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Systems of Conservation Laws of Mixed Type

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A local theory of weak solutions of first-order nonlinear systems of conservation laws is presented. In the systems considered, two of the characteristic speeds become complex for some achieved values of the dependent variable. The transonic "small disturbance" equation is an example of this class of systems. Some familiar concepts from the purely hyperbolic case are extended to such systems of mixed type, including genuine nonlinearity, classification of shocks into distinct fields and entropy inequalities. However, the associated entropy functions are not everywhere locally convex, shock and characteristic speeds are not bounded in the usual sense, and closed loops and disjoint segments are possible in the set of states which can be connected to a given state by a shock. With various assumptions, we show (1) that the state on one side of a shock plus the shock speed determine the state on the other side uniquely, as in the hyperbolic case; (2) that the "small disturbance" equation is a local model for a class of such systems; and (3) that entropy inequalities and/or the existence of viscous profiles can still be used to select the "physically relevant" weak solution of such a system.

I. INTRODUCTION

Let f, g, u denote n-dimensional vectors, with f, g smooth functilns of u, and u a function of x, y. A first-order system of nonlinear conservation laws

$$g(u)_x + f(u)_y = 0 \tag{1.1}$$

is said to be hyperbolic at a point u_0 if the values of λ such that the matrix $g_u(u_0) + \lambda f_u(u_0)$ is singular are all real. Strict hyperbolicity requires, in addition, that the $\lambda_i(u)$, $i = 1, 2,..., n$, called characteristic speeds, be distinct.

A solution of (1.1) assumes values $u \in D \subset \mathbb{R}^n$. Our concern here is with systems which are strictly hyperbolic in only part of D, i.e., $D = H \cup B \cup E$, where (1.1) is strictly hyperbolic in H, but two of the $\lambda_j(u)$ are a complex pair for $u \in E$. E will be called the elliptic region, analogous to the case $n = 2$. The

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boundary B between E and H is assumed to be a smooth manifold of dimension $n - 1$.

Such systems arise, for example, as descriptions of stationary, transonic fluid flow. As in the case of purely hyperbolic systems, smooth solutions of (1.1) are not expected in general (c.f. [18]). Weak solutions, possibly piecewise smooth but admitting jump discontinuities (shocks) are anticipated. An essential feature of mixed type nonlinear systems is the possibility of shocks between values of u one of which is in E and the other in H. Such discontinuities are routinely observed in transonic flow, but are not described by linear systems of mixed type, e.g., the Tricomi equation, or by purely hyperbolic nonlinear systems.

The primary objective of this paper is a local description of weak solutions of (1.1). Such weak solutions are in general not unique; an entropy condition is needed to determine which shocks are admissible on physical grounds. The classical entropy condition of [13] is obviously inappropriate for shocks connecting states in E with states in H. However, a number of ideas from the theory of hyperbolic systems can be extended to such systems of mixed type, such as genuine nonlinearity, classification into distinct fields (k-shocks), and the existence of entropy inequalities. There are also, of course, essential differences from the purely hyperbolic case, which include nonconvex entropy functions, unbounded characteristic speeds and shock speeds, and closed curves in the set of states which can be connected to a given state by a shock.

A brief outline of our discussion follows: in Sections 2 and 3, we briefly review some of the theory of nonlinear conservation laws, as applied to systems of the form (1.1). We also obtain some additional results; for example, we show that the system (1.1) is hyperbolic at any point where a locally convex entropy function exists. Nonetheless, entropy inequalities for nonconvex entropy functions are motivated by consideration of regularization of the system (1.1), as in the hyperbolic case [14]. With various assumptions, we show that if the state on one side of a shock and the shock speed are prescribed, there exists at most one state which can be on the other side of the shock.

In Section 5 we prove a representation theorem; with a number of assumptions, we show that in a small neighborhood of a point in $B(u = 0)$, the leading part of that portion of the system which is changing type corresponds to the transonic "small disturbance" problem [1]

$$\begin{aligned} (p^2)_x - q_y &= 0, \\ q_x - p_y &= 0, \end{aligned} \tag{1.2}$$

with x acting as a time-like variable. We note in passing that the pair, (1.2) has also been studied separately in the hyperbolic region $p > 0$, with y the time-like variable [21], as a problem arising in mechanics.

Thereafter we adopt a modest generalization of (1.2) as a prototype problem. In Section 6 we classify the possible discontinuities for such a problem, and prove theorems on the equivalence of different forms of entropy conditions: one-sided bounds on the spatial derivatives, entropy inequalites, and the existence or nonexistence of viscous profiles. These results can be applied to the construction of difference schemes for this type of problem, as it is easier to construct dif-difference schemes satisfying entropy inequalities than schemes satisfying one-sided bounds on the spatial derivatives [2, 10, 20]. Also, the results on existence of viscous profiles suggest some suitable forms of regularization for higher-order difference schemes.

II. Entropy Functions, Hyperbolicity, and Regularization

Throughout this paper, we shall assume that smooth solutions of (1.1) satisfy an additional scalar conservation law, of the form [5, 9, 14]

$$U(u)_x + V(u)_y = 0. \tag{2.1}$$

This will be the case if there exists a vector function $v(u)$ such that

$$g_u{}^T v = U_u\,, \qquad f_u{}^T v = V_u\,, \tag{2.2}$$

for scalar functions U, V. Where g_u^{-1}, f_u^{-1} exist we have, respectively,

$$v = U_g\,, \qquad v = V_f\,. \tag{2.3}$$

We next assume that the mapping $u \to v$ is locally invertible; i.e., that $v_u^{-1}(u)$ exists for all $u \in D$. Then (1.1) can be written in symmetric form with v as the dependent variable [5, 8]. For $u \in D$, let the scalar functions Φ, Ψ be given by

$$\Phi(v(u)) = v(u) \cdot g(u) - U(u), \qquad \Psi(v(u)) = v(u) \cdot f(u) - V(u); \tag{2.4}$$

then from (2.2) it follows that

$$\Phi_v = g, \qquad \Psi_v = f, \tag{2.5}$$

so that if u is a solution of (1.1), $v(u)$ satisfies

$$(\Phi_v(v))_x + (\Psi_v(v))_y = 0. \tag{2.6}$$

We note that in view of (2.5), the equivalence of the systems (1.1) and (2.6) holds even for weak solutions, and independently of any adopted entropy condition. For systems which change type, an entropy condition is obtained somewhat more naturally for the symmetric form (2.6).

The characteristic speeds λ_j and associated eigenvectors r_j, $j = 1, 2,..., n$, for the system (2.6) may be obtained from the mixed symmetric eigenvalue problem

$$\Phi_{vv} r_j + \lambda_j \Psi_{vv} r_j = 0. \tag{2.7}$$

A directional bias may be given to the system (1.1) by requiring that U be a locally convex function of g, possibly after a rotation in the $x - y$ plane. If $U_{gg} \geqslant 0$, then weak solutions of (1.1) which are limits of smooth solutions as $\epsilon \to 0+$ of a regularized problem such as

$$g(u_\epsilon)_x + f(u_\epsilon)_y = \epsilon\, \Delta g(u_\epsilon), \tag{2.8}$$

satisfy an entropy inequality [14]

$$U(u)_x + V(u)_y \leqslant 0, \tag{2.9}$$

in the sense of distributions. In this context, there is considerable freedom in the choice of the regularization term, i.e., the right side of (2.8). It suffices that the solutions of (2.8) be sufficiently smooth that multiplication of (2.8) by $U_g(u_\epsilon)$ be defined, and that the right side of (2.8), multiplied by $U_g(u_\epsilon)$ and integrated over the x, y plane, be nonpositive in the limit $\epsilon \to 0+$. However, a single second derivative of $g(u_\epsilon)$, or second derivatives of $U_g(u_\epsilon)$, etc., may be used in principle, depending on the particular problem under consideration.

With suitable hypotheses, (2.9) is equivalent to the classical entropy condition for shocks [13]. Assuming that g_u^{-1} exists, the independent variable x may be considered time-like. (In this context, the characteristic speeds λ_j given in (2.7) are the reciprocals of those as usually defined. This convention will also be adopted for shock speeds, for reasons of boundedness, which will become clear in Section 3.)

The following lemma shows how the directional bias is inherited by the symmetric form (2.6).

LEMMA 2.1. *If* $U_{gg}(u_0) > 0$ and $g_u^{-1}(u_0)$ exists, then $\Phi_{vv}(v(u_0)) > 0$.

Proof. From (2.3, 2.5), it follows easily that $\Phi_{vv} = U_{gg}^{-1}$.

Since Φ_{vv} and Ψ_{vv} are symmetric, however, an immediate consequence of (2.7) and Lemma 2.1 is

THEOREM 2.2. *If* $U_{gg}(u_0) > 0$ *and* $g_u^{-1}(u_0)$ *exists, then the system* (1.1) *is hyperbolic at* $u = u_0$.

Thus the local convexity of U cannot hold everywhere in systems which change type. The entropy inequality (2.9) may remain valid, however; such an example will be described in Section 6.

THEOREM 2.3. *Assume that the mapping $u \to v$ is invertible over D, and that as $\epsilon \to 0+$ the solutions v_ϵ of*

$$(\Phi_v(v_\epsilon))_x + (\Psi_v(v_\epsilon))_y = \epsilon \, \Delta v_\epsilon \tag{2.10}$$

converge boundedly almost everywhere, to a limit function v. Then (2.9) *holds, in the sense of distributions, with $u = u(v)$, whether or not U is locally convex.*

Proof. Let ξ be a non-negative C_0^∞ function of x, y; multiply (2.10) by ζv_ϵ and integrate with respect to x, y, over a region in the x, y plane containing the support of ζ. The result follows by the usual partial integrations, using (2.2) and (2.5).

In symmetric systems where hyperbolicity fails, Φ_{vv} must become singular. Thus if the mapping $u \to v$ is invertible and v_u^{-1} exists globally, it follows from (2.5) that g_u must become singular. In particular, the explicit case $g(u) = u$ is precluded under these conditions.

We also note that any negative semidefinite form of regularization may be used in (2.10), leading to the entropy inequality (2.9). It is not known what additional hypotheses are needed so that (2.9) uniquely determines weak solutions. Even in the purely hyperbolic case, it is known that different forms of such regularization can lead to different weak solutions obtained as limits [4, 22].

A system (1.1) may admit several additional conservation laws of the form (2.1), for different pairs U, V. To each such pair there corresponds a symmetric form (2.6) and an entropy inequality (2.9), obtained by regularization such as (2.10). The entropy inequalities (2.9) for different pairs U, V will in several, admit different shocks in weak solutions. In such cases, it is clear that some additional knowledge of the dissipation mechanisms is necessary to select the "physically correct" weak solution. This difficulty does not arise in the strictly hyperbolic, genuinely nonlinear case when the solutions are piecewise smooth and the shocks are sufficiently weak [14], because the entropy functions can be assumed to be locally convex for all assumed values of u.

III. Pairs of States Which Can Be Connected by a Shock

We specialize to a symmetric system

$$(\Phi_u(u))_x + (\Psi_u(u))_y = 0, \tag{3.1}$$

with u assuming values in D as described above. We assume $\Phi_{uu}(u) > 0$ and strict hyperbolicity for $u \in H$, but Φ_{uu} becomes singular and two of the characteristic speeds are equal for $u \in B$. For $u \in E$, two of the characteristic speeds are complex; we assume that the real characteristic speeds remain distinct.

We also assume that $\Psi_{uu}^{-1}(u)$ exists for all $u \in D$, which implies that the characteristic speeds, as obtained from (2.7), are bounded. With these assumptions, the real characteristic speeds can be consistently labeled. For $u \in H$, we simply take the λ_j in increasing order as usual. For $u_0 \in E$, we label the real $\lambda_j(u_0)$ by continuity, i.e., move u along a path γ into H, and assign to real $\lambda(u_0)$ the corresponding index in H. The continuity of λ along the path γ follows from the assumed existence of Ψ_{uu}^{-1}. It remains to show that such an assignment is independent of path. Suppose not, i.e., that for two paths γ_1, γ_2, different indices are obtained for the same characteristic speed $\lambda(u_0)$, $u_0 \in E$. Let $\tilde{\lambda}_1(u)$, $\tilde{\lambda}_2(u)$, denote the characteristic speeds at a point u on γ_1, γ_2, respectively, obtained by continuity, with $\tilde{\lambda}_1(u_0) = \tilde{\lambda}_2(u_0) = \lambda(u_0)$. Suppose that γ_1, γ_2 cross B at u_1, u_2, respectively; let $\lambda_\nu(u_1)$, $\lambda_\nu(u_2)$ denote the double characteristic speed at u_1, u_2. The real characteristic speeds are assumed distinct within E, and since only two of the characteristic speeds become equal in B, and these are complex in E, it is clear thet $\tilde{\lambda}_1(u_1) \neq \lambda_\nu(u_1)$, $\tilde{\lambda}_2(u_2) \neq \lambda_\nu(u_2)$. Therefore the only way that γ_1, γ_2 can lead to different indices for $\lambda(u_0)$ is for $\tilde{\lambda}_1(u_1) < \lambda_\nu(u_1)$ and $\lambda_2(u_2) > \lambda_\nu(u_2)$, or vice versa. But in such a case strict hyperbolicity fails in H, at some point along a line within $B \cup H$ connecting u_1, u_2.

We shall call a value of k passive if λ_k is real in E, and essential if $\lambda_k(u)$ becomes complex as u enters E. Since only two characteristic speeds (the ones corresponding to essential k) are equal in B, it is clear that the essential values of k are independent of u. Since the essential values are necessarily consecutive, we denote them by ν, $\nu + 1$.

The Rankine–Hugoniot relations, describing pairs of points u_+, u_- which can be connected by a shock of speed $\sigma(u_+, u_-)$, are given by

$$\Phi_u(u_+) - \Phi_u(u_-) + \sigma(u_+, u_-)(\Psi(u_+) - \Psi_u(u_-)) = 0. \tag{3.2}$$

The condition of genuine nonlinearity which we adopt is as follows: let u_+, u_- be any two states in D for which (3.2) is satisfied, for some scalar value of $\sigma(u_+, u_-)$; then $\sigma(u_+, u_-)$ is not to be a characteristic speed at u_+ or u_-. This conditions is briefly mentioned in [6] and analyzed in some detail in [16]. In general, it is stronger than the classical condition of [13], although for many systems it is equivalent. It is applicable to states in nonhyperbolic regions; indeed in purely elliptic regions a real shock speed will never be characteristic, even if the system is locally linear.

A number of implications follow from this assumption [13, 16]. Let $\Gamma(u_0)$ be the set of states in D which can be connected to u_0 by a shock, i.e., for which (3.2) is satisfied. Differentiating (3.2) ,we obtain

$$(\Phi_{uu}(u) + \sigma(u, u_0)\, \Psi_{uu}(u))\, u' = -\sigma'(u, u_0)(\Psi_u(u) - \Psi_u(u_0)) \tag{3.3}$$

for $u \in \Gamma(u_0)$, where primes denote differentiation within $\Gamma(u_0)$. From (3.3) and the assumption of genuine nonlinearity, it follows that $\sigma' \neq 0$ and that

$\Gamma(u_0)$ is composed of distinct 1-manifolds. For $u_0 \in H$, corresponding to each $k = 1, 2, \ldots, n$ there exist two 1-manifolds $\Gamma_k^+(u_0)$, $\Gamma_k^-(u_0) \subset \Gamma(u_0)$, which have u_0 as one endpoint. As $u \to u_0$ along $\Gamma_k^\pm(u_0)$, $\sigma(u, u_0) \to \lambda_k(u_0)$; $\sigma(u, u_0)$ increases (decreases) as u moves away from u_0 along $\Gamma_k^+(u_0)(\Gamma_k^-(u_0))$. Globally, $\sigma(u, u_0)$ always lies between $\lambda_k(u_0)$ and $\lambda_k(u)$, so that the boundedness of characteristic speeds implies boundedness of shock speeds. These results also hold for $u_0 \in B \cup E$, if k is passive.

For $u_0 \in H \cup B$, we assume explicitly that this is all of $\Gamma(u_0)$, i.e., that there are no detached branches. With this assumed form of genuine nonlinearity, in particular the requirement $\sigma' \neq 0$ in (3.3), this will be the case if there are no solutions of (3.2) with $u_+ = u_0$ and u_- on the boundary of D (or arbitrarily far from u_0), other than $u_- \in \Gamma_k^\pm(u_0)$ for some k.

With this form of genuine nonlinearity, the entropy inequality (2.9) and the classical entropy condition of [13] are equivalent, in the following sense:

THEOREM 3.1. *Let u_+, u_- satisfy* (3.2), *with $u_- \in \Gamma(u_+)$; let U, V be an entropy function pair, with U locally convex as a function of Φ_u, for u in the convex hull of that portion of $\Gamma(u_+)$ or $\Gamma(u_-)$ between u_+ and u_-. Then the entropy jump condition*

$$U(u_+) - U(u_-) + \sigma(u_+, u_-)(V(u_+) - V(u_-)) < 0 \tag{3.4}$$

and the entropy condition of [13] *are equivalent.*

This theorem is only a slight generalization of Theorem 3.1 of [17], and so a proof will be omitted.

For j, k both passive, any $u_0 \in D$, the proof of Theorem 2.1 of [16] and its corollary remain valid, and so $\Gamma_j(u_0)$, $\Gamma_k(u_0)$ are distinct, where $\Gamma_k(u) = \Gamma_k^+(u) \cup \Gamma_k^-(u)$. For k passive, it follows from [16] that

$$u \in \Gamma_k(u_0) \Leftrightarrow u_0 \in \Gamma_k(u). \tag{3.5}$$

For j and/or k essential, it follows from (3.3) that the direction of u' is unique at each point $u \in \Gamma(u_0)$. Thus $\Gamma_j(u_0)$ and $\Gamma_k(u_0)$ do not cross, but $\Gamma_\nu^+(u_0)$ and $\Gamma_{\nu+1}^-(u_0)$ might form a closed loop, if $u_0 \in H$, and $\Gamma_\nu^+(u_0)$, $\Gamma_{\nu+1}^-(u_0)$ both enter E. This in fact does typically occur; in such cases we consider $\Gamma_\nu^+(u_0)$ and $\Gamma_{\nu+1}^-(u_0)$ to both consist of the entire closed loop (except for the point $u = u_0$). For $u_0 \in E$ and j essential, $\Gamma_j(u_0)$ cannot have u_0 as an endpoint, because $\lambda_j(u_0)$ is complex. We consider $\Gamma_j^\pm(u_0)$, $u_0 \in E$, j essential, to be the connected subsets of $\Gamma(u_0)$ such that (3.5) and (3.8) below hold for all $u \in H$. It turns out that for $u_0 \in E$, $\Gamma_{\nu+1}^-(u_0)$ and $\Gamma_\nu^+(u_0)$ frequently do not exist, and $\Gamma_\nu^-(u_0)$, $\Gamma_{\nu+1}^-(u_0)$ lie within H and may overlap or coincide entirely.

We assume that there are no other detached branches of $\Gamma(u_0)$, $u_0 \in E$.

Typical curves for $\Gamma(u_0)$, corresponding to $n = 4$, are shown in Figs. 1–3.

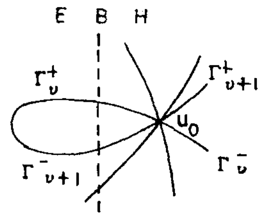

FIG. 1. $\Gamma(u_0)$, $u_0 \in H$.

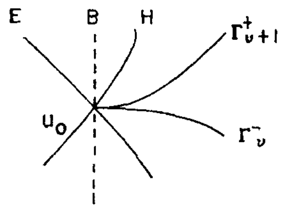

FIG. 2. $\Gamma(u_0)$, $u_0 \in B$.

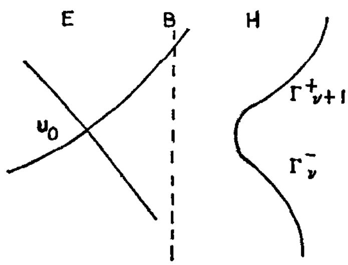

FIG. 3. $\Gamma(u_0)$, $u_0 \in E$.

To make these results precise, we first claim

LEMMA 3.2. *For* $u \in B$, $\lambda_\nu(u) = \lambda_{\nu+1}(u) = 0,$ (3.6)

the proof of which is deferred. For $u_0 \in H$, such that $\Gamma_\nu^+(u_0)$ and $\Gamma_{\nu+1}^-(u_0)$ from a closed loop, it follows from the assumption of genuine nonlinearity and Lemma 3.2 that

$$\lambda_\nu(u_0) < 0 < \lambda_{\nu+1}(u_0), \tag{3.7}$$

and thus that for such $u_0 \in H$, $u \in \Gamma_{\nu+1}^+(u_0)$,

$$0 < \lambda_{\nu+1}(u_0) < \lambda_{\nu+1}(u).$$

Thus $\lambda_{\nu+1}(u) \neq 0$, and $\Gamma_{\nu+1}^+(u_0)$ cannot enter E. A similar argument shows that $\Gamma_\nu^-(u_0)$ cannot enter E.

For all $u_0 \in H$, $u \in D$, we have [16]

$$u \in \Gamma^+_{\nu+1}(u_0) \Rightarrow u_0 \in \Gamma^-_{\nu+1}(u), \qquad u \in \Gamma_\nu^-(u_0) \Rightarrow u_0 \in \Gamma_\nu^+(u). \tag{3.8}$$

However, the converses need not hold $\Gamma_\nu^+(u_0)$ and $\Gamma^-_{\nu+1}(u_0)$ form a closed loop.

For $u \in \Gamma_\nu^+(u_0) \cup \Gamma^-_{\nu+1}(u_0)$, $u_0 \in H$, $\lambda_\nu(u_0) < \sigma(u, u_0) < \lambda_{\nu+1}(u_0)$ so that this part of $\Gamma(u_0)$ is distinct from all the others. Thus for $u_0 \in H$, $\Gamma_k^\pm(u)$ are all distinct, except that $\Gamma_\nu^+(u_0)$ and $\Gamma^-_{\nu+1}(u_0)$ may form a closed loop. For $u_0 \in B$, $\Gamma_\nu^+(u_0)$ and $\Gamma^-_{\nu+1}$ do not exist, and $\Gamma_\nu^-(u_0)$, $\Gamma^+_{\nu+1}(u_0)$ behave essentially as for $u_0 \in H$. The arguments are quite similar. We next show

Lemma 3.3. *For $u_0 \in E \cup B$ and k essential, $\Gamma_k(u_0)$ lies entirely within H.*

Proof. First we consider $u_0 \in B$. If $\Gamma_k(u_0)$ enters E, there will be some point $u_1 \in B$, $u_1 \in \Gamma_k(u_0)$. Since $\lambda_k(u_0) = \lambda_k(u_1) = 0$ by Lemma 3.2, the requirement that $\sigma(u_0, u_1)$ lies strictly between $\lambda_k(u_0)$ and $\lambda_k(u_1)$ will be violated.

For $u_0 \in E$, we first note that $\Gamma_k(u_0)$ and $\Gamma_j(u_0)$ cannot overlap or coincide for any passive value of j. Since (3.5) applies to all passive fields, we would then have the existence of points $u_1 \in H$ such that $u_0 \in \Gamma_j(u_1)$ and $u_0 \in \Gamma_k(u_1)$. This is also impossible by genuine nonlinearity. Thus if $\Gamma_k(u_0)$ enters E, with $u_0 \in E$, there exists a point $u_1 \in B$, $u_1 \in \Gamma_k(u_0)$. Clearly $u_0 \in \Gamma(u_1)$, and $u_0 \notin \Gamma_j(u_1)$ for any passive j. This in contradiction with the case $u_0 \in B$ discussed above.

We have thus shown the following:

Lemma 3.4. *Let $u, u_0 \in D$, with $u \in \Gamma(u_0)$. Then for some value of k, either*

$$u \in \Gamma_k^+(u_0) \qquad \textit{and} \qquad u_0 \in \Gamma_k^-(u), \tag{3.9}$$

or vice versa. The condition $u \in \Gamma_j(u_0)$, $u_0 \notin \Gamma_j(u)$ is possible only if $u_0 \in H$, $u \in H \cup B$, j is essential, $\Gamma_\nu^+(u_0)$ and $\Gamma^-_{\nu+1}(u_0)$ form a closed loop, and (3.9) (*or vice versa*) *holds with $j + k = 2\nu + 1$.*

Lemma 3.4 is the essential ingredient used to show that u_0 and $\sigma(u, u_0)$ determine $u \in \Gamma(u_0)$ uniquely.

Theorem 3.5. *For all $u_0 \in D$ and all real η, there exists at most one point $u \in \Gamma(u_0)$ such that $\sigma(u, u_0) = \eta$.*

Remark. As in the purely hyperbolic case, there are some values of η, for example the real characteristic speeds at u_0, for which no such point exists.

Proof. Suppose not, let u_1, u_2 be two such points. First assume that u_0, u_1, u_2 are all in $H \cup B$ or all in E. By genuine nonlinearity, $u_1 \in \Gamma_j(u_0)$, $u_2 \in \Gamma_k(u_0)$, with $j \neq k$; if u_0, u_1, u_2 are in E, then j, k are both passive as $\Gamma_\nu(u_0)$ and $\Gamma_{\nu+1}(u_0)$ lie entirely within H. Using Lemma 3.4, we may assume that $u_0 \in \Gamma_j(u_1)$,

$u_0 \in \Gamma_k(u_2)$; this will require picking j, k properly if j or k is essential and a closed loop is formed. Clearly $u_1 \in \Gamma(u_2)$. Since $\lambda_j(u)$, $\lambda_k(u)$ are real for $u = u_0, u_1, u_2$, it follows that $u_1 \in \Gamma_j(u_2)$ or $u_1 \in \Gamma_k(u_2)$. In the former case, we then also have $u_2 \in \Gamma_j(u_1)$, which is impossible, because u_0 is the unique point in $\Gamma_j(u_1)$ corresponding to shock speed η. Similarly, $u_1 \in \Gamma_k(u_2)$ is impossible, because u_0 is the unique point in $\Gamma_k(u_2)$ corresponding to speed η.

For one of the points in E and two in $H \cup B$, the same argument applies with $u_0 \in E$, $u_1, u_2 \in H \cup B$. In fact, the only failure of the above argument occurs when $u_0 \in H$, $u_1, u_2 \in E$; $u_1 \in \Gamma_\nu^+(u_0)$, $u_2 \in \Gamma_{\nu+1}^-(u_0)$, in the case where $\Gamma_\nu^+(u_0)$ and $\Gamma_{\nu+1}^-(u_0)$ do not form a closed loop. Because λ_ν, $\lambda_{\nu+1}$ are complex in E, it is possible in principle that u_1, u_2 are connected in one of the passive fields $\nu - 1$ or $\nu + 2$. Consider the case $u_2 \in \Gamma_{\nu-1}^-(u_1)$, $u_1 \in \Gamma_{\nu-1}^+(u_2)$. By genuine nonlinearity, we then have

$$\lambda_\nu(u_0) < \eta < \lambda_{\nu+1}(u_0), \tag{3.10}$$

$$\lambda_{\nu-1}(u_2) < \eta < \lambda_{\nu-1}(u_1). \tag{3.11}$$

Consider what happens to $\sigma(u, u_0)$, as u moves along $\Gamma_\nu^+(u_0)$ from u_0 towards u_1. For u near u_0, $\sigma(u, u_0) \approx \lambda_\nu(u) > \lambda_{\nu-1}(u)$. But for u near u_1, $\sigma(u, u_0) \approx \eta < \lambda_{\nu-1}(u)$. Thus $\sigma(u, u_0)$ becomes equal to $\lambda_{\nu-1}(u)$ at some point, violating genuine nonlinearity. The case where $u_1 \in \Gamma_{\nu+2}(u_2)$ is entirely similar, completing the proof.

IV. Proof of Lemma 3.2

Let $\phi = \Phi_{uu}$ and $\psi = \Psi_{uu}$ be partitioned as shown in (4.1) and let α represent (in generic sense) vectors of dimension $n - 1$. At a point $u_0 \in B$, take ϕ diagonal; one of the diagonal entries, taken to be ϕ_{11}, must be zero. (If two or more of the diagonal entries are zero, then the conclusion is obvious, as two eigenvectors corresponding to $\lambda = 0$ are easily constructed.)

$$\phi = \left(\begin{array}{c:c} \phi_{11} & \phi_{12} \\ \hdashline & \\ \varphi_{21} & \phi_{22} \end{array}\right), \qquad \psi = \left(\begin{array}{c:c} \psi_{11} & \psi_{12} \\ \hdashline & \\ \psi_{21} & \psi_{22} \end{array}\right) \tag{4.1}$$

We first claim that for $u_0 \in B$, ϕ diagonal with $\phi_{11} = 0$ and the other diagonal entries of ϕ positive, either the conclusion holds or else ψ_{11} must be zero. Suppose not, and suppose that the conclusion of the lemma is not true. Then there exist eigenvalues $\lambda_k(u)$, $\lambda_{k+1}(u)$ becoming equal and nonzero at $u = u_0 \in B$. and complex as u enters E. In this case $\lambda_i(u_0) = 0$ is of multiplicity one, with eigenvector

$r_i(u_0) = (1, 0, 0, ..., 0)^T$. Furthermore, since λ_i is distinct at u_0, $r_i(u)$ is continuous there. From (2.7), rewritten in present notation as

$$\phi(u)\, r_j(u) + \lambda_j(u)\, \psi(u)\, r_j(u) = 0, \tag{4.2}$$

it easily follows that for $j \neq i$, $\bar{r}_j(u) \cdot \psi(u)\, r_i(u) = 0$, and so for u close to u_0, $|\psi_{11}(u)| \geqslant c > 0$ implies that

$$r_j(u) = \begin{pmatrix} O(1) \\ \alpha \end{pmatrix}, \qquad \text{with} \quad \|\alpha\| \geqslant c > 0. \tag{4.3}$$

If λ_k is one of a complex pair of eigenvalues, then from (4.2) we also readily obtain

$$(\lambda_k(u)^{-1} - \bar{\lambda}_k(u)^{-1})(\bar{r}_k(u) \cdot \phi(u)\, r_k(u)) = 0. \tag{4.4}$$

But for u close to u_0, $|\phi_{11}|$, $\|\phi_{12}\|$, and $\|\phi_{21}\|$ are $o(1)$, whereas ϕ_{22} is strictly positive definite and bounded away from zero. Thus from (4.3), $\bar{r}_k \cdot \phi r_k > 0$ in (4.4), providing a contradiction.

Now move u to a point in H, a distance ϵ from u_0. The reciprocals of the eigenvalues λ_j are the stationary points of the functional

$$R(v) = \frac{v \cdot \psi v}{v \cdot \phi v} = \frac{a^2\psi_{11} + 2a(\psi_{12} \cdot \alpha) + \alpha \cdot (\psi_{22}\alpha)}{a^2\phi_{11} + 2a(\phi_{12} \cdot \alpha) + \alpha \cdot (\phi_{22}\alpha)}, \tag{4.5}$$

where a is a scalar and $v = (a, \alpha)^T$. By smoothness, we have $|\psi_{11}|$, $|\Phi_{11}|$ and $\|\Phi_{12}\| \leqslant O(\epsilon)$, and $\|\psi_{12}\|$ bounded away from zero by the invertibility of ψ. Setting $a = 1$ in (4.5), we have

$$R(v) = \frac{O(\epsilon) + 2\alpha \cdot \psi_{12} + O(\|\alpha\|^2)}{O(\epsilon + \|\alpha\|^2)}. \tag{4.6}$$

Clearly, we can choose α so that as $\epsilon \to 0$, $R(v)$ assumes arbitrarily large positive maximum and negative minimum values. These extrema are the reciprocals of the two eigenvalues approaching zero as $\epsilon \to 0$. Since the corresponding $\|\alpha\| \to 0$ as $\epsilon \to 0$, both of the corresponding eigenvectors approach $r_i(u_0)$, as expected.

Lemma 3.2 admits the following physical interpretation: in regions of the x, y plane where u is smoth and the system (3.1) changes type, the characteristic curves corresponding to the essential fields given by

$$dx/dy = \lambda_\nu\, , \lambda_{\nu+1}$$

become parallel to the y (space-like) axis, i.e., the effective signal speeds become infinite.

V. A Local Model

THEOREM 5.1. *Suppose that a smooth symmetric system* (3.1) *satisfies the following properties, for* $u \in D = H \cup B \cup E$, *where* B *is a smooth manifold of dimension* $n - 1$:

(i) *The system is strictly hyperbolic in* H, *with two characteristic speeds coinciding in* B *and becoming complex in* E.

(ii) Φ *is locally convex in* u, *for* $u \in H$.

(iii) *The system is genuinely nonlinear, in the sense that*

$$| r_j \cdot \nabla\lambda_j | \geqslant c \| r_j \|, \qquad c > 0, \text{ for all } j \text{ and all } u \in H \cup B. \tag{5.1}$$

(iv) Ψ_{uu}^{-1} *exists for all* $u \in D$.

Then in a small neighborhood of u_0, (*without loss of generality we take* $u_0 = 0$)

$$\Phi(u) = \tfrac{1}{3}(u^1)^3 + (1 + O((u^1)^2))\, O((u^2)^2 + \cdots + (u^n)^2) \tag{5.2}$$

$$\Psi(u) = -u^1u^2 + (1 + O(| u^1 |))\, O((u^2)^2 + \cdots + (u^n)^2) \tag{5.3}$$

after a suitable rotation and scaling of the dependent variables, where $u = (u^1, u^2, \ldots, u^n)^T$ *and* u^1 *is perpendicular to* B *at* u_0.

Several remarks precede the proof. From (5.2, 5.3), the leading terms in the first two variables correspond to a pair of equations of the form

$$\begin{aligned} ((u^1)^2)_x - u^2{}_y &= 0, \\ u^2{}_x - u^1{}_y - bu^2{}_y &= 0 \end{aligned} \tag{5.4}$$

for some constant b. However, the term $bu^2{}_y$ in (5.4) is in some sense a higher-order effect. Scaling the variables via

$$u^1 = \delta^{2/3}\tilde{u}^1, \qquad u^2 = \delta\tilde{u}^2, \qquad \tilde{x} = \delta^{1/3}\tilde{x}, \qquad y = \tilde{y},$$

which makes the $(\tilde{u}^1)^3$ and $(\tilde{u}^2)^2$ contributions to Φ in (5.2) the same order of magnitude, has the effect of multiplying b by $\delta^{1/3}$ in (5.4), with respect to the scaled variables. For $b = 0$, (5.4) is the small disturbance problem (1.2).

As noted above, the local convexity of Φ is equivalent to the existence of a locally convex entropy function for $u \in H$. The requirement (5.1) is slightly stronger than implied by the genuine nonlinearity assumption of Section 3. However, this assumption has a physical interpretation. If $r_j \cdot \nabla\lambda_j$ becomes zero for $u \in B$, then rarefaction waves may have unbounded spatial derivations away from their centers.

The assumption of invertible Ψ_{uu} implies bounded characteristic speeds, in the sense described above.

This theorem is essentially a corollary of Lemma 3.2. The facts that two of the characteristic speeds (as defined here) become zero as B is approached from H and that Φ is locally convex in H provide the thrust of the argument.

There are, of course, other interesting examples of systems in which hyperbolicity fails. In the system for stationary compressible fluid flow in two or three space dimensions, the passive fields are linearly degenerate, but much of the local behavior is as described here. Also, there are applications leading to systems with parabolic degeneracies [11].

Proof. Choose u so that $\phi = \Phi_{uu}$ is diagonal at $u = u_0$. Since two of the characteristic speeds are zero by Lemma 3.2, it follows that at least one of the diagonal entries, say ϕ_{11}, is zero. Our first cleim is that the other ϕ_{jj} are positive. If not, there will be two independent eigenvectors corresponding to $\lambda = 0$. At $u = u_0$, $\nabla\lambda_j$ will be perpendicular to B for each $\lambda_j = 0$, so that by taking an appropriate linear combination of the eigenvectors, (5.1) can be made to fail.

Thus the direction of u^1 is uniquely determined. From (4.2), it follows that the single eigenvector $r_\nu(u_0)$ corresponding to $\lambda = 0$ is in the u^1-direction. By assumption (iii), $\nabla\lambda_\nu(u_0)$, which is normal to B, has a nonzero first component.

We next claim that $\nabla\phi_{11}(u_0)$ is not identically zero. From (4.2), we may infer that

$$| \det \phi(u)| \geqslant O(u \cdot e)^2, \tag{5.5}$$

where e is the unit normal to B at u_0. Assumption (ii) requires

$$| \phi_{11}(u)| \geqslant O(\phi_{1j}^2(u)) \tag{5.6}$$

for any j. Thus $\nabla\phi_{11}(u_0)$ can vanish only if $\phi_{11} = O(\| u \|^2)$ and $\phi_{1j} = O(\| u \|)$ for some $j \neq 1$. In this case, the leading part of ϕ must be singular; otherwise, assumption (i) fails when $u \in H$ is replaced by $-u \in E$, u sufficiently close to u_0. However, if the leading part of ϕ is singular, then (5.5) fails.

Thus $\nabla\phi_{11}(u_0)$ is not identically zero, and indeed is also normal to B. We set

$$\phi_{11}(u) = cu^1 + \Sigma\beta_j u^j + O(\| u \|^2), \qquad c \neq 0, \tag{5.7}$$

and claim that all the β_j are zero. If not, the corresponding off-diagonal element $\phi_{1j} \sim \beta_j u^1$, and assumption (ii), in the context of (5.6), will fail at points in H where $cu^1 + \Sigma\beta_j u^j = \epsilon$, $| \epsilon |$ sufficiently small compated with $| u^1 |$. Thus u^1 is normal to B at u_0, and

$$\phi_{11}(u) = \text{const} \cdot u^1 + O(\| u \|^2). \tag{5.8}$$

Next we consider the off-diagonal components ϕ_{1j}, for $j \geqslant 2$. By assumption (ii), the only term which could appear is proportional to u^1. Such would imply a term of $O((u^1)^2 u^j)$ in Φ, which would imply a term of $O(u^j)$ in ϕ_{11}. This is inconsistent with (5.8) so $| \phi_{1j} | \leqslant O(\| u \|^2)$, for $j \geqslant 2$. This establishes (5.2).

To obtain (5.3), we first claim that $\psi_{11}(u_0) = 0$. Otherwise, there will not be two characteristic speeds λ_ν, $\lambda_{\nu+1}$ approaching zero as $u \in H$ approaches u_0. This may be inferred either from (4.5) or by partitioning ϕ, ψ as in (4.1) and writing out (4.2), obtaining after some simple computation

$$\phi_{11} + \lambda\psi_{11} = (\phi_{12} + \lambda\psi_{12})(\phi_{22} + \lambda\psi_{22})^{-1}(\phi_{12} + \theta\psi_{12}). \tag{5.9}$$

From (5.9), only one value of λ will approach zero as ϕ_{11}, $\phi_{12} \to 0$, unless ψ_{11} also approaches zero. Then from assumption (iv), the vector ψ_{12} (of dimension $n - 1$) is not zero at $u = u_0$. We may rotate and scale the coordinates $u^2,\dots,u^n$ so that $\psi_{12}(u_0) = (-1, 0, 0,\dots, 0)^T$, which establishes (5.3) and so completes the proof.

VI. A Prototype Problem

In view of the results of Sections 4 and 5, we adopt a prototype problem of the form

$$\begin{aligned} \theta'(p)_x - q_y &= 0, \\ q_x - p_y &= 0, \end{aligned} \tag{6.1}$$

which corresponds to

$$\Phi(p, q) = \theta(p) + \tfrac{1}{2}q^2, \ \Psi(p, q) = -pq. \tag{6.2}$$

Let $\theta(0) = \theta'(0) = \theta''(0) = 0$; then H, B, E correspond simply to $\theta''(p) > 0$, $=0$, <0, respectively. The assumption of genuine nonlinearity requires

$$\theta'''(p) > 0, \qquad p \geqslant 0, \tag{6.3}$$

so that $p > 0$ corresponds to H and $p = 0$ to B; in some of the results below we shall assume $\theta'''(p)$ positive for $p < 0$ as well. Throughout we assume $\theta''(p) < 0$ for $p < 0$, which thus corresponds to the region E.

Smooth solutions of (6.1) satisfy various additional conservation laws of form (2.1), some examples of which correspond to

$$U_1(p, q) = p\theta'(p) - \theta(p) + \tfrac{1}{2}q^2, \qquad V_1(p, q) = -pq; \tag{6.4}$$

$$U_2(p, q) = q\theta'(p), \qquad V_2(p, q) = -\tfrac{1}{2}q^2 - \theta(p); \tag{6.5}$$

$$U_3(p, q) = \int_0^p \theta(t)\,\theta''(t)\,dt + \tfrac{1}{2}q^2\theta'(p), \qquad V_3 = -q\theta(p) - \tfrac{1}{6}q^3. \tag{6.6}$$

The system (6.1) is in the symmetric form corresponding to U_1, V_1 and this is the entropy function pair generally identified with this system.

The characteristic speeds are easily obtained from (2.7), and satisfy

$$\lambda^2 = \theta''(p). \tag{6.7}$$

The Rankine–Hugoniot relations (3.2) are in this case

$$\theta'(p_+) - \theta'(p_-) = \sigma(u_+, u_-)(q_+ - q_-); \tag{6.8}$$

$$q_+ - q_- = \sigma(u_+, u_-)(p_+ - p_-); \tag{6.9}$$

where $u_\pm = (p_\pm, q_\pm)^T$. Combining (6.8, 6.9) we easily obtain

$$\sigma(u_+, u_-)^2 = \frac{\theta'(p_+) - \theta'(p_-)}{p_+ - p_-}, \tag{6.10}$$

from which we easily infer

THEOREM 6.1. *For the system* (6.1), *there are no shocks connecting two points in E, or a point in E with one in B.*

Consider now piecewise smooth solutions of (6.1), with a shock between two states u_+, u_-, on opposite sides of a curve in the x, y described by

$$\frac{dx}{dy} = \sigma(u_+, u_-). \tag{6.11}$$

An entropy condition is needed to determine which shocks are physically admissible; the commonly adopted one is

$$p_x < \infty, \tag{6.12}$$

i.e., p decreases across shocks. For u_+, u_- both in H, the condition (6.12) is equivalent to the classical entropy condition of [13]. The entropy jump condition (3.4), applied to the pair U_1, V_1, becomes

$$\begin{aligned} 0 &> U_1(u_+) - U_1(u_-) + \sigma(u_+, u_-)(V_1(u_+) - V_1(u_-)) \\ &= \tfrac{1}{2}(p_+ - p_-)(\theta'(p_+) + \theta'(p_-)) - \theta(p_+) + \theta(p_-) \\ &= \tfrac{1}{12}(p_+ - p_-)^3\, \theta'''(\xi), \end{aligned} \tag{6.13}$$

using (6.8, 6.9, 6.10) and the mean value theorem in the last step. Thus if we identify u_+, u_- by the convention that u_+ lies to the right (larger value of x) of u_-, we have

THEOREM 6.2. *Suppose that $\theta'''(p) > 0$ for almost alls p; then the entropy conditions* (6.12) *and* (2.9) (*or* (3.4)) *are equivalent.*

The equivalence of the entropy conditions thus remains valid in this case, even though U_1 is not locally convex (in the variables $\theta'(p)$, q) for $p \leqslant 0$.

Another method of applying an entropy condition is to require that the physically admissible shocks be limits of viscous profiles [6]. The existence of such viscous profiles has been extensively studied, especially for pairs of equations [3].

For the system (6.1) in the regularized form

$$\begin{aligned} \theta'(p)_x - q_y &= \epsilon \Delta p, \\ q_x - p_y &= \epsilon \Delta q \end{aligned} \tag{6.14}$$

the viscous profiles are solutions of the autonomous system

$$\begin{aligned} \dot{p} &= \theta'(p) - \theta'(p_+) - \sigma(u_+, u_-)(q - q_+), \\ \dot{q} &= q - q_+ - \sigma(u_+, u_-)(p - p_+), \end{aligned} \tag{6.15}$$

with boundary conditions

$$p(+\infty) = p_+, \qquad p(-\infty) = p_-, \qquad q(+\infty) = q_+, \qquad q(-\infty) = q_-,$$

where $p_\pm$, $q_\pm$, $\sigma(u_+, u_-)$ satisfy (6.8, 6.9). In (6.15), $\cdot$ denotes differentiation with respect to $t = x/\epsilon(1 + \sigma(u_+, u_-)^2)$. Again, freedom exists in the specific form of regularization. Replacing Δp, Δq in (6.14) by p_{xx}, q_{xx}, respectively, leaves (6.15) intact with a change of the scaling between x, t. However, the replacement of Δp, Δq by p_{yy}, q_{yy} is inappropriate, as $\sigma(u_+, u_-)$ can be zero if one state is in H and the other in E. Our next result is:

THEOREM 6.3. *Suppose that $\theta'''(p) > 0$ for almost all p; then the system* (6.15, 6.16) *has a solution if and only if $p_+ < p_-$.*

Theorem 6.3 shows that the existence or nonexistence of viscous profiles is equivalent to the entropy inequality (2.9) or to (6.12) as an entropy condition.

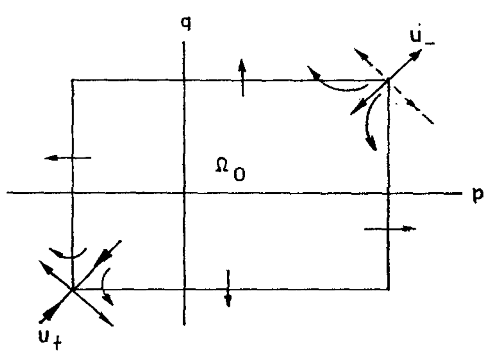

FIG. 4. Existence of viscous profile.

Proof. First we consider $p_+ < p_-$; from (6.9) $q_+ - q_-$ has the opposite sign from $\sigma(u_+, u_-)$. It is no real restriction to take $q_+ < q_-$ as the case $\sigma = 0$, $q_+ = q_-$ is trivial. Consider the rectangle Ω_0, with sides parallel to the p, q axes and opposite corners at u_+, u_-, as shown in Fig. 4. From (6.8, 6.9), it is readily seen that the vector field determined by the right side of (6.15) points out of Ω_0, at all points on the boundary except the two critical points u_+, u_-.

We next claim that with respect to the system (6.15), u_+ is a saddle and u_- a repulsive improper node. This is easily shown, using (6.10) and noting that

$$\theta''(p_+) < \sigma^2(u_+, u_-) < \theta''(p_-) \tag{6.17}$$

by hypothesis. It is also readily seen that the various eigenvectors are oriented as shown in Fig. 4. There are no other critical points in Ω_0, as may be seen either by direct calulation or by appeal to Theorem 3.5. Clearly the orbit entering u_+ from the interior of Ω_0 originates at u_-, which gives existence of a solution of (6.15), (6.16). The orbit is clearly unique up to translation. Also, there is no orbit from u_+ to u_-, so that there is no viscous profile if the boundary conditions are reversed.

Other forms of regularization of (6.1) are of course possible. Another such form, which is attractive in the construction of difference schemes of second- or third-order accuracy, is

$$\begin{aligned} \theta'(p)_x - q_y &= -\epsilon p_{xxxx}, \\ q_x - p_y &= 0. \end{aligned} \tag{6.18}$$

Limits of solutions of (6.18) will satisfy the entropy inequalities (2.9) or (3.4), for the entropy function pair U_1, V_1. This form of regularization has also been successfully applied to difference schemes for purely hyperbolic equations [17].

For the system (6.18), viscous profiles correspond to solutions of

$$\dddot{p} = -\theta'(p) + \theta'(p_+) + \sigma(u_+, u_-)^2(p - p_+), \tag{6.19}$$

with boundary conditions

$$p(+\infty) = p_+, \qquad p(-\infty) = p_-. \tag{6.20}$$

THEOREM 6.4. *Suppose that* $\theta'''(p) > 0$ *for almost all* p; *then there exists a solution of* (6.19, 6.20) *if and only if* $p_+ < p_-$.

Thus also for this form of regularization, the existence or nonexistence of viscous profiles in an entropy condition equivalent to the other forms discussed.

Proof. For $p_+ < p_-$, the existence of a solution follows from the existence theorem proved in [15]. For $p_+ > p_-$, multiply (6.19) by $\dot{p}$ and integrate, obtaining

$$-\int_{-\infty}^{\infty} (\ddot{p}(t))^2 \, dt = \tfrac{1}{12}(p_+ - p_-)^3 \, \theta'''(\xi) \tag{6.21}$$

by the same calculation leading to (6.13).

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