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CONF-880392--1

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LA-UR--88-1384

DE88 009113

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SUBMITTED TO: Proceedings of the 2nd International Conference on Hyperbolic Problems, March 14-18, 1988

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SOME SUPRACONVERGENT SCHEMES FOR HYPERBOLIC EQUATIONS ON IRREGULAR GRIDS

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SUMMARY

An analysis of the truncation error for finite difference schemes frequently shows an apparent loss of accuracy when a nonuniform grid is used. Some schemes exhibit the phenomenon of supraconvergence, that is, there is no loss of accuracy in the global error. We show that this is the case for smooth solutions of the color equation for an upstream conservative scheme, for two versions of the Lax-Wendroff scheme, and for a variant of the von Neumann-Richtmyer scheme for gas dynamics, if the latter three are stable.

1. INTRODUCTION

Finite difference equations seem to work best on uniform grids, but even in one dimension one might be forced into using an irregular grid. A typical example of this situation occurs when materials with very disparate densities occur side-by-side; for example, at an air water interface. If Lagrangian coordinates are used, with mass taking the place of length, then equal mass cells would require roughly 1000 times as many cells in the water as in the air. One could change the grid size gradually but rapidly near the interface so that there is a grid gradient rather than a severe jump. This helps but does not remove the error. Giles and Thompkins [1] analyze the wave propagation properties of a grid gradient. Noh [4] shows the bad things that can happen with an exponentially varying grid when an infinite strength shock is sent through it. In any event, there is definitely a point to considering difference equations on totally nonuniform grids.

It is very clear that the local truncation error of most difference schemes suffers if the grid is nonuniform. These local errors for many difference schemes have been examined by Turkel [6], and by Pike [5]. The point of our presentation here is that the *global* error is sometimes better behaved than the local error would indicate, a property that has been called *supraconvergence*. This is well-known for finite element methods, but is somewhat of a surprise for finite difference methods. This phenomenon was observed by Manteuffel and White [3] for a cell-centered scheme for $u'' = f$ which is inconsistent on an arbitrary grid but maintains global second order accuracy. This was extended by Kreiss *et al* [2] to higher order equations and other difference methods, not all of which have this so-called supraconvergence property. Numerov's method, for example, fails to maintain its fourth order accuracy on some grids.

We show that supraconvergence obtains for smooth solutions of the color equation for an upstream conservative scheme, for an edge-centered version of the Lax-Wendroff scheme (with mild mesh restrictions), for a cell-centered Lax-Wendroff scheme, and for a variant of the von Neumann-Richtmer scheme for gas dynamics, provided that the latter three are stable.

2. AN UPSTREAM CONSERVATIVE SCHEME

In [5] Pike gives an example of a conservative scheme which is inconsistent on a nonuniform grid yet which compares favorably with a first-order method. For the differential equation

$$u_t + u_x = 0$$

the difference equation is

$$LU \equiv \frac{U_i^{n+1} - U_i^n}{\Delta t} + \frac{U_i^n - U_{i-1}^n}{h_i} = 0,$$

where U_i^n is the grid function at the cell edge (x_i, t^n), and $h_{i-1/2} = x_i - x_{i-1}$, $h_i = \dfrac{h_{i+1/2} + h_{i-1/2}}{2}$. This conserves $\Sigma h_i U_i$.

The truncation error is

$$Lu = \frac{h_{i-1/2} - h_{i+1/2}}{2h_i} + O(h, \Delta t)$$

and therefore, in general, the scheme is inconsistent. In fact, it is *first-order accurate* if u(x,t) is smooth. To see this, note that

$$\frac{h_{i-1/2} - h_{i+1/2}}{2h_i} u_{x,i} = \frac{h_{i-1/2} u_{x,i-1} - h_{i+1/2} u_{x,i}}{2h_i} + O(h).$$

Defining the corrected error

$$e_i = u_i - U_i + \frac{1}{2} h_{i+1/2} u_{x,i}$$

we see that

$$Le = O(h, \Delta t)$$

since $u_{x,i}$ is a smooth function of t. This scheme is monotone and stable if $\Delta t / h_i \le 1$, in which case both e and $u - U$ are $O(h)$.

3. LAX-WENDROFF

3.1. An Edge-Centered Version

Pike also considered an edge-centered version of the Lax-Wendroff scheme. With the same notation as above the difference equation is then since half the cells have length h^2,

$$\left|\prod_{k=r}^{s}\frac{1-\nu_{k+1/2}}{1+\nu_{k+1/2}}\right| \le \left(\frac{1}{2}\right)^{\frac{s-r}{2}} = \rho^{s-r},\ \rho=\sqrt{\frac{1}{2}}.$$

In general, it is sufficient (but not necessary) that the fraction of intervals which have a bounded ratio with the smallest interval remain bounded from below as the mesh is refined. Of course, this only permits a correction which cancels the bad part of the truncation error. The actual global error depends on the stability of the scheme.

We still have to show that the divided time difference of the correction δ is $O(h^2)$. This is so, since

$$\frac{T_{i+1/2}^{n+1}-T_{i+1/2}^{n}}{\Delta t} = \frac{1}{2}h_{i+1/2}^{2}\frac{u_{xx,i+1}(t+\Delta t)-u_{xx,i+1}(t)}{\Delta t}$$

and we are assuming that u(x,t) is sufficiently smooth.

3.2. A Cell-Centered Version

We consider the particular solution $u(x,t)=(x-t)^2$. For a cell-centered Lax-Wendroff the full error equation is

$$Le = \frac{e_{i+1/2}^{k+1}-e_{i+1/2}^{k}}{\Delta t} + \frac{e_{i+1}^{\bullet}-e_{i}^{\bullet}}{h_{i+1/2}} - \frac{\Delta t}{2h_{i+1/2}}\left\{\frac{e_{i+3/2}-e_{i+1/2}}{h_{i+1}} - \frac{e_{i+1/2}-e_{i-1/2}}{h_i}\right\}$$

$$= \frac{1}{4}\left(h_{i+3/2}-h_{i-1/2}\right)$$

$$-\frac{\Delta t}{4}\left\{\frac{h_{i+3/4}-2h_{i+1/2}+h_{i-1/2}}{h_{i+1/2}}\right\}$$

where

$$e_{i+1}^{\bullet} = \frac{h_{i+3/2}e_{i+1/2}+h_{i+1/2}e_{i+3/2}}{h_{i+3/2}+h_{i+1/2}}.$$

Let $d_{i+1/2}=e_{i+1/2}-\frac{1}{4}h_{i+1/2}^{2}$. Since

$$\frac{h_{i+3/2}h_{i+1/2}^{2}+h_{i+1/2}h_{i+3/2}^{2}}{h_{i+3/2}+h_{i+1/2}} = h_{i+3/2}h_{i+1/2}$$

and

$$\frac{h_{i+3/2}^{2}-h_{i+1/2}^{2}}{h_{i+3/2}+h_{i+1/2}} = h_{i+3/2}-h_{i+1/2}$$

then

$$Ld = 0\ .$$

In general, a similar analysis shows that for sufficiently smooth solutions there is a correction which removes the low-order truncation terms.

4. VON NEUMANN-RICHTMYER

We consider the classic von Neumann-Richtmyer method without viscosity. The gas-dynamic equations in Lagrangian form are

$$v_t - u_x = 0,$$
$$u_t + p_x = 0,$$
$$e_t + p\,v_t = 0,$$
$$p = p\,(v,e).$$

The velocities u are set at the cell corners, and the volumes and energies v and e are set at the cell centers see Fig. 1.1. Thus, the difference equations are

$$\frac{V^{n+1/2}_{i+1/2} - V^{n-1/2}_{i+1/2}}{\Delta t^n} - \frac{U^n_{i+1} - U^n_i}{h_{i+1/2}} = 0, \tag{2a}$$

$$\frac{U^n_i - U^{n-1}_i}{\Delta t^{n-1/2}} + \frac{p\,(V^{n-1/2}_{i+1/2}, E^{n-1/2}_{i+1/2}) - p\,(V^{n-1/2}_{i-1/2}, E^{n-1/2}_{i-1/2})}{h_i} = 0, \tag{2b}$$

$$\frac{E^{n+1/2}_{i+1/2} - E^{n-1/2}_{i+1/2}}{\Delta t^n} \tag{2c}$$

$$+ \frac{\Delta t^{n-1/2} p\,(V^{n+1/2}_{i+1/2}, E^{n+1/2}_{i+1/2}) + \Delta t^{n+1/2} p\,(V^{n-1/2}_{i+1/2}, E^{n-1/2}_{n+1/2})}{\Delta t^n} \; \frac{V^{n+1/2}_{i+1/2} - V^{n-1/2}_{i+1/2}}{\Delta t^n} = 0.$$

where

$$h_{i+1/2} = x_i - x_{i-1}, \quad h_i = \frac{1}{2}(h_{i+1/2} + h_{i-1/2}),$$

and

$$\Delta t^{n+1/2} = t^{n+1} - t^n, \quad \Delta t^n = \frac{1}{2}(\Delta t^{n+1/2} + \Delta t^{n-1/2}).$$

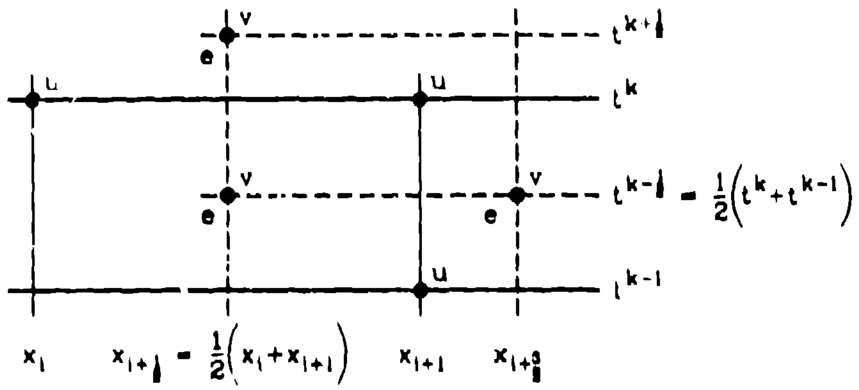

Fig. 1.1. Staggered grid.

Now we need to look at the detailed truncation error to see if the appropriate corrections can be made.

In eqs. (2a) and (2c) it is the uncentered time difference that contributes to the low order truncation error. Specifically, the form is

$$\frac{f(t^{n+1/2}) - f(t^{n-1/2})}{\Delta t^n} = f_t + \frac{1}{8} f_{tt} \frac{(\Delta t^{n+1/2})^2 - (\Delta t^{n-1/2})^2}{\Delta t^n} + O(\Delta^2),$$

where the functions on the right are evaluated at t^n. The second term on the right can be rewritten as

$$\frac{\frac{1}{8}(\Delta t^{n+1/2})^2 f_{tt}^{n+1/2} - \frac{1}{8}(\Delta t^{n-1/2})^2 f_{tt}^{n-1/2}}{\Delta t^n} + O(\Delta^2).$$

Thus there is a second-order correction $\bar{f}$ to f which satisfies

$$\frac{\bar{f}^{n+1/2} - \bar{f}^{n-1/2}}{\Delta t^n} = f_t + O(\Delta^2),$$

where

$$\bar{f}^{n+1/2} = f(t^{n+1/2}) - \frac{1}{8}(\Delta t^{n+1/2})^2 f_{tt}(t^{n+1/2}).$$

It follows immediately that, replacing f by v, the truncation error for eq. (2a) is $O(\Delta^2)$ for $\bar{\mathrm{v}}$, that is,

$$\frac{\bar{\mathrm{v}}_{i+1/2}^{n+1/2} - \bar{\mathrm{v}}_{i+1/2}^{n-1/2}}{\Delta t^n} - \frac{u_{i+1}^n - u_i^n}{h_{i+1/2}} = O(\Delta^2).$$

Since the average p used in eq. (2c) is already second-order accurate it follows in the same way that eq. (2c) is also second-order for $\bar{e}$ and $\bar{\mathrm{v}}$.

There remains the momentum equation (2b) to deal with. We first note that for any smooth second-order perturbations α and β

$$p(\bar{\mathrm{v}}_+ + \alpha_+, \bar{e}_+ + \beta_+) - p(\bar{\mathrm{v}}_- + \alpha_-, \bar{e}_- + \beta_-) = p(\bar{\mathrm{v}}_+, \bar{e}_+) - p(\bar{\mathrm{v}}_-, \bar{e}_-) +$$
$$\frac{\partial p}{\partial \mathrm{v}}(\alpha_+ - \alpha_-) + \frac{\partial p}{\partial e}(\beta_+ - \beta_-) + h_i O(\Delta^2),$$

where $\alpha_+ = \alpha_{i+1/2}$, etc. The low order part of the truncation error of the off centered $\partial p/\partial x$ in eq. (2b) is

$$\frac{1}{8} p_{xx} \frac{(h_{i+1/2})^2 - (h_{i-1/2})^2}{h_i}.$$

In order to cancel it we can take the corrections to be

$$\alpha_{i+1/2}^{n-1/2} = -\frac{1}{8}(h_{i+1/2})^2 \left[\frac{p_\mathrm{v}}{p_\mathrm{v}^2 + p_e^2} p_{xx}\right]_{i+1/2}^{n-1/2}$$

and

$$\beta_{i+1/2}^{n-1/2} = -\frac{1}{8}(h_{i+1/2})^2 \left[\frac{p_e}{p_\mathrm{v}^2 + p_e^2} p_{xx}\right]_{+1/2}^{n-1/2},$$

where we must assume that $p_\mathrm{v}^2 + p_e^2$ is bounded away from zero.

For the corrected exact solution given by

$$\hat{v} \equiv \bar{v} + \alpha,$$

$$\hat{e} \equiv \bar{e} + \beta,$$

and

$$\hat{u} \equiv u,$$

the local truncation error is $O(\Delta^2)$. Since these differ from the exact solution itself by $O(\Delta^2)$, stability would imply second-order accuracy.

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