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Hyperbolic Systems of Conservation Laws II*

P. D. LAX

1. Introduction

A conservation law is an equation in divergence form, i.e.

$$u_t + \sum_{j=1}^{3} \frac{\partial}{\partial x_j} f_j = 0.$$

It expresses the fact that the quantity of u contained in any domain G of x-space changes at a rate equal to the flux of the vector-field (f_1, f_2, f_3) into G:

$$\frac{d}{dt}\iiint_G u\, dx = \iint_{BG} f \cdot n\, dS.$$

Many physical laws are conservation laws; the quantities u and f depend on the variables describing the state of a physical system, and on their derivatives. In theories which ignore mechanisms of dissipation such as viscous stresses, heat conduction, ohmic loss, the conservation laws are of first order, i.e., the quantities u and f are functions of the state variables but not of their derivatives. In this paper we shall consider such systems of first order conservation laws in one space variable. The components of u shall be chosen as state variables so that the system is in the form

$$u_t + f_x = 0, \tag{1.1}$$

where u is a vector of n components and $f = f(u)$ a vector valued function of u.

When the differentiation in (1.1) is carried out, a quasilinear system of first order results:

$$u_t + A(u)u_x = 0, \qquad A = \operatorname{grad} f. \tag{1.1'}$$

The system of conservation laws (1.1) is called *hyperbolic* if as a quasilinear system it is hyperbolic, i.e., if the matrix $A = \operatorname{grad} f$ in (1.1′) has real and distinct eigenvalues for all values of the argument u.

The initial value problem consists of determining solutions u of (1.1)

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from their initial state $u(x, 0) = \phi$ for all future time. In this paper we shall try to develop an adequate theory of the initial value problem for systems of conservation laws.

At any time t_0, the function $u(x, t_0)$ describing the state of the system is required to possess certain properties. Denote by $\{\phi\}$ the set of these "permissible" states.[1] We wish to assign to each state ϕ of this set a solution $u(x, t) = S(t)\phi$ for all time $t \geqq 0$ with the following properties:

(i) $u(x, t) = S(t)\phi$ is a solution of the system of conservation laws (1.1).

(ii) The operators $S(t)$ map the set of permissible states intot iself.

(iii) The operators $S(t)$ form a one-parameter semigroup,

$$S(t_1+t_2) = S(t_1)S(t_2), \qquad t_1, t_2 \geqq 0, \qquad S(0) = I.$$

(iv) For each t, the operator $S(t)$ is continuous in some topology.

Solutions in the classical sense of the quasilinear hyperbolic system (1.1′) develop singularities (discontinuities) after a finite time, no matter how smooth their initial data, and cannot be continued as regular solutions. They can be continued, however, as solutions in a generalized sense; this kind of generalization is dictated by the integral version of a conservation law which states that the vector field

$$(u, f_1, f_2, f_3)$$

is divergence-free in space-time. Generalized (or weak) solutions are functions u for which the above vector field is divergence-free in a generalized sense.

A precise definition of weak solution is given in Section 2. It turns out (see e.g. [16] for examples) that weak solutions are not determined uniquely by their initial values, therefore, an additional principle is needed for selecting a relevant subclass. This matter is discussed in Sections 5 and 7.

In Section 2 we shall solve the initial value problem for a generalized solution of a *single* conservation law for a single unknown function by an explicit formula. This formula has been derived in [11] by Hopf for a quadratic conservation law and by the author [16] for the general case.

Section 3 contains a brief discussion of the "viscosity method", i.e. of obtaining solutions of systems of conservation laws as limits of solutions of parabolic equations as the coefficient of the dissipative term goes to zero. The results of Hopf [11], Cole [4], Olejnik [23, 24] and Ladyzhenskaya [13] are briefly described.

In Section 4 we prove that a difference scheme proposed by the author in [14] and [16] for solving the initial value problem converges for a particular

[1] These need not form a linear space.

conservation law. This result was announced in [16]. Recently Vvedenskaya [28] succeeded in proving the convergence of this scheme and of related ones for an arbitrary single conservation law.

In Section 5 we state the uniqueness theorems of Germain and Bader [9] and of Olejnik [25], and give a *nonuniqueness* theorem for the *backward* initial value problem.

In Section 6 we discuss the asymptotic behavior of solutions as t tends to infinity. Our results are related to those of Friedrichs [6], Lighthill [19], Hopf [11], Whitham [29] and Keller [12].

Sections 7—9 are about systems of conservation laws. In Section 7 a generalization of the entropy condition is derived and used in developing a theory of shocks. Section 8 contains a theory of simple waves for quasilinear systems. These theories are used in Section 9 to solve the Riemann initial value problem, i.e. the one for which the initial function is piecewise constant.

We show in particular that the equations of compressible flow (time dependent and steady supersonic) are examples of hyperbolic systems of conservation laws. Our studies in Sections 7—9 show that many important properties of such flows are shared by solutions of general systems. In particular, the well-known rule that across a weak shock entropy is constant up to terms of third order in shock strength appears as a special case of a general law. Friedrichs has observed in [7] that the equations of hydromagnetics with infinite conductivity (Lundquist model) furnish another example of systems of conservation laws.

Most of the results of this paper were presented at the 1954 Summer Symposium on Partial Differential Equations in Kansas and are contained in the Proceedings of that Symposium [17].

SINGLE CONSERVATION LAWS

2. An Explicit Formula

DEFINITION. The function $u(x, t)$ is a weak solution of the system of conservation laws (1.1) with initial values ϕ if u and $f(u)$ are integrable functions over every bounded set of the half-plane $t \geqq 0$ and the integral relation

$$\int_0^\infty \int_{-\infty}^\infty \{w_t u + w_x f(u)\} \, dx \, dt + \int_{-\infty}^\infty w(x, 0) \, \phi(x) \, dx = 0 \tag{2.1}$$

is satisfied for all smooth test vectors w which vanish for $|x| + t$ large enough.

This definition expresses the divergence-free character of the vector field (u, f) in the weak sense.

Smooth solutions of (1.1′) are weak solutions and, conversely, if u is a weak solution with continuous first derivatives, then it satisfies the differential equation (1.1′).

LEMMA 2.1. *If u is a piecewise continuous weak solution of* (1.1), then *across the line of discontinuity the jump relation*

$$s[u] = [f] \tag{2.2}$$

holds; here [ ] denotes the jump across the line of discontinuity, and s is the *velocity of propagation*[2] *of the discontinuity.*

Relation (2.2) is a generalization of the Rankine-Hugoniot relation; it expresses the fact that the component of the vector field (u, f) in the direction normal to the line of discontinuity is continuous across the line of discontinuity. It is easy to show that piecewise continuous generalized divergence-free vector fields have this property.

In this section we shall discuss single conservation laws; u and f denote scalar quantities. The conservation law

$$u_t+f_x = 0 \tag{2.3}$$

can be written as a quasilinear equation

$$u_t+a(u)u_x = 0, \tag{2.4}$$

where $a(u)$ denotes $f'(u)$. We shall require that (2.4) be *genuinely nonlinear*, i.e., that a, the coefficient of u_x, should vary with u in the sense that $a'(u)$ is not zero. Since $a = f'$, this means that $f'' \neq 0$, i.e., *the function $f(u)$ is either strictly convex or strictly concave.*

In [15], the author has defined the notion of genuine nonlinearity for quasilinear systems. We shall return to it in Section 8 of this paper.

Given a convex (concave) function $f(u)$ defined for all u we define the conjugate* function $g(s)$ by the relation

$$g(s) = \operatorname*{Max}_{u}\ (\mathrm{Min})\ \{us-f(u)\}.$$

Denote by $u = b(s)$ that value of u where the above maximum (minimum) occurs. It is easy to show that b is the inverse of $a = f'$ and also that b is the derivative of g.

$$b(a(u)) = u, \tag{2.5}$$

$$\frac{d}{ds}g(s) = b(s). \tag{2.6}$$

It is easy to show that $b(s)$ and $g(s)$ are uniquely defined on the range of $a(u)$, that $g(s)$ is convex (concave) there, and that $g(s)$ tends to infinity as s approaches the endpoints of the domain of definition of g.

[2]I.e., if the line of discontinuity is given by $t = t(\sigma)$, $x = x(\sigma)$, then $s = \dot{x}/\dot{t}$.

**Footnote added in proof:* The theory of conjugate functions (or Lengendre transformation) for functions of one variable has been developed by Mandelbrojt [30]; for functions of a finite number of variables the theory is due to Fenchel [31], and for functions over an infinite-dimensional space, to Friedrichs [32] and Courant-Hilbert (Vol. I, Chapter IV) and, in an abstract setting, to Hörmander [33].

With the aid of these auxiliary functions we shall assign to any bounded measurable initial function $\phi(x)$ a function $u(x, t)$ which will turn out to be a weak solution of our conservation law (2.3) with initial value $\phi(x)$.

First we define $\Phi(y)$ as the integral of ϕ:

$$\Phi(y) = \int_0^y \phi(\eta)\, d\eta.$$

Consider the expression

$$\Phi(y) + tg\left(\frac{x-y}{t}\right) \tag{2.7}$$

which, for fixed x and t, is a continuous function of y. One can easily show that it tends to plus (minus) infinity as y approaches the endpoints of its domain of definition. Therefore, it assumes a finite minimum (maximum) in the interior.

Following Hopf [11] we prove two lemmas:

LEMMA 2.2. *Let y_1 and y_2 be values where the function* (2.7) *assumes its minimum (maximum) for x_1, t and x_2, t, respectively. If $x_1 < x_2$ then $y_1 \leqq y_2$.*

Proof: Take the case when f—and thereby g—is convex. By definition of y_1 and y_2 as minimum points, we have

$$\Phi(y_1) + tg\left(\frac{x_1-y_1}{t}\right) \leqq \Phi(y_2) + tg\left(\frac{x_1-y_2}{t}\right)$$

and

$$\Phi(y_2) + tg\left(\frac{x_2-y_2}{t}\right) \leqq \Phi(y_1) + tg\left(\frac{x_2-y_1}{t}\right).$$

Adding the two inequalities we get

$$g\left(\frac{x_1-y_1}{t}\right) + g\left(\frac{x_2-y_2}{t}\right) \leqq g\left(\frac{x_1-y_2}{t}\right) + g\left(\frac{x_2-y_1}{t}\right).$$

Since g is a convex function, it follows from this inequality that $y_1 \leqq y_2$.

For fixed x, t, denote by $y^+(x, t)$ and $y^-(x, t)$ the largest and smallest value of y for which the function (2.7) assumes its minimum. By definition $y^-(x) \leqq y^+(x)$; on the other hand, we deduce from Lemma 2.2 that $y^+(x_1) \leqq y^-(x_2)$ for $x_1 < x_2$. From these facts it follows that y^- and y^+ can differ only at points of discontinuity. Since y^- and y^+ are nondecreasing functions of x, this can happen only at a denumerable number of points. So we have

LEMMA 2.3. *For given t, with the exception of a denumerable set of values of x, the function* (2.7) *assumes its minimum at a single point which we denote by $y_0(x, t)$.*

With the aid of y_0 we define

$$u(x, t) = b\left(\frac{x-y_0}{t}\right) \tag{2.8}$$

and assert

THEOREM 2.1. *The function u defined by* (2.8) *is a weak solution of the conservation law* (2.3) *with initial value ϕ. This solution of the initial value problem satisfies the properties* (i)—(iv) *stated in Section* 1.

Proof: We show first that if ϕ is a smooth function, then the function given by (2.8) coincides with the smooth solution of the initial value problem as long as the latter exists. For ϕ continuous the function (2.7) has a continuous first derivative; hence at the point where the minimum (maximum) occurs, the first derivative must vanish:

$$\phi(y_0) - g'\left(\frac{x-y_0}{t}\right) = 0$$

which by (2.6) and (2.7) is the same as

$$\phi(y_0) = b\left(\frac{x-y_0}{t}\right) = u(x, t).$$

Using (2.5) we get

$$\frac{x-y_0}{t} = a(\phi(y_0)).$$

These equations express the fact that the function $u(x, t)$ is constant along straight lines, and that the slope of the line issuing from the point $(y_0, 0)$ is $a(\phi(y_0))$. This is precisely what the differential equation

$$u_t + a(u)u_x = 0$$

asserts.

A systematic derivation of formula (2.8) for differentiable solutions of such a quasilinear equation can be given with the aid of a method of Bellman [1] for expressing solutions of nonlinear equations as suprema of solutions of linear ones.

Next we show that the functions given by formula (2.8) are weak solutions of (2.3).

We can write u as $\lim_{N\to\infty} u_N$, where

$$u_N(x, t) = \frac{\displaystyle\int_{-\infty}^{\infty} b\left(\frac{x-y}{t}\right) \exp\left\{-N\left[\Phi(y) + tg\left(\frac{x-y}{t}\right)\right]\right\} dy}{\displaystyle\int_{-\infty}^{\infty} \exp\left\{-N\left[\Phi(y) + tg\left(\frac{x-y}{t}\right)\right]\right\} dy}. \tag{2.9}$$

Denote the denominator in (2.9) by

$$V_N = \int_{-\infty}^{\infty} \exp\left\{-N\left[\Phi(y) + tg\left(\frac{x-y}{t}\right)\right]\right\} dy. \tag{2.10}$$

In view of relation (2.6) we can write (2.9) as

$$u_N = \frac{1}{N}\frac{(V_N)_x}{V_N} = \left(\frac{1}{N}\log V_N\right)_x = (U_N)_x . \tag{2.9'}$$

Likewise, we can write $f(u)$ as $\lim f_N$, where

$$f_N(x,t) = \frac{\int_{-\infty}^{\infty} f\left[b\left(\frac{x-y}{t}\right)\right]\exp\left\{-N\left[\Phi(y)+tg\left(\frac{x-y}{t}\right)\right]\right\}dy}{V_N} . \tag{2.11}$$

Using the definition of g it is easy to show that

$$f(b(s)) = sb(s)-g(s).$$

Substituting this into (2.11) we get

$$f_N = \frac{1}{N}\frac{(-V_N)_t}{V_N} = -(U_N)_t . \tag{2.11'}$$

From (2.9'), (2.11') we conclude that the vector fields (u_N, f_N) are divergence-free. Therefore, their limit $(u, f(u))$ is divergence-free in the generalized sense. This implies that our $u(x, t)$ is indeed a weak solution. From the relation

$$U_N = \frac{1}{N}\log V_N = \log (V_N)^{1/N}$$

and formula (2.10) for V_N, we can determine the limit of U_N.

$$U(x,t) = \lim_{N\to\infty} U_N = \min_y \left\{\Phi(y)+tg\left(\frac{x-y}{t}\right)\right\} . \tag{2.12}$$

2,12
Integrating ~~(4.6')~~ with respect to x and letting N tend to ∞, we obtain in the limit

$$U(x,t) = \int_{-\infty}^{x} u(\xi, t)\,d\xi.$$

Thus we have derived this

COROLLARY TO THEOREM 2.1. *The value of the minimum (maximum) of* (2.7) *is equal to the x-integral of the solution* $u(x, t)$ *defined by formula* (2.8).

We show now that as t tends to zero, $u(x, t)$ tends to ϕ in the weak sense.

Denote $\max_x (y_0(x,t)-x)$ by $\delta(t)$. Clearly

$$U(x,t) = \min_{x-\delta(t)<y<x+\delta(t)} \left\{\Phi(y)+tg\left(\frac{x-y}{t}\right)\right\} . \tag{2.12'}$$

Since $g(s)$ is a strictly convex function which tends to ∞ as $|s|$ tends to ∞, it follows that $g(s)/|s|$ also tends to infinity with increasing $|s|$. Since ϕ

was assumed bounded, $\Phi(y)$ is $O(y)$. From these facts, and the form of the function (2.7) it follows easily that $\delta(t)$ tends to zero with t.

Let $\eta(\delta)$ denote the oscillation of $\Phi(y)$ over an interval of length δ; since $\Phi(y)$ is uniformly continuous, $\eta(\delta)$ tends to zero with δ. Denote by m a lower bound for g; we have, from (2.12′), the following lower bound for U:

$$U(x, t) \geqq \Phi(x) - \eta(\delta) + mt.$$

On the other hand, according to the corollary to Theorem 2.1, the value of the function (2.7) at any point, in particular at $y = x$, is an upper bound for U:

$$U(x, t) \leqq \Phi(x) + tg(0).$$

These estimates show that $U(x, t)$ tends to $\Phi(x)$ uniformly as t tends to zero, i.e., that $u(x, t)$ tends to $\phi(x)$ *in the weak sense.*

Next we show that the class of solutions defined in Theorem 2.1 form a semigroup in t.

Proof: Let t_1, t_2 be two positive quantities, x any value. According to the corollary,

$$U(y, t_1) = \min_z \left[U(z, 0) + t_1 g\left(\frac{y-z}{t_1}\right)\right] \tag{2.13}$$

and

$$U(x, t_1+t_2) = \min_z \left[u(z, 0) + (t_1+t_2)\, g\left(\frac{x-z}{t_1+t_2}\right)\right]. \tag{2.14}$$

To exhibit the semigroup property we have to show that $U(x, t_1+t_2)$ can also be obtained as

$$\min_y \left[U(y, t_1) + t_2 g\left(\frac{x-y}{t_2}\right)\right]. \tag{2.15}$$

Substitute (2.13) into (2.15); we get

$$\min_{y,z} \left[U(z, 0) + t_1 g\left(\frac{y-z}{t_1}\right) + t_2 g\left(\frac{x-y}{t_2}\right)\right]. \tag{2.15′}$$

The minimum with respect to the variables y and z can be taken in any order. Evaluating it with respect to y we first get, by differentiating with respect to y,

$$b\left(\frac{y-z}{t_1}\right) - b\left(\frac{x-y}{t_2}\right) = 0.$$

Since b is a monotonic function, we must have

$$\frac{y-z}{t_1} = \frac{x-y}{t_2} = \frac{x-z}{t_1+t_2}.$$

Using this relation to eliminate y from (2.15′) we see that (2.15) is equal to (2.14).

The initial function ϕ can be any function which has a finite integral Φ; in particular, ϕ could be a δ-function or a sum of δ-functions. Such initial configurations represent point sources. Of course, in contrast to the linear theory, we cannot build up more general solutions by superimposing point sources.

Finally we show that the solutions constructed depend continuously on the initial data. In fact, we shall show that the dependence of u on ϕ is *completely continuous*:

THEOREM 2.2. *Let ϕ_n be a sequence of functions which converges weakly to a limit ϕ. Let u_n be the solution assigned to the initial value ϕ_n by formula* (2.8) *and let u be the solution assigned to ϕ. Then the sequence u_n converges to u at all points of continuity of u.*

Proof: Let Φ_n and Φ denote as before the integrals of ϕ_n and ϕ. Let (x, t) be a point where the function (2.7) has a unique minimum y_0. Since ϕ_n converges weakly to ϕ, Φ_n converges uniformly to Φ. Therefore the function

$$\Phi_n(y) + tg\left(\frac{x-y}{t}\right) \tag{2.7$_n$}$$

achieves its minimum at a point y_n which tends to y_0 as n tends to infinity. The rest follows from the explicit formula (2.8).

A different proof of a slightly weaker kind of compactness was given in [16].

The transformation $v = a(u)$ transforms solutions u of the quasilinear equation (2.4) into solutions v of the quadratic equation $v_t+vv_x = 0$. But it is easy to see that it does *not* map *discontinuous* weak solutions of (2.3) into weak solutions of the quadratic conservation law $v_t+\frac{1}{2}(v^2)_x = 0$. Another striking example of this noninvariance of weak solutions under nonlinear transformation occurs in the theory of compressible flows: A discontinuous flow which conserves mass, momentum and energy does not, in contrast to smooth flows, conserve entropy.

An explicit formula similar to (2.8) can be given for solutions with values prescribed along an arbitrary curve in the x, t-plane.

3. The Viscosity Method

For a quadratic conservation law, $f(u) = \frac{1}{2}u^2$, the sequence of approximations u_n used in formula (2.9) of the last section are solutions of the parabolic equation[3]

[3]This equation was first investigated by J. M. Burgers, see e.g. [2].

$$u_t+\tfrac{1}{2}uu_x = \lambda u_{xx}\,, \qquad \lambda = \frac{1}{2N}\,.$$

For nonquadratic f it is no longer possible to find such an explicit solution of the parabolic equation

$$u_t+f(u)_x = \lambda u_{xx}\,.$$

However, Olejnik succeeded [23, 24, 25] in proving that solutions of this parabolic equation with given initial value $u(x, 0) = \phi(x)$ tend to the weak solution of $u_t+f_x = 0$ given in the last section. A simpler proof was found recently [13] by Ladyzhenskaya.

4. The Method of Finite Differences

In [14] and [16] I have proposed a finite difference method for constructing weak solutions, with prescribed initial values, of systems of conservation laws. In addition to presenting numerical evidence, I stated there that I was able to give a rigorous proof of convergence in one case. We present now the details of that proof.

THEOREM 4.1. *Consider the single conservation law* (1.1) *with*

$$f(u) = -\log(a+be^{-u}), \qquad a+b = 1. \tag{4.1}$$

Replace in (1.1) *the time derivative by a forward difference and the space derivative by a left difference*[4]:

$$u(x, t+\Delta) = u(x, t)-\{f(u(x, t))-f(u(x-\Delta, t))\}. \tag{4.2}$$

Let u_Δ be the solution of this difference equation with bounded and measurable initial value $u(x, 0) = \phi(x)$. Then

$$\lim_{\Delta\to 0} u_\Delta = u(x, t) \tag{4.3}$$

exists for fixed t for almost all x, where u is given by (2.8)[5].

Proof: Define

$$U(x, t) = \sum_{k=-\infty}^{0} u_\Delta(x-k\Delta, t);$$

here we assume that $\phi(x)$ — and thereby $u(x, t)$ — is zero for large negative x. Write $x-k\Delta$ for x in (4.2) and sum with respect to k. Taking $f(0) = 0$ — which is true for our choice (4.1) — we get

$$U(x, t+\Delta) = U(x, t)-f(U(x, t)-U(x-\Delta, t)). \tag{4.4}$$

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[4]Here $\Delta t = \Delta x = \Delta$. Our scheme differs slightly (in an inessential way) from the one discussed in [16].

[5]An essential feature of the scheme presented here is that the exact derivative f_x in the conservation law is replaced by an exact difference.

We seek a nonlinear transformation

$$U = G(V)$$

which linearizes (4.4). The equation for V is

$$G(V(x, t+\Delta)) = G(V(x, t)) - f[G(V(x, t)) - G(V(x-\Delta, t))] \tag{4.4'}$$

which is linear if the identity

$$G(V) - f[G(V) - G(W)] = G(aV + bW) \tag{4.5}$$

holds for all V and W with some constants a and b. It can be shown that the only function G which satisfies an identity of this kind is

$$U = G(V) = \log V, \tag{4.6}$$

with $f(u)$ as given by (4.1). Equation (4.4') becomes in this case

$$V(x, t+\Delta) = aV(x, t) + bV(x-\Delta, t)$$

which has the well-known solution

$$V(x, t) = \sum_{l=0}^{n} \binom{n}{l} a^{n-l} b^{l} V(x - l\Delta, 0). \tag{4.7}$$

Thus

$$\begin{aligned} u_\Delta(x, t) &= U(x, t) - U(x-\Delta, t) = \log \frac{V(x, t)}{V(x-\Delta, t)} \\ &= \log \frac{\sum_{l=0}^{n} \binom{n}{l} a^{n-l} b^{l} V(x - l\Delta, 0)}{\sum_{l=0}^{n} \binom{n}{l} a^{n-l} b^{l} V(x - (l+1)\Delta, 0)}. \end{aligned} \tag{4.8}$$

Writing $l-1$ for the summation variable in the numerator and abbreviating $x-(l+1)\Delta$ by y, we get

$$u_\Delta(x, t) = \log \frac{\sum_{l=-1}^{n} \frac{t-(x-y)}{x-y} \cdot \frac{b}{a} e^{\Theta/\Delta}}{\sum_{l=0}^{n} e^{\Theta/\Delta}}, \tag{4.8'}$$

where

$$\Theta = \Delta \log \left[\binom{n}{l} a^{n-l} b^{l}\right] + \Delta U(y, 0). \tag{4.9}$$

Here we have made use of relation (4.6)

$$V(y, 0) = e^{U(y,0)}.$$

As Δ tends to 0 (i.e., n tends to ∞), $\Delta U(y, 0)$ tends to $\Phi(y)$ in measure.

We can evaluate the binomial coefficient $\binom{n}{l}$ asymptotically by Sterling's formula obtaining the following expression for (3.12):

$$\Theta = \Phi(y) - t\left[\log\left(1 - \frac{x-y}{t}\right) - \frac{x-y}{t}\log\left(\frac{t-x+y}{x-y}\cdot\frac{b}{a}\right)\right] + \text{error}, \tag{4.9$'$}$$

where the error tends to zero for almost all x. The expression (4.9′) has a unique maximum y_0 in y for all but a denumerable number of x.

By a well-known argument we conclude then that the limit of u_Δ as given by formula (4.8′) is equal to

$$\log\left[\frac{b}{a}\left(\frac{t}{x-y_0} - 1\right)\right].$$

This is formula (2.8) of Section 2.

I arrived at formula (2.8) by noting the similarity of the explicit formula for solutions derived by Hopf for $f(u) = \frac{1}{2}u^2$ to the one derived by me in the present case, and generalizing it.

It does not seem possible to solve explicitly the difference equation (4.2) for any choice of $f(u)$ other than (4.1). Vvedenskaya has succeeded recently [28] in proving convergence for any f by another method.

5. Uniqueness and Irreversibility

Weak solutions of conservation laws are not determined uniquely by their initial values. Therefore, some additional principle is needed for preferring some—such as the ones selected in Section 2—to others. In [16] I conjectured that there is just one way of assigning a weak solution u to each ϕ of a permissible class so that the properties (i)–(iv) of Section 1 are satisfied. Even if this proposition were true, it would not yield a practical criterion for picking out a preferred weak solution from many with the same initial value. A different type of criterion is that the preferred weak solution be obtainable by a specific limiting process such as the viscosity method. In order to avoid carrying out the limiting process, it is desirable to give an intrinsic characterization of such weak solutions. It is easy to show that if u is a piecewise continuous weak solution whose discontinuities occur along smooth arcs and which is a limit of solutions of parabolic equations, then the jump $[u]$ from left to right across each discontinuity curve is negative for f convex, positive for f concave. Germain and Bader have shown in [9] that there exists at most one weak solution with prescribed initial value whose jumps satisfy this condition. Their proof, given for the quadratic conservation law, can be extended without alteration to any single conservation law. Another proof has been given by A. Douglis (unpublished).

Olejnik has proved the following uniqueness theorem which completely characterizes weak solutions that are limits of solutions of parabolic equations.

Among all weak solutions of the conservation law (2.3), *f convex* (*concave*), *there exists exactly one whose x-difference quotients are bounded from above* (*below*):

$$\frac{u(x_1, t)-u(x_2, t)}{x_1-x_2} \leqq K(t) \qquad (\geqq K(t)).$$

The upper (*lower*) *bound may tend to plus* (*minus*) *infinity as t tends to zero.*

Olejnik's brief and elegant proof is contained in [25].

It is easy to verify that the solutions given in Section 2 have this property.

The condition on the sign of jumps of discontinuous weak solutions derived in the present section will be generalized, on the basis of the theory of mixed initial and boundary value problems, to systems of conservation laws in Section 7.

Theorem 2.2 asserts that the future is a completely continuous function of the past. It follows, therefore, that the past is a discontinuous function of the future. More than that: there exist bounded, piecewise continuous states ψ such that $\psi(x) \neq u(x, t_0)$, $t_0 > 0$, for any u of the class of solutions discussed in Section 2. On the other hand, there exist solutions in this class which are equal for $t \geqq t_0$ but differ for $t < t_0$. Concerning these the following curious result holds:

THEOREM 5.1. *The set of states* $\{\phi\}$, *for which the corresponding solutions are equal at* $t = T$, *is convex.*

Proof: Let ϕ_1 and ϕ_2 be two elements of such a set; we wish to show that $s\phi_1+(1-s)\phi_2$, $0 \leqq s \leqq 1$, also belongs to this set. Since the solutions corresponding to ϕ_1 and ϕ_2 were assumed to be equal, it follows from (4.2) that the functions $y_0(x, T)$, corresponding to these two different initial states, are equal for almost all x, i.e., that the two functions

$$\Phi_1(y) + Tg\left(\frac{x-y}{T}\right) \tag{5.1}$$

and

$$\Phi_2(y) + Tg\left(\frac{x-y}{T}\right) \tag{5.2}$$

take on their minimum (maximum) at the same value y for almost all x. If two functions assume their minimum at the same point, so does any linear combination of them with positive weights s and $1-s$. This completes the proof of the theorem.

6. Asymptotic Behavior for Large t

E. Hopf, in [11], discusses the asymptotic behavior for large t of solutions of the special conservation law $u_t+(\frac{1}{2}u^2)_x = 0$. We shall discuss related properties of solutions of an arbitrary conservation law (see also the discussion in Lighthill [19] and the articles of Whitham quoted there).

DEFINITION 6.1. A function $\phi(x)$ has mean value M if

$$\lim_{L\to\infty} \frac{1}{L}\int_a^{a+L} \phi(x)\,dx = M$$

uniformly in a.

THEOREM 6.1. *Let $u(x, t)$ be a bounded weak solution of a system of conservation laws whose initial value has a mean value. Then $u(x, t)$ has the same mean value for all t.*

Proof: It is easy to show that, if u is a weak solution,

$$\int u(x, t)\, w(x)\, dx$$

is independent of t for every smooth test function w. Taking w to be equal to $1/L$ in $a < x < a+L$, zero for $x < a-1$ and $a+L+1 < x$, smooth in between, and letting L tend to infinity we deduce our result.

One can show similarly that if $u(x, t)$ is bounded and integrable at $t = 0$, then $\int u(x, t)dx$ is time independent.

THEOREM 6.2. *Let u be the solution defined in Theorem 4.1 of the single nonlinear conservation law $u_t+f_x = 0$. Assume that the initial value of u has mean value M. Then $u(x, t)$ tends to M uniformly in x as t tends to infinity.*

Proof: The proof is based on the explicit formula, given in Section 4,

$$u(x, t) = b\left(\frac{x-y_0}{t}\right), \tag{6.1}$$

where $y_0 = y_0(x, t)$ is the value of y for which the function

$$\Phi(y)+tg\left(\frac{x-y}{t}\right) \tag{6.2}$$

achieves its minimum (maximum).

Take the case when the mean value M is zero[6], and denote the sound speed in the state $u = 0$ by

$$c = f'(0) = a(0).$$

From the relations (2.5), (2.6) between the functions a, b and g we see that $b(c)$ is zero, and $g(s)$ assumes its minimum (maximum) at $s = c$.

[6]This can be accomplished by introducing $\bar{u} = u-M$ as a new variable.

Introduce the abbreviation

$$y_1 = y_1(x, t) = x - ct. \tag{6.3}$$

We claim that, as t tends to infinity, the quantity

$$\left|\frac{y_1 - y_0}{t}\right| \tag{6.4}$$

tends to zero uniformly in x. We shall prove this by showing that otherwise the function (6.2) is, for large t, smaller at $y = y_1$ than at $y = y_0$, contrary to the definition of y_0.

We form the difference of the values of the function (6.2) at y_0 and y_1:

$$\begin{aligned}\Phi(y_0) + tg\left(\frac{x-y_0}{t}\right) - \Phi(y_1) - tg\left(\frac{x-y_1}{t}\right)& \\ = \Phi(y_0) - \Phi(y_1) + t\left\{g\left(c + \frac{y_1-y_0}{t}\right) - g(c)\right\}&.\end{aligned} \tag{6.5}$$

Since $g(s)$ is a strictly convex function which has its minimum at $s = c$, the quotient

$$\frac{g(c+\xi) - g(c)}{|\xi|}$$

is bounded away from zero if $|\xi|$ is, i.e., if to any positive δ there corresponds a positive $\varepsilon = \varepsilon(\delta)$ such that

$$g(c+\xi) - g(c) \geqq \varepsilon|\xi| \qquad \text{for } |\xi| > \delta. \tag{6.6}$$

Since ϕ has mean value zero, there is an $L = L(\varepsilon)$ such that

$$|\Phi(y_0) - \Phi(y_1)| \leqq \varepsilon|y_0 - y_1| \qquad \text{for } |y_0 - y_1| > L. \tag{6.7}$$

Inequalities (6.6), with $\xi = (y_1 - y_0)/t$, and (6.7) show that the quantity (6.5) is positive if $|y_0 - y_1|$ is greater than L and $|y_0 - y_1|/t$ is greater than δ. On the other hand, by the definition of y_0, (6.5) is nonpositive. Therefore $|y_1 - y_0|/t$ is less than δ for t greater than L/δ.

Writing formula (6.1) for u in the form

$$u(x, t) = b\left(c + \frac{y_1 - y_0}{t}\right)$$

and recalling that $b(c)$ is zero, we conclude that $U(x, t)$ tends to zero with $|y_1 - y_0|/t$.

Two interesting special cases of initial functions having mean values are: 1) ϕ is periodic, and 2) ϕ is zero outside of a finite interval. In both these cases one can estimate the rate at which u tends to zero as t increases, and even the asymptotic shape of u.

THEOREM 6.3. *Assume that $\phi(x)$ is periodic with period p, and has mean value zero, and assume that $\Phi(x)$, also periodic, has a unique minimum in each period. Then the maximum of $u(x, t)$ is asymptotically equal to*

$$\frac{\text{const. } p}{t},$$

where the value of the constant does not depend on ϕ. Furthermore the graph of $u(x, t)$ tends to the graph of a saw tooth function.

Proof: We start again by investigating the possible location of y_0, the value of y minimizing the function (6.2). Since the first term $\Phi(y)$ is periodic, y_0 must be within a period p of that value of y which minimizes the second term, i.e. $y = y_1$. In fact we must have

$$y_1 - \frac{p}{2} - \varepsilon \leqq y_0 \leqq y_1 + \frac{p}{2} + \varepsilon, \tag{6.8}$$

where ε tends to zero with t. Since the variation of the second term over this interval is small if t is large, we have

$$y_0 - \xi = \varepsilon \pmod{p}, \tag{6.9}$$

where ξ is the value for which the minimum of $\Phi(x)$ occurs. Clearly, the graph of

$$u(x, t) = b\left(c + \frac{y_1 - y_0}{t}\right)$$

is within ε of the graph of the curve we would obtain if in (6.8) and (6.9) the quantity ε were taken to be zero. Since b is nearly linear within the narrow range defined by (6.8), this latter curve is very nearly a saw tooth function; its maximum occurs at $y_0 = y_1 + p/2$ and its value is

$$b\left(a + \frac{p}{2t}\right) \approx b'(a)\,\frac{p}{2t}.$$

Observe that the amplitude of the asymptotic shape depends on the *frequency* but not the *amplitude* of the initial configuration. It is clear from our derivation that *the larger* the initial amplitude is, *the sooner* the asymptotic form becomes valid.

The law of decay of periodic disturbances in compressible flows has been investigated by Whitham [29], J. B. Keller [12] and Lighthill [19].

THEOREM 6.4. *Let $\phi(x)$ be an initial function which is zero outside of a finite interval, and denote by $u(x, t)$ the weak solution, given by formulas* (6.1), (6.2), *of the equation $u_t + f_x = 0$ with initial value ϕ. The asymptotic shape of $u(x, t)$ for large t is*

$$u(x, t) \approx \begin{cases} k\left(\dfrac{x}{t} - c\right) & \text{inside } ct-\alpha\sqrt{t} < x < ct+\beta\sqrt{t} \\ 0 & \text{outside} \end{cases} \tag{6.10}$$

in the sense that for large t every point of the graph of $u(x, t)$ is within a distance $o(\sqrt{t})$ of the graph of the function (6.10). *The constants in* (6.10) *have the following values*:

$$c = f'(0),$$

i.e. the speed of propagation of signals in the state $u = 0$,

$$k = \frac{1}{f''(0)},$$

$$\alpha = \sqrt{\frac{2m_1}{k}}, \qquad \beta = \sqrt{\frac{2m_2}{k}},$$

where

$$m_1 = \max_x \int_x^{-\infty} \phi(\xi)d\xi, \qquad m_2 = \max_x \int_x^{\infty} \phi(\xi)\,d\xi. \tag{6.11}$$

(*For f concave, the maxima in* (6.11) *are to be replaced by minima.*)

Observe that the maxima in (6.11) occur at the same point x_0, and that $m_2 - m_1 = \int_{-\infty}^{\infty} \phi(\xi)d\xi$ is time-invariant.

It follows in particular from Theorem 6.4 that the quantities m_1 and m_2 themselves are time-invariant. This also follows from the explicit formula (4.9) for $U(x, t)$. It would be desirable to find such additional time-invariants of solutions of systems of conservation laws.

On account of the shape of the curve (6.10), the asymptotic form of a finite disturbance is called an N-wave.

Proof: As before, we investigate the solution with the aid of the explicit formulas (6.1), (6.2) by locating asymptotically y_0 as a function of x, t. Denote as before $x-ct$ by y_1; we wish to investigate the position of the minimum of

$$\Phi(y) + tg\left(c + \frac{y_1 - y}{t}\right). \tag{6.12}$$

As we have shown before, $(y_1 - y_0)/t$ tends to zero. Therefore, in the range which is of interest to us, the function g is accurately represented by its Taylor expansion around c:

$$g(c) + \frac{k-\varepsilon}{2} s^2 < g(c+s) \leqq g(c) + \frac{k+\varepsilon}{2} s^2, \tag{6.13}$$

where ε can be taken arbitrarily small by taking t large enough. It is easy to show on the basis of the estimate (6.12) that

i) in the range $y_1 < -\sqrt{2m_1 t/(k-\varepsilon)}$, the minimum of the function (6.12) is reached at $y = y_1$,

ii) in the range $-\sqrt{2m_1 t/(k+\varepsilon)} < y_1 < \sqrt{2m_2 t/(k+\varepsilon)}$, the minimum of the function (6.12) is reached within ε of the point x_0 where the maxima in (6.11) are assumed,

iii) in the range $\sqrt{2m_2 t/(k-\varepsilon)} < y_1$, the minimum of (6.12) is reached for $y = y_1$.

According to formula (6.1), the solution is equal to

$$u(x, t) = b\left(c + \frac{y_1 - y_0}{t}\right).$$

Since $(y_1 - y_0)/t$ tends to zero, the function b is adequately represented by the first term in its Taylor expansion. This yields, with the above determination of y_0, formula (6.10).

The asymptotic form of the solution of a system of conservation laws whose initial state is zero outside of a finite interval is expected to consist of n distinct N-waves of width of the order $t^{1/2}$, height $t^{-1/2}$, each propagating with one of the characteristic speeds corresponding to the zero state. The propagation of N-waves in compressible flows has been investigated by Chandrasekhar [3], Friedrichs [6] and Whitham [29].

SYSTEMS OF CONSERVATION LAWS

7. The Entropy Condition

As we saw in Section 1, in order to develop a theory in the large for the initial value problem for systems of conservation laws, we need a criterion for selecting the physically relevant weak solutions. In this section we shall give such a criterion.

Suppose the system (1.1)

$$u_t + f_x = 0$$

consists of n scalar conservation laws in n unknowns. We assume that (1.1) is a hyperbolic system, i.e., that the associated quasilinear system (1.1′)

$$u_t + A(u)u_x = 0$$

is hyperbolic. This means that the matrix A has n real, distinct eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ which are functions of u and which are arranged in increasing order. The corresponding right and left eigenvectors of A are denoted by $r_1, r_2, \cdots, r_n$ and $l_1, l_2, \cdots, l_n$, and are also functions of u. Later on, we shall give a convenient normalization for them.

Throughout this section we shall consider piecewise smooth weak solutions of the system of conservation laws (1.1). These are, we recall from Lemma 2.1, completely characterized by requiring that the Rankine-Hugoniot condition

$$s[u] = [f]$$

hold across the lines of discontinuity. Let P be a point on a line of discontinuity C and let u_l and u_r be the values at P of the solution on the left and right side of the discontinuity C, respectively. Draw, issuing from P in the positive t-direction, those characteristics with respect[7] to the state u_l which stay to the left of C, and those with respect to the state u_r which stay to the right of C.

DEFINITION 7.1. **A jump discontinuity in a weak solution is called a shock if the total number of characteristics drawn in this fashion is $n-1$.**

The analytical expression for the shock requirement is: *for some index k, $1 \leq k \leq n$, the inequalities*

$$\begin{aligned} \lambda_{k-1}(u_l) < s < \lambda_k\ \ (u_l), \\ \lambda_k\ \ (u_r) < s < \lambda_{k+1}(u_r) \end{aligned} \tag{7.1}$$

hold. The characteristic speeds are assumed to be indexed increasingly; *k is the index of the shock.* Thus we see that there are n different kinds of shocks.

We have tacitly assumed in the definition of a shock that the characteristic speeds with respect to the state on the right and to the state on the left are either less or greater than the speed of the propagation of the discontinuity (shock speed). Nevertheless it could happen that the discontinuity C is a characteristic curve with respect to the state on one side. Such a discontinuity is called a *contact discontinuity,* and its nature will be analyzed in detail at the end of Section 8.

Example. The characteristic speeds of the equations of compressible flow are $q-c$, q, $q+c$, where q is the flow velocity and c the sound speed. There are three kinds of discontinuities possible. The second kind is a contact discontinuity; the first and third are shocks facing the left and the right, respectively.

A shock facing the right is, by 7.1 ~~(9.1)~~, characterized by the inequalities

$$\begin{aligned} q_l < s < q_l + c_l\,, \\ q_r + c_r < s. \end{aligned}$$

One consequence of these inequalities is that the shock speed s is greater than the flow speed on either side. Since distance is measured positively to the right, it follows that particles cross the shock from the right to the left.

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[7]We recall that, for a nonlinear equation, the characteristic speeds λ are functions of u.

Call the state to the right of the shock the front, to the left, the back. In this terminology the rest of the inequalities expresses the following rule:

Shock speed is supersonic with respect to the state in the front, subsonic with respect to the state in the back.

The same statement holds for shocks facing the right.

It is well known (see e.g. Courant-Friedrichs, Supersonic Flow and Shock Waves) that this property of shocks is entirely equivalent to the requirement that the entropy of particles increase upon crossing the shock.

The motivation for formulating the shock condition in terms of the number of characteristics is furnished by the theory of free boundary value problems, see Goldner [10]. According to that theory, the number of relations to be imposed on the states on the two sides must be equal to the number of characteristics impinging on the discontinuity curve C from either side. Since the Rankine-Hugoniot conditions represent (after elimination of s) $n-1$ conditions, there must be precisely $n-1$ characteristics impinging on C, as required by Definition 7.1.

Given a state u_l, we shall investigate now the set of all states u_r to which u_l can be connected by a k-shock on the right. Such states u_r are subjected to two kinds of conditions: the Rankine-Hugoniot relations, and the inequality embodied in the shock condition. The Rankine-Hugoniot conditions represent $n-1$ relations between u_l and u_r, so that, if u_l is kept fixed, the states u_r (at least the ones which are close enough to u_l) form a one-parameter family of states[8]:

$$u_r = u(\varepsilon), \qquad u(0) = u_l .$$

The shock speed s is also a function of the parameter ε:

$$s = s(\varepsilon).$$

We shall determine now the derivatives of $u(\varepsilon)$, $s(\varepsilon)$ with respect to ε at $\varepsilon = 0$. Differentiating the Rankine-Hugoniot relations once and putting $\varepsilon = 0$ we get

$$s\dot{u} = \dot{f} = \operatorname{grad} f\dot{u} = A\dot{u}.$$

This shows that if $\dot{u}$ is different from zero, $s(0)$ must be one of the eigenvalues of $A = A(u_l)$, say the k-th,

$$s(0) = \lambda_k(u_l), \tag{7.2}$$

and that $\dot{u}$ is parallel to the corresponding right eigenvector,

$$\dot{u}(0) = \alpha r_k(u_l). \tag{7.3}$$

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[8]It is not difficult to show that the permissible states u_r form a smooth one-parameter family if the condition stated in Definition 7.2 is satisfied.

By changing the parameter ε we can achieve that the constant of proportionality α is one, i.e.,

$$\dot{u}(0) = r_k(u_l). \tag{7.3'}$$

The normalization of r_k will be given below.

From now on we shall omit the subscript k. Differentiating the Rankine-Hugoniot relations once more we get, at $\varepsilon = 0$,

$$s\ddot{u} + 2\dot{s}\dot{u} = A\ddot{u} + \dot{A}\dot{u}.$$

Substituting our previous determination of s and $\dot{u}$ we can write this relation as

$$\lambda\ddot{u} + 2\dot{s}r = A\ddot{u} + \dot{A}r. \tag{7.4}$$

To determine $\dot{s}$ and $\ddot{u}$ we take the relation

$$\lambda r = Ar,$$

valid for all values of u, restrict u to the one-parameter family under consideration, and differentiate it with respect to ε. We then have

$$\lambda\dot{r} + \dot{\lambda}r = A\dot{r} + \dot{A}r. \tag{7.5}$$

Take the scalar product of both (7.4) and (7.5) with the k-th left eigenvector $l = l_k$ of A, and subtract the two relations:

$$\dot{\lambda}(0) = 2\dot{s}(0). \tag{7.6}$$

Subtracting (7.5) from (7.4), we find that $\ddot{u} - \dot{r}$ satisfies the equation

$$\lambda(\ddot{u} - \dot{r}) = A(\ddot{u} - \dot{r})$$

and therefore $\ddot{u} - \dot{r}$ is parallel to r:

$$\ddot{u}(0) = \dot{r} + \beta r. \tag{7.7}$$

By a change of parametrization, we can achieve that the constant β is zero,

$$\ddot{u}(0) = \dot{r} = \dot{u} \cdot \operatorname{grad} r = r \cdot \operatorname{grad} r. \tag{7.7'}$$

DEFINITION 7.2. The k-th characteristic field of a quasilinear system (1.1') is called genuinely nonlinear if grad λ_k and r_k are not orthogonal for any value of u:[9]

$$r \cdot \operatorname{grad} \lambda \neq 0 \qquad \text{for all } u.$$

[9]This condition ensures that the k-th characteristic speed depends truly on the state of the flow which produces "breaking" of waves, i.e. the development of cusps in the characteristic fields and corresponding discontinuities in the solution. I have shown in a previous publication [15], that if there is linear degeneration, i.e., if $r \cdot \operatorname{grad} \lambda$ is identically zero for, say, the k-th characteristic field, then there is no breaking of waves in the k-th mode of propagation. This condition is discussed further at the end of Section 8.

Assume now that the k-th characteristic field is genuinely nonlinear. We normalize the eigenvector r so that

$$r \cdot \operatorname{grad} \lambda \equiv 1. \tag{7.8}$$

With this determination of r we have from (7.3′) and (7.6)

$$\dot{\lambda}(0) = 1, \qquad \dot{s}(0) = \tfrac{1}{2}. \tag{7.6′}$$

It is easy to see, in consequence of (7.6′), that the discontinuity line with slope $s(\varepsilon)$ separating the state $u(0)$ on the left from $u(\varepsilon)$ on the right is a k-shock in the sense of our definition *if and only if ε is negative.* We can formulate this result as

THEOREM 7.1. *Any given state u can be connected to a one-parameter family of states $u_r = u(\varepsilon)$, $\varepsilon \leqq 0$, on the right through a k-shock, provided that the k-th family of characteristics is genuinely nonlinear. The parametrization is normalized by* (7.8), (7.3′) *and* (7.7′).

Relation (7.6) can be expressed as

THEOREM 7.2. *The shock speed, up to terms of order ϵ^2, is the arithmetic mean of the sound speeds in the front and the back.*

The discontinuities of solutions given by the explicit formula of Section 4 for single conservation laws are shocks. For, we have shown there that for a convex $f(u)$, the limit from the left is not less than the limit from the right,

$$u_l \geqq u_r .$$

The single characteristic speed $\lambda(u) = f'(u)$ is, by assumption, an increasing function of u; so we have

$$\lambda(u_l) \geqq \lambda(u_r),$$

while from the Rankine-Hugoniot relation

$$s(u_r - u_l) = f(u_r) - f(u_l)$$

and the convexity of f we conclude that the speed s lies between $f'(u_r)$ and $f'(u_l)$,

$$\lambda(u_l) \geqq s \geqq \lambda(u_r).$$

This is precisely the inequality characterizing shocks.

Ideally, the ultimate aim of the theory of shocks is to assign to every initial state a weak solution which exists for all time, and all whose discontinuities are shocks, and to show that this assignment is unique. So far, this has not been achieved, not even in the classical case of compressible flows although many special solutions are known. In Section 9, we construct weak solutions with shocks for very special initial data. Then we show that these solutions are unique, at least within a certain restricted class. Preparatory to this we present in the next section a theory of simple waves.

8. Simple Waves

In this section we shall describe a special class of solutions, called simple waves, of an arbitrary quasilinear hyperbolic system

$$u_t + A(u)u_x = 0. \tag{8.1}$$

It is not necessary for these equations to be conservation laws.

DEFINITION 8.1. A function $v = v(u_1, u_2, \cdots, u_n)$ *is* a k-Riemann invariant of the system (8.1) if it satisfies the condition

$$r_k \cdot \operatorname{grad} v = 0 \tag{8.2}$$

for all values of u, where r_k is the k-th right eigenvector of A.

Condition (8.2) is a single linear homogeneous equation for v as function of u. The classical theory of such equations[10] asserts

THEOREM 8.1. *There exist precisely $n-1$ independent Riemann invariants.*

By independence of functions we mean that their gradients are linearly independent. Since the gradient of a Riemann invariant is perpendicular to r, we see that *the gradients of $n-1$ independent Riemann invariants span the orthogonal complement of r.*

The equations of compressible flow are

$$\begin{aligned} \varrho_t + q\varrho_x + \varrho q_x &= 0, \\ q_t + qq_x + \frac{1}{\varrho}p_x &= 0, \\ S_t + \qquad qS_x &= 0, \end{aligned}$$

where ϱ is the density, S the entropy, p the pressure and q the flow velocity. The pressure is regarded as a function of ϱ and S, and $\sqrt{p_\varrho} = c$ is called the sound velocity.

The characteristic speeds are $q-c$, q and $q+c$; the right eigenvectors are $\begin{pmatrix} \varrho \\ -c \\ 0 \end{pmatrix}$, $\begin{pmatrix} p_s \\ 0 \\ -p_\varrho \end{pmatrix}$ and $\begin{pmatrix} \varrho \\ c \\ 0 \end{pmatrix}$. It is easy to see that the three pairs of Riemann invariants can be chosen as $\{S, q+h\}$, $\{q, p\}$ and $\{S, q-h\}$, where h is a function of ϱ and S, satisfying the relation $h_\varrho = c/\varrho$.

A *constant state* is a domain in the x, t-plane in which a solution is constant.

THEOREM 8.2. *Let G be a domain of the x, t-plane in which a smooth*

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[10] See e.g. Courant-Hilbert, Vol. II, Ch. II.

solution u of a quasilinear system is defined. Let C be a smooth arc in G such that u is constant on one side of C. Then, either u is constant also on the other side of C, or C is a characteristic arc.

COROLLARY. *A constant state in the x, t-plane is bounded by characteristics.*

Proof: If C is not characteristic then, according to the classical uniqueness theorem, there is precisely one way of continuing the solution as smooth solution across C. Since u can certainly be continued as a constant, our result follows.

The boundary of a constant state consists, therefore, of characteristics, necessarily straight lines. The solution on the other side of the boundary is characterized by

THEOREM 8.3. *If C is part of the boundary of a constant state and is a characteristic of the k-th field, then all k-Riemann invariants on the other side of C are constant.*

The following version of the proof is due to Friedrichs (see [8]). Let $v_1, v_2, \cdots, v_{n-1}$, denote $n-1$ independent k-Riemann invariants. Take the characteristic form of the equation (8.1),

$$l_j \frac{du}{dj} = 0, \qquad j = 1, 2, \cdots, n, \tag{8.3}$$

where l_j is the j-th left eigenvector of A, and d/dj abbreviates differentiation in the j-th characteristic direction,

$$\frac{d}{dj} = \frac{\partial}{\partial t} + \lambda_j \frac{\partial}{\partial x}.$$

The left and right eigenvectors of a matrix are biorthogonal:

$$l_j \cdot r_k = 0 \qquad \text{for } j \neq k.$$

Since the gradients of $n-1$ independent Riemann invariants span the orthogonal complement of r_k, the vectors l_j, $j \neq k$, can be expressed as linear combinations of the gradients of k-Riemann invariants:

$$l_j = \sum_{s=1}^{n-1} b_{js} \operatorname{grad} v_s, \qquad j \neq k.$$

The quantities b_{js} are functions of u. Substituting this expression into (8.3) we get

$$\sum_{s=1}^{n-1} b_{js} \operatorname{grad} v_s \frac{du}{dj} = 0$$

which is the same as

$$\sum_{s=1}^{n-1} b_{js} \frac{dv_s}{dj} = 0, \qquad j \neq k. \tag{8.4}$$

For a given solution u the coefficients b_{js} are given functions, and (8.4) is a linear hyperbolic system of $n-1$ equations for the $n-1$ unknowns $v_1, v_2, \cdots, v_{n-1}$ with characteristic speeds λ_j, $j \neq k$. Since C is no longer a characteristic of this system we conclude by the same uniqueness theorem as was used before that the functions v are constant on the other side of C, as asserted in our theorem.

DEFINITION 8.2. A solution in a domain of the x, t-plane for which all k-Riemann invariants are constant is called a k-simple wave.

According to this terminology Theorem 8.3 asserts that *a solution adjacent to a constant state is a simple wave.*

By making use of the k-th characteristic equation in (8.3) it is easy to show

THEOREM 8.4. *The characteristics of the k-th field in a k-simple wave are straight lines along which the solution is constant.*

We turn to *simple waves centered at the origin,* i.e., which depend on the ratio x/t only. Let $u(x, t)$ be a centered simple wave in the angle $a < x/t < b$, and assume that the corresponding characteristic field is genuinely nonlinear, i.e., that $r \cdot \operatorname{grad} \lambda$ is not zero; normalize r as before so that

$$r \cdot \operatorname{grad} \lambda \equiv 1.$$

Since u is supposed to be centered,

$$u(x, t) = h\left(\frac{x}{t}\right).$$

The function h must be determined so that

$$v_s(h) = c_s, \qquad s = 1, 2, \cdots, n-1, \tag{8.5}$$

the c_s being the assigned constant values of $n-1$ independent Riemann invariants.

According to Theorem 8.4, the lines $x/t = \xi =$ const. are characteristics. Therefore, the function h satisfies the relation

$$\lambda(h(\xi)) = \xi. \tag{8.6}$$

The assumption that $r \cdot \operatorname{grad} \lambda$ is not zero guarantees that (8.5), (8.6) lead to a uniquely determined function h. The relations (8.5) guarantee, as we saw before, that $n-1$ of the characteristic relations (8.3) are satisfied. Equation (8.6) guarantees that the lines $x/t =$ const. are characteristics of the field under consideration. Since along the lines $x/t =$ const. the solution is constant, we see that also the remaining characteristic equation is satisfied.

Denote by u_l and u_r the values of $u(x, t)$ on $x/t = a$ and $x/t = b$. For x/t less than a and greater than b we can define $u(x, t)$ to be equal to u_l and

u_r, respectively. The two constant states, connected through a centered simple wave, are subject to the restriction that they have the same k-Riemann invariants and that $\lambda_k(u_l)$ be less than $\lambda_k(u_r)$. Thus, we have

THEOREM 8.5. *Two states u_l and u_r can be connected from left to right by a centered k-simple wave if and only if they have the same k-Riemann invariants and*

$$\lambda_k(u_l) < \lambda_k(u_r). \tag{8.7}$$

The states u_r form a one-parameter family $u_r = u(\varepsilon)$, $u(0) = u_l$. By assumption, $v(u(\varepsilon))$ is a constant for all Riemann invariants v; differentiating at $\varepsilon = 0$ we get

$$\dot{u} \cdot \text{grad } v = 0.$$

Since the gradients of the Riemann invariants span the orthogonal complement of r, $\dot{u}(0)$ must be proportional to $r(u_l)$. We fix the parametrization so that the constant of proportionality is one:

$$\dot{u}(0) = r(u_l). \tag{8.8}$$

With this choice we have

$$\dot{\lambda} = \dot{u} \cdot \text{grad } \lambda = r \cdot \text{grad } \lambda = 1,$$

i.e., λ is an increasing function of ε at the origin. Therefore, condition (8.7) is satisfied only for positive values of ε.

The two halves of the one-parameter families of states to which a given state u_l can be connected by either a shock or a centered simple wave can be fitted together to form a single one-parameter family. Comparing (7.3′) with (8.8) we see that *the first derivative is continuous across* $\varepsilon = 0$.

Differentiate the relation $v(u(\varepsilon)) = \text{const.}$ a second time with respect to ε; we obtain at $\varepsilon = 0$

$$\ddot{u} \cdot \text{grad } v + \dot{u} \cdot (\dot{\text{grad}}\, r) = 0.$$

Substituting the expression (8.8) for $\dot{u}$ we have

$$\ddot{u} \cdot \text{grad } v + r \cdot (\dot{\text{grad}}\, v) = 0. \tag{8.9}$$

Next take the identities

$$r \cdot \text{grad } v = 0,$$

specialize u to $u = u(\varepsilon)$ and differentiate them with respect to ε:

$$\dot{r} \cdot \text{grad } v + r \cdot (\dot{\text{grad}}\, v) = 0. \tag{8.10}$$

Subtract (8.9) from (8.10); we get

$$(\ddot{u} - \dot{r}) \cdot \text{grad } v = 0.$$

Since the gradients of all the Riemann invariants span the orthogonal complement of r, it follows that

$$\ddot{u} = \dot{r} + \beta r.$$

But the parametrization of $u(\varepsilon)$ can be adjusted so that β is zero. Imagine that done; we then have

$$\ddot{u}(0) = \dot{r} = \dot{u} \cdot \operatorname{grad} r = r \cdot \operatorname{grad} r. \tag{8.11}$$

Compare (8.11) with (7.7′); they give identical values for $\ddot{u}$. *Hence our composite one-parameter family has continuous second derivatives even at* $\varepsilon = 0$.

An equivalent expression of this fact is

THEOREM 8.6. *The change in a k-Riemann invariant across a k-shock is of third order in ε, the magnitude of the shock.*

This is a well-known useful result for weak shocks in compressible flows. It appears here as a special case of a general law.

We turn now to characteristic fields which are linearly degenerate, i.e., in which $r_k \cdot \operatorname{grad} \lambda_k$ is identically zero. In our terminology, this fact is expressed by saying that λ_k is a k-Riemann invariant.

THEOREM 8.7. *If two nearby states u_l and u_r have the same k-Riemann invariants with respect to a degenerate field, then the Rankine-Hugoniot conditions are satisfied with $s = \lambda(u_l) = \lambda(u_r)$.*

Such a discontinuity is called a *contact discontinuity.*

Proof: We have to show that

$$s(u_r - u_l) = f(u_r) - f(u_l).$$

Imagine u_r and u_l connected by a differentiable one-parameter family of states $u(\varepsilon)$ for which all Riemann invariants are constant. This is certainly possible if u_r and u_l are nearby states. Since λ_k is one of the Riemann invariants, $\lambda_k(u(\varepsilon)) \equiv s$ for all ε. As before we conclude that $\dot{u}(\varepsilon)$, being orthogonal to the gradients of all Riemann invariants, is proportional to $r_k(u(\varepsilon))$; therefore

$$s\dot{u} = A\dot{u} \qquad \text{for all } \varepsilon.$$

Integrating this we obtain the Rankine-Hugoniot relation.

In a previous publication, [15], the following result was stated and its proof briefly sketched.

THEOREM 8.8. *Assume that in the quasilinear system* (8.1) *one of the characteristic fields, say the k-th, is degenerate, i.e., $r_k \cdot \operatorname{grad} \lambda_k \equiv 0$. Let $u(x, t)$ be a piecewise smooth solution of* (8.1), *and assume that u contains only a contact discontinuity, i.e., that the k-Riemann invariants are continuous across the* *discontinuity of u. Then, provided that the jump of u is not too large, u can be obtained as the limit of a sequence of smooth solutions.*

9. The Riemann Initial Value Problem

In this section we shall solve the initial value problem for initial states that are piecewise constant, with one jump discontinuity:

$$u(x, 0) = \phi(x) = \begin{cases} u_0 & \text{for } x < 0 \\ u_n & \text{for } 0 < x, \end{cases} \tag{9.1}$$

where u_0, u_n are constant vectors. The determination of the flow in a shock tube after the breaking of a diaphragm at $t = 0$ is such a problem; so is the problem of determining a flow after the head-on collision of two shocks, or the impinging of a shock on a contact discontinuity. Since such problems have been investigated by Riemann, we shall call this class of problems *Riemann's problem.*

As a preliminary observation we note that the solution $u(x, t)$ is a function of the ratio x/t. For, if $u(x, t)$ is a weak solution, with initial value (9.1), the function

$$u_\alpha = u(\alpha x, \alpha t),$$

α any positive constant, is also a weak solution; its discontinuities are shocks and it takes on the same initial value. Since presumably there is a unique solution, we must have $u_\alpha \equiv u$; this is the case if and only if u is a function of x/t only. Such a solution is called *centered at the origin* or, for brevity, centered.

Our solution to the initial value problem (9.1) consists of $n+1$ constant states $u_0, u_1, \cdots, u_n$; the constant states u_{k-1} and u_k are separated by a k-shock or a centered simple wave of the k-th kind, or, if the k-th field is linearly degenerate, by a contact discontinuity. The two end states u_0, u_n being given, we find the intermediate states u_1, u_2, etc. as follows:

According to Section 8, there is a one-parameter family of states u_1 that can be joined to u_0 on the right by waves of the first kind. Similarly, there is a one-parameter family of states u_2 which can be joined to u_1 on the right by a wave of the second kind, etc. Thus we get an n-parameter family of states

$$\begin{aligned} u_n &= u_n(u_0\,;\, \varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n), \\ u_0 &= u_n(u_0\,;\, 0, 0, \cdots, 0) \end{aligned} \tag{9.2}$$

that can be joined to u_0; the Jacobian is not zero at the origin since, according to (8.8), the derivative of u_n with respect to the k-th parameter is parallel to the k-th right eigenvector r_k, and these, by assumption, are linearly independent. Therefore by the implicit function theorem, a sufficiently small cube in ε-space is mapped in a one-to-one way onto a neighborhood of u_0. This can be summarized in

THEOREM 9.1. *Every state u_0, has a neighborhood such that, if u_n belongs to this neighborhood, the Riemann initial value problem* (9.1) *has a solution. This solution consists of $n+1$ constant states connected by centered waves. There is exactly one solution of this kind, provided the intermediate states are restricted to lie in a neighborhood of u_0.*

In order to find such a solution when the state u_n is *not* near u_0, i.e. for large values of the parameters ε_i, one must study the n-parameter family (9.2) and verify that u_n is in its range.

Within terms of order ε, the intermediate states $u_1, u_2, \cdots, u_{n-1}$ can be found by decomposing the initial discontinuity as follows:

$$u_n - u_0 = \sum_{k=1}^{n} \varepsilon_k r_k . \tag{9.3}$$

Then

$$u_j = u_0 + \sum_{k=1}^{j} \varepsilon_k r_k + o(\varepsilon). \tag{9.4}$$

The expression (9.4) resembles the solution of Riemann's initial value problem for a linear equation with constant coefficients. In that case, the solution consists of $n+1$ constant states separated by characteristics, and the intermediate states are given by (9.3) and

$$u_j = u_0 + \sum_{k=1}^{j} \varepsilon_k r_k . \tag{9.4'}$$

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