

What computer science has to say about the simulation hypothesis

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Abstract

The simulation hypothesis has recently excited renewed interest in the physics and philosophy communities. However, the hypothesis specifically concerns *computers* that simulate physical universes. So to formally investigate the hypothesis, we need to understand it in terms of computer science (CS) theory. In addition we need a formal way to couple CS theory with physics. Here I couple those fields by using the physical Church-Turing thesis. This allow me to exploit Kleene's second recursion, to prove that not only is it possible for us to be a simulation being run on a computer, but that we might be in a simulation that is being run on a computer – *by us*. In such a “self-simulation”, there would be two identical instances of us, both equally “real”. I then use Rice's theorem to derive impossibility results concerning simulation and self-simulation; derive implications for (self-)simulation if we are being simulated in a program using fully homomorphic encryption; and briefly investigate the graphical structure of universes simulating other universes which contain computers running their own simulations. I end by describing some of the possible avenues for future research. While motivated in terms of the simulation hypothesis, the results in this paper are direct consequences of the Church-Turing thesis. So they apply far more broadly than the simulation hypothesis.

He didn't know if he was Zhuang Zhou dreaming he was a butterfly, or a butterfly dreaming that he was Zhuang Zhou.

— Zhuangzi, chapter 2 (Watson translation [69])

1 Introduction

1.1 Background

It is broadly supposed in the physics community that the future values of any properties that we can experimentally measure concerning a system in our universe can be computed ahead of time on a Turing machine [57, 55]. Loosely speaking the supposition is that your laptop, sufficiently “souped up”, could be used to simulate the future value of any physical variable you could experimentally measure, if you just fed it sufficient information about the current state of the universe. Concretely, there has never been an experiment proposed where we would fully specify the initial condition of a closed system, would then measure the future value of a property of that system, and yet for some reason or other it was theoretically impossible for us to compute the future value of that property ahead of time on a sufficiently souped-up laptop.¹

Following much of the literature, I will refer to this supposition as the “physical Church-Turing thesis’ (PCT)’ ([57, 55] and references therein).² The PCT is particularly compelling if one adopts the “multiverse” perspective common in physics, which says that our universe *is* a mathematical structure, given by computing the logical consequences of a set of fundamental mathematical laws [66, 67, 35, 45, 42].

Under the PCT, it is impossible for us to experimentally rule out the possibility that the evolution of all properties of our physical universe that we can measure are the outputs of a program being run on some Turing machine (or even more powerful computational machine).³ More loosely, the PCT is consistent with the hypothesis that our universe *is* a simulation being run on some Turing machine [22]. This second possibility is known in the literature as the “simulation hypothesis”. In this paper, to avoid ontological issues, I will use the term “simulation hypothesis” more broadly, to refer to either of these two possibilities, without distinguishing them.

In addition to the PCT, another common supposition in the physics community is that our physical universe can contain a subsystem whose dynamics implements an arbitrary Turing machine (TM) [41]. In other words, the default stance in much of the physics community is not to rule out the possibility that a given

¹Throughout this paper, I will often refer to a “Turing machine” as this kind of device, defined in terms classical physics, even though the physical universe is quantum mechanical. There is a large literature on the “quantum Church Turing thesis”, applicable to quantum computers. The distinction between these two type of TM and associated form of the PCT are tangential to the focus of this paper, and therefore I will not engage with those distinctions. See Section A

²The precise formulation of the PCT is the subject of much debate. Here, for reasons of expediency, I simply define it to mean the supposition described just above.

³See Section A for formal definitions of Turing machines and their properties.

hypothesized dynamical system can exist in our universe, simply because that system has the computational power of a TM.⁴

In contrast to the PCT, I will refer to this second supposition as the “reverse physical Church-Turing thesis (RPCT)”. I will also sometimes use the shorthand “CT thesis” to refer to the two suppositions at once.⁵

Note that the RPCT raises the possibility that whether or not our universe is a simulation, our universe could contain within it a (physical subsystem implementing a) computer which is running a simulation of some other universe. Intriguingly, it would seem that combining this implication of the RPCT with the PCT would establish that our universe can contain a computer that is implementing a simulation *of our universe itself*.

Whether or not our universe does actually obey the PCT and / or the RPCT, there are many questions that they raise;

1. What subsystems in our physical universe validate the RPCT (if any)?
2. As a practical matter, assuming there are subsystems of our universe that validate the RPCT, how much of their computational power is accessible to us humans?
3. What are the implications of computer science (CS) theory for the limitations of any physical universe obeying the full CT thesis?
4. In particular, what are the implications of CS theory for the simulation hypothesis?

The first two questions in this list have previously been considered in the physics literature, relatively extensively [48, 47, 16, 27, 61]. There has also been work in the physics community that addresses the third question in the list [66, 67, 11, 1, 35]. In addition there has been some work in the physics community that focuses on the fourth question, investigating how we might experimentally determine whether the simulation hypothesis holds [15, 12, 10].⁶

⁴Indeed, there has been some very important work that considers whether determining certain properties of physical systems might *require* computational power that is equal to that of a (universal) TM [27, 61]. Crucially, these properties are not experimentally measurable though, which is why they do not contradict the PCT.

⁵The reader unfamiliar with TMs and associated concepts like Universal Turing machines (UTMs), instantaneous descriptions (ID) of the state of a TM and its tapes, prefix-free encodings, etc., should consult Section A for appropriate background.

⁶In addition to this work in the physics community, there has been a lot of work in the philosophy of science literature on the simulation hypothesis. See Section E, for a critical summary of this work, in particular discussing the problematic aspects of the attempts in that literature to assign a probability to the simulation hypothesis.

However, with one partial exception discussed in Section F, no work has even been done on using CS theory to investigate the simulation hypothesis. This is despite the fact that, after all, the simulation hypothesis concerns *computer* simulation. So one would expect that the very powerful results of CS theory are extremely important to understanding the simulation hypothesis. Phrased differently, if we don't investigate what CS theory has to say about fourth question, it seems likely that deeply important aspects of the simulation hypothesis will be missed.

One natural way to address this hole in the literature starts by formalizing both the PCT and the RPCT in a very general (and fully rigorous) way. This would allow us both to formalize what it would mean for our physical universe *to be simulated*, and what it would mean for some subsystem in our universe *to simulate* a universe. Armed with this machinery, we could then apply the powerful CS-theoretic results concerning TMs, to investigate question four. I adopt this approach in this paper.

As an example of the benefits of this approach, I show below how Kleene's second recursion theorem from CS theory can be used to formally investigate this possibility, that our physical universe could in theory simulate *itself*. On the other hand though, below I will also use a different theorem from CS theory, Rice's theorem, to derive impossibility results concerning simulation of one universe by another. (The reader unfamiliar with one or the other of these two theorems should read Section B.)

It is important to emphasize that the “simulation hypothesis” as popularly conceived is that there is some being running a computer program in some physical universe, and all observables in our universe are just a set variables in that program. What the program does is evolve those variables, from one time to another. In particular, in this popular conception of the simulation hypothesis, *we* are just such variables in a computer program, as are all our experimental apparatuses and observations.

While motivated in terms of that popular version of the simulation hypothesis, the actual results in this paper are far more general. In fact, they don't even require that there be such a being who is running a program on a computer that may be simulating some universe..

Rather, the analysis in this paper starts from the fact that all physical experiments are interactions between an experimenter and physical reality. If we assume the Church-Turing thesis, the responses of physical reality during such an interaction can be modeled as the results of a computation on a TM. The Church-Turing thesis also means that the behavior of we the experimenters can be modeled that way. One might also assume that we ourselves can run a Turing complete computer. This paper is simply an investigation of the formal consequences if adopt one and / or the other of these two assumptions. Those formal consequences

would apply to our actual universe if those two assumptions hold, i.e., they are a new set of (im)possibility results concerning the laws of physics.

1.2 General comments

It is important to emphasize that in this paper I do *not* assume that the PCT applies to our actual physical universe, and / or that the RPCT does. Nor do I make any assumptions about whether the simulation hypothesis does or does not hold for our universe. These issues are ultimately decided by the Hamiltonian of our actual universe. However, I do not even restrict attention to those universes that obey the laws of physics of our universe, as we currently understand those laws. The focus is instead more general, considering what CS theory has to say about any universe whose dynamics could (in theory at least) be simulated on a (Turing complete) computer, and / or a universe that contains a computer that could simulate the dynamics of a universe. (See [15, 12, 10] for some work in the physics community that touches on the issue of how we might experimentally determine whether the simulation hypothesis holds.)

In particular, in this paper I am only concerned with establishing the *mathematically necessary* properties of a universe V that can (contain a computer that can) simulate some universe V' . I do not investigate the *physical* possibility of such a computer under the laws of physics in our universe, as we currently understand those laws. Nor do I consider the even more narrow question of whether there *is* such a computer in our universe.

It is also important to emphasize that I will frequently refer to the computer in a universe as a universal Turing machine (UTM). By this I do not mean that it is a physical system consisting of a set of infinite tapes with associated heads, etc. Rather I just mean that it has the properties of a (U)TM, i.e., that it is computationally universal. I then choose to discuss any such computational system as though it were implemented as a (U)TM. So for example, the universe could be a running laptop with a memory that can be extended dynamically by an arbitrary finite amount, for an arbitrary number of times. This is the standard approach in the literature for justifying the application of Turing machine theory to our physical universe. (See Section 2.6 below for a detailed example of how a subset of the specific universe occupied by us humans fits into this framework.)

Furthermore, to adopt the terminology of logic and model theory, there is no semantics considered in this paper, in the sense of T-schema, etc. So there are no structures, definable relations, etc.. This reflects the implicit viewpoint of the simulation hypothesis itself. Formally, just like the simulation hypothesis itself, my use of the the physical Church-Turing thesis can be viewed as coupling syntax with semantics, but I have no more explicit consideration of semantics. (See [66, 67, 11, 1] for work in the physics literature that couples syntax and semantics in a

similar way, and the discussion of ontic structural realism in Section E for related, informal discussion.) Just as in the simulation hypothesis, I make no distinction between universes that are “real” and those that are “only simulations”.

As a final comment, the analysis in this paper only considers *deterministic* TMs. The simulator and simulatee are formalized as such TMs. However, many scenarios in CS theory involve probabilistic TMs, e.g., the complexity class **BPP**, interactive proof systems, cryptography, etc. In addition, one might argue that to extend the analysis in this paper to a quantum mechanical universe, we would need to allow for probabilistic dynamics in the simulated universe, e.g., if one adopts a Copenhagen interpretation of quantum mechanics.

One can easily extend the formalism in this paper to handle such stochasticity in the simulated TM and / or the simulator’s TM. To do so we just need to suppose that the deterministic TMs have an auxiliary, read-only input tape they can use, which is filled with an infinite string of random bits in addition to their other inputs. (This is the standard trick in CS theory for defining TMs that *behave* IID randomly even though they are purely deterministic TMs. See [6].)

1.3 Contributions

The focus of this paper is on the CS theoretic aspects of the simulation hypothesis.

I begin Section 2 by presenting the mathematical framework I will use in this paper. The simulation hypothesis in particular, and both the PCT and RPCT more generally, all involve a pair of dynamical systems, e.g., the simulating universe and the simulated universe. Accordingly, I next formally define “simulate” in a way that can apply to any pair of dynamical systems, with no restrictions to the laws of our particular universe. After that I formally define the PCT in terms of the mathematical framework I will use in this paper, as well as the RPCT. These definitions allow me to provide the first formalization of the simulation hypothesis, as well as the first fully general formalization of the PCT, applicable to arbitrary universes, not just ours. These definitions also provide the first fully general distinction of the PCT from the RPCT. For those readers who have not previously considering the PCT and / or RPCT in detail, I end this section by giving some pointers to the literature on physical systems that are computationally universal. I also introduce a new example of such a scenario, grounded in purely classical physics, and without requiring ability manipulate and observe real numbers of infinite precision.

In the next section I explicitly prove that if a universe V' obeys the PCT, and a universe V obeys the RPCT, then V can simulate V' . I end that section by describing several semi-formal arguments based on this result that one might suppose disprove the possibility of self-simulation, i.e., which prove that we could not be simulations in a computer that we ourselves run.

In Section 4 I use Kleene’s second recursion theorem to address these semi-formal arguments against self-simulation.⁷ I do this by using that theorem to prove that in fact we *could* be simulations in a computer that we ourselves run. Specifically, I show that if a universe V obeys both the PCT and the RPCT, then it can simulate itself, according to the formal definition of “simulate” provided in Section 2.2. I call this the *self-simulation lemma*. I then describe several important formal features of the self-simulation lemma, and present an example of how self-simulation might arise with advanced versions of our current laptops. I end this section by describing how self-simulation is a far deeper connection between an entity and itself than arises in all the earlier versions of self-reference considered in the mathematics literature.

In the following section, Section 5, I present several mathematical properties of the number of iterations taken to simulate one’s dynamics a given time Δt into the future. Some of these properties involve the implications if we require that the time taken to simulate Δt into one’s own future does not decrease with Δt , for any specific pair of values of Δt . Other properties involve the time-complexity of self-simulation, i.e., how much longer than a time Δt it takes to simulate a universe’s evolution up to a time Δt in the future.

The next section, Section 6 starts with a discussion of some of the peculiar philosophical implications of the simulation lemma, and especially of the self-simulation lemma, for notions of identity. In particular, that section contains a discussion of the fact that you self-simulating does not just mean that you create some doppelganger of yourself, a clone of yourself, which has autonomy and starts to evolve differently from you once it has been created. Self-simulation does not mean something akin to your stepping into a variant of the Star Trek transporter which creates a copy of you at some other location while the original you still exists.

Rather than such cloning of yourself, self-simulation means that you run a program on a computer which implements the exact same dynamics as your *entire* universe, the universe that contains both you and your computer. So in particular, that universe being simulated in a program running in your computer N contains an instance of you who, in this simulation, is running a program on a computer N' that simulates your entire universe, and so in particular simulates an instance of you who, in this simulation-within-a-simulation, is running a program on a computer N'' that simulates your entire universe, and so in particular ... Crucially, under the PCT, all those instances of you *are* you; it is meaningless to ask which of those instances “is the real you”, with the others being “just a copy”.⁸

⁷This theorem is just called “the recursion theorem” in CS theory; see Section B for a summary.

⁸As discussed in Section 5, an instance of your universe that you are simulating would be “time-delayed”, in that as you observed that simulation unfold, after it has run for a certain amount,

This section ends with a discussion of the philosophical quirks that would arise if the program being used to (self-)simulate a universe is encrypted as it runs, as in homomorphic computing [4]. In such situation only the being with a special decryption key can understand the result of that computation. In particular, I raise the issue of what would happen in this situation if that being running the computation – running our universe – were to lose the decryption key, and so no longer could understand or observe any aspect of our universe.

Returning to mathematics, the simulation and self-simulation lemmas allow us to define the “simulation graph”. This is the directed graph where each node is a universe containing a computer, and there is an edge from one node to another if the (universe identified with the) first node can simulate the (universe identified with the) second node. In Section 7 I present a preliminary investigation of this graph.

Then in Section 8 I discuss some of the mathematical properties that arise in both simulation and self-simulation, in addition to those raised by consideration of the simulation graph. Specifically, I use Rice’s theorem to establish that many of the mathematical questions one might ask concerning simulation and self-simulation are undecidable.

Next in Section 9 I discuss some of the very many open mathematical issues involving the simulation framework that I have not considered in this paper.

Finally, I begin Section 10 with a discussion of the implications of the results of this paper for arguments in some of the earlier semi-formal work on the simulation hypothesis. After that I present quickly mention some of the ways that the paradigm implicitly considered in this paper involving the classical Church-Turing thesis might be extended to apply to quantum and / or relativistic universes.

2 Preliminaries

2.1 Notation

My notation is conventional. The set of all positive integers is \mathbb{N} , and the set of non-negative integers is $\mathbb{Z}^+ = \mathbb{N} \cup \{0\}$. I write $|X|$ for the cardinality of any set X . In addition, for any set X I write X^* for the set of all finite strings of elements of X . Note that so long as X is finite, X^* is countable. As an important example, \mathbb{B}^* is the set of all finite bit strings. Since that set can be bijectively mapped to \mathbb{N} , I will follow convention and treat finite bit strings as positive integers and vice-versa, with the bijection implicit.

you would notice that it is running more slowly than your universe. In particular, the simulation unfolding would be the same as at delayed version of you. Of course, there is no experimentally testable way that the simulated version of you could notice this.

Section A reviews Turing machine theory, how each separate TM can be given its own index value from \mathbb{N} , etc. As discussed there in detail, I write $T^m(x)$ for the (possibly partial) function given by running the TM with index m on input x until it halts (with the function undefined for input x if it does not halt for that input). So m ranges over all natural numbers \mathbb{N} . Often I assume, implicitly or otherwise, that certain quantities are in the form of prefix-encoded bit strings. In particular, I use some standard bijective encoding function of all tuples of bit strings into a single bit string, indicated using angle brackets, $\langle ., . \rangle, \langle ., ., . \rangle$, etc. I assume that this encoding is non-decreasing in the number of arguments, i.e., for any finite set of bit strings $\{b(1), b(2), \dots, b(m)\}$, the length of the bit string $\langle b(1), b(2), \dots, b(m-1) \rangle$ is not greater than that of $\langle b(1), b(2), \dots, b(m) \rangle$.

Other important definitions and notations are in Section A , e.g., of partial functions, and of computable functions (here always assumed to be total). I also review the definition of “computational universality” in that appendix. In addition, there I review the definitions of a universal TM (UTM), and a prefix-free TM (the implementation of TMs I will often be assume in this paper, implicitly or otherwise). I also define the instantaneous description (ID) of a Turing machine (TM) there. In the main body of this paper I will assume that the non-blank alphabet of the TM, Λ , is just \mathbb{B} . So as described in Section A, we can take the state of the tape to be a finite string in \mathbb{B}^* , even though strictly speaking the definition of TMs assumes infinitely long tapes.

Unfortunately (as happens all too often), there are some conflicts in the literature concerning terminology for TMs. The reader should always check Section A for the specific definitions used in this paper. Furthermore, even if the reader is well-versed in TMs and the associated notation, and even if they are familiar with the recursion theorem, they should still read Section B, since in this paper I will use a slight extension of the recursion theorem, called the “total recursion theorem”.

2.2 Framework for analyzing CS theory of the simulation hypothesis

To connect with the PCT and CS theory more generally, I will consider universes that can be understood as evolving in discrete time, and that contain a subsystem that we will view as a “computer”. Since I will want to take that computer to be computationally universal (and therefore having exactly the computational power of a UTM), I will assume that it initially has some arbitrarily large finite number of states, where that state space can be enlarged dynamically, as needed, by an arbitrary amount.

I will also mostly be interested in cases where the computer simulates the

evolution of the state of the universe external to the computer (the **environment** of the computer) and/or its own evolution. To simplify the analysis of such scenarios, I will assume the state space of the environment is finite (though arbitrarily large), to ensure that its state at a particular time can be appropriately encoded on the input of a UTM. (This assumption can be relaxed, at the cost of complicating the analysis.) This leads me to write the state space of a (**computational**) **universe** as

$$V = W \times N \tag{1}$$

where W is initially finite and cannot be extended dynamically, whereas N is countably infinite.

The elements of W and N are written as w and n , respectively. The elements of V are written generically as $v = (v_W, v_N)$ or (w, n) . In general, the elements of W as well as N can be indexed as multi-dimensional variables, e.g., as bit strings. However, I never need to make such indexing explicit in this paper. Furthermore, sometimes it will be convenient to give subscripts to some of these quantities, to indicate the time (which I take to be discrete). For example, I will sometimes write $v_t = (w_t, n_t)$ for the time- t state of the universe.

W is the set of possible states of the computational universe external to a computer that it contains. N is the set of possible states of that computer. I will parameterize N as $\underline{N} \times R$. R is a finite set that represents the internal variables of the computer. So for example, in a conventionally represented TM, R would be the state of the TM's head, the position(s) of its pointer(s) on the tape(s), etc. $\underline{N} = \mathbb{B}^*$ is instead the state of the tape(s) in the case that the computer is represented as a TM, or it is the state of the memory in the case that the computer is represented as a RAM machine, or is represented as an appropriately modified laptop computer.^{9,10} From now on to simplify the language, I will refer to the “tape” of the UTM, singular, even if I am representing the case of a multi-tape TM. I will write the elements of \underline{N} as \underline{n} .

I will always assume that the initial, $t = 0$ state of R is some special initialized value, r^\emptyset . So n_0 is fully specified by \underline{n}_0 , the initial state of the tape of the TM. In addition, except when explicitly stated otherwise, whenever I refer to the state of a computer for times $t > 0$, I will only be interested in the state of the tape at that time. Therefore except where explicitly stated otherwise, I will treat N and

⁹The reason for writing the countably infinite space \underline{N} as the set of finite bit strings is so that we can physically implement it as a physical system that initially has a finite state space, but whose state space can be expanded by an arbitrarily large finite amount an arbitrary number of times as the system evolves. See Section 2.6.

¹⁰Strictly speaking, in a TM the pointer variables can have any value in \mathbb{N} , since the TM's head can point to an arbitrary position on the (countably infinite) tape. This would seem to contradict the definition that the pointer's value lies in the finite space R . This technical concern can be addressed several ways. Perhaps most simply, we could just have R infinitely extendable.

\underline{N} as identical, with the associated individual states written as $n_t = \underline{n}_t$. (The major exception to this convention occurs in Section 5.3, in which I need to consider r_t for some times t that are after initialization but before the TM has halted.)

To simplify the analysis in this paper, I do not assume that the models of physical universes that I investigate capture those universes *in toto*, e.g., if the state spaces of those entire universes are actually uncountably infinite. Similarly, I will not assume that a simulation running in a computer inside a universe models the dynamics of an entire universe. Rather throughout this paper the expression “universe” should be understood as shorthand for “possibly coarse-grained subset of a universe”. However, I do assume that the universe obeys deterministic dynamics, be it over the original state space (assuming it is countable) or some coarse-grained version of that state space. As an example, in Section 2.6 I present a detailed example of what such a “coarse-grained subset of a universe” could be for our particular physical universe.

I write the evolution of the universe from an initial, time-0 state (w_0, n_0) to a time t state as a vector-valued **evolution function** g of the initial state of that universe,

$$g(t, w_0, n_0) = (w_t, n_t) \tag{2}$$

Note that the image of g is the Cartesian product $W \times N = W \times (\underline{N} \times R)$. I use the usual notation for components of vector-valued functions. So in particular, $g(t, w_0, n_0)_{\underline{N}}$ is the \underline{N} -component of $g(t, w_0, n_0)$.

Unless specifically stated otherwise, from now on I restrict attention to universes whose evolution function g is computable (see Section A). Note that for “computable” to even be a potential property of g means that I am implicitly assuming that the outputs of g are actually single bit strings (or equivalently, single counting numbers) that encode a pair of bit strings, as in Eq. (2).

As shorthand, below I will often “propagate” accents on symbols concerning computational universes. For example, when I say below that “ V simulates V' ”, I will implicitly assume $V = W \times N$, and that “ $V' = W' \times N'$ ”. All universes, with whatever accents, are assumed to be defined for the set of times $\mathbb{Z}^+ = \{0, 1, \dots\}$.

2.3 What it means for one universe to simulate another

The term “simulation” has not been given a formal definition in any of the previous literature on the simulation hypothesis. One of the effects of this lack of formal rigor has been to prevent the exploitation of CS theory to investigate the simulation hypothesis. Another effect is to obscure some important distinctions between different types of simulation. (These are introduced below.) More directly though, this lack of formal rigor encourages confusion when one tries to

exploit CS theory to understand the simulation hypothesis. For example, “simulation” (and the associated term “bisimulation”) already has a formal definition in the CS theory of state transition systems [71]. However, this definition from CS theory does not describe what “simulation” is loosely understood to mean in the context of the simulation hypothesis. In this subsection I fill this hole in the literature, providing a fully formal definition of “simulation” as the term is used in the simulation hypothesis.

Intuitively, the idea is that V “simulates” V' if for all initial states v'_0 of V , and all amounts of time $\Delta t'$ into the future of V' that we might want to simulate the dynamics of V' , there is some associated initial state of N and of W , together with a time into the future of the simulating universe, Δt , such that the state of the computer N at that future time Δt gives us the desired state of V' at its future time $\Delta t'$. That triple of associated initial state of N and of W and a time Δt are functions of $v'_0, \Delta t'$.

Writing v'_0 as (w'_0, n'_0) , this requirement can be formalized as follows:

Definition 1. A universe $V = W \times N$ with evolution function g **simulates** the evolution of a universe $V' = W' \times N'$ with evolution function g' iff there exist three functions

$$\begin{aligned}\mathcal{T}(\Delta t', w'_0, n'_0) &\in \mathbb{N} \\ \mathcal{W}(\Delta t', w'_0, n'_0) &\in W \\ \mathcal{N}(\Delta t', w'_0, n'_0) &\in N\end{aligned}$$

such that for all $\Delta t' \in \mathbb{N}, w'_0 \in W', n'_0 \in N', t \geq \tau$,

$$g(t, \omega, \eta)_{\underline{N}} = \langle g'(\Delta t', w'_0, n'_0) \rangle \quad (3)$$

where as shorthand, $\tau := \mathcal{T}(\Delta t', w'_0, n'_0), \omega := \mathcal{W}(\Delta t', w'_0, n'_0), \eta := \mathcal{N}(\Delta t', w'_0, n'_0)$.

Note that the LHS of Eq. (3) is the second of the two components of the vector-valued function g , while the RHS is an encoding of both components of g' into a single variable. Note also the requirement in Definition 1 that Eq. (3) hold for all $t \geq \tau$, and so the state of the simulating computer N does not change after it completes its simulation of the future state of V' . This just means that I require that the simulating computer halts when it completes its simulation.

To simplify the analysis, I also require that the ID of the computer N be replaced *in toto* (i.e., uniquely, with all other variables fixed to some predefined, post-halting values) by the output of its simulation program. So for example, if N is a multi-tape prefix TM, this means that when N halts with the result of its simulation on its output tape, all the other tapes — the intermediate work tapes and the input tape — have been re-initialized to be all blanks.

I will sometimes say that V “can simulate” V' rather than say that it “simulates” V' . I also sometimes say that V simulates V' **for simulation functions** $\mathcal{T}(\cdot, \cdot, \cdot)$, $\mathcal{W}(\cdot, \cdot, \cdot)$, and $\mathcal{N}(\cdot, \cdot, \cdot)$ if Definition 1 holds for that particular triple of functions. In addition I say that V **computably** simulates V' if it simulates V' , and in addition the three functions $\mathcal{T}(\Delta t', w'_0, n'_0)$, $\mathcal{W}(\Delta t', w'_0, n'_0)$, $\mathcal{N}(\Delta t', w'_0, n'_0)$ are all computable. Unless specified otherwise, whenever I refer to “simulation” in this paper I implicitly assume it is computable.

Note that Definition 1 does not require that the simulating computer N calculates the future state of the universe V' by itself, independently of the initial state of the environment W outside of N , i.e., independently of the value of w_0 . (Formally, the function g on the LHS of Eq. (3) is not independent of ω , the initial state of W .) The reason for this flexibility is to allow the computer N to retrieve the specific information it needs to perform its simulation of the dynamics of the specific state v'_0 from those super-aliens who are running that computer N , and who exist in the environment of N, W .

In addition, since there are no restrictions in Definition 1 on the functions \mathcal{T} , \mathcal{W} and \mathcal{N} , the second and third arguments of g in Eq. (3) can vary arbitrarily. This means there are no *a priori* restrictions on how the full universe V can evolve. As a result, Eq. (3) allows any putative beings running the simulation computer N to intervene on the dynamics of that computer after it has started running, e.g., by overwriting the simulation program being run, in a completely arbitrary fashion. They can even “pull the plug early” on that computer, before it finishes its computation.

Note also that Definition 1 allows the initial state of the simulating computer, n_0 , to vary if we vary the time into the future, $\Delta t'$, that the simulating computer is calculating. Indeed, the definition allows n_0 to vary for different $\Delta t'$ even if N is simulating the future state of V' for all those values of $\Delta t'$ evolving from the same initial state $v'_0 \in V'$. Concretely, I, a super-alien, might use one program to compute the state v' of a universe V' at the time $\Delta t'_1$ into the future of that universe, and use a different program to compute the state of V' at a time $\Delta t'_2$ into the future of that universe. (However, this flexibility is circumscribed if we restrict attention to “time-consistent” universes, as discussed below in Section 5.3.)

I will informally use the term **cosmological universe** to refer to an entire physical universe obeying one set of laws of physics throughout, with one set of shared initial conditions, etc.¹¹ In general, V and V' might be portions of different cosmological universes, obeying different laws of physics. They could also be sub-regions of the same cosmological universe though (and therefore obey the same laws of physics). One way this could occur is if there are physically distinct re-

¹¹For simplicity, I sidestep the issue of multiverses with different physical constants but otherwise identical physical laws, e.g., arising from a shared inflation epoch.

gions of same universe. (See Section 2.6.) This case of multiple computational universes in the same cosmological universe will be important in the discussion of the simulation graph below in Section 7.

Definition 1 allows both w_0 and n_0 to vary if we change w'_0 , even if n'_0 is fixed. Often we are interested in a more restrictive notion of simulation, where for a fixed n'_0 , changing w'_0 does not change n_0 , even though changing w'_0 is allowed to change w_0 in general. Intuitively, this restrictive form of simulation corresponds to the case where the simulation program n_0 is fixed by n'_0 , and that simulating computer “reads in” the precise initial state of the entire universe it is simulating at some iteration $t > 0$ after it starts running, by appropriately interacting with the state of its own external universe at that time, w_t . As an example, this form of simulation would be met if N were a UTM, so that n_0 specifies the precise TM that N is implementing, while w'_0 is extra information that is “read in” at a later time t by that TM n_0 , from the associated state w_t .

We can formalize this special case of simulation with a simple extension of Definition 1:

Definition 2. Suppose that V simulates the evolution of V' for three functions $\mathcal{T}, \mathcal{W}, \mathcal{N}$. Then V **freely simulates** V' for $n'_0, \Delta t'$, if $\mathcal{N}(\Delta t', w'_0, n'_0)$ is independent of w'_0 .

Note that this definition allows the evolution of n' to depend on w'_0 , i.e., it does not restrict the interactions between the computer being simulated and its external world. Rather the restriction only affects initialization details concerning the universe doing the simulating.

Finally, the definitions above only stipulate that the simulating computer N eventually outputs the future state of V' at one specific time, $v_{\Delta t'}$. While sufficient for this paper, it is worth pointing out that we could extend these definitions to instead have N output an entire trajectory of L such future states for some finite number L . To do this we would replace the first arguments of \mathcal{T}, \mathcal{W} and \mathcal{N} with a vector $\vec{\Delta t'} \in \mathbb{N}^L$. We would also extend the definition of the evolution function, to have

$$g'(\vec{\Delta t'}, w'_0, n'_0) \tag{4}$$

be the states of V' at an entire sequence of times $\vec{\Delta t'}$ which arise if it starts at time 0 with the state (w'_0, n'_0) . Finally, we would modify the condition in Eq. (3) to say that V **trajectory-simulates** V' if

For all $\vec{\Delta t'} \in \mathbb{N}^L, w'_0 \in W', n'_0 \in N', t \geq \tau$,

$$g(t, \omega, \eta)_{\underline{N}} = \langle g'(\vec{\Delta t'}, w'_0, n'_0) \rangle \tag{5}$$

where $\tau := \mathcal{T}(\vec{\Delta t'}, w'_0, n'_0)$, $\omega := \mathcal{W}(\vec{\Delta t'}, w'_0, n'_0)$, $\eta := \mathcal{N}(\vec{\Delta t'}, w'_0, n'_0)$.

For simplicity in this paper I do not consider trajectory-simulation, focusing on single-moment simulation. However, all the results below still apply for trajectory-simulation, with minor terminological changes.

To minimize notation, in the sequel I will implicitly choose units of physical time so that under the dynamics of any universe I am considering, the physical computer in that universe takes one unit of (physical) time to run one iteration of the computational machine it is implementing. (For example, if that computational machine is a UTM, then each iteration of the UTM takes one unit of physical time.)

2.4 The Physical Church-Turing Thesis

Even restricting consideration to computers that can be described using classical physics, there are many different semi-formal definitions of the PCT in the physics literature [56, 55, 25, 26, 3]. If we extend consideration to include quantum computers [52], there are even more definitions [7, 51].

Whether in fact our particular cosmological universe obeys the PCT, be it the classical or quantum PCT, has been subject to endless argument [2, 54, 7, 55, 53]. In particular, some researchers have designed purely theoretical, contrived physical systems that are uncomputable in some sense or other [58, 27, 61] (see also [21]). This work has resulted in attempts to define the PCT to exclude the case of physical systems whose future is uncomputable but which cannot be constructed by us humans in a finite amount of time. This amounts to tightening the PCT to concern not just what systems can be simulated, but rather what systems can be constructed and then simulated.

In any case, as mentioned above, for the purposes of this paper, it does not matter whether some particular form of the PCT applies to *our* specific universe. What matters is the CS theory implications of universes simulating other universes, and in particular the implications if the PCT holds for such universes. Accordingly, for current purposes, I make the following (fully formal) definition:

Definition 3. *The Physical Church-Turing thesis (PCT) holds for universe V iff the evolution function $g(\cdot, \cdot, \cdot)$ of V is computable.*

Definition 3 is the first fully general definition of the PCT, even applicable to universes whose laws of physics differ from ours.

By Definition 3, if the PCT holds for V, there must be a UTM that (halts and) outputs the vector value of $g(\cdot, \cdot, \cdot)$, for all values of g's arguments. More precisely, there must be such a UTM that outputs the string $\langle g(\cdot, \cdot, \cdot) \rangle$ for all values of g's arguments if it receives the encoded version of those arguments as its input.

The analysis below implicitly assumes that all of the universes being discussed obey the PCT. This means that the analysis below does not apply to any universes that can contain computers capable of super-Turing computation [3]. In particular, the analysis would not apply to any such universes that are simulating our universe.

Much of the earlier literature on the “physical Church-Turing thesis” accords a prominent role to humans, and their abilities (or lack thereof), e.g., in arbitrarily configuring the initial state of physical systems, or in observing their subsequent physical state. There is no such role in Definition 3. All the PCT means in this paper is that evolving the computational universe V does not require a computational machine more powerful than a TM. In fact, it could be that that evolution can be calculated on a machine that is strictly weaker than a TM, e.g., a finite automaton, and it would still obey Definition 3. Furthermore, Definition 3 could still apply if V is a only sub-region of some cosmological universe, and machines more powerful than a TM are required to calculate the evolution of some other sub-region of that cosmological universe, different from V .

For these kinds of reasons, some readers might argue that Definition 3 doesn’t exactly capture any of the various properties that have been referred to as the “physical Church Turing thesis” in the literature. In some senses it more like one of the related concepts inspired by modern physics, e.g., some forms of ontic structural realism [30, 45, 44, 5] or the level IV multiverse [66, 67]. But for current purposes, we can ignore these terminological distinctions.

Finally, note that the set of spaces $W \times N$ is countably infinite, if we restrict attention to any and all finite W . Therefore the set of universes defined by the specification of such a space, together with an evolution function that obeys the PCT, is also countably infinite. Moreover, many of the considerations of the “simulation hypothesis” in the literature implicitly assume such a universe and evolution function. However, it is impossible to assign a uniform probability distribution to the set \mathbb{N} . This establishes the claim made in Section E, that it is impossible to assign a uniform probability distribution to the kind of universes often considered in the literature on the simulation hypothesis. This contradicts an assumption made in much of the philosophy literature on the simulation hypothesis.

2.5 The Reverse Physical Church-Turing Thesis

Loosely speaking, the PCT says that the dynamics of any universe that we are considering can be computed on a UTM. One can “reverse” the requirement that a universe obey the PCT, to produce the requirement that a universe contains a UTM in it. If it obeys such a reversed PCT, a universe could implement all TMs.

One might imagine that the reversed PCT could be formalized in a way that among other things requires that a universe’s computer N evolves independently

of the rest of that universe. However, in general that is not possible — we need to allow the beings running the computer to provide it information in order to characterize its computational power, which means that that computer N does not evolve autonomously as a UTM. So we cannot impose this simple version of a reverse PCT. On the other hand, we can require that N *effectively* implements a UTM. This is done as follows:

Definition 4. *The reverse physical Church-Turing (RPCT) holds for universe $V = W \times N$ with evolution function g iff there exist three functions*

$$\begin{aligned}\widehat{\mathcal{T}}(k, y) &\in \mathbb{N} \\ \widehat{\mathcal{W}}(k, y) &\in W \\ \widehat{\mathcal{N}}(k, y) &\in N\end{aligned}$$

such that

1. those three functions are well-defined for all pairs of TM index k and $y \in \mathbb{B}^*$ such that T^k halts on input string y ;
2. for all such (k, y) ,

$$g(\widehat{\mathcal{T}}(k, y), \widehat{\mathcal{W}}(k, y), \widehat{\mathcal{N}}(k, y))_N = T^k(y)$$

I say that three functions $\widehat{\mathcal{N}}, \widehat{\mathcal{T}}, \widehat{\mathcal{W}}$ all taking $\mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ **have the RPCT properties for g** if they obey the properties listed in Definition 4.

Note that there is no concern in Definition 4 for the form of the functions $\widehat{\mathcal{N}}, \widehat{\mathcal{T}}, \widehat{\mathcal{W}}$ for any argument (k, y) where T^k does not halt on input string y . They may even be undefined for such an argument. Moreover, *a priori*, the three functions in Definition 4 could be uncomputable for some (k, y) pair even when T^k does halt on input string y . Since no assumption is made in this paper that any universe / alien beings / computer actually *constructs* those functions, that wouldn't matter.

Nonetheless, it is an important special case when those three functions are partial computable, in a specific way. Abusing terminology, I say that the **computable** RPCT holds for V iff all three functions $\widehat{\mathcal{T}}(k, y), \widehat{\mathcal{W}}(k, y), \widehat{\mathcal{N}}(k, y)$ are computable over the set of all pairs of arguments (k, y) such that $T^k(y)$ halts. For simplicity, unless specified otherwise, throughout this paper I will assume that whenever the RPCT holds, in fact the computable RPCT holds.

Broadly speaking, the RPCT says that the system N operates like a UTM for all pairs (k, y) such that $T^k(y)$ is defined, where k and/or y may be encoded in some degrees of freedom in w_0 rather than directly in n_0 . That freedom to have k and

/ or y specified in the environment of the computer N allows that computer to do things like observe its environment to retrieve the input string for a computation from its environment before running that computation. (Note that this freedom is just like the property of free simulation, defined above.)

Exploiting this freedom, I say that the special case of **pristine** RPCT holds if for all k, y , the RPCT holds with

$$\widehat{N}(k, y) = k \tag{6}$$

$$\widehat{W}(k, y) = y \tag{7}$$

Eqs. (6) and (7) mean that the physical system implementing the UTM is initialized with the precise TM that it is supposed to implement, but not the actual data that TM will be run on. In general, that data (the value of y) would be transferred into the computer N at a subsequent iteration, after this initialization of the UTM, but before the UTM starts to run. As an example, this is what would occur if we were to initialize a laptop with a universe-simulating program, with the precise data that program is to run on fed into the laptop before starting the simulation program. (See Section 2.6.) In this paper, when assuming the RPCT I will not also implicitly assume that the *pristine* RPCT holds; it only holds if I explicitly say so.

Just as the PCT as defined in Definition 3 has no role for humans, the RPCT has no role for them. In particular, Definition 4 does *not* say that humans could configure the initial state of V to implement the dynamics of any desired TM. It simply says that there is some such initial state of V that could implement that dynamics.

The RPCT is accepted by many researchers, implicitly or otherwise. (Indeed, it is commonly confounded with the PCT.) For example, Scott Aaronson wrote in a blog post on Feb. 8, 2024 that “My personal belief is that... ‘yes,’ in some sense (which needs to be spelled out carefully!) the physical universe really is a giant Turing machine.” See also [16, 47, 58] and related literature for more formal considerations of (what amounts to) the RPCT.

Despite the popularity of the RPCT among researchers, even if the PCT holds in our specific universe, that does not mean that the RPCT has to hold as well. In particular, cosmological considerations could prevent it from holding [43]. So the results below specific to universes that obey the PCT and / or universes that obey the RPCT might not apply to our specific universe. However, approximations of the analysis might apply even if the PCT and/or RPCT don’t hold exactly in our universe, depending on how precisely they are violated.

Even without worrying about the laws of physics in our universe, one might suppose that the RPCT is logically impossible, and so cannot hold in *any* universe. After all, the RPCT requires that the set X of physical variables of a universe

decomposes as $X = W \times N$ where N is a physical system that implements a UTM. In other words, a proper subset of the spatial degrees of freedom of the universe constitutes a physical structure N capable of implementing any TM. But the PCT thesis supposes that such an N would be able to simulate the dynamics of *all* of X . So this N would have to be able to simulate *itself*, at the same time as it is also simulating the entire rest of X , outside of N . This would seem to imply a contradiction, that N is more computationally powerful than N is.

If this argument were valid, the RPCT would be impossible. And so in particular, no matter what the actual laws of physics, it would not be possible for us to be part of a simulation by a computer, if that computer were itself contained in our cosmological universe. In other words, if this argument were valid, it would be a proof that the simulation hypothesis must be wrong.

One might be suspicious of this argument though, since it is quite similar to the arguments that were made in the last century that no physical system can make an extra copy of itself without destroying itself. (These arguments that copying required destruction of the original were used to make the case that the common definition of life involving replication must be wrong, or at least deficient.) Responding to these arguments, Von Neumann designed his “universal constructor” in a cellular automaton setting. This demonstrated explicitly that it *is* possible for a system to copy itself without harming itself in the process.¹²

2.6 Sketch of how Newtonian universes might obey the PCT and RPCT

This section discusses how both the RPCT and PCT could be obeyed in our physical universe. I first mention the literature on this issue. Most of that literature presumes that the physical experimentalist can access real value variables to infinite precision. So I present a new example of a system that obeys both the PCT and the RPCT, which only involves Newtonian mechanics.

There is a very large literature showing how to design Newtonian physical systems that are computationally universal [48, 16, 58]. Those establish RPCT, and since Newtonian mechanics is computable, also the PCT, since we’re treating the entire universe, outside and inside of Y , as Newtonian.

Typically such examples from the literature rely on the ability of the person using a physical system as a computer to manipulate and observe real-valued physical variables in that system to arbitrarily fine precision. One might object that this

¹²It is interesting to note that Von Neumann’s proof of this property of his universal constructor is essentially identical to Kleene’s earlier proof of his second recursion theorem — and that it is hard to imagine that Von Neumann did not know of Kleene’s earlier result, despite not citing it in his work.

is beyond the capabilities of humanity, both now and in the foreseeable future. However, it is straight-forward to design a Newtonian system that is computationally universal but doesn't require the ability to access physical variables to arbitrary resolution. The rest of this subsection demonstrates one such design.

Fix some finite set of point masses evolving through a perfect vacuum under mutual gravitational attraction. Suppose there no other forces coupling them after the initial time $t = 0$. So in particular, after $t = 0$ there are no interactions between those point masses and the universe external to them. Let Γ be the phase space of that set of point masses.

Identify the initial IDs of a UTM at time 0 with the set $\mathcal{S}(0)$ of points in a countably infinite grid over Γ . Assume as well that at all of those grid points, at least one of the point masses has nonzero momentum. Write $\gamma_y(t)$ for the phase space position at time $t > 0$ of the system if it was at grid point $y \in \mathcal{S}(0)$ at time 0. Write the set of all phase space positions given by evolving the positions in $\mathcal{S}(0)$ forward from time 0 to time t as $\mathcal{S}(t)$.

Define the infinite set of discrete times $\{t_i\} = \mathbb{Z}^+$. Also, choose the grid points $\mathcal{S}(0)$ so that for any two distinct time-0 grid points, $y, y' \neq y \in \mathcal{S}(0)$, the associated curves they trace out as they evolve through Γ , $\{\gamma_y(t) : t > 0\}$, and $\{\gamma_{y'}(t) : t > 0\}$, never cross one another. (For example, one could do this by having different momentum vectors of the centers of mass of all the points in $\mathcal{S}(0)$.) This ensures that there is no overlap between any two sets $\mathcal{S}(t_i)$ and $\mathcal{S}(t_j)$ for $i \neq j$.

Now simply identify each of the initial grid points $y \in \mathcal{S}(0)$ with a different initial ID of some fixed UTM, and the subsequent points it goes through at the times t_i , $\gamma_y(t_i)$, with the subsequent IDs of the UTM that started with the ID y . Since $\mathcal{S}(t_i) \cap \mathcal{S}(t_j) = \emptyset$ for $i \neq j$, we are guaranteed that there is no ambiguity of what ID should be identified with a given position in Γ .

Despite this though, in general there will be some time $t_i > 0$ after which there is a growing set of grid points that are assigned the same ID as some other grid point. (This is due to merging of trajectories through the set of all IDs of the UTM.) There will also be times after which there will some IDs that do not correspond to any of the grid points. (This is due to the fact that some IDs of the UTM cannot arise after a certain number of iterations, no matter what input tape the UTM starts with.) Moreover, in general the same ID will be assigned to multiple points in Γ . Finally, note that if $\gamma_y(t_i)$ is assigned to an ID which is a halt state, then for all $j > i$, $\gamma_y(t_j)$ is assigned that same halting ID, $\gamma_y(t_i)$.

In any case, this system obeys the RPCT by construction, if we identify N with the UTM whose dynamics is given by the trajectories $\{\gamma_y(t_i) : y \in \mathcal{S}(0), t_i \in \mathbb{Z}^+\}$. More formally, for all k, y as defined in Definition 4, write $\langle k, y \rangle$ for the initial ID of our UTM (i.e., the initial contents of its input tape) that will cause it to compute $T^k(y)$ and then halt. It suffices to establish the RPCT by choosing

- $\widehat{\mathcal{T}}(k, y)$ is the halting time for our UTM to compute $T^k(y)$ when initialized to the ID $\langle k, y \rangle$;
- $\widehat{W}(k, y)$ is arbitrary;
- $\widehat{N}(k, y)$ is $\langle k, y \rangle$.

This establishes the RPCT, as claimed. Moreover, since Newtonian mechanics is computable and we're treating the entire universe, both inside and outside of Γ , as Newtonian, the PCT holds automatically, as claimed.

See Section D for discussion of some subtleties of this scheme for implementing a universe V that obeys the full CT using Newtonian mechanics.

3 The simulation lemma

In this section I first derive the simulation lemma. I then discuss some of its features, and in particular why it might seem to imply that self-simulation is impossible.

3.1 Proof of the simulation lemma

In this subsection I first state the simulation lemma formally and then prove it.

Lemma 1. *If a universe $V = W \times N$ obeys the RPCT, then it can simulate any universe V' that obeys the PCT.*

Proof. By Definition 3, since V' obeys the PCT, its evolution function g' is computable. Therefore there is an index $k \in \mathbb{N}$ such that

$$T^k(\langle \Delta t', w', n' \rangle) = \langle g'(\Delta t', w', n') \rangle$$

for all $\Delta t' \in \mathbb{N}, w' \in W', n' \in N'$. Therefore by Definition 4, since V obeys the RPCT, there are three functions $\widehat{\mathcal{T}}, \widehat{W}, \widehat{N}$ such that for any triple $(\Delta t', w', n')$ and associated finite string $y := \langle \Delta t', w', n' \rangle$,

$$\begin{aligned} g\left(\widehat{\mathcal{T}}(k, y), \widehat{W}(k, y), \widehat{N}(k, y)\right)_N &= T^k(y) \\ &= \langle g'(\Delta t', w', n') \rangle \end{aligned} \tag{8}$$

Next define

$$\mathcal{T}(\Delta t', w', n') := \widehat{\mathcal{T}}(k, \langle \Delta t', w', n' \rangle) \tag{9}$$

$$W(\Delta t', w', n') := \widehat{W}(k, \langle \Delta t', w', n' \rangle) \tag{10}$$

$$N(\Delta t', w', n') := \widehat{N}(k, \langle \Delta t', w', n' \rangle) \tag{11}$$

Plugging these three definitions into Eq. (8) and then comparing to Definition 1 completes the proof. \square

I will refer to Theorem 1 as the **simulation lemma**.

3.2 Why the simulation lemma might seem to preclude self-simulation

The simulation lemma tells us that if there are some super-sophisticated aliens in a universe V in which the RPCT holds, and if our universe obeys the PCT, then it's possible that we are simulations in a computer that the aliens are running. (Of course, we would never know it if they were.) On the other hand, suppose that as many have argued the PCT does *not* hold in our universe, because its evolution cannot be evaluated on a TM. In such a case, even if the RPCT holds in the universe that the aliens inhabit, there is no guarantee that we may be in a simulation that they are running. This formalizes the intuitive idea that our universe has to be "sufficiently simple, computationally speaking" in order for us to be simulations in a computer of some super-sophisticated aliens.

Note though that even if the conditions in the simulation lemma hold, that lemma doesn't in some sense specify what argument $(\Delta t', w'_0, n'_0)$ to simulate. In particular, it does not tell us what value n'_0 the computer N' that N is simulating would need to start with in order for that simulation to give the dynamics of V itself. To understand the implications of this, suppose that: i) the RPCT holds for our universe, so we have a computer N we can run to simulate the evolution of any universe that obeys the PCT; ii) our universe itself obeys the PCT. One might suppose that when these two conditions are met, Theorem 1 implies that we can run a simulation of our own universe, including ourselves. However, the simulation lemma provides no means for us to initialize our computer N in order to run a simulation of ourselves. It provides no way to have a universe V determine how to simulate *itself*.

In fact, if we take $V = V'$ in the proof of Theorem 1, then Eq. (11) becomes a fixed point equation:

$$n' := \widehat{N}(k, \langle \Delta t', w', n' \rangle) \quad (12)$$

We have no guarantee that the solution n' to this fixed point equation is computable. In fact, *a priori*, one might suspect that it is possible for there not to be any solution to Eq. (12) whatsoever, computable or otherwise. In light of Theorem 1, if that were the case, it would mean that the conditions for the simulation lemma cannot be met, i.e., that it is not possible for a universe V to obey both the PCT and the RPCT.

An associated concern is that Theorem 1 only establishes the possibility of V simulating V' . It does not establish the possibility of *free* simulation. So even if we can establish that there is a solution to the fixed-point equation Eq. (12), there might only be one, i.e., there might only be a very specific initial state of our own universe that we can simulate.

In fact, if a computer were to simulate the evolution of itself up to a certain time in the future, that means in particular that it would simulate a “copy” of itself running a computer which is simulating of a copy of itself running a computer which is simulating, etc., *ad infinitum*. In other words, one might expect that self-simulation would require an infinite regress of computers within simulations of computers. That in turn would suggest that the computer could never complete such a simulation of itself in finite time (assuming that the computer operates at a finite physical speed).

Finally, none of the analysis above establishes that it is even logically possible for a universe to obey both the PCT and the RPCT. It may be mathematically impossible to meet the conditions for Theorem 2.

In the next section I prove that these concerns do not in fact prevent a universe from simulating itself. In fact, not only is there a solution to Eq. (12), and not only is there an explicit algorithm to construct that solution, but that solution is computable, i.e., the algorithm that computes it is guaranteed to halt if implemented on a Turing machine. This means that it *is* possible for a universe to obey both the PCT and the RPCT. More interestingly, it means that we can construct a computer such that for any given finite time interval Δt and initial state of our universe outside of that computer, we can load a program onto that computer which is guaranteed to halt in finite time after having simulated the full state of our universe at the time Δt into the future — including in particular the state of that computer itself at that time. It also means that we could be inside a simulation being run on a computer that we ourselves are running (supposing our universe obeys both the PCT and RPCT of course).

4 The self-simulation lemma

In this section I begin by proving that despite the arguments in Section 3.2, in fact self-simulation *is* possible. The key is to have a time delay between the future moment for which the state of the universe is being simulated on the one hand, and on the other hand, the moment when the computer’s simulation of that future state completes.

After presenting that proof, I discuss certain important features of self-simulation. I end this section by discussing how the possibility of self-simulation differs from various forms of “self-reference” considered in the literature.

4.1 Proof of the self-simulation lemma

The proof of the simulation lemma just relies on elementary properties of TMs, and the assumptions that the PCT and RPCT both are obeyed. We need more than that to establish the possibility of free self-simulation. Specifically, we need to also use the total recursion theorem, and we need to strengthen the assumption that the RPCT holds into assuming that the pristine RPCT holds.

Lemma 2. *If both the PCT and the pristine RPCT hold for a universe $V = W \times N$, then for all Δt there exists $n_0 \in N$ such that V freely simulates V for $n_0, \Delta t$.*

Proof. Since the PCT holds for V , its evolution function $g(\cdot, \cdot, \cdot)$ is computable (not just partial computable). Therefore if we fix Δt , the first argument of $g(\cdot, \cdot, \cdot)$, the resultant dependence on the other two arguments is computable. If we now invoke the total recursion theorem (discussed in Section B), we see that there exists n^* such that

$$T^{n^*}(w_0) = g(\Delta t, w_0, n^*) \quad (13)$$

for all $w_0 \in W$, where both $g(\Delta t, w_0, n^*)$ and $T^{n^*}(w_0)$ halt for all inputs w_0 . Note that the TM index n^* will depend on both g and Δt in general.

Now apply our assumption that the pristine RPCT holds for V to the LHS of Eq. (13) and then plug in the RHS of that equation. This shows that there must exist a computable function $\widehat{\mathcal{T}}$ such that for all $w_0 \in W$,

$$g\left(\widehat{\mathcal{T}}(n^*, w_0), w_0, n^*\right)_N = T^{n^*}(w_0) \quad (14)$$

$$= g(\Delta t, w_0, n^*) \quad (15)$$

Finally, if we now define

$$\mathcal{T}(\Delta t', w'_0, n'_0) := \widehat{\mathcal{T}}(n'_0, w_0) \quad (16)$$

$$\mathcal{W}(\Delta t', w'_0, n'_0) = w'_0 \quad (17)$$

$$\mathcal{N}(\Delta t', w'_0, n'_0) = n'_0 \quad (18)$$

for all $\Delta t', w'_0, n'_0$, then Eq. (15) can be re-expressed as

$$g(\mathcal{T}(\Delta t', w'_0, n^*), \mathcal{W}(\Delta t', w'_0, n^*), \mathcal{N}(\Delta t', w'_0, n^*)) = g(\Delta t, w_0, n^*) \quad (19)$$

Plugging this into Definition 1 completes the proof that V simulates the evolution of V for $n_0 = n^*, \Delta t$. Since Eq. (15) holds for all w_0 , while n_0 is fixed to n^* , we see that in fact V freely simulates the evolution of V for $n^*, \Delta t$. \square

I will refer to Theorem 2 as the **self-simulation lemma**. It says that for any fixed Δt , there is an associated initial state of the computer such that for any initial state of the rest of the universe, w_0 , that computer is guaranteed to halt and to output the state of the entire universe at time Δt .

4.2 Important features of the self-simulation lemma

Recall the convention that “simulate” implicitly means “computably simulate”. So the self-simulation lemma not only says that there is an initial state of the computer, n_0 , for which the computer simulates the entire universe including itself. It says that n_0 is a computable function. Specifically, that value n_0 is the solution to Eq. (13), and so implicitly depends on the combination of the time into the future that we want to simulate, Δt , and the evolution function g (which is encoded as the index of a TM).

By definition, that means there is a halting program that constructs that n_0 . So we can run that program (on any UTM we like, whether a physical system in V or not) and be assured that it will eventually finish and tell us what *other* program n_0 to load into our computer N in order to simulate the evolution of our universe.

The evolution function g arising in the self-simulation lemma can be encoded as an integer (since it is a computable function, by hypothesis). Moreover, there is an implicit function given by the self-simulation lemma that takes the combination of Δt and g to the initial state of the self-simulating computer. (See comment just below Eq. (13) in the proof of Theorem 2.) As a notational shorthand, I will write that implicit function as $\mathcal{S} : \mathbb{N} \times \mathbb{N} \rightarrow N$. (\mathcal{S} stands for “self-simulation map”). As discussed above, \mathcal{S} is computable. In general, I will leave the second argument of \mathcal{S} implicit, and just write the TM whose existence is ensured by the self-simulation lemma as $\mathcal{S}(\Delta t)$. So the output of the self-simulating computer when predicting the state of its own universe at time Δt when the initial state of its environment is w_0 is $g(\Delta t, w_0, \mathcal{S}(\Delta t))$.

Example 1. Suppose you have a laptop which has a memory that can be extended dynamically by an arbitrary finite amount, an arbitrary number of times. You’ve got some program n_0 that was already loaded into the laptop at iteration 0. The first thing you then do is load into the input of that program (an encoding of) the current state of your environment, i.e., of the rest of the universe external to your laptop, which is w_0 . This changes n_0 to some new value, n_1 .

After this you physically isolate your laptop, from the rest of the universe, and start running it. The function $g(\Delta t, w_0, n_0)$ gives the joint state of your laptop and the universe external to it at the time Δt into the future. Note that in particular you, the being who is running your laptop, is part of the environment of that laptop. So your initial state is specified in the value w_0 .

The self-simulation lemma says that there is some program n_0 that your laptop could have started with such that under the dynamics of the universe, it is guaranteed to halt at some finite time $\widehat{\mathcal{T}}((n_0, w_0))$. At that time that it halts, its output would be the joint state of the universe external to your laptop and of the laptop itself at that time Δt . All other variables in the laptop other than this output have

been reinitialized, to the values they had before n_0 was loaded onto the laptop, e.g., to be all blanks.

Note that for any $k \in \mathbb{N}$, there are an infinite number of indices $i \in \mathbb{N}$ such that $T^i = T^k$. This means we can trivially modify the proof of Theorem 2 to establish that there are an infinite number of initial states of the computer, n_0 , such that that computer simulates the full universe, including itself. In Example 1, this is reflected in the fact that there are an infinite number of precise programs all of which perform the same computation as the program n_0 . Note though that in general, those different programs will require different numbers of iterations to complete their computations of the future state of the universe they are in.

Another important point is that the self-simulation lemma holds for any Δt . Whatever time Δt we pick to evolve our universe to, there is a program we can use to initialize the computer subsystem so that it will calculate the joint state of the universe at that time. So in particular, if $\mathcal{T}(n_0, w_0) > \Delta t$, then the computer calculates its own state at that time Δt , a state it had along the way while it was calculating what the joint state would be at that time.

It's also worth pointing out that the proof of the self-simulation lemma does not require the full power of the recursion theorem. That theorem applies to any function as long as it is partial computable. However, the self-simulation lemma only needs to use it for the specific case of the evolution function, which is in fact a total computable function.

Note as well that the self-simulation lemma “hard-codes” the time Δt into the program n_0 that will run on the computer. If we change Δt , then in general we will also change n_0 . This is what allows us to define the function $S(\Delta t)$ (for implicitly fixed g). Note though that Δt does not exist in the universe as a physical variable, in addition to the variables w and n . It is merely a parameter of the evolution function that determines how we wish to initialize the (physical) variables of the computer. So the cosmological universe does not perform that calculation of $S(\Delta t)$ for some physically specified Δt , and then use that value $S(\Delta t)$ to initialize N . See Section C for more discussion of this point.

4.3 Difference between self-simulation lemma and previous work

It's important to distinguish the self-simulation lemma from concepts considered in the earlier literature. First, the traditional versions of the simulation hypothesis discussed before did not involve the possibility that the beings running a simulating computer might be simulating their own universe, including themselves. Self-simulation is novel (*pace* informal musings like the one quoted in this paper's epigraph).

Note as well that the self-simulation lemma is not a reformulation of the old warhorse, central to so much of computer science theory and the foundations of math, of the concept of a mathematical object “referencing itself”. To clarify the difference, specify some uniquely decodable function f that takes two finite bit strings arguments. Consider any bit string c in the range of f . So $f^{-1}(c)$ is the decoding of the bit string c into an ordered pair of bit strings. Write those two bit strings as $f_1^{-1}(c)$ and $f_2^{-1}(c)$ respectively. So for all finite bit strings a, b ,

$$[f_1^{-1}(f(a, b)), f_2^{-1}(f(a, b))] = (a, b) \quad (20)$$

Suppose we represent a universal TM U as working on such encoded pairs of bit strings, by decoding them into a pair of bit strings, and then interpreting the first of those two bit strings as the specification of a TM which is to be run on the second of those two bit strings. So

$$U(c) = [f_1^{-1}(c)][f_2^{-1}(c)] \quad (21)$$

and therefore for all TMs T and strings w that we could feed into T ,

$$U(f(T, w)) = [f_1^{-1}(f(T, w))][f_2^{-1}(f(T, w))] \quad (22)$$

$$= T(w) \quad (23)$$

Now suppose we choose T in Eq. (23) to equal U itself with w arbitrary, as in the case of a UTM engaged in self-reference. Then if we apply Eq. (21) after having evaluated Eq. (23), we get

$$U(f(U, w)) = U(w) \quad (24)$$

$$= [f_1^{-1}(w)][f_2^{-1}(w)] \quad (25)$$

This is the output produced by whatever TM $f_1^{-1}(w)$ happens to be, if run on whatever input string $f_2^{-1}(w)$ happens to be. *A priori*, that output has no relation to the output of the UTM U run on an encoding of itself. (Recall that w is arbitrary.) Indeed, $f_1^{-1}(w)$ may not even halt when run on input string $f_2^{-1}(w)$.

Indeed, for U to simulate its actual behavior on its actual input string, we would need to feed U an infinitely long input string, defined by an infinite regress of instances of itself

$$f(f, f(f, (f, \dots))) \quad (26)$$

(One can see this simply by iteratively expanding w into $f(U, w)$.) It would be physically impossible for any physical computer \mathcal{D} implementing U to finish such a computation in finite (physical) time if it starts with this infinite string as its input. Moreover, there is no way to get information about the rest of the physical

universe *outside* of that computer \mathcal{D} into the input to \mathcal{D} (since such information would need to be appended at the end of the infinite string $f(f, f(f, (f, \dots)))$) None of these problems apply with the self-simulation lemma.

It is also important to distinguish the self-simulation lemma from the age-old observation that if i) the universe extends infinitely spatially; ii) if the physical constants (and more generally physical laws) do not vary across that infinite space; iii) if the initial conditions of the regions inside the backward local light cones across the full universe include all physically possible initial conditions; then somewhere else in this universe there is a copy of each of us, identical down to a fine level of detail. This simple statistical phenomenon does not result in *exact* identity between any of those copies of us and ourselves. More importantly, perhaps the most striking aspect of the self-simulation lemma is that if it applies to us, it would mean that *we* are running a program on a computer that is simulating ourselves, as we run that simulating program. There is as direct a coupling between ourselves and our indistinguishable copy as possible; one of us is directly controlling the other one of us, but those two versions of us are one and the same. (See Section 6.) Indeed, if the self-simulation lemma holds, then there is an infinite regress of copies of you, residing inside the successive layers of nested dolls of computers simulating computers.¹³ No such nested set of copies of an individual arises under the “age-old observation”.

Another function considered in the literature that in some respects resembles the function g of the self-simulation lemma is the Von Neumann constructor [68]. The Von Neumann constructor is a configuration of neighboring states in a particular cellular automaton that is capable of constructing an identical copy of itself as the cellular automaton evolves, resulting in two copies of the constructor.¹⁴ However, a Von Neumann constructor releases the copy of itself once it has created it, and that copy then has a completely independent existence, undergoing different dynamics. In fact, there is no time at which the Von Neumann constructor and that of the copy of itself that it constructs undergo identical dynamics. The Von Neumann constructor does not “run” the copy of itself that it makes, in the sense of the function g in the self-simulation lemma.

¹³See Section 5 for some of the mathematical properties that distinguish all of those nested simulations. Importantly, despite those distinguishing mathematical properties, if the Church-Turing thesis holds then *any inhabitant of one of those nested simulations experiences the exact same reality as their copy in the other nested simulations*. There is no experimental test that one can do to ascertain which nested simulation one inhabits, according to the Church Turing thesis.

¹⁴As an aside, the second recursion theorem of Kleene formally establishes how to design self-reproducing entities in a Turing machine. Despite Kleene’s renown, and the fact that he established this theorem many years before Von Neumann worked on his constructor, it seems that Von Neumann never emphasized Kleene’s precedence.

5 Time delay, cheating computers, and self-simulation

In this section I present a few of the particularly elementary mathematical properties of self-simulation.

5.1 Necessity of time delay in self-simulation

One might suspect that there must be a delay between the time Δt into the future that a self-simulating computer is simulating, and the time at which it completes that simulation. After all, suppose instead that $\mathcal{T}(\Delta t, w_0, \mathcal{S}(\Delta t)) = \Delta t$. This would imply that

$$n_{\Delta t} = (\langle w_{\Delta t}, n_{\Delta t} \rangle) \quad (27)$$

$$= (\langle w_{\Delta t}, \langle w_{\Delta t}, n_{\Delta t} \rangle \rangle) \quad (28)$$

$$= (\langle w_{\Delta t}, \langle w_{\Delta t}, \langle w_{\Delta t}, n_{\Delta t} \rangle \rangle \rangle) \quad (29)$$

$$\dots \quad (30)$$

By the pigeonhole principle, the lengths of the encoded strings in this sequence must grow infinitely long. Therefore $n_{\Delta t}$ would have to be an infinitely long string. But no UTM can write an infinite number of bits onto its output tape in a finite number of iterations. This implies that the equality cannot be satisfied.

One could make a counter-argument though. In many ways our cosmological universe evolves more like a parallel computer rather than a serial one, so that for example the state space of N could evolve like an infinite one-dimensional Cellular automaton. That would seem to allow an infinite number of operations to occur simultaneously — disproving the intuitive argument above. On the other hand, the computational implications if there were parallel rather than serial dynamics in our universe would result in some subtle problems. — see Section D. So this approach will not be considered here.

These arguments and counter-arguments are resolved in the following lemma:

Lemma 3. *For all cases where a computer V is simulating itself, and for all associated $\Delta t, w_0$,*

$$\mathcal{T}(\Delta t, w_0, \mathcal{S}(\Delta t)) > \Delta t$$

assuming $|W| > 1$.

Proof. Hypothesize that there is some $\Delta t, w_0$ such that

$$\mathcal{T}(\Delta t, w_0, \mathcal{S}(\Delta t)) \leq \Delta t \quad (31)$$

Then by Definition 1, for all $t \geq \mathcal{T}(\Delta t, w_0, \mathcal{S}(\Delta t))$,

$$g(t, \mathcal{W}(\Delta t, w_0, \mathcal{S}(\Delta t)), \mathcal{N}(\Delta t, w_0, \mathcal{S}(\Delta t))_{\underline{N}} = g(t, \mathcal{W}(\Delta t, w_0, \mathcal{S}(\Delta t)), \mathcal{S}(\Delta t))_{\underline{N}} \quad (32)$$

So in particular, we would have

$$g(\Delta t, \mathcal{W}(\Delta t, w_0, \mathcal{S}(\Delta t)), \mathcal{S}(\Delta t))_{\underline{N}} = \langle g(\Delta t, w_0, \mathcal{S}(\Delta t)) \rangle \quad (33)$$

Using the fact that we're doing *self*-simulation, $\mathcal{S}(\Delta t) = n_{\Delta t}$, and $\mathcal{W}(\Delta t, w_0, \mathcal{S}(\Delta t)) = w_0$. Plugging this into Eq. (33),

$$g(\Delta t, w_0, n_{\Delta t})_{\underline{N}} = \langle g(\Delta t, w_0, n_{\Delta t}) \rangle \quad (34)$$

However, again using the fact that we're doing self-simulation, $g(\Delta t, w_0, n_{\Delta t})_{\underline{N}}$ just equals $n_{\Delta t}$. Combining,

$$n_{\Delta t} = \langle g(\Delta t, w_0, n_{\Delta t}) \rangle \quad (35)$$

$$= \langle w_{\Delta t}, n_{\Delta t} \rangle \quad (36)$$

Next, recall from Section 2.1 that in this paper I assume the encoding $\langle \cdot, \cdot \rangle$ produces strings whose lengths are non-decreasing functions of the lengths of its two arguments. So if W has more than one state (and therefore $w_{\Delta t}$ is at least a bit long), $|\langle w_{\Delta t}, n_{\Delta t} \rangle| > |n_{\Delta t}|$. In this case Eq. (36) is a contradiction. So our hypothesis must be wrong, which establishes the lemma for the case $|W| > 1$. \square

Theorem 3 means that for all w_0 , $\mathcal{T}(\mathcal{S}(\Delta t), w_0, \Delta t)$ has no upper bound as Δt grows, and must always exceed Δt . It does not mean that $\mathcal{T}(n^*, w, \Delta t)$ is an everywhere increasing function of Δt though; $\mathcal{T}(\mathcal{S}(\Delta t), w_0, \Delta t) + 1$ can be less than $\mathcal{T}(\mathcal{S}(\Delta t), w_0, \Delta t)$, as long as it's greater than Δt .

One could of course forbid this possibility, simply by requiring that $\mathcal{T}(\mathcal{S}(\Delta t), w_0, \Delta t)$ is not decreasing as a function of Δt . Another way to address this issue is to modify Definition 1 so that the simulating computer does not just produce a single future state of the universe being simulated, but rather produces an entire (finite) sequence of future states of the simulated universe, in order, as discussed in Section 2.2. A related way to address this issue is discussed in the next subsection.

5.2 Restrictions to impose on the evolution function

It will sometimes be natural to place several additional restrictions on the evolution function. Such restrictions do not change the main results presented below in Sections 3 and 4. However, it is helpful to invoke them in certain parts of the subsequent analysis of those main results.

We begin with the following definition:

Definition 5. A universe V has a (*stationary*) **Markovian evolution function** g if for all $\Delta t > 0, w \in W, n \in N$,

$$g(\Delta t, w_0, n_0) = \gamma^{\Delta t}(w_0, n_0)$$

for some function $\gamma : W \times N \rightarrow W \times N$

The dynamics of a stationary Markovian universe can be expressed as a time-translation invariant function γ repeatedly running on its own output, i.e., as an iterated function system. In practice, we are often interested in universes whose dynamics is Markovian. In particular, we humans believe that our actual cosmological universe has this property.

Another restriction is especially natural to impose when considering universes that simulate themselves. This is the restriction that the computer in that universe and its environment do not interact after some iteration $k > 0$. Arguably, without this restriction, we have no assurances that the computer is *simulating* the future state of its environment from times k to Δt , rather than just observing the state of its environment at time Δt . We can capture this restriction in the following definition:

Definition 6. Fix Δt , and choose some integer k such that $0 < k < \Delta t$. A universe V with a Markovian evolution function is **shielded (for Δt after iteration k)** if for all w, w_0, n_0 ,

$$[\gamma^{\Delta t-k}(w, n_k)]_N \tag{37}$$

is independent of w , where as shorthand we defined

$$n_k := [\gamma^k(w_0, n_0)]_N \tag{38}$$

5.3 Cheating self-simulation and how to prevent it

There are several ways that the self-simulation lemma can be met that can be viewed as “cheating”. Perhaps the most egregious is the scenario in the following example:

Example 2. Suppose the computer’s initial state n_0 is blank, and does not evolve until time Δt . Then at time Δt , the value $w_{\Delta t}$ is copied onto the state of the computer (e.g., by the computer observing that state of the environment at that time, or some beings in the external universe overwriting the state of the computer at that time). This results in the computer state $n_{\Delta t+1} = w_{\Delta t}$, which was precisely the state of $v_{\Delta t}$.

The scenario described in Example 2 can be prevented by requiring that the computer is shielded for all times after some $k \ll \Delta t$. However, even if we require shielding, together with the associated requirement of Markovian evolution, the self-simulation lemma can still be satisfied if the computer essentially does nothing up to the time Δt , so that its state at that time does not depend on w_0 . This is illustrated in the following example.¹⁵

Example 3. *For simplicity, I describe this scenario as though the computer is a single-tape UTM. Suppose that the initial state n_0 of the tape of the TM is Δt . The first thing that happens is the tape is provided the initial state of the computer's environment, w_0 (implicitly followed by an infinite string of blanks). Suppose that in addition, there is a “counter” variable c that is initialized with the value 0 that is appended to w_0 on the tape. So the initial state of the tape is $\underline{n}_0 = (\Delta t, w_0, 0)$, the initial state of the full TM is $n_0 = [\Delta t, w_0, 0; r_0]$, and the initial state of the full universe is $v_0 = ([\Delta t, w_0, 0; r_0], w_0)$.*

The TM evolves shielded from its environment after this initialization. In all subsequent iterations up to Δt , \underline{n}_t does not change, while c increments by 1 in each of those iterations. Then when the counter reaches Δt , it stops incrementing. At this time the contents of the tape of the TM is $(\Delta t, w_0, \Delta t)$, with the other variables of the TM (its state and its head's position) having some value $r_{\Delta t}$. So the entire ID of the TM at this moment is $[\Delta t, w_0, \Delta t; r_{\Delta t}]$, and therefore the state of the full universe is $([\Delta t, w_0, \Delta t; r_{\Delta t}], w_{\Delta t})$.

Next the computer makes a second copy of w_0 and appends that together with $r_{\Delta t}$ to the end of its tape, so that it now has $[\Delta t, w_0, \Delta t, r_{\Delta t}, w_0]$ on its tape.¹⁶ It then computes $w_{\Delta t}$ from that copy of w_0 , so that at some later time $t_2 > \Delta t$ its tape is $[\Delta t, w_0, \Delta t, r_{\Delta t}, w_{\Delta t}]$.

At this time the state of the full universe is $v_{t_2} = ([\Delta t, w_0, \Delta t, r_{\Delta t}, w_{\Delta t}; r_{t_2}], w_{t_2})$. Note though that the value of the tape of the TM at t_2 is identical to the state of the entire universe at the earlier time, Δt . In other words, $n_{t_2} = g(\Delta t, w_0, n_0)$. We therefore satisfy the self-simulation lemma by choosing $\mathcal{T}(\Delta t, w_0, n_0) = t_2$.

One might argue that in a certain sense the algorithm used in Example 3 for self-simulation is a milder type of “cheating”, since the computer only increments

¹⁵As was mentioned in Section 2.2, in this subsection I need to explicitly write $n_t = (\underline{n}_t, r_t)$, rather than use the shorthand of identifying the state of the computer n_t with the state of its tape, \underline{n}_t which is used in most of the rest of this paper.

¹⁶Recall the discussion in Section 2.2 of the fact that it may prove convenient to augment our TMs with a special instruction that copies the contents of an arbitrarily large portion of V to another portion of V in a single iteration; that extra instruction could be used here, for calculational convenience, so that we don't have to account for the number of iterations it would require a non-augmented TM to copy over that entire input, working on only a single variable in the multi-dimensional state space W at a time.

a counter up to time Δt , not trying to evolve w_0 at all. However, we can nest that algorithm within itself an arbitrary number of times, so that w_0 is being evolved continually, not just after reaching the time Δt . This is illustrated in the following extension of Example 3, which shows how a shielded computer can produce a *finite subsequence* of an entire trajectory of states of the full universe, computing those states one after the other, in the proper time order:

Example 4. *As in Example 3, treat the computer as a single-tape TM. However, do not have any value Δt on the tape of the TM at t_0 . In addition, there is no counter variable. Other than that, we run the exact same algorithm as in Example 3, as though Δt had been set to 1, and there was no need for counter incrementing. So the first thing that happens is the state of the tape gets overwritten with the value w_0 . Suppose that this copy operation completed at some iteration $t_1 > 0$. So the full ID of the TM at t_1 is $[w_0; r_{t_1}]$, and the state of the universe then is $([w_0; r_{t_1}], w_{t_1})$.*

Next the TM appends r_{t_1} to the end of its tape, and then copies w_0 to after that. When this is done it computes w_{t_1} from the copy of w_0 , overwriting that copy. Supposing it completes this at $t_2 > t_1$, the state of its tape at t_2 is (w_0, r_{t_1}, w_{t_1}) , the full ID of the TM then is $[(w_0, r_{t_1}, w_{t_1}); r_{t_2}]$, and the state of the universe then is $v_{t_2} = [(w_0, r_{t_1}, w_{t_1}); r_{t_2}], w_{t_2}$. So the state of the TM at t_2 is the state that the entire universe had at $t_1 < t_2$.

At this point the TM appends the value r_{t_2} to its tape, and then appends a copy of the state w_{t_1} that was stored on its tape to the end of its tape. It then uses that copy to compute w_{t_2} . Assuming it completes that computation at iteration $t_3 > t_2$, at iteration t_3 the state of the tape is $(w_0, r_{t_1}, w_{t_1}, r_{t_2}, w_{t_2})$, the ID of the TM is $[(w_0, r_{t_1}, w_{t_1}, r_{t_2}, w_{t_2}); r_{t_3}]$, and the state of the full universe is $([(w_0, r_{t_1}, w_{t_1}, r_{t_2}, w_{t_2}); r_{t_3}], w_{t_3})$. In particular, the state of the tape at t_3 is the state of the full universe at $t_2 < t_3$.

The computer keeps repeating this process, without ever halting. (Or alternatively, it can halt after some arbitrary, pre-fixed number of iterations, using a counter variable to count iterations that is stored in R .) As it does so it computes the full states of the universe at the iterations $1, t_1, t_2, t_3, \dots$, outputting those computations at the iterations t_1, t_2, t_3, \dots , respectively, where $t_1 < t_2 < t_3 < \dots$

I refer to the procedure run in Example 4 as **(greedy) nested** simulation of a trajectory of states, with the sequence $\{t_1, t_2, \dots\}$ called the **simulation time sequence**.¹⁷ In the special case that nested simulation is applied by a universe to itself, as in Example 4, I refer to it (greedy) nested self-simulation as **(greedy) nested self-simulation**.

¹⁷The qualifier “greedy” reflects the fact that each successive computation is written to the tape of the TM as early as possible. Technically, greedy nested simulation means that for all t_i , $t_{i+1} - t_i$ is as small as possible.

Note that in nested self-simulation, while the number of iterations to compute the state of the universe at a time t is an increasing function of t , it is a partial function. Only a sub-sequence of the full trajectory of states of the universe defined by the simulation time sequence, $\{v_{t_1}, v_{t_2}, v_{t_3}, \dots\}$, is computed. This illustrates that the requirement of shielding does not totally prevent this particularly mild form of cheating. However, that requirement does restrict that kind of cheating to only work for a proper subset of all future times t . In other words, by requiring shielding we can ensure that only mild, “partial” cheating can occur.

As a final comment, note that nested self-simulation only places a sequence of pairs (r_{t_i}, w_{t_i}) onto the tape, never separating the two elements of such a pair in its output. In addition, recall from above we can almost always treat N as synonymous with \underline{N} , with Example 4 being the only instance in this paper in which we explicitly distinguish the \underline{N} and R components of N .

Given all this, one might think that we could absorb the variable R into the variable W , leaving only \underline{N} in the computer variable N . Doing that would change W into the environment of \underline{N} , not of $N = \underline{N} \times R$. Using this alternative form of W would have the advantage that it simplifies the notation of this paper.

However, Definition 6 requires that R and \underline{N} both lie in N . If we contradicted that definition by absorbing R into W , then in nested self-simulation the computer N (which would now only consist of the tape \underline{N}) would have to interact with its external environment for all iterations, never evolving autonomously. So we could not require that the computer be shielded from its environment. That’s why we don’t use this alternative form W in this paper.

6 Philosophical issues raised by the simulation and self-simulation lemmas

6.1 Who am I?

The simulation and self-simulation lemmas have some interesting philosophical aspects. Most obviously, suppose that the PCT and RPCT both apply in our particular computational universe. Then not only might we be a simulation in a computer run by aliens in a universe that our universe supervenes on — we and our entire universe might be a simulation in a computer *in our very universe*. (This is not an issue considered in the earlier literature on the simulation hypothesis.) It might be that we comprise a portion of the universe external to the simulation computer, i.e., our state *in toto* at time t is specified by w_t , and our dynamics is exactly given by g , and therefore we and our dynamics would also be exactly given by the dynamics of n . In other words, we would be both in the universe external to the computer, and in the simulation being run on the computer. And importantly,

there would be no possible experimental test we could perform that could distinguish “which of those two entities we are”. Our existence would be duplicated; we would *be* a duality, in all respects.

In that particular scenario, we are not the ones running the simulating computer. However, if we ever in the future gain the capability of building and running simulation computers, then by the self-simulation lemma, at that time we might even be simulations in a computer that we ourselves run! In such a situation, since the part of our universe containing us, W , is being reproduced in exact detail inside the computer that is simulating the dynamics of W , the version of us inside the computer is itself running a computer, that in turn is simulating us running a computer in exact detail.

In other words, we the humans running that physical computer inside of W , might “*be*” either those people running that physical computer — or we might be the people inside the physical computer who are evolving as the simulation runs, and who are indistinguishable from the “other” humans who are running that physical computer. By the RPCT, there is no conceivable physical test, no observable value, that could tell us which of those two dynamic processes “*is*” us. So in a non-Leibnizian, empirically meaningful sense of the term, we could “*be*” either one of those two evolving objects. We could even both, and would never know.

This conundrum raised by the self-simulation lemma concerning the concept of “identity” is in some senses reminiscent of the “the boat of Theseus” concept. Going beyond that concept though, the self-simulation lemma considers a situation where one object that is directly controlling the other, as both evolve. By construction we cannot distinguish between the possibilities that we are the controller object or we are the controlled, as both of them evolve. There is no such splitting of identity among two simultaneously evolving objects in the boat of Theseus scenario.

As a related point, loosely speaking, one can define “conscious experience” of a person as their thinking about their own thinking. If we adopt that definition, and modify the RPCT thesis appropriately, then we could use the associated modified version of the self-simulation lemma to establish the formal possibility of conscious experience. Rather than apply the original version of the lemma to a physical computer’s simulating itself as it simulates itself, we would apply this modified version of the lemma a physical brain that performs computations (“*thinks*”) about those computations (“*thoughts*”) it is performing.

6.2 Running (self-)simulation using fully homomorphic encryption

Another interesting set of issues arises if there is one universe $V = W \times N$ that simulates a second universe $V' = W' \times N'$, but that simulation is a fully homomorphic encrypted (FHE) version of the evolution function of that second universe.¹⁸ In other words, the program $n_0 \in N$ could be an FHE version of the program simulating V' that the beings who are running that computer N want it to compute. So those beings would need to use a decryption key to understand the result of their computer's simulation of the evolution of V' .

In this case, as a practical matter, if the simulator beings lost the decryption key, then they would not be able to read out the results of their simulation. This would be the case even though that simulation would in fact be perfectly accurate. Or to be more precise, those simulator beings *could* read the results of their simulation — but would require their expending a huge amount of computational resources to do so.¹⁹

On the other hand, since V' obeys the PCT there is no sense in which the beings being simulated could know that they are being produced in a simulation. Life would appear “normal” to them, with no “randomness” of any sort. So in particular, there is no way that they would be able to distinguish between being produced in a simulation being made via an FHE algorithm, or instead in some simulation that is easier to understand. As an example, suppose we are a simulation, so that the laws of physics we perceive are simply the evolution function g' of our universe. In this situation, we would not be able to distinguish the case

¹⁸Recall that in FHE you have an algorithm that runs on some encrypted data, producing a result that when decrypted is identical to the result you would get if ran an associated algorithm on the original, pre-encrypted data, without any encryption of any sort. So in order to use FHE encryption to run a program in an encrypted fashion, all you need to do is have the encrypted data specify that program, and have the algorithm running on the encrypted data be a UTM.

¹⁹As an aside, recall the common supposition that the sequence of events in our universe must have low Kolmogorov complexity, in order for it to contain a pattern that is evident to us, so that it “counts as having been generated by mathematical laws, rather than just being a lawless, random sequence”. Note though that running a program via FHE rather than running it directly does not change its Kolmogorov complexity. (Though to run it via an FHE *and then also decrypt the results* would result in a composite program with slightly larger Kolmogorov complexity.) So we could have a sequence of events that *appear* to be purely random, to us (and so would not “count as mathematical laws”), even though they have low Kolmogorov complexity. In such a situation they have low complexity, but *we* cannot distinguish them from a sequence of events with high Kolmogorov complexity. One might argue that even if low Kolmogorov complexity of the sequence is not a sufficient condition for it to be considered lawful, it is still a *necessary* condition for it to be considered lawful. However, Chaitin’s incompleteness theorem says it is impossible to prove that any sequence with Kolmogorov complexity above a very small value actually has that Kolmogorov complexity. So we can never prove that such a necessary condition is violated.

where we are being run on a simulation program that directly implements the laws of physics, in the straight-forward way (cf. Section 2.6), or are instead being run on a FHE version of the laws of physics. Moreover, in the latter case, if we could actually somehow see our universe’s evolution from the perspective of the beings who are running the program producing us, we would not be able to distinguish the laws of physics controlling our universe in that simulation from completely random noise. In this sense, the actual laws of physics in our universe might in fact be pure noise — and we would not be able to tell the difference.

Similar consequences arise if we consider self-simulation. We might be simulating ourselves, but have done so with an FHE version of ourselves. Similarly to the possibility discussed above, suppose this is the case, but that we have misplaced the decryption key. In this scenario, we are a simulation that we ourselves are running — but we cannot understand that simulation of ourselves, a simulation which *is* us.

7 The simulation graph

Section 6 contains a brief discussion of the philosophical issues that arise if we consider simulation involving more than two universes. Such situations are barely even considered in a cursory way in the existing literature. Yet, there are nontrivial mathematical properties of such situations. In particular, the graphical structure of universes simulating universes can be quite interesting.

I start in this section with some preliminary remarks concerning that graph. Then in Section 9 I discuss some other open mathematical questions.

7.1 The graph of simulations and self-simulations

I begin with the simple observation that simulation is a transitive relation:

Lemma 4. *If V simulates V' , and V' simulates V'' , then V simulates V'' .*

Proof. The proof parallels that of Theorem 1. By hypothesis, the evolution function g'' of V'' is computable, and there exist associated functions $\mathcal{T}_{V,V''}$, $\mathcal{W}_{V,V''}$, $\mathcal{N}_{V,V''}$ that obey the RPCT properties for V' for (a set K' that includes the number k' coding for) the TM $T^{k'}$ that implements the evolution function g'' , where the argument y' of $T^{k'}$ is set to $\langle \Delta t'', w_0'', n_0'' \rangle$ for any $\Delta t'' \in \mathbb{N}$.

Similarly, by hypothesis there exist functions $\mathcal{T}_{V,V'}$, $\mathcal{W}_{V,V'}$, $\mathcal{N}_{V,V'}$ that obey the RPCT properties for V for (a set K that includes the number k coding for) the TM T^k that implements the evolution function g' , where the argument y of T^k is set to $\langle \Delta t', w_0', n_0' \rangle$ for any $\Delta t' \in \mathbb{N}$. So in particular, V has this property where the

argument y of T^k is set to

$$\langle \mathcal{T}_{V',V''}(\Delta t''), \mathcal{W}_{V',V''}(w_0''), \mathcal{N}_{V',V''}(n_0'') \rangle$$

□

A (**bare**) **simulation graph** Γ is defined as a directed graph whose nodes are universes where there is an edge from V to V' iff V simulates V' . In light of the transitivity of simulation, for any node V in a simulation graph Γ , and any node V' on a directed path leading out of V , Γ must contain an edge from V to V' . In general though, if V' and V'' are two universes that are both descendants of V in the graph Γ , it need not be the case that one of them can simulate the other. In addition, due to the possibility of self-simulation, the simulation graph need not be a simple graph; it might contain edges that point to the same node that they came from.

Indeed, suppose we restrict attention to such a simulation graph universes where there is a directed edge from any node V in the graph to any other node $V' \neq V$. Then the simulation relation cannot be a partially ordered set. Given this, suppose that all of the nodes (universes) have a finite W and an evolution function that obeys the PCT. As pointed out in Section 2.4, the set of such universes is countably infinite, and so we cannot assign a uniform probability distribution to that set. However, since the elements of the set are not partially ordered in the simulation graph, we also cannot assign a Cantor measure over the elements of the set, if we wish to use the simulation relation to fix how to assign such a measure to the nodes in the graph. This establishes the claim made in the introduction, that it is not possible to use a Cantor measure to assign probabilities to a very naturally defined set of universes (at least, it's not possible if we try to use the simulation relation to fix the measure).

A bare simulation graph could contain computational universes that exist in the same cosmological universe (see Section 2.2), obeying the same laws of physics, the same initial conditions, etc. It might also contain computational universes that exist in different cosmological universes. These two cases can be intermingled as well.

Finally, suppose we restrict the simulation graph Γ to only contain universes that can simulate themselves. Suppose as well that we restrict the graph so that there is at most a single edge from any universe to V to V' (where V' may or may not differ from V). Then the directed edges in Γ form a reflexive, transitive relation, i.e., they form a preorder. In general, the relation provided by the edges in that simulation graph can include both pairs of nodes that are symmetric under the relation and pairs of nodes that are anti-symmetric under the relation. So while it is a preorder, those edges need not provide either an equivalence relation or a partially ordered set.

There exist many kinds of equivalence classes that could apply to a simulation graph, depending on what universes it contained. Most obviously, we could always divide those universes into equivalence classes where all computers in a class can simulate one another. If in addition the edges form a linear order, then all computers in a class can also simulate all computers in a class that is lower (according to the \leq ordering). But no universe can contain a computer that simulates a universe in a higher equivalence class.

7.2 Some types of simulation graphs

There are many specific types of simulation graphs that might be interesting to explore. As an example, any particular universe V might be running more than one simulation at once. Physically, this could occur by having one computer in that universe running multiple simulations simultaneously, just like a laptop runs multiple tasks simultaneously, by “swapping”. It could also occur by having multiple computers in the universe, all running simulations.²⁰ In such a situation, the fan-out from the node of that universe V in the simulation graph would be greater than 1.

Note that in general there might also be multiple edges coming *in* to each node. If we are such a node, that would mean that that we could simultaneously be the simulations being run by more than one set of beings in other cosmological universes. Another possibility is for there to be multiple beings in a single cosmological universe all of whom are running a simulation that is us.

There are several interesting special cases of this example, where the universe V is running simulations of multiple different universes, V_1, V_2, \dots simultaneously. Suppose that each of those different universes are running their own simulations, which in turn are running *their* own simulations, etc. In this case there is an extensive branching structure of simulation paths coming out of V . Moreover, it could be that as one runs down some of those simulation paths they eventually converge on our universe. In this special case, we are simultaneously being simulated in more than one computer.

As another interesting special case, there might be beings in a universe who are running us in their computer, and who themselves are a simulation in a computer that *we* are running. In such a case, the aliens would be a simulation running in a computer that they themselves control, just “one step removed”. In such a situation, we too would be self-simulating “one step removed”. We would be a

²⁰In terms of the formalism in this paper, the latter case would mean that “the” computer N of the universe is actually a set of multiple computers running independently, in parallel. The simulation lemma would apply directly, and the self-simulation lemma would also hold, where $S(\Delta t)$ is the initial joint state of all of the computers in the universe.

means for the aliens to “split” their ontological status in two, while also being a means for us to split *our* ontological status in two.

Taking this further, suppose we are being simulated by multiple other sets of aliens, but in addition we are running simulations that are us. We might even be running simulations of those beings who are running simulations of us. In the associated simulation graph there would be edges that are loops from us into us, and edges from those other universes into us, perhaps together with some edges from us into those other universes.

As a final comment, recall from Section 4.2 that when the conditions in Theorem 2 hold for a universe V , there are an infinite number of initial states of the computer, n_0 , such that that computer simulates what the full universe V , including itself, will be. That infinite degeneracy may have some interesting philosophical and mathematical implications. For example, suppose I run a computer which is simulating what my entire universe would some time Δt in my future, if my universe had my actual environment w_0 at $t = 0$. Suppose though that that simulation is for an initial state of the computer, n'_0 , that is computationally equivalent to n_0 , the actual simulation program I am using in that simulation — but where that program n'_0 actually differs from n_0 . In a certain sense, this would not be a perfect self-simulation, even though it would obey the self-simulation lemma. This possibility suggests replacing any single loop in the simulation graph (i.e., any single edge from a universe-node into itself) with a set of multiple such loops, distinguished from one another by the fact that they use different programs, all of which happen to be computationally equivalent.

7.3 Time-ordered and time-bounded simulation graphs

In general, a universe might only obey the RPCT partially, in that it can only simulate a subset $K \subset \mathbb{N}$ of all TMs T^k . A discussion of this case is presented just below, in Section 7.4.

For the moment though, consider a set of computable universes all of which obey the RPCT in full, for $K = \mathbb{N}$. The simulation graph for such a set of universes is trivial; it is a fully connected graph. However, even if we’re considering a set of computable universes all of which obey RPCT for $K = \mathbb{N}$, there might still be nontrivial structure in the graph given by placing an edge from V to V' only if V simulates V' sufficiently quickly. More formally, we can consider the **time-bounded simulation graph** in which there is an edge from V to V' iff V simulates V' with a function $\mathcal{T}(\Delta t', w', n')$ that obeys some bound in the worst-case over all pairs (w', n') in how fast it can grow as a function of $\Delta t'$. For example, one could consider the variant of a simulation graph given by only placing an edge from V to V' if for all pairs (w', n') , $\mathcal{T}(\Delta t', w', n')$ grows at most polynomially with $\Delta t'$. Other kinds of nontrivial simulation graphs arise if we consider the scaling

of the resources (time, memory, etc.) needed to compute the functions $\mathcal{T}, \mathcal{W}, \mathcal{N}$, or in the case of self-simulation, \mathcal{S} . (See also the discussion in Section 10 of time-minimal simulation functions, their scaling properties, and computational complexity theory.)

Next, consider any two nodes V, V' connected by a directed path in a (bare) simulation graph. $\mathcal{T}_V(\Delta t', w', n';)$ will differ from Δt . In this sense, $\mathcal{T}(\Delta t, w, n)$ provides a well-defined measure of “the speed of time of the dynamics of a universe”, a speed of time that varies among the different universes along any path descended from V . Note though that this speed of time is only defined for the universe being simulated, measured against time intervals in the universe doing the simulating. Moreover, the speed of time might differ depending on Δt , i.e., $\mathcal{T}(\Delta t, w, n)/\Delta t$ might vary depending on Δt , even for a fixed w, n .

In the case of self-simulation, we know from Theorem 3 that in fact the speed of time is always sped up in a universe being simulated by itself. Accordingly, I define a **time-ordered simulation graph** as any bare simulation graph where all edges $V \rightarrow V'$ are removed where for at least one triple $(\Delta t', w', n')$, $\mathcal{T}(\Delta t', w', n') < \Delta t$. (Whether or not $V = V'$, as in self-simulation.) We do not allow edges in which the speed of time slows down in time-ordered simulation graphs (though we allow time to speed up).

Suppose that in fact $\mathcal{T}(\Delta t', w', n') < \Delta t$ for all n', w' in all universes on the nodes of the graph. Then the time-ordered simulation graph cannot be cyclic. However, it could still “spiral”, in the sense that going along a directed path starting from a node v_1 could land on a node v_N that is an indistinguishable copy of (the universe evolving in) node v_1 , except that the speed of time in v_N is greater than that in v_1 .

7.4 Weak RPCT

In general we do not need to assume the full strength of the RPCT to prove a particular instance of either the simulation lemma or the self-simulation lemma. In the case of the former, we just need to assume that the simulating universe can implement the evolution function of the universe being simulated, i.e., can implement the particular TM specified by that evolution function of the universe being simulated. It is not necessary that it can implement the Turing machine of any universe there is. In the case of the latter, we just need to assume that the simulating universe can implement the (Turing machine specifying the) computable function $\mathcal{S}(\Delta t)$. We do not need the computer in the simulating universe to be computationally universal.

Accordingly, given any $K \subset \mathbb{N}$, I say that the **weak RPCT (for a set $K \subset \mathbb{N}$)** holds for universe V if Definition 4 holds for V after rather than the requirement

in Definition 4 that the functions $\widehat{\mathcal{T}}$, $\widehat{\mathcal{W}}$ and $\widehat{\mathcal{N}}$ have the RPCT properties for *all* k , we instead only require they those three functions have the RPCT for all $k \in K$ (and evolution function g of V).

The associated **weak simulation lemma** says that V can simulate a universe $V' = W' \times N'$ whose evolution function is given by the TM T^k if

1. V' obeys the PCT;
2. V obeys the weak RPCT for a set K that includes three functions $\widehat{\mathcal{T}}$, $\widehat{\mathcal{W}}$ and $\widehat{\mathcal{N}}$ that have the RPCT properties for all $(\Delta t', w' \in W', n' \in N')$, for an associated evolution function $g' = T^k$.

With obvious generalizations, we can weaken the RPCT further, by restricting the set of $w' \in W'$ and / or the set of Δt — which corresponds to restrictions on the set of y in Definition 4. Transitivity of simulation (Theorem 4) would still apply for a set of universes related this way. So the simulation graph for such a set of universes would again be a preorder.

Similarly, suppose that a universe obeys the weak RPCT for a set K , where the solution $S(\Delta t)$ for V to simulate itself a time Δt into the future lies in K . In this case the universe does not obey the full RPCT, but it still is able to simulate itself for that future time Δt . Accordingly, I call this the **weak self-simulation lemma** for universe V . As with the simulation lemma, we can further weaken the RPCT so that we limit the set of $w \in W$ and / or Δt for which V simulates itself.

One could extend the simulation graph by changing the definitions of the edges to include these weakened versions of the simulation and / or self-simulation lemmas. In particular, it might be of interest to investigate how the structure of the graph progressively changes as we progressively weaken those lemmas.

In a similar way, we could weaken the PCT, either instead of weakening the RPCT or in addition to weakening the RPCT. This would result in yet another pair of lemmas, and another extension of the simulation graph.

8 Implications of Rice's theorem for (self-)simulation

Rice's theorem, discussed in Section B, has some interesting implications for both simulation and self-simulation. To illustrate these, for simplicity, throughout this subsection I'm restricting attention to universes with a countably infinite W as well as a countably infinite N .

First, Rice's theorem tells us that the set of all computational universes V' that can be simulated by a fixed universe V is undecidable. More formally, fix some universe V that simulates at least one other universe. Define $A(V)$ as the collection of all TMs that compute the evolution function of universes V' that are simulated

by V . Note that every TM that lies in $A(V)$ must be total, since all evolution functions are. Therefore there are TMs that do not lie in $A(V)$ (e.g., all TMs that compute a partial function), as well as TMs that do lie in V (by definition of V and $A(V)$). Moreover, any two TMs that compute the exact same evolution function either both lie in $A(V)$ or both do not, i.e., membership in $A(V)$ does not depend on *how* the associated TM operates, only on the function it computes. Therefore by Rice's theorem, it is undecidable whether an arbitrary (total TM and associated) V' is a member of $A(V)$.

As a variant of this result, again fix V , and also fix some spaces W', N' . Define $B(V, (w'_0, n'_0), (w'_{\Delta t}, n'_{\Delta t}))$ as the collection of all TMs that compute the evolution function g' of some universe $V' = W' \times N'$ with the following two properties. First, g' sends (w'_0, n'_0) to $(w'_{\Delta t}, n'_{\Delta t})$. Second, V simulates V' for the specific initial condition (w'_0, n'_0) and the simulation time Δt . Again, it is undecidable whether an arbitrary TM lies in $B(V, (w'_0, n'_0), (w'_{\Delta t}, n'_{\Delta t}))$.

Flipping things around, Rice's theorems tells us that for any fixed universe V' , the set of all other universes V that can simulate V' is undecidable. More formally, fix V' , and define the property of TMs that the function they compute is the evolution function of a universe V that simulates V' . Then it is undecidable whether an arbitrary such (TM and associated) V has that property of simulating V' . An immediate consequence of this is that the set of all pairs of universes (V, V') such that V simulates V' is undecidable.

Rice's theorem also shows that:

1. The set of universes (and in particular evolution functions) that obey the RPCT is undecidable.
2. The set of universes that can simulate themselves is undecidable.
3. The set of universes V that can simulate a universe $V' \neq V$ that can in turn simulate any universe at all is undecidable.
4. The set of universes V that can simulate a universe $V' \neq V$ that can in turn simulate V is undecidable.
5. Restrict attention to some set \mathcal{V} of universes that can simulate themselves (e.g., because all the universes in \mathcal{V} obey both the PCT and the RPCT). The set of those universes in \mathcal{V} that can simulate itself for all $\Delta t \in \mathbb{N}$ using some specific function $S(\Delta t)$ is undecidable.
6. In particular, for any $k \in \{2, \dots\}$, the set of universes in \mathcal{V} that can simulate itself for an associated $S(\Delta t)$ for all $\Delta t \leq k$ but not for some $\Delta t > k$ is undecidable.

7. For any $k \in \{2, \dots\}$, the set of universes in \mathcal{V} that can simulate itself for an associated $\mathcal{S}(\Delta t)$ for all $\Delta t \geq k$ but not for some $\Delta t > k$ is undecidable.
8. The set of those universes in \mathcal{V} that can simulate itself for a time-ordered $\mathcal{S}(\Delta t)$ is undecidable.

There are some strange philosophical implications of these impossibility results, especially those that concern self-simulation. For example, it is possible that we are in a universe V that is simulating itself — but only up to some future time, after which it is impossible for the simulation to still be accurate. At that future time we would “split” into two versions of ourselves, which share an identical past: the simulating version of us, and the simulated version of us. The impossibility results above say that we can never be sure that this is not the case.

9 Mathematical issues raised by the self-simulation and simulation lemmas

There are many mathematical questions suggested by the self-simulation lemma that I am not considering in this paper. Most obviously, I have not considered the computational complexity of finding $\mathcal{S}(\Delta t)$, and its dependence on g , Δt , $|W|$, etc.

I also have not considered the relation between Δt and the physical time $\mathcal{T}(\Delta t, w_0, n_0)$ at which a computer N with initial state n_0 running a simulation finishes its calculation of the state of V at physical time Δt , when the initial state of the environment is w_0 . In particular, I have not considered how the minimal value (over all n_0) of $\mathcal{T}(\Delta t, w_0, n_0)/\Delta t$ might depend on g , w_0 , the value of Δt , etc. Associated questions, more in the spirit of computational complexity theory, would involve the scaling of

$$\min_{n_0} \max_{w_0} \frac{\mathcal{T}(\Delta t, w_0, n_0)}{\Delta t} \quad (39)$$

with $|W|$, for a fixed family of evolution functions $\{g_{|W|}\}$.

Next, define “non-greedy nested self-simulation” as the variant of nested self-simulation where for at least one t_i in a simulation time sequence computed by the computer, t_i , the gap $t_{i+1} - t_i$ is not minimal. Define the **density (of simulation times)** of an instance of nested self-simulation producing the simulation time sequence $\{t_1, t_2, \dots\}$ for initial condition $v_0 = (w_0, n_0)$ as

$$D(w_0, n_0) := \lim_{i \rightarrow \infty} \frac{i}{t_i} \quad (40)$$

Note that this is well-defined even if the instance of nested self-simulation produced by n_0 isn’t greedy.

One obvious question is what the properties of g and v_0 need to be for the density $D(w_0, n_0)$ to be well defined. A related question is whether in scenarios where the density is well-defined, greedy self-simulation maximizes it. (*A priori*, it could be that delaying some simulation times t_i allows denser subsequent simulation times.) A higher-level question is whether nested self-simulation, greedy or otherwise, maximizes the density of simulation times over the set of all possible TMs.

There are also interesting computational complexity issues concerning the three computable functions that define one universe’s simulating another. In particular, there are obvious extensions of the basic framework to concern not perfect simulation, but rather approximate simulation. This then immediately suggests investigating variants of simulation graphs, defined by the approximation complexity of (imperfect) simulation [6]. For example, one might consider simulation graphs where edges from V to V' must respect an upper bound on how fast the function $\mathcal{T}(\Delta t', w', n')$ would have to grow (as a function of $\Delta t'$), in order for the resultant simulation of V' ’s evolution to be at least a factor α within being exactly correct. As another example, one could consider such approximation complexity in the computation of the functions $\mathcal{T}, \mathcal{W}, \mathcal{N}$ themselves, rather than in the resultant behavior of \mathcal{T} .

Similarly, we can consider the average-case complexity of all the issues arising in the simulation framework. In this kind of approach we would again consider a variant of simulation graphs, this time defined by requiring all edges from one universe (labeled as V) to a potentially different universe (labeled as V') in the simulation graph must obey associated bounds. In this case though, those bounds would concern the average-case behavior of $\mathcal{T}(\Delta t', w', n')$ as a function of $\Delta t'$, or of the resources needed to compute the functions $\mathcal{T}, \mathcal{W}, \mathcal{N}$. We might also want to follow Levin in how precisely to define such “average-case complexity”.

There are many other open questions that involve slight variants of the framework introduced in this paper, in addition to those discussed in the main text. For example, there are some ways to refine the definition of simulation that might be worth pursuing. One of these is to define the “time-minimal” triple of simulation functions $(\mathcal{T}, \mathcal{W}, \mathcal{N})^{V,V'}$ used by V to simulate V' as the three such functions where for all triples $(\Delta t', w'_0, n'_0)$, the associated time to complete the simulation, $\mathcal{T}^{V,V'}(\Delta t', w'_0, n'_0)$, is minimal. So intuitively, this triple of simulation functions minimizes the time cost (in the complexity theory sense) for V to perform the computation of the future state of V' . Next define

$$\mathbb{T}^{V,V'}(\Delta t') := \max_{w'_0 \in W', n'_0 \in N'} \mathcal{T}^{V,V'}(\Delta t', w'_0, n'_0) \quad (41)$$

This is the worst-possible time to complete the simulation using the fastest simulating computer.

It might be of interest to investigate the scaling of $\mathbb{T}^{V,V'}(\Delta t')$ as a function of $|V'|$, the size of the state space of V' . (In the case of self-simulation, $V = V'$, and we might instead investigate the scaling of $\mathbb{T}^{V,V}(\Delta t')$ as a function of $|W|$.) In particular, to get a precise analogy with the concept of time complexity in computational complexity theory, we might want to investigate how that scaling depends on both g and g' .

Analogous issues would arise for a “space minimal” variant of simulation, involving the simulation function \mathcal{N} rather than \mathcal{T} . In particular, we could investigate how the scaling properties of the space-minimal cost depends on g, g' . This would provide an analogy with the concept of space complexity in computational complexity theory. In a similar way, it might be fruitful to view time-minimal and / or space-minimal versions of the RPCT.

10 Discussion

To my knowledge, Theorem 1 is the first fully formal statement of what has been informally referred to in the literature as the “simulation hypothesis”. Going further, it is also the first formal derivation of a set of sufficient conditions that ensure that the simulation hypothesis holds. Theorem 2 then goes further, and establishes sufficient conditions for a universe to have a computer that simulates itself, a possibility with strange philosophical consequences. These lemmas also lead to many interesting questions that are purely mathematical, e.g., concerning the simulation graph of universes simulating universes, the minimal time delay in self-simulation, etc.

There have been informal discussions in the literature attacking the simulation hypothesis on the grounds that each successive level of simulation within simulation would necessarily be computationally weaker than the one just above it. The idea is that due to this strict “weakening of computational power”, there is a deepest possible level of simulation, containing a species that is not computationally powerful enough to simulate any other species [17]. The argument is made that this deepest level would only be a finite number of levels below the one we inhabit.

This argument has been criticized for not considering the possibility that the computational power of the successive levels might *asymptote* at some weakest amount of power. In this case there would actually be an infinite number of levels below the one we inhabit, and none of them would be so weak as to be incapable of simulating yet a deeper level. The point is moot however; the self-simulation lemma disproves the starting supposition of the argument, that “each successive level of simulation within simulation would necessarily be computationally weaker than the one just above it”.

The self-simulation lemma also problematizes — perhaps fatally — the whole idea of assigning a probability to the possibility that “we are a simulation”. If in fact we are a *self*-simulation, then we would be both the simulation, and the simulator. Indeed, in some senses we would be an infinite number of simulations-within-simulations, all distinguishable by how slowly they evolve, but in all others ways completely identical. That raises the obvious question of how many instances of us we need to include in tallying up the number of cases in which we’re a simulation. Without answering that question, it is hard to see how to calculate the probability of our being a simulation.

Numerous open issues concerning simulation and mathematics are discussed in the text. There are also several open issues concerning simulation and the laws of physics as we currently understand them. In particular, it might be worth investigating extensions of the analysis in this paper to concern quantum mechanical and / or relativistic universes. One obvious question in this regard is whether the quantum no-cloning theorem means that self-simulation could never arise (at the quantum level) in our universe. If so, that might point to ways for the recursion theorem to be modified for quantum rather than classical computers.

Finally, it is worth re-emphasizing that while couched in the language of the simulation hypothesis, the analysis in this paper is far more general. If we assume the Church-Turing thesis, the responses of physical reality during any observation or experiment we might conduct can be modeled as the results of a computation on a TM. The Church-Turing thesis also means that the behavior of we the observers / experimenters can be modeled that way. One might also assume that we ourselves can run a Turing complete computer. This paper is simply an investigation of the formal consequences if we adopt one and / or the other of these two assumptions.

Acknowledgments:

I would like to thank the Santa Fe Institute for support, and Aram Ebtekar for a close read.

Statements

I declare no conflicts of interest occurred in the preparation of this manuscript. I declare that no ethical issues (e.g., concerning research on animals or human subjects) arose in the preparation of this manuscript.

APPENDICES

A Turing machines

Perhaps the most famous class of computational machines are Turing machines [39, 64, 6, 60]. One reason for their fame is that it seems one can model any computational machine that is constructable by humans as a Turing machine. A bit more formally, the *Church-Turing thesis* (CT) states that, “A function on the natural numbers is computable by a human being mechanically following an algorithm, ignoring resource limitations, if and only if it is computable by a Turing machine.” Note that it’s not even clear whether this is a statement about the physical world that could be true or false, or whether instead it is simply a definition, of what “mechanically following an algorithm” means [25]. In any case, the “physical CT” (PCT) modifies the CT to hypothesize that the set of functions computable with Turing machines includes all functions that are computable using mechanical algorithmic procedures (i.e., those we humans can implement) admissible by the laws of physics [7, 55, 58, 47].

In earlier literature, the CT and PCT were both only always described semi-formally, or simply taken as definitions of terms like “calculable via a mechanical procedure” or “effectively computable” (in contrast to the fully formal definition given in Section 2.4). Nonetheless, in part due to the CT thesis, Turing machines form one of the keystones of the entire field of computer science theory, and in particular of computational complexity [49]. For example, the famous Clay prize question of whether $P = NP$ — widely considered one of the deepest and most profound open questions in mathematics — concerns the properties of Turing machines. As another example, the theory of Turing machines is intimately related to deep results on the limitations of mathematics, like Gödel’s incompleteness theorems, and seems to have broader implications for other parts of philosophy as well [3]. Indeed, invoking the PCT, it has been argued that the foundations of physics may be restricted by some of the properties of Turing machines [11, 1].

Along these lines, some authors have suggested that the foundations of statistical physics should be modified to account for the properties of Turing machines, e.g., by adding terms to the definition of entropy. After all, given the CT, one might argue that the probability distributions at the heart of statistical physics are distributions “stored in the mind” of the human being analyzing a given statistical physical system (i.e., of a human being running a particular algorithm to compute a property of a given system). Accordingly, so goes the argument, the costs of generating, storing, and transforming the minimal specifications of the distributions concerning a statistical physics system should be included in one’s thermodynamic analysis of those changes in the distribution of states of the sys-

tem. See [19, 20, 74].

There are many different definitions of Turing machines that are computationally equivalent to one another, in that any computation that can be done with one type of Turing machine can be done with the other. In fact, the “scaling function” of one type of Turing machine, mapping the size of a computation to the minimal amount of resources needed to perform that computation by that type of Turing machine, is at most a polynomial function of the scaling function of any other type of Turing machine. (See for example the relation between the scaling functions of single-tape and multi-tape Turing machines [6].) The following definition will be useful for our purposes, even though it is more complicated than strictly needed:

Definition 7. A **Turing machine** (*TM*) is a 7-tuple $(R, \Lambda, b, v, r^\varnothing, r^A, \rho)$ where:

1. R is a finite set of **computational states**;
2. Λ is a finite **alphabet** containing at least three symbols;
3. $b \in \Lambda$ is a special **blank symbol**;
4. $v \in \mathbb{Z}$ is a **pointer**;
5. $r^\varnothing \in R$ is the **start state**;
6. $r^A \in R$ is the **accept state**; and
7. $\rho : R \times \mathbb{Z} \times \Lambda^\infty \rightarrow R \times \mathbb{Z} \times \Lambda^\infty$ is the **update function**. It is required that for all triples (r, v, T) , that if we write $(r', v', T') = \rho(r, v, T)$, then v' does not differ by more than 1 from v , and the vector T' is identical to the vectors T for all components with the possible exception of the component with index v ;²¹

r^A is often called the “halt state” of the TM rather than the accept state. (In some alternative, computationally equivalent definitions of TMs, there is a set of multiple accept states rather than a single accept state, but for simplicity I do not consider them here.) ρ is sometimes called the “transition function” of the TM. We sometimes refer to R as the states of the “head” of the TM, and refer to the third argument of ρ as a **tape**, writing a value of the tape (i.e., semi-infinite string of elements of the alphabet) as λ . The set of triples that are possible arguments to the update function of a given TM are sometimes called the set of **instantaneous**

²¹Technically the update function only needs to be defined on the “finitary” subset of $\mathbb{R} \times \mathbb{Z} \times \Lambda^\infty$, namely, those elements of $\mathbb{R} \times \mathbb{Z} \times \Lambda^\infty$ for which the tape contents has a non-blank value in only finitely many positions.

descriptions (IDs) of the TM. (These are sometimes instead referred to as “configurations”.) Note that as an alternative to Def. 7, we could define the update function of any TM as a map over an associated space of IDs.

Any TM $(R, \Lambda, b, v, r^\emptyset, r^A, \rho)$ starts with $r = r^\emptyset$, the counter set to a specific initial value (e.g. 0), and with λ consisting of a finite contiguous set of non-blank symbols, with all other symbols equal to b . The TM operates by iteratively applying ρ , if and until the computational state falls in r^A , at which time the process stops, i.e., any ID with the head in the halt state is a fixed point of ρ .

If running a TM on a given initial state of the tape results in the TM eventually halting, the largest blank-delimited string that contains the position of the pointer when the TM halts is called the TM’s **output**. The initial state of λ (excluding the blanks) is sometimes called the associated **input**, or **program**. (However, the reader should be warned that the term “program” has been used by some physicists to mean specifically the shortest input to a TM that results in it computing a given output.) We also say that the TM **computes** an output from an input. In general though, there will be inputs for which the TM never halts. The set of all those inputs to a TM that cause it to eventually halt is called its **halting set**. We write the output of a TM T run on an input x that lies in its halting set as $T(x)$.

Write the set of non-blank symbols of Λ as $\hat{\Lambda}$. Every λ on the tape of a TM that it might have during a computation in which it halts is a finite string of elements in Λ delimited by an infinite string of blanks. Accordingly, wolog we often refer to the state space of the tape as Λ^* , with the trailing infinite string of blanks implicit. Note that Λ^* is countably infinite, in contrast to Λ^∞ .

If a function is undefined for some elements in its domain, it is called a **partial** function. Otherwise it is a **total** function. In particular if a TM T does not halt for some of its inputs, so its halting set is a proper subset of its domain, then the map from its domain to outputs is a partial function, and if instead its halting set is its entire domain, it is a total function.

We say that a total function f from $(\Lambda \setminus \{b\})^*$ to itself is **recursive**, or **(total) computable**, if there is a TM with input alphabet Λ such that for all $x \in (\Lambda \setminus \{b\})^*$, the TM computes $f(x)$. If f is instead a partial function, then we say it is **partial recursive** (partial computable) if there is a TM with input alphabet Λ that computes $f(x)$ for all x for which $f(x)$ is defined, and does not halt for any other x . (The reader should be warned that in the literature, the term “computable” is sometimes taken to mean partial computable rather than total computable — and in some articles it is sometimes taken to mean either total computable or partial computable, depending on the context.)

An important special case is when the image of f is just \mathbb{B} , so that for all $s \in (\Lambda \setminus \{b\})^*$, $f(s)$ is just a single bit. In this special case, we say that the set of all $s : f(s) = 1$ is **decidable** if f is computable.

Famously, Turing showed that there are total functions that are not recur-

sive. In light of the CT, this result is arguably one of the deepest philosophical truths concerning fundamental limitations on human capabilities ever discovered. (See [25, 59, 56, 55].)

As mentioned, there are many variants of the definition of TMs provided above. In one particularly popular variant the single tape in Definition 7 is replaced by multiple tapes. Typically one of those tapes contains the input, one contains the TM’s output (if and) when the TM halts, and there are one or more intermediate “work tapes” that are in essence used as scratch pads. The advantage of using this more complicated variant of TMs is that it is often easier to prove theorems for such machines than for single-tape TMs. However, there is no difference in their computational power. More precisely, one can transform any single-tape TM into an equivalent multi-tape TM (i.e., one that computes the same partial function), as well as vice-versa [6, 39, 64].

To motivate an important example of such multi-tape TMs, suppose we have two strings s^1 and s^2 both contained in the set $(\Lambda \setminus \{b\})^*$ where s^1 is a proper prefix of s^2 . If we run the TM on s^1 , it can detect when it gets to the end of its input, by noting that the following symbol on the tape is a blank. Therefore, it can behave differently after having reached the end of s^1 from how it behaves when it reaches the end of the first $\ell(s^1)$ bits in s^2 . As a result, it may be that both of those input strings are in its halting set, but result in different outputs.

A **prefix (free) TM** is one in which this can never happen: there is no string in its halting set that is a proper prefix of another string in its halting set. The easiest way to construct such TMs is to have a multi-tape TM with a single read-only input tape whose head cannot reverse, and a write-only output tape whose head cannot reverse, together with an arbitrary number of work tapes with no such restrictions.²² I will often implicitly assume that any TM being discussed is such a multi-tape prefix TM.

Returning to the TM variant defined in Definition 7, one of the most important results in CS theory is that the number of TMs is countably infinite. This means that we can index the set of all TMs with \mathbb{N} , i.e., we can write the set of TMs as $\{T^k : k \in \mathbb{N}\}$. It also means that there exist **universal Turing machines** (UTMs), U , which can be used to emulate an arbitrary other TM T^k for any k . More precisely, we define a UTM U as one with the property that for any other TM T , there

²²It is not trivial to construct prefix single-tape TMs directly. For that reason it is common to use prefix three-tape TMs, in which there is a separate input tape that can only be read from, output tape that can only be written to, and work tape that can be both read from and written to. To ensure that the TM is prefix, we require that the head cannot ever back up on the input tape to reread earlier input bits, nor can it ever back up on the output tape, to overwrite earlier output bits. To construct a single-tape prefix TM, we can start with some such three-tape prefix TM and transform it into an equivalent single-tape prefix TM, using any of the conventional techniques for transforming between single-tape and multi-tape TMs.

is an invertible map f from the set of possible states of the input tape of T into the set of possible states of the input tape of U with the following properties: Both f and f^{-1} are computable, and if we apply f to any input string σ' of T to construct an input string σ of U , then:

1. U run on its input σ halts iff T run on its input σ' halts;
2. If U run on σ halts, and we apply f^{-1} to the resultant output of U , we get the output computed by T if it is run on σ' .

As is standard, I fix some (prefix free, computable, invertible) encoding that maps all finite sets of finite bit strings into a single bit string. Overloading notation, I write that encoding as $\langle \cdot \rangle$, $\langle \cdot, \cdot \rangle$..., depending on the number of strings in the set. Moreover, I require the input to a UTM to be encoded as $\langle k, x \rangle$ if it is emulating TM T^k running on input string x . However, sometimes for clarity of presentation I will leave the angle brackets implicit, and simply write the UTM U operating on input $\langle k, x \rangle$ as $U(k, x)$. In addition, as shorthand, if x is a vector whose components are all bit strings, I will write the encoded version of all of its components as $\langle x \rangle$.

Intuitively, the proof of the existence of UTMs just means that there exists programming languages which are “(computationally) universal”, in that we can use them to implement any desired program in any other language, after appropriate translation of that program from that other language. This universality leads to a very important concept:

Definition 8. *The Kolmogorov complexity of a UTM U to compute a string $\sigma \in \Lambda^*$ is the length of the shortest input string s such that U computes σ from s .*

Intuitively, (output) strings that have low Kolmogorov complexity for some specific UTM U are those with short, simple programs in the language of U . For example, in all common (universal) programming languages (e.g., C, Python, Java, etc.), the first m digits of π have low Kolmogorov complexity, since those digits can be generated using a relatively short program. Strings that have high (Kolmogorov) complexity are sometimes referred to as “incompressible”. These strings have no patterns in them that can be generated by a simple program. As a result, it is often argued that the expression “random string” should only be used for strings that are incompressible.

We can use the Kolmogorov complexity of prefix TMs to define many associated quantities, which are related to one another the same way that various kinds of Shannon entropy are related to one another. For example, loosely speaking, the conditional Kolmogorov complexity of string s conditioned on string s' , written as $K(s | s')$, is the length of the shortest string x such that if the TM starts with

an input string given by the concatenation xs' , then it computes s and halts. If we restrict attention to prefix-free TMs, then for all strings $x, y \in \Lambda^*$, we have [39]

$$K(x, y) \leq K(x) + K(x | y) + O(1) \leq K(x) + K(y) + O(1) \quad (42)$$

(where “ $O(1)$ ” means a term that is independent of both x and y). Indeed, in a certain technical sense, the expected value of $K(x)$ under any distribution $P(x \in \Lambda^*)$ equals the Shannon entropy of P . (See [39].)

Formally speaking, the set \mathbb{B}^* is a Cantor set. A convenient probability measure on this Cantor set, sometimes called the **fair-coin measure**, is defined so that for any binary string x the set of sequences that begin with σ has measure $2^{-|\sigma|}$. Loosely speaking, the fair-coin measure of a prefix TM T is the probability distribution over the strings in T ’s halting set generated by IID “tossing a coin” to generate those strings, in a Bernoulli process, and then normalizing.²³ So any string σ in the halting set has probability $2^{-|\sigma|}/\Omega$ under the fair-coin prior, where Ω is the normalization constant for the TM in question.

The fair-coin prior provides a simple Bayesian interpretation of Kolmogorov complexity: Under that prior, the Kolmogorov complexity of any string σ for any prefix TM T is just (the log of) the maximum over all input strings σ' of the joint probability of σ' and σ . (Strictly speaking, this result is only true up to an additive constant, given by the log of the normalization constant of the fair-coin prior for T .)

The normalization constant Ω for any fixed prefix UTM, sometimes called “Chaitin’s Omega”, has some extraordinary properties. For example, the successive digits of Ω provide the answers to *all* well-posed mathematical problems. So if we knew Chaitin’s Omega for some particular prefix UTM, we could answer every problem in mathematics. Alas, the value of Ω for any prefix UTM U cannot be computed by any TM (either U or some other one). So under the CT, we cannot calculate Ω . (See also [9] for a discussion of a “statistical physics” interpretation of Ω that results if we view the fair-coin prior as a Boltzmann distribution for an appropriate Hamiltonian, so that Ω plays the role of a partition function.)

It is now conventional to analyze Kolmogorov complexity using prefix UTMs, with the fair-coin prior, since this removes some undesirable technical properties that Kolmogorov complexity has for more general TMs and priors. Reflecting this, all analyses in the physics community that concern TMs assume prefix UTMs. (See [39] for a discussion of the extraordinary properties of such UTMs.)

²³Kraft’s inequality guarantees that since the set of strings in the halting set is a prefix-free set, the sum over all its elements of their probabilities cannot exceed 1, and so it can be normalized. However, in general that normalization constant is uncomputable, as discussed below. Also, in many contexts we can actually assign arbitrary non-zero probabilities to the strings outside the halting set, as long as the overall distribution is still normalizable. See [39].

Interestingly, for all their computational power, there are some surprising ways in which TMs are *weaker* than the other computational machines introduced above. For example, there are an infinite number of TMs that are more powerful than any given circuit, i.e., given any circuit C , there are an infinite number of TMs that compute the same function as C . Indeed, any single UTM is more powerful than *every* circuit in this sense. On the other hand, it turns out that there are circuit *families* that are more powerful than any single TM. In particular, there are circuit families that can solve the halting problem [6].

I end this appendix with some terminological comments and definitions that will be useful in the main text. It is conventional when dealing with Turing machines to implicitly assume some invertible computable map $h(\cdot)$ from \mathbb{Z} into Λ^* . Given such a map $h(\cdot)$, we can exploit it to implicitly assume an additional invertible map taking \mathbb{Q} into Λ^* , e.g., by uniquely expressing any rational number as one product of primes, a , divided by a product of different primes, b ; invertibly mapping those two products of primes into the single integer $2^a 3^b$; and then evaluating $h(2^a 3^b)$. Using these definitions, we say that a real number z is **computable** iff there is a recursive function f mapping rational numbers to rational numbers such that for all rational-valued accuracies $\epsilon > 0$, $|f(\epsilon) - z| < \epsilon$. We define computable functions from $\mathbb{Q} \rightarrow \mathbb{R}$ similarly.

B The recursion theorem and Rice's theorem

Any reader not already familiar with the theory of Turing machines should read Section A before this appendix.

Kleene's second recursion theorem [63, 38, 50, 8] can be stated as follows:

Theorem 5. *For any partial computable function $Q(x, y)$ there is a Turing machine with index e such that $T^e(x) = Q(x, e)$ for all x for which $Q(x, e)$ is defined.*

An elegant proof of an extended version of Kleene's second recursion theorem can be found in [50]. In terms of the notation in this paper, it proceeds as follow:

Proof. For any TM of the form $M(y, x)$ taking two arguments, define $\llbracket M(y, x) \rrbracket_x$ as the index e such that $T^e(x) = M(y, x)$ for all x , with y fixed to the indicated value.

Using this notation, define

$$S(t) := \llbracket T^t(t, x) \rrbracket_x \quad (43)$$

(Note that $S(\cdot)$ is a total computable function.) Using this definition, choose an index k such that

$$T^k(t, x) = Q(x, S(t)) \quad (44)$$

for all x, t . (At least one such index must exist since the RHS of Eq. (44) is a partial computable function.)

Finally, set $t = k$ in Eq. (44) and then plug in Eq. (43). The proof is completed by choosing

$$e := S(k) = \llbracket T^k(k, x) \rrbracket_x \quad (45)$$

□

In computer science theory, Kleene's second recursion theorem is just called “the recursion theorem”. It has played an extremely important role in computer science theory. For example it provides the underlying formal justification for Von Neumann's universal constructor, which in turn was extremely important for understanding the foundations of biology. More prosaically, it provides the formal justification for why computer viruses are possible (assuming we use computers that are Turing complete).

An important special case of the theorem is where $Q(., .)$ is a *total* computable function. In this case the theorem says that there must be an e such that $T^e(x) = Q(x, e)$ for all x , and therefore T is also total computable. So, we can restrict to total computable functions $Q(., .)$ and total TM's T^e , as a special case. I call this extension the “total recursion theorem” in the main text.

Kleene's second recursion theorem has elicited some interesting commentary. For example, [50] writes that

“The proof has always seemed too short and tricky, and some considerable effort has gone into explaining how one discovers it short of “fiddling around” ... Some of his students asked Kleene about it once, and his (perhaps facetious) response was that he just “fiddled around” — but his fiddling may have been informed by similar results in the untyped λ -calculus.”

Another interesting comment was made by Juris Hartmanis [31]:

“The recursion theorem is just like tennis. Unless you're exposed to it at age five, you'll never become world class.”

Interestingly, Hartmanis didn't encounter the recursion theorem until he was in his 20's — and yet went on to win the Turing award.

Rice's theorem is an extremely powerful theorem about computability which can be proven from Kleene's second recursion theorem. (This is shown in the wikipedia entry on Rice's theorem, for example.) Perhaps the simplest way to state it is the following:

Let G be any non-empty set of partial computable functions (e.g., represented as a set of bit strings that encode the TMs that compute those functions). Suppose I can design an algorithm that correctly determines whether any specific partial computable function f lies in G , i.e., suppose that membership in G is decidable. Rice's theorem says that if this is the case, then G must be the set of *all* partial computable functions, i.e., our algorithm must always produce the output, “yes”.

So if there is any TM T that computes a function that is not in G , then there must be a TM T' such that our algorithm cannot tell us whether the function that T' computes lies in G .

Intuitively, fix some property G of the functions that can be computed by TMs, and suppose we design an algorithm to decide whether the function computed by an arbitrary TM lies in G . Rice's theorem tells us that either G is the set of *all* such functions, or there are some TMs that our algorithm fails on.

An important special case is where G is restricted to a set of binary-valued partial functions. Since each of those functions is represented as a bit string, the set comprises a language. In this case Rice's theorem concerns the decidability of languages.

C Why Δt is not a physical variable

Recall that as mentioned in Section 4.2, Δt is not a physical variable, but rather it is a parameter of the evolution function. At first one might think that it makes more sense to have Δt be a physical variable in the universe, fixed in the value v_0 . The idea would be to design the framework so that if we change the value of this physical Δt from some t_1 to some $t_2 \neq t_1$, without changing any of the rest of the universe, then V would simulate V' for t_2 iterations into the future rather than t_1 . In addition, in this alternative approach g would only be an explicit function of w_0 and n_0 , and not of Δt .

This would be particularly problematic in the case of self-simulation though. In general, we are interested in evolving an arbitrary initial state of the universe an arbitrary time into the future. That initial state and that time into the future that interests us are completely independent. In particular, we might be interested in evolving the initial state of the universe to a future time that differs from whatever value the variable Δt specified in v_0 might have. So it would seem that we need *two* times into the future to be specified in this alternative framework.

There are also more formal problems with this alternative approach. Suppose that Δt were specified as the initial value of a component of w , the physical variable giving the universe external to the computer. In this approach, there would

be no way to have $\mathcal{W}(\Delta t', w'_0, n'_0) = w'_0$ (in order to get free simulation), while also having $\mathcal{T}(\Delta t', w'_0, n'_0) \neq \Delta t'$. Yet as described below, in fact it is impossible to have $\mathcal{T}(\Delta t', w'_0, n'_0) = \Delta t'$ for all $\Delta t'$ — the pristine RPCT could not hold if we imposed that requirement.

On the other hand, suppose that rather than having Δt be specified as a component of w_0 , we had the initial state of the computer be some string $\langle p_0, \Delta t \rangle$, where we want to view p_0 as a fixed “simulation program” that would run on the computer, taking Δt as input. In other words, suppose that Δt were always specified as part of the initial state of the computer, n , and g did not involve Δt directly. In this case, because g itself does not vary with Δt the recursion theorem would not just fix the initial simulation program we want the computer to run, but also the time Δt into the future we are running it. We would not be able to simulate the evolution of the universe to an arbitrary time in the future.²⁴

Ultimately, the way we are getting around these problems in the framework I’ve adopted is by having Δt just be a parameter of the evolution function, specifying how far into the future we want to simulate the evolution of the physical universe. and not a physical variable. So for example it does not need to be reproduced as the output of the computer if the cosmological universe is to simulate its own future.) A consequence of this workaround is that we need to hard-code n^* into the initial state of the program in a way that depends on Δt .²⁵

D Subtleties with the model in Section 2.6

Section 2.6 presented an example of how a portion of our actual universe could implement the purely formal model of universe V given in Section 2.2. In that example the computer was implemented as a UTM, replete with tapes, etc. This might seem a particularly awkward way of modeling a portion of our cosmological universe whose dynamics is computationally universal. After all, our universe “runs in parallel”, whereas a UTM is a serial system. This means that to implement such a UTM would require copious use of energy barriers and the like, to prevent the parallel nature of Hamilton’s equations from “leaking through”.

²⁴A subtlety is that the recursion theorem can in general be satisfied by more than one n^* — by an infinite number in fact. It is not clear though that there is a way to exploit this flexibility so that there is at least one n^* that satisfies the recursion theorem for all Δt . So for simplicity, this possibility is not considered in this paper.

²⁵Indeed, the way the derivation of the self-simulation lemma uses the recursion theorem can be viewed as a special case of the generalized parameter-dependent form of the recursion theorem given in [50]. The derivation of the self-simulation lemma uses that generalized form for the special case where the space of parameters is single-dimensional. So in the notation of that paper, here $m = 1$.

Given that, it might seem more reasonable to use an infinite one-dimensional cellular automata (CA [72, 36, 46, 29]) running a computationally universal rule, as a model of a universal computer that is purely parallel. Arguing against this though, one might object that such a CA performs an infinite number of operations in parallel in each iteration. A single conventional TM, operating on one cell per tape in each iteration, could not execute any such single iteration of a CA in finite time. So such an infinite CA is doing something that is beyond the ability of a Turing machine.

Despite this though, in point of fact one-dimensional CA are *not* viewed in the literature as more powerful than TMs. The reason is that that infinite number of parallel operations done by a CA does not provide it the ability to compute functions that cannot be computed by Turing machines. (E.g., one cannot use a one-dimensional CA to solve the halting problem.) Formally, this discrepancy is resolved by working with arbitrarily large — but finite — sub-strings of an infinite one-dimensional CA.

Another subtlety is that in the example in Section 2.6, the first thing that happens when V evolves is that w_0 is copied onto the input tape. That single operation could take an arbitrarily large number of iterations of the UTM, depending on the size of W . This would require adding some large constant dependent on $|W|$ to many of the calculations in this paper. To avoid having to consider this technical issue, it might be convenient to implicitly modify V so that N evolves as a conventional UTM that is augmented with a special instruction. That special instruction copies an arbitrarily number of bits from W into N in a single iteration. Similarly, it might be convenient to include a special instruction that copies an arbitrarily large number of bits from one portion of N to another portion. Whether to consider such a modified V or not is really just a matter of taste.

E Work in the philosophy of science literature on the simulation hypothesis

The modern iteration of the simulation hypothesis has many versions that vary in their precise details. Common to all of these versions is the idea that some portion of a physical universe, including some conscious reasoning agents that exist in that universe, might in fact be part of a simulation that is being run in a physical computer of some different, super-sophisticated reasoning agents, like some race of aliens [13, 33, 22].²⁶ Under the simulation hypothesis, those “conscious re-

²⁶A special case of this idea in which the “super-sophisticated reasoning agents” are our descendants, and so by simulating us they are simulating their ancestors, was the focus of some of the recent work on the simulation hypothesis [13]. Here I impose no such restrictions on the agents

soning agents” would just be variables evolving in a computer program running in that physical computer designed by that putative super-sophisticated race of aliens.

In particular, it might be that a portion of *our* physical universe is being simulated this way, and that in fact *we* are just variables evolving in some computer in that physical universe. The vague implication is that if we are in fact a simulation, then what we perceive as reality does not have any sort of “objective truth”, since it is just a simulation. What we think of as “reality” would in fact be a chimera, simply a play being put on by the super-sophisticated race of aliens. Crucially, that play could just as easily be put on by some other super-sophisticated race of aliens, in some other physical universe — what we perceive as reality is just the play, not the physical universe in which the play is being staged.

Much of philosophy of science, going back to Kant (at least), would be rendered moot if we are simulations. In particular, this would be the case for many of the flavors of “*** realism” (e.g., scientific realism), which suppose there “exists” some “real” physical truth which is not reducible to a mathematical formalism, a physical truth that instead is somehow “concrete”, and non-mathematical in its very essence — and so not just a computer algorithm. Indeed, the simulation hypothesis is consistent (in the formal logic sense) with all axiomatic systems for the laws of physics that have been offered. However, if true it would imply that there is no such “concrete”, non-mathematical reality. This establishes that it is impossible to establish the logical necessity of any of these flavors of realism, from all suggested axiomatic foundations to the laws of physics.

The central idea of the simulation hypothesis has been extended in an obvious manner, simply by noting that the aliens that simulate our universe might themselves be a simulation in the computer of some even more sophisticated species, and so on and so on, in a sequence of ever-more sophisticated aliens. Similarly, going the other direction, in the not too distant future it is conceivable that we would be able to produce our own simulation of a universe running in some future computer that we will create, a simulation complete with variables that constitute “conscious, reasoning agents”. Indeed, we might produce such a simulation in which the reasoning agents can produce their own simulated universe in turn, etc. If this were the case, then in the near future there might be a sequence of species’, each one with a computer running a simulation that produces the species just below it in the sequence, with us somewhere in the “middle” of that sequence.

The literature of the last few decades on the simulation hypothesis has focused almost exclusively on whether we, in our universe are such a simulation. This question is answered rather trivially if we adopt the view of ontic structural

simulating us. Indeed, the physical universe in which the those reasoning agents exist might not even be one that obeys the laws of physics of our (simulated) universe.

realism [45, 14], especially if that view is formalized in terms of Tegmark’s level IV multiverse [66, 67]: yes, in some universes we are a simulation, and no, in some other universes we are not. (See also [18, 55].)

Even if we subscribe to the idea of a level IV multiverse though, there are several research directions one can pursue. Some people have focused on ascribing probabilities to the specific hypothesis that we, in our universe, are programs in such a simulation [13, 22, 37]. In the language of the level IV multiverse, these researchers have asked what the relative probabilities are of the set of all universes in which we are programs in a simulation and the set of all universes in which we are not.

These investigations of relative probabilities all beg (many) questions about what such probabilities might actually mean, formally. (Just as there are such questions concerning the idea of ascribing probabilities to the universes in a level IV multiverse in general.) How does one ascribe a measure, obeying the Kolmogorov axioms based on some associated sigma algebra, to a collection of events each of which is a universe? Note that it is not even clear that this collection would be a set, rather than a proper class of some sort. Directly reflecting such problems, one could not experimentally assess any proposed value of such a probability, e.g., with a proper scoring rule.²⁷ See [70] for some related arguments.

In particular, many papers considering such probabilities assume, implicitly or otherwise, that there is some way to assign a uniform probability to all universes. However, it is proven in Section 2.4 that this is mathematically impossible in sets with the cardinality of the kind of universes considered in these other papers. One might imagine that rather than a uniform probability distribution, one could assign some sort of Cantor (“fair coin”) measure to the set of possible universes, as is done for example in algorithmic information theory. However, it is proven in Section 7 that it is impossible to assign a Cantor measure to a set of universes obeying a natural set of restrictions.

Yet others have considered issues concerning the simulation hypothesis that might best be characterized as “moral” in nature [34, 23, 28, 40]. This work considers potential normative implications if the simulation hypothesis holds, or even such implications that follow from the mere possibility that the simulation hypothesis holds.

²⁷One might try to use a Bayesian “degree of belief” interpretation of probability to circumvent this issue. However, without any decision that would be made based on the probability assigned, and associated loss function, one could not apply Bayesian decision theory. So it is hard to see how “degree of belief” is meaningful in this case.

F Semi-formal work that has been done on the CS theoretic aspects of the simulation hypothesis

The only work that has been done that is even tangentially related to the CS aspects of the simulation hypothesis was a set of semi-formal results presented in [73]. Some of those semi-formal results define “simulation” in terms of the relationship between languages at different levels of the time, polynomial, or exponential hierarchies of CS theory, and some of them define “simulation” in terms of the relationship between languages at different levels the arithmetic or analytic hierarchies of logic, or similar constructions. (See [24, 6, 32] for background on such hierarchies.) Other results in [73] define “simulation” in terms of the relationship between computational machines with different Turing degrees [65, 62]. Yet others consider the implications of Gödel’s second incompleteness theorem, defining simulation in terms of languages (in the logic theory sense) with nested sets of axioms.

None of these results in [73] consider what it means for a computational machine to simulate a *physical universe*, per se. There is no concern for “coupling” the mathematics of CS theory to the laws of physics of our universe. No work has been done to date concerning the simulation hypothesis that focuses on the dynamical system of an evolving physical universe, like ours.

References

- [1] Scott Aaronson, *Guest column: Np-complete problems and physical reality*, ACM Sigact News **36** (2005), no. 1, 30–52.
- [2] ———, *Quantum computing since democritus*, Cambridge University Press, 2013.
- [3] ———, *Why philosophers should care about computational complexity*, Computability: Turing, Gödel, Church, and Beyond (2013), 261–327.
- [4] Abbas Acar, Hidayet Aksu, A Selcuk Uluagac, and Mauro Conti, *A survey on homomorphic encryption schemes: Theory and implementation*, ACM Computing Surveys (Csur) **51** (2018), no. 4, 1–35.
- [5] Peter Mark Ainsworth, *What is ontic structural realism?*, Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics **41** (2010), no. 1, 50–57.
- [6] Sanjeev Arora and Boaz Barak, *Computational complexity: a modern approach*, Cambridge University Press, 2009.

- [7] Pablo Arrighi and Gilles Dowek, *The physical church-turing thesis and the principles of quantum theory*, International Journal of Foundations of Computer Science **23** (2012), no. 05, 1131–1145.
- [8] Jeremy Avigad, *Computability and incompleteness*, (2007).
- [9] John Baez and Mike Stay, *Algorithmic thermodynamics*, Mathematical Structures in Computer Science **22** (2012), no. 05, 771–787.
- [10] John D Barrow, *Living in a simulated universe*, na, 2007.
- [11] _____, *Godel and physics*, Kurt Gödel and the Foundations of Mathematics: Horizons of Truth (2011), 255.
- [12] Silas R Beane, Zohreh Davoudi, and Martin J. Savage, *Constraints on the universe as a numerical simulation*, The European Physical Journal A **50** (2014), no. 9, 148.
- [13] Nick Bostrom, *Are we living in a computer simulation?*, The Philosophical Quarterly **53** (2003), no. 211, 243–255.
- [14] Otávio Bueno, Steven French, and James Ladyman, *On representing the relationship between the mathematical and the empirical*, Philosophy of Science **69** (2002), no. 3, 497–518.
- [15] Tom Campbell, Houman Owhadi, Joe Sauvageau, and David Watkinson, *On testing the simulation theory*, arXiv preprint arXiv:1703.00058 (2017).
- [16] Robert Cardona, Eva Miranda, Daniel Peralta-Salas, and Francisco Presas, *Constructing turing complete euler flows in dimension 3*, Proceedings of the National Academy of Sciences **118** (2021), no. 19, e2026818118.
- [17] Sean Carroll, *Maybe we do not live in a simulation: The resolution conundrum*, Blog post, 2016.
- [18] Sean Carroll and Frank Wilczek, *Frank wilczek on the present and future of fundamental physics*, <https://www.preposterousuniverse.com/podcast/2021/01/18/130-frank-wilczek-on-the-present-and-future-of-fundamental-physics>.
- [19] Carlton M Caves, *Entropy and information: How much information is needed to assign a probability*, Complexity, Entropy and the Physics of Information (1990), 91–115.
- [20] _____, *Information and entropy*, Physical Review E **47** (1993), no. 6, 4010.

- [21] Gregory Chaitin, Francisco A Doria, and Newton CA Da Costa, *Goedel's way: Exploits into an undecidable world*, CRC Press, 2011.
- [22] David J Chalmers, *Reality+: Virtual worlds and the problems of philosophy*, (2022).
- [23] Vincent Conitzer, *A puzzle about further facts*, Erkenntnis **84** (2019), no. 3, 727–739.
- [24] S Barry Cooper, *Computability theory*, Chapman and Hall/CRC, 2017.
- [25] B Jack Copeland, *The church-turing thesis*, The Stanford Encyclopedia of Philosophy (2023).
- [26] B Jack Copeland and Oron Shagrir, *The church-turing thesis: logical limit or breached barrier?*, Communications of the ACM **62** (2018), no. 1, 66–74.
- [27] Toby S Cubitt, David Perez-Garcia, and Michael M Wolf, *Undecidability of the spectral gap*, Nature **528** (2015), no. 7581, 207–211.
- [28] Barry Dainton, *On singularities and simulations*, Journal of Consciousness Studies **19** (2012), no. 1-2, 42–85.
- [29] Codd E. F., *Cellular automata*, Academic Press, New York, 1968.
- [30] Steven French and James Ladyman, *In defence of ontic structural realism*, Scientific structuralism, Springer, 2010, pp. 25–42.
- [31] Al Gaulle, *Upson's familiar quotations. (1984-1985)*, Tech. report, Cornell University, 1985.
- [32] Petr Hájek, *Arithmetical hierarchy and complexity of computation*, Theoretical Computer Science **8** (1979), no. 2, 227–237.
- [33] Salah Hamieh, *On the simulation hypothesis and its implications*, Journal of Modern Physics **12** (2021), no. 5, 541–551.
- [34] Robin Hanson, *How to live in a simulation*, Journal of Evolution and Technology **7** (2001), no. 1, 3–13.
- [35] Piet Hut, Mark Alford, and Max Tegmark, *On math, matter and mind*, Foundations of Physics **36** (2006), no. 6, 765–794.
- [36] N. Israeli and N. Goldenfeld, *Computational irreducibility and the predictability of complex physical systems*, Physical review letters **92** (2004).

- [37] David Kipping, *An objective bayesian analysis of life's early start and our late arrival*, Proceedings of the National Academy of Sciences **117** (2020), no. 22, 11995–12003.
- [38] Stephen Cole Kleene, NG De Bruijn, J de Groot, and Adriaan Cornelis Zaanen, *Introduction to metamathematics*, vol. 483, van Nostrand New York, 1952.
- [39] M. Li and Vitanyi P., *An introduction to kolmogorov complexity and its applications*, Springer, 2008.
- [40] Abraham Lim, *Why we are not living in a computer simulation*, International journal for the study of skepticism **12** (2022), no. 4, 331–351.
- [41] Seth Lloyd, *Any nonlinear gate, with linear gates, suffices for computation*, Physics Letters A **167** (1992), no. 3, 255–260.
- [42] _____, *Ultimate physical limits to computation*, Nature **406** (2000), no. 6799, 1047–1054.
- [43] _____, *The universe as quantum computer*, A Computable Universe: Understanding and exploring Nature as computation (2013), 567–581.
- [44] Gordon McCabe, *Structural realism and the mind*, (2006).
- [45] Gordon McCabe et al., *Possible physical universes*, Zagadnienia Filozoficzne w Nauce (2005), no. 37, 73–97.
- [46] M. Mitchell, *An introduction to genetic algorithms*, MIT Press, Cambridge, MA, 1996.
- [47] Cristopher Moore, *Unpredictability and undecidability in dynamical systems*, Physical Review Letters **64** (1990), no. 20, 2354.
- [48] _____, *Generalized shifts: unpredictability and undecidability in dynamical systems*, Nonlinearity **4** (1991), no. 2, 199.
- [49] Cristopher Moore and Stephan Mertens, *The nature of computation*, Oxford University Press, 2011.
- [50] Yiannis N Moschovakis, *Kleene's amazing second recursion theorem*, Bulletin of Symbolic Logic **16** (2010), no. 2, 189–239.
- [51] Michael A Nielsen, *Computable functions, quantum measurements, and quantum dynamics*, Physical Review Letters **79** (1997), no. 15, 2915.

- [52] Michael A Nielsen and Isaac L Chuang, *Quantum computation and quantum information*, Cambridge university press, 2010.
- [53] Michael A Nielsen, Mark R Dowling, Mile Gu, and Andrew C Doherty, *Quantum computation as geometry*, Science **311** (2006), no. 5764, 1133–1135.
- [54] Ian Parberry, *Knowledge, understanding, and computational complexity*, Optimality in biological and artificial networks?, Routledge, 2013, pp. 141–160.
- [55] Gualtiero Piccinini, *The physical church–turing thesis: Modest or bold?*, The British Journal for the Philosophy of Science (2011).
- [56] Gualtiero Piccinini and Corey Maley, *Computation in physical systems*, (2010).
- [57] ———, *Computation in physical systems*, Stanford Encyclopedia of Philosophy (2021).
- [58] Marian Boykan Pour-El and Ian Richards, *Noncomputability in models of physical phenomena*, International Journal of Theoretical Physics **21** (1982), no. 6, 553–555.
- [59] Panu Raattkainen, *On the philosophical relevance of godel’s incompleteness theorems*, Revue internationale de philosophie **59** (2005), no. 4, 513–534.
- [60] John E Savage, *Models of computation*, vol. 136, Addison-Wesley Reading, MA, 1998.
- [61] Naoto Shiraishi and Keiji Matsumoto, *Undecidability in quantum thermalization*, Nature communications **12** (2021), no. 1, 1–7.
- [62] Richard A Shore, *The turing degrees: an introduction*, Forcing, iterated ultrapowers, and Turing degrees, World Scientific, 2016, pp. 39–121.
- [63] Michael Sipser, *Introduction to the theory of computation*, ACM Sigact News **27** (1996), no. 1, 27–29.
- [64] ———, *Introduction to the theory of computation*, vol. 2, Thomson Course Technology Boston, 2006.
- [65] Robert I Soare, *Turing computability: Theory and applications*, vol. 300, Springer, 2016.

- [66] Max Tegmark, *Is “the theory of everything” merely the ultimate ensemble theory?*, Annals of Physics **270** (1998), no. 1, 1–51.
- [67] _____, *The mathematical universe*, Foundations of Physics **38** (2008), no. 2, 101–150.
- [68] John Von Neumann, Arthur Walter Burks, et al., *Theory of self-reproducing automata*, (1966).
- [69] Burton Watson et al., *Zhuangzi: Basic writings*, Columbia University Press, 2003.
- [70] Brian Weatherson, *Are you a sim?*, The Philosophical Quarterly **53** (2003), no. 212, 425–431.
- [71] Wikipedia, *State transition systems*, December 2023.
- [72] Stephen Wolfram, *Cellular automata as models of complexity*, Nature **311** (1984), no. 5985, 419–424.
- [73] David H. Wolpert, *What can we know about that which we cannot even imagine?*, 2024, In press in “New Frontiers in Science”, Marilena Streit-Bianchi and Vittorio Gorini (Ed.).
- [74] W. H. Zurek, *Algorithmic randomness and physical entropy*, Phys. Rev. A **40** (1989), 4731–4751.