

Rigorous and Generalized Proof of Security of Bitcoin Protocol with Bounded Network Delay

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Abstract

A proof of the security of the Bitcoin protocol is made rigorous, and simplified in certain parts. A computational model in which an adversary can delay transmission of blocks by time Δ is considered. The protocol is generalized to allow blocks of different scores and a proof within this more general model is presented. An approach used in a previous paper that used random walk theory is shown through a counterexample to be incorrect; an approach involving a punctured block arrival process is shown to remedy this error. Thus, it is proven that with probability one, the Bitcoin protocol will have infinitely many honest blocks so long as the fully-delayed honest mining rate exceeds the adversary mining rate.

I. INTRODUCTION

In 2009, Satoshi Nakamoto introduced the Bitcoin protocol [1], a permissionless distributed ledger whose security is based on honest nodes having over 50% of hashing power. Nakamoto proved that such a protocol was secure against the private double-spend attack, but future papers recognized there are other possible attacks (including, in particular, selfish mining attacks [2] and balance attacks [3]).

In [4] and *et al.* and [5], proofs of the security of the bounded-delay network model are given, but the bounds are not tight. In [6] the authors provide a tight bound of the security of the Bitcoin protocol, but do not prove the security against any attack.

Security of the Bitcoin protocol against *all* types of attacks was finally proven in [7] and [8]. However, the proof in [7] has a small but important error in its analysis. The error involves assuming the sequences of random variables representing the difference in adversary chain length and honest chain length is a random walk: in reality, it is *not* a random walk, which we prove in a counterexample Appendix A. The alternative proof [8] also proves the security of Bitcoin. However, the proof is long and non-intuitive in parts.

In work that was concurrent with [7] paper, [8] also proves the security of Bitcoin. However, this proof is complicated and non-intuitive.

Our approach is also important for the proof of another protocol called Merged Bitcoin which is introduced in a companion paper. Hence, for this paper, we generalize the Bitcoin model. In our case, we allow blocks of different types, and each of these types may have different point values.

We define the fully-delayed score growth rate of honest nodes as the rate of growth of honest blocks when the blocks are subject to full network delay Δ , which we call λ_h . The score growth rate of all adversary blocks is defined as λ_a . The security region is then proven to be $\lambda_a < \lambda_h$.

In order for our paper to be mostly self-contained, we reproduce some of the results of [7]. However, we do this for our generalized Bitcoin model in which different types of blocks may have different point values. The paper changes the approach of [7] in three significant ways. First, the proof herein is generalized to the multi-block type model, allowing the proof to be used for Merged Bitcoin [9]. Second, instead of considering the probability that a block arriving at a time τ_j^h is a Nakamoto block (a block that stays in the chain forever), we consider the probability that an interval is a *Nakamoto interval*. This avoids the complication that, conditioned on a particular time, block arrivals occurring before an honest block arrival j may not be independent of the honest arrival time. Finally, we resolve the issue with the former paper involving random walks, in which an erroneous assumption about block arrivals in the prior paper is remedied with a punctured arrival process technique.

II. MODELS AND DEFINITIONS

A. The protocol

We consider the Bitcoin protocol as defined in [1], with one small modification. We allow miners to mine blocks of different types, where each type of block may have a different score. Miners, both honest and dishonest, can mine blocks of any of these types. The score of a subchain is the total score of all the blocks that form the chain. The fork-choice rule is to mine on the chain with the highest score, with ties broken in an arbitrary way. This generalization does not considerably change the analysis. However, in a companion paper, we use this general proof for a generalization of the Bitcoin algorithm called Merged Bitcoin.

B. The Bounded Delay Network Model

We consider the Δ -bounded delay network model¹

In this model, the adversary can delay the transmission of honest blocks up to time Δ . The adversary may also delay transmission of its own blocks to any honest miner by any time. However, once transmitted to a single honest miner, the dishonest miner can only delay this information by time Δ to all the other miners. This is meant to model a case where honest miners are constantly transmitting their view of the blockchain to the public network.

C. The Arrival Processes

For each block type, each honest miner has a fixed block-rate for each block type, where the arrivals of each block type is a Poisson process. We presume that each honest miner has a small total fraction of the mining power. Hence, in any interval of length Δ , we can presume that any honest miner does not mine more than one block. Let each block type have some score $c_i > 0$. Let the combined honest block-rate (measured in blocks per second) for all the honest miners for blocks of type i be h_i . Hence, total score growth rate of such a process when there are no delays and no adversary blocks is $\sum c_i h_i$.

The adversary is not subject to network delays, and we presume the adversary can see all blocks mined by honest miners as soon as they arrive. Let the block-rate (measured in blocks per second) for the adversary for blocks of type i be b_i . Hence, the score growth rate of the adversary mining by itself is given by $\sum c_i b_i$.

III. NAKAMOTO BLOCKS STAY IN CHAIN FOREVER

In this section we will introduce the idea of a Nakamoto interval, which is adapted from [7]. we will prove that Nakamoto intervals have an honest block (called a Nakamoto block) which stays in the canonical chain forever. In a following section, we will show that these intervals occur at any point with probability greater than 0.

A. Fully-Delayed Growth Rate is Minimal Growth Rate

Let us first define some notation. Borrowing from [7], we let $\mathcal{T}_h(t)$ be the fictitious honest tree composed of all the honest blocks that were mined since the genesis block in the order they arrived, subject to each of them facing a network delay of Δ . As in [7], $\mathcal{T}_h(0)$ is the genesis block.

Consider the mother tree $\mathcal{T}(t)$, which is the tree of *all* blocks, honest or dishonest, public or private, that exist connected to the genesis block. Each block of that tree has a score, which is the score of the chain that leads to it, starting at the genesis block.

¹This has also been called the asynchronous network with Δ -bounded delays [5], the Δ -synchronous model [7]. We use the term Δ -bounded delay model to emphasize that the model is *not* synchronous, but rather has a delay that could potentially be very large.

Definition 1. The *chain score* of a block j is the score of the chain that starts from the genesis block and ends at block j . Using this terminology, the fork choice rule is to mine on top of the visible block with the highest chain score.

Definition 2. A *delay schedule* for a set of arrival times is a schedule of delays for each honest block and adversary block. For a given honest block mined by a miner j , the delay schedule is a set of delays for each other honest miner, which is some time in $[0, \Delta]$. For each adversary blocks, the delay schedule includes a time in $[0, \infty]$ from when the block is mined by the adversary to when it is first broadcast to an honest block. It also includes a time in $[0, \Delta]$ for each other honest miner, that indicates how much it is delayed to all the other honest blocks.

We shall see in this section that the only delay schedule we need to consider is the fully-delayed schedule, in which all honest blocks are delayed by Δ to the other miners.

Definition 3. The *fully-delayed honest chain* is the hypothetical honest chain that is produced when no adversary blocks are added and each honest block is delayed the maximum time Δ to all other honest blocks.

Lemma 4. *The score of the highest score honest block grows at least as fast as the fully-delayed honest chain.*

Proof: The analogous lemma was proven in [6] for the Bitcoin case and we generalize the proof for our case. First, let us assume that the only arrivals included are honest block arrivals (we shall consider the case of adversary arrivals being added to the tree later in the proof). Suppose there exists a block that was not fully delayed to all other blocks, which we shall call a *non-delayed block*.

Let us call the tree that contains at least one node that was not fully delayed \mathcal{T}_{prior} . We shall produce a new tree, $\mathcal{T}_{delayed}$, formed using the same blocks and same arrival times, in which this non-delayed block is fully delayed. We shall show that the chain score of each block in this new tree is less than or equal to its score in \mathcal{T}_{prior} , for all times t .

Let us consider a block j arriving at τ_j^h that was not fully-delayed to future blocks. Consider all the honest blocks that arrived within time Δ from τ_j^h . Some of these may be mined on top of block j . Consider a particular such block, block m . If block m was mined on block j , this means that the chain-score of block j was the highest that that miner saw when the block was mined. Now let's consider the tree produced with the exact same delay schedule, except block j was delayed fully. With this new delay schedule, for all blocks within Δ of block j , their chain score is now either the same or less. Either these blocks were mined on block j , which they now no longer can be, or they were mined on a prior block. If they previously had been mined on a prior block, then nothing changes by adding this delay. If they were mined on block j , they now must be mined on a block (that they can still see) with chain score less than or equal to j (because j was the highest score chain that it saw previously).

Thus, by adding this delay, all blocks up to $\tau_j^h + \Delta$ have chain score less than or equal to their original score.

Now we show by induction that from this time onward, the chain score of all blocks after this are less than or equal to their score in \mathcal{T}_{prior} . First, let us consider the first block that arrives after $\tau_j^h + \Delta$. From its viewpoint, it can only see a subset of all blocks that arrived before it. But all blocks that arrived before it have chain score less than or equal to their score in \mathcal{T}_{prior} . This new block will be added to the highest score visible to its miner each of which are less than or equal to their score in \mathcal{T}_{prior} . Hence the score of this block is less than or equal to its score in \mathcal{T}_{prior} .

Now assume that all blocks up to block k have chain score less than or equal to \mathcal{T}_{prior} . Then, block $k+1$ has a view of a subset of these blocks, each of which have score less than or equal to their score in \mathcal{T}_{prior} . Then, block $k+1$ will be added to one of these blocks, and thus it will have score less than or equal to what it had in \mathcal{T}_{prior} .

This procedure of adding full delays to blocks that were not fully delayed can be applied until the delay schedule has only full delays, and this can only maintain or decrease the score of the canonical chain.

A similar proof by induction can show that removing any adversary block from the tree can only maintain or decrease the score of all honest blocks. Hence, the minimum score tree is produced when honest blocks are fully-delayed and no adversary blocks are injected into the chain. ■

Using the same proof by induction method, we can also show the following lemma:

Lemma 5. *Removing any honest block from the tree can only decrease or maintain the score of the canonical chain in the view of any honest node.*

B. First key result: Score growth rates are additive

Definition 6. We define $S_h(a, b)$ to be the growth in score of a fully-delayed honest tree starting at time a and ending at time b .

We also define:

Definition 7. We define $S_{min}(t)$ as the minimum score of the highest score honest block at time t that is visible to all honest miners, minimized over all possible attack strategies.

Lemma 8. *Consider any two times t_1 and t_2 , $t_2 \geq t_1 + 2\Delta$. For all such times:*

$$S_{min}(t_2) \geq S_h(t_1 + \Delta, t_2 - \Delta) + S_h(t_1). \quad (1)$$

In words, the above lemma says that the score at time t_2 is at least the score growth of the fictional fully delayed honest chain from $t_1 + \Delta$ to $t_2 - \Delta$, plus the score at t_1 .

Proof: By Lemma 4, we know that maximal delays minimize the score of the canonical chain in the view of all honest nodes at all times. Since network delay is at most Δ , at time t_2 , all nodes will see all blocks mined up to time $t_2 - \Delta$. From the time $t_1 + \Delta$ to $t_2 - \Delta$, if no blocks arrive in the time $[t_1, t_1 + \Delta]$, the score of the honest chain will increase by $S_h(t_1 + \Delta, t_2 - \Delta)$. By Lemma 5, we know that if blocks are deleted from a chain it can only maintain or *decrease* the score of the chain in the view of any node in the future. Hence, the score grows *at least* by $S_h(t_1 + \Delta, t_2 - \Delta)$ (the amount it would have grown if there were zero arrivals in the Δ seconds after t_1). ■

C. Nakamoto Block Definition

We define a few events that are properties of a set of arrival times, and an honest block j .

Adapting a definition from [10], we define:

Definition 9. An honest block is a *loner* if no honest blocks occur in the time Δ before and after it.

Note that all loners appear in the fully-delayed honest chain that has no dishonest blocks added.

We now adapt the notion of a Nakamoto block from [7] by considering a Nakamoto interval.

Definition 10. An interval of length $2q > 0$ centered at a time τ_q is *q-loner* if (1) a single honest block arrives in this interval, (2) there are no other honest blocks mined in the *honest loner interval* $[\tau_q - q - \Delta, \tau_q + q + \Delta]$, and (3) there are no dishonest blocks in the *dishonest loner interval* $[\tau_q - q - 2\Delta, \tau_q + q + 2\Delta]$. The honest block that occurs in a *q-loner* interval is called a *Nakamoto block*.

We shall see why we have included the 2Δ terms in the dishonest loner interval later in the proof. Note that the block that arrives in the length $2q$ Nakamoto interval is a loner. Hence it appears in the hypothetical fully-delayed honest chain.

Let L_q represent the event that a particular length $2q$ interval is a loner interval.

We now define the arrival time properties central to being a Nakamoto block:

Definition 11. (Honest chain, ending at τ_q , dominates from all honest blocks in the past) : Let τ_i^h be the arrival time of the i th honest block. Let E_1 be the event that fully-delayed honest score growth in the

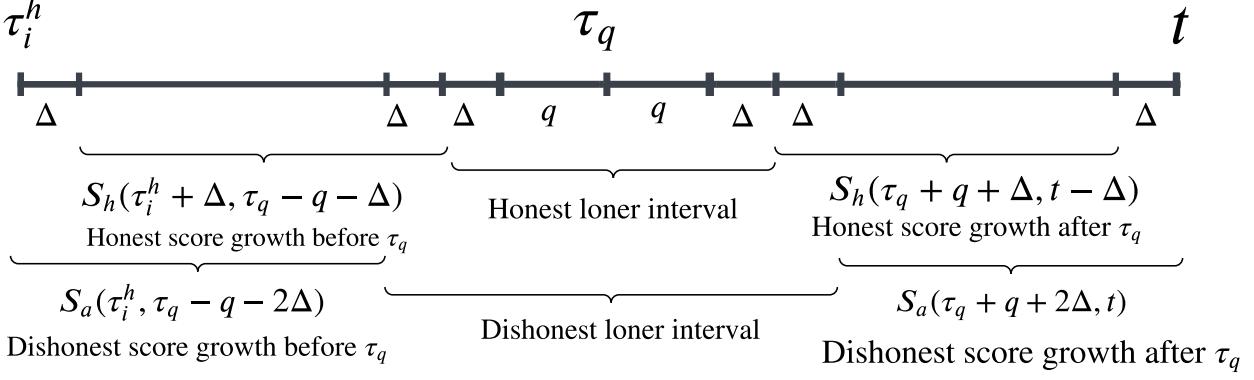


Fig. 1. Diagram representing the past and future honest and adversary growth intervals used in the definitions of E_1 and E_2 . Observe two things. First, the dishonest and honest growth intervals are the same length, even though they start and end at times offset by Δ . Second, the honest score growth intervals do not overlap the honest loner interval; similarly, the dishonest score growth intervals do not intersect the dishonest loner interval. Hence, the event of being a loner interval and the events E_1 and E_2 are independent. Our proof depends on showing that with probability greater than 0, when in the security region, honest score growth exceeds the dishonest score growth for all score growth intervals around a particular time τ_q .

interval $[\tau_i^h + \Delta, \tau_q - \Delta - q]$ is greater than the adversary score growth in the interval $[\tau_i^h, \tau_q - q - 2\Delta]$ for all i in which $\tau_q - q - 2\Delta > \tau_i^h$:

$$E_1 := [S_h(\tau_i^h + \Delta, \tau_q - q - \Delta) > S_a(\tau_i^h, \tau_q - q - 2\Delta) \text{ for all } i \text{ such that } \tau_q - q - 2\Delta > \tau_i^h.]$$

See Figure 1 for a diagram of these intervals, which is labeled with the relevant expressions for score growth in these intervals.

Definition 12. (Honest chain, starting at τ_q , dominates at all times in the future) Let E_2 be the event that the honest score growth in the interval $[\tau_q + q + \Delta, t - \Delta]$ is greater than the adversary score growth in the interval $[\tau_q + q + 2\Delta, t]$, for all $t > \tau_q + 2\Delta + q$. Symbolically:

$$E_2 := [S_h(\tau_q + q + \Delta, t - \Delta) > S_a(\tau_q + q + 2\Delta, t) \text{ for all } t > \tau_q + q + 2\Delta].$$

As for the definition above, refer to Figure 1 to see these intervals labeled, as well as the expressions for score growth during these intervals.

We can now define a Nakamoto interval and its associated Nakamoto block:

Definition 13. An interval of length $2q$ in which the event

$$L_q \cap E_1 \cap E_2$$

occurs is called a *Nakamoto interval*. The honest block that arrives in that interval is called a *Nakamoto block*.

D. Nakamoto Blocks Stay in Chain Forever Proof

We now prove a key result of the proof of the security of the Bitcoin algorithm: that Nakamoto blocks stay in the chain forever. This was also proven in [7], but we reproduce a complete proof here so that our result can be self-contained, and also because our definition of a Nakamoto block is slightly different.

Lemma 14. *If the event*

$$L_j \cap E_1 \cap E_2$$

occurs for some time interval centered at τ_q , then the block that arrives in that interval will stay in the canonical chain forever.

The key ideas are that E_1 means that at the time of the Nakamoto block arrival, no adversary chain can dominate. E_2 implies that at no time after the honest block arrival can the adversary dominate. L_j implies that the canonical chain must contain this block because no other honest or dishonest block arrived within a time close to it. The proof makes these intuitive observations formal.

Proof: Let τ_h be the time of arrival of the Nakamoto block within the length $2q$ interval, and call this block j . Suppose this block does not stay in the chain forever. Then there is the earliest time $t^* \geq \tau_h$ in which there is a chain which dominates the chain containing block j in the view of at least one honest node. Also, this dishonest chain has the most recent honest parent *before* honest block j (which could be the genesis block).

If the most recent honest parent of this alternate chain was *not* before honest block j , then either (1) it was mined on a block that descended from j , or (2) it was mined on an honest block that was produced *after* block j that did not descend from j . In the case of (1), then this chain cannot remove block j from the canonical chain, because block j is part of the chain. In the case of (2), let k be the honest block (occurring after block j), upon which this dishonest chain was mined. Since block j is a loner, the miner producing block k must have seen block j , and this implies that there was another earlier subchain which dominated block j that was produced after block j (otherwise an honest node would not have mined on it). But we are considering the *earliest* time t^* in which an alternate chain dominates block j . This cannot occur, because then the ancestor of honest block k would be the tip of a chain that dominates the chain containing block j , and t^* would not be the earliest time that an alternate chain dominates the chain containing block j .

Note that this t^* must be after $\tau_q + q + 2\Delta$ since we assumed L_j , which means no adversary block arrived in the interval $[\tau_q - q - 2\Delta, \tau_q + q + 2\Delta]$, and given E_1 , no adversary chain can dominate the honest chain at exactly time $\tau_q - q - 2\Delta$.

Suppose that the adversary chain is one starting at the i th honest block. By Lemma 8, and the fact that j is a loner, the score of the tree containing block j is at least:

$$S_{\min}(t^*) \geq S(\tau_i^h) + S_h(\tau_i^h + \Delta, t^* - \Delta)$$

Also, the dishonest blocks can only be mined on top of honest block i *after* i is produced. Hence, the increase in score of the dishonest chain on top of block i must be at most $S_a(\tau_i^h, t^*)$. Recall that we are considering the earliest honest block i upon which the dominating chain was mined, and so this competing chain must only contain adversary blocks. Thus, its score must be at most

$$S(\tau_i^h) + S_a(\tau_i^h, t^*).$$

This means that at this time t^* , if the adversarial chain is to replace the chain containing block j , it must be that:

$$\begin{aligned} S(\tau_i^h) + S_h(\tau_i^h + \Delta, t^* - \Delta) &\leq S(\tau_i^h) + S_a(\tau_i^h, t^*) \\ S_h(\tau_i^h + \Delta, t^* - \Delta) &\leq S_a(\tau_i^h, t^*). \end{aligned} \tag{2}$$

However, we have assumed E_1 and E_2 have occurred. This means that for this particular i , from E_1 :

$$S_h(\tau_i^h + \Delta, \tau_q - q - \Delta) > S_a(\tau_i^h, \tau_q - q - 2\Delta)$$

and because no adversary blocks arrived within $2\Delta + q$ of j :

$$S_h(\tau_i^h + \Delta, \tau_q - q - \Delta) > S_a(\tau_i^h, \tau_q). \tag{3}$$

As well, from E_2 , and that no dishonest blocks arrived within $2\Delta + q$ of τ_q , we have that for this particular time t^* :

$$S_h(\tau_q + q + \Delta, t^* - \Delta) > S_a(\tau_q + q + 2\Delta, t^*) = S_a(\tau_q, t^*). \quad (4)$$

But, the growth of the fully-delayed honest chain from $(\tau_i^h + \Delta)$ to $(t^* - \Delta)$ is at least the sum of its growth in the interval $[\tau_i^h + \Delta, \tau_q - q - \Delta]$ and the interval $[\tau_q + q + \Delta, t^* - \Delta]$. Symbolically:

$$S_h(\tau_i^h + \Delta, t^* - \Delta) \geq S_h(\tau_i^h + \Delta, \tau_q - q - \Delta) + S_h(\tau_q + q + \Delta, t^* - \Delta).$$

Using this, and substituting the right side of this inequality with the right sides of inequalities (3) and (4), we get

$$S_h(\tau_i^h + \Delta, t^* - \Delta) > S_a(\tau_i^h, \tau_q) + S_a(\tau_q, t^*) = S_a(\tau_i^h, t^*),$$

which contradicts (2). Hence, $E_1 \cap E_2 \cap L_j$ implies that the interval centered at τ_q is a Nakamoto interval and the block j that arrives during that interval stays in the canonical chain forever. ■

IV. PROOF THAT A THE n TH HONEST BLOCK IS A NAKAMOTO BLOCK WITH PROBABILITY GREATER THAN 0

In this section we provide the main contribution of this paper. Specifically, we show that if the adversary and honest blockrates are within the security region which we will define, then there is an a-priori probability greater than 0 that any given length $2q$ interval has a Nakamoto block. In the first subsection, we show that a quantity called the average fully delayed growth rate, λ_h , exists. We show in Appendix A that an approach used by [7] has a flaw, specifically that a process claimed to be a random walk is not a true random walk. In this section, we show our contribution, which uses random walk theory on a punctured arrival process which is, in fact, a true random walk.

Note that [7] provides an alternative proof of this theorem in Appendix C.2, which is based on the ergodic properties of arrival times. However, the authors neither precisely define the variables assumed to possess these properties nor specify what those properties are, let alone prove that the variables satisfy them. Consequently, we do not consider this alternative proof sufficient. In this paper, we resolve these shortcomings.

A. Proof That Fully Delayed Average Growth Rate Exists

We can now proceed to show that there exists a fully-delayed average growth rate.

Definition 15. An honest block (that is not the mother block) forms a Δ -gap if no other honest blocks occur within time Δ after this block.

We order the Δ gaps by the time in which they arrive.

Let $T(0)$ be the time from the mother block to end of the first Δ -gap. Let $T(n)$ ($n \geq 1$) be the time from the end of n th Δ -gap to the end of the $(n+1)$ th Δ -gap. Note that these random variables have finite expected value.

Let $S(0)$ be the score of the fully delayed honest chain from the mother block to the end of the first Δ -gap. Let $S(n)$ be the score of the fully delayed chain from the end of the n th to the end of the $(n+1)$ th Δ -gap. Since these are bounded by a Poisson growth process, these random variables also have finite expected value.

Lemma 16. *Let $S(t)$ be the score of the fully-delayed honest chain at time t . Then there exists a constant λ_h in which, almost surely, $\lim_{t \rightarrow \infty} \frac{S(t)}{t} = \lambda_h$. Moreover, for any $\epsilon > 0$, for sufficiently large t , $E\left(\frac{S(t)}{t}\right) \geq \lambda_h - \epsilon$.*

Proof: Note that after the arrival of the first Δ -gap, the random variables $S(n)$ and $T(n)$ are independent and identically distributed.

First, observe the following:

$$\lim_{t \rightarrow \infty} \frac{S(t)}{t} = \lim_{n \rightarrow \infty} \frac{S(0) + \sum_{i=1}^n S(i)}{T(0) + \sum_{i=1}^n T(i)} = \lim_{n \rightarrow \infty} \frac{\frac{S(0)}{n} + \frac{\sum_{i=1}^n S(i)}{n}}{\frac{T(0)}{n} + \frac{\sum_{i=1}^n T(i)}{n}} = \frac{\lim_{n \rightarrow \infty} \frac{S(0)}{n} + \lim_{n \rightarrow \infty} \frac{\sum_{i=1}^n S(i)}{n}}{\lim_{n \rightarrow \infty} \frac{T(0)}{n} + \lim_{n \rightarrow \infty} \frac{\sum_{i=1}^n T(i)}{n}}.$$

By the Strong Law of Large Numbers, $\lim_{n \rightarrow \infty} \frac{\sum_{i=1}^n S(i)}{n}$ and $\lim_{n \rightarrow \infty} \frac{\sum_{i=1}^n T(i)}{n}$ approach the expected value of $S(n)$ and $T(n)$, respectively, and $\frac{S(0)}{n}$ and $\frac{T(0)}{n}$ approach 0, all with probability 1. Hence, $\lim_{t \rightarrow \infty} \frac{S(t)}{t}$ approaches $\frac{E(S(i))}{E(T(i))}$, with probability 1.

For the second part of the Lemma, since $\frac{S(t)}{t}$ approaches λ_h almost surely, it also does so in probability. Hence, for any $\epsilon_1 > 0$:

$$\lim_{t \rightarrow \infty} P\left(\frac{S(t)}{t} \leq \lambda_h - \epsilon_1\right) = 0.$$

Hence, for any $\epsilon_1 > 0$ and $\epsilon_2 > 0$, there is sufficiently large t , such that $P\left(\frac{S(t)}{t} \leq \lambda_h - \epsilon_1\right) < \epsilon_2$. Thus,

$$E\left(\frac{S(t)}{t}\right) \geq (1 - \epsilon_2)(\lambda_h - \epsilon_1) + \epsilon_2(0) = (1 - \epsilon_2)(\lambda_h - \epsilon_1)$$

where we use the fact that for all t , the minimum value of $\frac{S(t)}{t}$ is 0, for all events in the sample space. Hence, choose ϵ_1 and ϵ_2 to be small enough such that $(1 - \epsilon_2)(\lambda_h - \epsilon_1) > \lambda_h - \epsilon$, and thus, for any $\epsilon > 0$ and large enough t , $E\left(\frac{S(t)}{t}\right) \geq \lambda_h - \epsilon$. ■

Definition 17. We call the constant λ_h in the proof above the *fully-delayed average growth rate*, or more simply the *average growth rate*.

Note that we do not actually compute what this rate is for the generalized multiple-score model. In the standard Bitcoin case, this growth rate is shown to be $\lambda_h = \frac{h}{1+\Delta h}$, where h is the average honest block-rate. If blocks can have different scores, and blocks of each score have different growth rates, this value may be different. We do not concern ourselves with computing these values in this paper, but in the companion paper we provide some bounds for this growth rate [9].

B. The punctured arrival process

In this section we will prove that, with probability greater than 0, the score growth-rate of the fully-delayed honest chain stays close to or above average for all time, starting from any particular time.

We shall consider a *punctured* arrival process. In this process, we run the fully-delayed delay schedule, but then, in spacings of B seconds, we *puncture* the block arrivals. For simplicity, and without loss of generality, we consider starting at time $t = 0$. In our punctured arrival process, we allow all blocks through in the interval $[0, B]$ and then delete all blocks in the interval $[B, B + \Delta]$. Then we let all blocks through in the interval $[B + \Delta, 2B + \Delta]$, and then delete those blocks in the interval $[2B + \Delta, 2B + 2\Delta]$. The process where we do this forever is called the *punctured arrival process* and the chain that this produces the *punctured chain*. See Figure 2 for a diagram of the punctured arrival process.

Definition 18. Let $S_B(t)$ be the score of the punctured chain at time t and call it the *punctured chain score*.

We also define:

Definition 19. Let $S(t)$ be the score of the non-punctured, fully-delayed chain, starting at time $t = 0$, called the *fully-delayed score*.

By Lemma 5, deleted blocks can only reduce the score, hence:

$$S(t) \geq S_B(t). \tag{5}$$

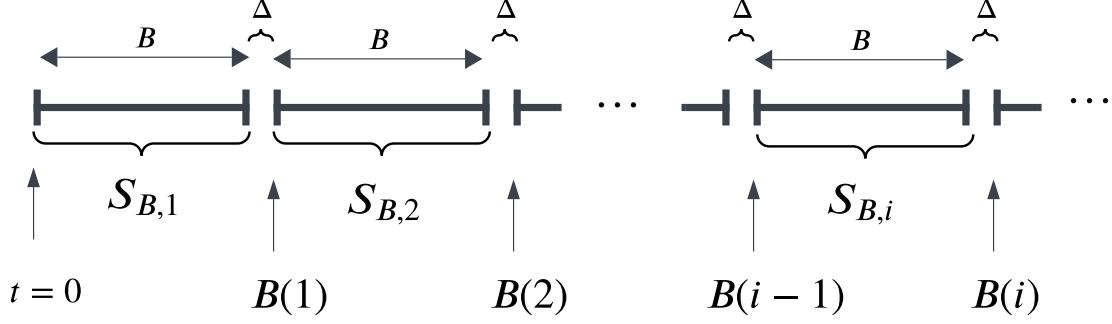


Fig. 2. Figure representing the punctured arrival process. B is the time-length of the part of each interval that is not deleted. Without loss of generality, we start this process at a time $t = 0$. Recall that Δ is the maximum delay of the Δ -delay model, and also is the length of each puncture. In this process, all honest blocks that arrive during these length Δ intervals are deleted. $B(i)$ is the i th total punctured score, which is the total score of the fully-delayed chain after the honest blocks in the punctured intervals are deleted. $S_{B,i}$ is the i th punctured interval score, which is the score growth of the fully-delayed honest blocks in each length B interval. Observe that, due to the length Δ deletion, each $S_{B,i}$ is independent and identically distributed.

At the end of the n th punctured section, the time is $t = Bn + n\Delta$. Hence, we introduce a new random process:

Definition 20. Let $B(n) = S_B(nB + n\Delta)$, which is a random variable that represents the score of the punctured chain at the end of the n th puncture, which we call the *n*th total punctured score.

We also define:

Definition 21. Let $S_{B,i}$ be the random variable representing the score of the fully-delayed honest chain produced in the length B interval i , and call this the *i*th punctured interval score.

Due to the Δ -gaps, and the fact that these are properties solely of arrival times within these intervals, each $S_{B,i}$ is independent and identically distributed. Also due to the Δ -gaps, the honest miners will see all the chain produced in the prior intervals.

Hence, the n th total punctured score is given by:

$$B(n) = \sum_{i=1}^n S_{B,i}.$$

Using the above equation, and substituting the definition of $B(n)$ into (5) gives us, for each integer $n > 0$:

$$S(Bn + \Delta n) \geq B(n) = \sum_{i=1}^n S_{B,i}$$

Hence, the non-punctured score at the end of each interval is at least the sum of all the punctured interval scores up to that point.

Lemma 22. For any $\epsilon > 0$, and for all sufficiently large B ,

$$P\left(\left[\frac{B(n)}{n} \geq B\lambda_h - B\epsilon\right] \text{ for all } n > 0\right)$$

is some number $p > 0$.

Proof: First, due to Lemma 16, we can choose B such that $E(S_{B,i}) > B\lambda_h - B\epsilon$ for this ϵ . We let

$$Z_B[n] = \sum_{i=1}^n (S_{B,i} - (B\lambda_h - B\epsilon))$$

be a random walk. Note that it is a random walk because each of the $S_{B,i}$ s are independent and identically distributed. The expected drift of this random walk is greater than 0 since we have chosen B so that $E(S_{B,i}) > B\lambda_h - B\epsilon$. Thus, by random walk theory, there is a probability greater than 0 that this random walk goes above 0 after the first step and then always stays above 0. Which means there is a probability greater than 0 that

$$\sum_{i=1}^n (S_{B,i} - (B\lambda_h - B\epsilon)) > 0 \text{ for all integers } n > 0$$

This event is equivalent to:

$$\sum_{i=1}^n S_{B,i} > \sum_{i=1}^n (B\lambda_h - B\epsilon) \text{ for all integers } n > 0$$

Substituting the definition of $B(n)$ on the left and summing up n constants on the right gives us:

$$B(n) > Bn\lambda_h - nB\epsilon \text{ for all integers } n > 0$$

has probability $p > 0$. Dividing both side by n proves the lemma. ■

For the lemma below, recall that $S(t)$ is the score of the fully delayed, unpunctured chain at time t

Lemma 23. *For any $\epsilon > 0$, there is a B sufficiently large, so that for each point in time $t_n = Bn + \Delta n$ ($n > 0$, $n \in \mathbb{N}$), there is a probability $q > 0$ such that $\frac{S(t_n)}{t_n} > \lambda_h - \epsilon$.*

Proof: First, choose a sufficiently small ϵ_1 and a B_1 so that

$$\frac{\Delta\lambda_h}{B_1 + \Delta} + \frac{B_1\epsilon_1}{B_1 + \Delta} < \epsilon.$$

This is possible because the left of the inequality approaches 0 as ϵ_1 approaches 0 and B_1 approaches infinity. Then, choose a B_2 such that for this ϵ_1 :

$$E(S_{B_2,i}) > B_2\lambda_h - B_2\epsilon_1$$

(which is possible because of Lemma 22): Finally, let $B = \max(B_1, B_2)$ so that:

$$\frac{\Delta\lambda_h}{B + \Delta} + \frac{B\epsilon_1}{B + \Delta} < \epsilon \tag{6}$$

and

$$E(S_{B,i}) > B\lambda_h - B\epsilon_1.$$

Because of (5), Lemma 22, and $Bn = t_n - \Delta n$, with probability $p > 0$, at each $t_n = Bn + \Delta n$, for our choice B and ϵ_1 :

$$\begin{aligned} S(t_n) &\geq \sum_{i=1}^n S_{B,i} \geq Bn\lambda_h - nB\epsilon_1 = (t_n - \Delta n)\lambda_h - nB\epsilon_1 \\ &= t_n\lambda_h - \Delta n\lambda_h - nB\epsilon_1. \end{aligned}$$

We divide by $t_n = Bn + \Delta n$:

$$\frac{S(t_n)}{t_n} > \lambda_h - \frac{\Delta n\lambda_h}{Bn + \Delta n} - \frac{nB\epsilon_1}{Bn + \Delta n} = \lambda_h - \left(\frac{\Delta\lambda_h}{B + \Delta} + \frac{B\epsilon_1}{B + \Delta} \right) > \lambda_h - \epsilon$$

where in the final inequality we use our choice of ϵ_1 and B so that $\left(\frac{\Delta\lambda_h}{B + \Delta} + \frac{B\epsilon_1}{B + \Delta} \right) < \epsilon$ from (6). ■

Lemma 24. *There is a probability $q > 0$ such that*

$$\frac{S(t)}{t} > \lambda_h - \epsilon$$

for all t after the first arrival and for any $\epsilon > 0$.

Proof: First we show that this is true for all t sufficiently large. We know from Lemma 23, that at points in time $t_n = Bn + \Delta n$, ($n > 0, n \in \mathbb{N}$), and any $\epsilon_1 > 0$, that there is some probability greater than 0 that:

$$\frac{S(t_n)}{t_n} > \lambda_h - \epsilon_1.$$

Choose $\epsilon_1 = \frac{\epsilon}{2}$.

We assume pessimistically that this occurs but that all increases in $S(t)$ occur exactly at the times $t_n = Bn + \Delta n$. Thus, for all time in the interval $B + \Delta$ after the t_n s, the honest chain has no arrivals. But, this allows us to assume that

$$S(t) \geq (Bn + \Delta n)\lambda_h - (Bn + \Delta n)\frac{\epsilon}{2}$$

for all $t \in [Bn + \Delta n, B(n+1) + \Delta(n+1)]$. Dividing by t on the left side, and dividing by the maximum value of t in each interval on the right side, we conclude that for all t in this interval:

$$\begin{aligned} \frac{S(t)}{t} &\geq \frac{(Bn + \Delta n)\lambda_h - (Bn + \Delta n)\frac{\epsilon}{2}}{B(n+1) + \Delta(n+1)} = \\ \frac{(Bn + \Delta n)(\lambda_h - \frac{\epsilon}{2})}{(n+1)(B + \Delta)} &\geq \frac{n(B + \Delta)(\lambda_h - \frac{\epsilon}{2})}{(n+1)(B + \Delta)} = \frac{n}{n+1} \left[\lambda_h - \frac{\epsilon}{2} \right]. \end{aligned}$$

Observe that $\frac{n}{n+1}$ approaches 1 from below, so for sufficiently large n , this expression can be made as close as possible to $\lambda_h - \frac{\epsilon}{2}$. So choose n so that it is within $\frac{\epsilon}{2}$ of $\lambda_h - \frac{\epsilon}{2}$. Hence, with probability greater than 0, for any $\epsilon > 0$, there is sufficiently large n , (and thus for all t greater than some t'), such that:

$$\frac{S(t)}{t} > \lambda_h - \frac{\epsilon}{2} - \frac{\epsilon}{2} = \lambda_h - \epsilon. \quad (7)$$

Now we prove this is true for all t after the first arrival.

Let t' be the minimum t for which the inequality 7 holds. To show that the lemma holds for all t , we simply need to show that $\frac{S(t)}{t} > \lambda_h - \epsilon$ for all t after first arrival in $[t, t']$. Note that when ϵ is fixed, t' is a fixed constant. Hence, the probability $\frac{S(t)}{t} > \lambda_h - \epsilon_3 > 0$ after the first arrival and until t' is simply some constant greater than 0. Combining this with the argument above for $t > t'$, and the lemma follows. ■

Lemma 25. *With probability greater than 0, the number of adversary block arrivals stays close to or below average. That is, with probability greater than 0, for any $\epsilon > 0$ and $t_2 > t_1$:*

$$S_a(t_2) - S_a(t_1) \leq (\lambda_a + \epsilon)(t_2 - t_1).$$

Proof: Without loss of generality, let $t_1 = 0$. Divide time into small intervals of time δ . In each interval the average number of adversary arrivals is $\lambda_a \delta$. Define $X[n]$ as the adversary score in the n th such interval, and $S[n] = \sum X[n]$ as the score at time $t = \delta n$. Let $\epsilon > 0$ and define:

$$Z[n] = \sum [X[n] - (\lambda_a + \epsilon) \delta]$$

Note that $Z[n]$ forms a random walk since each $[X[n] - (\lambda_a + \epsilon) \delta]$ is independent and identically distributed. Finding the expected value of each step of the random walk gives us:

$$\begin{aligned} E([X[n] - (\lambda_a + \epsilon) \delta]) &= \lambda_a \delta - \lambda_a \delta - \epsilon \delta \\ &= -\epsilon \delta < 0. \end{aligned}$$

Hence, the random walk has negative drift, and by random walk theory, with probability greater than 0, $Z[n]$ becomes negative and never returns to 0. Hence, with probability greater than 0, for all $n > 0$:

$$\begin{aligned}\sum [X[n] - (\lambda_a + \epsilon) \delta] &< 0 \\ S[n] &< (\lambda_a + \epsilon) \delta n\end{aligned}$$

But $S[n]$ is just $S_a(t)$ for $t = \delta n$. The lemma is then implied by taking the limit as δ approaches 0. ■

C. Using this to bound the event probabilities

Lemma 26. If $\lambda_h > \lambda_a$, then for any honest arrival time τ_q , there is some chance greater than 0 that event E_2 , rewritten below, occurs:

$$E_2 := [S_h(\tau_q + q + \Delta, t - \Delta) > S_a(\tau_q + q + 2\Delta, t) \text{ for all } t > \tau_q + q + 2\Delta]. \quad (8)$$

Proof: We rewrite our condition as:

$$\lambda_h - \frac{\epsilon}{2} > \lambda_a + \frac{\epsilon}{2}. \quad (9)$$

for some $\epsilon > 0$. We shall use this ϵ in the remainder of the proof. ■

Let A_1 be the event that the honest, fully-delayed chain score stays within $\frac{\epsilon}{2}$ or above average for all time greater than $\tau_q + q + \Delta$. Symbolically, this is:

$$\begin{aligned}A_1 = S_h(\tau_q + q + \Delta, t - \Delta) &\geq (t - \tau_q - q - 2\Delta)(\lambda_h - \frac{\epsilon}{2}) \\ \text{for all } t &> \tau_q + q + \Delta.\end{aligned} \quad (10)$$

Let A_2 be the event that the adversary chain has with $\frac{\epsilon}{2}$ or below average rate of arrival for all times greater than $\tau_q + q + 2\Delta$.

$$A_2 = S_a(\tau_q + q + 2\Delta, t) \leq (t - \tau_q^h - 2\Delta)(\lambda_a + \frac{\epsilon}{2})$$

We note from Lemma 24 that A_1 occurs with probability greater than 0. As well, from Lemma 25, A_2 occurs with probability greater than 0.

Using 9 and multiplying both sides by $(t - \tau_q - q - 2\Delta)$:

$$(t - \tau_q - q - 2\Delta)(\lambda_h - \frac{\epsilon}{2}) > (t - \tau_q - q - 2\Delta) \left(\lambda_a + \frac{\epsilon}{2} \right).$$

If A_1 occurs, then, for the left side of this inequality:

$$S_h(\tau_q + q + \Delta, t - \Delta) > (t - \tau_q - q - 2\Delta)(\lambda_h - \frac{\epsilon}{2})$$

If A_2 occurs, then, for the right side of this inequality:

$$(t - \tau_q - q - 2\Delta)(\lambda_a + \frac{\epsilon}{2}) \geq S_a(\tau_q + q + 2\Delta, t).$$

Combining these gives us

$$S_h(\tau_q + q + \Delta, t - \Delta) > S_a(\tau_q + q + 2\Delta, t).$$

Since A_1 and A_2 each occur with probability greater than 0, and they are properties independent arrival processes (the adversary and honest arrival process), their intersection also occurs with probability greater than 0. Since

$$E_2 = A_1 \cap A_2,$$

therefore E_2 also occurs with probability greater than 0.

Lemma 27. The event E_1 occurs with probability greater than 0.

Proof: In this case, one can show that there is a probability greater than 0 that honest block arrivals stay close to or above average for all intervals beginning before $\tau_q - q - \Delta$ and ending at $\tau_q - q - \Delta$. This involves dividing time up in punctured intervals of size B , and then recognizing the sum of the score growth in these punctured intervals as a random walk. This can show that at the endpoints of these intervals there is positive probability that honest score growth rate is close to or above average. We follow the same proof as above to show that all the time between these endpoints also have this property with positive probability. The rest of the proof is symmetrical to the proof above, so we omit it for conciseness. ■

Lemma 28. *For any $q > 0$, the event that any given interval of size $2q$ centered at a time $\tau_q > q + 2\Delta$ is a Nakamoto interval (and thus contains an honest block that stays in the chain forever), has probability greater than 0.*

Proof: Note L_q occurs with probability greater than 0, since it is a property of arrival times in a finite interval. Event E_1 is a property of arrival times of honest blocks outside the honest loner interval and of dishonest blocks outside the dishonest loner interval. Hence, L_q , E_2 , and E_2 are independent. Therefore:

$$P(\text{Nakamoto interval}) = P(L_q \cap E_1 \cap E_2) = P(L_q)P(E_1)P(E_2) > 0.$$

■

V. THE BOOTSTRAP ARGUMENT

The previous section does not actually prove that a chain will have honest blocks if $\lambda_h > \lambda_a$. It merely states that the n th block arrival will be in the chain forever with probability greater than 0. However, it does not prove that these events are independent, and thus it may be that with probability greater than 0 (but less than 1) there are no honest blocks in the canonical chain. However, in this section we follow the proof in [7] to show that the probability that there are not Nakamoto blocks in an interval of length t scales exponentially to zero in length t . We simplify the approach and use an induction argument, where the base case depends on Lemma 28. This shows that, with probability 1, there are infinitely many honest blocks in the canonical chain whenever $\lambda_h > \lambda_a$.

In the following section, we use the symbols A and c to represent arbitrary constants greater than 0. Hence, if used in different expressions, they do not necessarily represent the same value.

We first define a few terms used for this section.

Definition 29. The *score* of a sub-chain is the total score of all the blocks containing it, starting from the mother block, and ending at the tip. In the Bitcoin algorithm, each block has score 1.

Definition 30. A subchain D is said to *dominate* another subchain W at a particular time t if subchain D has a greater score at that time *and* does not contain the tip of subchain W .

Definition 31. Conflicted: If block is not a loner it is said to be *conflicted*.

Definition 32. Overtakable: A target block is *overtakable* if there exists a set of arrival times of adversary blocks and a prior honest block, such that if the dishonest blocks formed a chain starting at the prior honest block, it would dominate the fictional fully-delayed honest chain containing the target block that started at the prior honest block.

Definition 33. Insecure: A block that is either conflicted or overtakable is considered *insecure*.

Definition 34. Secure: A block which is an honest block that is not conflicted and is not overtakable by a dishonest subchain, is called *secure*. note that Nakamoto blocks are necessarily secure.

Note that overtakable does not mean *overtaken* by a particular adversary strategy.

Definition 35. The *time-length* of an adversary chain is the time since the most recent honest block (including potentially the mother block) on the adversary chain.

Definition 36. Let $B_{a,b}$ be the event that there are no Nakamoto blocks that arrive in the time interval $[a, b]$, in which $0 < a \leq b$.

We shall now consider a hypothetical interval $[s, s+t]$, $s, t > 0$.

Definition 37. Let B be the event that there is at least one honest block in the interval $[s, s+t]$ that is overtakable by an adversary chain of time-length at least \sqrt{t} .

Lemma 38. *The probability of event B scales to zero at least as fast as $A \exp(-c\sqrt{t})$. More generally, the probability that there is at least one block in a fixed interval of length s that is overtakable by a length t or longer adversary chain is less than $A \exp(-ct)$.*

Proof: This flows from a Chernoff bound argument. The details of the argument are given in Appendix B. ■

Observe from elementary probability theory that:

$$\begin{aligned} B_{s,s+t} &= (B \cap B_{s,s+t}) \cup (B^C \cap B_{s,s+t}) \\ &\subseteq B \cup (B^C \cap B_{s,s+t}) \end{aligned} \tag{11}$$

where the second line arises from weakening the condition in the first term.

Now consider the event $(B^C \cap B_{s,s+t})$. By definition this is the event that no single block in the interval $[s, s+t]$ is overtakable by a \sqrt{t} or longer adversary chain, and all are insecure. If all are insecure, *and* none are overtakable by a long chain, this is the same event as each block is either conflicted *or* overtakable by a \sqrt{t} or smaller adversary chain.

Definition 39. Locally secure: A block that is neither conflicted nor overtakable by a \sqrt{t} or shorter time-length chain is called *locally secure*. Otherwise it is called *locally insecure*.

Definition 40. Let Q_i be the event that honest block i is overtakable by a \sqrt{t} or smaller chain.

Definition 41. Let C_i be the event that the i th honest block is conflicted.

$$(B^C \cap B_{s,s+t}) = \bigcap_{j: \tau_j \in [s, s+t]} (C_j \cup Q_j)$$

This is simply the AND over the events that the j th block in our interval of interest is locally insecure.

A. Bootstrap argument step one: Dividing interval in segments of length \sqrt{t}

1) Part one: Decomposing the Big AND into Independent Events and Applying Independence: Let us divide the interval $[s, s+t]$ into $\lfloor \sqrt{t} \rfloor$ segments of length \sqrt{t} .

Let T_i be the i th such subinterval.

Now, the above conjunction over all honest blocks can be broken up as an AND over blocks in each length \sqrt{t} sub-interval, as well as the blocks arriving in the remaining $\sqrt{t} - \lfloor \sqrt{t} \rfloor$ time. Symbolically:

$$\begin{aligned} (B^C \cap B_{s,s+t}) &= \bigcap_{i=1}^{\lfloor \sqrt{t} \rfloor} \left[\bigcap_{j: \tau_j \in T_i} (C_j \cup Q_j) \right] \\ &\quad \bigcap_{j: \tau_j \in [\lfloor \sqrt{t} \rfloor, \sqrt{t} - \lfloor \sqrt{t} \rfloor]} (C_j \cup Q_j) \end{aligned}$$

where we note that $\bigcap_{j: \tau_j \in T_i} (C_j \cup Q_j)$ is an AND over all honest arrival times τ_j in the interval T_i .

But this is a subset of the AND over every third sub-interval:

$$(B^C \cap B_{s,s+t}) \subseteq \bigcap_{k=1}^{\left\lfloor \frac{\lfloor \sqrt{t} \rfloor}{3} \right\rfloor} \left[\bigcap_{j:\tau_j \in T_{3k-1}} (C_j \cup Q_j) \right].$$

Let N_k be the event $\bigcap_{j:\tau_j \in T_{3k-1}} (C_j \cup Q_j)$ (this is the event in the square brackets in the expression above). This is the event that the $(3k - 1)$ th subinterval has all blocks locally insecure. In other words, it is the event that all loners in the $(3k - 1)$ th subintervals are overtakable by an adversary chain of length less than \sqrt{t} .

Then we can rewrite the expression above as as:

$$(B^C \cap B_{s,s+t}) \subseteq \bigcap_{k=1}^{\left\lfloor \frac{\lfloor \sqrt{t} \rfloor}{3} \right\rfloor} [N_k].$$

Note that we use of the floor function in the expression $\left\lfloor \frac{\lfloor \sqrt{t} \rfloor}{3} \right\rfloor$ because $\frac{\lfloor \sqrt{t} \rfloor}{3}$ may not be divisible by 3. Now we can go back to Expression 11:

$$B_{s,s+t} \subseteq B \bigcup \left[\bigcap_{k=1}^{\left\lfloor \frac{\lfloor \sqrt{t} \rfloor}{3} \right\rfloor} [N_k] \right]$$

This implies:

$$P(B_{s,s+t}) \leq P(B) + P\left(\bigcap_{k=1}^{\left\lfloor \frac{\lfloor \sqrt{t} \rfloor}{3} \right\rfloor} [N_k]\right). \quad (12)$$

Note that each $(3k - 1)$ th subinterval represented in N_k has a subinterval of length \sqrt{t} before and after it that is not neighboring the subinterval of a different N_k . Thus, each N_k is solely a property of honest and adversary arrival times in its own or neighboring subintervals. Therefore, these probabilities are independent and identically distributed. Hence:

$$P(B_{s,s+t}) \leq P(B) + P(N_1)^{\left\lfloor \frac{\lfloor \sqrt{t} \rfloor}{3} \right\rfloor}. \quad (13)$$

If all blocks in a subinterval are locally insecure, then none are secure. Thus, the event that all blocks are locally insecure is a subset of the event that none are secure. But we know that the probability that there are no secure blocks in an interval of length at least $q > 0$ is less than 1 (From Lemma 28). Hence, $P(N_1) = \rho$ for some $\rho < 1$, and hence $P(N_1)^{\left\lfloor \frac{\lfloor \sqrt{t} \rfloor}{3} \right\rfloor} = \rho^{\left\lfloor \frac{\lfloor \sqrt{t} \rfloor}{3} \right\rfloor} \leq A \exp(-c(\sqrt{t}))$ for arbitrary constants A and c .

From Lemma 38, $P(B) \leq A \exp(-c\sqrt{t})$ for arbitrary constants A and c .

Combining these two observations with expression 13 above gives us:

$$\begin{aligned} P(B_{s,s+t}) &\leq P(B) + P(N_1)^{\left\lfloor \frac{\lfloor \sqrt{t} \rfloor}{3} \right\rfloor} \\ &\leq A \exp(-ct^{1/2}). \end{aligned}$$

B. Step 2: The Induction Argument

We have shown above that for $k = 1$:

$$P(B_{s,s+t}) \leq A \exp(-ct^{1/2}).$$

We shall assume our induction hypothesis that for some integer k , for t sufficiently large,

$$P(B_{s,s+t}) \leq A \exp(-ct^{\frac{k}{k+1}}). \quad (14)$$

Note that the case of $k = 1$ is proven in the section above.

We shall use the same arguments as Section V-A, with four differences. First, we divide the length t interval into $\left\lfloor t^{\frac{1}{k+2}} \right\rfloor$ subintervals, each of length $t^{\frac{1}{k+2}}$. Second, we define local insecurity as the event that a block is either conflicted or is overtakable by a length $t^{\frac{1}{k+2}}$ or smaller adversary subchain. Third, we define N_q as the event that all honest blocks in the $(3q - 1)$ th sub-interval are locally insecure. Fourth, we define B as the event that there is at least one honest block in the interval $[s, s+t]$ that is overtakable by an adversary chain of time-length at least $t^{\frac{1}{k+2}}$. From this we can conclude that:

$$P(B_{s,s+t}) \leq Ae^{-ct^{\frac{k+1}{k+2}}} + (P(N_q)) \left[\underbrace{\left[\begin{array}{c} \left\lfloor t^{\frac{1}{k+2}} \right\rfloor \\ \hline 3 \end{array} \right]}_{\text{length}} \right]. \quad (15)$$

If all blocks in a subinterval are locally insecure, then none are secure. Thus, the event that all blocks in an interval are locally insecure is a subset of the event that none in that sub-interval are secure. But from our induction hypothesis, this probability scales exponentially in the length of the interval raised to $\frac{k}{k+1}$. The length of the interval is $t^{\frac{1}{k+2}}$ and hence, it should decrease exponentially in

$$\left(t^{\frac{1}{k+2}} \right)^{\frac{k}{k+1}} = t^{\frac{k}{k+2}}$$

and thus:

$$P(N_q) \leq A \exp(-ct^{\frac{k}{k+2}}).$$

Plugging into 15 we get:

$$\begin{aligned} P(B_{s,s+t}) &\leq Ae^{-ct^{\frac{k+1}{k+2}}} + A \left(\exp(-ct^{\frac{k}{k+2}}) \right) \left[\underbrace{\left[\begin{array}{c} \left\lfloor t^{\frac{1}{k+2}} \right\rfloor \\ \hline 3 \end{array} \right]}_{\text{length}} \right] \\ &\leq Ae^{-ct^{\frac{k+1}{k+2}}} + A \exp(-ct^{\frac{k+1}{k+2}}) \\ &\leq A \exp \left(ct^{\left(\frac{k+1}{k+2} \right)} \right) \end{aligned}$$

Therefore this bound is true for all integers $k > 1$.

As k approaches infinity, $\left(\frac{k+1}{k+2} \right) \rightarrow 1$, and thus

$$P(B_{s,s+t}) \leq A \exp(-ct^{1-\epsilon})$$

for any $\epsilon > 0$ and some constants A and c greater than 0. This argument proves the main theorem of our paper:

Theorem 42. *The probability that any sub-interval has no honest blocks that stay in the chain forever goes to zero exponentially in time-length t when $\lambda_h > \lambda_a$.*

Corollary 43. *The probability that there are infinitely many honest blocks in the canonical chain is 1 if $\lambda_h > \lambda_a$.*

Proof: Divide time into intervals, indexed by i , of length $t_i = i$. Let M_i be the event that interval i has no honest blocks that stay in the chain forever. We know from above that $P(M_i) \leq A \exp(ci^{1-\epsilon})$ for large enough i .

Observe that, since these probabilities are exponential, therefore $\sum_{i=1}^{\infty} P(M_i) < \infty$. By the Borel-Cantelli lemma [11], [12], the event that these insecure intervals occur infinitely often has probability 0. Hence, with probability one, there will be a time after which all these intervals have a secure block. Hence, the canonical chain has infinitely many honest blocks with probability 1 when $\lambda_h > \lambda_a$. ■

INSECURITY REGION

Theorem 44. *If $\lambda_a > \lambda_h$, then Merged Bitcoin is insecure, and there exists an attack in which the canonical chain, with 100% probability has all dishonest blocks.*

Proof: The attack that will work is the private mining attack. The adversary simply mines a chain in private and then reveal it whenever it has score greater than the highest score honest block. The reader should observe that by the law of large numbers, between every reveal, with probability one, eventually the adversary chain will dominate the honest chain. ■

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APPENDIX A PRIOR PAPER FLAW

We will prove using a counterexample a flaw with the prior proof in [7].

The authors of [7] state in Appendix C.1: “ $S[n]$ [represents the] difference between the increase in D_h and the number of adversary arrivals.” In this case, D_h is the depth of an honest, fully-delayed chain.

The claim is equivalent to $S[n] = S[n - 1] + X_n$ for some X_n s that are independent and identically distributed. This implies that $S[n] - S[n - 1] = X_n$.

The authors continue: “Hence, $S[n]$ simply counts the difference between the number of honest and adversary arrivals when there are n arrivals in total. In this case, $S[n]$ jumps up by 1 when there is an honest arrival, and, goes down by 1 when there is an adversary arrival.”

The authors then apply the well-known result that a random walk with positive drift leaves the origin once and never returns with probability greater than 0.

Based on the stated definition of $S[n]$, X_n is a random variable that tracks the next block in the hypothetical race between a *fully-delayed* honest chain and the adversary chain. It is defined as:

$$X_n = \begin{cases} +1 & \text{nth block is honest} \\ -1 & \text{nth block dishonest} \end{cases}$$

We shall prove that these random variables are not independent. Therefore the conclusions drawn in [7] do not flow from the arguments.

Indeed, a generalized random walk does not necessarily behave like a simple random walk. For example, consider a process that starts out deterministically leaving the origin, returning, and then continuing on forever as a normal random walk with positive drift. This process is similar to a simple random walk, and the average drift of this process is clearly positive, but with probability one it leaves the origin and *always* returns.

We can fix this issue with a punctured arrival technique, but we shall now prove that the X_n s defined above are not independent.

A. Counterexample

For this section, let b be the dishonest mining rate and h be the honest mining rate (before there are any delays).

We consider an honest chain growing when subject to full-delays by the adversary, and consider the classic Bitcoin case of all blocks being the same score.

Consider the probability that there is a dishonest arrival in the time Δ after a particular non-orphaned honest block of the fully-delayed honest chain. The arrivals of these dishonest blocks is a Poisson process with parameter b . Applying the well-known probability distribution of number of arrivals of a Poisson process with parameter b in time interval Δ :

$$P(\text{dishonest block in } \Delta \text{ after honest block}) = 1 - P(0 \text{ blocks in time } \Delta) = 1 - e^{-\Delta b}$$

We are in the security region considered when:

$$b < \frac{h}{1 + \Delta h}.$$

Let Δ and h be very large, (say, 100).

Now, in this regime, let b be very close to but within the security region boundary:

$$b \approx \frac{h}{1 + \Delta h} = \frac{1}{\frac{1}{h} + \Delta} \approx \frac{1}{\Delta}$$

where the approximation comes from h and Δ each being much larger than 1.

Then:

$$P(\text{at least 1 dishonest block in } \Delta \text{ after honest block}) \approx 1 - e^{-\Delta \frac{1}{\Delta}} = 1 - e^{-1} \approx 0.63$$

Note that if any dishonest block arrives in the time Δ after an honest block of the fully-delayed honest tree, then the first of these blocks will be the next block. It doesn't matter if other honest blocks arrive since they will be orphaned due to the network delay. Thus, the probability that the next block is dishonest given that the last block is honest is *at least* the probability that a dishonest block arrives in time Δ .

Combining this with the approximation above, this implies that:

$$P(X_n = -1 | X_{n-1} = +1) \geq 0.63.$$

For the X_n s to be independent, we require that $P(X_n = -1 | X_{n-1} = +1) = P(X_n = -1)$. However, since we are in a security region, the probability that the n th block is dishonest must be less than 50 percent. Thus $P(X_n = -1) < 0.5$, and thus the X_n s are not independent. Hence, the sequence $S[n]$ is not a random walk and presumptions about the drift of a random walk made in [7] do not necessarily hold.

APPENDIX B THE CHERNOFF BOUND ARGUMENT

As in previous sections, for this appendix, for simplicity of notation, we will continue to use the convention that A and c are arbitrary positive constants (and not necessarily the same constant in every expression).

We shall prove that the event B (that there is at least one honest block in the interval $[s, s + t]$ that is overtakable by an adversary chain of time-length at t') scales as $Ae^{-ct'}$.

Suppose the blockrate for honest non-delayed blocks for each block type i is h_i , and the score of each block type is c_i . Hence, the non-delayed chain has score growth rate $\lambda' = \sum c_i h_i$. We let the fully-delayed growth rate of blocks of type i be h_d^i (note that, by the same argument as Lemma 16, this growth rate exists). Hence, the fully-delayed honest score growth rate is given by $\lambda_h = \sum c_i h_d^i$.

We let the adversary score growth rate be λ_a . We consider a region in which $\lambda_h > \lambda_a$ (the security region of the protocol).

Probability that score growth of one block type is close to or below average scales as $A \exp(-ct')$

We consider the probability that, in the fully delayed chain, blocks of a particular type have number of arrivals close to or below average. We show that this probability goes to 0 exponentially in the time window t .

Without loss of generality, let us consider blocks of type 1. Let Y_d be the random variable representing the number of blocks of this type that arrive in the fully delayed chain in a length of time t' . Let μ_d be the average number of arrivals of blocks of this type in the fully-delayed chain in this interval. Let h_d the fully-delayed blockrate for type 1 blocks (in blocks per second) in the full chain, (where in this subsection we suppress the indices i for notational clarity).

Note 1: In any particular finite interval of length t' , the expected number of blocks of this type is close to but may not be *exactly* $h_d t'$ (as this can depend on, for example, blocks arriving within Δ of the end of the interval). However, for the same reasoning as Lemma 22, because h_d is the *limit* of these average arrival rates as t' approaches infinity, then for any $\delta > 0$, and *for sufficiently large t'* , $(1 - \delta)h_d t' \leq \mu_d$. We shall use this observation later in our proof.

Directly applying the Chernoff bound formula for Y_d gives us for all $y < \mu_d$ and $s < 0$:

$$P(Y_d \leq y) \leq \frac{E[e^{sY_d}]}{e^{sy}}$$

(as a reference, see [13]).

We do not have a formula for the moment generating function ($E[e^{sY_d}]$) of Y_d . However, let Y be the number of arrivals of type 1 blocks without delays in this interval. This process has average number of arrivals $\mu = t'h_i$. Obviously, since no blocks get erased in the non-delayed process:

$$\begin{aligned} Y_d &\leq Y \\ sY_d &\leq sY. \end{aligned}$$

Hence:

$$e^{sY_d} \leq e^{sY}$$

and thus

$$E(e^{sY_d}) \leq E(e^{sY}).$$

Substituting this bound into the formula above, and using the well-known formula for the $E(e^{sY})$ [13]:

$$P(Y_d \leq y) \leq \frac{e^{\mu(e^s - 1)}}{e^{sy}}.$$

Now we simply let $s = \ln(y/\mu_d) < 0$ into the formula. This gives us:

$$P(Y_d \leq y) \leq \frac{\exp(\mu (\exp(\ln(y/\mu_d)) - 1))}{\exp(\ln(y/\mu_d)y)} = \frac{\exp(\mu \left(\frac{y}{\mu_d} - 1 \right))}{\exp(\ln(y/\mu_d)y)}$$

Now let $y = (1 - \delta)h_d t'$ (which for all δ and sufficiently large t' is less than μ_d , and so is a valid choice for y):

$$P(Y_d \leq (1 - \delta)h_d t') = \frac{\exp(\mu \left(\frac{(1-\delta)h_d t'}{\mu_d} - 1 \right))}{\exp \left(\ln \left(\frac{(1-\delta)h_d t'}{\mu_d} \right) ((1 - \delta)h_d t') \right)}$$

which by inspection we see scales as $A \exp(-ct' \ln t) \leq A \exp(-ct')$.

Hence:

$$P(Y_d \leq (1 - \delta)h_d t') \leq A \exp(-ct').$$

A. Union Bound Argument for Adversary Chain Dominating Event

Above, we showed that the probability that a particular block type has below average rate of arrivals in our length t' interval scales as $A \exp(-ct')$.

A basic Chernoff bound argument can also show that the probability that the adversary has arrivals less than or equal to $((1 + \delta)\lambda_a t'$ scales as $A \exp(-ct')$.

Recall we are in a region in which $\lambda_h > \lambda_a$. Hence,

$$(1 - \delta)t'\lambda_h > (1 + \delta)t'\lambda_a \quad (16)$$

for sufficiently small δ . Let us consider such a δ below.

Observe that if all honest block types have $(1 - \delta)h_d^i t'$ or greater arrivals in this interval, and the adversary has less than $(1 + \delta)\lambda_a t'$, then:

$$\begin{aligned} \text{honest chain score growth in interval} &\geq \sum_{i=1}^n (1 - \delta)c_i h_d^i t' = \\ &= (1 - \delta)t' \sum_{i=1}^n c_i h_d^i \\ &= (1 - \delta)t'\lambda_h \\ &> (1 + \delta)t'\lambda_a \end{aligned}$$

where in the last line we use 16. But we assumed adversary arrivals are less than $(1 + \delta)t'\lambda_a$.

Hence, in order for the adversary chain to dominate in this case, at least one of the block types must have less than average growth rate, or the adversary has more than average growth rate. Each of these occur with probability that scales at least as $A \exp(-ct')$. By union bound, the probability of such a dominating event occurring also scales as $A \exp(-ct')$.

B. The Integration Argument: The Probability any Block in Interval is Dominated by Long Adversary Chain

Let $C_{t'}^{t_s}$ be the event that there is an adversary chain with these three properties: (1) it starts at an honest block at time t_s (start time), (2) it is length greater than t' , (time length), and (3), it can overtake an alternate honest chain that starts at time t_s .

Let H_{t_s} be the event that there is an honest block that arrives in a small interval around time t_s .

Once we have the exponential scaling for one event, bounding that the probability of a length t' or greater catch-up occurs to a block in the interval $[s, s+t]$ is easy. Note from above that if $\lambda_h > \lambda_a$, then:

$$P(C_{t'}^{t_s} | H_{t_s}) \leq A \exp(-ct').$$

As well, $P(H_{t_s}) \leq adt$ for some constant a .

To find the probability that a block in the target interval is dominated by a long adversary chain, we integrate over a number of variables, and use the fact that the integral of an exponential function is also an exponential function.

First, t_3 will index the possible arrival times of honest blocks in the interval $[s, s+t]$.

Second, let t_2 index the length of the dominating chain, which can vary from t' to ∞ .

Finally, let t_1 index the possible start time of the dominating chain, which can vary from $t_3 - t_2$ to t_3 .

$$\begin{aligned} P(\text{catch up of length } t' \text{ or greater of block in interval } [s, s+t]) &\leq \\ \int_s^{s+t} \int_{t'}^{\infty} \int_{t_3-t_2}^{t_3} P(C_{t_2}^{t_1} | H_{t_1}) P(H_{t_1}) dt_1 dt_2 dt_3 &\leq \\ \int_s^{s+t} \int_{t'}^{\infty} \int_{t_3-t_2}^{t_3} A e^{-ct'} adt_1 dt_2 dt_3 &\leq A e^{-ct'}. \end{aligned}$$