Formalism and proofs

A.3.1 Construction of $\tilde{\mathcal{R}}$

First, we prove the existence of $\tilde{\mathcal{R}}$ as described above. Take a point $p_0 \in \Gamma$ in the causal past of $\Gamma \cap \partial K$ (i.e. before the observer enters the warp drive) and its timelike tangent vector along Γ , $\mathbf{n}_0 \in T_{p_0}M$. Since \mathcal{R} is simply connected and Riemann-flat, a coordinate system $(\tau, x, y, z) : \mathcal{R} \rightarrow \mathbb{R}$ can be constructed such that associated tangent vector to the coordinate τ has the same components as \mathbf{n}_0 , and that the metric takes the Minkowskian form²⁰. Γ is the path $(\tau, \mathbf{0}), \tau \in \mathbb{R}$. Now define $\tilde{r} > 0$ by

$$\tilde{r}^2 \equiv \inf \{x^2(q) + y^2(q) + z^2(q) : q \in \partial \mathcal{R}\}. \quad (\text{A.1})$$

By openness of \mathcal{R} and compactness of K , we must have²¹ $\tilde{r} > 0$. Then we simply define

$$\tilde{\mathcal{R}} \equiv \{q \in \mathcal{R} : x^2(q) + y^2(q) + z^2(q) < \tilde{r}^2\}, \quad (\text{A.2})$$

and our construction is complete. $\tilde{\mathcal{R}}$ is a subset of \mathcal{R} , is clearly open as it is the preimage of an open set under a continuous mapping, and in these coordinates, it is manifest that $\tilde{\mathcal{R}}$ is isometric (not just homeomorphic like \mathcal{R}) to an infinitely-long, Riemann-flat 3+1 cylinder.

A.3.2 Construction of the time coordinate and reference frame outside K

For a schematic diagram displaying the following decomposition of M , see Figure 6. A reference frame, in Minkowski space, can be determined from the choice of only one timelike vector at some point, and extending it by parallel transport to make a vector field \mathbf{n} on the whole of Minkowski space. Since Minkowski space is flat and simply connected, this extension is unique. There will also exist a timelike coordinate t that one can derive from this, satisfying $-dt = \mathbf{n}$, with the minus sign a convention ensuring that \mathbf{n} is future-pointing, that is, $n^0 > 0$.

To create a global reference frame on the flat region F , we can similarly take the tangent vector of Γ , \mathbf{n}_0 , at p_0 , and define a vector at another point $q \in F$ by parallel-transporting the vector at p along a path contained in F . Since F is Riemann-flat and simply connected, the resulting vector at q is independent of the path chosen. Using this method, we can define a unique vector field

$$\mathbf{n} : p \in F \rightarrow T_p F.$$

This gives us a global reference frame on F , set to be at rest with respect to the observer following Γ before entering K , which allows us to quantify the change in the observer's 4-velocity after passing through K , thus determining whether a rest frame transition has occurred. Again, we can take a coordinate $t : F \rightarrow \mathbb{R}$ such that $-dt = \mathbf{n}$, choosing t such that

$$0 = \inf \{t(\tilde{\mathcal{R}} \cap K)\}, \quad (\text{A.3})$$

and we define

$$T \equiv \sup \{t(\tilde{\mathcal{R}} \cap K)\}. \quad (\text{A.4})$$

Finally, we choose some $\tilde{K} \supset K$ such that the intersection of $\tilde{\mathcal{R}}$ and $\partial \tilde{K}$ has $t(\tilde{\mathcal{R}} \cap \partial \tilde{K}) = \{0, T\}$, that is, we extend K to \tilde{K} so that the boundary of the section of $\tilde{\mathcal{R}}$ contained in \tilde{K} has $t = 0$ at one end and

²⁰ τ has been named as such since it corresponds to the proper time of a passenger following Γ .

²¹Technically, \mathcal{R} could become arbitrarily thin as $\tau \rightarrow +\infty$ or $\tau \rightarrow -\infty$, but since this is safely far away from K , one could simply choose $\mathcal{R}' \supset \mathcal{R}$ such that this is not the case.

$t = T$ at the other. We also de-specify t on $\tilde{K} \setminus K$ so that we are free to choose t as we like inside $\tilde{\mathcal{R}} \cap \tilde{K}$. $\tilde{\mathcal{R}}$ and \tilde{K} will be the main regions we shall study from now on.

It is crucial that we do not extend \mathbf{n} along a path inside \mathcal{R} , as the point of this analysis is to see how the 4-velocity of an observer following Γ changes relative to the *background* reference frame defined by \mathbf{n} in F . If we extended it along a path *inside* \mathcal{R} , we would ensure that the observer following Γ remains at rest with respect to the local reference frame defined by \mathbf{n} , and the possible discontinuity in the normal vector field from extending \mathbf{n} to $\mathcal{R} \cup F$ would instead occur in F .

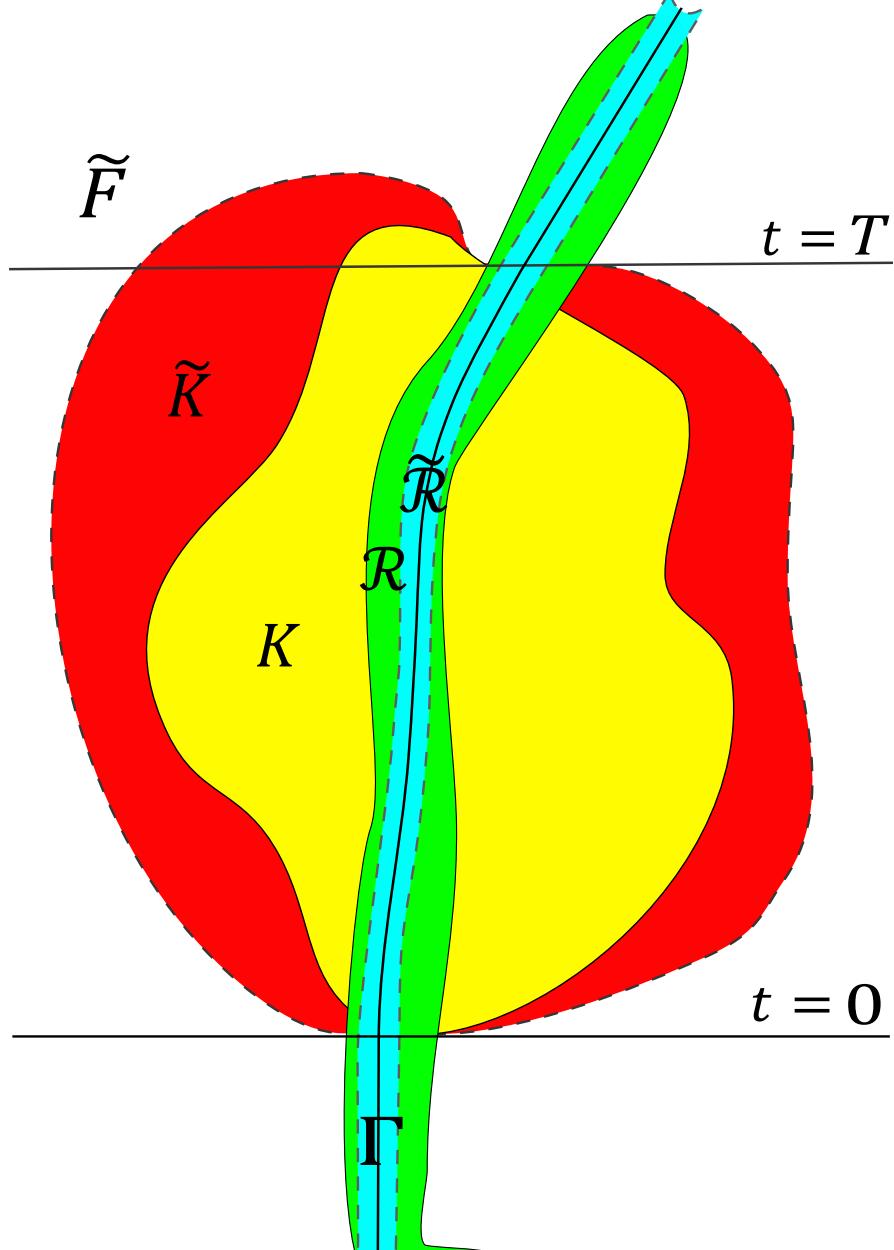


Figure 6: Schematic diagram showing Γ , \mathcal{R} , $\tilde{\mathcal{R}}$, K , \tilde{K} and \tilde{F} . $\tilde{K} \supset K$ is such that the boundary $\partial\tilde{K} \cap \tilde{\mathcal{R}}$ is composed of two sections, one at $t = 0$ and the other at $t = T$, the ‘entry’ and ‘exit’ of \tilde{K} .

A.3.3 Construction of the time coordinate inside $\tilde{\mathcal{R}}$

In $\tilde{F} \equiv M \setminus \tilde{K}$, the time coordinate is already specified. How may we choose it inside \tilde{K} , and in particular $\tilde{\mathcal{R}} \cap \tilde{K}$? One may imagine that since $\tilde{\mathcal{R}}$ is flat, one can analytically extend t inside $\tilde{\mathcal{R}} \cap \tilde{K}$, to create a continuous time coordinate with a constant associated normal vector inside $\tilde{\mathcal{R}}$. However, this is not in general possible, a consequence of the flat region $\tilde{F} \cup \tilde{\mathcal{R}}$ not being simply connected. There is no guarantee that the parallel transport of \mathbf{n} defined at some point with $t < 0$ along a path traversing $\tilde{\mathcal{R}}$ will match \mathbf{n} on the other side of K .

So what freedom do we have to choose t inside $\tilde{\mathcal{R}}$? What is the most natural choice of t ? If it were *always* possible to choose a time coordinate t such that within $\tilde{\mathcal{R}}$ we have $N = N(t)$, this would imply that rest frame transitions are impossible²², contrary to the example in Section 2.2. Therefore, we must search for something weaker. The next-best option is to have $\tilde{R}_t \equiv \tilde{\mathcal{R}} \cap \Sigma_t$ extrinsically flat (a locally-Minkowskian time coordinate), which in turn implies that it is also intrinsically flat by the Gauss equation. This is given in general as follows:

$$\gamma_\rho^\alpha \gamma_\sigma^\beta \gamma_\mu^\gamma \gamma_\nu^\delta {}^{(4)}R_{\alpha\beta\gamma\delta} = {}^{(3)}R_{\rho\sigma\mu\nu} + K_{\rho\mu} K_{\sigma\nu} - K_{\rho\nu} K_{\sigma\mu}, \quad (\text{A.5})$$

where $\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$. Since the ambient space $\tilde{\mathcal{R}}$ is flat and the extrinsic curvature $K_{\mu\nu}$ is assumed to vanish, we are left with

$${}^{(3)}R_{\rho\sigma\mu\nu} = 0.$$

We will now show that such a choice of t always exists by explicit construction. Once such a t has been chosen, we will be able to see exactly what the restrictions on the associated lapse N are.

Take p_0 again and, using the same method with parallel transport, construct a vector field on $\tilde{\mathcal{R}}$ from the normal vector \mathbf{n} at this point:

$$\mathbf{n}_1 : p \in \tilde{\mathcal{R}} \rightarrow T_p \tilde{\mathcal{R}}.$$

In exactly the same way, choosing some point in Γ after it has traversed \tilde{K} , we can construct a second vector field:

$$\mathbf{n}_2 : p \in \tilde{\mathcal{R}} \rightarrow T_p \tilde{\mathcal{R}}.$$

These vector fields are the extension of the global reference frame on \tilde{F} inside $\tilde{\mathcal{R}}$, with \mathbf{n}_1 from the past of \tilde{K} and \mathbf{n}_2 from the future. As explained above, there is no reason that they must agree inside $\tilde{\mathcal{R}}$.

Now we reuse the Minkowskian coordinate system (τ, x, y, z) inside $\tilde{\mathcal{R}}$, choosing $\tau = 0$ to coincide with $t = 0$. In the coordinate basis we have $n_1^\mu = (1, \mathbf{0})$, and we now define σ such that

$$\begin{aligned} n_2^\mu &= (\gamma, \gamma\sigma), \\ \gamma &\equiv (1 - \sigma^2)^{-\frac{1}{2}}, \quad \sigma \equiv |\sigma| < 1. \end{aligned} \quad (\text{A.6})$$

σ is a constant vector, so $d\sigma = 0$, where d denotes the exterior derivative. This implies $\exists \phi : \tilde{\mathcal{R}} \rightarrow \mathbb{R}$ such that $\sigma = d\phi$. Looking at the expressions for $\mathbf{n}_1, \mathbf{n}_2$ and σ , we see

$$\mathbf{n}_2 = \gamma(\mathbf{n}_1 + \sigma) \iff \mathbf{n}_2 = \gamma(-d\tau + d\phi), \quad (\text{A.7})$$

a coordinate-independent equation. Now define

$$\tau_0 \equiv \inf \{\tau(p) : p \in \tilde{\mathcal{R}} \cap t^{-1}(T)\},$$

recalling that

$$\tilde{\mathcal{R}} \cap t^{-1}(T) \subset \partial \tilde{K},$$

²²At least for warp drives with a finite region of flat space inside them, that is, warp drives with a non-point-like interior.

that is, $\tilde{\mathcal{R}} \cap t^{-1}(T)$ is the ‘exit’ of the compact region \tilde{K} inside $\tilde{\mathcal{R}}$. ϕ is only specified up to an arbitrary constant, so we take it that $\phi = 0$ somewhere in the boundary of \tilde{R}_T , with $\phi \geq 0$ inside \tilde{R}_T .

Consider the section of the hypersurface $t = T$ inside $\tilde{\mathcal{R}}$, the ‘exit’ of \tilde{K} . This has a constant normal

$$(n_2)_\mu = (-\gamma, \gamma\sigma),$$

so using usual Euclidean geometry, it has the planar equation

$$-\tau + \sigma \cdot \mathbf{x} = \text{constant}. \quad (\text{A.8})$$

Thus, we see that the infimum of τ on this surface, τ_0 , will be reached at the (limiting) point where $\sigma \cdot \mathbf{x}$ is minimised. Since $d\phi = \sigma$, we can also see that

$$\phi = \phi_0 + \sigma \cdot \mathbf{x}, \quad (\text{A.9})$$

for some $\phi_0 \in \mathbb{R}$. Therefore we see that minimising τ within the surface $t = T$ is equivalent to minimising ϕ . Since we set ϕ such that its minimum on \tilde{R}_T is zero, for any point where $\tau = \tau_0$ and $t = T$, $\phi = 0$. Also, since hypersurfaces of constant τ have a fixed boundary in these coordinates (independent of τ) and ϕ does not depend on τ either, this means $\phi \geq 0$ everywhere inside $\tilde{\mathcal{R}}$.

Now it is time to choose the time coordinate t inside $\tilde{\mathcal{R}}$. There are many possible choices that give a valid time coordinate, continuous at $t = 0$ and $t = T$ with \tilde{R}_t being extrinsically flat, so here we give just one, to prove existence:

$$t = \frac{T}{\tan \sigma} \tan \left(\frac{\sigma \tau}{\tau_0 + \phi} \right), \quad (\text{A.10})$$

or if $\sigma = 0$, we simply take

$$t = T \left(\frac{\tau}{\tau_0} \right), \quad (\text{A.11})$$

as in that case $\phi \equiv 0$. See Figure 7 for a diagram showing this choice of t . This satisfies the necessary conditions:

1. $\tau = 0 \implies t = 0$ and $(\tau = \tau_0, \phi = 0) \implies t = T$.
2. The hypersurfaces \tilde{R}_t are extrinsically flat.
3. $t = T \implies -dt$ is parallel to \mathbf{n}_2 , and thus the *normalised* normal vector field associated to this choice of t is continuous, and t is a legitimate time coordinate.

The first of these points follows immediately from (A.10) and (A.11). The second can be seen by observing that a hypersurface in $\tilde{\mathcal{R}}$ with $t = \text{constant}$ is described by

$$\tau = k(\tau_0 + \phi), \quad (\text{A.12})$$

for some $k > 0$. Since ϕ takes the form $\phi = \phi_0 + \sigma \cdot \mathbf{x}$, \tilde{R}_t is isometric to a section of a 3-dimensional plane in Minkowski space, and clearly has no extrinsic curvature. The last point follows from calculating $-dt$, and finding

$$-dt \propto -(\tau_0 + \phi) d\tau + \tau d\phi,$$

where the constant of proportionality is strictly positive. Using this and (A.7), the condition for $-dt \propto \mathbf{n}_2$ is that $\tau_0 + \phi = \tau$. But this is precisely the condition that $t = T$ in (A.10) and (A.11). Therefore, our time coordinate t is continuous with a continuous, normalised, everywhere-timelike normal vector field $\mathbf{n} = -N dt$, where $N : \tilde{F} \cup \tilde{\mathcal{R}} \rightarrow (0, \infty)$ is the associated lapse, and the construction of our time coordinate is complete.

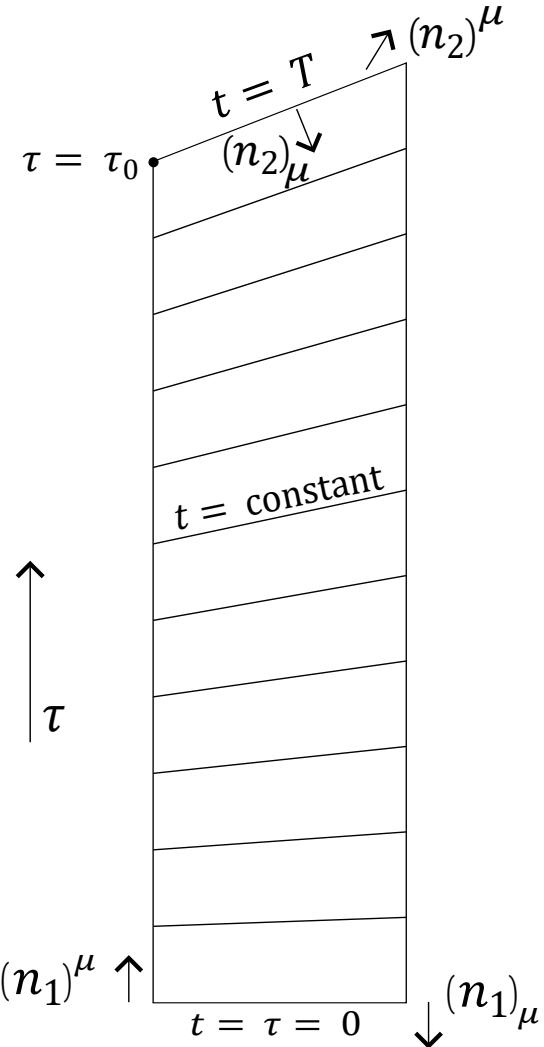


Figure 7: A choice of t that is locally Minkowskian, in the region $\tilde{\mathcal{R}} \cap \tilde{K}$. The vector fields \mathbf{n}_1 and \mathbf{n}_2 have constant components throughout $\tilde{\mathcal{R}}$.

A.3.4 Construction of local coordinate system in $\tilde{\mathcal{R}}$

Now that we have vanishing extrinsic curvature of \tilde{R}_t , it is time to see what this means. Choose a (x', y', z') on $\tilde{\mathcal{R}}$ such that in the ADM formalism, its metric can be written as

$$ds^2 = -N^2 dt^2 + \delta_{ij}(dx'^i - \beta'^i dt)(dx'^j - \beta'^j dt), \quad (\text{A.13})$$

for some shift vector β' . This must be possible since the spatial slices are intrinsically flat, so we can take the induced metric to be δ_{ij} . The extrinsic curvature tensor of a hypersurface $t = \text{constant}$ in this metric is

$$K_{ij} = \frac{1}{N} \partial_{(i} \beta'_{j)}. \quad (\text{A.14})$$

Therefore, the vanishing of the extrinsic curvature implies that β' is a Killing vector of the hypersurfaces. Now we apply the coordinate transformation

$$\begin{aligned} t &\rightarrow t, \\ x' &\rightarrow \tilde{x}(t, x', y', z'), \\ y' &\rightarrow \tilde{y}(t, x', y', z'), \\ z' &\rightarrow \tilde{z}(t, x', y', z'), \end{aligned}$$

to some small open region inside $\tilde{\mathcal{R}}$, setting

$$\begin{aligned} (\partial_t \tilde{x}, \partial_t \tilde{y}, \partial_t \tilde{z}) &= \beta', \\ (\tilde{x}(t_0, x', y', z'), \tilde{y}(t_0, x', y', z'), \tilde{z}(t_0, x', y', z')) &= (x', y', z'), \end{aligned} \tag{A.15}$$

for some $0 < t_0 < T$. The region must be small in order to ensure that the integral curves of β' do not go so far as to leave $\tilde{\mathcal{R}}$. However, since this argument can be applied to any part of $\tilde{\mathcal{R}}$, this is not problematic. One can calculate that this means the new metric $g_{\mu\nu}$ has $g_{0\mu} = (-N^2, 0, 0, 0)$. Without calculation, we also know that the spatial metric must be preserved, as we are shifting points along a Killing vector field of the hypersurfaces, under which the spatial metric is invariant.

Noting that since the time coordinate t did not change, the lapse N did not change either, we see that the metric now becomes

$$ds^2 = -N^2 dt^2 + d\tilde{x}^2 + d\tilde{y}^2 + d\tilde{z}^2.$$

Since here $K_{\mu\nu} = 0$, one can see from the Ricci equation (3.26) that there is at least one component of the Riemann tensor proportional to $\partial_i \partial_j N$ for each pair $\{i, j\} \in \{1, 2, 3\}^2$. Requiring that \mathcal{R} is flat therefore means that the most general form that the lapse may take in this coordinate system is linear in the spatial coordinates, giving

$$N(t, \tilde{\mathbf{x}}) = p(t) + \mathbf{q}(t) \cdot \tilde{\mathbf{x}},$$

for some p, \mathbf{q} dependent only on t .

Looking at Figure 7, it is now clear exactly how rest frame transitions work geometrically. We simply have to choose a geometry inside $\tilde{\mathcal{R}}$ such that the extension of \mathbf{n} inside $\tilde{\mathcal{R}}$ from the past and the future do not agree, differing by a vector characterised by σ . Then free-falling observers will exit $\tilde{\mathcal{K}}$ with a 3-velocity of $-\sigma$.

B Tables of symbols

Due to the large number of functions and quantities represented by different symbols throughout the paper, we provide a summary of the most important ones in this appendix for ease of reference.

General

The following symbols appear in several sections, with very similar or identical meanings.

u	Speed of destination rest frame as measured in passenger's initial rest frame
γ_u	Lorentz factor for speed u
Σ_t	Hypersurface of constant t
\mathbf{n}, n^μ	Future-pointing unit normal vector to Σ_t
N	Lapse function associated to t such that $\mathbf{n} = -N dt$
β	Shift vector
γ_{ij}	Induced metric on hypersurface Σ_t
Γ_{RFT}	Inextendible timelike geodesic corresponding to a rest frame transition

Section 1: Introduction

$\zeta(t)$	Path in z followed by Alcubierre drive
$v(t)$	Velocity of Alcubierre drive, given by $\partial_t \zeta(t)$
f	Shape function of Alcubierre drive
$\alpha(\tau)$	Worldline of Alcubierre drive centre, a geodesic parameterised by proper time

Section 2: Geodesic rest frame transitions

T_{flat}	Time at which warp drive vanishes
$\mathbf{r}(t)$	Path followed by warp drive centre
$\mathbf{v}(t, \mathbf{x})$	Velocity vector field of warp drive; 3-velocity of warp drive's passenger at $(t, \mathbf{r}(t))$
$s(t, \mathbf{x})$	Function appearing in definition of lapse N
T_1, τ_1	Coordinate and proper time at which warp drive starts transition ($T_1 = \tau_1$)
T_2, τ_2	Coordinate and proper time at which warp drive finishes transition ($T_2 \neq \tau_2$)
$a(t)$	Transition function for shift vector β ; decreases smoothly from 1 to 0 in $T_1 < t < T_2$
$b(t)$	Transition function for lapse N ; takes form of bump function between $T_1 < 0 < T_2$
$\lambda(t)$	Function of a ; analogous to Lorentz factor
$\chi_1(\tau)$	Parametrisation of first part of geodesic inside warp drive by proper time, before transition ($0 \leq \tau < T_1$)
$\chi_2(\tau)$	Parametrisation of second part of geodesic inside warp drive by proper time, during transition ($T_1 \leq \tau < T_2$)
$\chi(\tau)$	Parametrisation of combination of χ_1 and χ_2 by proper time; also a geodesic

Section 3: Creating a closed timelike geodesic

Every symbol describing the outgoing warp drive has a counterpart with a hat. These are the corresponding symbols describing the returning warp drive in the CTC spacetime, M_{CTC} . There are two other changes:

1. The critical times, $t = 0, T_1, T_2$, now become $\hat{t} = \hat{T}_0, \hat{T}_1, \hat{T}_2$ respectively.
2. $\hat{\mathbf{v}}$ finishes such that the warp drive is at rest in S , so for $\hat{t} \geq \hat{T}_1$, $\hat{\mathbf{v}} = (0, 0, -u)$.

$h_{\mu\nu}, \hat{h}_{\mu\nu}$	Perturbations to Minkowski metric in outgoing and returning warp drives' domains respectively
T_{finish}	Time when second warp drive returns, as measured in S
$\bar{v}^z, \hat{\bar{v}}^z$	Average speeds (taken as positive) of both warp drives along their respective z axes in S and \hat{S} respectively
C	Section of t axis between $t = T_{finish}$ and $t = 0$, assuming $T_{finish} < 0$
Γ_{CTC}	Union of outgoing and returning geodesics at warp bubble centre and C ; itself a geodesic
$\epsilon, \hat{\epsilon}$	Amount of proper time elapsed since start of transition
$(CTC) R_{\rho\sigma\mu\nu}$	Riemann tensor on CTC spacetime
$R_{\rho\sigma\mu\nu}, \hat{R}_{\rho\sigma\mu\nu}$	Riemann tensors within respective domains of warp drives
$\gamma_{\mu\nu}, \hat{\gamma}_{\mu\nu}$	Projection operators for hypersurfaces $\Sigma_t, \hat{\Sigma}_{\hat{t}}$
$K_{\mu\nu}, \hat{K}_{\mu\nu}$	Extrinsic curvature tensors for hypersurfaces $\Sigma_t, \hat{\Sigma}_{\hat{t}}$

The specific example in subsection 3.4 adds the following symbols:

q	Bump function with $0 \leq q(x) \leq 1$, $\text{supp } q = (0, 1)$ and $\int_0^1 q(x) dx = 1$; here chosen as $q(x) = 140x^3(1-x)^3$ for $0 < x < 1$
$q^{(-1)}$	Primitive of q , a ‘smooth step function’; $q^{(-1)}(x) = 0$ for $x \leq 0$, $q^{(-1)}(x) = 1$ for $x \geq 1$
$q^{(-2)}$	Second primitive of q ; $q^{(-2)}(x) = 0$ for $x \leq 0$
r_1	Radius of flat region inside warp drives
r_2	Radius of warp drives as measured by external observers in their respective frames
t_1, \hat{t}_1	Times at which warp drives change from acceleration to deceleration
t_2, \hat{t}_2	Actual finish times of transition, slightly less than T_2, \hat{T}_2 , to avoid collision of warp drives
α	Acceleration parameter
κ	Constant associated with the choice of q ; $\kappa = q^{(-2)}(1)$
ω	T_1/T_2 , fraction of the outgoing journey spent before the transition starts; same for both outgoing and returning sections

Section 4: The weak energy condition with a non-unit lapse

ρ	Eulerian energy density
ϕ_i	Momentum flux of energy-momentum tensor; $\phi_i = G_{ni}$
ψ	Primitive of β in the curl-free case; $\beta = \nabla \psi$

Appendix A: General analysis of rest frame transitions

M	Entire spacetime
K	Compact region outside which M is Riemann-flat
F	Complement of K ; Riemann-flat
\mathcal{R}	Flat interior of warp bubble
Γ	Inextendible timelike geodesic inside \mathcal{R}
$\tilde{\mathcal{R}}$	Subset of \mathcal{R} isometric to infinite 3+1 cylinder with $\Gamma \subset \tilde{\mathcal{R}}$
$\tilde{\mathcal{R}}_t$	Hypersurface inside $\tilde{\mathcal{R}}$ of time t
T	Time at which observer in warp drive exits $\tilde{\mathcal{R}}$
\tilde{K}	Superset of K such that $\tilde{\mathcal{R}} \cap \partial \tilde{K}$ is composed of two disjoint hypersurfaces, one at $t = 0$ and one at $t = T$
\tilde{F}	Complement of \tilde{K} ; Riemann-flat
\mathbf{n}_1	Extension of \mathbf{n} inside $\tilde{\mathcal{R}}$ from past of $\tilde{K} \cap \tilde{\mathcal{R}}$
\mathbf{n}_2	Extension of \mathbf{n} inside $\tilde{\mathcal{R}}$ from future of $\tilde{K} \cap \tilde{\mathcal{R}}$
σ	Proportional to spatial part of \mathbf{n}_2 , when viewed in (τ, x, y, z) coordinates
ϕ	Primitive of σ , i.e. $d\phi = \sigma$
τ_0	Infimum of τ on surface $t = T$

References

- [1] Miguel Alcubierre. The warp drive: hyper-fast travel within general relativity. *Classical and Quantum Gravity*, 11(5):L73–L77, may 1994. arXiv:gr-qc/0009013, doi:10.1088/0264-9381/11/5/001.
- [2] Barak Shoshany. Lectures on Faster-than-Light Travel and Time Travel. *SciPost Phys. Lect. Notes*, 10, 2019. arXiv:1907.04178, doi:10.21468/SciPostPhysLectNotes.10.
- [3] Francisco S. N. Lobo, editor. *Wormholes, Warp Drives and Energy Conditions*. Springer International Publishing, 2017. doi:10.1007/978-3-319-55182-1.

- [4] Serguei Krasnikov. *Back-in-Time and Faster-than-Light Travel in General Relativity*. Springer International Publishing, 2018. doi:10.1007/978-3-319-72754-7.
- [5] Erik Curiel. A Primer on Energy Conditions. In Dennis Lehmkuhl, Gregor Schiemann, and Erhard Scholz, editors, *Towards a Theory of Spacetime Theories*, pages 43–104. Springer New York, New York, NY, 2017. arXiv:1405.0403, doi:10.1007/978-1-4939-3210-8_3.
- [6] Jessica Santiago, Sebastian Schuster, and Matt Visser. Generic warp drives violate the null energy condition. *Physical Review D*, 105(6), 2022. doi:10.1103/physrevd.105.064038.
- [7] Chris Van Den Broeck. A ‘warp drive’ with more reasonable total energy requirements. *Classical and Quantum Gravity*, 16(12):3973–3979, 1999. doi:10.1088/0264-9381/16/12/314.
- [8] Pedro F. González-Díaz. Warp drive space-time. *Physical Review D*, 62(4), 2000. doi:10.1103/physrevd.62.044005.
- [9] Allen E. Everett. Warp drive and causality. *Phys. Rev. D*, 53:7365–7368, 1996. URL: <https://link.aps.org/doi/10.1103/PhysRevD.53.7365>, doi:10.1103/PhysRevD.53.7365.
- [10] Francisco S. N. Lobo. Closed timelike curves and causality violation. 2010. URL: <https://arxiv.org/abs/1008.1127>, doi:10.48550/ARXIV.1008.1127.
- [11] Jacob Hauser and Barak Shoshany. Time Travel Paradoxes and Multiple Histories. *Phys. Rev. D*, 102, Sep 2020. arXiv:1911.11590, doi:10.1103/PhysRevD.102.064062.
- [12] Barak Shoshany and Jared Wogan. Wormhole Time Machines and Multiple Histories. *Gen. Relativ. Gravit.*, 55(44), 2023. arXiv:2110.02448, doi:10.1007/s10714-023-03094-8.
- [13] Barak Shoshany and Zipora Stober. Time Travel Paradoxes and Entangled Timelines. 2023. arXiv:2303.07635, doi:10.48550/arXiv.2303.07635.
- [14] Barak Shoshany. OGRe: An Object-Oriented General Relativity Package for Mathematica. *Journal of Open Source Software*, 6(65):3416, 2021. doi:10.21105/joss.03416.
- [15] Barak Shoshany. OGRePy: An Object-Oriented General Relativity Package for Python. 2024. arXiv:2409.03803, doi:10.48550/arXiv.2409.03803.
- [16] José Natário. Warp drive with zero expansion. *Classical and Quantum Gravity*, 19(6):1157–1165, 2002. doi:10.1088/0264-9381/19/6/308.
- [17] Martin Bojowald. *Hamiltonian formulation of general relativity*, pages 17–112. Cambridge University Press, 2010. doi:10.1017/CBO9780511921759.003.
- [18] Erik W. Lentz. Breaking the warp barrier: Hyper-fast solitons in einstein-maxwell-plasma theory, 2020. URL: <https://arxiv.org/abs/2006.07125>, doi:10.48550/ARXIV.2006.07125.
- [19] Alexey Bobrick and Gianni Martire. Introducing physical warp drives. *Classical and Quantum Gravity*, 38(10):105009, 2021. doi:10.1088/1361-6382/abdf6e.
- [20] Shaun D B Fell and Lavinia Heisenberg. Positive energy warp drive from hidden geometric structures. *Classical and Quantum Gravity*, 38(15):155020, 2021. doi:10.1088/1361-6382/ac0e47.
- [21] Ken D. Olum. Superluminal travel requires negative energies. *Physical Review Letters*, 81(17):3567–3570, 1998. doi:10.1103/physrevlett.81.3567.