On Dispersive Difference Schemes. I

JONATHAN GOODMAN AND PETER D. LAX
Courant Institute

Introduction

It is well known that solutions of nonlinear hyperbolic equations, in one space variable, of which

$$(1) \qquad u_t + uu_x = 0$$

is the best known and best beloved example, break down after a finite elapse of time. It is also known that solutions of (1) can be continued beyond the time of the breakdown as solutions in the integral sense of the conservation law

$$(1)' \qquad u_t + \left(\tfrac{1}{2}u^2\right)_x = 0.$$

These solutions in the integral sense contain discontinuities, the mathematical representation of shock waves; they are uniquely determined by their initial data provided that the discontinuities are constrained to satisfy an entropy condition.

It is further known that discontinuous solutions of (1)′ in the integral sense can be obtained as strong limits—L^1 and bounded a.e.—of smooth solutions of the viscous equation

$$(2) \qquad u_t + uu_x = \mu u_{xx}, \qquad \mu > 0,$$

as μ tends to zero. They can also be obtained as strong limits of solutions of difference equations approximating (1)′ that contain a sufficient amount of artificial viscosity, such as the one proposed and used by von Neumann and Richtmyer [10], Friedrichs and Lax, Godunov, etc. The solutions constructed as such limits satisfy the entropy condition.

On the other hand, it is known that solutions of dispersive approximations to equation (1) behave quite differently. Specifically, it is known (see [4], [12]) that solutions $u(x, t; \varepsilon)$ of the KdV equation

$$(3) \qquad u_t + uu_x + \varepsilon^2 u_{xxx} = 0,$$

$$(4) \qquad u(x,0) = u_0(x),$$

behave as follows, as ε tends to 0:

As long as the solution of equation (1) with initial value (4) has a smooth solution, $u(x, t; \varepsilon)$ tends uniformly to that smooth solution. However, when t

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exceeds the time when the solution of (1), (4) breaks down, $u(x, t; \varepsilon)$ behaves in an oscillatory manner over some x-interval; as ε tends to zero, the amplitude of these oscillations remains bounded but does not tend to zero, and its wave length is of order ε. Clearly, $u(x, t; \varepsilon)$ does not tend strongly to a limit as ε tends to zero; however, $u(x, t; \varepsilon)$ converges weakly. It is easy to show that this weak limit —in the sense of distribution—which we denote as $\bar{u}$,

$$(5) \qquad d\text{-}\lim_{\varepsilon \to 0} u(x, t; \varepsilon) = \bar{u}(x, t),$$

does *not* satisfy the conservation law (1)′ in the integral sense. For rewrite (3) in conservation form:

$$(3)' \qquad u_t + \left(\tfrac{1}{2}u^2\right)_x + \varepsilon^2 u_{xxx} = 0.$$

It follows from (5) that the d-limit of the first term is $\bar{u}_t$, and that of the third term zero; therefore, the middle term also converges, and its limit is $(\frac{1}{2}\overline{u^2})_x$, where

$$(5)' \qquad \overline{u^2} = d\text{-}\lim_{u \to 0} u^2(x, t; \varepsilon).$$

So we get from (3)′ that

$$(6) \qquad \bar{u}_t + \left(\tfrac{1}{2}\overline{u^2}\right)_x = 0.$$

It is a well-known fact that if $u(\varepsilon)$ tends weakly but <u>not</u> strongly to $\bar{u}$, then the weak limit of $u^2(\varepsilon)$ is *not* the square of the weak limit of $u(\varepsilon)$:

$$(7) \qquad \overline{u^2} \neq \bar{u}^2;$$

in fact,

$$(7)' \qquad \overline{u^2} > \bar{u}^2.$$

The explicit formulas that have been derived in [4] for $\bar{u}$ and $\overline{u^2}$ bear out these inequalities. Setting this into (6) we deduce that

$$(6)' \qquad \bar{u}_t + \left(\tfrac{1}{2}\bar{u}^2\right)_x \neq 0,$$

unless $\overline{u^2}$ happens to differ from $\bar{u}^2$ by a constant; the explicit formulas for $\bar{u}$ and $\overline{u^2}$ show that this is not the case.

The aim of this paper is to study the corresponding phenomena of oscillatory behavior and weak, but not strong, convergence of solutions of difference approximations to (1)′ that are *dispersive*. The particular approximation we study here is semi-discrete, i.e., continuous in time, discrete in x:

$$\frac{d}{dt}U_k + U_k\frac{U_{k+1} - U_{k-1}}{2\Delta} = 0, \tag{8}$$

$$U_k(0) = u_0(k\Delta). \tag{8$'$}$$

To see the dispersive nature of this approximation we use Taylor's expansion:

$$\frac{U_{k+1} - U_{k-1}}{2\Delta} \simeq u_x + \tfrac{1}{6}\Delta^2 u_{xxx}. \tag{9}$$

Setting this into (8) we obtain

$$u_t + uu_x + \tfrac{1}{6}\Delta^2 uu_{xxx} \simeq 0,$$

an equation very much like the KdV equation (3) as long as u does not change sign. If this analogy is valid, we can expect equation (8) to have solutions that oscillate with a wave length $O(\Delta)$, i.e., on mesh scale.

The purpose of this study is to show that the analogy is indeed valid, and that solutions of (8) behave analogously in their dependence on Δ as solutions of (3) do in their dependence on ε. As usual, the study of the difference equation is trickier than that of the differential equation.

The organization of this paper is as follows: in Section 1 we show that, for t less than the breakdown time of a solution of (1) with given initial data (4), the solution of (8) with the same initial data converges to the solution of (1), provided the initial data are sufficiently smooth.

In Section 2 we construct a special class of oscillatory solutions of (8) and show their weak convergence. These weak limits gratifyingly fail to satisfy the conservation law (1)′.

In Section 3 we show that beyond breakdown time for a solution of (1) the corresponding solution of (8) does not converge strongly. Using a simple transformation we change (8) to a system studied by Kac and van Moerbeke [3] and shown by them to be completely integrable; this system was reduced by Moser [8] to the Toda lattice, in the form due to Flaschka [2]. The integrals can be used to bound solutions of (8).

In Section 4 we present ample numerical evidence for the oscillatory character of solutions of (8) beyond breakdown time, and for their weak convergence to nonsolutions of (1).

In a subsequent publication we hope to use the complete integrability of (8) as discussed in [7] to furnish rigorous proofs for these experimental findings.

The oscillatory nature of solutions of dispersive difference schemes was discovered, accidentally, by von Neumann [9] in 1944 in a calculation of compressible flows with shocks in one space dimension, employing centered difference schemes. The solutions—calculated using punched card equipment—contained, as expected, a shock; but, unexpectedly, it contained post shock oscillations on mesh scale. Von Neumann suggested that these mesh-scale oscillations in velocity are to be interpreted as heat energy produced by the irreversible action of the shock wave, and conjectured that, as Δx, Δt tend to zero, the solutions of the difference equations converge weakly to exact discontinuous solutions of the equations governing the flow of compressible fluids. If the simplified model (1)′ is a guide to the equations of compressible flow, then there is reason to doubt the validity of von Neumann's surmise. A rigorous proof either way is a long way off; further analysis is contained in [5] and [6].

1. Convergence to Smooth Solutions

For convenience we take all functions of x to be periodic with period 2π, and choose $\Delta = 2\pi/N$, N integer. This makes U_k periodic in k with period N.

We rely on a basic convergence theorem of Strang. As stated in [11], the theorem deals with explicit difference schemes, but the proof applies to semi-discrete schemes as well.

Let

$$u_t + Cu_x = 0 \tag{1.1}$$

be a quasilinear hyperbolic system of partial differential equations. Here $C(u)$ is a matrix-valued function of u, depending smoothly on its arguments.

We consider a semi-discrete approximation to (1.1) of the form

$$\frac{d}{dt}U_k + F(U_{k-1}, U_k, U_{k+1}, \Delta) = 0 \tag{1.2}$$

that is consistent with (1.1). We assume that F is a smooth function of its arguments.

CONVERGENCE THEOREM OF STRANG.[1] *Let $u(x, t)$ be a smooth, x-periodic solution of the hyperbolic system* (1.1). *Assume that the approximation* (1.2) *to* (1.1) *is linearly l^2 stable around $u(x, t)$.*

Choose the initial data of U_k to coincide with those of u:

$$U_k(0) = u(k\Delta, 0), \tag{1.2$'$}$$

then $U_k(t)$ converges uniformly to $u(k\Delta, t)$ as $\Delta \to 0$, for $0 \leqq t \leqq T$. Furthermore,

$$|U_k(t) - u(k\Delta, t)| \leqq O(\Delta^p), \tag{1.3}$$

where p is the order of accuracy of the scheme (1.2) *for equation* (1.1).

The *linearization* of (1.2) is

$$\frac{d}{dt}\eta_k + \sum_{-1}^{1} F_j \eta_{k+j} = 0, \tag{1.4}$$

[1] Strang's theorem applies to any number of space variables, and to any finite stencil in the approximation (1.2).

where F_j is the gradient of the function F with respect to its $(j+2)$-nd argument, $j = -1, 0, 1$, evaluated at

$$u((k-1)\Delta, t), u(k\Delta, t), u((k+1)\Delta, t).$$

l^2 *stability* means that solutions of (1.4) are uniformly bounded in the l^2 norm:

$$\sum_1^N \eta_k^2(t) \leqq K(t) \sum_1^N \eta_k^2(0), \tag{1.4$'$}$$

where $K(t)$ is *independent* of Δ.

We turn now to equation (1); let u be a smooth solution. We linearize (8) around u, obtaining the specialization of (1.4) to our case:

$$\frac{d}{dt}\eta_k + u_k \frac{\eta_{k+1} - \eta_{k-1}}{2\Delta} + \eta_k \frac{u_{k+1} - u_{k-1}}{2\Delta} = 0, \tag{1.5}$$

where

$$u_k = u(k\Delta, t). \tag{1.5$'$}$$

Multiply (1.5) by η_k and sum:

$$\frac{d}{dt}\frac{1}{2}\sum \eta_k^2 + \sum u_k \frac{\eta_k \eta_{k+1} - \eta_{k-1}\eta_k}{2\Delta} + \eta_k^2 \frac{u_{k+1} - u_{k-1}}{2\Delta} = 0.$$

Summing by parts the middle term, we get, using the abbreviation

$$H(t) = \frac{1}{2}\sum_1^N \eta_k^2, \tag{1.6}$$

that

$$\frac{d}{dt}H + \sum \frac{u_k - u_{k+1}}{2\Delta}\eta_k \eta_{k+1} + \sum \frac{u_{k+1} - u_{k-1}}{2\Delta}\eta_k^2 = 0. \tag{1.7}$$

Since $u(x, t)$ is a smooth function of x, it follows from (1.5)$'$ and (1.7) that

$$\frac{d}{dt}H \leqq 2LH,$$

where L is a Lipschitz constant for $u(x, t)$. From this stability (1.4)$'$ follows with

$$K(t) = \exp 2Lt.$$

We appeal now to Strang's convergence theorem to conclude:

THEOREM 1.1. *Let $u(x, t)$ be a smooth x-periodic solution of* (1) *for* $0 \leqq t \leqq T$, $U_k(t)$ *the solution of* (8), (8)′, *with* $u_0(x) = u(x, 0)$. *Then* $U_k(t)$ *converges uniformly to* $u(k\Delta, t)$ *as* $\Delta \to 0$, *for* $0 \leqq t \leqq T$.

We recall that the order of accuracy of a scheme (1.2) approximating (1.1) is the smallest integer p such that

$$|C(u)u_x - F(u(x-\Delta), u(x), u(x+\Delta), \Delta)| \leqq O(\Delta^p). \tag{1.8}$$

For the approximation (8) to equation (1) it follows from (9) that the order of accuracy is 2. Therefore, by (1.3),

$$|u(k\Delta, t) - U_k(t)| \leqq O(\Delta^2), \tag{1.9}$$

for $0 \leqq t \leqq T$. This is born out by the numerical calculations reported in Section 4.

2. A Class of Oscillatory Solutions

We choose the number N of mesh points per period to be even. We represent U_k as

$$U_k = V_k + (-1)^k W_k, \tag{2.1}$$

V_k and W_k of period N.

Inserting (2.1) into (8) gives

$$\frac{d}{dt}(V_k + W_k) + (V_k + W_k)\left(\frac{V_{k+1} - V_{k-1}}{2\Delta} - \frac{W_{k+1} - W_{k-1}}{2\Delta}\right) = 0 \tag*{(2.2)$_+$}$$

for k even, and

$$\frac{d}{dt}(V_k - W_k) + (V_k - W_k)\left(\frac{V_{k+1} - V_{k-1}}{2\Delta} + \frac{W_{k+1} - W_{k-1}}{2\Delta}\right) = 0 \tag*{(2.2)$_-$}$$

for k odd.

We require now that $(2.2)_+$ and $(2.2)_-$ hold for k both even and odd. The resulting equations can be interpreted as a semi-discrete approximation to the following system of partial differential equations:

$$\begin{aligned} &(v + w)_t + (v + w)(v_x - w_x) = 0, \\ &(v - w)_t + (v - w)(v_x + w_x) = 0. \end{aligned} \tag{2.3}$$

Introducing

$$v + w = a, \qquad v - w = b, \tag{2.4}$$

we can rewrite (2.3) as

$$\begin{aligned} a_t + ab_x &= 0, \\ b_t + ba_x &= 0. \end{aligned} \tag{2.5}$$

Clearly, (2.5) is a hyperbolic system if and only if $ab > 0$, in particular if a and b are positive. In view of (2.1), (2.4), this is related to $u > 0$. If this condition is satisfied by the initial data of a, b, then it will continue to be satisfied by the solution $a(x, t)$, $b(x, t)$, at least until the first shock forms.

We denote

$$V_k + W_k = A_k, \qquad V_k - W_k = B_k, \tag{2.6}$$

and rewrite $(2.2)_\pm$ as

$$\frac{d}{dt}A_k + A_k\frac{B_{k+1} - B_{k-1}}{2\Delta} = 0, \tag*{$(2.7)_+$}$$

$$\frac{d}{dt}B_k + B_k\frac{A_{k+1} - A_{k-1}}{2\Delta} = 0. \tag*{$(2.7)_-$}$$

We claim that if A_k and B_k are positive at $t = 0$, then they are positive at all future t. To see this we add the two equations $(2.7)_+$ and $(2.7)_-$; the sum of the second terms is a perfect difference:

$$\frac{A_kB_{k+1} + A_{k+1}B_k}{2\Delta} - \frac{A_{k-1}B_k + A_kB_{k-1}}{2\Delta}. \tag{2.8}$$

Therefore when we sum from $k = 1$ to N the sum of (2.8) is zero, and we get

$$\frac{d}{dt}\sum A_k + B_k = 0,$$

i.e.,

$$\sum A_k(t) + B_k(t) \tag{2.9}$$

is a conserved quantity.

If we divide $(2.7)_+$ by A_k and sum from $k = 1$ to N, we get likewise

$$\sum \frac{1}{A_k}\frac{d}{dt}A_k = 0,$$

i.e.,

$$\prod A_k(t) \tag*{$(2.10)_+$}$$

is a conserved quantity. Similarly,

$$(2.10)_- \qquad \prod B_k(t)$$

is a conserved quantity.

It follows from these conservation laws that if $A_k(t)$, $B_k(t)$ are initially positive, they can never change sign. For otherwise one or several of them would have to approach zero at some time t while all the others remain positive. It follows then from $(2.10)_\pm$ that another A_j or B_j would have to approach ∞; but this contradicts the conservation of (2.9). This argument shows that the system $(2.7)_\pm$ of ordinary differential equations has solution for all time.

THEOREM 2.1. *Let $a(x,t)$, $b(x,t)$ be a smooth, x-periodic solution of* (2.5) *for $0 \leqq t \leqq T$, a and b both positive. Let A_k, B_k be solutions of* (2.7) *whose initial values are equal to $a(k\Delta,0)$, $b(k\Delta,0)$. Then $A_k(t)$, $B_k(t)$ converge uniformly to $a(k\Delta,t)$, $b(k\Delta,t)$ for $0 \leqq t \leqq T$.*

Proof: We appeal to Strang's convergence theorem. We have to verify that the linearization of (2.7) around a, b is l^2 stable. The linearization is

$$(2.11)_+ \qquad \frac{d}{dt}\eta_k + a_k\frac{\zeta_{k+1}-\zeta_{k-1}}{2\Delta} + \eta_k\frac{b_{k+1}-b_{k-1}}{2\Delta} = 0,$$

$$(2.11)_- \qquad \frac{d}{dt}\zeta_k + b_k\frac{\eta_{k+1}-\eta_{k-1}}{2\Delta} + \zeta_k\frac{a_{k+1}-a_{k-1}}{2\Delta} = 0,$$

where

$$a_k = a(\Delta k, t), \qquad b_k = b(k\Delta, t).$$

Multiply $(2.11)_+$ by $b_k\eta_k$, $(2.11)_-$ by $a_k\zeta_k$, and add them; the sum of the first terms can be written as

$$(2.12)_1 \qquad \frac{1}{2}\frac{d}{dt}\left[b_k\eta_k^2 + a_k\zeta_k^2\right] - \frac{1}{2}\eta_k^2\frac{d}{dt}b_k - \frac{1}{2}\zeta_k^2\frac{d}{dt}a_k.$$

Using (2.8) we can write the sum of the second term as

$$(2.12)_2 \qquad \frac{1}{2\Delta}a_kb_k[\eta_k\zeta_{k+1} + \eta_{k+1}\zeta_k - \eta_{k-1}\zeta_k - \eta_k\zeta_{k-1}].$$

Since $a(x,t)$ and $b(x,t)$ are Lipschitz continuous in x, the sum of the third terms is less than

$$(2.12)_3 \qquad K(\eta_k^2 + \zeta_k^2),$$

independent of k, Δ and t. We sum (2.12) from $k = 1$ to N; in $(2.12)_2$ we sum by parts. Altogether we get

$$\frac{1}{2}\frac{d}{dt}\sum b_k\eta_k^2 + a_k\zeta_k^2 \leqq K'\sum\eta_k^2 + \zeta_k^2. \tag{2.13}$$

Introduce the abbreviation

$$H = \frac{1}{2}\sum b_k\eta_k^2 + a_k\zeta_k^2;$$

since for $0 \leqq t \leqq T$, a_k and b_k are bounded from below by a positive constant, we can rewrite (2.13) as

$$\frac{d}{dt}H \leqq K''H.$$

This shows that

$$H(t) \leqq \exp K''tH(0).$$

This proves l^2 stability of the linearized system, and so by Strang's theorem we conclude that Theorem 2.1 holds.

If we switch back to the original variables v, w, and V, W we see that $V_k(t)$ tends to $v(k\Delta, t)$, $W_k(t)$ to $w(k\Delta, t)$.

We extend now U_k, V_k and W_k to continuous arguments by setting

$$\begin{aligned} &U_\Delta(x,t) = U_k(t), \\ &V_\Delta(x,t) = V_k(t) \quad \text{for} \quad (k - \tfrac{1}{2}\Delta < x < (k + \tfrac{1}{2})\Delta, \\ &W_\Delta(x,t) = W_k(t). \end{aligned} \tag{2.14}$$

Similarly, we set

$$E_\Delta(x) = (-1)^k \quad \text{for} \quad (k - \tfrac{1}{2})\Delta < x < (k + \tfrac{1}{2})\Delta. \tag{2.14$'$}$$

It follows from (2.1) that

$$U_\Delta = V_\Delta + E_\Delta W_\Delta. \tag{2.15}$$

We have seen above that V_Δ and W_Δ converge uniformly to v and w, respectively. It follows from (2.1)′ that E_Δ tends weakly to zero. Setting these into (2.15) we conclude that U_Δ tends weakly to v:

$$d\text{-}\lim U_\Delta = v. \tag{2.16}$$

Denote by $\mathbf{T}_\Delta$ translation in x by Δ:

$$(\mathbf{T}_\Delta U)(x) = U(x+\Delta). \tag{2.17}$$

It follows from the definition (2.14)′ of E_Δ that

$$\mathbf{T}_\Delta E_\Delta = -E_\Delta. \tag{2.18}$$

Apply $\mathbf{T}_\Delta$ to (2.15); then from the above we get

$$\mathbf{T}_\Delta U_\Delta = \mathbf{T}_\Delta V_\Delta - E_\Delta \mathbf{T} W_\Delta.$$

Multiply this by (2.15); using (2.18), we obtain

$$\begin{aligned} U_\Delta \mathbf{T}_\Delta U_\Delta = {} & V_\Delta \mathbf{T}_\Delta V_\Delta - W_\Delta \mathbf{T}_\Delta W_\Delta \\ & + E_\Delta [W_\Delta \mathbf{T}_\Delta V_\Delta - V_\Delta \mathbf{T}_\Delta W_\Delta]. \end{aligned}$$

Taking the weak limit as $\Delta \to 0$ we conclude, as above, that

$$d\text{-}\lim U_\Delta \mathbf{T}_\Delta U_\Delta = v^2 - w^2. \tag{2.19}$$

Using the notation U_Δ we can rewrite (8) as an equation in conservation form:

$$\frac{d}{dt} U_\Delta + \frac{U_\Delta \mathbf{T}_\Delta U_\Delta - (\mathbf{T}_\Delta^{-1} U_\Delta) U_\Delta}{2\Delta} = 0.$$

We take the limit of this equation in the distribution sense; we obtain, taking (2.16) and (2.19) into account,

$$v_t + \tfrac{1}{2}(v^2 - w^2)_x = 0. \tag{2.20}$$

Unless w is independent of x, it follows that

$$v_t + \left(\tfrac{1}{2} v^2\right)_x \neq 0. \tag{2.20$'$}$$

This proves

THEOREM 2.1. *Equation* (8) *has solutions of the form* (2.1), *oscillating on mesh scale, whose weak limit does not satisfy the conservation law* (1)′.

3. Complete Integrability

Let N be an even positive integer, as in Section 2. Let A denote an N-vector of real numbers:

$$A = (A_1, \cdots, A_N).$$

We extend A_k to be defined for all k and be periodic with period N:

$$A_{k+N} = A_k.$$

The operator $\mathbf{T}$ is translation:

$$(\mathbf{T}A)_k = A_{k+1}. \tag{3.1}$$

Clearly, $\mathbf{T}$ maps periodic sequences into periodic sequences. We shall use the abbreviation

$$\mathbf{T}A = A_+. \tag{3.2$_+$}$$

The inverse of $\mathbf{T}$ is $\mathbf{T}^{-1}$, translation in the opposite direction. We abbreviate

$$\mathbf{T}^{-1}A = A_-. \tag{3.2$_-$}$$

Following Kac and van Moerbeke [2], we associate with each vector A the operators

$$\mathbf{L} = A\mathbf{T} + A_-\mathbf{T}^{-1}, \tag{3.3}$$

where A acts as the multiplication operator. L is a symmetric, cyclic Jacobi matrix with zero diagonal. We associate with A a second operator

$$\mathbf{B} = AA_+\mathbf{T}^2 - A_-A_{--}\mathbf{T}^{-2}, \tag{3.4}$$

which is antisymmetric. Here

$$A_{--} = \mathbf{T}^{-1}A_- = \mathbf{T}^{-2}A.$$

The commutator $[\mathbf{B}, \mathbf{L}] = \mathbf{BL} - \mathbf{LB}$ is

$$[\mathbf{B}, \mathbf{L}] = A(A_+^2 - A_-^2)\mathbf{T} + A_-(A - A_{--})\mathbf{T}^{-1}. \tag{3.5}$$

Consider the differential equation

$$\frac{d}{dt}\mathbf{L} + [\mathbf{B}, \mathbf{L}] = 0. \tag{3.6}$$

Componentwise this reads as

$$\frac{d}{dt}A_k + A_k(A_{k+1}^2 - A_{k-1}^2) = 0. \tag{3.6$'$}$$

Kac and van Moerbeke show that this system is completely integrable. In [8], Moser reduced this system to Flaschka's form of the Toda lattice, known to be completely integrable.

It follows from (3.6) that, for any positive integer p,

$$\frac{d}{dt}\mathbf{L}^p + [\mathbf{B}, \mathbf{L}^p] = 0. \tag{3.6$''$}$$

This implies that

$$\operatorname{Tr} \mathbf{L}^p \tag{3.7}$$

are conserved quantities for solutions of (3.6). These quantities are nontrivial for p even: $2, 4, \cdots, N$.

In particular,

$$\mathbf{L}^2 = AA_+\mathbf{T}^2 + \left(A^2 + A_-^2\right)\mathbf{T} + A_-A_{--}\mathbf{T}^{-2}$$

and

$$\mathbf{L}^4 = AA_+A_{++}A_{+++}\mathbf{T}^4 + AA_+\left(A_-^2 + A^2 + A_+^2 + A_{++}^2\right)\mathbf{T}^2$$
$$+\left[A^2A_+^2 + A_-^2A_{--}^2 + \left(A^2 + A_-^2\right)^2\right]\mathbf{T} + \cdots.$$

Thus, by (3.7), the quantities

$$\operatorname{Tr} \mathbf{L}^2 = \sum A_k^2 + A_{k-1}^2 = 2\sum A_k^2 \tag{3.7$'$}$$

and

$$\operatorname{Tr} \mathbf{L}^4 = \sum 2A_k^2A_{k+1}^2 + \left(A_k^2 + A_{k-1}^2\right)^2 \tag{3.7$''$}$$

are conserved, etc.

Multiply (3.6)$'$ by $2A_k$; abbreviating

$$A_k^2 = U_k, \tag{3.8}$$

we obtain

$$\frac{d}{dt}U_k + 2U_k(U_{k+1} - U_{k-1}) = 0. \tag{3.8$'$}$$

This equation is, except for a numerical coefficient, the same as our equation (8). The correct numerical factor can be obtained by rescaling t or U_k.

Conversely, every positive solution $\{U_k\}$ can be reduced via a rescaling and the transformation (3.8) to a positive solution $\{A_k\}$ of (3.6)$'$.

It is easy to show that every solution of (8) whose initial values are positive remains positive for all t:

We rewrite (8) in conservation form,

$$\frac{d}{dt}U_k + \frac{U_kU_{k+1} - U_{k-1}U_k}{2\Delta} = 0; \tag{3.9}$$

we obtain, summing from $k = 1$ to N, and using the periodicity of U_k, that

$$\frac{d}{dt}\sum U_k = 0.$$

Hence

$$\sum U_k \tag{3.10}$$

is a conserved quantity. In view of (3.8), this is the same as (3.7)′.

If we divide (3.9) by U_k, we obtain another conservation form,

$$\frac{d}{dt}\log U_k + \frac{U_{k+1} - U_{k-1}}{2\Delta} = 0. \tag{3.11}$$

Summing from $k = 1$ to N, and using the periodicity of U_k, we deduce that

$$\frac{d}{dt}\sum \log U_k = 0;$$

hence,

$$\prod U_k \tag{3.12}$$

is a conserved quantity.

The positivity of $U_k(t)$ for all t can be deduced from these two conservation laws by an argument identical to the one used in Section 2 to show the positivity of $A_k(t)$ and $B_k(t)$ for all t.

The conservation of (3.10) and the positivity of $U_k(t)$ imply that all the $U_k(t)$ are bounded from above by M,

$$M = \sum U_k(0). \tag{3.13}$$

The uniform boundedness of $U_k(t)$ guarantees that every solution of (3.9) which has positive data exists for all time.

We turn now to the question of convergence of the approximate solutions. We use the notation introduced in Section 2:

$$U_\Delta(x, t) = U_k(t) \quad \text{for} \quad (k - \tfrac{1}{2}\Delta) < x < (k + \tfrac{1}{2})\Delta. \tag{3.14}$$

We denote the shift operator by $\mathbf{T}_\Delta$:

$$\mathbf{T}_\Delta U(x) = U(x + \Delta). \tag{3.15}$$

Using (3.10) and the above notation we conclude that

$$\int U_\Delta\, dx = \Delta \sum U_k \tag{3.16}$$

is a conserved quantity.

Using (3.8), the conservation of (3.7)″ can be restated in the above notation as the conservation of

$$\Delta \sum U_k U_{k+1} + \tfrac{1}{2}(U_k + U_{k-1})^2 = \int \left[U_\Delta \mathbf{T}_\Delta U_\Delta + \tfrac{1}{2}(U_\Delta + \mathbf{T}_\Delta U_\Delta)^2 \right] dx. \tag{3.17}$$

Since U_Δ is positive, it follows from the above that, for all t,

$$\int U_\Delta^2(x,t)\, dx \leqq 3 \int U_\Delta^2(x,0)\, dx. \tag{3.18}$$

Entirely similarly, we can deduce from the conservation of $\operatorname{Tr} \mathbf{L}^8$ that, for all t,

$$\int U_\Delta^4(x,t)\, dx \leqq \text{const.} \int U_\Delta^4(x,0)\, dx. \tag{3.18$'$}$$

LEMMA 3.1. *Let U_Δ denote the solution of* (3.9) *for given, smooth initial data. Then every sequence $\{\Delta\}$ tending to zero contains a subsequence such that U_Δ and $U_\Delta \mathbf{T}_\Delta U_\Delta$ both converge weakly with respect to the Hilbert space norm*

$$\int_0^T \int V^2(x,t)\, dx\, dt = \|V\|_T^2 \tag{3.19}$$

for any T.

Proof: For fixed T this is a standard consequence of the weak sequential compactness of bounded sets in Hilbert space, combined with inequalities (3.18) and (3.18)′. To get simultaneous weak convergence for all T we choose a denumerable sequence $T_n \to \infty$ and apply a diagonal process.

THEOREM 3.2. *Let U_Δ denote the solution of* (3.9) *for given, smooth initial data. Let T denote a time exceeding the breakdown time for the given initial data; then no subsequence of U_Δ can converge strongly in the norm* (3.19).

Proof: Suppose on the contrary that

$$s\text{-}\lim U_\Delta = \overline{U}. \tag{3.20}$$

From this it follows easily that

$$L^1\text{-}\lim U_\Delta \mathbf{T}_\Delta U_\Delta = \overline{U}^2. \tag{3.21}$$

Using (3.18)′ we further conclude from (3.20) that

$$L^1\text{-}\lim U_\Delta (\mathbf{T}_\Delta U_\Delta)^2 = \overline{U}^3, \tag{3.22}$$

and

$$L^1\text{-}\lim U_\Delta (\mathbf{T}_\Delta U_\Delta)(\mathbf{T}_\Delta^2 U_\Delta) = \overline{U}^3. \tag{3.22$'$}$$

We write now the conservation law (3.9) in the notation (3.14), (3.15) as

$$\frac{d}{dt} U_\Delta + \frac{1}{2\Delta}\left[U_\Delta \mathbf{T}_\Delta U_\Delta - \left(\mathbf{T}_\Delta^{-1} U_\Delta\right) U_\Delta\right] = 0.$$

We take the limit of this equation in the sense of distributions; we get, using (3.20) and (3.21), that

$$\overline{U}_t + \left(\tfrac{1}{2}\overline{U}^2\right)_x = 0. \tag{3.23}$$

Next we derive another conservation law from (3.9), corresponding to the conserved quantity (3.17). A brief calculation shows that every solution of (3.9) satisfies

$$\begin{aligned}
&\frac{d}{dt}\left[U_k^2 + U_{k+1}U_k + U_k U_{k-1}\right] \\
&\quad + \frac{1}{2\Delta}\left[U_{k+2}U_{k+1}U_k + U_{k+1}^2 U_k\right. \\
&\qquad\quad \left. + U_{k+1}U_k^2 - U_k U_{k-1} U_{k-2} - U_k^2 U_{k-1} - U_k U_{k-1}^2\right] = 0.
\end{aligned}$$

In the notation of (3.14), (3.15) this can be written as

$$\begin{aligned}
&\frac{d}{dt}\left[U_\Delta^2 + U_\Delta \mathbf{T}_\Delta U_\Delta + U_\Delta \mathbf{T}_\Delta^{-1} U_\Delta\right] + \frac{1}{2\Delta}\left[(\mathbf{T}_\Delta^2 U_\Delta)(\mathbf{T}_\Delta U_\Delta) U_\Delta - U_\Delta\left(\mathbf{T}_\Delta^{-1} U_\Delta\right)\mathbf{T}_\Delta^2 U_\Delta\right] \\
&\quad + \frac{1}{2\Delta}\left[(\mathbf{T}_\Delta U_\Delta)^2 U_\Delta - U_\Delta^2 \mathbf{T}_\Delta U_\Delta\right] + \frac{1}{2\Delta}\left[(\mathbf{T}_\Delta U_\Delta) U_\Delta^2 - U_\Delta (\mathbf{T}_\Delta U_\Delta)^2\right] = 0.
\end{aligned}$$

We take now the limit in the sense of distributions of the above equation. Using (3.21), (3.22) and (3.22)′ we get

$$(3\overline{U}^2)_t + (2\overline{U}^3)_x = 0. \tag{3.24}$$

Equations (3.23) and (3.24) are equivalent as differential equations, i.e., they have the same smooth solutions; but they are incompatible for discontinuous solutions. For T exceeding breakdown time, $\overline{U}$ is a discontinuous solution of (3.23); therefore it cannot satisfy (3.24) in the distribution sense. This shows that (3.20) is untenable.

We remark that when U_Δ tends weakly but not strongly to $\overline{U}$, $U_\Delta \mathbf{T}_\Delta U_\Delta$ could, unlike U_Δ^2, very well tend weakly to $\overline{U}^2$. If that were the case, the weak limit $\overline{U}$ would satisfy (3.23) in the distribution sense. We believe that the weak limit of $U_\Delta \mathbf{T}_\Delta U_\Delta$ is in fact different from $\overline{U}^2$; the numerical calculations confirm this.

4. Numerical Experiments

We describe some numerical experiments with the difference scheme (8). The initial data $u_0(x)$ is taken to be the positive periodic function $u_0(x) = \cos(x) + \frac{3}{2}$. We solve the ordinary differential equations (8) using the four-stage Runge-Kutta method, (see [1], p. 346). The Runge-Kutta method, with fixed CFL ratio, is dissipative of high frequencies; we have to take quite a small time step to get full time accuracy.

For this problem a shock first forms at time $t = 1$. Table 1 illustrates the second-order accuracy of the scheme at time $t = \frac{2}{10}\pi \simeq 0.628$ before the breakdown time. Similar results hold in the L^2 and L^∞ norms. This agrees with the conclusion of Theorem 1.1, inequality (1.9).

Figure 1 illustrates the oscillations that have appeared by time $t = 0.8\pi \simeq 2.51$. These do not seem to be of the type constructed in Section 2. Each frame contains a plot of the "raw" solution U and a smoothed solution $V = \mathbf{S}_\Delta^k U$, where $\mathbf{S}_\Delta = \frac{1}{4}\mathbf{T}_\Delta^{-1} + \frac{1}{2}\mathbf{I} + \frac{1}{4}\mathbf{T}_\Delta$ in the notation of Section 3.

In both figures, k is increased with N. In view of the central limit theorem, the effective number of grid points over which smoothing is performed is proportional to $\sqrt{k}$.

Figure 3 contains plots of the discrete indefinite integral of the oscillatory solution. This is motivated by the work of Venakides [12]. Suppose that $u^\varepsilon(x, t)$ satisfies KdV: $u_t^\varepsilon + u^\varepsilon u_x^\varepsilon = \varepsilon u_{xxx}^\varepsilon$. Venakides proved that $u^\varepsilon \rightharpoonup u$ weakly (for some function $u(x, t)$) by showing that $u^\varepsilon(x, t) = (\partial/\partial x)v^\varepsilon(x, t)$, where $v^\varepsilon(x, t) \to v(x, t)$ strongly and $u = (\partial/\partial x)v$. This was an improvement over Lax

Table I. e_N is the l_1 error using N points at time $t = 2\pi/10$.

N	e_N	e_{2N}/e_N
10	2.95×10^{-2}	5.47
20	5.39×10^{-3}	3.66
40	1.47×10^{-3}	3.90
80	3.78×10^{-4}	4.03
160	9.36×10^{-5}	4.005
320	2.34×10^{-5}	

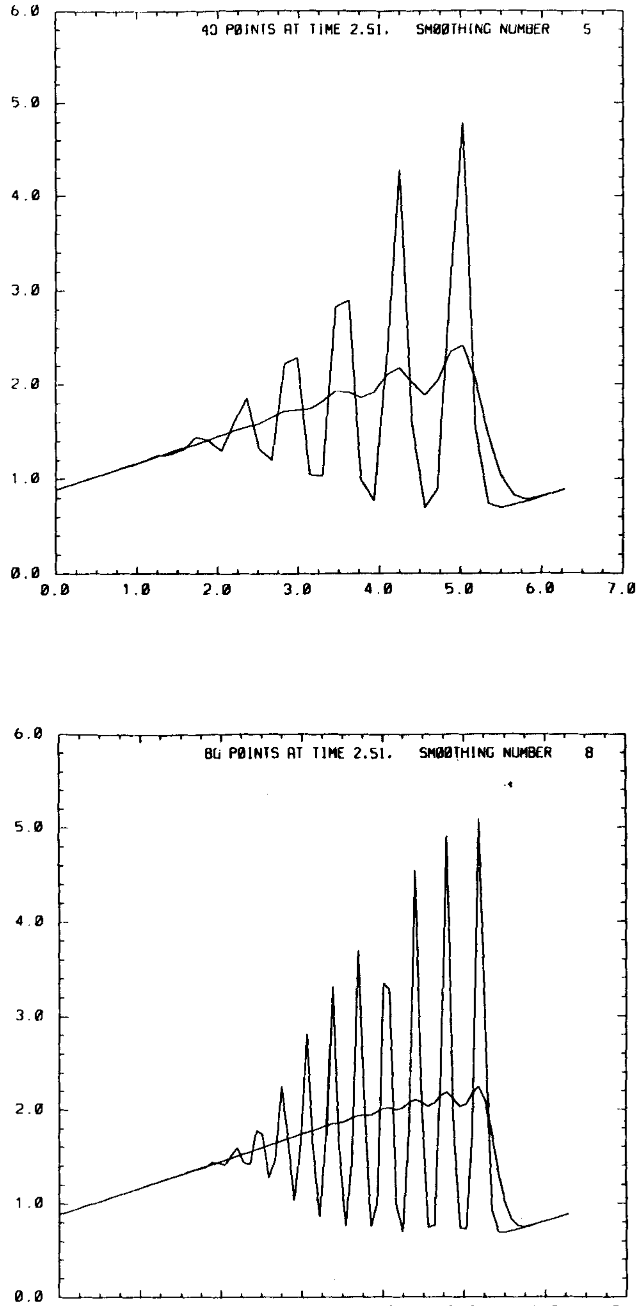
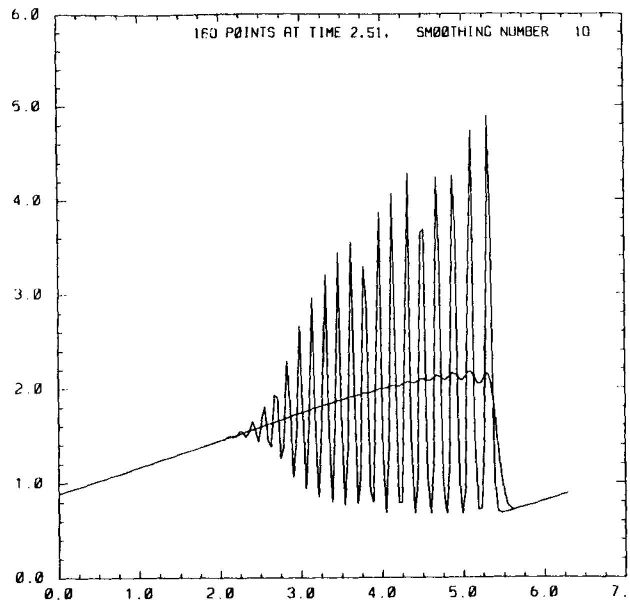


Figure 1. Oscillatory solutions after the breakdown time. The smoother curve is S^k operating on the raw solution where k is the smoothing number. All curves are piecewise linear interpolants of the N discrete values.



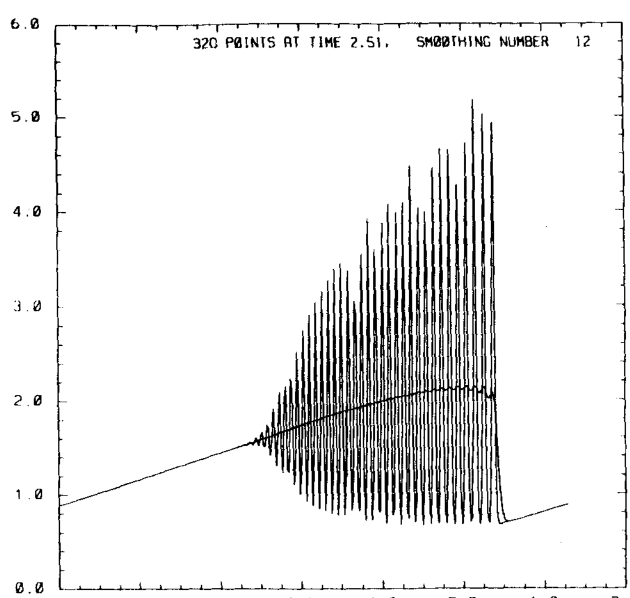
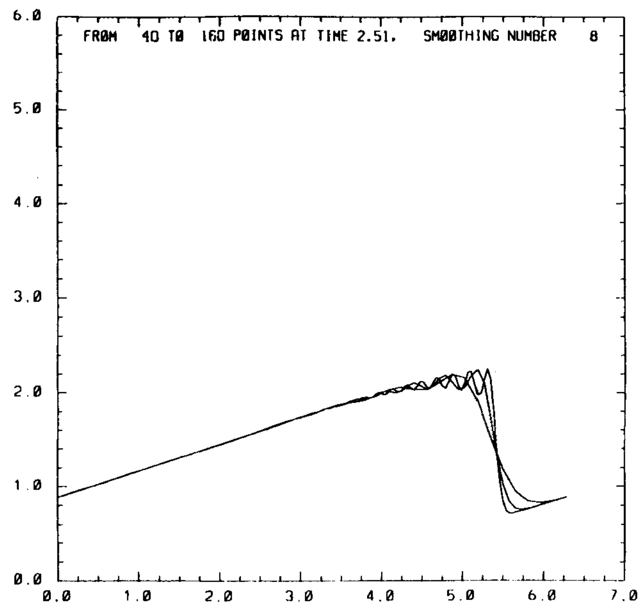


Figure 1. Continued



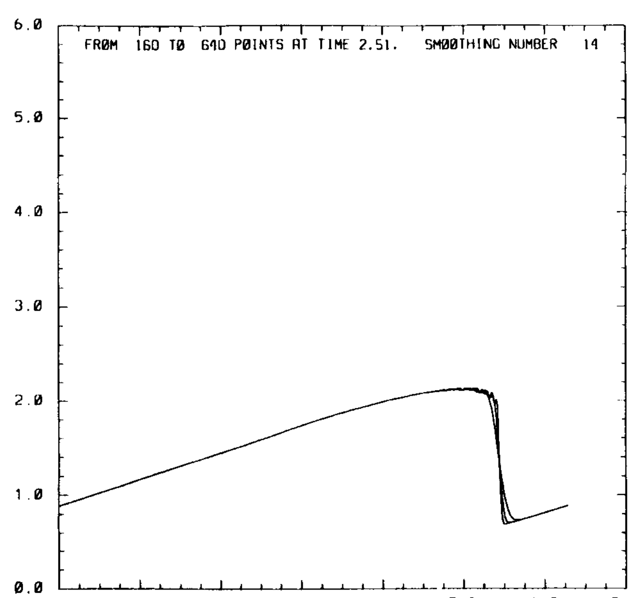


Figure 2. Superposition of several smoothed curves. The steeper and more oscillatory curves have larger N.

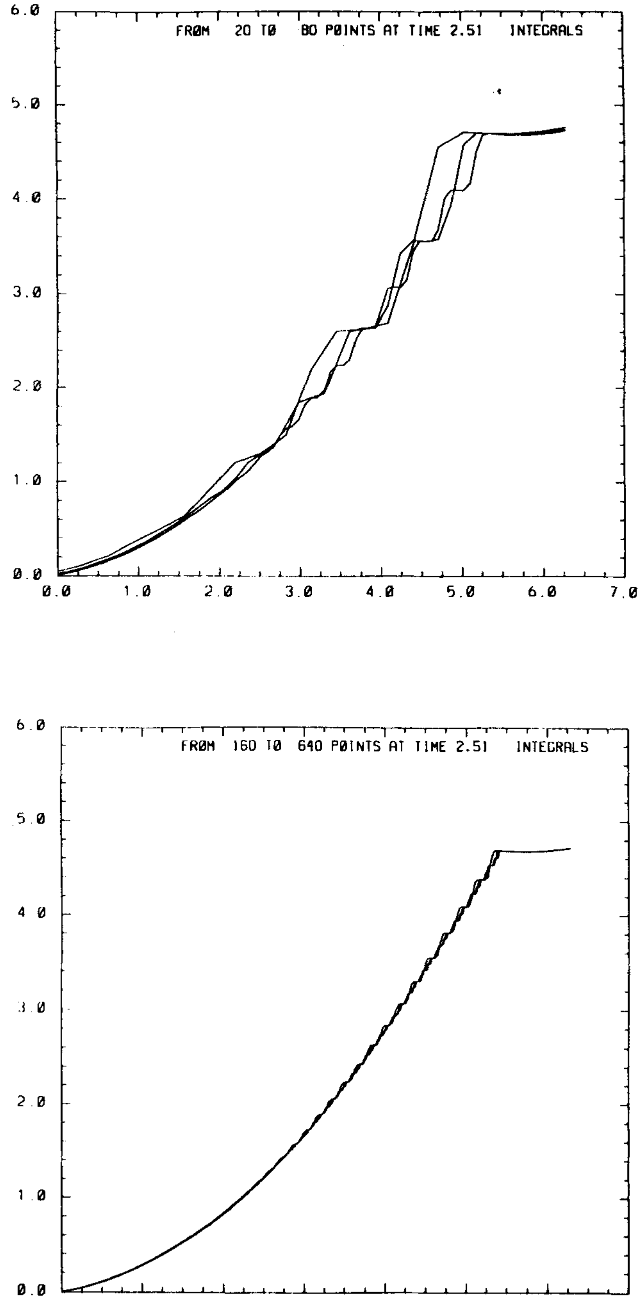


Figure 3. Indefinite integrals of the oscillatory solutions illustrating weak convergence (see text). The curves are piecewise linear interpolants of the data points. Curves with more corners correspond to larger N.

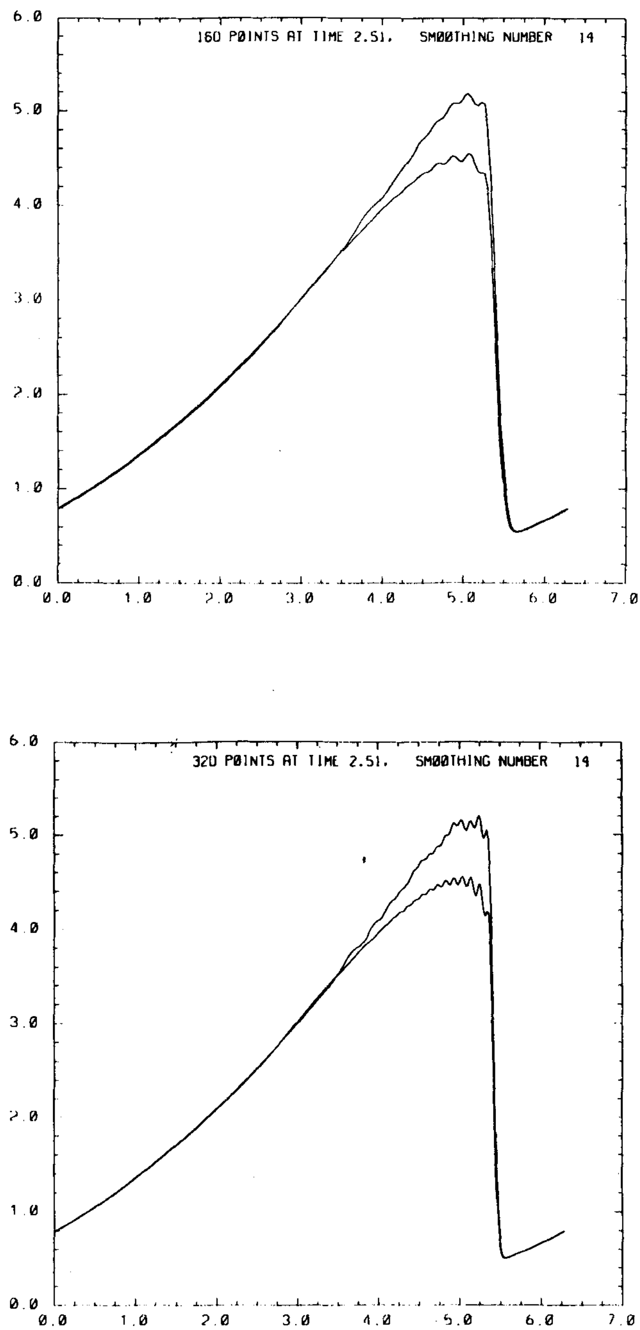


Figure 4. Weak limits of oscillatory solutions are not weak solutions of Burgers' equation. Plots of smoothings of $U_\Delta T_\Delta U_\Delta$ and of the square of the smoothing of U_Δ for the same N and t. The two are clearly different in the oscillatory region.

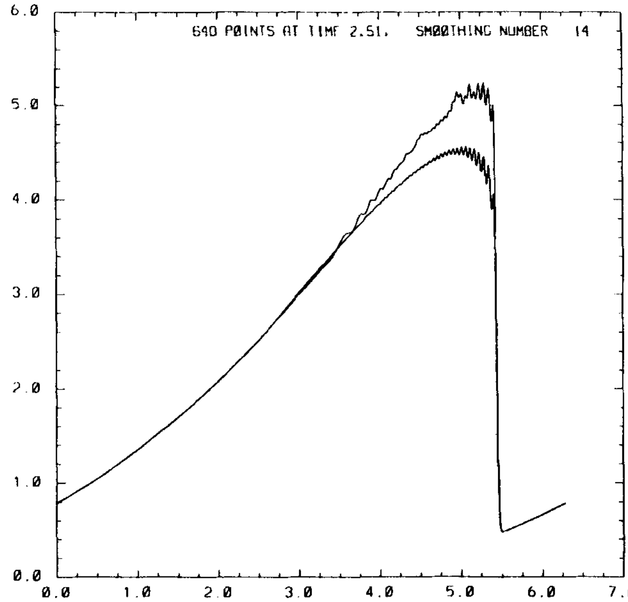


Figure 4. Continued

and Levermore [4] who needed to integrate twice instead of once. This motivates us to compute a discrete imitation of the indefinite integral, namely

$$V_k(t) = \Delta \sum_{j=1}^{k} \left(U_j(t) - \tfrac{3}{4}\right).$$

The $\frac{3}{4}$ is to make the plots more readable and obviously has no effect on weak or strong convergence. Figure 3 clearly demonstrates the convergence of $V_k(t)$ to a function $v(x_k, t)$ as $\Delta \to 0$, in precise analogy with Venakides' result for KdV.

Figure 4 provides evidence that the weak limit of the U_Δ is not a distributional solution of Burgers' equation, $u_t + (\frac{1}{2}u^2)_x = 0$. As is explained in Section 2, if $U_\Delta \rightharpoonup \overline{U}$ and $\overline{U}$ satisfies Burgers' equation the weak limit of $U_\Delta \mathbf{T}_\Delta U_\Delta$ is equal to $\overline{U}^2$. However, the plots of $\mathbf{S}_\Delta^k(U_\Delta \mathbf{T}_\Delta U_\Delta)$ and of $(\mathbf{S}_\Delta^k U_\Delta)^2$, for large k, clearly show that the two are not equal in the oscillatory region.

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