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Systems of Conservation Equations with a Convex Extension

K. O. FRIEDRICHS AND P. D. LAX

Courant Institute of Mathematical Sciences, New York University, New York, N.Y. 10012

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ABSTRACT **We discuss first-order systems of nonlinear conservation laws which have as a consequence an additional conservation law. We show that if the additional conserved quantity is a convex function of the original ones, the original system can be put into symmetric hyperbolic form. Next we derive an entropy inequality, which has also been suggested by I. Kružhkov, for discontinuous solutions of the given system of conservation laws.**

Systems of first-order nonlinear partial differential equations are conservation equations if they can be written in the form

$$\partial_t u^j + \partial_k f^{j,k} = 0, \qquad j = 1,\ldots,m. \tag{1}$$

Here ∂_t and ∂_k stand for partial differentiation with respect to the time t and the space coordinates x^k, summation with respect to k is implied. The $f^{j,k}$ are functions of the quantities $\{u^1, \ldots, u^m\} = u$. If the quantities u^j, as functions of the x^k and t, satisfy Eq. **1**, the integrals

$$\int\int \ldots u^j(t,x)dx^1dx^2\ldots \text{ are constant in } t. \tag{2}$$

Therefore the u^j will be referred to as the conserved quantities.

The question is, when is a new conservation law,

$$\partial_t U + \partial_k F^k = 0, \tag{3}$$

with U and F^k functions of u, a consequence of the old ones?

Denoting partial differentiation with respect to u^l by a subscript l, and writing (**1**) and (**3**) as

$$\partial_t u^j + f_l^{j,k}\partial_k u^l = 0, \qquad \text{and} \tag{1$'$}$$

$$U_j\partial_t u^j + F_l^k\partial_k u^l = 0, \tag{3$'$}$$

we see that (**3**)$'$ follows from (**1**)$'$ if and only if

$$U_j f_l^{j,k} = F_l^k. \tag{4}$$

We assume that (**4**) is satisfied; differentiating this relation with respect to u^h we get

$$U_{j,h}f_l^{j,k} + U_j f_{l,h}^{j,k} = F_{l,h}^k.$$

The second term on the left and the right side are symmetric in the subscripts l and h. Therefore, so is the first term,

$$U_{j,h}f_l^{j,k} = U_{j,l}f_h^{j,k}. \tag{5}$$

Multiplying (**1**)$'$ by $U_{j,h}$ and summing with respect to j we get

$$U_{j,h}\partial_t u^j + U_{j,h}f_l^{j,k}\partial_k u^l = 0. \tag{6}$$

This system is equivalent to (**1**) if the matrix $(U_{j,h})$ is nonsingular. The noteworthy feature of Eq. **6** is that it is *symmetric;* i.e., the matrices which multiply $\partial_t u$ and $\partial_k u$ are symmetric, by (**5**).

If the coefficient matrix of $\partial_t u$ is *positive definite,* a symmetric system is called *symmetric hyperbolic.* The importance of this concept is that *Cauchy's initial value problem* for symmetric hyperbolic systems *is well posed;* that is, the values of their solution on a surface t = const., or on any other space-like surface may be prescribed as arbitrary smooth functions of x. A unique solution of the equation will then exist at least in a neighborhood of the initial surface.

The system (**6**) is evidently symmetric hyperbolic if the matrix $U_{j,h}$ is positive definite, i.e., if the matrix of second derivatives of U with respect to the u^j is positive definite; in other words, if U is *convex* as function of the u_k.

Thus we have shown: *If a system of conservation laws* (**1**) *implies a new conservation law* (**3**) *such that the new conserved quantity U is a convex function of the original conserved quantities u^j, the initial value problem for Eq.* **1** *is well posed.*

We observe that in case Eq. **1**$'$ is already symmetric,

$$f_l^{j,k} = f_j^{l,k},$$

we can derive from it a new conservation law with

$$U = {}^1\!/_2\sum_i (u^i)^2, \qquad F^k = u^j f^{j,k} - g^k,$$

where g^k satisfies $g_l^k = f^{l,k}$. Clearly U is convex.

We remark that the conclusion derived above is also valid for nonhomogeneous conservation laws, i.e., equations of the form

$$\partial_t u^j + \partial_k f^{j,k} = h^j,$$

where the h^j are functions of $u^1,\ldots,u^m$.

Symmetric hyperbolic systems are not the only ones for which the initial value problem is properly posed; for example, this problem is also well posed for all strictly hyperbolic systems, i.e., systems with real and distinct characteristics. The requirement of distinctness of the characteristics, an unnatural one for mathematical physics, is not needed in the symmetric hyperbolic case.

Most basic equations of mathematical physics can be written as systems of conservation laws (**1**) that have a convex extension (**3**). This is, for example, the case of the equations of Maxwellian electromagnetism, of elasticity, of the dynamics of compressible fluids in Eulerian form, and of magneto-fluid-dynamics, both nonrelativistic and relativistic.

The original conserved quantities in these equations may be the densities of mass and momentum, but may also be other quantities. The role of new conserved quantity will frequently be played by the energy density; but possibly also by the density of the negative entropy. Circumstances under which this is so will be indicated at the end of this note. Details about convexly extensible systems of conservation laws in mathematical physics will be described in a future publication, in which also the relationship of the Lagrangian formalism with the present one will be discussed.

In handling concrete problems it would be awkward if one forced oneself to take the original conserved quantities u^j as unknowns. If the u^j as well as the $f^{j,h}$ are considered functions of other unknowns $v = \{v^h\}$, one will postulate the existence of multipliers $\lambda^j(v)$—in place of U_j—such that

$$\lambda^j(\partial_t u^j + \partial_k f^{j,k}) \equiv \partial_t U + \partial_k F^k.$$

Then the matrix $(\partial\lambda^j/\partial v^k)(\partial u^j/\partial v^l)$—in place of $U_{j,h}$—will serve as the symmetrizer and the convexity condition is replaced by the condition that the matrix

$$(\partial^2 U/\partial v^h\partial v^l - \lambda^j\partial^2 u^j/\partial v^h\partial v^l) \text{ is positive definite.} \qquad (\#)$$

This remark is helpful if one wants to take the derived conserved quantity U as an original one and the negative of one of the original ones, $-u^m$ say, as the new derived conserved quantity. Since the new multipliers are $\{\lambda^1/\lambda^m,\ldots,\lambda^{m-1}/\lambda^m,-1/\lambda^m\}$, the new convexity condition is that the matrix

$$\left(-\partial^2 u^m/\partial v^h\partial v^l - (\lambda^m)^{-1}\sum_{j=1}^{m-1}\lambda^j\partial^2 u^j/\partial v^h\partial v^l - (\lambda^m)^{-1}\partial^2 U/\partial v^h\partial v^l\right)$$

should be positive definite. This is the same as condition (#) if the factor λ^m is positive. Thus, the quantities U and u^m may be switched if $\lambda^m > 0$. In fluid dynamics one may take U and u^m as the densities of energy and entropy; then λ^m is the temperature and hence positive. The switch between energy and negative entropy is therefore permitted.

We must return to our statement that the initial problem of our conservation Eq. **1** have a unique solution in a neighborhood of the initial surface. The latter qualification is necessary inasmuch as, in general, continuous solutions of nonlinear equations exist only for a limited time interval. Such solutions can, nevertheless, be continued for all future time as discontinuous solutions which satisfy the conservation laws in the integrated sense (**2**). Across a discontinuity surface these integrated laws imply the jump relations

$$[u^j]n_t + [f^{j,k}]n_k = 0, \qquad (7)$$

in which $n = (n_t,n_k)$ is the normal to the discontinuity surface in (t,x) − space and $[f]$ denotes the jump in f across the discontinuity surface in the direction of n. It is well known that discontinuous solutions which satisfy the conservation laws in the form (**7**), in addition to (**1**), are not determined uniquely by their initial values. The physically relevant ones are singled out as the limits of solutions of modified equations that incorporate dissipative forces. We shall indicate how to single out the physically relevant solutions directly, without such a limit process, provided the equations imply an additional convex conservation law.

The modified equations* will be taken in the form

$$u_t^j + \partial_k f^{j,k} = \epsilon\Delta u^j, \qquad \epsilon > 0, \qquad (8)$$

in which $\Delta = \partial_k\partial_k$ is the spatial Laplace operator. The term $\epsilon\Delta u^j$ may be regarded as representing viscous forces. Multiplying Eq. **8** by U_j and summing with respect to j we get

$$\partial_t U + \partial_k F^k = \epsilon U_j\Delta u^j. \qquad (9)$$

From the identity

$$\Delta U = U_j\Delta u^j + U_{j,l}\partial_k u^j\partial_k u^l$$

and the convexity of U, we deduce that

$$\Delta U \geq U_j\Delta u^j.$$

Since ϵ is positive, it follows that the solutions of (**9**) satisfy the inequality

$$U_t + \partial_k F^k \leq \epsilon\Delta U. \qquad (10)$$

We now let ϵ tend to zero and suppose that a sequence of solutions $u^j(x,t,\epsilon)$ of (**8**) tends, as $\epsilon \to 0$, boundedly a.e. to a limit function $u^j(x,t)$. Since then $\epsilon\Delta u^j$ tends to 0 in the sense of distributions, the limit function satisfies the conservation Eq. **1**. At the same time letting $\epsilon \to 0$ in Eq. **10** gives

$$U_t + \partial_k F^k \leq 0. \qquad (11)$$

From this inequality we draw two conclusions concerning the quantities U and F^k as functions of the limit functions $U_l(x,t)$: A) The integral

$$\int U dx \qquad (12)$$

is a decreasing function of t.

B) Across a surface at which U is discontinuous the

* It has been shown in [2] that also other dissipative limiting procedures lead to inequality (**11**).

inequality

$$[U]n_t + [F^k]n_k < 0 \tag{13}$$

holds.

Inequality (**13**) together with the jump condition (**7**), in addition to Eq. **1**, presumably determine the solution of the initial value problem uniquely for all time. Under appropriate circumstances this can be proved to be true. For a more detailed discussion, see a later publication [1].

If the derived conserved quantity U is taken as the energy density, the quantity F^k is the energy flux across the discontinuous surface. Eq. **13** then implies that the energy decreases across such a surface. This may be interpreted by saying that energy is lost by dissipation.

Actually, energy is not lost but is converted into internal energy. In some processes, such as the flow of compressible fluids, one must keep track of this conversion; in these cases the total energy must be taken as one of the original conserved quantities. Negative entropy is then a derived conserved quantity; it is indeed a convex function of the other conserved quantities, as follows from the remark after formula (#). Now, inequality (**13**) stipulates that the entropy increases when the fluid crosses a shock. This entropy inequality has also been suggested by Kružhkov [2].

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