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ON A CLASS OF HIGH RESOLUTION TOTAL-VARIATION-STABLE FINITE-DIFFERENCE SCHEMES*

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With appendix by Peter D. Lax

Abstract. This paper presents a class of explicit and implicit second order accurate finite-difference schemes for the computation of weak solutions of hyperbolic conservation laws. These highly nonlinear schemes are obtained by applying a nonoscillatory first order accurate scheme to an appropriately modified flux. The so derived second order accurate schemes achieve high resolution, while retaining the robustness of the original first order accurate scheme.

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1. Introduction. In this paper, we consider numerical solutions of the initial value problem for hyperbolic conservation laws:

$$u_t+f(u)_x=0, \qquad u(x,0)=u_0(x), \qquad x\in D. \tag{1.1a}$$

Here $u(x,t)$ is an m-vector of unknowns and $f(u)$, the flux, is a C^1 function of m components. $u_0(x)$, the initial data, is assumed to be a function of bounded total variation in D. We assume that $u_0(x)$ is either of compact support in $D=(-\infty,\infty)$ or periodic in a finite interval D. (1.1a) can also be written in a matrix form:

$$u_t+A(u)u_x=0, \qquad A(u)=f_u. \tag{1.1b}$$

Equation (1.1) is called hyperbolic if all eigenvalues $\{a^k(u)\}$ of $A(u)$ are real and the set of its right eigenvectors $\{r^k(u)\}$ is complete.

It is well known that solutions to (1.1) may develop shocks and contact discontinuities even when the initial data are smooth. To allow for discontinuous solutions, one admits weak solutions which satisfy the system (1.1) in the sense of distribution theory, i.e.,

$$\int_0^\infty\int[\phi_t u+\phi_x f(u)]\,dx\,dt+\int\phi(x,0)u_0(x)\,dx=0, \tag{1.2}$$

where $\phi(x,t)$ is any C^∞ test function.

Here, and elsewhere in this paper, we leave limits of x-integration and summation unspecified, to be interpreted according to the nature of the initial data $u_0(x)$ and its domain of definition D. Similarly, the test functions $\phi(x,t)$ are also assumed to have the same x-behavior as the initial data.

We assume that the system (1.1) possesses an *entropy function* $U(u)$, defined as follows:

(i) U satisfies

$$U_u f_u=F_u \tag{1.3a}$$

where F is some other function called *entropy flux*.

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(ii) U is a convex function of u.

Furthermore, we assume that weak solutions $u(x, t)$ of (1.1) that satisfy the entropy inequality

$$U(u)_t + F(u)_x \leqq 0 \tag{1.3b}$$

in the sense that

$$-\int_0^\infty \int [\phi_t U(u) + \phi_x F(u)]\, dx\, dt - \int \phi(x, 0) U(u_0(x))\, dx \leqq 0 \tag{1.3c}$$

for all nonnegative smooth test functions, $\phi(x, t)$, are uniquely determined by their initial data (see [7] and [8] for more details).

Next we consider finite-difference (FD) approximations $v(x, t)$ to weak solutions of (1.1). These are piecewise-constant functions defined on a rectangular grid:

$$v(x, t) = v_j^n, \qquad (x, t) \in \left((j - \tfrac{1}{2})h, (j + \tfrac{1}{2})h\right) \times [n\tau, (n+1)\tau); \tag{1.4a}$$

h is the spatial mesh size and τ is the time step.

The values v_j^n for $n \geqq 0$ are generated from the average values of the initial data

$$v_j^0 = \frac{1}{h} \int_{(j-1/2)h}^{(j+1/2)h} u_0(x)\, dx \tag{1.4b}$$

by

$$v_j^{n+1} = (S \cdot v^n)_j \tag{1.4c}$$

where S is some FD scheme.

In this paper we consider numerical schemes of the form

$$L \cdot v^{n+1} = R \cdot v^n \tag{1.4d}$$

where L and R are centered finite-difference (FD) operators that depend on $(2l + 1)$ and $(2r + 1)$ points, respectively.

If $l = 0$, and $L = I$, the identity operator, then $S = R$ in (1.4c) and the scheme is *explicit*; in this case, initial data with compact support, as well as periodic ones, are appropriate. If $l > 0$, then the scheme is *implicit*, and $S = L^{-1}R$. The definition of L^{-1}, the inverse of L, necessarily involves boundary conditions; to avoid the complication of treating boundaries, we consider only periodic initial data for implicit schemes.

Let us denote the dependence of the FD approximation (1.4) on the grid, assuming $\tau = O(h)$, by $v_h(x, t)$ and consider the limiting process $h \to 0$ in the strip $0 \leqq t \leqq T$. If there exists $u(x, t)$ such that

$$\lim_{h \to 0} v_h(x, t) = u(x, t), \qquad 0 \leqq t \leqq T, \quad x \in D, \tag{1.5}$$

we say that the scheme (1.4) is *convergent*.

The subject of this paper is the design of second order accurate schemes (in the sense of truncation analysis) for the computation of weak solutions of the nonlinear initial value problem (1.1). In general it is difficult to directly relate the particular form of discretization used in the scheme to the limiting process of convergence. Fortunately, when the partial differential equation is linear ($A = A(x, t)$ in (1.1b)), and the numerical scheme is linear, then the convergence of the numerical approximation is implied by the consistency and stability of the scheme. Furthermore, in the constant coefficient case (A = const. in (1.1b)), L_2-stability of the scheme can be easily found as a consequence of the von Neumann condition. However, L_2-stability is necessary, but not sufficient, for convergence of *nonlinear* schemes. This has led to the design of "artificial viscosity" methods for the computation of the weak solution of the nonlinear problem (1.1) (see [12]). These schemes have been developed according to the following guidelines: (i) Design a second order accurate scheme so that its linearized version is L_2-stable. (ii) Add numerical dissipation to damp oscillations and to control "nonlinear instabilities". Since the addition of numerical dissipation brings about loss of information, the designer of such numerical schemes finds himself in a position where he has to compromise accuracy to achieve stability, or vice versa.

The goal of this paper is to show a way to avoid this "uncertainty principle"; to do so we have to depart from the linear theory and "go nonlinear".

The first step is to replace the requirement of linearized L_2-stability by that of stability in the sense of uniformly bounded total variation in x, and to add the requirement of consistency with the conservation law (1.1) and its entropy inequality (1.3); this implies the convergence of the scheme to the unique weak solution of (1.1) (see § 2).

Next we use this convergence theorem to design schemes that are both robust and accurate. The construction guidelines are as follows: (i) Design a scheme that is consistent with the conservation law and its entropy inequality and is total-variation stable. (ii) Extract from it as much numerical dissipation as possible subject to the requirements in (i).

The resulting schemes are second order accurate (in the sense of truncation error analysis) and *truly nonlinear*, i.e., they are nonlinear even in the constant coefficient case.

Numerical experiments with these schemes are reported in [5], [15]–[17].

The organization of the rest of this paper is as follows: § 2: Convergence theorem; § 3: Basic theory of total-variation-diminishing (TVD) schemes; § 4: Second order accurate TVD schemes; § 5: Extension to systems of conservation laws; § 6: Application to steady state calculations; § 7: Appendix by Peter Lax.

2. Convergence theorem. In this section we describe the convergence theorem of [7] and outline its proof.

First we define the notions of consistency and stability to be used in this theorem.

We say that a finite-difference scheme is *consistent with the conservation law* (1.1) if it can be written in the form

$$v_j^{n+1} = v_j^n - \lambda(\bar{f}_{j+1/2} - \bar{f}_{j-1/2}), \qquad \lambda = \frac{\tau}{h}, \tag{2.1a}$$

where

$$\bar{f}_{j+1/2} = \bar{f}(v_{j-r+1}^n, \cdots, v_{j+r}^n; v_{j-l+1}^{n+1}, \cdots, v_{j+l}^{n+1}), \tag{2.1b}$$

the numerical flux, is a Lipschitz continuous function that satisfies

$$\bar{f}(u, \cdots, u; u, \cdots, u) = f(u); \tag{2.1c}$$

$f(u)$ is the flux of the system of conservation laws (1.1).

Similarly, we say that the numerical scheme S (1.4c) is *consistent with the entropy inequality* (1.3) of the system of conservation laws (1.1) if there exists a numerical entropy flux $\bar{F}_{j+1/2}$ that is consistent with $F(u)$ (in the same sense as (2.1c)), such that

$$U(v_j^{n+1}) \leqq U(v_j^n) - \lambda(\bar{F}_{j+1/2} - \bar{F}_{j-1/2}). \tag{2.2}$$

We turn now to consider a sequence of numerical approximations $v_h(x, t)$, $x \in D$, $0 \leqq t \leqq T$, with $h \to 0$ and $\tau = O(h)$. We say that the numerical scheme is *total-variation* (TV) *stable* if the total variation in x of $v_h(x, t)$,

$$TV\ (v_h(x, t)) = \sum_j |v_{j+1}^n - v_j^n|, \tag{2.3}$$

is uniformly bounded in t and h; here n is the integer part of t/τ.

THEOREM 2.1 (convergence). *Let us assume that the scheme* (1.4) *is consistent with the conservation law* (2.1) *and its entropy inequality* (2.2). *If the resulting numerical approximation is* TV *stable, then the scheme is convergent and its limit is the unique weak solution of* (1.1) *that satisfies the entropy inequality* (1.3).

In [7] this theorem is stated and proved for explicit schemes defined on an irregular moving mesh. The initial data considered there are of bounded functions in $(-\infty, \infty)$ that have bounded total variation. The limit (1.5) in this theorem is in the sense of bounded, L_1^{loc} convergence.

In the present paper we consider both explicit and implicit schemes. To avoid treatment of boundary conditions we assume periodicity, unless otherwise stated. Examination of the proof of the convergence theorem in [7] shows that it carries over to the present situation with only minor changes.

LEMMA 2.2. *If an* FD *scheme is consistent with the conservation law* (2.1) *and is total-variation stable* (2.3), *then it is also stable in the norm*

$$\|v\| = \text{TV}\ (v) + \max_j |v_j|. \tag{2.4}$$

Proof. We prove this lemma for periodic boundary conditions. Since the assumptions of this lemma hold for each of the components, we prove this lemma componentwise, using scalar notation. The assumption of total-variation stability means that there exists a constant B, not depending on h or n, such that

$$\text{TV}\ (v^n) \leqq B \cdot \text{TV}\ (v^0)$$

for all n. By the definition of total variation, we have

$$\text{TV}\ (v^n) = \sum_j |v_{j+1}^n - v_j^n| \geqq \max_j v_j^n - \min_j v_j^n;$$

thus

$$\max_j v_j^n - \min_j v_j^n \leqq B \cdot \text{TV}\ (v^0). \tag{2.5a}$$

The combination of periodic boundary conditions and consistency with the conservation law (2.1) implies that for all $n \geqq 0$

$$\sum_j v_j^n = \sum_j v_j^0. \tag{2.5b}$$

Since

$$\min_j v_j^n \leqq \frac{1}{J} \sum_j v_j^n \leqq \max_j v_j^n, \tag{2.5c}$$

where J is the number of mesh points in a period, we get from (2.5a)–(2.5c) that

$$\min_j v_j^n \geqq -B \cdot \text{TV}\ (v^0) + \max_j v_j^n \geqq -B \cdot \text{TV}\ (v^0) + \frac{1}{J} \sum_j v_j^0,$$

$$\max_j v_j^n \leqq B \cdot \text{TV}\ (v^0) + \min_j v_j^n \leqq B \cdot \text{TV}\ (v^0) + \frac{1}{J} \sum_j v_j^0;$$

hence

$$\max_j |v_j^n| \leqq B \cdot \mathrm{TV}(v^0) + \left|\frac{1}{J}\sum_j v_j^0\right| \leqq B \cdot \mathrm{TV}(v^0) + \max_j |v_j^0|,$$

and therefore

$$\|v^n\| \leqq \hat{B}\|v^0\|, \qquad \hat{B} = \max(2B, 1),$$

which is the assertion of this lemma.

We note that the proof of Lemma 2.2 combines the uniform bound of the maximal oscillation of the numerical solution (due to TV stability) with that of its mean value (due to the conservation form) to conclude uniform boundedness of the approximation. This argument remains valid for explicit schemes with initial data of compact support; it can also be extended to other boundary conditions.

LEMMA 2.3 (Glimm). *If a numerical scheme satisfies the assumptions of Lemma* 2.2, *then for all n and m*

$$|v_h(\cdot, t_m) - v_h(\cdot, t_n)|_{L_1} \leqq O(|t_m - t_n|).$$

Proof. See [7, Lemma 3.6]; note that the numerical flux (2.1) is assumed to be Lipschitz continuous.

We turn now to outline the proof of Theorem 2.1. The first part of it follows the same reasoning as Glimm's convergence proof in [1].

Proof of the convergence theorem (Theorem 2.1). By Lemma 2.2, the FD scheme is uniformly bounded in the maximum norm. Since the functions $v_h(x, t)$ have uniformly bounded total variation in x and are uniformly bounded, by compactness we deduce that for any fixed t, a subsequence converges in the L_1-norm. By a diagonal argument, we can construct a subsequence that is L_1-convergent for, say, all rational values of t. Using Lemma 2.3 we conclude that this subsequence converges in the norm

$$\max_t \|v(\cdot, t)\|_{L_1}$$

to a limit that we denote by $u(x, t)$. Certainly, $u(x, t)$ is uniformly bounded and has a uniformly bounded total variation in x. By [8, Thm. 1.1] (which is a straightforward extension of the Lax–Wendroff theorem in [11]) $u(x, t)$ is a weak solution of (1.1) that satisfies its entropy inequality (1.3).

Finally, we claim that not only a subsequence of $v_h(x, t)$, but the whole sequence, converges to $u(x, t)$. For if not, then by compactness we could select a subsequence that tends to a limit $\bar{u} \neq u$. Now $\bar{u}$ and u are both weak solutions of (1.1) that satisfy its entropy inequality (1.3); furthermore, they both have the same prescribed initial data $u_0(x)$. Therefore, by the uniqueness of the initial value problem assumed in § 1, we conclude that $\bar{u} \equiv u$. This proves that the scheme is convergent.

Note that, because of the indirect nature of the convergence proof, we have no error estimates.

3. Basic theory of TVD schemes. In this section we consider the scalar conservation law ($m = 1$ in (1.1))

$$u_t + f(u)_x \equiv u_t + a(u)u_x = 0. \tag{3.1}$$

Equation (3.1) states that the solution is constant along the characteristic lines $dx/dt = a(u)$. Hence, if the initial data are smooth, the total variation in x of the solution remains constant in time, as long as the characteristic lines do not intersect. Even when a shock forms due to convergence of characteristics, the solution at each point

(x, t) remains well defined in terms of a backward drawn characteristic line, connecting it to the initial data. Since the initial data in the domain of dependence of the shock curve are dissipated into it, the formation of a shock may actually decrease the total variation in x.

In view of this behavior of the total variation we are interested in designing numerical approximations to the scalar conservation law (3.1) that are total-variation diminishing (TVD).

In this section we review the theory that has been developed in [5] for explicit TVD[1] schemes, and extend it to implicit ones.

We define the total variation of a mesh function v to be

$$\mathrm{TV}(v) = \sum_{j=-\infty}^{\infty} |\Delta_{j+1/2} v| \tag{3.2}$$

where $\Delta_{j+1/2} v = v_{j+1} - v_j$. (We use the general notation convention

$$\Delta_{j+1/2} b = b_{j+1} - b_j \tag{3.3}$$

for any mesh function b). We say that a numerical scheme

$$v^{n+1} = S \cdot v^n \tag{3.4a}$$

is TVD if

$$\mathrm{TV}(v^{n+1}) \leqq \mathrm{TV}(v^n); \tag{3.4b}$$

here v^n denotes an approximation to the solution of (3.1) at $t = n\tau$, where τ is the time step.

In this paper we consider the one-parameter family of schemes in conservation form

$$L \cdot v^{n+1} = R \cdot v^n \tag{3.5a}$$

where L and R are the following finite-difference operators:

$$(L \cdot v)_j = v_j + \eta\lambda(\bar{f}_{j+1/2} - \bar{f}_{j-1/2}), \tag{3.5b}$$

$$(R \cdot v)_j = v_j - (1-\eta)\lambda(\bar{f}_{j+1/2} - \bar{f}_{j-1/2}). \tag{3.5c}$$

Here $\bar{f}$ is a numerical flux that is consistent with $f(u)$ in (3.1). The scheme (3.5) can be written in the form (2.1) with $\bar{\bar{f}} = (1-\eta)\bar{f}^n_{j+1/2} + \eta \bar{f}^{n+1}_{j+1/2}$, and therefore is consistent with the conservation law in the sense of (2.1). The scheme (3.5) with $\eta = 0$ is the explicit forward-Euler scheme; for $\eta > 0$ (2.5) is implicit, thus $S = L^{-1}R$ in (3.4a). The scheme (3.5) with $\eta = 1$ is the implicit backward-Euler scheme.

LEMMA 3.1. *If R is a total-variation diminishing* (TVD) *operator, i.e., if*

$$\mathrm{TV}(R \cdot v) \leqq \mathrm{TV}(v), \tag{3.6a}$$

and L is a total-variation increasing (TVI) *operator, i.e., if*

$$\mathrm{TV}(L \cdot v) \geqq \mathrm{TV}(v), \tag{3.6b}$$

then the numerical scheme (3.4a), *with $S = L^{-1}R$, is* TVD.

Proof. $\mathrm{TV}(v^{n+1}) \leqq \mathrm{TV}(L \cdot v^{n+1}) = \mathrm{TV}(R \cdot v^n) \leqq \mathrm{TV}(v^n)$.

In this section we consider 3-point schemes, i.e., $\bar{f}_{j+1/2}$ in (3.5) is of the form

$$\bar{f}_{j+1/2} = \bar{f}(v_j, v_{j+1}) \tag{3.7a}$$

[1] In [5] we use the term TVNI (total variation nonincreasing) instead of TVD.

where

(3.7b) $$\bar{f}(u, u) = f(u).$$

Assuming the numerical flux in (3.7) to be Lipschitz continuous, we can write it in the general form

(3.8) $$\bar{f}_{j+1/2} \equiv \bar{f}(v_j, v_{j+1}) = \frac{1}{2}(f_j + f_{j+1}) - \frac{1}{2\lambda} q_{j+1/2} \Delta_{j+1/2} v$$

where $q_{j+1/2} = q(v_j, v_{j+1})$ is some bounded function and $\Delta_{j+1/2} v$ is (3.3). The notation $f_i = f(v_i)$ is used throughout this paper.

Another general form to describe a Lipschitz continuous numerical flux in (3.7) is

(3.9a) $$\bar{f}_{j+1/2} \equiv \bar{f}(v_j, v_{j+1}) = f_j - \frac{1}{\lambda} \tilde{C}^+_{j+1/2} \Delta_{j+1/2} v = f_{j+1} - \frac{1}{\lambda} \tilde{C}^-_{j+1/2} \Delta_{j+1/2} v,$$

where $\tilde{C}^\pm_{j+1/2} = \tilde{C}^\pm(v_j, v_{j+1})$ are some bounded functions. Comparing (3.8) with (3.9a), we get

(3.9b) $$\tilde{C}^\pm_{j+1/2} = \tfrac{1}{2}(q_{j+1/2} \mp \nu_{j+1/2})$$

where $\nu_{j+1/2}$ is the mean value local CFL number

(3.10a) $$\nu_{j+1/2} = \lambda a_{j+1/2},$$

(3.10b) $$a_{j+1/2} = \frac{\Delta_{j+1/2} f}{\Delta_{j+1/2} v}.$$

Using (3.9a) for the numerical fluxes in (3.5b) and (3.5c), we see that the finite-difference operators L and R can be expressed in the form

(3.11a) $$(L \cdot v)_j = v_j - \eta(\tilde{C}^+_{j+1/2} \Delta_{j+1/2} v - \tilde{C}^-_{j-1/2} \Delta_{j-1/2} v),$$

(3.11b) $$(R \cdot v)_j = v_j + (1-\eta)(\tilde{C}^+_{j+1/2} \Delta_{j+1/2} v - \tilde{C}^-_{j-1/2} \Delta_{j-1/2} v).$$

Next we formulate sufficient conditions for a finite difference operator to be TVD (3.6a) or TVI (3.6b).

LEMMA 3.2. *Let Z be a finite difference operator of the form*

(3.12a) $$(Z \cdot v)_j = v_j + C^+_{j+1/2} \Delta_{j+1/2} v - C^-_{j-1/2} \Delta_{j-1/2} v$$

where $\Delta_{i+1/2} v$ *is* (3.3).

(a) *If for all* j

(3.12b) $$C^\pm_{j+1/2} \geqq 0, \qquad C^+_{j+1/2} + C^-_{j+1/2} \leqq 1,$$

then Z *is* TVD.

(b) *If for all* j

(3.12c) $$-\infty < C \leqq C^\pm_{j+1/2} \leqq 0,$$

then Z *is* TVI.

Proof. Denote $w_j = (Z \cdot v)_j$ and subtract (3.12a) at j from (3.12a) at $j+1$ to get

(3.13a) $$\Delta_{j+1/2} w = (1 - C^+_{j+1/2} - C^-_{j+1/2}) \Delta_{j+1/2} v + C^+_{j+3/2} \Delta_{j+3/2} v + C^-_{j-1/2} \Delta_{j-1/2} v.$$

The inequalities (3.12b) are exactly the conditions for the coefficients of $\Delta_{j+k+1/2} v$, $k = 0, \pm 1$ in (3.13a) to be nonnegative. Hence taking the absolute value of (3.13a) and using the triangle inequality we get

(3.13b) $$|\Delta_{j+1/2} w| \leqq (1 - C^+_{j+1/2} - C^-_{j+1/2}) |\Delta_{j+1/2} v| + C^+_{j+3/2} |\Delta_{j+3/2}| + C^-_{j-1/2} |\Delta_{j-1/2} v|;$$

summing (3.13b) over j, assuming v to be periodic or of compact support, we obtain

$$\begin{aligned} \mathrm{TV}(Z\cdot v) &= \sum_j |\Delta_{j+1/2}w| \\ &\leqq \sum_j (1-C^+_{j+1/2}-C^-_{j+1/2})|\Delta_{j+1/2}v| + \sum_j C^+_{j+3/2}|\Delta_{j+3/2}v| + \sum_j C^-_{j-1/2}|\Delta_{j-1/2}v| \\ &= \sum_j |\Delta_{j+1/2}v| = \mathrm{TV}(v); \end{aligned}$$

this completes the proof of part (a).

To prove part (b) we rewrite (3.13a) as

$$\text{(3.13c)}\quad \Delta_{j+1/2}w + (-C^+_{j+3/2})\Delta_{j+3/2}v + (-C^-_{j-1/2})\Delta_{j-1/2}v = (1-C^+_{j+1/2}-C^-_{j-1/2})\Delta_{j+1/2}v$$

and observe that (3.6c) is the condition for the coefficients of $\Delta_{j+k+1/2}$, $k=0,\pm 1$ in (3.13c) to be nonnegative. Hence taking the absolute value of (3.13c) and using the triangle inequality, we get

$$\text{(3.13d)}\quad |\Delta_{j+1/2}w| - C^+_{j+3/2}|\Delta_{j+3/2}v| - C^-_{j-1/2}|\Delta_{j-1/2}v| \geqq (1-C^+_{j+1/2}-C^-_{j+1/2})|\Delta_{j+1/2}v|.$$

Summing (3.13d) over j, we obtain

$$\begin{aligned} \mathrm{TV}(Z\cdot v) - \sum_j (C^+_{j+1/2}+C^-_{j+1/2})|\Delta_{j+1/2}v| &= \sum_j |\Delta_{j+1/2}w| - \sum_j C^+_{j+3/2}|\Delta_{j+3/2}v| - \sum_j C^-_{j-1/2}|\Delta_{j-1/2}v| \\ &\geqq \sum_j (1-C^+_{j+1/2}-C^-_{j+1/2})|\Delta_{j+1/2}v| \\ &= \mathrm{TV}(v) - \sum_j (C^+_{j+1/2}+C^-_{j+1/2})|\Delta_{j+1/2}v|. \end{aligned}$$

Comparing the two extreme parts of the above relation we conclude that

$$\mathrm{TV}(Z\cdot v) \geqq \mathrm{TV}(v);$$

this completes the proof of Lemma 3.2.

Next we use Lemma 3.2 to formulate a sufficient condition for the schemes (3.5) to be TVD.

THEOREM 3.3. *If $q_{j+1/2}$ in* (3.8) *satisfies*

$$\text{(3.14)}\qquad |\nu_{j+1/2}| \leqq q_{j+1/2} \leqq \frac{1}{1-\eta},$$

where $\nu_{j+1/2}$ is the mean value CFL *number* (3.10), *then the scheme* (3.5) *with* (3.8) *is* TVD.

Proof. Comparing (3.11) with (3.12a) we see that both the finite difference operators L and R can be written in the form (3.12a) with

$$\text{(3.15a)}\qquad C^{\pm} = (1-\eta)\tilde{C}^{\pm} \quad \text{for } R,$$

$$\text{(3.15b)}\qquad C^{\pm} = -\eta\tilde{C}^{\pm} \quad \text{for } L.$$

Using Lemma 3.2 we conclude from (3.9b), (3.14) and (3.15) that the operator L in (3.5) is TVI, while the operator R is TVD. Therefore, it follows from Lemma 3.1 that the scheme (3.5) is TVD.

In the following we consider the class of schemes, $S_1(\eta)$, that is defined by (3.5) and (3.8) with

(3.16a) $$q_{j+1/2}=Q(\nu_{j+1/2};\varepsilon_{j+1/2})$$

where $Q(x;\varepsilon)$ is either

(3.16b) $$Q(x;\varepsilon)=\begin{cases}\varepsilon, & |x|<\varepsilon,\\ |x|, & |x|\geqq\varepsilon,\end{cases}$$

or

(3.16c) $$Q(x;\varepsilon)=\begin{cases}\dfrac{1}{2}\left(\dfrac{x^2}{\varepsilon}+\varepsilon\right), & |x|<\varepsilon,\\ |x|, & |x|\geqq\varepsilon;\end{cases}$$

here $\varepsilon_{j+1/2}=\varepsilon(v_j, v_{j+1})\geqq 0$. Note that

(3.16d) $$Q(x;0)=|x| \quad\text{and}\quad Q(x;\varepsilon)\geqq\varepsilon/2.$$

Whenever we are not concerned with the ε-dependence we shall use $Q=Q(x)$. For $\varepsilon>0$ both (3.16b) and (3.16c) are positive approximations to $|x|$; $Q(x)$ in (3.16c) is continuously differentiable, but it is only Lipschitz continuous in (3.16b). If $0\leqq\varepsilon<1/(1-\eta)$ in (3.16b)–(3.16c) then $q_{j+1/2}$ in (3.16a) satisfies the inequality (3.14) for all $|\nu_{j+1/2}|\leqq 1/(1-\eta)$. Therefore, it follows immediately from Theorem 3.3 that

COROLLARY 3.4. *The scheme* $S_1(\eta)$, $0\leqq\eta\leqq 1$, *is* TVD *under the* CFL-*like restriction*

(3.17) $$\lambda\max|a_{j+1/2}|\leqq\frac{1}{1-\eta}.$$

Observe that the backward-Euler implicit scheme, $\eta=1$ in (3.5), is unconditionally TVD, while the forward-Euler explicit scheme, $\eta=0$, is TVD under the CFL condition of 1.

The set of TVD schemes contains the set of monotone schemes (see [5], [18], [19]). However, unlike the case of monotone schemes (see [6]), the fact that a scheme is TVD does not automatically imply that it is consistent with the entropy inequality (2.2). For example, the first order accurate upstream differencing (3.5) and (3.8) with $q_{j+1/2}=|\nu_{j+1/2}|$ (i.e. $\varepsilon=0$ in (3.16b)–(3.16c)) admits stationary "expansion shocks" (see [8] and [9]).

In [3] and [9, Appendix A] we represent the numerical flux (3.8) with (3.16) as the flux across $x=0$ in an approximate solution to the Riemann IVP

(3.18a) $$u_0(x)=\begin{cases}v_j, & x<0,\\ v_{j+1}, & x>0.\end{cases}$$

(3.18b) $$w\left(\frac{x}{t};v_j,v_{j+1}\right)=\begin{cases}v_j, & \dfrac{x}{t}<a_{j+1/2},\\ v_{j+1}, & \dfrac{x}{t}>a_{j+1/2},\end{cases}$$

where $a_{j+1/2}$ is (3.10b). The fact that (3.18b) approximates the solution to (3.18a) by a shock, regardless of the entropy condition, is responsible for the stationary "expansion shocks" in the upstream differencing schemes mentioned above. Therefore, we consider the following modification of (3.18b):

$$w\left(\frac{x}{t}; v_j, v_{j+1}\right) = \begin{cases} v_j, & \dfrac{x}{t} < a_{j+1/2} - \delta, \\ v_{j+1/2}\left(\dfrac{x}{t}\right), & a_{j+1/2} - \delta < \dfrac{x}{t} < a_{j+1/2} + \delta, \\ v_{j+1}, & a_{j+1/2} + \delta < \dfrac{x}{t}. \end{cases} \tag{3.18c}$$

(3.16b) corresponds to taking $v_{j+1/2}(x/t)$ to be the constant state

$$v_{j+1/2}\left(\frac{x}{t}\right) \equiv \frac{1}{2}(v_j + v_{j+1}), \tag{3.18d}$$

while (3.16c) corresponds to the linear function

$$v_{j+1/2}\left(\frac{x}{t}\right) = \frac{1}{2}(v_j + v_{j+1}) + \frac{x/t - a_{j+1/2}}{2\delta}(v_{j+1} - v_j); \tag{3.18e}$$

δ has the dimension of velocity and

$$\varepsilon = \lambda\delta \tag{3.18f}$$

in (3.16).

In [9, Appendix A] we show that if δ in (3.18c) is chosen so that the fan $|x/t - a_{j+1/2}| < \delta$ covers the domain of influence of the initial discontinuity (3.18a), then the approximate solution in (3.18c)–(3.18d) is consistent with the entropy inequality of the Riemann problem. We present there an expression for $\delta = \delta(v_j, v_{j+1})$ that in addition to the domain of influence condition also satisfies that $\delta = 0$ if the solution to the Riemann problem (3.18a) is a shock.

Using the representation of the forward-Euler scheme, $\eta = 0$ in (3.5), (3.8) and (3.16b), as a Godunov-type scheme corresponding to the Riemann solver (3.18c) (see [8] and [9]), we deduce that it is consistent with the entropy inequality (2.2). Since this scheme is also consistent with the conservation law (2.1) and it is TVD under the CFL restriction (3.12) of 1, we conclude by Theorem 2.1 that

COROLLARY 3.5. *The explicit forward-Euler scheme,* S_1 ($\eta = 0$), *is convergent and its limit is the unique weak solution of* (3.1).

We note that the modification of $|x|$ in (3.16b) is equivalent to adding an *entropy viscosity* term with a coefficient ε to the upstream differencing scheme. Choosing $\varepsilon = \varepsilon_{j+1/2}$ locally by the domain of influence condition mentioned above, we get:

$$\text{(i)} \qquad \varepsilon_{j+1/2} = O(|v_{j+1} - v_j|), \tag{3.19a}$$

$$\text{(ii)} \qquad \varepsilon_{j+1/2} = 0 \quad \text{if (3.18a) is a shock.} \tag{3.19b}$$

Hence the added entropy viscosity term is $O(|\Delta v|^3)$ and, unlike the standard artificial viscosity terms, it vanishes in the neighborhood of shocks.

The schemes in the class (3.5) with (3.8) are all first order accurate in both time and space, except for $\eta = \frac{1}{2}$, where the scheme is second order accurate in time.

In the next section we present a rather general technique to convert a 3-point first order accurate TVD scheme of the form (3.5), (3.8) into a 5-point second order accurate (in both time and space, or just space) TVD scheme of the same generic form. This second stage of our design of high-resolution TVD schemes capitalizes on the fact that the exact solution to (3.1) is TVD due to the phenomenon of propagation along characteristics, and is independent of the particular form of the flux $f(u)$ in (3.1). Similarly, the original first order accurate scheme is TVD subject only to the CFL restriction (3.17), independently of the particular form of the flux. Thus to achieve second order accuracy while retaining the TVD property, we use the original TVD scheme with an appropriately modified flux $f+(1/\lambda)g$. The requirements on g are: (i) The characteristic speed $\Delta g/\Delta v$ should be uniformly bounded; (ii) The modified scheme should be second order accurate almost everywhere. The details of the construction of such a g are given in the next section.

We would like to point out that the resulting second order accurate scheme has to be *truly nonlinear*, i.e. nonlinear even in the constant coefficient case, since *linear* second order accurate TVD schemes do not exist (see [5]).

4. Second order accurate TVD schemes. In this section we convert the first order accurate TVD schemes of Corollary 3.4 into second order accurate ones by using a technique developed in [5].

The basic idea is to use the first order accurate scheme with an appropriately modified flux $f+(1/\lambda)g$, i.e.

$$\text{(4.1a)} \qquad \lambda\bar{f}_{j+1/2}=\frac{\lambda}{2}(f_j+f_{j+1})+\frac{1}{2}(g_j+g_{j+1})-\frac{1}{2}Q(\nu_{j+1/2}+\gamma_{j+1/2})\Delta_{j+1/2}v$$

where

$$\text{(4.1b)} \qquad \gamma_{j+1/2}=\frac{\Delta_{j+1/2}g}{\Delta_{j+1/2}v},$$

thus preserving the TVD property of the scheme.

The construction of g to be used in (4.1) is done in two steps: First we use truncation error analysis to find the Taylor expansion of g for which (3.5) with (4.1) is a second order accurate approximation to (3.1). Then we apply a regularization procedure to the point values of g to ensure that $\gamma_{j+1/2}$ in (4.1b) is bounded.

The first step is summarized in the following lemma:

LEMMA 4.1. *If g in* (4.1) *has the Taylor expansion*

$$\text{(4.2a)} \qquad g=h\sigma(\nu)v_x+O(h^2)$$

where $\nu=\lambda a$ and

$$\text{(4.2b)} \qquad \sigma(\nu)=\tfrac{1}{2}Q(\nu)+(\eta-\tfrac{1}{2})\nu^2,$$

then (3.5) *is a second order accurate approximation to* (3.1).

Proof. The function $Q(x)$ (3.16) is at least Lipschitz continuous, therefore,

$$\begin{aligned}|Q(\nu_{j+1/2}+\gamma_{j+1/2})-Q(\nu_{j+1/2})|\,|\Delta_{j+1/2}v| &\leq \text{const.}\,|\gamma_{j+1/2}|\,|\Delta_{j+1/2}v|\\ &=\text{const.}\,|g_{j+1}-g_j|=O(h^2).\end{aligned}$$

Hence the Taylor expansion to $O(h^2)$ around $x_{j+1/2}$ of (4.1a) can be written as

$$\text{(4.3a)} \qquad \lambda\bar{f}_{j+1/2}=\left[\lambda f+g-\frac{h}{2}Q(\nu)v_x\right]_{j+1/2}+O(h^2).$$

Using (4.2) we get

$$\text{(4.3b)} \qquad \lambda\bar{f}_{j+1/2}=[\lambda f+h(\eta-\tfrac{1}{2})\nu^2v_x]_{j+1/2}+O(h^2).$$

It follows therefore that

$$\lambda(\bar{f}^n_{j+1/2}-\bar{f}^n_{j-1/2})=[\tau f_x+(\eta-\tfrac{1}{2})\tau^2(a^2v_x)_x]^n_j+O(h^3),$$
$$\lambda(\bar{f}^{n+1}_{j+1/2}-\bar{f}^{n+1}_{j-1/2})=\lambda(\bar{f}^n_{j+1/2}-\bar{f}^n_{j-1/2})+\tau^2(f_{xt})^n_j+O(h^3),$$

where we have assumed the $O(h^2)$ terms in the above expansions to be at least Lipschitz continuous and $\tau=O(h)$.

Using these expansions in (3.5) we get by rearranging terms that

$$\begin{aligned} v^{n+1}_j&=v^n_j-[\tau f_x+(\eta-\tfrac{1}{2})\tau^2(a^2v_x)_x]^n_j-\eta\tau^2(f_{tx})^n_j+O(h^3)\\ &=[v-\tau f_x+\tfrac{1}{2}\tau^2(a^2v_x)_x]^n_j+O(h^3), \end{aligned}$$

which completes the proof of the lemma.

We remark that if $\sigma(\nu)$ in (4.2b) is taken to be

$$\sigma(\nu)=\tfrac{1}{2}Q(\nu), \tag{4.4a}$$

then (4.3b) becomes

$$\lambda\bar{f}_{j+1/2}=\lambda f_{j+1/2}+O(h^2), \tag{4.4b}$$

which shows that the modified scheme is second order accurate in space; this observation will be used in dealing with steady state solutions.

The essential part of the second step is given in the following lemma:

LEMMA 4.2. *Let* $\bar{g}$ *be defined by*

$$\bar{g}_j=s\cdot\max\{0,\min(\sigma_{j+1/2}|\Delta_{j+1/2}v|,s\cdot\sigma_{j-1/2}\Delta_{j-1/2}v)\} \tag{4.5a}$$

where

$$s=\operatorname{sgn}(\Delta_{j+1/2}v) \tag{4.5b}$$

and $\sigma_{j+1/2}=\sigma(\nu_{j+1/2})$ *is* (4.2b). *Then*

(4.6) (a) $$\bar{g}=h\sigma(\nu)v_x+O(h^2),$$

(4.7) (b) $$|\bar{\gamma}_{j+1/2}|=\left|\frac{\Delta_{j+1/2}\bar{g}}{\Delta_{j+1/2}v}\right|\leqq\sigma(\nu_{j+1/2}).$$

(*Note that* $\sigma(\nu)\geqq 0$ *for* $|\nu|\leqq 1/(1-\eta)$.)

Proof. (a) See [5], or verify directly.

(b) Note that the sequence $\bar{g}_j$ in (4.5a) cannot change sign without vanishing at the transition point. Consequently

$$\begin{aligned} |\bar{g}_{j+1}-\bar{g}_j|&\leqq\max(|\bar{g}_j|,|\bar{g}_{j+1}|)\\ &\leqq\max[\min(\sigma_{j+1/2}|\Delta_{j+1/2}v|,\sigma_{j-1/2}|\Delta_{j-1/2}v|),\\ &\qquad\qquad\min(\sigma_{j+3/2}|\Delta_{j+3/2}v|,\sigma_{j+1/2}|\Delta_{j+1/2}v|)]\\ &\leqq\sigma_{j+1/2}|\Delta_{j+1/2}v|. \end{aligned}$$

Comparing the two extreme parts of the above inequality, we get (4.7). Hence the characteristic speed $\bar{\gamma}_{j+1/2}$ is uniformly bounded.

The most general g in (4.1) to satisfy the requirements of our construction is of the form

$$g=\bar{g}+\hat{g} \tag{4.8a}$$

where $\bar{g}$ is (4.5) and $\hat{g}$ is subject to the requirements

(4.8b) (i) $$\hat{g} = O(h^2),$$

(4.8c) (ii) $$\hat{\gamma}_{j+1/2} = \Delta_{j+1/2}\hat{g}/\Delta_{j+1/2}v \quad \text{is uniformly bounded.}$$

We denote the class of schemes (3.5) with (4.1) where g is (4.5) with (4.8) by $S_2(\eta)$, and recall that they are obtained by applying the first order accurate schemes $S_1(\eta)$ to a modified flux. The following theorem summarizes the results of our construction technique.

THEOREM 4.3. *The scheme* $S_2(\eta)$ *is a second order accurate approximation to* (3.1) *and is* TVD *for all* $0 \leqq \eta < 1$ *under the* CFL-*like restriction*

(4.9a) $$\max_j |\nu_{j+1/2} + \gamma_{j+1/2}| \leqq \frac{1}{1-\eta}.$$

Proof. It follows from (4.8b) and (4.6) that g in (4.8a) satisfies (4.2). In [5] we show that the coefficients in the $O(h^2)$ term in (4.2) may be discontinuous, but only at a finite number of points. Hence by Lemma 4.1 the scheme is second order accurate.

That the scheme is TVD follows from its being the same TVD scheme of Corollary 3.4, except that it is applied to the flux $f + (1/\lambda)g$.

Exactly as in (3.9), the numerical flux (4.1) can be rewritten in the form

(4.9b) $$\lambda \bar{f}_{j+1/2} = \lambda f_j + g_j - \tilde{C}^+_{j+1/2}\Delta_{j+1/2}v = \lambda f_{j+1} + g_{j+1} - \tilde{C}^-_{j+1/2}\Delta_{j+1/2}v$$

where

(4.9c) $$\tilde{C}^\pm_{j+1/2} = \tfrac{1}{2}[Q(\nu+\gamma) \mp (\nu+\gamma)]_{j+1/2}.$$

Therefore we conclude in the same way as for Corollary 3.4 that the modified scheme is TVD under the modified CFL restriction (4.9a); this completes the proof of Theorem 4.3.

Unless otherwise stated we take in (4.8) $\hat{g} \equiv 0$; hence, from this point on, $g \equiv \bar{g}$ (4.5) and $\gamma \equiv \bar{\gamma}$.

We remark that by (4.7)

$$|\nu + \gamma| \leqq |\nu| + |\gamma| \leqq |\nu| + \sigma(\nu);$$

hence the CFL restriction on $|\nu + \gamma|$ (4.9) can be expressed in terms of a CFL restriction on $|\nu|$:

(4.10a) $$\max_j |\nu_{j+1/2}| \leqq \mu.$$

where μ is the largest value of x for which the following inequality holds:

(4.10b) $$|x| + \sigma(x) \leqq \frac{1}{1-\eta};$$

$\sigma(x)$ is (4.2b).

We observe that in the two important cases of $\eta = 0$ and $\eta = 1$, the restriction (4.10) is identical to the original one (3.17), i.e., $\mu = 1$ for $\eta = 0$ (see [5]) and $\mu = \infty$ for $\eta = 1$.

Next we discuss a particular property of the numerical flux (4.1) with (4.5). The dependence of g_j on v in (4.5) is of the form

(4.11a) $$g_j = g(v_{j-1}, v_j, v_{j+1}).$$

$\bar{f}_{j+1/2}$ in (4.1) depends on g_j and g_{j+1}; therefore its dependence on v is of the form

$$\bar{f}_{j+1/2} = \bar{f}(v_{j-1}, v_j, v_{j+1}, v_{j+2}). \tag{4.11b}$$

Since (3.5) depends on $\bar{f}_{j-1/2}$ and $\bar{f}_{j+1/2}$, the modified scheme becomes a 5-point scheme.

The consistency relation of a numerical flux of a genuinely 5-point scheme is

$$\bar{f}(v, v, v, v) = f(v).$$

The numerical flux of our second order accurate TVD scheme, however, has a consistency relation which is similar to that of the original 3-point scheme (3.7), in the following sense:

PROPERTY 4.4. *$\bar{f}_{j+1/2}$ (4.11b), the numerical flux (4.1) with (4.5), satisfies*

$$\bar{f}(u, v, v, w) = f(v) \tag{4.11c}$$

for all v and for all u and w.

Proof. The particular form (4.11c) is equivalent to setting $v_j = v_{j+1} = v$ in (4.1) and (4.5); thus $\Delta_{j+1/2}v = 0$. In this case $g(u, v, v) = g(v, v, w) = 0$ for (4.5) in the form (4.11a), or equivalently $g_j = g_{j+1} = 0$ in (4.1). Since $\nu_{j+1/2}$ (3.10) and $\gamma_{j+1/2}$ (4.1b) are bounded (see (4.7)), it follows that $Q(\nu_{j+1/2} + \gamma_{j+1/2})\Delta_{j+1/2}v = 0$ (independent of the particular value assigned to $\nu_{j+1/2}$ for $\Delta_{j+1/2}v = 0$). Hence $\bar{f}_{j+1/2} = \frac{1}{2}[f(v_j) + f(v_{j+1})] = f(v)$, which completes the proof of Property 4.4.

Having Property 4.4 in mind, we shall loosely refer to this 5-point modified scheme as "essentially a 3-point scheme". This property is the reason for the ease of handling of the extra numerical boundary conditions, and for the tridiagonal structure of the linearized left-hand side of the implicit schemes (to be discussed later).

Now that we have shown that the modified second order accurate scheme is consistent with the conservation law (3.1), we conclude by Theorem 4.3 and the first part of the proof of the convergence theorem that

COROLLARY 4.5. *The second order accurate* TVD *scheme of Theorem* 4.3 *has a convergent subsequence, the limit of which is a weak solution of* (3.1).

Numerical experiments reported in [5] indicate that the second order accurate TVD scheme of this section (with an appropriate choice of ε in (3.16), (3.18)) is consistent with the entropy inequality; however, at present we do not have a rigorous proof for this fact. Consequently we cannot use the second part of Theorem 2.1 to conclude full convergence. We console ourselves with the following:

COROLLARY 4.6. *The second order accurate* TVD *scheme of Theorem* 4.3 *is convergent in the constant coefficient case.*

This follows immediately from the uniqueness of the initial value problem in the constant coefficient case.

Again, we would like to point out that the second order accurate TVD scheme is nonlinear even in the constant coefficient case, and therefore Corollary 4.6 cannot be obtained via a standard linear L_2-stability argument.

5. Systems of conservation laws. In this section we describe how to extend the new second order accurate scalar schemes of § 4 to systems of conservation laws (1.1). Our extension technique is a somewhat generalized version of the procedure suggested by Roe in [13].

Let

$$R(u) = (r^1(u), \cdots, r^m(u)) \tag{5.1a}$$

be an $m \times m$ matrix the columns of which are the right eigenvectors of the Jacobian matrix $A(u)$ (1.1b). Since the set of right eigenvectors of $A(u)$ is assumed to be complete, $R(u)$ (5.1a) is invertible. We note that the rows $l^1(u), \cdots, l^m(u)$ of $R^{-1}(u)$ constitute the bi-orthonormal system of left-eigenvectors of $A(u)$,

$$l^i r^j = \delta_{ij}, \tag{5.1b}$$

and that

$$R^{-1}AR = \Lambda, \qquad \Lambda_{ij} = a^i \delta_{ij}, \tag{5.1c}$$

where $\{a^i(u)\}$ are the eigenvalues of $A(u)$.

It is shown in [10] that the initial value Riemann problem for (1.1)

$$u_0(x) = \begin{cases} u_L, & x < 0, \\ u_R, & x > 0, \end{cases} \tag{5.2a}$$

where $|u_R - u_L|$ is small, has a unique solution which consists of $m+1$ constant states

$$u_L = u^0, u^1, \cdots, u^m = u_R \tag{5.2b}$$

separated by m centered waves. If the k-characteristic field is genuinely nonlinear ($a_u^k r^k \neq 0$), then the k-wave separating u^{k-1} and u^k is either a shock or a rarefaction, depending on whether the characteristic field is convergent or divergent; if the k-field is linearly degenerate ($a_u^k r^k \equiv 0$) then the k-wave is exclusively a contact discontinuity. The intermediate states $\{u^j\}$ satisfy

$$u^k - u^{k-1} = \tilde{\alpha}^k r^k(u_L) + O(|u_R - u_L|^2) \tag{5.2c}$$

where

$$\tilde{\alpha}^k = l^k(u_L)(u_R - u_L). \tag{5.2d}$$

Unlike the scalar case, the total variation in x of the solution to the system of equations (1.1) is not necessarily a monotonic decreasing function of time, and it may actually increase at moments of interaction between waves.

In the following we describe how to extend our new scalar TVD scheme to systems of conservation laws so that the resulting scheme is TVD for the "locally frozen" constant-coefficient system. To accomplish that we define at each point a *local* system of characteristic fields, and apply our scheme "scalarly" to each of the fields.

Let $v_{j+1/2}$ denote some symmetric average of v_j and v_{j+1}, i.e.

$$v_{j+1/2} = V(v_j, v_{j+1}) \tag{5.3a}$$

where

$$V(u, v) = V(v, u), \qquad V(u, u) = u, \tag{5.3b}$$

and let $a_{j+1/2}^k$, $r_{j+1/2}^k$ and $l_{j+1/2}^k$ denote the respective quantities of $A(v_{j+1/2})$. Let $\alpha_{j+1/2}^k$ be defined by the relation

$$v_{j+1} - v_j = \sum_{k=1}^{m} \alpha_{j+1/2}^k r_{j+1/2}^k; \tag{5.4a}$$

it follows from (5.1b) that

$$\alpha_{j+1/2}^k = l_{j+1/2}^k (v_{j+1} - v_j). \tag{5.4b}$$

$\alpha^k_{j+1/2}$ is exactly the jump of the k-wave in the constant-coefficient Riemann problem

(5.4c) $$u_t + A(v_{j+1/2})u_x = 0, \qquad u(x,0) = \begin{cases} v_j, & x<0, \\ v_{j+1}, & x>0; \end{cases}$$

it is an $O(|\Delta_{j+1/2}v|^2)$ approximation to the size of the k-wave in (5.2).

We now extend the scalar scheme of Theorem 4.3 to general systems of conservation laws as follows:

(5.5a) $$v_j^{n+1} + \eta\lambda(\bar{f}^{n+1}_{j+1/2} - \bar{f}^{n+1}_{j-1/2}) = v_j^n - (1-\eta)\lambda(\bar{f}^n_{j+1/2} - \bar{f}^n_{j-1/2})$$

where

(5.5b) $$\bar{f}_{j+1/2} = \frac{1}{2}\left\{f_j + f_{j+1} + \frac{1}{\lambda}\sum_{k=1}^m [g_j^k + g_{j+1}^k - Q(\nu^k_{j+1/2} + \gamma^k_{j+1/2})\alpha^k_{j+1/2}]r^k_{j+1/2}\right\},$$

(5.5c) $$g_j^k = s \cdot \max[0, \min(\sigma(\nu^k_{j+1/2})|\alpha^k_{j+1/2}|, \sigma(\nu^k_{j-1/2})\alpha^k_{j-1/2}\cdot s)], \qquad s = \operatorname{sgn}(\alpha^k_{j+1/2});$$

(5.5d) $$\gamma^k_{j+1/2} = \frac{\Delta_{j+1/2}g^k}{\alpha^k_{j+1/2}},$$

(5.5e) $$\nu^k_{j+1/2} = \lambda a^k_{j+1/2};$$

here, $\alpha^k_{j+1/2}$ is (5.4), and as before, $Q(x)$ is (3.16) and $\sigma(x)$ is (4.2b).

LEMMA 5.1. *The finite-difference scheme* (5.5) *is a globally second order accurate approximation to smooth solutions of* (1.1).

Proof. For each k, exactly as in the scalar case, we have

$$g_j + g_{j+1} = 2\sigma(\nu_{j+1/2})\alpha_{j+1/2} + O(h^2)$$

and

$$Q(\nu_{j+1/2} + \gamma_{j+1/2})\alpha_{j+12} = Q(\nu_{j+1/2})\alpha_{j+1/2} + O(h^2);$$

hence

$$\begin{aligned} g_j + g_{j+1} - Q(\nu_{j+1/2} + \gamma_{j+1/2})\alpha_{j+1/2} &= [2\sigma(\nu_{j+1/2}) - Q(\nu_{j+1/2})]\alpha_{j+1/2} + O(h^2) \\ &= 2(\eta - \tfrac{1}{2})(\nu_{j+1/2})^2\alpha_{j+1/2}. \end{aligned}$$

Using (5.4a) we get

$$\begin{aligned} \frac{1}{2}\sum_{k=1}^m [g_j^k + g_{j+1}^k - Q(\nu^k_{j+1/2} + \gamma^k_{j+1/2})\alpha^k_{j+1/2}]r^k_{j+1/2} \\ &= (\eta - \tfrac{1}{2})\lambda^2 \sum_{k=1}^m [(a^k)^2\alpha^k r^k]_{j+1/2} + O(h^2) \\ &= (\eta - 1)\lambda^2[A(v_{j+1/2})]^2\Delta_{j+1/2}v + O(h^2). \end{aligned}$$

Thus, the numerical flux (5.5b) satisfies

$$\lambda\bar{f}_{j+1/2} = [\lambda f + h(\eta - \tfrac{1}{2})\lambda^2(A^2u_x)]_{j+1/2} + O(h^2),$$

which is the same as (4.3b). The rest of the proof is identical to that of Lemma 4.1.

We turn now to consider the constant coefficient case $A(u) \equiv A = \text{const.}$ in (1.1). In this case the eigenvalues $\{a^k\}$ and the eigenvectors $\{l^k\}$ and $\{r^k\}$ are also constant, and the pure IVP decouples into m scalar constant coefficient problems for the characteristic variables, i.e.,

$$w^k = l^k v, \tag{5.6a}$$

$$w_t^k + a^k w_x^k = 0. \tag{5.6b}$$

The total variation in x of each of the characteristic variables is diminishing in time. We show now that the same is true for (5.5), i.e. the finite-difference scheme also decouples into m scalar ones for the characteristic variables. To see that, we multiply (5.5a)–(5.5b) from the left by l^i and note that

$$l^i(f_j + f_{j+1}) = l^i A(v_j + v_{j+1}) = a^i(w_j^i + w_{j+1}^i);$$

for each of the characteristic variables (dropping the superscript), we get

$$w_j^{n+1} + \eta\lambda(\tilde{f}_{j+1/2}^{n+1} - \tilde{f}_{j-1/2}^{n+1}) = w_j^n - (1-\eta)\lambda(\tilde{f}_{j+1/2}^n - \tilde{f}_{j-1/2}^n) \tag{5.7a}$$

where

$$\tilde{f}_{j+1/2} = l\bar{f}_{j+1/2} = \frac{1}{2}\left\{a(w_j + w_{j+1}) + \frac{1}{\lambda}[g_j + g_{j+1} - Q(\nu_{j+1/2} + \gamma_{j+1/2})\alpha_{j+1/2}]\right\}. \tag{5.7b}$$

Since

$$\alpha_{j+1/2} = \Delta_{j+1/2} w, \tag{5.7c}$$

it follows that (5.7) with (5.5c)–(5.5e) is identical in the constant coefficient case to the second order accurate scalar TVD scheme of Theorem 4.3. Hence it is TVD with $\mathrm{TV}(w) = \sum_j |\alpha_{j+1/2}|$ and convergent under the CFL restriction (4.9). We conclude that

THEOREM 5.2. *The second order accurate scheme* (5.5) *in the constant coefficient case is* TVD *and convergent, where*

$$\mathrm{TV}(v) = \sum_j \sum_{k=1}^{m} |\alpha_{j+1/2}^k|, \tag{5.8}$$

subject to the CFL *restriction*

$$\max_{j,k} |\lambda a^k + \gamma_{j+1/2}^k| \leqq \frac{1}{1-\eta}. \tag{5.9}$$

We note that the particular form of averaging in (5.3) does not enter into the considerations of Theorem 5.2. However, if we require the scheme (5.5) for $m = 1$ to be identical to the scalar scheme of § 4, then we have to choose (5.3) so that $\nu_{j+1/2}$ in (5.5e) is the same as the mean value CFL number (3.10). This can be accomplished by taking the eigenvalues $a_{j+1/2}^k$ and the eigenvectors $l_{j+1/2}^k$ and $r_{j+1/2}^k$ in (5.5) to be those of $A(v_j, v_{j+1})$, where $A(u, v)$ is Roe's mean value Jacobian. This matrix satisfies (see [14]):

(i) $f(v) - f(u) = A(u, v)(v - u)$;

(ii) $A(u, u) = A(u)$;

(iii) $A(u, v)$ has real eigenvalues and a complete set of eigenvectors.

The existence of a mean value Jacobian that satisfies (iii) globally is related to the symmetrizability of (1.1) and can be constructed via the entropy function (see [4] and [8]). Roe in [14] constructs a mean value Jacobian for the Euler equations of gas dynamics of the form $A(u, v) = A(V(u, v))$, where $V(u, v)$ (5.3) is some particular average.

The numerical flux (5.5b) approximates the flux across $x = 0$ in the Riemann problems (5.2) and (5.4); therefore, the scheme (5.5) can be interpreted as a second order accurate Godunov-type scheme (see [3]).

6. Applications to steady-state calculations. Steady-state solutions (satisfying $f_x = 0$) for the initial value problem of (1.1) are either a constant state (for the periodic case) or a stationary shock (for the infinite domain). Hence the practical steady-state problem involves a forcing term (i.e. $f(u)_x = H(u, x)$) and appropriate boundary conditions to maintain the richer structure of a steady-state solution.

In this section we consider the calculation of a steady-state solution through a time-consistent approach. We choose initial data $u(x, 0) = u_0(x)$ such that $u(x, t)$ satisfies

$$\lim_{t\to\infty} u(x, t) = u_\infty(x), \tag{6.1}$$

where $u_\infty(x)$ is the steady state solution, and advance it in time using a scheme of the form (3.5a); the calculation is terminated when $\|v^{n+1} - v^n\|$ falls below some specified tolerance. Hence this time-consistent approach involves two *separate* limiting processes: (i) The limit $n \to \infty$ with fixed h and τ, which is the stationary solution of the finite difference scheme, $\lim_{n\to\infty} v^n = v_*$; (ii) The limit $h \to 0$ of v_*, which is the steady-state solution $u_\infty(x)$ of the partial differential equation.

Indulging in linear thinking, we may consider the time-consistent approach to steady state to be computationally equivalent to the decay to zero of the solution of (1.1) to the initial data $u(x, 0) = u_\infty(x) - u_0(x)$, where $u_\infty(x)$ is the steady-state solution (6.1). From this point of view, the use of TVD schemes for such calculations is most appropriate, as these schemes are particularly suited to handling processes of decay.

The rate of decay associated with the pure initial value problem (1.1) is $O(t^{-1/2})$ in the infinite domain, and $O(t^{-1})$ in the periodic case (see [2] and [10]). Hence the use of an *explicit* scheme in a time-consistent approach to steady state may turn out to be prohibitively expensive. Therefore, it makes good computational sense to use the implicit backward-Euler scheme, $\eta = 1$ in (5.5), which is unconditionally TVD (see (5.9)). This enables us to take very large time steps (that are guaranteed not to increase the oscillation of the numerical solution) and thus obtain fast convergence to the stationary solution v_* of the finite difference scheme. The spatial second order accuracy allows v_* to be a highly resolved yet nonoscillatory approximation to the steady-state solution $u_\infty(x)$.

We turn now to consider the second order accurate, TVD, modified backward-Euler scheme in the scalar case:

$$v_j^{n+1} + \lambda(\bar{f}_{j+1/2}^{n+1} - \bar{f}_{j-1/2}^{n+1}) = v_j^n; \tag{6.2a}$$

$\bar{f}_{j+1/2}^{n+1}$ is the following numerical flux calculated at the $(n+1)$ time level:

$$\bar{f}_{j+1/2} = \frac{1}{2}\left\{f_j + f_{j+1} + \frac{1}{\lambda}[g_j + g_{j+1} - Q(\nu_{j+1/2} + \gamma_{j+1/2})\Delta_{j+1/2}v]\right\}; \tag{6.2b}$$

$Q(x)$ is (3.16),

$$\Delta_{j+1/2}v = v_{j+1} - v_j,$$

$$\nu_{j+1/2} = \frac{\lambda\,\Delta_{j+1/2}f}{\Delta_{j+1/2}v},$$

$$\gamma_{j+1/2} = \frac{\Delta_{j+1/2}g}{\Delta_{j+1/2}v}, \tag{6.2c}$$

$$g_j = s \cdot \max\{0, \min[\sigma(\nu_{j+1/2})|\Delta_{j+1/2}v|, s \cdot \sigma(\nu_{j-1/2})\Delta_{j-1/2}v]\}$$

and

$$\sigma(\nu)=\begin{cases}\frac{1}{2}[Q(\nu)+\nu^2] & \text{for time-dependent problems,}\\ \frac{1}{2}Q(\nu) & \text{for steady-state problems.}\end{cases} \tag{6.2d}$$

It follows from (4.9b)–(4.9c) that the scheme (6.2) can be expressed, as in (3.11), by

$$(L\cdot v^{n+1})_j \equiv v_j^{n+1}+(C_{j+1/2}^{+})^{(n+1)}\Delta_{j+1/2}v^{n+1}-(C_{j-1/2}^{-})^{(n+1)}\Delta_{j-1/2}v^{n+1}=v_j^n \tag{6.3a}$$

where by (3.15b) and (4.9b)

$$C^{\pm}=-\tilde{C}^{\pm}=\tfrac{1}{2}[\pm(\nu+\gamma)-Q(\nu+\gamma)]. \tag{6.3b}$$

We note that since $Q(x)$ in (3.16) is such that $Q(x)\geqq|x|$, therefore $C^{\pm}$ in (6.3b) satisfy

$$C^{\pm}\leqq 0. \tag{6.4}$$

If we take $Q(x)=|x|$ (i.e. $\varepsilon=0$ in (3.16)) then $C^{\pm}=\pm(\nu+\gamma)^{\mp}$, which corresponds to upstream differencing with respect to the characteristic field $(\nu+\gamma)$. (On the right-hand side of the above equality we have used the functional convention $b^{+}=\max(b,0)$, $b^{-}=\min(b,0)$, not to be confused with the notation $C^{\pm}$ on the left-hand side.) Hence $Q(x)$ with $\varepsilon\geqq 0$ in (6.2) corresponds to upstream differencing supplemented by a viscosity term with coefficient ε.

The backward-Euler scheme (6.3) is the only one in the class of schemes (3.5) to have the identity operation as its right-hand side. Since it is the requirement for the right-hand side to be a TVD operator that limits the CFL number, it is also the only scheme in the class (3.5) that is unconditionally TVD.

The scheme (6.2) is implicit, i.e., the value of v^{n+1} is obtained as the solution of a system of *nonlinear* algebraic equations. This system consists of the equation (6.2) for all j in the computational domain, supplemented by appropriate boundary conditions. (To simplify our presentation we assume periodic boundary conditions throughout this section.) The iteration needed to solve this system of nonlinear equations, at each time step, say, by Newton's method may completely wipe out the savings gained by the ability to take larger time steps.

To overcome this obstacle, we consider a linearized version of (6.3) in which the coefficients $(C^{\pm})^{(n+1)}$ are replaced by $(C^{\pm})^{(n)}$, i.e.,

$$(L\cdot v^{n+1})_j \equiv v_j^{n+1}+(C_{j+1/2}^{+})^{(n)}\Delta_{j+1/2}v^{n+1}-(C_{j-1/2}^{-})^{(n)}\Delta_{j-1/2}v^{n+1}=v_j^n \tag{6.5a}$$

or

(6.5b)

$$(L\cdot v^{n+1})_j \equiv (C_{j-1/2}^{-})^{(n)}v_{j-1}^{n+1}+(1-C_{j-1/2}^{-}-C_{j+1/2}^{+})^{(n)}v_j^{n+1}+(C_{j+1/2}^{+})^{(n)}v_{j+1}^{n+1}=v_j^n.$$

(6.5) is a *tridiagonal system* of linear equations; furthermore, the nonpositivity of the coefficients $C^{\pm}$(6.4) implies that (6.5b) is *diagonally dominant*. Therefore it follows immediately that

LEMMA 6.1. *L, the finite difference operator on the left-hand side of* (6.5), *is unconditionally invertible.*

The nonpositivity of $C^{\pm}$ in (6.4) also implies that the operator L in (6.5) is TVI (see part (b) of Lemma 3.2). Since the right-hand side of (6.5) is the identity operator, it follows immediately from Lemma 3.1 that

LEMMA 6.2. *The linearized backward-Euler implicit scheme* (6.5) *is unconditionally* TVD.

We note that the linearized scheme (6.5) is not in conservation form and therefore should not be used to approximate time-dependent solutions. The following theorem shows that it is, however, a suitable scheme for the calculation of steady-state solutions.

THEOREM 6.3. *If v_* is the limit $n \to \infty$, with h and τ fixed, of* (6.5), *i.e. if* $\lim_{n\to\infty} v^n = v_*$, *then v_* satisfies*

$$\bar{f}_{j+1/2}(v_*) - \bar{f}_{j-1/2}(v_*) = 0 \tag{6.6}$$

where $\bar{f}_{i+1/2}(v_*)$ is the numerical flux (6.2b) *computed at v_*.*

Proof. Rewrite (6.5a) in the form

$$\begin{aligned} d_j + (C^+_{j+1/2})^{(n)}\Delta_{j+1/2}d - (C^-_{j-1/2})^{(n)}\Delta_{j-1/2}d \\ = (C^+_{j+1/2})^{(n)}\Delta_{j+1/2}v^n - (C^-_{j-1/2})^{(n)}\Delta_{j-1/2}v^n \end{aligned} \tag{6.7a}$$

where

$$d_j = v_j^{n+1} - v_j^n \tag{6.7b}$$

and observe that by (4.9b) the right-hand side of (6.7a) is equal to $-\lambda(\bar{f}^n_{j+1/2} - \bar{f}^n_{j-1/2})$, where $\bar{f}_{j+1/2}$ is (6.2b), i.e.

$$d_j + (C^+_{j+1/2})^{(n)}\Delta_{j+1/2}d - (C^-_{j-1/2})^{(n)}\Delta_{j-1/2}d = -\lambda(\bar{f}^n_{j+1/2} - \bar{f}^n_{j-1/2}). \tag{6.7c}$$

Next we note that v^n is a finite-dimensional vector, and therefore, the convergence $v^n \to v_*$ is componentwise. Hence $\lim_{n\to\infty} d_j = 0$ for all j and (6.6) follows from (6.7c).

We turn now to describe the extension of (6.5) to systems of conservation laws. As in § 5, our only design guideline is the requirement that the scheme will be TVD in the constant coefficient system case. To ensure that the steady state solution is consistent with the conservation form in the sense of (6.6), we use the form (6.7c) rather than (6.5). Using the extension technique of § 5, we get

$$d_j + K^+_{j+1/2}\Delta_{j+1/2}d - K^-_{j-1/2}\Delta_{j-1/2}d = -\lambda(\bar{f}^n_{j+1/2} - \bar{f}^n_{j-1/2}) \tag{6.8a}$$

where the numerical flux $\bar{f}^n$ is (5.5b)–(5.5c) with $\sigma(x)$ given by (6.2d); $K^\pm$ are $m \times m$ matrices given by

$$K^\pm_{j+1/2} = R_{j+1/2}\psi^\pm_{j+1/2}R^{-1}_{j+1/2} \tag{6.8b}$$

where $R_{j+1/2} = R(v_{j+1/2})$ (5.1a) and $\psi^\pm$ are the diagonal matrices

$$(\psi^\pm_{j+1/2})_{i,k} = \tfrac{1}{2}[\pm(\nu^k_{j+1/2} + \gamma^k_{j+1/2}) - Q(\nu^k_{j+1/2} + \gamma^k_{j+1/2})]\delta_{i,k}. \tag{6.8c}$$

The scheme (6.8) can also be written in the operator form

$$(I + K^+_{j+1/2}\Delta_{j+1/2} - K^-_{j-1/2}\Delta_{j-1/2})(v^{n+1} - v^n) = -\lambda(\bar{f}^n_{j+1/2} - \bar{f}^n_{j-1/2}) \tag{6.9}$$

where $I_{m\times m}$ is the identity matrix.

Next we consider the constant coefficient case and note that (6.9) differs from (5.5) with $\eta = 1$. The reason is that (5.5) is nonlinear even in the constant coefficient case, and (6.9) is its linearized version. However, since (6.9) decouples into m scalar schemes of the form (6.5) for the characteristic variables, we conclude by Lemma 6.2 and Theorem 2.1 that

THEOREM 6.4. *The second order accurate implicit scheme* (6.8)–(6.9) *in the constant coefficient case is unconditionally* TVD (5.8) *and convergent.*

In [17] we present numerical experiments with scheme (6.9) and compare it to conventional artificial viscosity methods. Altogether we find the performance of this new scheme to be quite pleasing.

Summary. A typical complaint about the application of artificial viscosity schemes to the solution of nonlinear problems is that they are either not robust or not accurate enough, or both.

In this paper we propose a setting for the design of schemes for nonlinear problems, in which one can obtain both robustness and accuracy at the same time.

We use the convergence Theorem 2.1 to *rigorously* settle the question of "robustness". Under the guidelines of this theorem, we consider the class of schemes that are total-variation stable and consistent with the conservation law and its entropy inequality.

We present a technique to convert first order accurate schemes in this class into second order ones. We remark that 2 seems to be the maximal order of accuracy of total-variation *diminishing* schemes, since monotonicity requirements seem to necessitate the occasional deterioration of the truncation error to $O(h^2)$.

The theory of this class of schemes is still in its infancy. Much more work is needed to formulate necessary and sufficient conditions for uniform boundedness of the total variation, as well as for consistency with the entropy inequality.

7. Appendix by Peter D. Lax. We give here a potentially useful general criterion for a linear difference operator to diminish total variation. We start with the general observation that over Z, the l^∞ and l^1 norms are dual to each other with respect to the l^2 duality. It follows then that if S is any operator mapping functions on Z into functions on Z, and if S^* is the transpose of S with respect to the l^2 scalar product, then the following relation holds between the norms of S and S^*:

$$|S|_{l^1} = |S^*|_{l^\infty}.$$

We take now S to be a difference operator:

$$S = \sum a_j T^j, \tag{A1}$$

T translation on Z, a_j functions on Z. Then

$$S^* = \sum T^{-j} a_j = \sum a_j^{(j)} T^{-j}, \tag{A1*}$$

where

$$a_j^{(j)}(k) = a_j(k+j). \tag{A2}$$

Now the l^∞ norm of a difference operator is easily determined:

$$|S^*|_{l^\infty} = \max_k \sum_j |a_j^{(j)}(k)| \tag{A3}$$

so $|S^*|_{l^\infty} \leqq 1$ if and only if

$$\sum_j |a_j(k+j)| < 1 \quad \text{for all } k. \tag{A4}$$

This is in particular the case if

$$a_j \geqq 0 \quad \text{and} \quad \sum_j a_j(k+j) \leqq 1 \quad \text{for all } k. \tag{A4$'$}$$

Thus, (A4) is the necessary and sufficient condition for S given by (A1) to diminish the l^1 norm.

Next we show how to use this to deduce a condition that a linear difference operator

$$R = \sum b_j T^j \tag{A5}$$

diminish total variation. By definition, for any function v defined on Z,

$$\text{TV } v = |(I-T)v|_{l^1}. \tag{A6}$$

We suppose now that R preserves the constant function. This is the case if and only if

$$\sum_j b_j(k) = 1 \quad \text{for all } k. \tag{A7}$$

Then there is an operator S such that

$$(I - T)R = S(I - T). \tag{A8}$$

To see this we express both sides as difference operators:

$$\begin{aligned}(I-T)R &= \sum b_j T^j - \sum T b_j T^j \\ &= \sum b_j T^j - \sum b_j^{(-1)} T^{j+1} = \sum (b_j - b_{j-1}^{(-1)}) T^j.\end{aligned}$$

Similarly

$$S(I-T) = \sum (a_j T^j - a_j T^{j+1}) = \sum (a_j - a_{j-1}) T^j.$$

Therefore (A8) holds if and only if

$$a_j - a_{j-1} = b_j - b_{j-1}^{(-1)},$$

i.e.

$$a_j(k) - a_{j-1}(k) = b_j(k) - b_{j-1}(k-1). \tag{A9}$$

From these equations we can determine a_j:

$$a_j(k) = \sum_{-\infty}^{j} b_i(k) - \sum_{-\infty}^{j-1} b_i(k-1). \tag{A10}$$

We claim that if the operator R is explicit, i.e. if $b_j = 0$ for $|j| > J$, then so is the operator S, for (A10) shows that then $a_j = 0$ for $j < -J$; combining (A10) with (A7) we see that $a_j = 0$ also for $j > J$.

The same argument shows that if R is implicit, i.e. $b_j \to 0$ as $j \to \pm\infty$, the same is true for the operator R.

We have shown:

The operator R given by (A5), satisfying condition (A7), is total-variation diminishing if and only if the operator S, whose coefficients are given by (A10), satisfies condition (A4).

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