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ADAPTIVE FINITE ELEMENT METHODS FOR DIFFUSION AND CONVECTION PROBLEMS

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We give a survey of recent results obtained together with K. Eriksson on adaptive h-methods for the basic linear partial differential equations of elliptic, parabolic and hyperbolic type. Our adaptive algorithms are based on a posteriori error estimates leading to reliable methods, and comparison with sharp a priori error estimates is made to prove efficiency of the procedures.

1. Introduction

In this note we give a survey of some recent results on adaptive finite element methods obtained in collaboration with Eriksson, (see [1–8]). As model problems we shall consider the heat equation including the corresponding stationary Poisson equation representing diffusion-dominated problems, and also linear convection-dominated convection–diffusion problems. Together, these problems cover the basic linear partial differential equations of parabolic, elliptic and (first order) hyperbolic type. In each of these cases our goal is to solve the following problem (Problem A): Given a norm $\|\cdot\|$, a tolerance $\mathrm{TOL} > 0$, and a piecewise polynomial finite element discretization of a certain type (e.g., piecewise polynomials of a certain given degree), design an algorithm for constructing a mesh T with (nearly) minimal number of degrees of freedom, such that

$$\|u - U\| \leq \mathrm{TOL}\,, \tag{0.1}$$

where u is the exact solution and U is the finite element solution on the mesh T. Clearly, Problem A has two ingredients. First, we want the adaptive algorithm to be *reliable* in the sense that the error control (0.1) is guaranteed. Secondly, we want the algorithm to be *efficient* in the sense that the constructed mesh is nowhere overly refined. Note that for definiteness our criterion for efficiency is a minimal number of degrees of freedom. Of course, in practice depending on the particular implementation, mesh generator, solution techniques, etc., we may accept a certain over-refinement.

Adaptive codes are now entering into applications and adaptivity may be expected to become a standard feature of finite element software in future. Quantitative error control is of obvious interest in applications, and efficient techniques for adaptive local refinement or orientation of the mesh opens fascinating possibilities of computing accurate solutions to complex problems involving different scales, such as problems in fluid mechancis with boundary layers and shocks, crack problems in solid mechanics, semiconductor problems, reaction–diffusion problems, etc.

Our adaptive algorithms are based on a posteriori error estimates of the form

$$\|u - U\| \leq \mathscr{E}(U, h, \text{data}) , \tag{0.2}$$

where as indicated the error bound $\mathscr{E}$ depends on the computed solution U, on the mesh size h of the corresponding mesh T and the data of the problem. Here h is a function giving the local mesh size in space and time. Starting from (0.2) we have the following *adaptive method* for error control in the $\|\cdot\|$-norm to the tolerance TOL: Find a mesh T, with mesh function h and corresponding approximate solution U, with minimal number of degrees of freedom such that $\mathscr{E}(U, h, \text{data}) \leq \text{TOL}$. Since U depends on h, this is a (complex) non-linear minimization problem. To solve this minimization problem approximately we design an *adaptive algorithm* usually of the following form: Given a first coarse mesh T_0, construct successively meshes T_j, $j = 1, \ldots, J$, with corresponding mesh functions h_j and approximate solutions U_j, with minimal number of degrees of freedom, such that

$$\mathscr{E}(U_{j-1}, h_j, \text{data}) \leq \theta \,\text{TOL} , \tag{0.3}$$

until $\mathscr{E}(U_J, h_J, \text{data}) \leq \text{TOL}$, which is the *stopping criterion*. Here θ is a factor $(\theta \sim 1)$ influencing the total number of steps J required to reach the stopping criterion. Since U_{j-1} is given in (0.3), the minimization problem in h_j is easy to solve approximately by seeking to *equidistribute* the element contributions to the global quantity $\mathscr{E}$. Note that we consider here adaptive forms of the so called h-method, where the quantity determined adaptively is the local element size. More generally, it is of interest to develop methods where the mesh orientation and stretching, and the degee of the piecewise polynomials are also determined adaptively.

Since our adaptive algorithms are based on a posteriori error estimates, it follows that the algorithms are reliable in the above sense; if the stopping criterion $\mathscr{E}(U, h, \text{data}) \leq \text{TOL}$ is satisfied, then by (0.2) we will have $\|u - U\| \leq \text{TOL}$ and the error will be within the given tolerance. Concerning the efficiency of the adaptive algorithms more or less precise results may be stated. Ideally, one would like to prove that the final mesh generated by the algorithm is close to the *optimal* mesh, which we take to be the mesh with fewest degrees of freedom such that the approximate solution is within the given tolerance. In certain cases it is possible to actually prove such a precise result (up to constants of moderate size), while in other cases we obtain weaker results. In general, to prove efficiency we rely on sharp a priori error estimates, where the error $\|u - U\|$ is estimated by a quantity $E(u, h)$ depending on the exact solution and the mesh size h. In the elliptic and parabolic case we prove that the a posteriori quantity $\mathscr{E}(U, h, \text{data})$ may be estimated by a constant times the a priori quantity $E(u, h)$, which proves efficiency in a weak sense, and may be sharpened through various localization results to prove efficiency in a strict sense for certain problems. Note that by proving that the a posteriori bounds may be estimated by a constant times the a priori bounds, it follows in particular that by decreasing the mesh size it is possible to realize the stopping criterion under some assumption on the nature of the exact solution, which is not evident from the start.

Summing up so far, our adaptive algorithms are based on (sharp) a posteriori error estimates leading to reliable methods, and comparison with sharp a priori error estimates is used to prove efficiency in a more or less precise way. The error estimates are based on a representation of the error in terms of the solution of a certain dual problem. This error representation is fundamental in our approach to adaptivity since it gives information on the *structure* of the global error as composed of contributions from individual elements, which gives the basis for the design of the adaptive algorithm. Basically, the error estimates are obtained by using the orthogonality properties of the Galerkin method and standard finite element interpolation estimates, together with appropriate stability estimates for the dual problem. In the case of the a posteriori error estimates the dual problem is a continuous problem, while for the a priori estimates the dual problem is discrete. In this framework there is a close analogy between the a priori and a posteriori error estimates which appears to be fundamental. In both cases the stability of the dual problem is the critical issue. In general, the stability of the continuous dual problem connected with the a posteriori estimates is easier to tackle by analytical tools than that of the discrete dual problem related to the a priori estimates, and thus in many cases the a posteriori estimate is easier to prove than the a priori estimate, contrary to a common opionion that a posteriori estimates are more difficult to obtain. For more general problems (e.g., nonlinear problems or problems with variable coefficients) the stability of the continuous dual problem cannot be accurately estimated by analytical means (in particular, one has to determine approximately the size of certain constants involved) and in these cases one has to build in a computational estimate of the stability of the dual problem as a part of the adaptive process. For certain problems and norms, numerical experiments show that this is feasible, while for more general problems more work is required to obtain reliable and accurate computational estimates of the stability of the dual problem. We note that this is a fundamental problem which has to be faced, analytically or computationally, since the stability of the dual problem reflects the error propagation properties of the given equation.

For parabolic problems we use the *Discontinuous Galerkin* method based on a space–time finite element discretization with basis functions continuous in space and discontinuous in time. The time step and the space discretization may vary from one time level to the next and it is also possible to use more general space–time meshes with the time steps being variable also in space. For hyperbolic type problems we use the *Streamline Diffusion* method (SD-method) again with space–time finite elements in the time-dependent case.

We now briefly comment on the difference in our approach to adaptivity as compared to the pioneering work by Babuška, see e.g. [9], and the related work by Bank [10] and Ewing [11]. First, in the work by Babuška et al., the emphasis is on elliptic problems with error control in the energy norm, while we consider also other norms and also parabolic and hyperbolic problems. Secondly, Babuška et al. seek to construct a posteriori error bounds which are very precise in the sense that the quotient between the estimated and the actual global error (the *effectivity index*) tends to one as the mesh size tends to zero. For this purpose elaborate a posteriori estimates based on solving local problems are used. However, in our approach we set the goal lower in this respect, and we use simpler possibly less precise estimates and accept (depending on the difficulty of the problem, the chosen norm and the tolerance) effectivity indices in the range, say, from 1 to 3. In contrast we are able to attack more general problems and we are not restricted to only energy norms. A further difference is that we seek to obtain adaptive algorithms which we can prove to be efficient in a more or less precise way. Let us note that we should distinguish between the concept of effectivity index and the efficiency of the adaptive algorithm. Even if the effectivity index is close to one, which says that we are able to estimate the global error on a given mesh very accurately, it is not clear that the underlying mesh is close to the optimal mesh related to the corresponding tolerance level; the given mesh may be locally overly refined and there is no way we can detect this by only looking at the effectivity index.

To sum up, in our approach we do not seek to achieve effectivity indices necessarily very close to one, but we seek adaptive algorithms for a general class of problems with error control in a variety of norms and we seek to prove that the algorithms are efficient in the sense that almost optimal not overly refined meshes are generated. Note that for parabolic problems our adaptive methods seem to be the first to give reliable and efficient error control in $L_\infty(L_2)$, i.e., the maximum norm in time and L_2 in space. For hyperbolic problems our results appear to give the first adaptive methods based on a posteriori error estimates.

After [6] was completed we discovered that our a posteriori error estimates in the energy norm (H^1-norm) for the Poisson equation are analogous to those presented in Abdalass [12] and Verfürth [13] for the Stokes problem. We also learned that similar a posteriori error estimates for the Poisson equation were already considered in 1979 by Bank [14]. These estimates are based on estimating the H^{-1}-norm of the residual of the finite element solution in terms of a weighted L_2-norm of the residual over the element interiors and the jumps of the normal derivatives of the finite element solution across interelement boundaries (using the orthogonality relation built in the Galerkin method). In this approach (which is very simple and natural) the residual of the finite element solution is separately estimated in the interior of the elements and on element boundaries, which leads to effectivity indices not necessarily very close to one. The development in the early and mid eighties with mathematical emphasis, however, took a different route concentrating on methods with effectivity index close to one. Our interest in a posteriori estimates of the indicated form is motivated by their simplicity and the possibility of handling problems of different nature and different norms. For pioneering work on adaptive methods with emphasis on engineering aspects, see also [15, 16].

An outline of the remainder of this article is as follows. In Section 1 we present the discretization methods for our model problems of elliptic, parabolic and hyperbolic type. In Section 2 we present the a priori and a posteriori error estimates, in Section 3 we state the corresponding adaptive algorithms and in Section 4 we discuss their reliability and efficiency. In Section 5 we indicate the structure of the proofs of the a posteriori and a priori error estimates and finally in Section 6 we present the results of some numerical experiments.

1. The discretization methods

For simplicity we shall restrict our considerations to some standard model problems of elliptic, parabolic and hyperbolic type, namely, to find u such that

$$-\Delta u(x) = f \quad \text{in } \Omega\,, \qquad u(x) = 0 \quad \text{on } \Gamma\,, \tag{1.1}$$

$$u_t - \Delta u = f \quad \text{in } \Omega \times \mathbb{R}^+\,, \qquad u = 0 \quad \text{on } \Gamma \times \mathbb{R}^+\,, \qquad u(\cdot\,, 0) = u_0 \quad \text{in } \Omega\,, \tag{1.2}$$

$$\beta \cdot \nabla u + \alpha u - \operatorname{div}(\varepsilon \nabla u) = f \quad \text{in } \Omega\,, \qquad u = g \quad \text{on } \Gamma\,, \tag{1.3}$$

$$u_t + \beta \cdot \nabla u + \alpha u - \operatorname{div}(\varepsilon \nabla u) = f \quad \text{in } \Omega \times \mathbb{R}^+ , \qquad u = g \quad \text{on } \Gamma \times \mathbb{R}^+ ,$$
$$u(\cdot, 0) = u_0 \quad \text{in } \Omega , \tag{1.4}$$

respectively. Here Ω is a bounded polygonal domain in $\mathbb{R}^2$ with boundary Γ, $\mathbb{R}^+ = (0, \infty)$, Δ is the usual Laplacian, $u_t = \partial u / \partial t$, $\beta = (\beta_1, \beta_2)$ is a given velocity field, α is a given absorption coefficient, $\varepsilon \geq 0$ a given (small) diffusion coefficient, all coefficients possibly depend on x and t, and the functions f, g and u_0 are given data. We note that when $\varepsilon = 0$, then the boundary condition $u = g$ in (1.3) is imposed only on the inflow part of the boundary $\Gamma_- = \{x \in \Gamma \mid n(x) \cdot \beta(x) < 0\}$, where $n(x)$ is the outward unit normal to Γ at $x \in \Gamma$, and similarly for (1.4).

For the discretization of these problems with respect to the space variable $x = (x_1, x_2)$, let Σ be the class of all finite element discretizations (h, T, S) defined as follows:

(i) h is a positive function in $C^1(\bar{\Omega})$, such that

$$|\nabla h(x)| \leq \mu \quad \forall x \in \bar{\Omega} , \tag{1.5a}$$

(ii) $T = \{K\}$ is a triangulation of Ω into triangles K of diameter h_K, such that

$$c_1 h_K^2 \leq \int_K \mathrm{d}x \quad \forall K \in T , \tag{1.5b}$$

and associated with the function h through

$$c_2 h_K \leq h(x) \leq h_K \quad \forall x \in K \quad \forall K \in T , \tag{1.5c}$$

where c_1 and c_2 are given positive constants,

(iii) S is the set of all continuous functions on $\bar{\Omega}$ which are linear in x on each $K \in T$ and vanish on $\partial\Omega$.

As indicated, in the adaptive process we need to construct for a given mesh function h satisfying (1.5a) a corresponding mesh T satisfying (1.5b,c). In our implementations we have for this purpose used two mesh generators: one based on successive subdivision of one triangle into four similar triangles by joining the midpoints of the sides of the given triangle and introducing 'transition triangles' divided into two subtriangles connecting zones with different mesh size, and another 'front generator' which constructs a mesh with given local mesh size by adding elements at a 'front' coinciding initially with the boundary and sweeping the region, cf. [15].

The stationary elliptic problem (1.1) may now be approximated in the usual way: Let $(h, T_h, S_h) \in \Sigma$ and find $U \in S_h$ such that

$$(\nabla U, \nabla v) = (f, v) \quad \forall v \in S_h , \tag{1.6}$$

where $(\cdot, \cdot)$ denotes the usual inner product in $[L_2(\Omega)]^d$, $d = 1, 2$. By introducing the L_2 projection operator $P_h : L_2(\Omega) \to S_h$ defined by $(P_h w, v) = (w, v)$ $\forall v \in S_h$, and the discrete Laplacian $\Delta_h : H^1(\Omega) \to S_h$ defined by $(\Delta_h w, v) = -(\nabla w, \nabla v)$ $\forall v \in S_h$, we may write (1.6) equivalently as $-\Delta_h u_h = P_h f$, which has a more obvious resemblance with (1.1).

Let us now turn to the time-dependent parabolic problem (1.2). For a full discretization of this problem with the Discontinuous Galerkin method we consider partitions $0 = t_0 < t_1 < \cdots < t_n < \cdots$ of $\mathbb{R}^+$ into subintervals $I_n = (t_{n-1}, t_n)$ of length $k_n = t_n - t_{n-1}$, and associate with each such time interval a space discretization $(h_n, T_n, S_n) \in \Sigma$. For q a nonnegative integer we define $V_{qn} = \{v \mid v = \Sigma_{j=0}^{q} t^j \varphi_j,\ \varphi_j \in S_n\}$, and discretize (1.2) as follows: Find U such that for $n = 1, 2, \ldots, U|_{\Omega \times I_n} \in V_{qn}$ and

$$\int_{I_n} \{(U_t, v) + (\nabla U, \nabla v)\}\, \mathrm{d}t + ([U]_{n-1}, v_{n-1}^+) = \int_{I_n} (f, v)\, \mathrm{d}t \quad \forall v \in V_{qn}\,, \tag{1.7}$$

where $[w]_n = w_n^+ - w_n^-$, $w_n^{+(-)} = \lim_{s \to 0+(-)} w(t_n + s)$ and $U_0^- = u_0$. With $f = 0$, (1.7) is equivalent to the sub-diagonal $(q+1, q)$-Padé scheme of order of accuracy $2q+1$, see [18].

REMARK. Note that in the discretization (1.7) the space and time steps may vary in time and that the space discretization may be variable also in space, whereas the time steps k_n are kept constant in space. Clearly, optimal mesh design requires the time steps to be variable also in space. Now, it is easy to extend the method (1.7) to admit time steps which are variable in space simply by defining

$$V_{qn} = \{v \mid v(x, t) = \sum_i v_i(t) \chi_i(x)\}\,,$$

where $\{\chi_i\}$ is a basis for S_n and the coefficients v_i now are piecewise polynomial of degree q in t, without continuity requirements, on partitions of I_n which may vary with i. The discrete functions may now be discontinuous also inside the 'slab' $\Omega \times I_n$. The Discontinuous Galerkin method again takes the form (1.7) with the difference that the term $([U]_{n-1}, v_{n-1}^+)$ is replaced by a sum over all jumps of U in $\Omega \times [t_{n-1}, t_n)$ and further the discontinuities of U are discarded in the integral involving U_t. Adaptive methods for the Discontinuous Galerkin method in this generality are considered in [7].

Finally, we consider the convection diffusion problems (1.3) and (1.4). For the discretization of these problems we shall as indicated use the SD-method which is a variant of a standard Galerkin finite element methods obtained by two basic modifications: a 'streamline' modification of the test functions (in the stationary case) from v to $v + \delta(\beta \cdot \nabla v + \alpha v)$ where $\delta \sim h$, and a second modification obtained by adding an artificial viscosity term with viscosity coefficient proportional to h^α (with $\frac{3}{2} < \alpha < 2$) and the residual of the finite element solution. The streamline SD-method is the first general finite element method for (first order) hyperbolic equations which combines good stability with high order accuracy. Convergence results are available for linear scalar convection–diffusion problems, for the incompressible Euler and Navier–Stokes equations, for scalar conservation laws in several dimensions, and also (entropy) consistency results for e.g. the compressible Euler and Navier–Stokes equations (see [19–22]). With $q = 1$ the SD-method for (1.3) may be formulated as follows in the case $g = 0$ and $\varepsilon > 0$: Find $U \in S_h$, such that

$$(\beta \cdot \nabla U + \alpha U, v + \delta(\beta \cdot \nabla v + \alpha v)) + (\hat{\varepsilon} \nabla U, \nabla v) = (f, v + \delta(\beta \cdot \nabla v + \alpha v)) \quad \forall v \in S_h\,, \tag{1.8}$$

where

$$\delta = C_1 \max\left(h - \frac{\varepsilon}{|\beta|}, 0\right) \Big/ |\beta| ,$$

$$\hat{\varepsilon} = \hat{\varepsilon}(U) = \max(\varepsilon, C_2 h^{\nu} |\beta \cdot \nabla U + \alpha U - f|) ,$$

where the C_i and ν are positive constants with $\frac{3}{2} < \nu < 2$. In the computations we normally choose ν close to 2.

For the time-dependent problem (1.4) the SD-method reads as follows using the notation of (1.7) again with $q = 1$ and assuming that $g = 0$ and $\varepsilon > 0$; Find U, such that for $n = 1, 2, \ldots, U|_{\Omega \times I_n} \in V_{1n}$ and

$$\begin{aligned} &\int_{I_n} \{(U_t + \beta \cdot \nabla U + \alpha U, v + \delta(v_t + \beta \cdot \nabla v + \alpha v))\} \, dt \\ &\quad + \int_{I_n} (\hat{\varepsilon} \hat{\nabla} U, \hat{\nabla} v) \, dt + ([U_{n-1}], v_{n-1}^{+}) \\ &\quad = \int_{I_h} (f, v + \delta(v_t + \beta \cdot \nabla v + \alpha v)) \, dt \quad \forall v \in V_{1n} , \end{aligned} \tag{1.9}$$

where

$$U_0^- = u_0 ,$$

$$\delta = C_1 \max\left(h - \frac{\varepsilon}{|(1, \beta)|}, 0\right) \Big/ |(1, \beta)| ,$$

$$\hat{\varepsilon} = \hat{\varepsilon}(U) = \max\left(\varepsilon, C_2 h^{\nu}\left(|U_t + \beta \cdot \nabla U + \alpha U - f| + \left|\frac{[U_{n-1}]}{k_n}\right|\right)\right) \quad \text{in } \Omega \times I_n ,$$

$$\hat{\nabla} v = \left(\frac{\partial v}{\partial t}, \frac{\partial v}{\partial x_1}, \frac{\partial v}{\partial x_2}\right) ,$$

with the C_i and ν as above.

2. A priori and a posteriori error estimates

In this section we state a priori and a posteriori error estimates for the discretization methods (1.6)–(1.9). By $\|\cdot\|$ we denote the $L_2(\Omega)$-norm and $D^S u = (\Sigma_{|\alpha|=s} |D^{\alpha} u|^2)^{1/2}$. For the stationary elliptic problem (1.1) we have the following a priori estimate.

THEOREM 2.1. Let $f \in L_2(\Omega)$ and let u and U be the solutions of (1.1) and (1.6), respectively. Then for $m = 1$ and 2 there exists a constant C depending only on the constants c_1 and c_2 in (1.5), such that

$$\|D^{2-m}(u - U)\| \leq C \|h^m D^2 u\| , \tag{2.1}$$

where for $m = 2$ we assume that μ is sufficiently small and Ω is convex.

REMARK 2.1. Note the way in which the local mesh size $h(x)$ enters in these error estimates showing that large second derivatives of u may be compensated for by a (locally) small mesh size so as to control the quantity $\|h^m D^2 u\|$, $m = 1, 2$, bounding the error. This indicates the possibility of adaptively choosing the mesh size to control the error if $D^2 u$ may be computationally estimated, an idea which was explored in [4]. In this note, however, we will follow a related but different adaptive strategy directly based on a posteriori error estimates.

REMARK 2.2. Note further that the estimate (2.1) is optimal in the sense that there exists a constant c such that 'for most u' (e.g., if $D^\alpha u$, $|\alpha| = 2$, is roughly constant on each element),

$$\inf_{v \in S_h} \|D^{2-m}(u - v)\| \geq c\|h^m D^2 u\| , \quad m = 1, 2 ,$$

which indicates that error control based on (2.1) should be efficient.

The error estimate (2.1) with $m = 1$ is classical, whereas with $m = 2$ the estimate in the present generality can be found in [2]. For quasi-uniform partitions (corresponding to taking h constant) the case $m = 2$ is well-known. Let us further remark that (2.1) may also be derived for a (convex or nonconvex) domain Ω with smooth boundary, in the case $m = 2$ with the constant C depending on Ω.

To state the a posteriori estimate for the stationary elliptic problem (1.1) underlying the adaptive algorithm we need some notation. With each side $\tau \in \partial K \cap \Omega$ of a triangle $K \in T_h$ we associate a vector n_τ of length one normal to τ and define for $v \in S_h$

$$\left[\frac{\partial v}{\partial n_\tau}\right] = \lim_{s \to 0^+} (\nabla v(x + sn_\tau) - \nabla v(x - sn_\tau)) \cdot n_\tau , \quad x \in \tau ,$$

that is, $[\partial v / \partial n_\tau]$ is the jump across τ in the normal component of ∇v. We define for $v \in S_h$ the piecewise constant quantity $D_h^2 v$ by

$$D_h^2 v = \max_{\tau \in \partial K \cap \Omega} \left|\left[\frac{\partial v}{\partial n_\tau}\right]\right| \Big/ h_K \quad \text{on } K \in T_h ,$$

where as indicated only the sides τ in the interior of Ω occur, which may be viewed as a discrete counterpart of $|D^2 v|$.

We may then state the following a posteriori error estimates for the stationary elliptic problem (1.1) given in [6].

THEOREM 2.2. There are constants α_m and β_m only depending on the constants c_1 and c_2, such that if $f \in L_2(\Omega)$ and u and U are the solutions of (1.1) and (1.6), respectively, then for $m = 1, 2$,

$$\|D^{2-m}(u - U)\| \leq \alpha_m \|h^m f\| + \beta_m \|h^m D_h^2 U\| , \tag{2.2}$$

where in the case $m = 2$, we assume that Ω is convex.

REMARK 2.4. Note that without further analysis the amount of information in the a posteriori estimate (2.2) is not obvious. Clearly, if we compute U using (1.6), then we may bound the error using (2.2) by evaluating the right-hand sides of these estimates. If these quantities turn out to be sufficiently small, then we may be satisfied and quit. However, without further analysis it is conceivable that the right-hand sides of (2.2) would always be large and then these estimates would be useless. In fact, a posteriori error estimates of the form (2.2) may be derived also for unstable methods and in such cases the right-hand side quantities could be large regardless of the mesh size. In our case we shall prove that in fact (2.2) is sharp, and thus may be useful in practice, by comparison with the optimal a priori estimates (2.1).

REMARK 2.5. If $f \in H^2(\Omega)$, then the f-terms in (2.2) or (2.3) may be replaced by $\alpha_3 \|h^{4-m} D^2 f\|$.

Let us next state optimal a priori estimates for the parabolic problem (1.2) given in [6]. For simplicity we assume that Ω is convex. The estimates may be extended to general domains with smooth boundary with the constants C depending on Ω.

THEOREM 2.3. Let u be the solution of (1.2) and U that of (1.7), suppose μ is small enough and assume that for each n one of the following two assumptions holds:

$$S_n \subset S_{n-1}\,, \tag{2.4a}$$

$$\bar{h}_n^2 \leq \gamma k_n\,, \tag{2.4b}$$

where $\bar{h}_n = \max_{x \in \bar{\Omega}} h_n(x)$ and γ is sufficiently small and that for all n, $k_n \leq C k_{n+1}$. Then there exist constants C only depending on c_1 and c_2 (if Ω is convex) such that for $q = 0, 1$ and $N = 1, 2, \ldots$

$$\|u - U\|_{I_N} \leq C L_N \max\nolimits_{1 \leq n \leq N} E_{qn}(u)\,, \tag{2.5a}$$

and for $q = 1$ and $N = 1, 2, \ldots,$

$$\|u(t_N) - U_N^-\| \leq C L_N \max\nolimits_{1 \leq n \leq N} E_{2n}(u)\,, \tag{2.5b}$$

where

$$L_N = \frac{1}{4}\left(\ln \frac{t_N}{k_N} + 1\right)^{1/2},$$

$$E_{qn}(u) = \min_{j \leq q+1} k_n^j \|u_t^{(j)}\|_{I_n} + \|h_n^2 D^2 u\|_{I_n}\,, \quad q = 0, 1, 2\,,$$

with

$$u_t^{(1)} = u_t\,, \quad u_t^{(2)} = u_{tt}\,, \quad u_t^{(3)} = \Delta u_{tt} \text{ and } \|w\|_{I_n} = \max_{t \in I_n} \|w(t)\|\,.$$

REMARK 2.6. Note that (2.5) states that the Discontinuous Galerkin method (1.7) is of order $q+1$ globally in time and of order $2q+1$ at the discrete time levels t_n for $q=0,1$. Further, the estimates (2.5a, b) are optimal in the sense that for some positive constant c

$$\inf_{v \in V_{qn}} \|u - v\|_{I_n} \geq c E_{qn}(u), \quad q = 0, 1, 2, \tag{2.6}$$

if here, in the definition of $E_{qn}(u)$, we put $u_t^{(3)} = u_{ttt}$ and restrict the variation of $u_t^{(3)}$ and $D^\alpha u$ for $|\alpha| = 2$ as in Remark 2.2. Note that for the 'super approximation' result (2.5b) it is relevant to compare with approximation in V_{2n}.

REMARK 2.7. With quasi-uniform space-meshes with $h_n(x) \sim \bar{h}_n$ we expect to have $\bar{h}_n^2 \sim k_n$ for $q = 0$ and $\bar{h}_n^2 << k_n$ if $q = 1$, since the Discontinuous Galerkin method is of second order in space and of order $2q + 1$ in time, $q = 0, 1$. Thus, in particular for $q = 1$ the condition (2.4b) does not appear to be restrictive and in fact allows a considerable variation of $h_n(x)$. In certain extreme situations, however, e.g. with initial data u_0 highly concentrated in space, (2.4b) may impose a restriction on the mesh. It is possible that (2.4b) may be weakened to a condition of the form $\bar{h}_n^2 \leq \gamma K_n$, where $K_n = t_{n^*} - t_{n-1}$ and $S_m = S_n$ for $m = n, n+1, \ldots, n^*$.

We now state a posteriori estimates for the parabolic problem (1.2) (see [6]). Again, we assume that Ω is convex, but generalizations to smooth non-convex domains are possible (cf. Remark 2.10).

Theorem 2.4. Let u be the solution of (1.2) and U that of (1.7), and suppose Ω is convex. Then for $N \geq 1$, we have for $q = 0$

$$\|u(t_N) - U_N^-\| \leq \max_{1 \leq n \leq N} \mathscr{E}_{0n}(U), \tag{2.7a}$$

and for $q = 1$

$$\|u(t_N) - U_N^-\| \leq \max_{1 \leq n \leq N} \mathscr{E}_{2n}(U), \tag{2.7b}$$

where

$$\begin{aligned}
\mathscr{E}_{0n}(U) &= C_1 \|h_n^2 f\|_{I_n} + C_2 \int_{I_n} \|f\| \, dt + C_3 \|h_n^2 D_n^2 U_n^-\| + C_4 \|[U_{n-1}]\| \\
&\quad + C_5 \|h_n^2 [U_{n-1}]/k_n\|^*, \\
\mathscr{E}_{2n}(U) &= C_6 \|h_n^2 f\|_{I_n} + C_7 k_n^2 \int_{I_n} \|f_{tt}\| \, dt + C_8 \|h_n^2 D_n^2 U\|_{I_n} \\
&\quad + \min(C_9 \|[U_{n-1}]\|, C_{10} k_n \|\Delta_n P_n [U_{n-1}]\|) + C_{11} \|h_n^2 [U_{n-1}]/k_n\|^*,
\end{aligned}$$

where $D_n^2 = D_{h_n}^2$ and an asterisk indicates that the corresponding term is present only if $S_n \not\supseteq S_{n-1}$. Further the C_i are constants given by

$$\begin{aligned}
&C_1 = \alpha_2 L, \quad C_2 = L + 2, \quad C_3 = \beta_2(L + 2), \quad C_4 = L + 1, \\
&C_5 = \alpha_2(L + \exp(-1)), \quad C_6 = \alpha_2(L + 2), \quad C_7 = \gamma_3(\gamma_1 L + \gamma_0 + 1), \\
&C_8 = 2\beta_2(L + 1), \quad C_9 = C_4, \quad C_{10} = \gamma_2 L + \gamma_1, \quad C_{11} = C_5, \quad L = \max_N L_N,
\end{aligned}$$

where α_2 and β_2 are certain constants depending on c_1 and c_2 related to approximatations by functions in S_h, and the γ_i are absolute constants related to one-dimensional approximation by linear functions (*see Section* 6).

REMARK 2.8. The term $C_1\|h_n^2 f\|_{I_n}$ in $\mathscr{E}_{0n}$ and $\mathscr{E}_{2n}$ may be replaced by $\bar{C}_1\|h_n^4 D^2 f\|_{I_n}$, the term $C_2\int_{I_n}\|f\|\,\mathrm{d}t$ in $\mathscr{E}_{0n}$ by $\bar{C}_2 k_n\int_{I_n}\|f_t\|\,\mathrm{d}t$ and $C_7 k_n^2\int_{I_n}\|f_{tt}\|\,\mathrm{d}t$ in $\mathscr{E}_{2n}$ by $\bar{C}_7 k_n^3\int_{I_n}\|\Delta f_{tt}\|\,\mathrm{d}t$ with modified constants $\bar{C}_i$.

REMARK 2.9. The comments of Remark 2.4 are also relevant for the a posteriori estimate (2.7). By comparison with the optimal a priori estimate (2.5) we can prove that (2.7) is sharp and thus may be used as a basis for an efficient adaptive algorithm.

REMARK 2.10. In the general case with the boundary of Ω smooth, (some of) the constants C_i should be replaced by constants $\hat{C}_i = C_S C_i$, where C_S is a stability constant depending on Ω defined by

$$C_S = \sup_{\substack{v\in H_0^1(\Omega)\cap H^2(\Omega)\\ v\neq 0}} \frac{\|D^2 v\|}{\|\Delta v\|}\,.$$

The approximation constants α_2 and β_2 (depending on c_1 and c_2) and the absolute constants γ_i entering in the C_i, may be estimated once and for all (values of these constants used in our numerical computations are given in Section 9), while the stability constant C_S in general depends on Ω. It is possible that a relevant value of C_S may be found by computing the quotient $\|D_h^2 v\|/\|\Delta_h v\|$ for some properly chosen $v\in S_h$. The a posteriori estimates may be generalized also to problems with variable coefficients or nonlinear problems (see [7]). In this case the C_i should be replaced by $\hat{C}_i = C_S(u)C_i$, where $C_S(u)$ is a 'stability constant' depending on Ω and the coefficients, and also 'mildly' on u. It is likely that such constants may be estimated through the adaptive procedure, cf. [1, 5, 7].

REMARK 2.11. One can prove direct analogues of Theorems 2.1–2.4, replacing the $L_2(\Omega)$-norm by the $L_p(\Omega)$-norm, $1\leq p\leq\infty$, (see [1]).

Finally, we shall state some a priori and a posteriori error estimates for the SD-method for the convection–diffusion problems (1.3) and (1.4). We start with an a priori error estimate from [19] for the stationary problem (1.3), for simplicity with α bounded below by a positive constant. Further, for notational simplicity we consider the convection dominated case with $\varepsilon < h$. The estimate can easily be extended to a general ε to give estimates analogous to (2.2) in the case $\varepsilon = 1$.

THEOREM 2.5. Suppose there are positive constants κ_i, such that $\kappa_0 < \alpha(x) < \kappa_1$, $x\in\bar{\Omega}$, and suppose the velocity β is smooth. If the exact solution u of (1.3) belongs to $W^{1,\infty}(\Omega)$, then there exists a constant C, such that if $\varepsilon < h$, then

$$\|\delta^{1/2}(\beta\cdot\nabla(u-U))\| + \|\hat{\varepsilon}^{1/2}\nabla(u-U)\| + \|u-U\| \leq C\|h^{3/2}D^2 u\|\,,$$

where U is the solution of (1.8).

We now state an a posteriori error estimate for the SD-method (1.8) for the stationary problem (1.3) from [7]. For simplicity we shall compare the computed solution U with the solution $\hat{u}$ of a perturbed continuous convection–diffusion problem obtained by replacing ε by $\hat{\varepsilon}(U)$ in (1.3). It is also possible in model cases to estimate the perturbation error $\|u-\hat{u}\|$ in terms of $\hat{\varepsilon}(U)-\varepsilon$, U and f, see [8]. In general, we expect $\|u-\hat{u}\|$ to be dominated by $C\|\hat{u}-U\|$, so that control of $\|\hat{u}-U\|$ suffices. In the adaptive algorithm for (1.8) to be presented below, we also have the option of including the requirement $\hat{\varepsilon}=\varepsilon$, corresponding to resolution of all details of the exact solution, in which case on the final mesh $\hat{u}=u$, see Section 3. For simplicity we assume that the coefficients α and β are constant.

THEOREM 2.6. There is a constant C, such that

$$\|\hat{u}-U\| \leqslant C\|\min(1, \hat{\varepsilon}^{-1}h^2)R(U)\| + \max_{\Gamma_-} \hat{\varepsilon}^{1/2}\,, \tag{2.8}$$

where

$$R(U) = |\beta \cdot \nabla U + \alpha U - f| + |\mathrm{div}_h(\hat{\varepsilon}\nabla U)|\,,$$

$$|\mathrm{div}_h(\hat{\varepsilon}\nabla U)| = \max_{\tau\in\partial K\cap\Omega} \max_{\tau} \left|\left[\hat{\varepsilon}\,\frac{\partial U}{\partial n_\tau}\right]\right| \Big/ h_K \quad \text{on } K\in T\,.$$

We also state the following analogue of Theorem 2.6 for the SD-method (1.9) for the time dependent problem (1.4), again assuming for simplicity that α and β are constant.

THEOREM 2.7. There is a constant C, such that

$$\|\hat{u}-U\|_{L_2(Q)} \leqslant C\|\min(1, \hat{\varepsilon}^{-1}h^2)R(U)\|_{L_2(Q)} + \max_{Q_-} \hat{\varepsilon}^{1/2}\,, \tag{2.9}$$

where $Q=\Omega\times I$, with $I=(0,T)$ a given time-interval, $Q_-=(\Gamma_-\times I)\cup(\Omega\times\{0\})$,

$$R(U) = |U_t + \beta\cdot\nabla U + \alpha U - f| + |\mathrm{div}_{h_n}(\hat{\varepsilon}\nabla U)| + \left|\frac{[U]}{k_n}\right| \quad \text{on } \Omega\times I_n\,.$$

REMARK 2.12. Note that (2.8) has essentially (if $\nu=2$) the form $\|\hat{u}-U\| \leqslant C\|\min(R(U),1)\|$ which should be compared with the a posteriori estimate for the standard Galerkin method for (1.3) corresponding to choosing $\delta=0$ and $\hat{\varepsilon}=\varepsilon$ in (1.8): $\|u-U\| \leqslant C\|R(U)\|$. In a situation with boundary or internal layers $\|R(U)\|\to\infty$ as $h\to 0$ (until $h<\varepsilon$) and then adaptive error control is not possible for the standard Galerkin method (unless $h<\varepsilon$), cf. Section 4.

3. Adaptive algorithms

In this section we present the adaptive algorithms based on the a posteriori error estimates stated above, considering first the elliptic problem (1.1). Starting from the a posteriori error estimate (2.2) we have the following algorithm for control of $\|D^{2-m}(u-U)\|$, $m=1,2$: Given an initial triangulation T_0, determine successively triangulations T_j with N_j elements and mesh functions h_j and corresponding approximate solutions U_j, $j=1,\ldots,J$, such that h_j is maximal under the condition

$$\alpha_m \|h_j^m f\|_{L_2(K)} + \beta_m \|h_j^m D^2_{h_{j-1}} U_{j-1}\|_{L_2(K)} \leq \frac{\theta \mathrm{TOL}}{\sqrt{N_{j-1}}}\,, \quad K \in T_{j-1}\,, \tag{3.1}$$

where J is the smallest integer such that

$$\alpha_m \|h_J^m f\| + \beta_m \|h_J^m D^2_{h_{J-1}} U_J\| \leq \mathrm{TOL}\,. \tag{3.2}$$

Further, θ is a parameter (here $\theta \sim \frac{1}{2}$), through which we may monitor the total number of steps J required to achieve (3.2). Normally, we may expect to have $J \sim 2$–5. Notice that (3.1) seeks to equidistribute the contribution form each element to the global error bound $\alpha_m \|h^2 f\| + \beta_m \|D_h^2 u\|$.

For the parabolic problem the a posteriori error estimates (2.7a, b) have the form

$$\|u_N - U_N^-\| \leq \max_{n \leq N} \mathscr{E}_n(U, h_n, k_n, f)\,, \tag{3.3}$$

where $\mathscr{E}_n$ is a quantity related to time step n. The adaptive algorithm based on (3.3) for control of $\max_{n \leq N} \|u_n - U_n^-\|$ has the following form: For $n = 1, 2, \ldots, N$, construct a mesh S_n with N_n elements and mesh function h_n, a time step k_n and corresponding approximate solutions U_n on $\Omega \times I_n$, such that

$$\mathscr{E}_n(U_n, h_n, k_n, f) = \mathrm{TOL}\,,$$

and N_n / k_n is (nearly) minimal. To solve the minimization problem we again seek to equidistribute the contributions from the elements in the space–time discretization of $\Omega \times I_n$. For a precise statement of the adaptive algorithm in this case, see Example 6.2.

For the stationary hyperbolic problem (1.3) discretized by the SD-method (1.8), we may design two adaptive algorithms: (i) one algorithm based on (2.8) and (ii) one algorithm based on (2.8) together with the additional requirement that the mesh is refined until $\hat{\varepsilon} = \varepsilon$. In the case (i) the adaptive algorithm is obtained by replacing (3.1) by

$$Ch_j \min(1, \hat{\varepsilon}_{j-1} h_{j-1}^2) R(U_{j-1}) \leq \frac{\theta \mathrm{TOL}}{\sqrt{N_{j-1}}} \quad \text{on } K \in T_{j-1}\,, \tag{3.4a}$$

$$Ch_j^{\nu/2} R(U_{j-1})^{1/2} \leq \mathrm{TOL} \quad \text{if } K \cap \Gamma \neq \emptyset\,. \tag{3.4b}$$

In the case (ii) we also add the requirement that

$$Ch_j^{\nu} R(U_{j-1}) \leq \varepsilon\,. \tag{3.4c}$$

The stopping criterions are obvious. With proper normalization it appears that $\hat{\varepsilon} = \varepsilon$ corresponds to resolving all scales of the continuous solution, see Section 4.

Extension to the time-dependent hyperbolic problem is obtained by replacing Ω by $\Omega \times I$ according to (2.9), and also here we may add the requirement $\hat{\varepsilon} = \varepsilon$.

REMARK 3.1. For error control in the maximum norm ($L_\infty(\Omega)$-norm), e.g., for the Poisson equation, the adaptive algorithm has the form (see [2])

$$C(\|h_j^2 f\|_{L_\infty(K)} + \|h_j^2 D_{h_{j-1}}^2 U_{j-1}\|_{L_\infty(K)}) = \text{TOL} \quad \text{on } K \in T_{j-1} .$$

4. Reliability and efficiency

We recall that our adaptive algorithms are based on a posteriori error estimates of the form $\|u - U\| \le \mathscr{E}(U, h, \text{data})$ and that the stopping criterion is $\mathscr{E}(U, h, \text{data}) \le \text{TOL}$, which guarantees that if the stopping criterion is satisfied, then the error will be within the given tolerance and thus the adaptive algorithm is *reliable*.

Next, we consider the *efficiency* of our adaptive algorithms. To prove the efficiency in a precise way we need to prove that the final mesh produced by the adaptive algorithm is close to the optimal mesh, which is the mesh with fewest degrees of freedom such that the correponding approximate solution is within the tolerance. It is possible to show that, e.g., for the Poisson equation with error control in the maximum norm by using localization techniques to prove that (see [2]) on a mesh produced by the adaptive algorithm, we have for $x \in \Omega$,

$$\max_{|y-x| \le Ch} |h^2(y) D^2 u(y)| \ge c\,\text{TOL} .$$

This result proves essentially that for all $x \in \Omega$ the local interpolation error is bounded below by a constant times the tolerance and thus that the mesh is nowhere overly refined. In L_2-norms efficiency in this precise sense is more difficult to prove and in such cases it may be of interest to prove efficiency in a weaker sense. We present a simple result of this type for the Poisson equation, stating that the a posteriori error bounds may be estimated by (sharp) a priori error bounds (see [6]).

THEOREM 4.1. Under the assumption of Theorem 2.1 there is a constant C, such that for $m = 1, 2$,

$$\alpha_m \|h^m f\| + \beta_m \|h^m D_h^2 U\| \le C \|h^m D^2 u\| .$$

From this result it follows by Remark 2.2 that in general the global L_2 or H^1-interpolation error on a mesh produced by the adaptive algorithm is not essentially below the given tolerance, which indicates efficiency in a certain sense (but does not necessarily exclude local over-refinement). A somewhat different indication on efficiency also follows from Theorem 4.1, namely that an optimal mesh for which $C\|h^m D^2 u\| = \text{TOL}$, will (up to a constant) be accepted by the stopping criterion of the adaptive algorithm. In particular it follows that it is possible to satisfy the stopping criterion for any tolerance, e.g., if $\|D^2 u\|$ is finite.

For the parabolic problem (1.2) one can prove an efficiency result localized in time corresponding to Theorem 4.1 stating that for almost all time steps n the interpolation error $E_{0n}(q = 0)$ or $E_{2n}(q = 1)$ is not essentially below the given tolerance on meshes generated by the adaptive algorithm, see [6].

Also for the hyperbolic model problems (1.3) and (1.4) certain results indicating efficiency of our adaptive algorithms are available. Let us give an outline of these results for the SD-method (1.8) for the stationary problem (1.3). Typically, the exact solution u of (1.3) is piecewise smooth with a boundary layer of width $O(\varepsilon)$ at the outflow boundary $\Gamma_+ = \Gamma\backslash\Gamma_-$ and internal layers of width $O(\sqrt{\varepsilon})$ along streamlines of the velocity field β, e.g., if the inflow boundary data is discontinuous. In the typical case the continuous solution u thus has features on the three different scales $O(1)$, $O(\sqrt{\varepsilon})$ and $O(\varepsilon)$ in smooth regions, internal layers and outflow layers, respectively. Let us now first consider the adaptive algorithm (3.3) for L_2-norm control based on the a posteriori bound (2.8). In this case theoretical and computational results indicate that the algorithm will produce a mesh with mesh size of order $O(\mathrm{TOL}^2)$ at the outflow boundary, $O(\mathrm{TOL}^{8/3})$ at an internal layer, and of order $O(\mathrm{TOL})$ in regions where the exact solution is smooth. This follows using Theorem 2.6 from localization results showing that the width of the numerical outflow layer is $O(h)$ and the width of the internal numerical layer is $O(h^{3/4})$, (see [19, 20]) and the fact that the integrand in (2.8) will be of order $O(1)$ in the layers, and by Theorem 2.5 of order $O(h)$ in regions where the exact solution is smooth. Altogether, these results indicate that the algorithm for L_2-norm control will produce a mesh with correct mesh size close to layers and possibly slight over-refinement in smooth regions, since there the a priori error estimate indictes $O(h^{3/2})$ accuracy, while the a posteriori estimate only gives $O(h)$. Notice in particular that the algorithm is able to handle a problem with both boundary and interior layers and smooth parts with a balanced attention to all features. Depending on the tolerance level chosen and ε, the algorithm may resolve internal layers (if $\mathrm{TOL} \leqslant O(\varepsilon^{3/16})$) and also outflow layers (if $\mathrm{TOL} \leqslant O(\varepsilon^{1/2})$).

Next, we add an indication to refine if $\hat{\varepsilon} > \varepsilon$. In an outflow layer we will have $\hat{\varepsilon} = O(h)$ if $\nu = 2$, and thus $\hat{\varepsilon} = \varepsilon$ will require $h = O(\varepsilon)$ which corresponds to resolution of the outflow layer of width $O(\varepsilon)$. In an internal layer, we will have $\hat{\varepsilon} = O(h^{3/2})$ if $\nu = 2$, and thus $\hat{\varepsilon} = \varepsilon$ will require $h = O(\varepsilon^{2/3})$, which again corresponds to resolution of the internal layer of width $O(\varepsilon^{1/2})$ since the width of the numerical layer is $O(h^{3/4})$. Of course, the stated results are qualitative in nature and are only valid up to constants, but indicate that with proper normalization the requirement $\hat{\varepsilon} = \varepsilon$ imposes resolution of all scales of the continuous solution.

5. The structure of the proofs of the a priori and a posteriori error estimates

In this section we briefly outline the structure of the proofs of our a priori and a posteriori error estimates. We start from a continuous problem with the following variational formulation: Find $u \in V$, such that

$$B(u, v) = L(v) \quad \forall v \in V\,, \tag{5.1}$$

where $B(\cdot\,,\cdot)$ is a continuous bilinear form on $V \times V$, L is a continuous linear form on V and V is a Hilbert space (e.g., $H_0^1(\Omega)$ in the case of (1.1) and (1.3)). Next, we consider a corresponding discrete problem: Given a finite element space $V_h \subset V$, find $V \in V_h$ such that

$$B(U, v) = L(v) \quad \forall v \in V_h\,. \tag{5.2}$$

To prove an a posteriori error estimate in a norm $\|\cdot\|$ related to the scalar product $(\cdot,\cdot)$, let $\varphi \in V$ be the solution of the continuous dual problem: Find $\varphi \in V$, such that

$$B(w, \varphi) = (w, u - U) \quad \forall w \in V, \tag{5.3}$$

a problem which we assume to be uniquely solvable. Choosing now $w = u - U$ in (5.3) we have, using (5.1),

$$\|u - U\|^2 = B(u - U, \varphi) = B(u, \varphi) - B(U, \varphi) = L(\varphi) - B(U, \varphi),$$

which gives the following error representation formula using (5.2):

$$\|u - U\|^2 = L(\varphi - \tilde{\phi}) - B(U, \varphi - \tilde{\phi}), \tag{5.4}$$

where $\tilde{\phi} \in V_h$ is an interpolant of φ. The idea is now to establish a stability result for the dual problem (5.3) of the form

$$|||\varphi||| \leq C\|u - U\|, \tag{5.5}$$

where the norm $|||\cdot|||$ is as strong as possible, and then estimate $\varphi - \tilde{\phi}$ in a weighted norm (as strong as possible), with weight depending on (a negative power of) the mesh size h, in terms of $C|||\varphi|||$. Inserting this estimate into (5.4) and dividing by $\|u - U\|$ gives an a posteriori error estimate of the form

$$\|u - U\| \leq \mathscr{E}(U, h, L),$$

where $\mathscr{E}(U, h, L)$ depends on U, the mesh size h and the data L. Clearly, the stability estimate (5.5) for the continuous dual problem (5.3) is the critical ingredient; in particular we have to find, analytically or computationally, a reasonable approximation of the best constant in (5.5)

The a priori estimate is obtained by introducing the following discrete dual problem: Find $\phi \in V_h$, such that

$$B(w, \phi) = (w, \tilde{U} - U),$$

where $\tilde{U} \in V_h$ is an interpolant of u. Choosing here $w = \tilde{U} - U \in V_h$ we get, using (5.1) and (5.2),

$$\|\tilde{U} - U\|^2 = B(\tilde{U} - U, \phi) = B(\tilde{U} - u, \phi), \tag{5.6}$$

from which we obtain an estimate for $\|\tilde{U} - U\|$ using a (strong) stability estimate for ϕ again of the form (5.5) (but with a different norm $|||\cdot|||$) and standard interpolation error estimates for $\tilde{U} - u$.

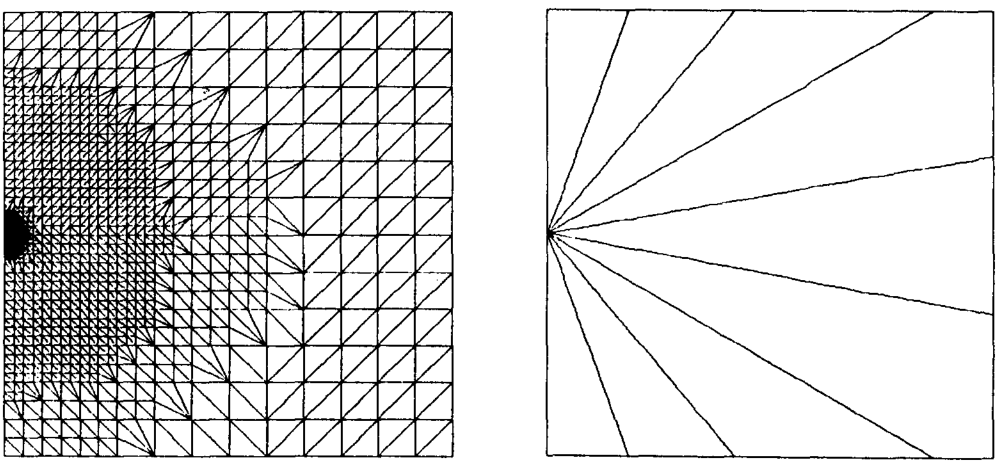
Summing up, the proofs of the a posteriori and a priori error estimates are based on error representation formulas of the form (5.4) and (5.6) together with strong stability estimates for the associated continuous and discrete dual problems, and standard interpolation theory is used to estimate $\varphi - \tilde{\phi}$ and $u - \tilde{U}$, respectively. The right-hand side of (5.4) is clearly related to the *residual* of the discrete solution U, while the right-hand side of (5.6) may be viewed as a *truncation error*. For the concrete implementation of the above approach, we refer to [1–8, 23–25].

6. Numerical results

In this section we present the results of some numerical experiments. Here each mesh T_j is obtained from a previous mesh T_{j-1}, starting with a given coarse mesh T_0, by either local refinement dividing certain triangles (fathers) into four similar triangles (sons) by connecting the midpoints of the sides, or local unrefinement replacing a group of four sons by their common father. In particular, the minimal mesh size of the mesh T_j is half of that of T_{j-1}.

EXAMPLE 6.1. We consider the Poisson equation (1.1) on the square $(-1, 1)^2$ with $f = 0$ and exact solution $u(x_1, x_2) = \arctan(x_2/(x_1 + 1))$ with nonzero boundary conditions with a discontinuity at $(-1, 0)$. In Fig. 1 we give the final mesh produced by (3.1) in the case $m = 2$ choosing $\alpha_2 = 0.15$, $\beta_2 = 0.3$ and TOL $= 0.01$, together with the estimated and actual L_2-error on the successive meshes.

EXAMPLE 6.2. We consider the following adaptive algorithm for the Discontinuous Galer-



LEVEL	NODES	L2-NORM ERROR	APPROX. L2-ERR
1	16	0.26991E+00	0.65300E+00
2	49	0.19159E+00	0.20221E+00
3	169	0.95874E-01	0.10269E+00
4	472	0.47943E-01	0.51627E-01
5	773	0.24135E-01	0.27006E-01
6	924	0.12262E-01	0.14872E-01
7	1080	0.61471E-02	0.99555E-02

Fig. 1. Laplace equation with discontinuous boundary data. Final mesh and level curves of approximate solution. Actual and estimated L_2-error and number of nodes on the sequence of meshes. TOL $= 0.01$.

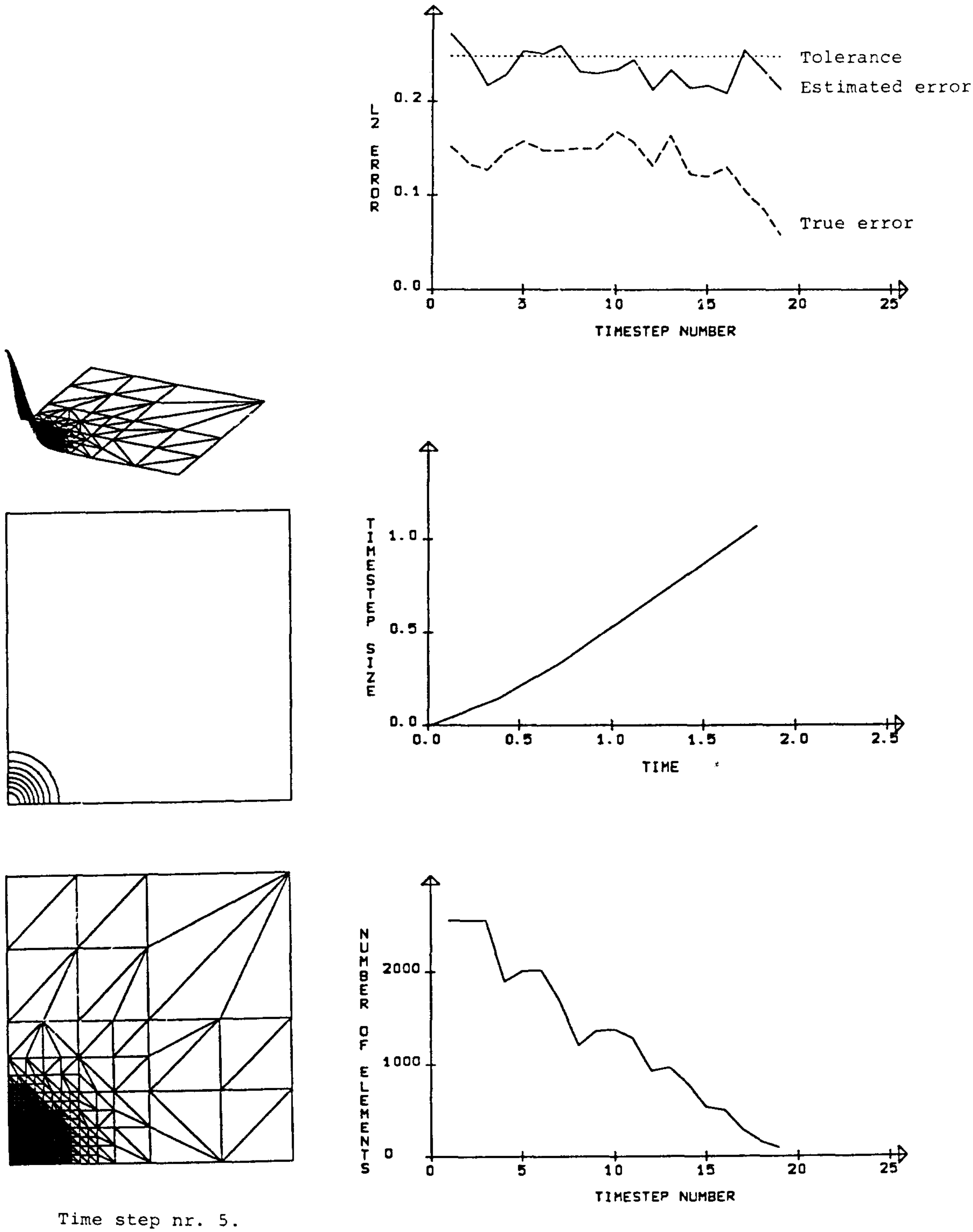


Fig. 2. Heat equation with approximate delta-function as initial data. Actual and estimated L_2-error and number of elements in space as functions of the timestep, and the size of the timestep as function of time. The space mesh and level curves and elevation of approximate solution at time step 5. TOL = 0.25.

kin method (1.7) for the parabolic problem (1.2) based on the a posteriori error estimate (2.7): For $n = 1, 2, \ldots, N$, given an initial triangulation $T_{n,0}$ and an initial time step $k_{n,0}$, determine successively triangulations $T_{n,j}$ with $N_{n,j}$ elements and mesh functions $h_{n,j}$, time steps $k_{n,j}$ and corresponding approximate solutions $U_{n,j}$ defined on $\Omega \times I_n$, $j = 1, \ldots, J$, such that with $h_{n,j}$ and $k_{n,j}$ maximal

$$C_6 \max_{I_{nj}} \|h_{n,j}^2 f\|_{L_2(K)} + C_8 \max_{I_{nj}} \|h_{n,j}^2 D_{n,j-1}^2 U_{n,j-1}\|_{L_2(K)}$$

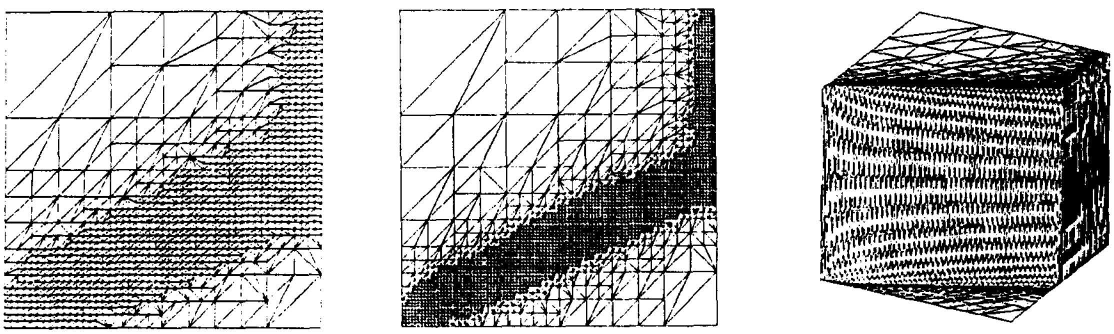
$$+ C_{11} \|h_{n,j}^2 [U_{n,j-1}]/k_{n,j-1}\|_{L_2(K)} = \frac{\theta \mathrm{TOL}}{2\sqrt{N_{n,j-1}}}, \quad \forall K \in T_{n,j-1},$$

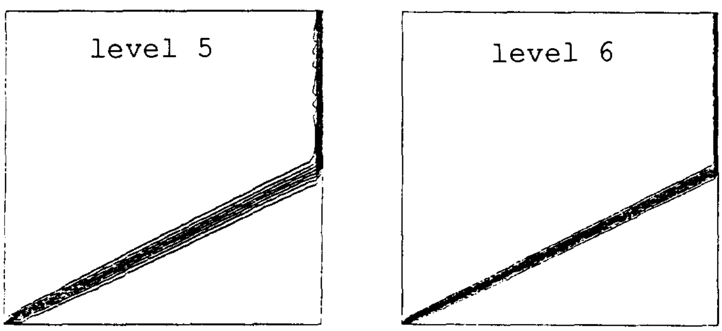
$$k_{n,j}^3 (C_7 \|f_{tt}\|_{I_{n,j}} + \min(C_{10} \|\Delta_{n,j-1} P_{n,j-1} [U_{n,j-1}]_{n-1}/k_{n,j-1}^2\|,$$

$$C_9 \|[U_{n,j-1}]_{n-1}/k_{n,j-1}^3\|)) = \frac{\mathrm{TOL}}{2} \quad \text{if } q = 1,$$

$$k_{n,j} (C_2 \|f\|_{I_{n,j}} + C_4 \|[U_{n,j-1}]_{n-1}/k_{n,j-1}\|) = \frac{\mathrm{TOL}}{2} \quad \text{if } q = 0,$$

where $C_2 = 3$, $C_6 = 0.15$, $C_7 = 1/36$, $C_8 = 0.3$, $C_9 = 2$, $C_{10} = 1/6$ and $C_{11} = 0.2$. We choose the



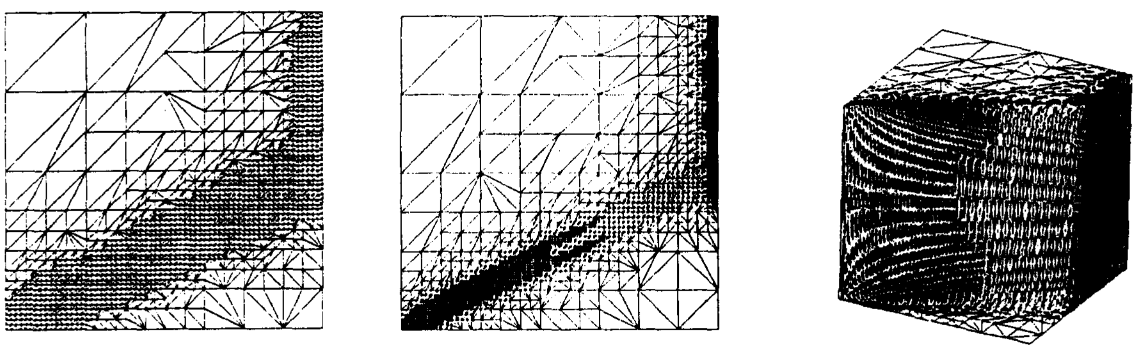


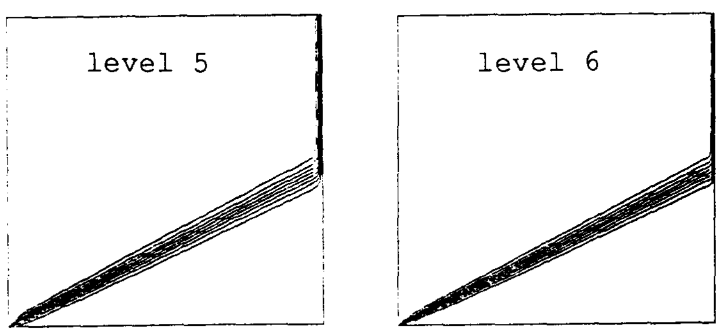
LEVEL	NODES	L2-NORM ERROR	APPROX. L2-ERR
1	16	0.29189E+00	0.29654E+00
2	46	0.21329E+00	0.23207E+00
3	142	0.16379E+00	0.20038E+00
4	382	0.12156E+00	0.16341E+00
5	1027	0.90524E-01	0.12973E+00
6	2418	0.67763E-01	0.10291E+00

Fig. 3. Stationary convection–diffusion problem with internal and boundary layer. Adaptivity based on (3.4a,b), $\varepsilon = 10^{-6}$, TOL = 0.1. Estimated and actual L_2-error and number of nodes on sequence of meshes (level 1 corresponds to the initial mesh). Elevation of final solution. Display of mesh and level curves of approximate solution for indicated levels.

initial data u_0 to be an 'approximate deltafunction' at $x = 0$: $u_0 = 250\exp(-|x|^2/250)$ and $\Omega = (0,1)^2$. We give in Fig. 2 the sequence of time steps, the number of elements in the triangulation on each time interval, and the $L_2(\Omega)$-error $\|u(t_n) - U_n^-\|$, $n = 1, 2, \ldots$, in the case $q = 1$, together with the space mesh at time step 5. We notice that the actual error is approximately constant in time and slightly below the given tolerance.

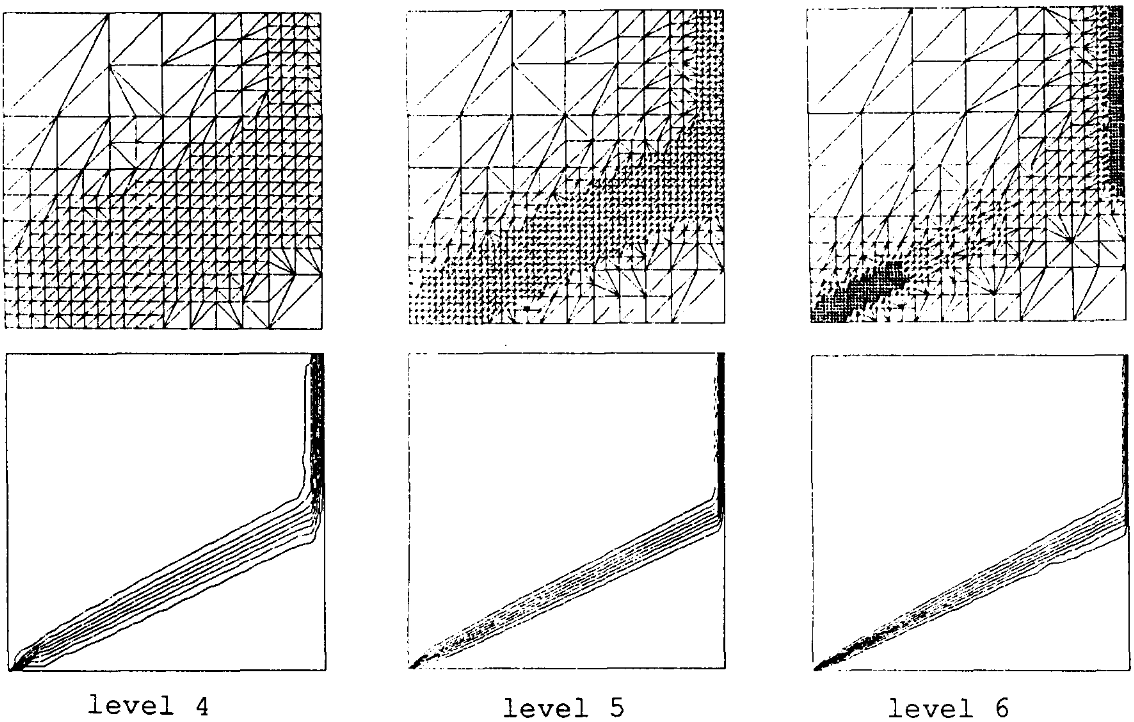
EXAMPLE 6.3. We now give some results for the adaptive algorithms for the streamline diffusion methods (1.8) for the stationary hyperbolic problem (1.3) based on (i) (3.4a,b) and (ii) (3.4a,b) together with the additional refinement criterion $\hat{\varepsilon} = \varepsilon$ corresponding to (3.4c). We consider a problem with both internal layer and outflow layer, and with $\Omega = (0,1)^2$, $f = 0$, $\beta = (2,1)$, $\alpha = 0$, $u(0, x_2) = 1$ for $0 < x_2 \leqslant 1$, $u(x_1, 1) = 1$ for $0 \leqslant x_1 < 1$ and $u(x_1, x_2) = 0$ if $x_1 = 1$ or $x_2 = 0$. In Figs. 3–6 we give some results with the algorithms (i) and (ii) and varying ε and TOL. The constants C_i in (1.8) were chosen as follows: $C_1 = 0.5$, $C_2 = 0.7$. Note that the width of the layer refinement of mesh T_j is related to the width of the numerical layer of the approximate solution U_{j-1} on mesh T_{j-1}. This is the reason why the width of the refinement of T_j appears to be too large as compared to the width of the numerical layer of the solution U_j. Note also that the L_2-error is computed, for simplicity, by comparison with the exact solution corresponding to $\varepsilon = 0$, which means that the stated L_2-error is not precise for refined meshes and relatively large ε.

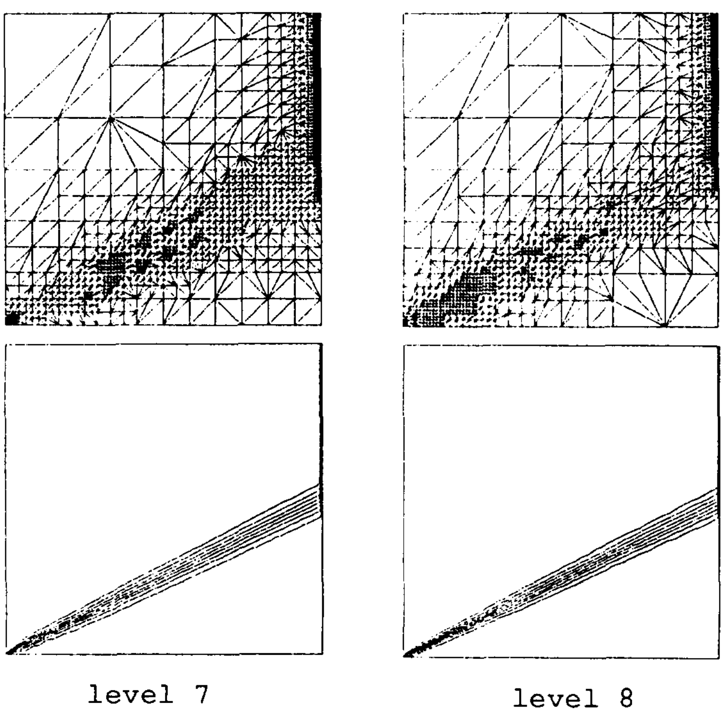


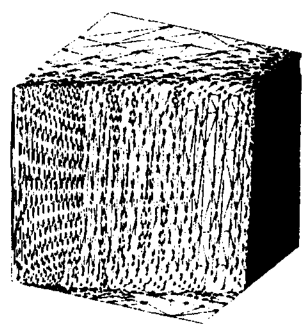


LEVEL	NODES	L2-NORM ERROR	APPROX. L2-ERR
1	25	0.25618E+00	0.26409E+00
2	78	0.20051E+00	0.22927E+00
3	235	0.15016E+00	0.18838E+00
4	639	0.11510E+00	0.11603E+00
5	1383	0.93323E-01	0.57050E-01
6	1973	0.82704E-01	0.45399E-01

Fig. 4. As in Fig. 3 with now $\varepsilon = 10^{-4}$, TOL $= 0.05$.







LEVEL	NODES	L2-NORM ERROR	APPROX. L2-ERR.
1	16	0.30011E+00	0.29531E+00
2	46	0.21934E+00	0.23643E+00
3	151	0.16960E+00	0.20248E+00
4	364	0.12859E+00	0.14998E+00
5	835	0.10133E+00	0.73539E-01
6	902	0.91251E-01	0.10336E+00
7	1478	0.90112E-01	0.62991E-01
8	2135	0.82937E-01	0.64859E-01

Fig. 5. As in Fig. 3 with now the adaptivity based on (3.4a–c), i.e., with the extra requirement $\hat{\varepsilon} = \varepsilon$; $\varepsilon = 10^{-3}$, TOL = 0.15.

References

[1] K. Eriksson, Adaptive finite element methods based on optimal error estimates for linear elliptic problems, Technical report, Department of Mathematics, Chalmers University of Technology, 1987.

[2] K. Eriksson, Adaptive finite element methods for parabolic problems II: A priori error estimates in $L_\infty(L_2)$ and $L_\infty(L_\infty)$, Technical report, Department of Mathematics, Chalmers University of Technology, 1988.

[3] K. Eriksson, Error estimates for the $H_0^1(\Omega)$ and $L_2(\Omega)$ projections onto finite element spaces under weak mesh regularity assumptions, Department of Mathematics, Chalmers University of Technology, 1988.

[4] K. Eriksson and C. Johnson, An adaptive finite element method for linear elliptic problems, Math. Comp. 50 (1988) 361–383.

[5] K. Eriksson and C. Johnson, Error estimates and automatic time step control for non-linear parabolic problems, I, SIAM J. Numer. Anal. 24 (1987) 12–23.

[6] K. Eriksson and C. Johnson, Adaptive finite element methods for parabolic problems I: A linear model problem, SIAM J. Numer. Anal. (to appear).

[7] K. Eriksson and C. Johnson, Adaptive finite element methods for parabolic problems III: Time steps variable in space, IV: A nonlinear problem, to appear.

[8] K. Eriksson and C. Johnson, Adaptive streamline diffusion finite element methods for convection–diffusion problems, Technical report, Department of Mathematics, Chalmers University of Technology, 1990.

[9] I. Babuška and W.C. Rheinboldt, Reliable error estimation and mesh adaptation for the finite element method, in: Computational Methods in Nonlinear Mechanics (North-Holland, New York, 1980) 67–108.

[10] R. Bank, Analysis of a local a posteriori error estimate for elliptic equations, in: I. Babuška et al., eds., Accuracy Estimates and Adaptive Refinements in Finite element Computations (Wiley, New York, 1986).

[11] D. Ewing, Adaptive mesh refinements in large-scale fluid flow simulation, in: I. Babuška et al., eds., Accuracy Estimates and Adaptive Refinements in Finite element Computations (Wiley, New York, 1986).

[12] E.M. Abdalass, Resolution performante du probléme de Stokes par mini-element, amillages auto-adaptifs et methodes multigrilles-applications, Thése de 3me cycle, Ecole Central de Lyon, 1987.

[13] R. Verfürth, A posteriori error estimators for the Stokes equations, Numer. Math. (1989).

[14] R. Bank, Personal communication, 1989.

[15] R. Löhner, K. Morgan and O.C. Zienkiewicz, Adaptive grid refinement for the compressible Euler equations, in: I. Babuška et al., eds., Accuracy Estimates and Adaptive Refinements in Finite element Computations (Wiley, New York, 1986).

[16] J.T. Oden, L. Demkowicz, T. Strouboulis and P. Devoo, Adaptive methods for problems in solid and fluid mechanics, in: I. Babuška et al., eds., Accuracy Estimates and Adaptive Refinements in Finite element Computations (Wiley, New York, 1986).

[17] P. Hansbo, Adaptivity and streamline diffusion procedures in the finite element methods, Thesis, Chalmers University of Technology, 1989.

[18] K. Eriksson, C. Johnson and V. Thomée, Time discretization of parabolic problems by the Discontinuous Galerkin method, RAIRO MAN 19 (1985) 611–643.

[19] C. Johnson, U. Nävert and J. Pitkäranta, Finite element methods for linear hyperbolic problems, Comput. Methods Appl. Mech. Engrg. 45 (1984) 285–312.

[20] C. Johnson, A. Schatz and L. Wahlbin, Crosswind smear and pointwise errors in streamline diffusion finite element methods, Math. Comp. 49 (1987) 25–38.

[21] C. Johnson, A. Szepessy and P. Hansbo, On the convergence of shock-capturing streamline diffusion finite element methods for conservation laws, Math. Comp. (to appear).

[22] A. Szepessy, Convergence of the streamline diffusion finite element method for conservation laws, Thesis, Department of Mathematics, Chalmers University of Technology, 1989.

[23] C. Johnson, Error estimates and adaptive time step control for a class of one step methods for stiff ordinary differential equations, SIAM J. Numer. Anal. 25 (1988) 908–926.

[24] C. Johnson, Y.-Y. Nie and V. Thomée, An a posteriori error estimate and automatic time step control for a backward Euler discretization of a parabolic problem, SIAM J. Numer. Anal. (to appear).

[25] J. Lennblad, An adaptive finite element method for a linear parabolic problem. Technical report, Department of Mathematics, Chalmers University of Technology, 1988.

[26] G. Lippold, Error estimates and step-size control for the approximate evolution of a first order evolution equation, Preprint, Akademie der Wissenschaffen der Karl-Weierstrass-institute für Matematik, Berlin, 1988.