Fluid dynamics in the warp drive spacetime geometry

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Abstract

The Alcubierre warp drive metric is a spacetime geometry featuring a spacetime distortion, called warp bubble, where a massive particle inside it acquires global superluminal velocities, or warp speeds. This work presents solutions of the Einstein equations for the Alcubierre metric having fluid matter as gravity source. The energy-momentum tensor considered two fluid contents, the perfect fluid and the parametrized perfect fluid (PPF), a tentative more flexible model whose aim is to explore the possibilities of warp drive solutions with positive matter density content. Santos-Pereira et al. (2020) already showed that the Alcubierre metric having dust as source connects this geometry to the Burgers equation, which describes shock waves moving through an inviscid fluid, but led the solutions back to vacuum. The same happened for two out of four solutions subcases for the perfect fluid. Other solutions for the perfect fluid indicate the possibility of warp drive with positive matter density, but at the cost of a complex solution for the warp drive regulating function. Regarding the PPF, solutions were also obtained indicating that warp speeds could be created with positive matter density. Weak, dominant, strong and null energy conditions were calculated for all studied subcases, being satisfied for the perfect fluid and creating constraints in the PPF quantities such that positive matter density is also possible for creating a warp bubble. Summing up all results, energy-momentum tensors describing more complex forms of matter, or field, distributions generate solutions for the Einstein equations with the warp drive metric where negative matter density might not be a strict precondition for attaining warp speeds.

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1. INTRODUCTION

It is well known that in general relativity particles can travel globally at superluminal speeds, whereas locally they cannot surpass the light speed. The *warp drive spacetime geometry* advanced by Alcubierre [1] uses these physical properties to propel material particles at superluminal speeds. It creates a limited spacetime distortion, called *warp bubble*, such that the spacetime is contracted in front of it and expanded behind the bubble as it moves along a geodesic. This warp drive metric is such that a particle trapped inside this bubble would locally move at subluminal speeds, whereas the bubble with the particle inside acquires global superluminal velocities, or *warp speeds*. In the seminal paper, Alcubierre also concluded that the warp metric would imply the violation of the energy conditions since it appeared that negative energy density would be required for the creation of the warp bubble.

Since this original work many authors have contributed in our understanding of the theoretical details of the Alcubierre warp drive metric and possible feasibility of matter particles acquiring warp speeds. Ford and Roman [2] applied quantum inequalities to calculate the amount of negative energy required to transport particles at superluminal speeds. They concluded that such energy requirements would be huge, so the amount of negative energy density necessary for the practical construction of a warp bubble would be impossible to achieve. Pfenning and Ford [3] also used quantum inequalities to calculate the necessary bubble parameters and energy for the warp drive viability, reaching at an enormous amount of energy, ten orders of magnitude greater than the mass-energy of the entire visible Universe, also negative. Hiscock [4] computed the vacuum energy-momentum tensor (EMT) of a reduced two dimensional quantized scalar field of the warp drive spacetime. He showed that in this reduced context that the EMT diverges if the apparent velocity of the bubble is greater than the speed of light. Such divergence is connected to the construction of an horizon in this two dimensional spacetime. Due to the semiclassical effects, the superluminal travel via warp drive might unfeasible. For example, to an observer within the warp drive bubble, the backward and forward walls look like the horizon of a white hole and of a black hole, respectively, resulting with a Hawking radiation.

The issue of superluminal speeds of massive particles traveling faster than photons has also been studied by Krasnikov [5], who argued that this would not be possible if some conjectures for globally hyperbolic spacetimes are made. He described some spacetime topologies and their respective need of the tachyon existence for the occurrence of travel at warp speeds. This author also advanced a peculiar spacetime where superluminal travel would be possible without tachyons, named as the *Krasnikov tube* by Everett and Roman [6], who generalized the metric designed by Krasnikov by proposing a tube in the direction of the particle's path providing a connection between Earth and a distant star. Inside the tube the spacetime is flat and the lightcones are opened out in order to allow the one direction superluminal travel. For the Krasnikov tube to work they showed that huge quantities of negative energy density would also be necessary. Since the tube does not possess closed timelike curves, it would be theoretically possible to design a two way non-overlapping tube system such that it would work as a time machine. In addition, the EMT is positive in some regions. Both the metric and the obtained EMT were thoroughly analyzed in Refs. [7, 8].

A relevant contribution to warp drive theory was made by van de Broeck [9], who demon-

strated that a small modification of the original Alcubierre geometry greatly diminishes to a few solar masses the total negative energy necessary for the creation of the warp bubble distortion, a result that led him to hypothesize that other geometrical modifications of this type could further reduce in a dramatic fashion the amount of energy necessary to create a warp drive bubble. Natario [10] designed a new warp drive concept with zero expansion by using spherical coordinates and the X-axis as the polar one. Lobo and Visser [11, 12] discussed that the center of the warp bubble, as proposed by Alcubierre, need to be massless (see also [13, 14]). A linearized model for both approaches was introduced and it was demonstrated that for small speeds the amassed negative energy inside the warp field is a robust fraction of the particle's mass inside the center of the warp bubble. Lee and Cleaver [15, 16] have looked at how external radiation might affect the Alcubierre warp bubble, possibly making it energetically unsustainable, and how a proposed warp field interferometer could not detect spacetime distortions. Mattingly et al. [17, 18] discussed curvature invariants in the Natario and Alcubierre warp drives.

In a previous paper [19] we have considered some of these issues, but from a different angle. Since the Alcubierre metric was not advanced as a solution of the Einstein equations, as it was originally proposed simply as an ad hoc geometry designed to create a spacetime distortion such that a massive particle inside it travels at warp speeds, whereas locally it never exceeds the light speed, the basic question was then the possible types of matter-energy sources capable of creating a warp bubble. To answer this question the Einstein equations have to be solved with some form of EMT as source. The simplest one to start with is incoherent matter. Following this line of investigation, we showed that the dust solutions of the Einstein equations for the warp drive metric implied in vacuum, that is, such distribution is incapable of creating a warp bubble, nevertheless, the Burgers equation appeared as part of the solution of the Einstein equations. In addition, since the Burgers equation describes shock waves moving in an inviscid fluid, it was also found that at these shock waves behave as plane waves [24–28].

In this paper we generalize the results obtained in Ref. [19] by following the next logical step, that is, investigating perfect fluid as EMT source for the Alcubierre metric. We also propose a slightly generalized perfect fluid EMT, called here *parametrized perfect fluid* (PPF), in order to produce a tentatively more flexible model such that the pressure may have different parameter values. The aim is to see if more flexible EMT distributions could relax the original requirement that warp speeds could only be achieved by means of negative matter density.

For the perfect fluid EMT solutions we found that two out of four subcases turn out to be the dust solution of Ref. [19] where both the matter density and pressure vanish, but the Burgers equation also appears as a result of the solutions of the Einstein equations [29]. Two other subcases, however, indicate that warp speeds are possible with positive matter density, but at the cost of a complex solution for the warp metric regulating function. Weak, dominant, strong and null energy conditions were calculate for both EMTs and all perfect fluid solutions satisfy them. In the case of the PPF, two out of four solutions give rise to a nonlinear equation of state linking various pressures to the matter density. Other solutions produced results where a nonvanishing pressure occurs with a vanishing matter density, condition considered unphysical and then dismissed. The solutions also produced

parameters and equations of state related to pressure and inequalities that satisfy all the energy conditions. These results indicate that energy-momentum tensors describing more complex forms of matter distributions generate solutions for the Einstein equations with the warp drive metric where negative matter density might not be a strict precondition.

The plan for the paper is as follows. In Section 2 we briefly review the Alcubierre warp drive theory and present the relevant equations and all nonzero components of the Einstein tensor for the warp drive metric. In Section 3 the Einstein equations are then solved and solutions presented for the warp drive metric having a perfect fluid gravity source. In Section 4 the non-zero components of the Einstein tensor in the warp drive geometry are written in terms of the PPF EMT. Solutions for this more flexible EMT are also obtained and studied in all subcases. Section 5 presents the EMT divergence of both the perfect fluid and PPF, whereas Section 6 discusses the energy conditions for the two types of EMTs. Section 7 provides further discussions on the results presented in the previous sections, and Section 8 depicts our conclusions.

2.   EINSTEIN EQUATIONS

We shall start this section by brief reviewing the Alcubierre warp drive metric. Subsequently, the nonzero components of the Einstein tensor of this metric will also be explicitly shown. The expressions presented in this section form the basic set of equations required for the next sections.

2.1.   The Alcubierre warp drive geometry

The start up geometry advanced in Ref. [1] may be written as follows,

$$ds^2 = - \left(\alpha^2 - \beta_i \beta^i \right) dt^2 + 2\beta_i \, dx^i \, dt + \gamma_{ij} \, dx^i \, dx^j \ , \tag{2.1}$$

where $d\tau$ is the proper time lapse, α is the lapse function, β^i is the spacelike shift vector and γ_{ij} is the spatial metric for the hypersurfaces.[1] The lapse function α and the shift vector β^i are functions to be determined, whereas γ_{ij} is a positive-definite metric on each of the spacelike hypersurfaces, for all values of time, a feature that makes the spacetime globally hyperbolic [21, 22].

Alcubierre [1] assumed the following particular parameter choices for Eq. (2.1),

$$\alpha = 1, \tag{2.2}$$

$$\beta^1 = -v_s(t) f\left[r_s(t) \right], \tag{2.3}$$

$$\beta^2 = \beta^3 = 0, \tag{2.4}$$

$$\gamma_{ij} = \delta_{ij}. \tag{2.5}$$

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[1] Throughout this paper Greek indices will range from 0 to 3, whereas the Latin ones indicate the spacelike hypersurfaces and will range from 1 to 3.

Hence, the *Alcubierre warp drive metric* is given by,

$$ds^2 = -\left[1 - v_s(t)^2 f(r_s)^2\right] dt^2 - v_s(t) f(r_s)\, dx\, dt + dx^2 + dy^2 + dz^2 \;, \qquad (2.6)$$

where $v_s(t)$ is the velocity of the center of the bubble moving along the curve $x_s(t)$. This is given by the following expression,

$$v_s(t) = \frac{dx_s(t)}{dt} \;. \qquad (2.7)$$

The function $f(r_s)$ is the warp drive *regulating function.* It describes the shape of the warp bubble, which is given by the following expression [1],

$$f(r_s) = \frac{\tanh\left[\sigma(r_s + R)\right] - \tanh\left[\sigma(r_s - R)\right]}{2\tanh(\sigma R)} \;, \qquad (2.8)$$

where σ and R are parameters to be determined. The variable $r_s(t)$ defines the distance from the center of the bubble $[x_s(t), 0, 0]$ to a generic point (x, y, z) on the surface of the bubble, given by the following equation,

$$r_s(t) = \sqrt{[x - x_s(t)]^2 + y^2 + z^2}. \qquad (2.9)$$

From the above one can see that the motion is one-dimensional, since the x-coordinate is the only one perturbed by the function $x_s(t)$.

2.2. Einstein tensor components

Let us now adopt Alcubierre's original notation by assuming

$$\beta = -\beta^1 = v_s(t) f(r_s) \qquad (2.10)$$

in Eq. (2.3). Then, the nonzero components of the Einstein tensor for the metric (2.6) are given by the following expressions:

$$G_{00} = -\frac{1}{4}(1 + 3\beta^2)\left[\left(\frac{\partial\beta}{\partial y}\right)^2 + \left(\frac{\partial\beta}{\partial z}\right)^2\right] - \beta\left(\frac{\partial^2\beta}{\partial y^2} + \frac{\partial^2\beta}{\partial z^2}\right), \qquad (2.11)$$

$$G_{01} = \frac{3}{4}\beta\left[\left(\frac{\partial\beta}{\partial y}\right)^2 + \left(\frac{\partial\beta}{\partial z}\right)^2\right] + \frac{1}{2}\left(\frac{\partial^2\beta}{\partial y^2} + \frac{\partial^2\beta}{\partial z^2}\right), \qquad (2.12)$$

$$G_{02} = -\frac{1}{2}\frac{\partial^2\beta}{\partial x\partial y} - \frac{\beta}{2}\left(2\frac{\partial\beta}{\partial y}\frac{\partial\beta}{\partial x} + \beta\frac{\partial^2\beta}{\partial x\partial y} + \frac{\partial^2\beta}{\partial t\partial y}\right), \qquad (2.13)$$

$$G_{03} = -\frac{1}{2}\frac{\partial^2\beta}{\partial x\partial z} - \frac{\beta}{2}\left(2\frac{\partial\beta}{\partial z}\frac{\partial\beta}{\partial x} + \beta\frac{\partial^2\beta}{\partial x\partial z} + \frac{\partial^2\beta}{\partial t\partial z}\right), \qquad (2.14)$$

$$G_{11} = -\frac{3}{4}\left[\left(\frac{\partial\beta}{\partial y}\right)^2 + \left(\frac{\partial\beta}{\partial z}\right)^2\right], \tag{2.15}$$

$$G_{12} = \frac{1}{2}\left(2\frac{\partial\beta}{\partial y}\frac{\partial\beta}{\partial x} + \beta\frac{\partial^2\beta}{\partial x\partial y} + \frac{\partial^2\beta}{\partial t\partial y}\right), \tag{2.16}$$

$$G_{13} = \frac{1}{2}\left(2\frac{\partial\beta}{\partial z}\frac{\partial\beta}{\partial x} + \beta\frac{\partial^2\beta}{\partial x\partial z} + \frac{\partial^2\beta}{\partial t\partial z}\right), \tag{2.17}$$

$$G_{23} = \frac{1}{2}\frac{\partial\beta}{\partial z}\frac{\partial\beta}{\partial y}, \tag{2.18}$$

$$G_{22} = -\left[\frac{\partial^2\beta}{\partial t\partial x} + \beta\frac{\partial^2\beta}{\partial x^2} + \left(\frac{\partial\beta}{\partial x}\right)^2\right] - \frac{1}{4}\left[\left(\frac{\partial\beta}{\partial y}\right)^2 - \left(\frac{\partial\beta}{\partial z}\right)^2\right], \tag{2.19}$$

$$G_{33} = -\left[\frac{\partial^2\beta}{\partial t\partial x} + \beta\frac{\partial^2\beta}{\partial x^2} + \left(\frac{\partial\beta}{\partial x}\right)^2\right] + \frac{1}{4}\left[\left(\frac{\partial\beta}{\partial y}\right)^2 - \left(\frac{\partial\beta}{\partial z}\right)^2\right]. \tag{2.20}$$

3. PERFECT FLUID

Besides incoherent matter, or dust, already studied in Ref. [19], the simplest matter-energy distribution to be considered as gravity source for the possible creation of a warp bubble, and then warp speeds, is the perfect fluid. Hence, this section will discuss matter content solutions of the Einstein equations considering a perfect fluid matter source EMT for the Alcubierre metric.

3.1. Perfect fluid content solutions

The EMT for a perfect fluid may be written as follows,

$$T_{\alpha\beta} = (\mu + p)\, u_\alpha u_\beta + p\, g_{\alpha\beta}, \tag{3.1}$$

where μ is the matter density, p is the fluid pressure, $g_{\alpha\beta}$ is the metric tensor and u_α is the 4-velocity of an observer inside the fluid. Perfect fluids have no shear stress, rotation, heat conduction or viscosity, nevertheless this ideal fluid provides a more complex matter content than simple dust [20], allowing us to study if a warp bubble can be created with this gravity source and how the respective gravity field equations solutions can be understood.

For the metric (2.6) the perfect fluid EMT assumes the following form,

$$T_{\alpha\beta} = \begin{pmatrix} \mu + \beta^2 p & -\beta p & 0 & 0 \\ -\beta p & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}. \tag{3.2}$$

Let us now use Eqs. (2.11) to (2.20) with the EMT above in the Einstein equations. Substituting components $G_{11} = 8\pi T_{11}$ and $G_{01} = 8\pi T_{01}$ into $G_{00} = 8\pi T_{00}$, after some algebra and

simplifications we may write the following expression,

$$T_{00} + 2\beta T_{01} + \frac{1}{3}(3\beta^2 - 1)T_{11} = 0 \ . \tag{3.3}$$

Substituting the values for the EMT components $T_{00} = \mu + \beta^2 p$, $T_{01} = -\beta p$, $T_{11} = p$ Eq. (3.3) results in the following expression,

$$p = 3\mu. \tag{3.4}$$

This is an equation of state for the Alcubierre metric having a perfect fluid gravity source EMT.

The component G_{23} is zero since $T_{23} = 0$. This case leads us to either $\partial\beta/\partial y$, or $\partial\beta/\partial z$, or both, equal to zero. Let us now analyze these possibilities and its consequences.

Case 1: $\left[\dfrac{\partial\beta}{\partial z} = 0\right]$ As β does not depend on z, the Einstein tensor components G_{13}, G_{23} and G_{03} are identically zero. Substituting this case into $G_{11} = 8\pi T_{11}$, where G_{11} is given by Eq. (2.15) and $T_{11} = p$, it follows immediately the result below,

$$\frac{3}{4}\left(\frac{\partial\beta}{\partial y}\right)^2 = -8\pi p. \tag{3.5}$$

Substituting $(\partial\beta/\partial y)^2$ above into $G_{01} = 8\pi T_{01}$, where G_{01} is given by Eq. (2.12) and $T_{01} = -\beta p$ from Eq. (3.2), as well as $G_{12} = 8\pi T_{12}$ and $G_{02} = 8\pi T_{02}$, where $T_{12} = 0$ and $T_{02} = 0$, the Einstein equations are reduced to the following equations,

$$p = 3\mu, \tag{3.6}$$

$$\left(\frac{\partial\beta}{\partial y}\right)^2 = -\frac{32}{3}\pi p = -32\pi\,\mu, \tag{3.7}$$

$$\frac{\partial^2\beta}{\partial y^2} = 0, \tag{3.8}$$

$$\frac{\partial^2\beta}{\partial x\partial y} = 0, \tag{3.9}$$

$$2\frac{\partial\beta}{\partial y}\frac{\partial\beta}{\partial x} + \frac{\partial^2\beta}{\partial t\partial y} = 0, \tag{3.10}$$

$$\frac{\partial^2\beta}{\partial t\partial x} + \beta\frac{\partial^2\beta}{\partial x^2} + \left(\frac{\partial\beta}{\partial x}\right)^2 = -\frac{64}{3}\pi p = -64\pi\mu, \tag{3.11}$$

$$\frac{\partial^2\beta}{\partial t\partial x} + \beta\frac{\partial^2\beta}{\partial x^2} + \left(\frac{\partial\beta}{\partial x}\right)^2 = -\frac{128}{3}\pi p = -128\pi\mu \ . \tag{3.12}$$

Eq. (3.7) implies that $\partial\beta/\partial y$ must be constant, since the pressure p is assumed constant. Eq. (3.8) also shows that β must be a linear function of the y-coordinate, which means that β must have a possible additional dependence of arbitrary functions on t

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and x. Both expressions in Eqs. (3.11) and (3.12) constitute the same homogeneous partial differential equation, but with different inhomogeneous parts, so the solution of the inhomogeneous equation is not unique, unless the pressure p is zero. Then, considering these points and Eqs. (3.7) and (3.10), it follows that,

$$\frac{\partial \beta}{\partial y} \frac{\partial \beta}{\partial x} = 0, \tag{3.13}$$

which means that either of these partial derivatives, or both, vanish. Let us discuss both possibilities below.

Case 1a: $\left[\dfrac{\partial \beta}{\partial z} = 0 \ \text{ and } \ \dfrac{\partial \beta}{\partial x} = 0\right]$ For this case the set of partial differential equations from Eqs. (3.6) to (3.12) simplify to,

$$p = 3\mu, \tag{3.14}$$

$$\frac{\partial \beta}{\partial y} = \pm\sqrt{-32\pi\mu}. \tag{3.15}$$

The above equations mean that the matter density μ must be negative or zero for a non complex solution of the Einstein equations, and β must be a function of both t and x coordinates only. The equation above is readily integrated, yielding

$$\beta(t, y) = \pm\sqrt{-32\pi\mu}\, y + g(t) \;, \tag{3.16}$$

where $g(t)$ is a function to be determined by the boundary conditions.

Case 1b: $\left[\dfrac{\partial \beta}{\partial z} = 0 \ \text{ and } \ \dfrac{\partial \beta}{\partial y} = 0\right]$ In this case the pressures vanishes, since $\partial\beta/\partial y = 0$, and the set of partial differential equations (3.6) to (3.12) simplify to,

$$p = 3\mu = 0, \tag{3.17}$$

$$\frac{\partial^2 \beta}{\partial t \partial x} + \beta\frac{\partial^2 \beta}{\partial x^2} + \left(\frac{\partial \beta}{\partial x}\right)^2 = 0. \tag{3.18}$$

Therefore, for $p = 0$ the equation of state $p = 3\mu$ implies zero matter density as well, which reduces the solution to the dust case and then vacuum. This also leads to the appearance of shock waves as plane waves, since $\beta = \beta(x, t)$ and the field equations are reduced to the Burgers equation, as studied in Ref. [19]. This is the case because Eq. (3.18) can be written in the following form,

$$\frac{\partial \beta}{\partial t} + \frac{1}{2}\frac{\partial}{\partial x}(\beta^2) = h(t). \tag{3.19}$$

Here $h(t)$ is a generic function to be determined by boundary conditions. In its homogeneous form, where $h(t) = 0$, it takes the conservation form of the inviscid Burgers equation.

$$\frac{\partial \beta}{\partial t} + \frac{1}{2}\frac{\partial}{\partial x}(\beta^2) = 0 \;. \tag{3.20}$$

See Ref. [19] for details of the Burgers equation in this context.

Case 2: $\left[\dfrac{\partial \beta}{\partial y} = 0\right]$ As β does not depend on the y-coordinate, it is easy to see that G_{12}, G_{23} and G_{02} are identically zero. In addition, considering this case into $G_{11} = 8\pi T_{11}$, it follows immediately that,

$$-\frac{3}{4}\left(\frac{\partial \beta}{\partial z}\right)^2 = 8\pi p. \tag{3.21}$$

Substituting Eq. (3.21) in $G_{01} = 8\pi T_{01}$, where G_{01} is given by Eq. (2.12), $T_{01} = -\beta p$ from Eq. (3.2), and inserting the component $G_{13} = 8\pi T_{13}$ into the component $G_{03} = 8\pi T_{03}$, where $T_{13} = T_{03} = 0$, after some algebra we reach at the following expressions,

$$p = 3\mu, \tag{3.22}$$

$$\left(\frac{\partial \beta}{\partial z}\right)^2 = -\frac{32}{3}\pi p, \tag{3.23}$$

$$\frac{\partial^2 \beta}{\partial z^2} = 0, \tag{3.24}$$

$$\frac{\partial^2 \beta}{\partial x \partial z} = 0, \tag{3.25}$$

$$2\frac{\partial \beta}{\partial z}\frac{\partial \beta}{\partial x} + \frac{\partial^2 \beta}{\partial t \partial z} = 0, \tag{3.26}$$

$$\frac{\partial^2 \beta}{\partial t \partial x} + \beta\frac{\partial^2 \beta}{\partial x^2} + \left(\frac{\partial \beta}{\partial x}\right)^2 = -\frac{64}{3}\pi p, \tag{3.27}$$

$$\frac{\partial^2 \beta}{\partial t \partial x} + \beta\frac{\partial^2 \beta}{\partial x^2} + \left(\frac{\partial \beta}{\partial x}\right)^2 = -\frac{128}{3}\pi p. \tag{3.28}$$

Eq. (3.24) shows that β is a linear function with respect to the z-coordinate. Eq. (3.23) implies that $\partial\beta/\partial z$ must be constant since the pressure p is assumed to be a constant. This means that all second partial derivatives of $\partial\beta/\partial z$ must vanish. Eqs. (3.27) and (3.28) are the same homogeneous partial differential equation, but both right hand side of theirs are different. Hence, the solution of the non-homogeneous equation is not unique, unless the pressure p is zero. Considering Eq. (3.26), this case also unfolds in two possibilities, since,

$$\frac{\partial \beta}{\partial z}\frac{\partial \beta}{\partial x} = 0, \tag{3.29}$$

and either or both are zero. Let us now analyze each subcase.

Case 2a: $\left[\dfrac{\partial \beta}{\partial y} = 0 \text{ and } \dfrac{\partial \beta}{\partial x} = 0\right]$ For this case Eqs. (3.22) to (3.28) yield,

$$p = 3\mu\,, \tag{3.30}$$

$$\frac{\partial \beta}{\partial z} = \pm\sqrt{-32\pi\mu}\,, \tag{3.31}$$

$$\frac{\partial \beta}{\partial z} = \pm\sqrt{\pm 96\pi\mu}\,. \tag{3.32}$$

Eq. (3.31) means that the matter density μ must be negative or zero for a non complex solution of the Einstein equations. Eq. (3.32) allows for possible positive matter density. In addition, β has its dependence reduced to $\beta = \beta(z,t)$. If the matter density μ is assumed constant, then the above expressions can be integrated, yielding

$$\beta(z,t) = \pm\sqrt{-32\pi\mu}\, z + \bar{g}(t)\,, \tag{3.33}$$

$$\beta(z,t) = \pm\sqrt{\pm 96\pi\mu}\, z + \bar{h}(t)\,. \tag{3.34}$$

where $\bar{g}(t)$ and $\bar{h}(t)$ are arbitrary functions to be determined by boundary conditions.

Case 2b $\left[\dfrac{\partial\beta}{\partial y} = 0 \text{ and } \dfrac{\partial\beta}{\partial z} = 0\right]$ For this subcase Eq. (3.23) implies zero pressure, and the set of partial differential equations (3.22) to (3.28) simplify to,

$$p = 3\mu = 0, \tag{3.35}$$

$$\frac{\partial^2\beta}{\partial t\partial x} + \beta\frac{\partial^2\beta}{\partial x^2} + \left(\frac{\partial\beta}{\partial x}\right)^2 = 0\,, \tag{3.36}$$

where the last equation is the result of the only nonzero Einstein tensor components G_{22} and G_{33}. These results are the same as Case 1b above, that is, the dust solution for the Alcubierre warp drive metric that results in the Burgers equation (3.19) and its inviscid form given by Eq. (3.20), as well as shock waves as plane waves [19].

□

Table I summarizes all cases and their respective results of the Einstein equations with the Alcubierre warp drive metric having a perfect fluid matter content as gravity source.

3.2.   Discussion

Cases 1b and 2b are simply the dust case already studied in Ref. [19], apparently being unable to generate a warp bubble since this is a vacuum solution, although it connects the warp metric to the Burgers equation and then to shock waves in the form of plane waves.

Cases 1a and 2a share the same equation of state $p = 3\mu$, but the coordinate dependencies are different, since $\beta = \beta(y,t)$ and $\beta = \beta(z,t)$, respectively. For β to be a real valued function the matter density μ must be negative in Case 1a. From Eqs. (3.16) and (3.33) we have assumed a constant matter density, which means a straightforward integration, but μ can be also a function of both t and y coordinates in the Case 1a, or a function of both t and z coordinates in the Case 2a.

Inasmuch as the matter density must be negative for real solutions, one could think of defining the total mass-energy density as follows,

$$\mu(t, x^j) = \mu^+ + a(t, x^j)\mu^- \leq 0, \tag{3.37}$$
$$x^j = y, z,$$

where μ^+ is the positive portion of the matter density of the perfect fluid and μ^- its negative portion that would allow the warp bubble to exist. $a(t, x^j)$ would be a regulating function

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Case	Condition	Results
1) $\dfrac{\partial\beta}{\partial z}=0$	1a) $\dfrac{\partial\beta}{\partial x}=0$	$p=3\mu$ $\beta=\beta(y,t)$ $\dfrac{\partial\beta}{\partial y}=\pm\sqrt{-32\pi\mu}$
	1b) $\dfrac{\partial\beta}{\partial y}=0$	$p=3\mu=0$ $\beta=\beta(x,t)$ $\dfrac{\partial\beta}{\partial t}+\dfrac{1}{2}\dfrac{\partial}{\partial x}(\beta^2)=h(t)$ $\rightarrow$ this is the dust solution of Ref. [19]
2) $\dfrac{\partial\beta}{\partial y}=0$	2a) $\dfrac{\partial\beta}{\partial x}=0$	$p=3\mu$ $\beta=\beta(z,t)$ $\dfrac{\partial\beta}{\partial z}=\pm\sqrt{-32\pi\mu}$ $\dfrac{\partial\beta}{\partial z}=\pm\sqrt{\pm96\pi\mu}$
	2b) $\dfrac{\partial\beta}{\partial z}=0$	$p=3\mu=0$ $\beta=\beta(x,t)$ $\dfrac{\partial\beta}{\partial t}+\dfrac{1}{2}\dfrac{\partial}{\partial x}(\beta^2)=h(t)$ $\rightarrow$ this is the dust solution of Ref. [19]

TABLE I: Summary of all solutions of the Einstein equations with the Alcubierre warp drive metric having perfect fluid EMT as mass-energy source.

that depends on both time t and space x^j coordinates, being related to the shape and location of the bubble. Remembering that $x^j=y$ for the Case 1a and $x^j=z$ for the Case 2a, since $\mu(t,x^j)$ must be negative there would be a restriction for the positive and negative portions of the matter density in Eq. (3.37).

It might be argued that there is no problem in complex solutions for β, but in the warp drive scenario $\beta=v_s(t)f(r_s)$ determines both the velocity and the shape of the bubble, so either the velocity $v_s(t)$ or the regulating function $f(r_s)$ of the bubble shape must be complex. A complex velocity has no physical meaning, but a complex regulating function could be acceptable if we only consider its real part. Thus, in such situation the formation of a warp bubble capable of generating warp speeds could still be possible in the presence of a perfect fluid positive matter density EMT as gravity source.

Nevertheless, caution is required here because if the result of integrating β turns out to be purely imaginary it is not clear what the bubble shape being represented by an imaginary function means. Therefore, in principle it seems reasonable to start with β being a real function and the warp bubble requiring a perfect fluid with negative mass-energy density

for warp speeds to be physically viable. But, it seems to us that this point remains open to debate.

Considering the results above it is apparent that a perfect fluid EMT generated a more complex set of solutions of the Einstein equations than the dust one, then it is conceivable that viable warp speeds are also possible in more complex EMTs. One such possibility will be discussed next.

4. PARAMETRIZED PERFECT FLUID

Let us propose a generalization of the perfect fluid EMT having seven quantities, namely the mass-energy density μ, the β function and five different pressures A, B, C, D and p. The last quantity D is a momentum density parameter. In the perfect fluid of Eq. (3.2) the pressure denoted by p are all the same in the EMT, a constraint that has been relaxed here. Let us call the perfect fluid generalization with the quantities above as the *parametrized perfect fluid* (PPF). Its respective EMT may be written as below,

$$T_{\alpha\sigma} = \begin{pmatrix} \mu + \beta^2 p & -\beta D & 0 & 0 \\ -\beta D & A & 0 & 0 \\ 0 & 0 & B & 0 \\ 0 & 0 & 0 & C \end{pmatrix}. \tag{4.1}$$

The quantities A, B, C, D and p will not be assumed as constants, but rather as functions of the spacetime coordinates (t, x, y, z).

This is clearly a more flexible EMT than the perfect fluid, and it is being proposed here as a tentative model in order to explore the consequences of more complex EMTs in terms of generating possible positive matter solutions of the Einstein equations with the warp drive metric without the caveats of the perfect fluid solutions discussed above. It is a tentative proposal for a more flexible, or toy, model for the possible creation of warp bubbles, and then warp speeds. Section 7.2 provides more details on the physics of this specific fluid proposal.

The nonzero components of the Einstein equations for the PPF are given by the following expressions,

$$-\frac{1}{4}(1 + 3\beta^2)\left[\left(\frac{\partial\beta}{\partial y}\right)^2 + \left(\frac{\partial\beta}{\partial z}\right)^2\right] - \beta\left(\frac{\partial^2\beta}{\partial y^2} + \frac{\partial^2\beta}{\partial z^2}\right) = 8\pi(\mu + \beta^2 p), \tag{4.2}$$

$$\frac{3}{4}\beta\left[\left(\frac{\partial\beta}{\partial y}\right)^2 + \left(\frac{\partial\beta}{\partial z}\right)^2\right] + \frac{1}{2}\left(\frac{\partial^2\beta}{\partial y^2} + \frac{\partial^2\beta}{\partial z^2}\right) = -8\pi\beta D, \tag{4.3}$$

$$-\frac{1}{2}\frac{\partial^2\beta}{\partial x\partial y} - \frac{\beta}{2}\left(2\frac{\partial\beta}{\partial y}\frac{\partial\beta}{\partial x} + \beta\frac{\partial^2\beta}{\partial x\partial y} + \frac{\partial^2\beta}{\partial t\partial y}\right) = 0, \tag{4.4}$$

$$-\frac{1}{2}\frac{\partial^2\beta}{\partial x\partial z} - \frac{\beta}{2}\left(2\frac{\partial\beta}{\partial z}\frac{\partial\beta}{\partial x} + \beta\frac{\partial^2\beta}{\partial x\partial z} + \frac{\partial^2\beta}{\partial t\partial z}\right) = 0, \tag{4.5}$$

$$-\frac{3}{4}\left[\left(\frac{\partial\beta}{\partial y}\right)^2 + \left(\frac{\partial\beta}{\partial z}\right)^2\right] = 8\pi A, \tag{4.6}$$

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$$\frac{1}{2}\left(2\frac{\partial\beta}{\partial y}\frac{\partial\beta}{\partial x}+\beta\frac{\partial^2\beta}{\partial x\partial y}+\frac{\partial^2\beta}{\partial t\partial y}\right) = 0, \tag{4.7}$$

$$\frac{1}{2}\left(2\frac{\partial\beta}{\partial z}\frac{\partial\beta}{\partial x}+\beta\frac{\partial^2\beta}{\partial x\partial z}+\frac{\partial^2\beta}{\partial t\partial z}\right) = 0, \tag{4.8}$$

$$\frac{1}{2}\frac{\partial\beta}{\partial z}\frac{\partial\beta}{\partial y} = 0, \tag{4.9}$$

$$-\left[\frac{\partial^2\beta}{\partial t\partial x}+\beta\frac{\partial^2\beta}{\partial x^2}+\left(\frac{\partial\beta}{\partial x}\right)^2\right]-\frac{1}{4}\left[\left(\frac{\partial\beta}{\partial y}\right)^2-\left(\frac{\partial\beta}{\partial z}\right)^2\right] = 8\pi B, \tag{4.10}$$

$$-\left[\frac{\partial^2\beta}{\partial t\partial x}+\beta\frac{\partial^2\beta}{\partial x^2}+\left(\frac{\partial\beta}{\partial x}\right)^2\right]+\frac{1}{4}\left[\left(\frac{\partial\beta}{\partial y}\right)^2-\left(\frac{\partial\beta}{\partial z}\right)^2\right] = 8\pi C. \tag{4.11}$$

Substituting Eqs. (4.6) and (4.3) into Eq. (4.2) it follows immediately that,

$$\mu = \beta^2(2D-A-p)+\frac{A}{3}\,. \tag{4.12}$$

This expression shows that the fluid density not only depends on the pressure components A and p, but also on the momentum component D and the warp bubble, since it varies with the shift vector β and, hence, the bubble movement. So, the bubble shape modifies the fluid density in a local way, a result that may imply an analogy with classical fluid dynamics and shock waves in fluids with global velocity greater than the speed of sound in that medium.

Applying this analogy to the warp drive, it may mean that the warp bubble plays the role of a shock wave in a fluid that moves with apparent velocity greater than the speed of light for an outside observer far away from the bubble, this being a result of the nonlinearity of the Einstein equations. The bubble modifies the fluid density which then causes the bubble motion. This classical relativistic fluid analogy may be a physical argument for a mechanism which accounts for the great amount of energy necessary for the feasibility of the warp drive.

After some algebra on the set of Einstein equations above they can be rewritten as below,

$$\beta^2(2D-A-p)+\frac{A}{3} = \mu, \tag{4.13}$$

$$\frac{\partial^2\beta}{\partial x\partial y} = 0, \tag{4.14}$$

$$\frac{\partial^2\beta}{\partial x\partial z} = 0, \tag{4.15}$$

$$\left(\frac{\partial\beta}{\partial y}\right)^2+\left(\frac{\partial\beta}{\partial z}\right)^2 = -\frac{32}{3}\pi A, \tag{4.16}$$

$$\left(\frac{\partial\beta}{\partial y}\right)^2-\left(\frac{\partial\beta}{\partial z}\right)^2 = 16\pi(C-B), \tag{4.17}$$

$$2\frac{\partial\beta}{\partial y}\frac{\partial\beta}{\partial x}+\frac{\partial^2\beta}{\partial t\partial y} = 0, \tag{4.18}$$

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$$2\frac{\partial\beta}{\partial z}\frac{\partial\beta}{\partial x} + \frac{\partial^2\beta}{\partial t\partial z} = 0, \tag{4.19}$$

$$\frac{\partial\beta}{\partial z}\frac{\partial\beta}{\partial y} = 0, \tag{4.20}$$

$$\frac{\partial}{\partial x}\left[\frac{\partial\beta}{\partial t} + \frac{1}{2}\frac{\partial}{\partial x}(\beta^2)\right] = -32\pi(B+C), \tag{4.21}$$

$$\frac{\partial^2\beta}{\partial y^2} + \frac{\partial^2\beta}{\partial z^2} = 16\pi\beta(A-D). \tag{4.22}$$

Eq. (4.20) shows that the solutions for the set of differential equations above have similar alternative cases as in the perfect fluid solutions, that is, either $\partial\beta/\partial z = 0$ and/or $\partial\beta/\partial y = 0$. Both situations and their respective subcases will be discussed next.

Case 1: $\left[\dfrac{\partial\beta}{\partial z} = 0\right]$ The set of equations (4.13) to (4.22) simplify to,

$$\mu = \beta^2(2D - A - p) + \frac{A}{3}, \tag{4.23}$$

$$\frac{\partial^2\beta}{\partial x\partial y} = 0, \tag{4.24}$$

$$\left(\frac{\partial\beta}{\partial y}\right)^2 = -\frac{32}{3}\pi A = 16\pi(C-B), \tag{4.25}$$

$$2\frac{\partial\beta}{\partial y}\frac{\partial\beta}{\partial x} + \frac{\partial^2\beta}{\partial t\partial y} = 0, \tag{4.26}$$

$$\frac{\partial}{\partial x}\left[\frac{\partial\beta}{\partial t} + \frac{1}{2}\frac{\partial}{\partial x}(\beta^2)\right] = -32\pi(B+C). \tag{4.27}$$

$$\frac{\partial^2\beta}{\partial y^2} = 16\pi\beta(A-D). \tag{4.28}$$

From Eq. (4.25) a relation between the pressures A, B and C are straightforward,

$$B = C + \frac{2}{3}A. \tag{4.29}$$

From Eqs. (4.24) and (4.25) it is easy to verify that A and $C-B$ do not depend on the x-coordinate. In addition, for real solutions A must be negative, assuming only real values that are equal to or smaller than zero. Differentiating Eq. (4.25) with respect to x yields,

$$2\frac{\partial\beta}{\partial y}\frac{\partial^2\beta}{\partial x\partial y} = 0 \implies \frac{\partial\beta}{\partial y} = 0 \quad \text{and/or} \quad \frac{\partial^2\beta}{\partial x\partial y} = 0, \tag{4.30}$$

and inserting the result $\partial\beta/\partial y = 0$ into Eq. (4.26) it follows that,

$$\frac{\partial^2\beta}{\partial t\partial y} = 0, \tag{4.31}$$

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The result $\partial\beta/\partial y = 0$ means that $A = D$ from Eq. (4.28) and $C = B$ from Eq. (4.25). Hence, the set of equations from Eq. (4.23) to Eq. (4.27) simplify for $\partial\beta/\partial y = 0$,

$$\mu = \beta^2(D - p) + \frac{D}{3} = \beta^2(A - p) + \frac{A}{3}, \qquad (4.32)$$

$$\frac{\partial}{\partial x}\left[\frac{\partial\beta}{\partial t} + \frac{1}{2}\frac{\partial}{\partial x}(\beta^2)\right] = -64\pi B = -64\pi C. \qquad (4.33)$$

From the result $\partial^2\beta/\partial x\partial y = 0$ of Eq. (4.30) the set of equations (4.23) to (4.28) is recovered. Therefore, Eqs. (4.30) show that this case separates itself in two conditions, either $\partial\beta/\partial x = 0$ or $\partial\beta/\partial y = 0$. Next we analyze both conditions.

Case 1a: $\left[\dfrac{\partial\beta}{\partial z} = 0 \;\text{ and }\; \dfrac{\partial\beta}{\partial x} = 0\right]$ Setting Eq.(4.27) equal to zero then $B = -C$, and the set of equations (4.23) to (4.28) simplify to,

$$\mu = \beta^2(2D - A - p) + \frac{A}{3}, \qquad (4.34)$$

$$B = -C = \frac{1}{3}A, \qquad (4.35)$$

$$\left(\frac{\partial\beta}{\partial y}\right)^2 = 32\pi C, \qquad (4.36)$$

$$\frac{\partial^2\beta}{\partial y^2} = 16\pi\beta(A - D). \qquad (4.37)$$

For this case, there is an equation of state given by Eq. (4.34), β is a function of time and y coordinates and must be found by solving both Eqs. (4.36) and (4.37) in terms of the pressures A, C and D. Note that Eq. (4.35) relates the pressures A, B and C. The EMT for this case may be written as follows,

$$T_{\alpha\sigma} = \begin{pmatrix} \beta^2(2D - A) + A/3 & -\beta D & 0 & 0 \\ -\beta D & A & 0 & 0 \\ 0 & 0 & A/3 & 0 \\ 0 & 0 & 0 & -A/3 \end{pmatrix}. \qquad (4.38)$$

One should also note that for the T_{00} component from Eq. (4.38) to be of positive value the following inequality must hold,

$$\beta^2 > \frac{A}{3(A - 2D)}. \qquad (4.39)$$

Case 1b: $\left[\dfrac{\partial\beta}{\partial z} = 0 \;\text{ and }\; \dfrac{\partial\beta}{\partial y} = 0\right]$ Since Eq. (4.25) is equal to zero, then $B = C$. Similarly for Eq. (4.28) it is clear that $A = D$. Consequently, the set of equations (4.23) to (4.28)

simplify to,

$$\mu = \beta^2(2D - A - p) + \frac{A}{3}, \tag{4.40}$$

$$B = C + \frac{2}{3}A, \tag{4.41}$$

$$B = C, \tag{4.42}$$

$$A = D, \tag{4.43}$$

$$\frac{\partial}{\partial x}\left[\frac{\partial \beta}{\partial t} + \frac{1}{2}\frac{\partial}{\partial x}(\beta^2)\right] = -64\pi B. \tag{4.44}$$

But, from Eq. (4.25) we have that $A = 0$ because $B = C$. So, from Eq. (4.32) we have that $\mu = -\beta^2 p$, and one is left with a non homogeneous Burgers equation. This case reduces the above set of equations to

$$\mu = -\beta^2 p, \tag{4.45}$$

$$B = C, \tag{4.46}$$

$$\frac{\partial}{\partial x}\left[\frac{\partial \beta}{\partial t} + \frac{1}{2}\frac{\partial}{\partial x}(\beta^2)\right] = -64\pi B. \tag{4.47}$$

Eq. (4.45) represents an equation of state between matter density μ and the pressure p. The pressures B and C are functions of the spacetime coordinates (t, x, y, z) and from Eq. (4.46) they are equal. Eq. (4.47) is a non homogeneous Burgers equation, since its right hand side cannot be readily integrated. It might mean that there is no conservation law that can describe the warp drive for the PPF EMT. The only possible way leading to conservation law is for B being a constant, which then allows a straightforward integration. However, since this is not the case, namely, all the pressures from the EMT are not, necessarily, constant functions, it is necessary to determine this functions through the boundary conditions. The EMT for the Case 1b case then yields,

$$T_{\alpha\sigma} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & B & 0 \\ 0 & 0 & 0 & B \end{pmatrix}. \tag{4.48}$$

The only non vanishing components of the PPF EMT above are T_{22} and T_{33}. This case recovers the perfect fluid Case 1b, that is, the dust EMTs when one chooses $p = B = 0$. For $B \neq 0$, one would have to solve Eq. (4.47) to determine how the bubble moves in this type of fluid spacetime. Again, negative matter density emerges from the Einstein equation solutions for this specific choice of EMT.

Nevertheless, this is a rather peculiar EMT, since the T_{00} component is zero, but the equation of state (4.45) remains and only two diagonal terms are not zero. In addition,

it is contradictory with the perfect fluid solution because the PPF reduces to the perfect fluid under the condition

$$p = A = B = C = D, \tag{4.49}$$

but in this solution $A = 0$, but $B \neq 0$. So, we discard this case as unphysical.

Case 2: $\left[\dfrac{\partial \beta}{\partial y} = 0 \right]$ The set of equations (4.13) to (4.22) simplify to,

$$\mu = \beta^2 (2D - A - p) + \frac{A}{3}, \tag{4.50}$$

$$\frac{\partial^2 \beta}{\partial x \partial z} = 0, \tag{4.51}$$

$$\left(\frac{\partial \beta}{\partial z} \right)^2 = -\frac{32}{3} \pi A = -16\pi(B - C), \tag{4.52}$$

$$2 \frac{\partial \beta}{\partial z} \frac{\partial \beta}{\partial x} + \frac{\partial^2 \beta}{\partial t \partial z} = 0, \tag{4.53}$$

$$\frac{\partial}{\partial x} \left[\frac{\partial \beta}{\partial t} + \frac{1}{2} \frac{\partial}{\partial x} (\beta^2) \right] = -32\pi(B + C) , \tag{4.54}$$

$$\frac{\partial^2 \beta}{\partial z^2} = 16\pi\beta(A - D). \tag{4.55}$$

Eq. (4.52) straightforwardly implies the following relationship between pressures A, B and C,

$$B = C + \frac{2}{3} A. \tag{4.56}$$

Eqs. (4.51) and (4.52) indicate that A and $B - C$ do not depend on the x-coordinate. In addition, for real solutions A must be negative or zero. Finally, Eq. (4.51) shows that that this case also unfolds in two subcases: $\partial \beta / \partial z = 0$ and/or $\partial \beta / \partial x = 0$.

Case 2a: $\left[\dfrac{\partial \beta}{\partial y} = 0 \ \text{ and } \ \dfrac{\partial \beta}{\partial x} = 0 \right]$ Clearly $B = -C$ as a consequence of Eq. (4.54). Then, the set of equations Eqs. (4.50) to (4.55) simplifies to,

$$\mu = \beta^2 (2D - A - p) + \frac{A}{3}, \tag{4.57}$$

$$B = -C = \frac{A}{3}, \tag{4.58}$$

$$\left(\frac{\partial \beta}{\partial z} \right)^2 = 32\pi C, \tag{4.59}$$

$$\frac{\partial^2 \beta}{\partial z^2} = 16\pi\beta(A - D). \tag{4.60}$$

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The EMT for this case is given by the following expression,

$$T_{\alpha\sigma} = \begin{pmatrix} \beta^2(2D-A)+A/3 & -\beta D & 0 & 0 \\ -\beta D & A & 0 & 0 \\ 0 & 0 & -A/3 & 0 \\ 0 & 0 & 0 & A/3 \end{pmatrix},$$ (4.61)

which is almost equal to Case 1a (Eq. 4.38), apart from the change of signs in the components T_{22} and T_{33}. Both have the same T_{00} component and hence, the same equation of state, and the condition for T_{00} to be of positive is given by,

$$\beta^2 > \frac{A}{3(A-2D)}.$$ (4.62)

Besides, this expressions determines another inequality that T_{00} must follow to be positive,

$$A - 2D > 0.$$ (4.63)

Case 2b: $\left[\dfrac{\partial\beta}{\partial y} = \mathbf{0} \ \text{ and } \ \dfrac{\partial\beta}{\partial z} = \mathbf{0}\right]$ The set of equations Eqs. (4.50) to (4.55) immediately simplify to,

$$\mu = \beta^2(2D-A-p) + \frac{A}{3},$$ (4.64)

$$B = C - \frac{2}{3}A,$$ (4.65)

$$B = C,$$ (4.66)

$$A = D = 0,$$ (4.67)

$$\frac{\partial}{\partial x}\left[\frac{\partial\beta}{\partial t} + \frac{1}{2}\frac{\partial}{\partial x}(\beta^2)\right] = -64\pi B = -64\pi C .$$ (4.68)

Therefore $\mu = -\beta^2 p$, a non homogeneous Burgers equation is also present, and the expressions above are reduced to

$$\mu = -\beta^2 p,$$ (4.69)

$$B = C,$$ (4.70)

$$\frac{\partial}{\partial x}\left[\frac{\partial\beta}{\partial t} + \frac{1}{2}\frac{\partial}{\partial x}(\beta^2)\right] = -64\pi B,$$ (4.71)

which are the same results as obtained in Case 1b, as well as the EMT, that is,

$$T_{\alpha\sigma} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & B & 0 \\ 0 & 0 & 0 & B \end{pmatrix}. \qquad (4.72)$$

□

Again, because this solution cannot fulfill the requirement given by Eq. (4.49), in fact it contradicts it since one cannot have both $A = 0$ and $B \neq 0$, this solution is, similarly to Case 1b, dismissed as unphysical.

Table II summarizes the Einstein equations solutions obtained as a result of the PPF EMT in the Alcubierre warp drive metric.

4.1. Discussion

As in the perfect fluid situations, Cases 1b and 2b are the same for the PPF, producing the same equations of state, coordinate dependency for the β function and a non-homogeneous Burgers equation. As stated above, only for a constant pressure B that a conservation law could possibly emerge from the Burgers equation (4.71) that is common to both cases. Otherwise this expression cannot be readily integrated unless further boundary conditions are met (see Section 5.3 below). Nevertheless, because the solutions of these two cases cannot satisfy the requirement established by Eq. (4.49) for reducing the PPF to the perfect fluid, they are dismissed as unphysical.

Regarding Cases 1a and 2a, the only physically plausible solutions remaining for the PPF EMT, they must obey the same inequalities (4.62) and (4.63) so that the T_{00} component be positive in both Eqs. (4.38) and (4.61). The function β depends on the y coordinate in the former case and the z one in the later, and, respectively, are a result of the integration of the following differential equations,

$$\beta \frac{\partial \beta}{\partial y} = \frac{C}{A - D}, \qquad (4.73)$$

$$\beta \frac{\partial \beta}{\partial z} = \frac{C}{A - D}. \qquad (4.74)$$

However, the quantities A, C and D are function of the coordinates, thus integrating the expressions above require further conditions, such as the ones respectively discussed in Sections 5.1 and 5.2 below.

4.1.1. Equations of state

The perfect fluid solutions have equations of state given by [Ref. 30, p. 14],

$$p = p(\mu) = (\gamma - 1)\mu, \quad \dot{\gamma} = \frac{d\gamma}{dt} = 0, \qquad (4.75)$$

Ordinary fluids can be approximated with $1 \leq \gamma \leq 2$, where incoherent matter, or dust, corresponds to $\gamma = 1$, and radiation to $\gamma = \frac{4}{3}$.

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Case	Conditions	Results
1) $\dfrac{\partial \beta}{\partial z} = 0$	1a) $\dfrac{\partial \beta}{\partial x} = 0$	$\mu = \beta^2(2D - A - p) + \dfrac{A}{3}$ $\beta = \beta(t, y)$ $B = -C = \dfrac{A}{3}$ $\left(\dfrac{\partial \beta}{\partial y}\right)^2 = 32\pi C,$ $\dfrac{\partial^2 \beta}{\partial y^2} = 16\pi \beta(A - D)$
	1b) $\dfrac{\partial \beta}{\partial y} = 0$	$\mu = -\beta^2 p$ $\beta = \beta(t, x)$ $B = C$ $A = D = 0$ $\dfrac{\partial}{\partial x}\left[\dfrac{\partial \beta}{\partial t} + \dfrac{1}{2}\dfrac{\partial}{\partial x}(\beta^2)\right] = -64\pi B$ $\rightarrow$ solution *dismissed* as unphysical
2) $\dfrac{\partial \beta}{\partial y} = 0$	2a) $\dfrac{\partial \beta}{\partial x} = 0$	$\mu = \beta^2(2D - A - p) + \dfrac{A}{3}$ $\beta = \beta(t, z)$ $B = -C = \dfrac{A}{3}$ $\left(\dfrac{\partial \beta}{\partial z}\right)^2 = 32\pi C$ $\dfrac{\partial^2 \beta}{\partial z^2} = 16\pi \beta(A - D)$
	2b) $\dfrac{\partial \beta}{\partial z} = 0$	$\mu = -\beta^2 p$ $\beta = \beta(t, x)$ $B = C$ $A = D = 0$ $\dfrac{\partial}{\partial x}\left[\dfrac{\partial \beta}{\partial t} + \dfrac{1}{2}\dfrac{\partial}{\partial x}(\beta^2)\right] = -64\pi B$ $\rightarrow$ solution *dismissed* as unphysical

TABLE II: Summary of all solutions of the Einstein equations for the Alcubierre warp drive metric having the parametrized perfect fluid (PPF) EMT. The cases, conditions and their respective designations are the same as in Table I

In the analyzes above we found that for the perfect fluid content solution Cases 1a and 2a for the warp drive metric the equation of state is given by $p = 3\mu$ (see Table I), which corresponds to $\gamma = 4$. However, they do not allow a simple expressions for the equation of state similar to Eq. (4.75) for the PPF solutions 1a and 2a (see Table II). The PPF content solution for the Cases 1b and 2b were dismissed as unphysical,

5. EMT DIVERGENCE OF THE PERFECT FLUID AND PPF

In this section we will investigate the associated conservation laws to the Einstein equations under a warp drive spacetime by means of the usual condition that the EMT divergence must be zero for both the perfect fluid and PPF. We shall start with the EMT for the PPF because from its very definition the perfect fluid can be recovered by setting the equality (4.49).

For the fluids discussed here, setting $T^{\alpha\sigma}{}_{;\sigma} = 0$ in the EMT (4.1) results in the following expressions,

$$-\frac{\partial \beta}{\partial x}(D + \mu) - \frac{\partial \mu}{\partial t} - \beta \left[\frac{\partial D}{\partial x} + \frac{\partial \mu}{\partial x} + \frac{\partial \beta}{\partial t}(2p + A - 3D) \right]$$

$$+\beta^2 \left[\frac{\partial D}{\partial t} - \frac{\partial p}{\partial t} + 3\frac{\partial \beta}{\partial x}(D - p) \right] + \beta^3 \left(\frac{\partial D}{\partial x} - \frac{\partial p}{\partial x} \right) = 0, \qquad (5.1)$$

$$\frac{\partial A}{\partial x} + \frac{\partial \beta}{\partial t}(D - A) + \beta \left[3\frac{\partial \beta}{\partial x}(D - A) + \frac{\partial D}{\partial t} - \frac{\partial A}{\partial t} \right] + \beta^2 \left(\frac{\partial D}{\partial x} - \frac{\partial A}{\partial x} \right) = 0, \qquad (5.2)$$

$$\frac{\partial B}{\partial y} + \beta \frac{\partial \beta}{\partial y}(D - A) = 0, \qquad (5.3)$$

$$\frac{\partial C}{\partial z} + \beta \frac{\partial \beta}{\partial z}(D - A) = 0. \qquad (5.4)$$

5.1. Case 1a: $\left[\dfrac{\partial \beta}{\partial z} = 0 \text{ and } \dfrac{\partial \beta}{\partial x} = 0 \right]$

Eqs. (5.1) to Eq. (5.4) are reduced to the ones below,

$$-\frac{\partial \mu}{\partial t} - \beta \left[\frac{\partial D}{\partial x} + \frac{\partial \mu}{\partial x} + \frac{\partial \beta}{\partial t}(2p + A - 3D) \right] + \beta^2 \left(\frac{\partial D}{\partial t} - \frac{\partial p}{\partial t} \right)$$

$$+\beta^3 \left(\frac{\partial D}{\partial x} - \frac{\partial p}{\partial x} \right) = 0, \qquad (5.5)$$

$$\frac{\partial A}{\partial x} + \frac{\partial \beta}{\partial t}(D - A) + \beta \left(\frac{\partial D}{\partial t} - \frac{\partial A}{\partial t} \right) + \beta^2 \left(\frac{\partial D}{\partial x} - \frac{\partial A}{\partial x} \right) = 0, \qquad (5.6)$$

$$\frac{\partial B}{\partial y} + \beta \frac{\partial \beta}{\partial y}(D - A) = 0, \qquad (5.7)$$

$$\frac{\partial C}{\partial z} = 0. \qquad (5.8)$$

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The results above concern the PPF. These four equations together with the five ones shown in the respective results of Table II means an overdetermined system for the six unknowns μ, p, A, B, C, D.

The perfect fluid is recovered by setting Eq. (4.49). Hence, Eqs. (5.5) to (5.8) become,

$$-\frac{\partial \mu}{\partial t} - \beta \left(\frac{\partial p}{\partial x} + \frac{\partial \mu}{\partial x} \right) = 0, \tag{5.9}$$

$$\frac{\partial p}{\partial x} = 0, \tag{5.10}$$

$$\frac{\partial p}{\partial y} = 0, \tag{5.11}$$

$$\frac{\partial p}{\partial z} = 0, \tag{5.12}$$

and, thus, the pressure does not depend on the spatial coordinates. In addition, Eq. (5.9) reduces to the expression below,

$$\frac{\partial \mu}{\partial t} + \beta \frac{\partial \mu}{\partial x} = 0, \tag{5.13}$$

which is the continuity equation, where μ plays the role of the fluid density and β is the flow velocity vector field. It is worth mentioning that for a constant density the fluid has incompressible flow, then all the partial derivatives of β in terms of the spatial coordinates vanish and the flow velocity vector field has null divergence, this being a classical fluid dynamics scenario, and the local volume dilation rate is zero.

5.2.   Case 2a: $\left[\dfrac{\partial \beta}{\partial y} = 0 \text{ and } \dfrac{\partial \beta}{\partial x} = 0 \right]$

Equations (5.1) to (5.4) simplify to the following expressions:

$$-\frac{\partial \mu}{\partial t} - \beta \left[\frac{\partial D}{\partial x} + \frac{\partial \mu}{\partial x} + \frac{\partial \beta}{\partial t}(2p + A - 3D) \right]$$
$$+ \beta^2 \left(\frac{\partial D}{\partial t} - \frac{\partial p}{\partial t} \right) + \beta^3 \left(\frac{\partial D}{\partial x} - \frac{\partial p}{\partial x} \right) = 0, \tag{5.14}$$

$$\frac{\partial A}{\partial x} + \frac{\partial \beta}{\partial t}(D - A) + \beta \left(\frac{\partial D}{\partial t} - \frac{\partial A}{\partial t} \right) + \beta^2 \left(\frac{\partial D}{\partial x} - \frac{\partial A}{\partial x} \right) = 0, \tag{5.15}$$

$$\frac{\partial B}{\partial y} = 0, \tag{5.16}$$

$$\frac{\partial C}{\partial z} + \beta \frac{\partial \beta}{\partial z}(D - A) = 0. \tag{5.17}$$

Assuming Eq. (4.49) the perfect fluid is recovered, resulting in the already discussed Eqs. (5.9) to (5.12), as well as the continuity equation (5.13).

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5.3. Cases 1b and 2b: $\left[\dfrac{\partial \beta}{\partial y} = 0 \ \text{and} \ \dfrac{\partial \beta}{\partial z} = 0\right]$

It follows from Table I that conditions 1b and 2b for the perfect fluid imply $\mu = p = 0$, which also means $A = B = C = D = 0$. Then, Eqs. (5.1) to (5.4) of the null EMT divergence are are immediately satisfied for the perfect fluid.

Regarding the PPF, we have already seen that in these cases the solutions were dismissed as unphysical. Nonetheless, it is worth analyzing the resulting expressions to show that they lead to trivial cases or to the dust solution already studied in Ref. [19].

Table II shows the following solutions for the PPF EMT considering Cases 1b and 2b: $A = D = 0$, $B = C$, and $\mu = -\beta^2 p$. Hence, Eqs. (5.1) to (5.4) are reduced to the following expressions,

$$-\frac{\partial \mu}{\partial t} - \beta \left(\frac{\partial \mu}{\partial x} + 2p\frac{\partial \beta}{\partial t}\right) - \beta^2 \frac{\partial p}{\partial t} - \beta^3 \frac{\partial p}{\partial x} = 0, \tag{5.18}$$

$$\frac{\partial B}{\partial y} = 0, \tag{5.19}$$

$$\frac{\partial C}{\partial z} = \frac{\partial B}{\partial z} = 0. \tag{5.20}$$

Eqs. (5.19) and (5.20) imply that B does not depend on both the y and z coordinates. Inserting the state equation $\mu = -\beta^2 p$ (see Table II) into Eq. (5.18) yields

$$p\beta^2 \frac{\partial \beta}{\partial x} = 0. \tag{5.21}$$

This expression can be separated in three subcases: $\beta = 0$, $p = 0$, $\partial \beta/\partial x = 0$. According to Eq. (4.71) the last subcase implies $B = 0$ and, hence, there is no Burgers equation. Let us discuss below the consequences of each of these subcases.

5.3.1. Subcase $[\beta = 0]$

This solution reduces the warp drive metric (2.6) into a Minkowski flat spacetime. There is no warp bubble and, so, no warp drive.

5.3.2. Subcase $[B = 0]$

Remembering Eqs. (4.48) and (4.72) this means that all EMT components are zero, even though, the equation of state $\mu = -\beta^2 p$ remains. But, since there is no EMT both μ and p must vanish, hence this case reduces itself to the dust solution [19].

5.3.3. Subcase $[p = 0]$

Considering the equation of state $\mu = -\beta^2 p$ then the matter density is also zero. The EMT for the PPF assumes the form given by Eq. (4.72), which implies in pressure without

matter density, an unphysical situation that should either be dismissed or assumed to be just the dust solution.

6.   ENERGY CONDITIONS

The energy conditions are well known inequalities discussed by Hawking and Ellis [23] that may be applied to the matter content in physical systems as boundary conditions and to test if the energy of such systems follows positive constraint values.

This section aims at obtaining these conditions for the Alcubierre warp drive geometry considering the previously discussed EMTs for both the perfect fluid and PPF. Our focus will be on the main classical inequalities, namely, the *weak*, *dominant*, *strong* and *null energy conditions*. Our analysis starts with the PPF EMT defined in Eq. (4.1) in order to constrain its quantities so that the inequalities are satisfied for each of the four just named conditions. Then the same analysis for the perfect fluid is performed by reducing the results to the choice set by Eq. (4.49), but considering only the physically plausible cases.

6.1.   Weak Energy Condition (WEC)

The WEC requires that at each point of the spacetime the EMT must obey the following inequality,

$$T_{\alpha\sigma}\, u^\alpha u^\sigma \geq 0. \tag{6.1}$$

This is valid for any timelike vector $\mathbf{u}\,(u_\alpha u^\alpha < 0)$ as well as any null vector $\mathbf{k}\,(k_\alpha k^\alpha = 0$; see Section 6.4 below). For an observer with a unit tangent vector $\mathbf{v}$ at some point of the spacetime, the local energy density measured by any observer is non-negative [23]. Considering the Eulerian (normal) observers from [1] with 4-velocity given by the expressions below

$$u^\alpha = (1, \beta, 0, 0)\,, \quad u_\alpha = (-1, 0, 0, 0), \tag{6.2}$$

and the EMT, given by Eq. (4.1), computing the expression $T_{\alpha\sigma}\, u^\alpha u^\sigma$ for the WEC yields,

$$\begin{aligned} T_{\alpha\sigma}\, u^\alpha u^\sigma &= T_{00}\, u^0 u^0 + 2T_{01}\, u^0 u^1 + T_{11}\, u^1 u^1 \\ &= \mu + \beta^2 (p - 2D + A)\,. \end{aligned} \tag{6.3}$$

Let us now calculate the WEC for all perfect fluid and PPF Einstein equations solutions obtained in the previous sections as referred to in Table II.

6.1.1.   *Cases 1a and 2a*

Let us start by considering the PPF. Substituting the equation of state (4.12) into Eq. (6.3) yields,

$$T_{\alpha\sigma}\, u^\alpha u^\sigma = \frac{A}{3} \geq 0. \tag{6.4}$$

So, the pressure A must positive. Considering that both cases led to Eq. (4.58), the pressure C must be negative. Besides, for the state equation (4.12) the density μ becomes positive if

$$A + p \leq 2D . \tag{6.5}$$

This inequality implies that it is possible for the fluid density to account for both negative and positive values in a local way depending on how the momentum and pressure components relate to each other in the spacetime.

Remembering Eq. (4.49), the resulting inequality (6.4) indicates that the WEC is satisfied with the perfect fluid EMT for a positive pressure p. Considering Eqs. (3.30) and (3.31) that would mean a complex solution for β, possibility already envisaged in Section 3.2. Note that this is a general result for the perfect fluid with a cross term solution.

6.1.2. *Cases 1b and 2b*

It has been shown above that in these two cases the PPF solutions result in $A = D = 0$ and $\mu = -\beta^2 p$, which substituted in Eq. (6.3) yield $T_{\alpha\sigma} u^\alpha u^\sigma = 0$. Hence, the WEC is not violated for the PPF.

For the perfect fluid, Table I shows that in these two cases the perfect fluid solutions reduce to the dust content solution for the warp drive metric, which trivially satisfies the WEC [Ref. 19, p. 6].

6.2. Dominant Energy Condition (DEC)

The DEC states that for every timelike vector u_a the following inequality must be satisfied,

$$T^{\alpha\beta} u_\alpha u_\beta \geq 0, \quad \text{and} \quad F^\alpha F_\alpha \leq 0, \tag{6.6}$$

where $F^\alpha = T^{\alpha\beta} u_\beta$ is a non-spacelike vector. This condition means that for any observer the local energy density is non-negative and the local energy flow vector is non-spacelike. In any orthonormal basis the energy dominates the other components of the EMT,

$$T^{00} \geq |T^{ab}|, \text{ for each } a, b. \tag{6.7}$$

Hawking and Ellis [23] suggested that this condition must hold for all known forms of matter and that it should be the case in all situations.

Evaluating this condition for PPF, the first DEC inequality is given by the following expression,

$$T^{\alpha\beta} u_\alpha u_\beta = T^{00} = \beta^2 (A - 2D + p) + \mu \geq 0, \tag{6.8}$$

whereas the second inequality $F^\alpha F_\alpha \leq 0$ yields,

$$\begin{aligned} (T^{\alpha\beta} u_\beta)(T_{\alpha\beta} u^\beta) = &- \mu^2 - A^2\beta^4 - \beta^4 p^2 + A^2\beta^2 - (4\beta^4 - \beta^2)D^2 \\ &+ 2(2A\beta^4 - A\beta^2)D - 2(A\beta^2 - 2\beta^2 D)\mu \\ &- 2(A\beta^4 - 2\beta^4 D + \beta^2 \mu)p \leq 0. \end{aligned} \tag{6.9}$$

As before, let us analyze next the solutions of the Einstein equations considering each group of subcases as referred to in Table II.

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6.2.1. Cases 1a and 2a

The two important results for these two cases are Eqs. (4.57) and (4.58). Substituting them into the two DEC inequalities above result in the following expressions,

$$T^{\alpha\beta}\, u_\alpha u_\beta = \frac{A}{3} \geq 0, \tag{6.10}$$

$$(T^{\alpha\beta}\, u_\beta)(T_{\alpha\beta}\, u^\beta) = \left[(A-D)\beta + \frac{A}{3}\right]\left[(A-D)\beta - \frac{A}{3}\right] \leq 0. \tag{6.11}$$

Eq. (6.10) is equal to Eq. (6.4), so this dominant energy condition is the same as the weak one. Eq. (6.11) separates in two inequalities,

$$-\frac{A}{3\,(D-A)} \leq \beta \leq \frac{A}{3\,(D-A)}, \tag{6.12}$$

$$-\frac{A}{3\,(A-D)} \leq \beta \leq \frac{A}{3\,(A-D)}, \tag{6.13}$$

which can be rewritten as

$$|\beta| \leq \frac{1}{3}\left|\frac{A}{D-A}\right|. \tag{6.14}$$

In addition to the inequalities above, Eq. (6.11) also provides the following solutions,

$$\beta = \pm\frac{A}{3(A-D)}. \tag{6.15}$$

Equation (6.14) shows that β is upper bounded, but since both A and D have no restraints, β can be greater than unity. Remembering that $\beta = v_s(t)f[r_s(t)]$, the dominant energy condition is not violated for cases where the apparent warp bubble speed is greater than the speed of light. This result also depends on the relation between the pressure component A and the momentum component D as can be seen in Eqs. (6.12) and (6.13). As an example suppose that $A = D+1$ in Eq. (6.12), so that

$$-\frac{D+1}{3} \leq v_s(t)f[r_s(t)] \leq \frac{D+1}{3}. \tag{6.16}$$

The symmetry on the negative and positive values that β may assume means the direction in which the warp bubble is moving on the x-axis on the spatial hyper surface. Note that, depending on the value of D, the warp bubble may assume the speed $v_s(t)$ greater than the speed of light.

Regarding the perfect fluid, choosing as in Eq. (4.49) the DEC is given by the following expressions,

$$T^{\alpha\beta}\, u_\alpha u_\beta = \frac{p}{3} \geq 0, \tag{6.17}$$

$$(T^{\alpha\beta}\, u_\beta)(T_{\alpha\beta}\, u^\beta) = p^2 \geq 0, \tag{6.18}$$

which means that for the DEC to be satisfied a positively valued matter density is enough (see Table I), although this also means a complex result for β (see Section 3.2).

In these cases the PPF solutions, although considered unphysical, produced $B = C$, $A = D = 0$ and $\mu = -\beta^2 p$ (see Table II). So, as shown by the expressions below the DEC is immediately satisfied,

$$T^{\alpha\beta}\, u_\alpha u_\beta = T^{00} = 0, \tag{6.19}$$

$$(T^{\alpha\beta}\, u_\beta)(T_{\alpha\beta}\, u^\beta) = 0\;, \tag{6.20}$$

The perfect fluid is the dust solution for both cases (see Table I), which is a vacuum solution and satisfies the inequalities above trivially.

6.3. Strong Energy Condition (SEC)

For any timelike vector u^α the EMT must obey the following inequality for the SEC be valid,

$$\left(T_{\alpha\beta} - \frac{1}{2}T\, g_{\alpha\beta}\right) u^\alpha u^\beta \geq 0, \tag{6.21}$$

This requirement is stronger than the WEC and only makes sense in general relativistic framework because this theory is governed by the Einstein equations. These conditions imply that gravity is always attractive.

To obtain expression for the SEC let us start with the scalar $T = g^{\alpha\beta}T_{\alpha\beta}$, where $g^{\alpha\beta}$ is given by the Alcubierre warp drive metric and $T_{\alpha\beta}$ is the EMT for the PPF (Eq. 4.1). The result is shown below,

$$T = g^{\alpha\beta}T_{\alpha\beta} = -\mu + A + B + C + \beta^2(2D - p - A). \tag{6.22}$$

Substituting the equation of state (4.12) into the expression above yields,

$$T = \frac{2}{3}A + B + C. \tag{6.23}$$

As shown in Section 6.1 above, Eq. (4.12) implies that $T_{\alpha\sigma}u^\alpha u^\sigma = A/3$ for $u^\sigma = (1, \beta, 0, 0)$. Inasmuch as $g_{\alpha\sigma}u^\alpha u^\sigma = -1$, then the SEC are given by the following expression,

$$\left(T_{\alpha\sigma} - \frac{1}{2}T\, g_{\alpha\sigma}\right) u^\alpha u^\sigma = \frac{2A}{3} + \frac{(B+C)}{2} \geq 0. \tag{6.24}$$

The particular cases, as summarized in Table II, can now be discussed.

6.3.1. Cases 1a and 2a

These cases resulted in the constraint $B = -C = A/3$. Eq. (6.24) may then be rewritten in the form below,

$$\left(T_{\alpha\sigma} - \frac{1}{2}T\, g_{\alpha\sigma}\right) u^\alpha u^\sigma = \frac{2A}{3} = 2B \geq 0\;. \tag{6.25}$$

Clearly, for the SEC be satisfied it is only necessary that the pressure component A must be non negative.

Regarding the perfect fluid its respective SEC reads as,

$$\left(T_{\alpha\sigma} - \frac{1}{2}T\,g_{\alpha\sigma}\right)u^{\alpha}u^{\sigma} = \frac{2p}{3} \geq 0. \tag{6.26}$$

So, the SEC can be satisfied by the perfect fluid in these subcases by requiring a positive pressure p.

6.3.2. Cases 1b and 2b

In these cases the PPF solutions stated that $A = D = 0$ and $B = C$. Hence, the SEC satisfied under the constraint below,

$$\left(T_{\alpha\sigma} - \frac{1}{2}T\,g_{\alpha\sigma}\right)u^{\alpha}u^{\sigma} = B \geq 0, \tag{6.27}$$

despite providing unphysical solutions. The perfect fluid solutions are just the dust solution that is a vacuum solution [19], so satisfying the SEC trivially.

6.4. Null Energy Condition (NEC)

The SEC and WEC are satisfied in the limit of the null observers with 4-velocities **k**. To satisfy the NEC the EMT must follow the inequality below,

$$T_{\alpha\sigma}\,k^{\alpha}k^{\sigma} \geq 0, \text{ for any null vector } k^{\alpha}. \tag{6.28}$$

To calculate the NEC let us suppose the following null vector below,

$$k^{\alpha} = (a, b, 0, 0), \tag{6.29}$$

where the relation between a and b can be obtained by imposing the condition

$$k_{\alpha}k^{\alpha} = 0, \tag{6.30}$$

that leads to

$$a^2\beta^2 - 2ab\beta - a^2 + b^2 = 0, \tag{6.31}$$

whose roots for a/b yield,

$$\frac{a}{b} = \frac{1}{\beta + 1} \text{ and } \frac{a}{b} = \frac{1}{\beta - 1}. \tag{6.32}$$

Considering the results above, the NEC may be written as follows,

$$T_{\alpha\sigma}\,k^{\alpha}k^{\sigma} = T_{00}k^0k^0 + 2T_{01}k^0k^1 + T_{11}k^1k^1, \tag{6.33}$$

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where $T_{\alpha\sigma}$ is the PPF EMT of Eq. (4.1). So, the null condition is given by the following expression,

$$T_{\alpha\sigma}\, k^\alpha k^\sigma = a^2\beta^2 p - 2ab\beta D + b^2 A + a^2\mu \geq 0, \tag{6.34}$$

Substituting the two roots in Eqs. (6.32) into Eq. (6.34) results in expressions below,

$$\frac{b^2}{(\beta+1)^2}\left[\mu - \beta^2(2D - A - p) + 2\beta(A - D) + A\right] \geq 0, \tag{6.35}$$

$$\frac{b^2}{(\beta-1)^2}\left[\mu - \beta^2(2D - A - p) - 2\beta(A - D) + A\right] \geq 0. \tag{6.36}$$

Next, as previously, we shall proceed to investigate first the Cases 1a and 2a and then 1b and 2b ones, as defined in Tables I and II, for the PPF and the perfect fluid.

6.4.1. Cases 1a and 2a

Considering the PPF first, the equation of state (4.57) is a solution of both cases (see Table II), so the resulting NEC given by Eqs. (6.35) and (6.36) simplify to the following expressions,

$$\frac{b^2}{(\beta+1)^2}\left[\frac{4A}{3} + 2\beta(A - D)\right] \geq 0, \tag{6.37}$$

$$\frac{b^2}{(\beta-1)^2}\left[\frac{4A}{3} - 2\beta(A - D)\right] \geq 0, \tag{6.38}$$

whose solutions for β yield the NEC for the PPF EMT, as below,

$$\beta \geq \frac{2}{3}\frac{A}{D - A} \quad \text{and} \quad \beta \neq -1, \tag{6.39}$$

$$\beta \leq \frac{2}{3}\frac{A}{A - D} \quad \text{and} \quad \beta \neq 1. \tag{6.40}$$

Remembering that $\beta = v_s(t)f(r_s)$, then Eq. (6.39) states that the warp bubble can not assume the negative sign of the speed of light, but it may assume values above it. To verify that, we have to choose $A = D - 1 > 2/3$ and $\beta > 1$. Eq. (6.40) states that β cannot assume the exact value of the speed of light and it is bounded by a superior value which can be greater than the speed of light. To have this, one just have to choose for example $A = D + 1 > 2/3$.

For the perfect fluid, according to Eq. (4.49) all pressures are equal. Hence, substituting both values for a/b found in Eq. (6.34) into Eqs. (6.35) and (6.36) they then simplify to expressions below,

$$\frac{b^2}{(\beta+1)^2}\,(\mu + p) \geq 0, \tag{6.41}$$

$$\frac{b^2}{(\beta-1)^2}\,(\mu + p) \geq 0. \tag{6.42}$$

These results mean that for the NEC is satisfied for the perfect fluid provided that

$$\mu + p \geq 0. \tag{6.43}$$

Remembering that $p = 3\mu$ for the perfect fluid in theses two cases, then the NEC is fulfilled if the matter density does not have negative values.

6.4.2. Cases 1b and 2b

These cases, although set as unphysical, we already concluded that the PPF relates its pressures by the following expressions: $A = D = 0$ and $B = C$. Therefore, Eqs. (6.35) and (6.36) yield,

$$\frac{b^2}{(\beta + 1)^2} \left(\mu + \beta^2 p\right) \geq 0, \tag{6.44}$$

$$\frac{b^2}{(\beta - 1)^2} \left(\mu + \beta^2 p\right) \geq 0. \tag{6.45}$$

So, the NEC establishes that $\mu + \beta^2 p \geq 0$, which is readily satisfied since in these cases we already concluded that $\mu = -\beta^2 p$, provided that $\beta \neq -1$ for the former and $\beta \neq 1$ for the latter. Regarding the perfect fluid, these cases reduce to the dust solution studied in [19], which is a vacuum solution and, therefore, immediately satisfies the NEC.

Table III summarizes the results of all energy conditions for the PPF and perfect fluid EMTs with the Alcubierre warp drive metric.

7. FURTHER DISCUSSIONS

This section discusses some points, and raises others, all related to the physics of the warp drive as suggested by the results presented in the previous sections. It aims at offering some thoughts that may be important in fostering further understanding on how a superluminal travel can be achieved.

7.1. Regulating Function and the Burgers Equation

The regulating function (2.8) describes the shape of the warp bubble, but it is not uniquely determined. However, the integration the Einstein equations in both the perfect fluid and the PPF led to the appearance of generic functions in the dynamic equations which may end up connected to the regulating function, a situation that adds to its nonuniqueness. This means that physically feasible superluminal speeds will require the specification of these generic functions, possibly by boundary conditions. We shall show below an example of this situation using the dust solution.

Considering Eqs. (2.3) and (2.10) the partial derivative of the Burgers equation (3.19) yields,

$$\frac{\partial \beta}{\partial t} = f \frac{d^2 x_s}{dt^2} + v_s \frac{\partial f}{\partial r_s} \frac{\partial r_s}{\partial t}, \tag{7.1}$$

where the simplified notation $f = f[r_s(t)]$ and $v_s = v_s(t)$ was adopted. Since,

$$\frac{\partial \beta}{\partial x} = v_s \frac{\partial f}{\partial r_s} \frac{\partial r_s}{\partial x}, \tag{7.2}$$

the Burgers equation becomes,

$$f \frac{d^2 x_s}{dt^2} + v_s \frac{\partial f}{\partial r_s} \frac{\partial r_s}{\partial t} + v_s^2 f \frac{\partial f}{\partial r_s} \frac{\partial r_s}{\partial x} = h(t). \tag{7.3}$$

The partial derivative of Eq. (2.8) may be written as follows,

$$\frac{\partial f}{\partial r_s} = \frac{\sigma}{2 \tanh(\sigma R)} \left\{ \mathrm{sech}^2[\sigma(r_s + R)] - \mathrm{sech}^2[\sigma(r_s - R)] \right\}. \tag{7.4}$$

Cases (refer to Tables I and II)	Energy Conditions	Results
1a and 2a	Weak	Perfect fluid: $p \geq 0$ PPF: $A \geq 0$
	Dominant	Perfect Fluid: $p \geq 0$ PPF: $A \geq 0$ and $\|\beta\| \leq \frac{1}{3} \left\| \frac{A}{D - A} \right\|$
	Strong	Perfect fluid : $p \geq 0$ PPF : $A \geq 0$
	Null	Perfect fluid : $\mu + p \geq 0$ PPF : $\beta \geq \frac{2}{3} \frac{A}{D - A}$ and $\beta \neq -1$ PPF : $\beta \leq \frac{2}{3} \frac{A}{A - D}$ and $\beta \neq 1$
1b and 2b	Weak	immediately satisfied for perfect fluid and PPF.
	Dominant	immediately satisfied for perfect fluid and PPF.
	Strong	Perfect fluid: immediately satisfied. PPF: $B \geq 0$
	Null	immediately satisfied for perfect fluid and PPF.

TABLE III: Summary of all energy conditions results for the perfect fluid and PPF EMTs with the Alcubierre warp drive spacetime geometry.

The partial derivatives of r_s yield,

$$\frac{\partial r_s}{\partial t} = \pm \frac{dx_s}{dt} = \pm v_s(t), \tag{7.5}$$

$$\frac{\partial r_s}{\partial x} = \pm 1, \tag{7.6}$$

and remembering that in the dust case $\beta = \beta(x,t)$, then $r_s(t)$ is given by (see Eq. 2.9),

$$r_s(t) = \sqrt{[x - x_s(t)]^2} = |x - x_s(t)|. \tag{7.7}$$

Considering the expressions above, Eq. (7.3) may be rewritten as follows,

$$f \frac{d^2 x_s}{dt^2} \pm v_s^2 \frac{\sigma F}{2 \tanh(\sigma R)} \pm v_s^2 f \frac{\sigma F}{2 \tanh(\sigma R)} = h(t), \tag{7.8}$$

where

$$F(r_s) \equiv \text{sech}^2[\sigma(r_s + R)] - \text{sech}^2[\sigma(r_s - R)]. \tag{7.9}$$

Remembering that the regulating function $f[r_s(t)]$, as defined by Eq. (2.8), can be approximated by a top hat function when $\sigma \gg R$ (see Ref. [19], §2.1), in this limit Eq. (7.8) takes the following form inside the bubble,

$$\frac{d^2 x_s}{dt^2} = h(t), \tag{7.10}$$

where $f = 1$. Outside the bubble $f = 0$ and then $h(t) = 0$, which means plane shock waves described by the inviscid Burgers equation [19].

Hence, the nonuniqueness of the shift vector $\beta = v_s(r_s)f[r_s(t)]$ arises from the fact that not only the function $h(t)$ is arbitrary, but also because the regulating function $f[r_s(t)]$ only requires a top hat behavior with null values outside the bubble. So, any well behaved function that respects such constraints may be part of a solution of the Burgers equation. Moreover, from the energy conditions calculated for the PPF, as summarized in Table III, one can see that β plays a fundamental role in Cases 1a and 2a for the null and dominant energy conditions to be satisfied, which adds further physical constraints to its behavior. So, it is clear that the nonlinearity of the Einstein equations imply that the generic functions appearing in their integration become entangled with the regulating function in a non trivial manner.

To extend this analysis to the perfect fluid and PPF contents require further constraints on the shift vector, something which at this stage would be done in an entirely arbitrary manner. Perhaps in the future that can be done under more physically plausible reasoning.

7.2. Anisotropic Fluids

The PPF proposed in Section 4 aimed at offering an alternative EMT for solving the Einstein equations endowed with the warp drive metric. As we shall see below, The PPF can actually be seen as an anisotropic fluid with heat flux [31].

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In general relativity the energy momentum tensor $T_{\mu\nu}$ represents the source of energy and momentum, where T_{00} is the flow of energy across a surface of constant time (energy density), T_{0i} is the energy flux across a surface in the i direction (constant x^i), T_{i0} is the momentum density and T_{ij} is the momentum flux. If we choose a comoving frame of reference that moves with the same velocity as the fluid this means that particles in this fluid will have zero velocity and the flux of energy will be only through the flux of heat, and the momentum flux will be via some sort of dissipative phenomena such as viscosity, thermal radiation or even some sort of electromagnetic type of radiation.

The general stress-energy tensor of a relativistic fluid can be written in the form below [31, 32],

$$T^{\alpha\beta} = \mu u^\alpha u^\beta + p h^{\alpha\beta} + u^\alpha q^\beta + u^\beta q^\alpha + \pi^{\alpha\beta}, \tag{7.11}$$

where

$$h_{\alpha\beta} = g_{\alpha\beta} + u_a u_b, \tag{7.12}$$

projects tensors onto hypersurfaces orthogonal to u^α, μ is the matter density, p is the fluid static pressure, q^α is the heat flux vector and $\pi^{\alpha\beta}$ is the viscous shear tensor. The world lines of the fluid elements are the integral curves of the four-velocity vector u^α. The heat flux vector and viscous shear tensor are transverse to the world lines, that is,

$$q_a u^a = 0, \quad \pi_{ab} u^b = 0. \tag{7.13}$$

In terms of coordinates we can write the energy momentum tensor for a general fluid as

$$T_{\alpha\beta} = \begin{pmatrix} \varepsilon & q_a \\ q_b & \pi_{ab} \end{pmatrix}, \tag{7.14}$$

where q_a is the three-vector heat flux vector, ε is a scalar function and π_{ab} is a three by three matrix viscous stress tensor, which is symmetric and traceless. Both q_a and π_{ab} have respectively three and five linearly independent components. Together with the density μ and the static pressure p, this makes a total of ten linearly independent components which is the number of linearly independent components in a four-dimensional symmetric rank two tensor. We noticed that the Einstein tensor components are highly non-linear for the warp drive metric and the off diagonal terms require those free parameters for a non over determined solution. For a non curved metric, i.e., the Minkowski metric $\eta_{\alpha\beta}$, the energy momentum tensor for a perfect fluid with anisotropic pressures can be written as

$$T_{\alpha\beta} = \begin{pmatrix} \mu & 0 & 0 & 0 \\ 0 & p_x & 0 & 0 \\ 0 & 0 & p_y & 0 \\ 0 & 0 & 0 & p_z \end{pmatrix}. \tag{7.15}$$

Isotropic static pressure means that $p_x = p_y = p_z = p$. The perfect fluid has no heat flux or dissipative phenomena, then $(q^\alpha = 0, \pi^{\alpha\beta} = 0)$. This special case with dust content is the well-known EMT, that is,

$$T^{\alpha\beta} = \mu u^\alpha u^\beta + p h^{\alpha\beta} = (\mu + p) u^\alpha u^\beta + p g^{\alpha\beta}. \tag{7.16}$$

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The PPF proposed in Section 4 has the matrix form given by Eq. (4.1), which may be broken down as the sum of a perfect fluid in a warp drive background as given by Eq. (3.2), and a dissipative fluid with the heat flux four-vector given by, $q_\alpha = -\frac{1}{2}(q_0, q_1, 0, 0)$. Hence,

$$q_\alpha u_\beta + q_\beta u_\alpha = \begin{pmatrix} q_0 & q_1 & 0 & 0 \\ q_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{7.17}$$

since the four-velocity for the moving frame is $u_\alpha = (-1, 0, 0, 0)$. The isotropic term is given by,

$$\pi_{\alpha\beta} = \begin{pmatrix} \pi_{00} & 0 & 0 & 0 \\ 0 & \pi_{01} & 0 & 0 \\ 0 & 0 & \pi_{02} & 0 \\ 0 & 0 & 0 & \pi_{03} \end{pmatrix}, \tag{7.18}$$

and

$$\mu + \beta^2 p \to (\mu + \pi_{00} + q_0) + \beta^2 p\,. \tag{7.19}$$

So, the four parameters of the PPF are as follows,

$$A = \pi_{01} + p\,, \tag{7.20}$$

$$B = \pi_{02} + p\,, \tag{7.21}$$

$$C = \pi_{03} + p\,, \tag{7.22}$$

$$D = p - \frac{q_1}{\beta}\,. \tag{7.23}$$

The tensor $\pi_{\alpha\beta}$ must be traceless, giving us one more equation to solve for the free parameters (anisotropic pressures)

$$\pi_{00} + \pi_{01} + \pi_{02} + \pi_{03} = 0 \tag{7.24}$$

From the above one can see that the warp drive metric endowed with the PPF allows the study of fluid anisotropy coupled with possible dissipative effects that could lead to a warp drive bubble. Perfect fluids are well known to be part of solutions for the Einstein equations, this being the case of the standard FLRW cosmological model that accounts for the expansion, isotropy and spatial homogeneity of the universe. On the other hand, anisotropic imperfect fluids offer a more complex source of gravitational effects, presenting dissipative processes, shear and bulk viscous pressures, interaction between fluids, radiation processes, electromagnetic interaction and even collision between particles, charged or not. These fluids are even known to avoid the big bang singularity in cosmological models [33–37]. MacCallum [38] discussed various ways of generating anisotropy such as the presence of electromagnetic fields, the presence of viscous terms and the anisotropic stresses due to the anisotropic expansion of a cloud of collisionless particles. Another way to account for viscosity, heat and energy flux is the interaction of two or more fluids [39, 40].

7.3. Other aspects of warp drive physics

The points presented above regarding the physical feasibility of superluminal travel with the Alcubierre spacetime geometry by no means exhaust this discussion. Several issues remain open, with each of them deserving separate studies that are beyond the scope of this paper, since in here we focused on the basic properties of the Einstein equations' solutions of the warp drive metric with fluid content. With respect to the open issues, one can point out the amount of mass-energy density, exotic or not, necessary for the feasibility for the warp drive in the context of both the perfect fluid and the PPF, as well as quantum effects and the question of stability or instability in our solutions. These issues deserve further investigations and are the subject of ongoing research.

8. CONCLUSIONS

In this work we have analyzed the Einstein equations for the Alcubierre warp drive metric having as gravity source two types of energy-momentum tensors (EMT) for fluid, namely the perfect fluid and the parametrized perfect fluid (PPF). The latter is defined by allowing the EMT pressure components of the perfect fluid to be different from one another and dependent on all coordinates.

After obtaining the components of the Einstein tensor for the warp drive metric we calculated the dynamic equations for both fluids by solving the respective Einstein equations. Solutions were found in the form of various equations of state, and, by further imposing the null divergence for the EMTs, new constraints were also found for the various variables. The weak, dominant, strong and null energy conditions were also calculated, which implied further constraints upon the free quantities for the EMTs. For the perfect fluid these were on the matter density μ and pressure p. For the PPF that occurred on the pressure components p, A, B, C, the matter density for the fluid μ and the momentum component D.

We found two main groups of solutions subcases possessing different conditions for each EMT. For the perfect fluid, one solution may be interpreted as requiring that the warp bubble can only be viable with negative matter density. The alternative interpretation is that the warp bubble is possible with positive matter density, but in this case the regulating function $f(r,s)$, which shapes the warp bubble, becomes a complex function. This comes from results allowing the possibility that the function $\beta = v_s(t)f(r_s)$ may have complex solutions once the matter density is positive. Other results in both fluids reinforce our earlier finding in Ref. [19] that the warp bubble necessary for generating superluminal velocities, or warp speeds, can be interpreted as a shock wave from classical fluid dynamics theory.

We named four cases arising from the solutions of the Einstein equations: 1a, 1b, 2a and 2b. However, the Cases 1a and 2a are very similar, or equal, to each other, the same happening to Cases 1b and 2b. For this reason they were grouped together in the tables that summarize all results.

Specifically, Cases 1b and 2b for the perfect fluid reduced the solutions to the ones found for dust content already studied in Ref. [19], that is, a vacuum solution unable to create a warp bubble, but which connects the warp metric to the inviscid Burgers equation, also

yielding β as a function of t and x coordinates. The null EMT divergence is satisfied and a continuity equation was also found (Eq. 5.13).

Cases 1b and 2b for the PPF resulted in a equation of state of the form $\mu = -\beta^2 p$, coordinate dependency of the β function became $\beta = \beta(x,t)$ and a non-homogeneous Burgers equation (4.71) emerged. However, the pressures were constrained in such a way that this PPF EMT solution for the warp drive was dismissed as unphysical.

Cases 1a and 2a for the perfect fluid produced an equation of state relating pressure and matter density given by $p = 3\mu$. The β function dependencies became $\beta = \beta(y,t)$ in the former case and $\beta = \beta(z,t)$ in the latter, and both produced a differential equation for β that either requires negative density for β to be a real valued function, or a positive matter density which then leads to a complex solution for β, which in turn leads to a complex regulating function $f(r_s)$ whose possible real part would then be related to a physically viable warp bubble.

Cases 1a and 2a for the PPF resulted in an equation of state relating almost all quantities, in the form $\mu = \beta^2(2D - A - p) + A/3$ which is valid for both cases. Coordinate dependency on β became, respectively, resulted in $\beta = \beta(y,t)$ and $\beta = \beta(z,t)$. The function β is also governed by the first order differential equations (4.73) and (4.74), respectively.

The null EMT divergence $T^{\alpha\sigma}_{\ ;\sigma} = 0$ were calculated, producing further sets of very nonlinear differential equations constraining all quantities in the PPF which could be used, in principle, to determine all pressures and matter density in this fluid. For the perfect fluid, Cases 1a and 2a are reduced to a continuity equation (5.13) including the function β, which is interpreted as playing the role of the flow velocity vector field. Cases 1b and 2b reduced the PPF to either the trivial condition of Minkowski flat spacetime with no warp drive, or an EMT with all components being zero, that is a vacuum case.

It has already been seen in Ref. [19] that the Burgers equation appears connected to the dust solution of the warp drive metric, which is in fact a vacuum solution. Cases 1b and 2b of the perfect fluid became reduced to the dust solution, and a non-homogeneous form of the Burgers equation appears in these respective cases for the PPF, although the whole solutions in these cases were dismissed as unphysical. The solutions that presented themselves as the most plausible ones for creating warp speeds, 1a and 2a for both the perfect fluid and PPF, do not generate a Burgers equation.

The weak, dominant, strong and null energy conditions were also studied in the context of the perfect fluid and PPF energy-momentum tensors for a warp drive metric. The resulting expressions were found to satisfy all conditions in the perfect fluid EMT. Regarding the PPF, specific expressions constraining its EMT quantities were obtained in order to satisfy these energy conditions, but they do not necessarily lead to the conclusion that negative matter density is always necessary for viable warp speeds, particularly because in the PPF the pressure A must assume values equal to zero or positive.

Summing up, the results of this paper indicate that warp speeds might be physically viable in the context of positive matter density as some solutions of the Einstein equations for both fluids keep open this possibility. Nevertheless, such a situation creates the additional issue in the perfect fluid context concerning the meaning of a possible complex regulating function in the warp metric, a result that may be interpreted as a caveat, or major stumbling block. Such difficulty does not appear to happen in the PPF scenario, although this fluid

was considered here mainly as a hypothetical model whose aim was to investigate whether or not new possibilities arise in the solutions of the Einstein field equations considering more complex energy-momentum tensors having the Alcubierre warp drive metric. On this front it seems then that the initial conclusions about the unphysical nature of warp drive, or the impossibility of generating warp speeds, may not be not as stringent as initially thought, or, perhaps, not valid at all.

Acknowledgments

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