Uniqueness of Solutions to Hyperbolic Conservation Laws

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1. Introduction. We consider strictly hyperbolic systems of conservation laws,

$$u_t + f(u)_x = 0, \qquad -\infty < x < \infty. \tag{1.1}$$

Here $u = u(x, t) \in R^n$ and f is a smooth nonlinear mapping from R^n to R^n. The condition of strict hyperbolicity requires that the Jacobian ∇f of f have n real and distinct eigenvalues:

$$\lambda_1(u) < \cdots < \lambda_n(u).$$

By a *weak solution* we shall mean an element of $L^\infty \cap BV(\Omega)$, $\Omega \subset R^2$, which satisfies (1.1) in the sense of distributions. Here $BV(\Omega)$ denotes the space of functions which have locally bounded total variation in the sense of Cesari; *i.e.* the space of $L^1_{\mathrm{loc}}(\Omega)$ functions whose first order partial derivatives are locally finite Borel measures, *cf.* Section 2.

It is well known that the Cauchy problem for (1.1) does not have in general globally defined smooth solutions; the nonlinear structure of the eigenvalues leads to the development of discontinuities in the solution. On the other hand, uniqueness is lost within the broader class of weak solutions; it is possible for many weak solutions to share the same initial data. Thus the problem arises of identifying the class of stable weak solutions.

In this connection, we recall that a function $\eta : \mathcal{D} \to R$ defined on an open domain $\mathcal{D} \to R^n$ is an *entropy* for (1.1) with *entropy flux* $q : \mathcal{D} \to R$ if all smooth solutions with range in $\mathcal{D}$ satisfy an additional conservation law of the form

$$\eta(u)_t + q(u)_x = 0. \tag{1.2}$$

Friedrichs and Lax [13] have observed that most of the conservative systems which result from continuum mechanics (in one and in several space dimensions) are endowed with a globally defined strictly convex entropy. We mention among others the equations of shallow water waves, fluid dynamics, magneto-fluid dynamics, elasticity in certain special cases and the general symmetric hyperbolic system; these systems are recorded in [9] along with their corresponding entropies and constitutive assumptions. In addition Lax [26] has

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shown how to construct locally defined strictly convex entropies for arbitrary strictly hyperbolic systems (1.1) of two equations and globally defined strictly convex entropies for a broad class of systems (1.1) of two equations.

For the class of systems (1.1) endowed with a strictly convex entropy η, Lax [26] and Kruzkov [24] postulate the following *entropy criterion*: a weak solution u with range in $\mathscr{D}$ is *admissible* if

$$\eta(u)_t + q(u)_x \leq 0 \tag{1.3}$$

in the sense of distributions. For the equations of fluid dynamics, minus the classical entropy density serves as η and minus the classical entropy flux density serves as q. In the setting of fluid dynamics, Equation (1.2) expresses the fact that entropy is conserved in smooth flows: the entropy of each fluid particle remains constant during its motion. The entropy inequality (1.3) expresses the second law of thermodynamics: the entropy of each fluid particle must increase upon crossing a shock front. We note that the entropy criterion has been postulated in order to characterize the stable solutions to a specific class of systems, the class of systems each of whose eigenvalues λ_j is either genuinely nonlinear or linearly degenerate in the sense of Lax [25], *i.e.* for each j either

$$r_j \cdot \nabla\lambda_j \neq 0 \quad \text{or} \quad r_j \cdot \nabla\lambda_j \equiv 0 \tag{1.4}$$

where $r_j = r_j(u)$ denotes the right eigenvector of ∇f corresponding to λ_j. This class of systems includes the aforementioned equations of shallow water waves, fluid dynamics, magneto-fluid dynamics and in certain special cases elasticity. For systems with eigenvalues satisfying (1.4), Lax [26] has shown that, in the case of solutions with moderate oscillation, the entropy criterion is equivalent to the Lax shock conditions [26] which govern the number and type of characteristics impinging on a shock wave. We note that the Lax shock conditions are necessary for the stability of the solution in the linearized sense and that they are designed to single out the stable weak solutions to systems with eigenvalues satisfying (1.4) independently of the existence of a strictly convex entropy. It is known that for arbitrary strictly hyperbolic systems, the entropy criterion is not sufficiently powerful to rule out all unstable solutions and a stronger criterion is needed. We refer the reader to the work of Oleinik [34, 36], Wendroff [43, 44], Dafermos [4, 5] and Liu [27, 28] concerning admissibility criteria for more general systems. In this paper we shall restrict our attention to systems of equations (1.1) with a strictly convex entropy and eigenvalues of the form (1.4) and consider the uniqueness problem for weak solutions satisfying the entropy criterion. We refer the reader to the work of Oleinik [34, 36], Vol'pert [42], Keyfitz [23] and Kruzkov [24] for results on the uniqueness of solutions to scalar conservation laws.

In view of the existence theory for conservation laws, it is natural to pose the uniqueness problem in the space $L^\infty \cap BV$. We recall that the Glimm difference scheme generates globally defined solutions to the Cauchy problem which lie in $L^\infty \cap BV$; convergence has been established by Glimm [14] for general strictly hyperbolic systems of n equations with initial data having small total variation and by several authors [1, 7, 8, 20, 29, 30, 32, 33] for special classes of systems with initial data having large total variation. Furthermore, Lax [26] has shown that the solution constructed by the Glimm scheme satisfies the entropy criterion.

In this framework the uniqueness problem may be formulated as follows. Consider a system of equations (1.1) with a strictly convex entropy and eigenvalues of the form (1.4). Let

$$\mathcal{S}(T) = \{(x, t) : 0 \le t < T\}$$

and let $K = K\{\mathcal{S}(T)\}$ denote the class of admissible weak solutions defined on the strip $\mathcal{S}(T)$. The problem is to prove that if u and v are two solutions in K whose initial data coincide at almost all x then u and v coincide at almost all (x, t) in $\mathcal{S}(T)$. The sense in which a weak solution assumes its initial data is discussed in Section 2. In this paper we consider a somewhat less general problem. We consider the class PL of admissible piecewise Lipschitz solutions and the problem of proving that each solution in PL is unique within K. More precisely, by $PL = PL\{\mathcal{S}(T)\}$ we mean the subclass of solutions in K with the following property: for each t in $[O, T)$ there exists a set of isolated points $x_j = x_j(t)$ such that the restriction $u(\cdot, t)$ of u to each interval (x_j, x_{j+1}) is a Lipschitz function of x; the dependence of the Lipschitz constant on the interval (x_j, x_{j+1}) as well as the dependence of the partition points x_j on t is arbitrary. We note that PL forms a broad subclass of K; PL contains for example the classical piecewise smooth solutions, *i.e.* solutions consisting of isolated shock waves, contact discontinuities, centered and noncentered rarefaction waves and compression waves and their interactions. In particular, PL contains the classical solution of the Riemann problem [25]. We shall comment further on the relationship between PL and K below. We refer the reader to Greenberg [17, 18, 19] for the construction and analysis of interactions in piecewise smooth solutions and to Oleinik [35], Rozhdestvenskii [37], Godunov [16], Hurd [21, 22] and Liu [27, 28] for results which establish that certain types of piecewise smooth solutions are equal if their data are equal. We refer the reader to Douglas [10] and Lyapidevskii [31] for stability results within certain special classes of solutions.

In this paper we first consider the uniqueness problem for genuinely nonlinear systems to two equations. We recall that (1.1) is said to be genuinely nonlinear if all of its eigenvalues are genuinely nonlinear. For such systems we prove that for every state $\tilde{u} \in R^2$ there exists a constant $\delta > 0$ depending only on $\tilde{u}$ and f with the following property. If $w \in PL$, $u \in K$, $|w(\cdot, \cdot) - \tilde{u}|_\infty < \delta$, $|u(\cdot, \cdot) - \tilde{u}|_\infty < \delta$ and $w(x, 0) = u(x, 0)$ for almost all x then $w = u$ for almost all (x, t) in $\mathcal{S}(T)$. Here the L^∞-norm is taken over the strip $\mathcal{S}(T)$. We note that the restriction to solutions with small oscillation is not essential for the argument. By the same method we consider the quasilinear wave equation

$$
(1.5) \qquad \begin{aligned} u_t^1 + p(u^2)_x &= 0 \\ u_t^2 - u_x^1 &= 0 \end{aligned}
$$

with $p' < 0$ and $p'' > 0$ and establish uniqueness of solutions with arbitrarily large oscillation: we prove that if w and u are arbitrary solutions of (1.5) which lie in PL and K respectively and whose initial data coincide at almost all x then w and u coincide at almost all (x, t). Under the hypotheses $p' < 0$ and $p'' > 0$, (1.5) represents a class of genuinely nonlinear systems having a globally defined strictly convex extropy; this class includes the equations of shallow water waves, isentropic fluid dynamics and the equations of certain thin elastic beams. Our interest in (1.5) stems in part from the fact that it serves as the prototype for the broad class of genuinely nonlinear systems of two equations introduced by Smoller and Johnson [41], *cf.* Section 2. In the case of arbitrary genuinely nonlinear systems of two (or more) equations, it will be necessary to supplement the entropy inequality (1.3) with additional inequalities in order to rule out all unstable solutions. It is currently an open problem to determine what these inequalities should be.

We note that the uniqueness problem for conservation laws is a local problem in space-time. We establish the corresponding local version of the theorems above with the aid of the notion of generalized characteristic introduced by Dafermos [6], *cf.* Theorem 6.3. Our approach is applicable in both situations to systems of n equations but it appears that an additional *a priori* estimate will be needed to treat the case $n > 2$.

For systems of equations with linearly degenerate eigenvalues we establish certain preliminary results. We prove, for example, that for systems of two equations and solutions with small oscillation the classical solution to the Riemann problem is unique within K, *cf.* Theorem 6.4.

For general systems the classes PL and K can be compared in terms of the shock sets of their member solutions. We recall that with each function u in BV there is associated a set of points $\Gamma = \Gamma(u)$ at which u experiences an approximate jump discontinuity [11, 42]; in the context of weak solutions the set of jump points $\Gamma(u)$ is referred to as the shock set of the solution u. We can prove that if u is a solution in K of a genuinely nonlinear system of two equations and if Ω is an open domain which does not intersect $\Gamma(u)$, then u is Lipschitz continuous on any compact subdomain of Ω after a possible modification on a set of measure zero. The proof will be published elsewhere. We conjecture that this result generalizes to genuinely nonlinear systems of n equations. Thus, in the case of genuinely nonlinear systems of two equations, PL is essentially the class of admissible weak solutions with isolated shock waves. It would be of interest to establish uniqueness in the case where both u and w contain accumulating shock waves, *i.e.* where both u and w are arbitrary solutions in K.

In Section 3 we establish some preliminary results on the stability of solutions. Consider an arbitrary system of conservation laws (1) which possesses a strictly convex entropy η and let $L = L\{\mathcal{S}(T)\}$ denote the class of Lipschitz solutions defined on $\mathcal{S}(T)$. We prove that solutions in L are L^2-stable relative to perturbations in K: if $w \in L$, $u \in K$ and $0 \leq t < T$ then for every $M > 0$

$$(1.6) \quad \int_{|x| \leq M} |w(x, t) - u(x, t)|^2 dx \leq c_2 \int_{|x| \leq M + c_1 t} |w(x, 0) - u(x, 0)|^2 dx$$

where the constant c_1 depends on f and the L^∞ norms of u and w while the constant c_2 depends on f, T, the L^∞ norms of u and w and Lip w. As before the ranges of u and w are assumed to lie within the domain of definition of η. Strictly speaking, the estimate (1.6) holds if u is replaced by its symmetric mean $\bar{u}$: $\bar{u}$ is obtained from u by mollifying with radically symmetric approximations of the δ-function and is defined H_1 almost everywhere, *cf.* Section 2. Throughout this paper we shall assume that each given weak solution u is replaced by its symmetric mean $\bar{u}$. In particular, we conclude from the uniqueness theorems above that $\bar{u}$ and w coincide at H_1 almost all points (x, t) in $\mathcal{S}(T)$.

We emphasize that the proof of (1.6) does not require any assumptions on system (1) beyond the smoothness of f and the existence of a strictly convex entropy η. We also establish the corresponding version of (1.6) for systems of conservation laws in several space dimensions

$$u_t + \sum_{j=1}^{n} \partial f^j(u)/\partial x_j = 0$$

which are endowed with a strictly convex entropy $\eta : \mathcal{D} \to R$ and entropy-fluxes $q_j : \mathcal{D} \to R$, *i.e.* for systems with the property that all smooth solutions with range in $\mathcal{D}$ satisfy an additional conservation law of the form

$$\eta(u)_t + \sum_{j=1}^{n} \partial q^j(u)/\partial x_j = 0.$$

This class of systems includes the equations of fluid-dynamics, magneto-fluid dynamics and the general symmetric hyperbolic systems in several space dimensions. Returning to systems in one space dimension, we note that in general the solution operator for (1.1) is at best Hölder continuous in L^2. We conjecture that if u and v are arbitrary solutions in $K\{\mathcal{S}(T)\}$ then

$$(1.7) \quad \int_{|x| \leq M} |u(x, t) - v(x, t)|^2 dx \leq c_4 \left\{ \int_{|x| \leq M + c_3 t} |u(x, 0) - v(x, 0)|^2 dx \right\}^{1/2}$$

where the constant c_3 depends only on f and the L^∞ norms of u and v while c_4 depends only on f, T and L^∞ norms of u and v. It would be interesting to study the solution operator in L^p, $1 \leq p < \infty$.

The role of the entropy criterion in conservation laws is illuminated by comparison with other admissibility criteria. In [4, 5] Dafermos postulates an entropy rate criterion which identifies the relevant solutions as those which dissipate entropy at the highest possible rate, *c.f.* Section 7. The entropy rate criterion was introduced by Dafermos for the purpose of dealing with systems having general eigenvalues. We note that the entropy rate criterion is stronger than the entropy criterion: any solution in $L^\infty \cap BV$ which satisfies the entropy rate criterion necessarily satisfies the entropy criterion. It is interesting to consider the converse, *i.e.* to consider a solution in $L^\infty \cap BV$ which satisfies the entropy criterion and determine in what class of solutions it satisfies the entropy rate criterion. In this direction we first prove that for arbitrary systems of n equations with a strictly convex entropy, all Lipschitz continuous solutions satisfy the entropy rate criterion relative to a broad class G_1 of solutions in $L^\infty \cap BV$, *cf.* Theorem 7.1. The class G_1 requires a very mild form of finite propagation speed from its member solutions. In this regard we note that an arbitrary weak solution need not possess the property of finite propagation speed for waves; indeed, it is easy to construct non-constant weak solutions with identically constant initial data. Secondly, we prove that all solutions in PL for the quasilinear wave equation with $p' < 0$ and $p'' > 0$ satisfy the entropy rate criterion relative to a fairly large class G_2 of solutions in $L^\infty \cap BV$, *cf.* Theorem 7.2. We note that Theorem 7.2 should be regarded as a preliminary result; we expect that G_2 can be substantially enlarged.

Next, we shall comment on certain aspects of the uniqueness proof. Consider a system of n conservation laws (1.1) with a strictly convex entropy $\eta : \mathcal{D} \to R$ and entropy flux $q : \mathcal{D} \to R$. Let (u, v) denote an arbitrary pair of weak solutions defined on some region Ω with range contained in $\mathcal{D}$. We estimate the distance between u and v in L^2 of the space variable. For this purpose we associate with each pair of weak solutions (u, v) a Borel measure $\gamma = \gamma(u, v)$ defined on Ω as follows: γ is the divergence of a vector-field (α, β) obtained from the quadratic part of η,

$$\gamma = \alpha(u, v)_t + \beta(u, v)_x$$

where $u = u(x, t)$, $v = v(x, t)$ and α and β are the mappings

$$\alpha(u, v) = \eta(u) - \eta(v) - \nabla\eta(v)(u - v) : \mathcal{D} \times \mathcal{D} \to R$$

$$\beta(u, v) = q(u) - q(v) - \nabla\eta(v)\{f(u) - f(v)\} : \mathcal{D} \times \mathcal{D} \to R.$$

We note that α and β form an n-parameter family of strictly convex entropies $\alpha(\cdot, v)$ and entropy fluxes $\beta(\cdot, v)$ with $v \in \mathcal{D}$ serving as the parameter; as observed by Dafermos [3] for each fixed $v \in \mathcal{D}$, the quadratic part of η at v, *i.e.* α, is an entropy in u with entropy flux β. This observation is also useful in studying the large-time decay of solutions [9].

According to the generalized Green's theorem for measures [11, 42], the γ-measure of (the density points of) a set with finite parameter equals the flux of the vector-field

$$\{\alpha(u(x, t), v(x, t)), \beta(u(x, t), v(x, t))\}$$

across the essential boundary of the set. In the case where u and v are defined on the strip $\mathscr{S}(T)$ and have compact support in x,

$$(1.8) \quad \gamma\{\mathscr{S}(T)\} = \int_{-\infty}^{\infty} \alpha(u(x, t), v(x, t))dx - \int_{-\infty}^{\infty} \alpha(u(x, 0), v(x, 0))dx$$

for $0 \leq t < T$. The integral of α at time t is equivalent to the square of the spatial L^2 norm of $u - v$ by virtue of the strict convexity of η. Given a pair of solutions $u \in K\{\mathscr{S}(T)\}$ and $w \in PL\{\mathscr{S}(T)\}$ whose initial data coincide at almost all x, we prove uniqueness by applying (1.8) with $v = \tilde{w}$ where $\tilde{w}$ is an (apparent) modification of w and then establishing a singular integral inequality for

$$(1.9) \qquad \int_{-\infty}^{\infty} \alpha(u(x, t), \tilde{w}(x, t))dx$$

where the singularity is nonintegrable but mild relative to the rate at which (1.9) approaches zero as t approaches zero; it follows that $u = \tilde{w} = w$ at almost all (x, t) in $\mathscr{S}(T)$. We note that the integral of α does not satisfy a standard Gronwall inequality since the solution operator is at best Hölder continuous in L^2. The Hölder continuity of the solution operator is also reflected in the fact that γ is not absolutely continuous with respect to 2-dimensional Lebesgue measure; the support of γ contains sets with finite 1-dimensional Hausdorff measure whose γ-measure depends on the associated pointwise values of the solutions. The corresponding local uniqueness problem is treated by applying γ to appropriate trapezoidal regions of the $x - t$ plane bounded by characteristics in the sense of Dafermos.

In the case of arbitrary pairs of weak solutions (u, v), one may regard γ as measuring the dissipation and dispersion in the solution u relative to the solution v. The γ-measure of an arbitrary Borel set B is determined by the amount of entropy which u dissipates in B relative to v, by the geometry of the joint shock set $\Gamma(u, v) \equiv \Gamma(u) \cup \Gamma(v)$ and by the rates of focusing and spreading of characteristics in u and v. The connection between those quantities is expressed through certain decompositions of γ. The primary decomposition is obtained by restricting γ to $\Gamma(u, v)$ and to the complementary region $\Gamma^c(u, v)$ of approximately continuous flow:

$$\gamma = \gamma|_{\Gamma(u,v)} + \gamma|_{\Gamma^c(u,v)}.$$

We shall discuss below the structure of each of these restrictions and their relationship to the stability of shock waves and rarefaction waves. We shall do so first in the general case where (u, v) forms an arbitrary pair of weak solutions and then describe the application of these results to the case where $u \in K$ and $v \in PL$.

Several preliminary remarks are in order concerning the measurement of dissipation in a weak solution. By a *shock wave* in a weak solution u we shall mean a Borel measurable subset of $\Gamma(u)$. If E is a shock wave then the total entropy dissipated by E is given by $\theta_u(E)$ where

$$\theta_u \equiv \eta(u)_t + q(u)_x.$$

We shall refer to a shock wave E as *admissible* if the restriction of the measure θ_u to E is non-positive, *i.e.* if $\theta_u(B) \leq 0$ for all Borel subset $B \subset E$. In general, the θ_u-measure of a Borel set B equals the total entropy dissipated by all of those shock waves of u which lie in B; it is not difficult to prove that θ_u is concentrated on $\Gamma(u)$:

$$\theta_u(B) = \theta_u(B \cap \Gamma(u)).$$

The relationship between the geometry of a shock wave E and the amount of entropy which it dissipates is expressed by Green's theorem [11, 42].

$$\theta_u(E) = \int_E \nu_t[\eta] + \nu_x[q]dH_1. \tag{1.10}$$

Here H_1 denotes 1-dimensional Hausdorff measure, $\nu = (\nu_x, \nu_t)$ denotes the unit normal to E in the sense of Federer [11] and bracket denotes the jump in the enclosed quantity across E in the direction of ν:

$$[\eta] = [\eta](P) = \eta\{\ell_\nu u(P)\} - \eta\{\ell_{-\nu}u(P)\}$$

where $\ell_{\pm\nu}u(P)$ denote the approximate limits of u at $P \in E$ with respect to the half-planes

$$H_{\pm\nu}(P) = \{Q : \langle Q - P, \pm \nu\rangle \geq 0\}.$$

We shall normalize the direction of ν by requiring that $\nu_x < 0$ and refer to

$$u_\ell \equiv \ell_\nu u \quad \text{and} \quad u_r \equiv \ell_{-\nu}u$$

as the left and right hand approximate limits of u at P. With this convention, bracket denotes the jump from left to right across a shock. We note that in an arbitrary weak solution $\nu_x \neq 0$ at H_1 almost all points of $\Gamma(u)$.

It follows from (1.10) that the rate of dissipation of entropy at $P \in E$ is given by

$$\tau[\eta] - [q]$$

where $\tau \equiv -\nu_t(P)/\nu_x(P)$ denotes the speed of propagation of E at P.

Indeed

$$\theta_u(E) = \int_E \tau[\eta] - [q]dt$$

where $dt \equiv -\nu_x dH_1$. In the proof of uniqueness an important role is played by the function

$$d : R \times R^{2n} \to R$$

$$d(\tau, u_\ell, u_r) = \tau[\eta] - [q].$$

We shall refer to d as the *dissipation function* for u and to θ_u as the *dissipation measure*.

More generally, one may regard the restriction of $\gamma(u, v)$ to the shock set $\Gamma(u)$ as measuring the dissipation in u relative to the solution v, *cf.* Sections 3 and 4. If E is a shock wave in u then the γ-measure of E is influenced by the limiting values of v along E. It follows from Green's theorem that

$$\gamma(E) = \int_E D(\tau, u_\ell, u_r, v_\ell, v_r)dt$$

where v_ℓ and v_r denote the approximate left and right hand limits of v along E and D denotes the mapping

$$D : R \times R^{2n} \times R^{2n} \to R$$

$$D(\tau, u_\ell, u_r, v_\ell, v_r) = \tau[\alpha] - [\beta]$$

where $[\alpha] = \alpha(u_\ell, v_\ell) - \alpha(u_r, v_r)$, *etc.* In general, the restriction of γ to the joint shock set $\Gamma(u, v)$ is determined by the relative dissipation of entropy and by the geometry of waves in $\Gamma(u)$ relative to those in $\Gamma(v)$. We show that the restriction of γ to $\Gamma(u, v)$ admits a decomposition into the relative dissipation measure $\theta_u - \theta_v$ and a measure μ which involves the relative speeds of propagation of waves in $\Gamma(u)$ and $\Gamma(v)$. We study the decomposition

$$\gamma|_{\Gamma(u,v)} = \theta_u - \theta_v + \mu$$

in order to understand how the relative dissipation and relative speed of propagation of pairs of shock waves (one in u the other in v) influence the time evolution of the spatial L^2 norm of $u - v$. For this purpose we associate with each shock wave S_u in a weak solution u a subset $\mathcal{S}_u$ of the product of space-time with a state space of dimension $2n + 1$:

$$\mathcal{S}_u \equiv \{(x, t, \tau(x, t), u_\ell(x, t), u_r(x, t) : (x, t) \in S_n\} \subset R^2 \times R^{2n+1} \equiv \mathscr{E}.$$

Given a pair of weak solutions (u, v) we consider sets of the form

$$\mathcal{S}_u \times \mathcal{S}_v \subset \mathscr{E} \times \mathscr{E}$$

together with their "projections"

$$P_2(\mathcal{S}_u) \cup P_2(\mathcal{S}_v) \quad \text{and} \quad P_{2n+1}(\mathcal{S}_u) \times P_{2n+1}(\mathcal{S}_v)$$

onto R^2 and $R^{2n+1} \times R^{2n+1}$ respectively. Here,

$$P_2 : \mathscr{E} \to R^2 \quad \text{and} \quad P_{2n+1} : \mathscr{E} \to R^{2n+1}$$

denote projection onto the first two and last $2n + 1$ variables respectively, *e.g.* $P_2(\mathcal{S}_u) = S_u$. We first note that the γ-measure of pairs of shock waves $S_u \cup S_v$ does not have a distinguished sign even if S_u and S_v are admissible. However we show that if S_u and S_v are admissible then the projections

$$P_{2n+1}(\mathcal{S}_u) \times P_{2n+1}(\mathcal{S}_v)$$

lie near a special class of subsets of $R^{2n+1} \times R^{2n+1}$ for which the time derivative D of γ is non-positive. In order to describe this special class it is convenient to introduce the projection

$$P_{4n+1} : R^{2n+1} \times R^{2n+1} \to R^{4n+1}$$

$$P_{4n+1}(\tau, u_\ell, u_r; \sigma, v_\ell, v_r) = (\tau, u_\ell, u_r, v_\ell, v_r),$$

where σ denotes the speed of shock waves in v. We associate with each point of $P_{2n+1}(\mathcal{S}_u)$ and $P_{2n+1}(\mathcal{S}_v)$ rates of dissipation $d(\tau, u_\ell, u_r)$ and $d(\sigma, v_\ell, v_r)$ and we associate with each point

$$(\tau, u_\ell, v_\ell, v_r) \in P_{4n+1}\{P_{2n+1}(\mathcal{S}_u) \times P_{2n+1}(\mathcal{S}_v)\}$$

the rate $D(\tau, u_\ell, u_r, v_\ell, v_r)$. Then using the decomposition (1.11) we show that D is determined by the relative rate of dissipation and the relative speed of propagation $\tau - \sigma$ modulo an error term:

(1.11) $D(\tau, u_\ell, u_r, v_\ell, v_r) = d(\tau, u_\ell, u_r) - d(\sigma, v_\ell, v_r) \pm \alpha(v_\ell, v_r)(\tau - \sigma) + \delta_\pm$

where the error terms δ_+ and δ_- vanish if $u_r = v_r$ and $u_\ell = v_\ell$ respectively, *cf.* Section 4. We shall refer to a point

$$A = (\tau, u_\ell, u_r; \sigma, v_\ell, v_r) \in R^{2n+1} \times R^{2n+1}$$

as *attracting* if u_ℓ and u_r as well as v_ℓ and v_r are connected by admissible shock waves of the same kind such that either $u_r = v_r$ or u_ℓ, *cf.* Section 4. We shall call a subset of $R^{2n+1} \times R^{2n+1}$ attracting if each of its points is attracting. We show that if A is an attracting point then

(1.12) $D\{P_{4n+1}(A)\} \leq -\text{const.}\ |u_\ell - v_\ell|^2$ or $D\{P_{4n+1}(A)\} \leq -\text{const.}\ |u_r - v_r|^2$

accordingly as $u_r = v_r$ or $u_\ell = v_\ell$. The properties of the constants are discussed in Section 4. Thus, at an attracting point D reduces to the quadratic part of a concave function d. We then show that the projections $P_{2n+1}(\mathcal{S}_u)$ and $P_{2n+1}(\mathcal{S}_v)$ lie near the class of attracting sets if $P_2(\mathcal{S}_u)$ and $P_2(\mathcal{S}_v)$ are admissible. Both facts taken together express the stability of admissible shock waves relative to small perturbations. We note that corresponding results hold for shock waves in several space dimensions.

In order to relate the γ-measure of pairs of shock waves $S_u \cup S_v$ to the values of its time derivative D on sets of the form

$$P_{4n+1}\{P_{2n+1}(\mathcal{S}_u) \cup P_{2n+1}(\mathcal{S}_v)\}$$

we proceed as follows. Let us suppose that u is an arbitrary weak solution defined on Ω and that v is an arbitrary function on $L^\infty \cap BV(\Omega)$. For convenience we shall refer to any Borel subset of $\Gamma(v)$ as a shock wave in v, *etc.* Now, in the case where a pair of shock waves S_u and S_v are equal, *i.e.* occupy the same position in space-time, we note that

$$\gamma(S_u \cup S_v) = \gamma(S_u) \equiv \int_{S_u} D(\tau, u_\ell, u_r, v_\ell, v_r)dt$$

and that

$$\gamma(S_u \cup S_v) \leq 0$$

if $P_{2n+1}(\mathcal{S}_u) \times P_{2n+1}(\mathcal{S}_v)$ is an attracting set. In general one is presented with pairs of shock waves S_u and S_v which are not equal, but the effect of the projection P_{4n+1} is to deform the wave S_v, so to speak, until it occupies the same position in space-time as S_u. Given a pair of solutions $u \in K$ and $w \in PL$, the operator P_{4n+1} may be realized as a mapping which associates with the pair (u, w) an approximate solution $\tilde{w}$. The function $\tilde{w}$ is obtained by continuing the solution w across the sides of its shock waves *via* the solution operator of the Cauchy problem. The continuation process leads to a two-sheeted surface which contains the solution surface associated with w. The function $\tilde{w}$ is obtained from this two-sheeted extension by making a single-valued selection according to P_{4n+1} in such a way that the dominant waves of u and $\tilde{w}$ occupy the same position in space-time. Using (1.12) and the fact that the projection $P_{2n+1}(\mathcal{S}_u) \times P_{2n+1}(\mathcal{S}_{\tilde{w}})$ lies near the class of attracting sets we show that

$$\gamma(u, \tilde{w})\{\Gamma(\tilde{w})\} \leq 0. \tag{1.13}$$

The estimate (1.13) is the first of the two main steps in the derivation of the singular integral inequality for the integral of $\alpha(u, \tilde{w})$.

Next, we shall discuss the connection between γ and the stability of rarefaction waves. As we remarked above, one may regard the restriction of γ to the shock set $\Gamma(u)$ as measuring the dissipation in u relative to the solution v. In general, if $E \subset \Gamma(u)$ then $\gamma(E)$ depends upon the limiting values of v along E. However, if v is nearly stationary on E in the sense that v is approximately continuous on E (as opposed to having a jump discontinuity) then the γ-measure of E reduces to the θ_u-measure of E, *cf.* Lemma 3.1: if

$$E \cap \Gamma(v) = \varnothing$$

then

$$\gamma(E) = \theta_u(E).$$

This reduction property is essentially equivalent to the fact that γ is expressed as the divergence of the vector-field obtained from the quadratic part of η. More generally, if the solution v is approximately continuous on an arbitrary Borel set B then the restriction of γ to B splits into two mutually singular measures, the dissipation measure θ_u and a measure $\pi = \pi(u, v)$ which couples u and v quadratically with a weight depending upon the geometry of characteristics in v:

$$\gamma|_B = \theta_u|_B + \pi(u, v) \tag{1.14}$$

where

$$\pi(u, v) = q(u, v)v_x|_B$$

and $q = O(u - v)^2$. The L^2-stability of Lipschitz solutions is an immediate consequence of the decomposition (1.14): if $w \in L\{\mathcal{S}(T)\}$ and $u \in K\{\mathcal{S}(T)\}$ then

$$\gamma(u, w) \leq \pi(u, w) \leq \text{const.}\ |u - w|^2 dxdt$$

since $w_x \in L^\infty$ and a standard Gronwall inequality follows for the integral of $\alpha(u, w)$. An analogous result holds in several space dimensions.

If the solution v is merely approximately continuous on B then the strength of the coupling between u and v as measured by γ depends upon the rate of focusing of characteristics in v. For simplicity we shall comment on the situation where v is a similarity solution $v = v(x/t)$ consisting of a centered j-rarefaction wave separating two constant states; in this case

$$V_x = \begin{Bmatrix} 0 & \text{if } x/t \leq a \\ r_j(v(x/t))/t & \text{if } a \leq x/t \leq b \\ 0 & \text{if } b \leq x/t \end{Bmatrix}.$$

We first note that in the case of a single genuinely nonlinear equation, formal arguments using characteristics indicate that the solution operator for admissible solutions is Lipschitz continuous at those data points which generate centered rarefaction waves. Indeed, it follows from the results of Section 6 that if u and v are two solutions of a single genuinely nonlinear equation with $u \in K\{\mathcal{S}(T)\}$ and v consisting of a centered rarefaction wave separating two constant states then

$$\int_{-\infty}^{\infty} |u(x, t) - v(x, t)|^2 ds \leq \text{const.} \int_{-\infty}^{\infty} |u(x, 0) - v(x, 0)|^2 dx \tag{1.5}$$

where the constant depends only on f and the L^∞-norm of u and v. Although (1.15) does not hold in general for systems of equations it leads one to conjecture that the coupling between u and v as measured by γ on a centered j-rarefaction wave in v depends on the orientation of the vectors $u(x, t) - v(x, t) \in R^n$ and that favorable coupling occurs in the direction corresponding to the eigenvector r_j. This turns out to be the case, *cf.* Section 6, and we deduce that

$$\gamma \leq \theta_u + \frac{\text{const.}}{t} \sum_{k \neq j} c_k^2 dxdt + \text{const.}\ |u - v|^2 dxdt \tag{1.16}$$

where the components c_k are defined by

$$u - v = \sum c_k r_k(v).$$

Thus, while the rate of focusing of characteristics in v produces the non-integrable term $1/t$ the only coupling coefficients which enter are those which correspond to the complementary directions r_k, $k \neq j$. We then show using a specially constructed family of non-convex entropies that if u and v are solutions to a system of two equations with the same initial data then the dissipation measure θ_u nearly balances the second term of (1.16) leading to a singular integral inequality with a factor of the form ϵ/t where ϵ is small.

2. Preliminaries. We shall begin with several remarks concerning the classes of solutions and systems which we shall treat in subsequent sections. We recall that a real valued function $u = u(y)$ defined on a region $\Omega \subset R^n$ is an element of $BV(\Omega)$ if it is locally integrable and if its gradient is a locally finite Borel measure:

$$\int u\nabla\phi dxdt = -\int \phi d\mu \tag{2.1}$$

for all $\phi \in C_0^\infty(\Omega)$ where μ is a Borel measure on Ω such that

$$|\mu(\Omega')| < \infty \tag{2.2}$$

for all compact subsets $\Omega' \subset \Omega$. More generally, a function u defined on $\Omega \subset R^n$ with values in R^n is an element of $BV(\Omega)$ if each of its components satisfies (2.1) and (2.2). We note that $u \in BV(\Omega)$, $\Omega \subset R^2$ if and only if the local total variation in x is locally integrable in t while the local total variation in t is locally integrable in x. For example, $u \in BV(\mathscr{S}(T))$, if and only if

$$\int_0^T \underset{(-M,M)}{TV}\ u(\cdot, t)dt < \infty \quad \text{and} \quad \int_{-M}^{M} \underset{(0,T)}{TV}\ u(x, \cdot)dx < \infty \tag{2.3}$$

for all M. A similar result holds for BV functions on $\Omega \subset R^n$.

As we remarked in the introduction, it is convenient to replace a given weak solution u by its symmetric mean $\bar{u}$:

$$\bar{u} \equiv \lim_{n\to\infty} \psi_n * u$$

where $\{\psi_n\}$ is a sequence of radially symmetric functions approximating the δ-function [42]. We recall that $\bar{u}$ is defined H_1 almost everywhere and that u and $\bar{u}$ coincide H_2 almost everywhere. If u is a weak solution defined on $\mathscr{S}(T)$ then it follows from the form of the system (1.1) that for all N, $\bar{u}$ is an absolutely continuous function of t with values in $L^1(-N, N)$. In particular, the weak solution $\bar{u}$ assumes its initial data in L^1_{loc}. We shall assume throughout that we are dealing with the symmetric mean of the solution, *i.e.* we shall assume that $u = \bar{u}$.

When studying solutions with large oscillation, we shall restrict our attention to systems in the Smoller-Johnson class [41]. By the Smoller-Johnson class we mean the class of genuinely nonlinear systems of two equations with the following properties:

$$\mathrm{A}_1 \qquad \frac{\partial f^1}{\partial u^2}\cdot\frac{\partial f^2}{\partial u^1} < 0$$

$$\mathrm{A}_2 \qquad \ell_j \cdot \nabla^2 f(r_k, r_k) > 0 \quad \text{if} \quad k \neq j$$

A_3 For each v,

$$\{u : (u^2 - v^2)[f^1(u) - f^1(v)] = (u^1 - v^1)[f^2(u) - f^2(v)]\} \text{ is connected.}$$

Here $u = (u^1, u^2), f = (f^1, f^2)$ and ℓ_j and r_j denote the left and right eigenvectors of ∇f corresponding to λ_j normalized so that

$$r_j \cdot \nabla\lambda_j > 0, \qquad \ell_j \cdot r_j > 0.$$

We note that the Smoller-Johnson class contains the quasilinear wave equation (1.5) if $p' > 0$ and $p'' < 0$.

For systems in the Smoller-Johnson class, it is not difficult to show that the Lax shock conditions are equivalent to the entropy inequality: if the triple (τ, u_ℓ, u_r) satisfies the Rankine-Hugoniot relations,

$$R(\tau, u_\ell, u_r) \equiv \tau[u] - [f] = 0$$

then

$$\tau[\eta] - [q] < 0$$

if and only if

$$\lambda_j(u_\ell) < \tau < \lambda_j(u_r) \tag{2.4}$$

for some j. Here $[u] = u_\ell - u_r$, *etc.* We note that under the assumptions A_1–A_3, condition (2.4) is equivalent to the full Lax shock conditions, *i.e.* if $j = 1$ then (2.4) implies that $\tau < \lambda_2(u_r)$ and if $j = 2$, then (2.4) implies that $\tau > \lambda_1(u_\ell)$.

The assumptions A_1–A_3 guarantee certain basic properties for the system (1.1). Assumption A_1 implies that (1.1) is strictly hyperbolic. Assumption A_2 is equivalent to the Glimm-Lax condition that the interaction of two shocks of the same kind produces a shock of that kind and a centered rarefaction wave of the opposite kind. Assumption A_3 together with A_1 and A_2 implies that for each v the projection of the set of all solutions (τ, u, v) of the Rankine-Hugoniot relations onto the $u_1 - u_2$ plane consists of four globally defined smooth curves emanating from v *cf.* [41]. Two of the curves (usually referred to as shock wave curves) consist of the projection of states (τ, u_ℓ, u_r) which satisfy (2.4); the other two consist of the projection of states (τ, u_ℓ, u_r) which satisfy the reverse inequality

$$\lambda_j(u_r) < \tau < \lambda_j(u_\ell).$$

For additional background we refer the reader to the work of Smoller [38, 39, 40] concerning the Riemann problem for systems of the above type.

In this paper we employ the concept of a generalized characteristic introduced by Dafermos [6]. A (generalized) j-characteristic in a weak solution u is defined as a trajectory of the equation

$$\dot{x} = \lambda_j\{u(x, t)\} \tag{2.5}$$

where (2.5) is interpreted in the sense of Fillipov [12]. Thus, a j-characteristic is a Lipschitz continuous curve $(x(t), t)$ whose speed of propagation $\dot{x}(t)$ lies for almost all t between the essential minimum and the essential maximum of $\lambda_j(u(\cdot, t))$ at the point $(x(t), t)$. We also employ the notions of the minimal and maximal forward j-characteristic through a point (x_0, t_0) which are defined as the lower and upper envelopes of the set of all solutions to (2.5) in the sense of Fillipov which pass through (x_0, t_0).

3. Lipschitz solutions.

In this section we shall establish the L^2 stability of Lipschitz solutions and a form of finite propagation speed for waves. In this connection we recall that if the initial data $w_0(x)$ are Lipschitz continuous then the Cauchy problem for (1.1) has a solution w which is defined and Lipschitz continuous on a strip $\mathcal{S}(T)$ where T depends only on f, $|w_0|_\infty$ and Lip w_0. We shall assume throughout that system (1.1) is endowed with a smooth entropy-entropy flux pair (η, q) where η is strictly convex and that the range of each solution considered lies within the domain of definition of η and q unless otherwise stated.

Theorem 3.1. *Suppose that $w \in L\{\mathcal{S}(T)\}$ and $u \in K\{\mathcal{S}(T)\}$. If $0 \leq t < T$ then*

$$(3.1) \quad \int_{|x| \leq M} |u(x, t) - w(x, t)|^2 dx \leq c_2 \int_{|x| \leq M + c_1 t} |u(x, 0) - w(x, 0)|^2 dx$$

where the constant c_1 depends on f and the L^∞-norms of u and w while the constant c_2 depends on f, T, the L^∞-norms of u and w and Lip $w(x, 0)$.

After proving Theorem 3.1 we shall establish the following more refined version of finite propagation speed for waves. Suppose that $u \in K$ and that the restriction of its initial data $u(x, 0)$ to the interval (a, b) is equal almost everywhere to a Lipschitz function $\tilde{u}_0$. Let $x_M^n(t)$ denote the maximal forward n-characteristic through $(a, 0)$ and let $x_m^1(t)$ denote the minimal forward 1-characteristic through $(b, 0)$.

Theorem 3.2. *There exists a constant $T > 0$ and a solution $\tilde{u}$ which is defined and Lipschitz continuous on*

$$\{(x, t) : x_M^n(t) \leq x \leq x_m^1(t), 0 \leq t < T\}$$

such that $\tilde{u}(x, 0) = \tilde{u}_0(x)$ for all x and $\tilde{u}(x, t) = u(x, t)$ almost everywhere in x for each t in $[0, T)$. Here, the constant T depends only on f, $|\tilde{u}_0|_\infty$ and Lip $\tilde{u}_0$.

We shall begin by studying the restriction of $\gamma(u, v)$ to the set $\Gamma^c(v)$ in the general setting where u and v are arbitrary weak solutions. For this purpose we shall consider the mappings

$$d : R \times R^n \rightarrow R \quad \text{and} \quad D : R \times R^{2n} \times R^{2n} \rightarrow R$$

given by

$$D(\tau, u_\ell, u_r) \equiv \tau(\eta) - [q] = \tau[\eta(u_\ell) - \eta(u_r)\} - q(u_\ell) + q(u_r)$$

$$D(\tau, u_\ell, u_r, v_\ell, v_r) \equiv \tau[\alpha] - [\beta]$$
$$= \tau\{\alpha(u_\ell, v_\ell) - \alpha(u_r, v_r)\} - \beta(u_\ell, v_\ell) + \beta(u_r, v_r).$$

We shall say that two states u_ℓ and u_r in R^n are *connected by a shock wave with speed* τ if the Rankine-Hugoniot relations are satisfied, *i.e.*, if

$$R(\tau, u_\ell, u_r) \equiv \tau(u_\ell - u_r) - f(u_\ell) + f(u_r) = 0.$$

The following lemma and corollary establish the reduction property for γ.

Lemma 3.1. *If u_ℓ and u_r are connected by a shock wave with speed τ then for all v*

$$D(\tau, u_\ell, u_r, v, v) = d(\tau, u_\ell, u_r).$$

Corollary 3.1. *If u and v are arbitrary weak solutions and $E \subset \Gamma(u) \cap \Gamma^c(v)$ then*

$$\gamma(E) = \theta_u(E). \tag{3.2}$$

Proof of Lemma 3.1 and Corollary 3.1. It follows from the definitions that

$$D(\tau, u_\ell, u_r, v, v) = d(\tau, u_\ell, u_r) - \nabla\eta(v)R(\tau, u_\ell, u_r).$$

Therefore

$$\gamma(E) = \int_E D(\tau, u_\ell, u_r, v, v)dt = \int_E d(\tau, u_\ell, u_r)dt = \theta_u(E)$$

if $E \subset \Gamma(u) \cap \Gamma^c(v)$.

The restriction of γ to domains of approximate continuity for both u and v is quadratic in the difference $u - v$ with a weight depending on the geometry of characteristics in v.

Lemma 3.2. *Suppose that u and v are arbitrary weak solutions. If $E \subset \Gamma^c(u) \cap \Gamma^c(v)$ then*

$$\gamma(E) = -\iint_E \nabla^2\eta(v)\, Qf(u, v)v_x \tag{3.3}$$

where Qf denotes the quadratic part of f at v;

$$Qf = f(u) - f(v) - \nabla f(v)(u - v).$$

Proof. We note that (η, q) is an entropy-entropy flux pair if and only if

$$\nabla\eta\, \nabla f = \nabla q. \tag{3.4}$$

Indeed, if u is a smooth solution then

$$\eta(u)_t + q(u)_x = \nabla\eta\, u_t + \nabla q\, u_x = (-\nabla\eta\,\nabla f + \nabla q)u_x$$

and the right hand side vanishes for all smooth u if and only if (3.4) holds. Now, restricting γ to the set $\Gamma^c(u) \cap \Gamma^c(v)$ we obtain by the chain rule [42]

$$\begin{aligned}\gamma = \alpha_t + \beta_x &= \alpha_u u_t + \alpha_v v_t + \beta_u u_x + \beta_v v_x \\ &= \{\beta_u - \alpha_u \nabla f(u)\}u_x + \{\beta_v - \alpha_v \nabla f(v)\}v_x.\end{aligned}$$

The coefficient of the measure u_x vanishes since α and β form an entropy-entropy flux pair in u and a short calculation shows that the coefficient of v_x equals

$$-\{f(u) - f(v)\}\nabla^2\eta(v) + (u - v)\nabla^2\eta(v)\nabla f(v).$$

Since the matrix $\nabla^2\eta \nabla f$ is symmetric [13], the coefficient of v_x may be rewritten in the desired form

$$-\{f(u) - f(v) - (u - v)\nabla f^t(v)\}\nabla^2\eta(v).$$

This completes the proof of Lemma 3.2.

We conclude from Corollary 3.1 and Lemma 3.2 that the restriction of γ to shock-free domains in v may be decomposed into the mutually singular sum of the dissipation measure θ_u and the measure $-\nabla^2\eta\, Q\, v_x$, *i.e.* if $E \subset \Gamma^c(v)$ then

$$\gamma(E) = \theta_u(E) - \iint_E \nabla^2\eta(v)\, Qf(u, v)v_x. \tag{3.4}$$

Proof of Theorem 3.1. Fix $M > 0$ and consider domains of the form

$$\Omega(t) = \{(x, s) : |x| \le \ell(s), 0 \le s \le t\}, \qquad 0 \le t < T,$$

where $\ell(s) = M + c_1(T - s)$ and the constant c_1 will be chosen below. It follows from Green's theorem for measures [11, 42] that

$$\begin{aligned}\gamma\{\Omega(t)\} &= \int_{\partial\Omega} \nu_\tau\alpha + \nu_x\beta\, d H_1 \\ &= \int_{\partial\Omega_\ell} \nu_t\alpha + \nu_x\beta\, d H_1 + \int_{|x| \le \ell(t)} \alpha(t)dx - \int_{|x| \le \ell(0)} \alpha(0)dx.\end{aligned}$$

Since η is strictly convex, the ratio β/α is bounded if the arguments lie in a compact set and we may choose the constant c (depending only on f and the L^∞-norm of u and w) so large that

$$\nu_t\alpha + \nu_x\beta = \alpha(\nu_t + \nu_x\beta/\alpha) \ge 0$$

on the lateral boundary $\partial\Omega_\ell$. Now, it follows from (3.4) that

$$\gamma \le -\nabla^2\eta\, Q\, w_x \le \text{const.}\, |u - w|^2 dxdt$$

and we conclude that the function

$$\phi(t) \equiv \int_{|x| \le \ell(t)} \alpha(t)dx$$

satisfies an integral inequality of the form

$$\phi(t) \leq \phi(0) + \text{const.} \int_0^t \phi(t)dt.$$

This completes the proof of Theorem 3.1.

The proof of Theorem 3.2 is somewhat technical and may be read after reading the remaining sections of the paper. In order to prove Theorem 3.2 we shall establish a preliminary lemma. Suppose that $u \in K\{\mathcal{S}(T)\}$ and that its initial data $u(x, 0)$ coincides almost everywhere on $(-\infty, 0)$ with a Lipschitz continuous function $\tilde{u}_0$. Consider the Cauchy problem with initial data of the form

$$w_0(x) = \begin{Bmatrix} \tilde{u}_0(x) & \text{if} \quad x \leq 0 \\ \tilde{u}_0^- & \text{if} \quad x \geq 0 \end{Bmatrix}$$

where

$$\tilde{u}_0^- = \lim_{x \uparrow 0} \tilde{u}_0(x).$$

This problem has a solution w which is defined and Lipschitz continuous in some strip $\mathcal{S}(T_w)$. We shall show that for each $\delta > 0$ the solutions u and w coincide almost everywhere to the left of the ray

$$x(t) = (\lambda^- - \delta)t$$

where $\lambda^- \equiv \lambda_1(\tilde{u}_0^-)$ provided that t is sufficiently small and that u has the appropriate limiting behavior at the origin. This is made precise as follows. Let

$$u_\delta(t) = u\{(\lambda^- - \delta)t - 0, t\}.$$

We note that u_δ exists for almost all t since the restriction $u(\cdot, t)$ of u to almost all lines $t = \text{const.}$ is a classical function of bounded variation having limits $u(x_0 \pm 0, t)$ at each point x_0.

Lemma 3.3. *If*

$$\text{ess} \lim_{t \to 0} u_\delta(t) = \tilde{u}_0^-$$

then there exists a constant $T > 0$ depending only on u and δ such that u and w coincide at almost all points (x, t) satisfying

$$x < (\lambda^- - \delta)t, \qquad 0 \leq t < T.$$

Remark. If the initial data $u(x, 0)$ coincides at almost all x with a Lipschitz function $\tilde{u}_0$, defined on $(0, \infty)$ then an analogous result holds for rays propagating at speeds slightly faster than $\lambda_n\{\tilde{u}_0^+\}$ where

$$\tilde{u}_0^+ = \lim_{x \downarrow 0} \tilde{u}_0(x).$$

We recall that

$$\operatorname{ess}\lim_{t\to 0} u_\delta(t) = \bar{u}_0^-$$

if and only if for every $\epsilon > 0$ there is t_ϵ such that

$$\{t : |t| \le t_\epsilon \quad \text{and} \quad |u_\delta(t) - \bar{u}_0^-| > \epsilon\}$$

has Lebesgue measure zero.

Proof of Lemma 3.3. Applying Green's theorem to the regions

$$\Omega(t) = \{(x, s) : x < x(s),\ 0 \le s < t\}$$

yields the identity

$$\int_{-\infty}^{x(t)} \alpha(t)dx + g(t) = \gamma\{\Omega(t)\}$$

where

$$g(t) \equiv -\int_0^t (\lambda^- - \delta)\alpha\{u_\delta(t), w(x(t), t)\} - \beta\{u_\delta(t), w(x(t), t)\}dt.$$

If we prove that $g(t)$ is non-negative for small t then it follows from (3.4) that the integral of α satisfies a Gronwall inequality. Thus, we need only show that

$$p(u, v) \equiv (\lambda^- - \delta)\alpha(u, v) - \beta(u, v)$$

is non-positive if the states u and v are sufficiently close to $\bar{u}_0^-$. Now it follows from Taylor's theorem that

$$\alpha(u, v) = \frac{1}{2}\nabla^2\eta(v)(u - v)^2 + 0(u - v)^3$$

$$\beta(u, v) = \frac{1}{2}\nabla^2\eta(v)\nabla f(v)(u - v)^2 + 0(u - v)^3,$$

and therefore

$$g = -\delta\alpha + \frac{1}{2}\nabla^2\eta(v)\{\lambda^- - \nabla f(v)\}(u - v)^2 + 0(u - v)^3.$$

If u and v are sufficiently close to $\bar{u}_0^-$ then $\lambda^- - \lambda_1(v)$ is small and

$$g \le -\delta/2\alpha + \frac{1}{2}\nabla^2\eta(v)\{\lambda_1(v) - \nabla f(v)\}(u - v)^2.$$

We observe that the matrix $\nabla^2\eta\{\lambda_1 - \nabla f\}$ is non-positive. Indeed, it follows from the symmetry of $\nabla^2\eta\nabla f$ that

$$\lambda_k r_j^t\nabla^2\eta r_k = r_j^t\nabla^2\eta\nabla f r_k = r_k^t\nabla^2\eta\nabla f r_j = \lambda_j r_k^t\nabla^2\eta r_j = 0$$

if $k \ne j$. This completes the proof of the lemma.

Proof of Theorem 3.2. We shall first recall a known estimate on Lipschitz continuous solutions. Let $\pi(u)$ denote the matrix whose j^{th} column is the normalized eigenvector $r_j(u)$:

$$\nabla f\, r_j = \lambda_j\, r_j, \qquad |r_j| = 1.$$

Consider a solution u which is defined and Lipschitz continuous on some strip $\mathscr{S}(T)$. Let

$$L\, u(t) = |\pi^{-1}\{u(\cdot, t)\}\, u_x(\cdot, t)|_\infty.$$

It is known that there exists a constant c depending only on f, T and $|u|_\infty$ such that

$$L\, u(t) \leq e^{ct}\, L\, u(0)/\{1 - ct\, L\, u(0)\} \tag{3.5}$$

if $t \leq 1/c\, L\, u(0)$. We recall that the Lipschitz norm of a function is equal to the L^∞-norm of its distributional gradient; it is somewhat more convenient here to work with $L\, u$ rather than with Lip $u(\cdot, t)$. The estimate (3.5) is proved by considering the diagonalized system satisfied by $z(x, t) \equiv \pi^{-1}(u)u_x$ and deriving an integral inequality for the quantity

$$|z(t)| \equiv \sum_j |z_j(\cdot, t)|_\infty = L\, u(t)$$

by integrating along characteristics. One may first consider C^2 solutions and then obtain the general case by passing to the limit.

In the paragraphs below we shall assume for simplicity that all solutions are defined on domains contained in the strip $\mathscr{S}(1)$. Then for a given system, the constant c appearing in (3.5) depends only on the L^∞-norm of the solution. Thus, one obtains the following uniform estimate on the growth rate of the Lipschitz norm of the solution: fix $k > 0$ and suppose that u is an admissible weak solution which is defined on the strip

$$\{(x, t) : t_1 \leq t < t_2\}$$

and satisfies there $|u|_\infty \leq N$. If $t_2 - t_1 < 1/c(N)k$ and

$$L\, u(\tau) \leq h(\tau, c, k)$$

then

$$L\, u(t) \leq h(t, c, k)$$

provided that $t_1 \leq \tau \leq t < t_2$ where

$$h(s, c, k) \equiv \epsilon^{cs}k/\{1 - cks\}.$$

The basic idea of the proof of Theorem 3.2 is to consider the largest domain on which the solution is Lipschitz continuous and show that it necessarily contains the region bounded by the extreme characteristics. For simplicity in terminology we shall call a measurable function Lipschitz continuous if it is equal almost everywhere to an everywhere defined function which is Lipschitz continuous in the standard sense. Let u be a solution in $K\{\mathscr{S}(1)\}$ and suppose that its initial data $u_0(x)$ are Lipschitz continuous on the interval $I_0 = (-\infty, 0)$. We shall prove that u is Lipschitz continuous to the left of the minimal 1-characteristic $x = x(t)$ passing through the origin. Let $|\cdot|_\infty(I)$ denote the L^∞-norm over the set I and put

$$N = 2|u_0|_\infty(I_0) \quad \text{and} \quad k = |\pi^{-1}(u_0)u_0'|_\infty(I_0).$$

Let $I(t) = (-\infty, \tilde{y}(t))$ denote the largest open interval with the property that the restriction of $u(\cdot, t)$ to $I(t)$ is Lipschitz continuous and satisfies

$$L\{u(t), I(t)\} \equiv |\pi^{-1}(u)u_x|_\infty\{I(t)\} \leq h(t, c, k).$$

We shall show that if $t \leq 1/c(N)k$ then $I(t)$ is nonempty and $\tilde{y}(t) \geq x(t)$. To this end, we shall first prove that

$$\tilde{y}(t) \geq -\text{const. } t, \tag{3.6}$$

for small t. Consider the Cauchy problem with initial data

$$v_0(x) = \begin{Bmatrix} u_0(x) & \text{if} \quad x \leq 0 \\ u_0(0-0) & \text{if} \quad x \geq 0 \end{Bmatrix}. \tag{3.7}$$

There exists a solution v to the problem (3.7) which is defined and Lipschitz continuous on $\mathscr{S}(T_v)$ where T_v depends only on $|v_0|_\infty$ and Lip v_0. If T_v is chosen so small that

$$|v|_\infty\{\mathscr{S}(T_v)\} \leq N,$$

then Theorem 3.1 guarantees that for each t in $[0, T_v)$, the solutions u and v coincide at almost all x satisfying $x \geq -\text{const. } t$ where the constant depends only on N. Since

$$|\pi^{-1}(v_0)v_0'|_\infty \leq |\pi^{-1}(u_0)u_0'|_\infty \equiv k$$

it follows that

$$L\ v(t) \equiv |\pi^{-1}(v)v_x|_\infty \leq h(t, c, k)$$

and therefore (3.6) holds for $0 \leq t < T_v$ with a constant depending only on N.

In order to prove that $\tilde{y}(t) \geq x(t)$, we shall assume for simplicity that $\sup_v \lambda_1(v) < 0$ and show that

$$y(t) \equiv \inf\{\tilde{y}(s) : 0 \leq s < t\}$$

is defined and Lipschitz continuous on the interval $[0, 1/c(N)k)$ and satisfies there $y(t) \geq x(t)$. In view of the monotonicity of $y(t)$ it suffices to show that

$$y(t_2) - y(t_1) \geq -\text{const. } (t_2 - t_1),\ t_2 > t_1 \tag{3.8}$$

if $t_2 - t_1$ is sufficiently small. The lower bound (3.8) may be proved in exactly the same way as (3.6): one need only replace the Cauchy problem (3.7) by the

Cauchy problem at time level $t = t_1$ with data

$$v(x, t_1) = \left\{ \begin{array}{ll} u(x, t_1) & \text{if } x \le y(t_1) \\ u\{y(t_1) - 0, t_1\} & \text{if } x > y(t_1) \end{array} \right\}.$$

The same constant will serve in both (3.6) and (3.8).

In order to prove that $y(t) \ge x(t)$ we shall show that for each $\delta > 0$

$$\dot{y}(t) \ge \min \lambda_1\{u(y(t), t)\} - \delta \tag{3.9}$$

for almost all t in $[0, T_v)$. Here, we denote the essential minimum of a function $g(\cdot, t)$ at the point y by

$$\min g(y, t) \equiv \inf_{\epsilon} \operatorname*{ess\,min}_{(y-\epsilon, y+\epsilon)} g(\cdot, t).$$

It follows from (3.9) that $y(t) \ge x_\delta(t)$ where x_δ denotes the minimal solution [12] of the forward initial value problem

$$\dot{x} = \lambda_1\{u(x(t), t)\} - \delta, \qquad x(0) = 0.$$

After passing to a subsequence we may conclude that x_δ converges uniformly to the minimal 1-characteristic $x(t)$ and that $y(t) \ge x(t)$ for $0 \le t < T_v$.

In order to prove (3.9) we observe that the conditions

$$\dot{y}(t)[u] - [f] = 0 \quad \text{and} \quad \dot{y}(t)[\eta] - [q] \le 0$$

hold for almost all t where $[u] = u(y(t) - 0, t) - u(y(t) + 0, t)$, *etc*. Let

$$J = \{t : 0 \le t < T_v, \qquad [u] \ne 0\}.$$

It follows from the entropy inequality that

$$\dot{y}(t) \ge \lambda_1\{u(y(t) + 0, t)\}$$

almost everywhere on J. Therefore, (3.9) holds almost everywhere on J and we need only prove (3.9) almost everywhere on J^c. To this end, let us consider the set

$$J_1 = \{t \in J^c : \dot{y}(t) < \lambda_1\{u(y(t), t)\} - \delta\}.$$

If J_1 has measure zero the proof is finished. If not, there exists a point t_1 in J_1 which has non-zero density with respect to 1-dimensional Hausdorff measure. We shall reach a contradiction as follows. Let

$$\lambda^* = \lambda_1\{u(y(t_1), t_1)\}.$$

For almost all c_1 and c_2 the restrictions of u to the lines

$$\{(x, t) : t = c_1\} \quad \text{and} \quad \{(x, t) : x = (\lambda^* - \delta/3)t + c_2\}$$

are classical functions of bounded variation whose points of continuity and discontinuity as a function of one variable correspond to points of approximate continuity and approximate jump discontinuity in (x, t). We conclude that there exist points (y_2, t_2) with $y_2 = y(t_2)$, $t_2 \in J_1$ which lie arbitrarily close to the point $(y(t_1), t_1)$ and at which u has the following limiting behavior:

$$u(y_2 - 0, t_2) = \lim_{t \downarrow t_2} u\{(\lambda^* - \delta/3)(t - t_2) + y_2, t_2\}.$$

Since we may assume without loss of generality that u is Lipschitz continuous in the standard sense to the left of the curve $(y(t), t)$, we may choose one such point with the additional property that

$$|\lambda_1\{u(y_2, t_2)\} - \lambda_1\{u(y_1, t_1)\}| < \delta/2. \tag{3.10}$$

At such a point the speed of propogation of $y(t)$ is strictly less than that of the ray

$$x = (\lambda^* - \delta/3)(t - t_2) + y_2, \qquad t \geq t_2.$$

We may now apply Lemma 3.3 to obtain the desired contradiction. Consider the Cauchy problem at time level $t = t_2$ with data

$$v(x, t_2) = \begin{Bmatrix} u(x, t_2) & \text{if} \quad x \leq y_2 \\ u(y_2 - 0, t_2) & \text{if} \quad x > y_2 \end{Bmatrix}.$$

This problem has a solution $v(x, t)$ which is defined and Lipschitz continuous on some band $[t_2, t_3)$. By Lemma 3.3 there exists a constant $\epsilon > 0$ such that u and v coincide at almost all points (x, t) which satisfy

$$x < (\lambda_1^* - \delta/3)(t - t_2), \qquad 0 \leq t - t_2 < \epsilon.$$

Thus it follows from the definition of $y(t)$ that

$$y(t) \geq (\lambda_1^* - \delta/3)(t - t_2)$$

for $0 \leq t - t_2 < \epsilon$. This completes the proof of Theorem 3.2.

4. Shock waves. In this section we study the restriction of γ to the joint shock set $\Gamma(u, v)$ and then establish the uniqueness of admissible shock waves in piecewise Lipschitz solutions to genuinely nonlinear systems of two equations.

Consider a genuinely nonlinear system (1.1) of two equations and suppose that w is a solution on $\mathcal{S}(T)$ which is Lipschitz continuous on either side of an admissible j-shock wave passing through the origin. More precisely, assume that w is Lipschitz continuous in regions of the form

$$\{(x, t) : x < y(t), 0 \leq t < T\} \quad \text{and} \quad \{(x, t) : x > y(t), 0 \leq t < T\}$$

where $y \in C^1[0, T)$, $y(0) = 0$ and

$$R(\dot{y}, u^-, u^+) = 0$$

$$\lambda_j(u^+) < \dot{y} < \lambda_j(u)$$

where $u^{\pm} = u(y(t) \pm 0, t)$. (We note that the assumptions above imply that the speed of propagation $\dot{y}(t)$ is a Lipschitz function of t.) It is not difficult to prove that there exists a solution w of the above form if one is given initial data $w_0(x)$ which are Lipschitz continuous on each of the intervals $(-\infty, 0)$ and $(0, \infty)$ and if the limiting values at the origin

$$w_0^{\pm} = w_0(0 \pm 0)$$

are connected by a sufficiently weak admissible j-shock wave, *i.e.* if there exists a number τ such that

$$R(\tau, w_0^-, w_0^+) = 0$$

$$\lambda_j(w_0^+) < \tau < \lambda_j(w_0^-).$$

In the case where (1.1) belongs to the Smoller-Johnson class one need not require that w_0^+ and w_0^- are close.

Theorem 4.1. *For every $\bar{u} \in R^2$ there exists a constant $\delta > 0$ depending only on f and $\bar{u}$ with the following property. If $u \in K\{\mathscr{S}(T)\}$, $|u - \bar{u}|_\infty < \delta$, $|w - \bar{u}|_\infty < \delta$ and $u(x, 0) = w(x, 0)$ for almost all x then $u = w$ for almost all (x, t) in $\mathscr{S}(T)$.*

Theorem 4.2. *Suppose that (1.1) is a system in the Smoller-Johnson class. If $u \in K\{\mathscr{S}(T)\}$ and $u(x, 0) = w(x, 0)$ for almost all x then $u = w$ for almost all (x, t) in $\mathscr{S}(T)$.*

Remarks. If (1.1) is an arbitrary strictly hyperbolic system of two equations then there exists in a neighborhood of each point $\bar{u} \in R^2$ a smooth strictly convex entropy [26]. If (1.1) lies in the Smoller-Johnson class then there exists in a neighborhood of each compact set in R^2 a smooth strictly convex entropy [26]. In proving Theorem 4.2 one chooses an entropy whose domain of definition contains the range of both u and w.

We shall begin with several lemmas which describe the structure of $\gamma(u, v)$ on the joint shock set $\Gamma(u, v)$ in the case where u and v are arbitrary weak solutions to a system of n equations. As always we assume that (1.1) has a smooth entropy-entropy flux pair (η, q) with η strictly convex and that the range of solutions considered lies within the domain of definition of η and q.

Lemma 4.1. *Suppose that v_ℓ and v_r are connected by a shock wave with speed σ. Then, for all u*

$$D(\sigma, u, u, v_\ell, v_r) = -d(\sigma, v_\ell, v_r) - [\nabla\eta]R(\sigma, u, v_\ell)$$

where $[\nabla\eta] = \nabla\eta(v_\ell) - \nabla\eta(v_r)$.

Proof. By definition

$$D = \sigma(\alpha_\ell - \alpha_r) - \beta_\ell + \beta_r$$

where

$$\alpha_\ell = \eta(u) - \eta(v_\ell) - \nabla\eta(v_\ell)(u - v_\ell)$$

$$\beta_\ell = q(u) - q(v_\ell) - \nabla\eta(v_\ell)\{f(u) - f(v_\ell)\}.$$

Thus,

$$\alpha_\ell - \alpha_r = -\eta_\ell + \eta_r - \nabla\eta(v_\ell)\{u - v_\ell\} + \nabla\eta(v_r)\{u - v_r\}$$

$$\beta_\ell - \beta_r = -q_\ell + q_r - \nabla\eta(v_\ell)\{f(u) - f(v_\ell)\} + \nabla\eta(v_r)\{f(u) - f(u_r)\}$$

where $\eta_\ell = \eta(v_\ell)$, $\eta_r = \eta(v_r)$, *etc*. Hence,

$$D(\sigma, u, u, v_\ell, v_r) = -d(\sigma, v_\ell, v_r) - \nabla\eta(v_\ell)\{\sigma(u - v_\ell) - f(u) + f(v_\ell)\} + \nabla\eta(v_r)\{\sigma(u - u_r) - f(u) - f(u_r)\}. \tag{4.2}$$

The identity (4.1) follows from (4.2) and the Rankine-Hugoniot relations

$$\sigma v_\ell - f(v_\ell) = \sigma v_r - f(v_r).$$

We conclude from Lemma 4.1 that the restriction of γ to the set $\Gamma^c(u) \cap \Gamma(v)$ splits into two measures, the negative dissipation for v, $-\theta_v$, and a measure which couples the limiting values of u and v on $\Gamma^c(u) \cap \Gamma(v)$:

Corollary 4.1. *Suppose that u and v are arbitrary weak solutions. If $E \subset \Gamma^c(u) \cap \Gamma(v)$ then*

$$\gamma(E) = -\theta_v(E) - \int_E [\nabla\eta(v)]R(\sigma, u, v_\ell)dH_1.$$

Remark. In general the coupling term $[\nabla\eta]R$ does not have a distinguished sign and is only first order in the difference between u and v:

$$[\nabla\eta]R = 0(|v_\ell - v_r)|\min\{|u - v_r|, |u - v_\ell|\}$$

since $R(\sigma, u, v_\ell) = R(\sigma, u, v_r)$.

Corollary 4.1 is useful in estimating the γ-measure of a pair of nearby shock waves. Consider a shock wave S_u which propagates through points of approximate continuity in v and a shock wave S_v in v which propagates through points of approximate continuity in u, *i.e.* consider sets S_u and S_v satisfying

$$S_u \subset \Gamma(u) \cap \Gamma^c(v), \qquad S_v \subset \Gamma^c(u) \cap \Gamma(v).$$

The restriction of γ to $S_u \cup S_v$ may be decomposed into the relative dissipation measure $\theta_u - \theta_v$ and a measure which couples the values of u and v along S_v: if $E \subset S_u \cup S_v$ then

$$\gamma(E) = \theta_u(E) - \theta_v(E) - \int_{E \cap S_v} [\nabla\eta(v)]R(\sigma, u, v_\ell)dH_1. \tag{4.3}$$

Special interest attaches to the case where the restriction of u to S_v coincides with one of the limiting values of u on S_u: assume that for each t_0 in the interval

(t_1, t_2) the sets S_u and S_v intersect the line $t = t_0$ at points P and Q respectively and that either

$$(4.4) \qquad u(P) = u_\ell(Q) \quad \text{or} \quad u(P) = u_r(Q).$$

In this situation the coefficient of the measure H_1 in (4.3) may be expressed in terms of the relative speed of propagation of S_u and S_v.

Lemma 4.2. *Suppose that u and u_r are connected by a shock wave with speed τ while v_ℓ and v_r are connected by a shock wave with speed σ. Then*

$$(4.5a) \qquad D(\sigma, u_r, u_r, v_\ell, v_r) = -d(\sigma, v_\ell, v_r) - [\nabla\eta(v)]\{R(\sigma, u_\ell, v_\ell) + (\tau - \sigma)(u_\ell - u_r)\}$$

$$(4.5b) \qquad D(\sigma, u_\ell, u_\ell, v_\ell, v_r) = -d(\sigma, v_\ell, v_r) - [\nabla\eta(v)]\{R(\sigma, u_r, v_r) - (\tau - \sigma)(u_\ell - u_r)\}.$$

Proof. Substituting $u = u_\ell$ in (4.1) yields

$$D(\sigma, u_\ell, u_\ell, v_\ell, v_r) = -d(\sigma, v_\ell, v_r) - [\nabla\eta(v)]R(\sigma, u_\ell, v_\ell)$$

we obtain (4.5b) using the identity

$$R(\sigma, u_\ell, v_\ell) = R(\sigma, u_r, v_r) - (\tau - \sigma)(u_\ell - u_r),$$

(4.5a) is proved in a similar way.

Therefore, if (4.4) holds the γ-measure of a pair of shock waves $S_u \cup S_v$ may be expressed in terms of the relative dissipation in (u, v) and the relative speed of propagation of $S_u \cup S_v$ modulo an error term: if $E \subset S_u \cup S_v$ and (4.4) holds then

$$\gamma(E) = \theta_u(E) - \theta_v(E) + \int_{E \cap S_v} [\nabla\eta(v)](\tau - \sigma)(u_\ell - u_r)ds + \text{error},$$

where

$$\text{error} = \int_{E \cap S_v} [\nabla\eta]R\, dH_1.$$

We note that in general the state $u(P)$ does not lie close to either $u_\ell(Q)$ or $u_r(Q)$ and the γ-measure of $S_u \cup S_v$ is not bounded by the Euclidean distance between S_u and S_v in physical space. However by associating with each shock wave S_u in a weak solution u the set

$$\mathcal{S}_u = \{(x, t, \tau, u_\ell, u_r) : (x, t) \in S_u\} \subset R^2 \times R^{2n+1} \equiv \mathcal{E}.$$

described in the introduction, one may regard the second term in (4.5a, b) as measuring the distances between the sets $P_{2n+1}(\mathcal{S}_u)$ and $P_{2n+1}(\mathcal{S}_v)$ in the state space R^{2n+1}. In this connection, we establish the following lemma.

Lemma 4.3. *Suppose that u_ℓ and u_r are connected by a shock wave with speed τ while v_ℓ and v_r are connected by a shock wave with speed σ. Then*

$$\begin{aligned}(4.6a)\quad D(\tau, u_\ell, u_r, v_\ell, v_r) &= d(\tau, u_\ell, u_r) - d(\sigma, v_\ell, v_r)\\ &\quad - \alpha(v_\ell, v_r)(\tau - \sigma) - [\nabla\eta(v)]R(\tau, u_\ell, v_\ell).\end{aligned}$$

$$\begin{aligned}(4.6b)\quad D(\tau, u_\ell, u_r, v_\ell, v_r) &= d(\tau, u_\ell, u_r) - d(\sigma, v_\ell, v_r)\\ &\quad + \alpha(v_\ell, v_r)(\tau - \sigma) - [\nabla\eta(v)]R(\tau, u_r, v_r).\end{aligned}$$

In order to establish the decompositions (4.6a) and (4.6b) we write the function D as the sum of two transition functions, one in which v is stationary the other in which u is stationary.

Proof. Let $\alpha_{\ell,\ell} = \alpha(u_\ell, v_\ell)$, $\alpha_{\ell,r} = \alpha(u_\ell, v_r)$, *etc.* Then

$$\begin{aligned}D &= \{\tau(\alpha_{\ell,\ell} - \alpha_{r,\ell}) - \beta_{\ell,\ell} + \beta_{r,\ell}\} + \{\tau(\alpha_{r,\ell} - \alpha_{r,r}) - \beta_{r,\ell} + \beta_{r,r}\}\\ &= d(\tau, u_\ell, u_r) + \{(\tau - \sigma)(\alpha_{r,\ell} - \alpha_{r,r}) + \sigma(\alpha_{r,\ell} - \alpha_{r,r}) - \beta_{r,\ell} + \beta_{r,r}\}\\ &= d(\tau, u_\ell, u_r) + (\tau - \sigma)(\alpha_{r,\ell} - \alpha_{r,r}) + D(\sigma, u_r, u_r, v_\ell, v_r).\end{aligned}$$

Substituting the expression (4.1) for $D(\sigma, u_r, u_r, v_\ell, v_r)$ we obtain

$$\begin{aligned}D &= d(\tau, u_\ell, u_r) - d(\sigma, v_\ell, v_r)\\ &\quad + (\tau - \sigma)\{\alpha_{r,\ell} - \alpha_{r,r} - [\nabla\eta(v)](u_\ell - u_r)\} - [\nabla\eta(v)]R(\sigma, u_\ell, v_\ell).\end{aligned}$$

A short calculation shows that

$$\alpha_{r,\ell} - \alpha_{r,r} - [\nabla\eta(v)](u_\ell - u_r) = -\alpha(v_\ell, v_r) - [\nabla\eta(v)](u_\ell - u_r).$$

Thus,

$$\begin{aligned}(4.7)\qquad D &= d(\tau, u_\ell, u_r) - d(\sigma, v_\ell, v_r)\\ &\quad - \alpha(v_\ell, v_r)(\tau - \sigma) - (\tau - \sigma)[\nabla\eta(v)](u_\ell - v_\ell)\\ &\quad - [\nabla\eta(v)]R(\sigma, u_\ell, v_\ell).\end{aligned}$$

The identity (4.6a) follows by factoring out $[\nabla\eta(v)]$ from (4.7) and making use of the relation

$$(\sigma - \tau)(u_\ell - v_\ell) + R(\sigma, u_\ell, v_\ell) = R(\tau, u_\ell, v_\ell).$$

The identity (4.6b) is proved in a similar way.

We may reformulate Lemma 4.3 as follows. Let u be an arbitrary weak solution defined on Ω and v arbitrary function in $L^\infty \cap BV(\Omega)$. If $E \subset \Gamma(u, v)$ and if almost all limiting states $v_\ell(P)$ and $v_r(P)$, $P \in E$, are connected by a shock wave with speed, say, $\sigma(P)$ then

$$(4.8)\qquad \gamma(E) = \int_E D(\tau, u_\ell, u_r, v_\ell, v_r)dt$$

where $\tau = -\nu_t/\nu_x$ denotes the speed of propagation of E, $dt = -\nu_x d\, H_1$, and D is given by either (4.6a) or (4.6b) with σ replaced by $\sigma(P)$. As always ν is normalized by the requirement that $\nu_x < 0$. Later, we shall employ the representation (4.8) with v replaced by an approximate solution. Although D does not have a distinguished sign in general we show that D is negative definite on a special class of points. We shall refer to a point

$$A = \{\tau,\, u_\ell,\, u_r,\, \sigma,\, v_\ell,\, v_r) \in R^{2n+1} \times R^{2n+1}$$

as *attracting* if the following three properties hold:

$$R(\tau,\, u_\ell,\, u_r) = 0 = R(\sigma,\, v_\ell,\, v_r)$$

$$\lambda_j(u_r) < \tau < \lambda_j(u_\ell) \quad \text{and} \quad \lambda_j(v_r) < \sigma < \lambda_j(v_\ell), \quad \text{for some } j.$$

Either $u_\ell = v_\ell$ or $u_r = v_r$. We shall refer to a subset of $R^{2n+1} \times R^{2n+1}$ as attracting if each of its points is attracting.

Lemma 4.4. *For every $\bar{u}$ in the domain of definition $\mathcal{D}$ of η and q there exists a neighborhood $N(\bar{u}) \subset \mathcal{D}$ with the following property. Suppose that A is an attracting point such that u_ℓ, u_r, v_ℓ and v_r lie in $N(\bar{u})$. If $u_\ell = v_\ell$ then*

$$D\{P_{4n+1}(A)\} \leq \text{const. } |u_r - v_r|(\tau - \sigma)^2. \tag{4.9}$$

If $u_r = v_r$ then

$$D\{P_{4n+1}(A)\} \leq \text{const. } |u_\ell - v_\ell|(\tau - \sigma)^2, \tag{4.10}$$

where the constants are positive and depend only on $\bar{u}$ and f.

Proof. Suppose that $u_\ell = v_\ell$. Fixing u_ℓ and the wave index j we may regard u_r as a function of the wave speed τ *via* the Rankine-Hugoniot relations if u_r is close to u_ℓ:

$$R(\tau,\, u_\ell,\, u_r(\tau)) = 0.$$

Consider the function

$$\phi(\tau) \equiv d(\tau,\, u_\ell,\, u_r(\tau)).$$

A straightforward computation shows that

$$\dot{\phi}(\tau) = \alpha(u_\ell,\, u_r(\tau)).$$

Now, if $u_\ell = v_\ell$ it follows from (4.6a) that

$$D(\tau,\, u_\ell,\, u_r(\tau),\, v_\ell,\, v_r) = \phi(\tau) - \phi(\sigma) - \dot{\phi}(\sigma)(\tau - \sigma).$$

A short calculation shows that

$$\ddot{\phi}(\tau) = -\nabla^2\eta(u)\{\dot{u},\, u_\ell - u\}.$$

Since η is strictly convex, we conclude that ϕ is a concave function of the wave speed τ. This completes the proof of (4.9). The case $u_r = v_r$ is treated in a similar fashion.

Remark. For systems (1.1) in the Smoller-Johnson class the corresponding result holds if we merely assume that u_ℓ, u_r, v_ℓ and v_r lie in a compact subset of R^2.

One may regard (4.9) and (4.10) as expressing the stability of shock waves relative to a special class of perturbations, namely those which share a common end state. The stability of shock waves relative to general perturbations is discussed below. We shall establish Theorems 4.1 and 4.2 with the aid of an approximate solution $\tilde{w}$. Suppose that u is a solution in $K\{\mathscr{S}(T)\}$ whose initial data coincide with $w(x, 0)$ at almost all x. The approximate solution $\tilde{w}$ is constructed from w and u by a two-step procedure. The first step envolves the extension of w across the shock wave

$$S_w = \{(y(t), t) : 0 \leq t < T\}.$$

Consider the restrictions $w^\pm$ of w to the right and left of S_w, *i.e.* to the regions

$$\Omega^+ = \{(x, t) : x > y(t), 0 \leq t < T\} \quad \text{and} \quad \Omega^- = \{(x, t) : x < y(t), 0 \leq t < T\}.$$

Since S_w is non-characteristic we may employ the classical existence theory to extend w^+ across S_w to the left: there exists $\epsilon^+ > 0$ and a solution $\tilde{w}^+$ which is defined and Lipschitz continuous in the region

$$\tilde{\Omega}^+ = \{(x, t) : x > y(t) - \epsilon^+, 0 \leq t < T\}$$

and which coincides on S_w with the restriction of w to the right side of S_w:

$$\tilde{w}^+(P) = w_r(P) \qquad \text{if} \quad P \in S_w.$$

Similarly, there exists $\epsilon^- > 0$ and a solution $\tilde{w}^-$ which is defined and Lipschitz continuous in the region

$$\tilde{\Omega}^- = \{ (x, t) : x < y(t) + \epsilon^+, 0 \leq t < T\}$$

and which coincides on S_w with the restriction of w to the left side of S_w:

$$\tilde{w}^-(P) = w_\ell(P) \qquad \text{if} \quad P \in S_w.$$

Both $\tilde{w}^+$ and $\tilde{w}^-$ may be constructed by extending S_w slightly below the x-axis and putting, say constant data along the extended part.

The approximate solution $\tilde{w}$ is defined in terms of $\tilde{w}^\pm$ and u in the following way. There are two cases to consider accordingly as S_w is a shock of the first or second kind. If S_w is a 1-shock let

$$S_u^1 = \{(x_1(t), t) : 0 \leq t < T\}$$

denote the minimal forward 1-characteristic in u and choose T_0 so small that restriction of S_u^1 to the interval $0 \leq t < T_0$ lies within the domains of definition of $\tilde{w}^\pm$. We define

$$\tilde{w}(x, t) = \left\{ \begin{array}{ll} u(x, t) & \text{if} \quad x < x_1(t) \\ \tilde{w}^+(x, t) & \text{if} \quad x > x_1(t) \end{array} \right\}$$

for $(x, t) \in \mathcal{S}(T_0)$. If S_w is a 2-shock let

$$S_u^2 = \{(x_2(t), t), 0 \le t < T\}$$

denote the maximal forward 2-characteristic in u and choose T_0 so small that the restriction of S_u^2 to the interval $0 \le t < T_0$ lies within the domains of definition of $\tilde{w}^{\pm}$. In this case, we define

$$\tilde{w}(x, t) = \left\{ \begin{array}{ll} \tilde{w}^-(x, t) & \text{if } \; x < x_2(t) \\ u(x, t) & \text{if } \; x > x_2(t) \end{array} \right\}$$

for $(x, t) \in \mathcal{S}(T_0)$.

Proof of Theorem 4.1. We shall first show that $u = \tilde{w}$ at almost all (x, t) in $\mathcal{S}(T_0)$. By the generalized Green's theorem

$$\gamma\{\mathcal{S}(t)\} = \int_{-\infty}^{\infty} \alpha\{u(x, t), \tilde{w}(x, t)\}dx$$

where $\gamma = \gamma(u, \tilde{w})$. It follows from (3.4) that

$$\gamma\{\mathcal{S}(t) \cap S_u^c\} \le \text{const.} \int_0^t \int_{-\infty}^{\infty} |u - \tilde{w}|^2 dxdt,$$

where $S_u = S_u^j$. In order to prove that u and $\tilde{w}$ coincide almost everywhere, we need only show that

$$\gamma\{\mathcal{S}(t) \cap S_u\} \le 0, \tag{4.11}$$

for $0 \le t \le T_0$. Indeed, the approximate solution $\tilde{w}$ was constructed in order that the $\gamma(u, \tilde{w})$-measure of the dominant wave S_u would be non-positive.

For concreteness let us assume that S_w is a 1-shock. Since u and $\tilde{w}$ coincide by definition to the left of S_u, we have $u_\ell = \tilde{w}_\ell$ and

$$\gamma\{\mathcal{S}(t) \cap S_u\} = \int_0^t D(\tau, u_\ell, u_r, \tilde{w}_\ell, \tilde{w}_r)dt = \int_0^t -\tau\alpha_{r,r} + \beta_{r,r}dt$$

where $\alpha_{r,r} = \alpha(u_r, \tilde{w}_r)$ and $\beta_{r,r} = \beta(u_r, \tilde{w}_r)$. We first note that $u_r(P)$ and $\tilde{w}_r(P)$ coincide at H_1 almost all points P of $S_u \cap S_w$. Indeed, it follows from Theorem 3.2 that u and w coincide on

$$\{(x, t) : x < \min[x(t), y(t)]\}$$

where $x(t)$ denotes the minimal 1-characteristic in u. Since the state on the right side of a shock wave is uniquely determined by the state on the left together with the speed of propagation it follows that $u_r(P)$ and $w_r(P)$ coincide at H_1 almost all points P of $S_u \cap S_w$. At such points $w_r(P) = \tilde{w}_r(P)$ by construction. Thus the quantity

$$-\tau\alpha_{r,r} + \beta_{r,r}$$

vanishes at H_1 almost all points of $S_u \cap S_w$. Therefore, in order to prove (4.11) we need only show that

$$\gamma\{\mathcal{S}(t) \cap S_u \cap S_w^c\} \leq 0$$

for $0 \leq t < T_0$. Since $S_u \cap S_w^c$ is a relatively open subset of S_u we may restrict our attention to the open components: for concreteness we shall assume that

$$\Delta(t) \equiv y(t) - x(t) > 0$$

if $t \in (t_1, t_2)$, $t_2 < T_0$ and $\Delta(t_1) = 0$ and show that

$$\gamma\{\mathcal{S}(t_1, t) \cap S_u\} \leq 0 \tag{4.12}$$

for $t_1 < t < t_2$ where

$$\mathcal{S}(t_1, t) = \{(x, t) : t_1 < t < t_2\}.$$

To this end we proceed as follows. Let

$$u_\ell = u_\ell(P) \qquad \tilde{w}_\ell = \tilde{w}_\ell(P) \qquad w_\ell = w_\ell(Q)$$

$$u_r = u_r(P) \qquad \tilde{w}_r = \tilde{w}_r(P) \qquad w_r = w_r(Q)$$

where $P = (x(t), t)$, $Q = (y(t), t)$ and $t_1 < t < t_2$. We recall that $u_\ell = \tilde{w}_\ell$.

Proposition. *If the parameter δ appearing in the statement of Theorem 4.1 is sufficiently small then*

$$D(\tau, u_\ell, u_r, \tilde{w}_\ell, \tilde{w}_r) \leq -\text{const.}\,|\tilde{w}_r - u_r|^2 + \text{const.}\,|w_\ell - u_\ell|^2 + \text{const.}\,|\tilde{w}_r - w_r|^2 \tag{4.13}$$

where the constants are positive.

Remark. The estimate (4.13) provides a sense in which the projections $P_{2n+1}(\mathcal{S}_u) \times P_{2n+1}(\mathcal{S}_w)$ associated with pairs of shock waves S_u and S_w lie near the class of attracting sets.

Proof of proposition. We shall establish a result which is slightly more general than the one stated in the proposition. Fix the state $\bar{u}$ and let $B_\delta(\bar{u})$ denote the ball of radius δ centered at $\bar{u}$. We claim that there exists a constant δ depending only on $\bar{u}$ and f with the following property. Suppose that (w_ℓ, w_r), $(\tilde{w}_\ell, \tilde{w}_r)$ and (u_ℓ, u_r) are three pairs of states in $B_\delta(\bar{u})$ such that $\tilde{w}_\ell = u_\ell$ and such that u_ℓ and u_r as well as w_ℓ and w_r are connected by admissible 1-shock waves. If $|w_\ell - w_r| \geq \epsilon > 0$ then

$$D(\tau, u_\ell, u_r, \tilde{w}_\ell, \tilde{w}_r) \leq -c_1|\tilde{w}_r - u_r|^2 + c_2|w_\ell - u_\ell|^2 + c_2|\tilde{w}_r - w_r|^2 \tag{4.14}$$

where the constants c_1 and c_2 depend only on ϵ, *cf.* Figure 1. As usual, τ denotes the wave speed corresponding to u_ℓ and u_r, *i.e.*

$$R(\tau, u_\ell, u_r) = 0.$$

If the system (1.1) lies in the Smoller-Johnson class then the corresponding global result holds: if the three pairs lie in a compact subset of the domain of definition of η and q and have the stated properties then (4.14) holds with constants c_1 and c_2 depending only on ϵ.

The estimate (4.13) follows immediately from (4.14) by noting that for a solution w of the specified form

$$|w_\ell(Q) - w_r(Q)| \geq \epsilon$$

for some $\epsilon > 0$.

Since D is smooth we may assume in proving (4.14) that without loss of generality $|w_\ell - u_\ell|$ and $|\tilde{w}_r - u_r|$ are small. In this situation there exists a state w_r' which lies on the 1-shock wave curve through u_ℓ and whose wave speed τ' satisfies

$$\sigma - \lambda_1(w_\ell) = \tau' - \lambda(u_\ell)$$

where σ denotes the wave speed of the pair (w_ℓ, w_r), *i.e.*

$$R(\sigma, w_\ell, w_r) = 0 \quad \text{and} \quad R(\tau', u_\ell, w_r') = 0.$$

The idea is to regard $D \equiv -\tau\alpha_{r,r} + \beta_{r,r}$ as a perturbation of

$$D' \equiv -\tau\alpha(u_r, w_r') + \beta(u_r, w_r').$$

To this end, we observe that

$$\alpha(u_r, \tilde{w}_r) = \alpha(u_r, w_r') + \alpha_w(u_r, \theta)(\tilde{w}_r - w_r')$$

where θ lies on the line segment joining $\tilde{w}_r$ and w_r'. Since $\alpha_w(u_r, u_r) = 0$, we have

$$|\alpha_w(u_r, \theta)| \leq c|\theta - u_r|.$$

This fact together with a similar estimate on β yields

$$|D - D'| \leq c|\theta - u_r|\,|\tilde{w}_r - w_r'|. \tag{4.15}$$

A simple application of the triangle inequality shows that

$$|\theta - u_r| \leq |\tilde{w}_r - w_r| + |w_r' - u_r| + c|w_\ell - u_\ell|$$

$$|\tilde{w}_r - w_r'| \leq |\tilde{w}_r - w_r| + c|w_\ell - v_\ell|.$$

Therefore,

$$|\theta - u_r||\tilde{w}_r - w_r'| \leq c\mu|w_r' - u_r|^2 + \frac{c}{\mu}\{|\tilde{w}_r - w_r|^2 + |w_\ell - u_\ell|^2\}. \tag{4.16}$$

Since, $D' \leq -c|w_r' - u_\ell||u_r - w_r'|^2$ it follows from (4.15) and (4.16) that

$$D \leq -c|w_r' - u_\ell||u_r - w_r'|^2 + c\mu|u_r - w_r'|^2 + \frac{c}{\mu}\{|\tilde{w}_r - w_r|^2 + |w_\ell - u_\ell|^2\}.$$

Since the coefficient of the leading term satisfies

$$|w_r' - u_\ell| \leq c|w_\ell - w_r| + c\,|w_\ell - u_\ell|$$

we have for an appropriate choice of μ

$$D \le -c\,|w_\ell - w_r||u_r - w_r'|^2 + c\{|\tilde{w}_r - w_r|^2 + |w_\ell - u_\ell|^2\}.$$

The desired estimate (4.14) follows by observing that

$$|u_r - w_r| \le |u_r - w_r'| + |w_r' - w_r| \le |u_r - w_r'| + c|u_\ell - w_\ell|$$

and hence

$$-|u_r - w_r|^2 \le -\frac{1}{2}|u_r - w_r|^2 + c|u_\ell - w_\ell|^2.$$

This completes the proof of (4.14).

Continuing with the proof of (4.12) we observe

$$D \le -c_1|\tilde{w}_r - u_r|^2 + c_3\,\Delta^2(t) \tag{4.17}$$

since

$$|\tilde{w}_r(P) - w_r(Q)| \le \text{const.}\ \Delta(t) \quad \text{and} \quad |u_\ell(P) - w_\ell(Q)| \le \text{const.}\ \Delta(t).$$

We may relate the separation distance $\Delta(t)$ to the deviation of the limiting states on the sides of the S_u and S_w with the aid of the Rankine-Hugoniot relations: using the fact that

$$\Delta(t) \le \int_{t_1}^{t_2} |y'(t) - x'(t)|dt$$

together with the smooth dependence of the shock speeds on the corresponding limiting states we may deduce in a straightforward way that

$$\Delta(t) \le \text{const.} \int_{t_1}^{t_2} |u_r - \tilde{w}_r|dt. \tag{4.18}$$

Combining (4.17) and (4.18) yields

$$D \le -g(t) + (t - t_1)\int_{t_1}^{t} \text{const.}\ g(\xi)d\xi$$

where $g(\xi) = c_1|u_r - \tilde{w}_r|^2(\xi)$. Dropping the factor of $(t - t_1)$ and integrating D with respect to t yields

$$\int_{t_1}^{t} D\,dt \le -h(t) + \int_{t_1}^{t} \text{const.}\ h(s)ds \le -h(t) + \text{const.}\ (t - t_1)h(t) \le 0$$

for t near t_1 since

$$h(t) \stackrel{\text{def.}}{=} \int_{t_1}^{t} g(t)dt$$

is non-decreasing. Thus, if we restrict our attention to a strip $\mathscr{S}(T_0')$ where T_0' is sufficiently small we conclude that (4.12) holds for $0 \le t < T_0'$ and therefore that u and $\tilde{w}$ coincide almost everywhere on $\mathscr{S}(T_0')$. Since T_0' depends only on the magnitude of the shock wave S_w we may continue the argument to deduce that (4.12) holds on $[0, T_0)$ and that u and $\tilde{w}$ coincide almost everywhere on $\mathscr{S}(T_0)$.

Lastly, we shall show that u and w coincide almost everywhere on $\mathscr{S}(T)$. It follows from the analysis above that both u and w are Lipschitz continuous on the complement of $S_u^1 \cup S_w$. Thus the symmetric part γ_s of γ,

$$\gamma_s = \{\alpha(u, w) + \alpha(w, u)\}_t + \{\beta(u, w) + \beta(w, u)\}_x$$

vanishes on $S_u \cup S_w$ and is bounded on $\{S_u' \cup S_w\}^c$ by a tame measure:

$$|\gamma_s(E)| \leq \iint_E \text{const.}\, |u - w|^2 dxdt$$

if $E \subset \{S_u^1 \cup S_w\}^c$. Theorem 4.1 follows by an application of Gronwall's inequality to the function

$$\int_{-\infty}^{\infty} \alpha(u(x, t), w(x, t))dx.$$

This completes the proof of Theorem 4.1. The proof of Theorem 4.2 is virtually identical.

Remark. If one is interested in working with a measure which is symmetric in u and v, then it will be necessary to employ a measure which does not reduce to the dissipation measure when one solution is constant. Such measures are more singular than the divergence of the vector-field obtained from the quadratic part of entropy.

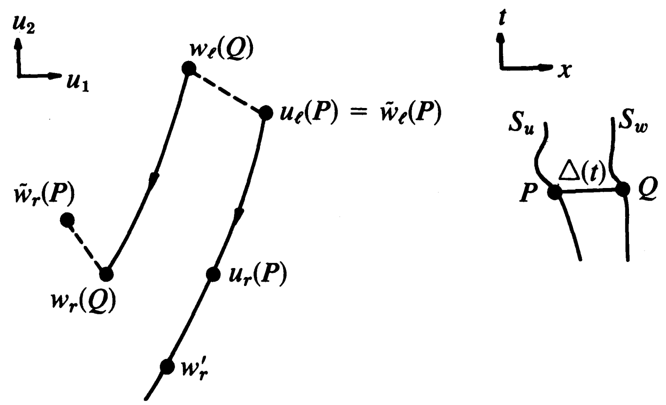

Figure 1.

5. Rarefaction waves. In this section we shall establish the uniqueness of centered rarefaction waves. Suppose that w is a solution in $K\{\mathscr{S}(T)\}$. Consider the set

(5.1) $$\Omega_j = \{(x, t) : x_m^j(t) \leq x \leq x_M^j(t); 0 \leq t < T\}$$

where x_m^j and x_M^j denote the minimal and maximal j-characteristics emanating from the origin. We shall say that Ω_j is a *centered Lipschitz j-rarefaction wave* if the following two conditions hold: $x_m^j(t) < x_M^j(t)$ for $0 < t \leq T$ and the restriction of the measure u_x to Ω has the form

$$u_x = \sum_{k=1}^{n} \alpha_k \, r_k(u(x, t)) dxdt \tag{5.2}$$

where $dxdt$ denotes 2-dimensional Lebesgue measure and $\alpha_k = \alpha_k(x, t)$ $k = 1, 2, \cdots, n$, are measurable functions satisfying

$$-\text{const.} \leq \alpha_j \leq \text{const.}/t \quad \text{and} \quad |\alpha_k| \leq \text{const.} \qquad \text{if} \quad k \neq j.$$

For the purposes of this paper it would suffice to impose the weaker condition that coefficient of α_j be bounded from below by an integrable function of t. We note that a classical centered j-rarefaction is a smooth similarity solution $u = u(x/t)$ which is defined in a domain of the form $a < x/t < b$ and satisfies there

$$u_x = r_j(u(x/t))/t.$$

The classical centered j-rarefaction wave can be generated by solving the Riemann problem with initial states having the same j-Riemann invariants [25].

Consider a solution w of a genuinely nonlinear system of two equations which is defined in a strip $\mathscr{S}(T)$ and has the following properties: w is Lipschitz continuous in each of the regions

$$\{(x, t) : x \leq x_m^j(t), 0 \leq t < T\} \quad \text{and} \quad \{(x, t): x \geq x_M^j(t), 0 \leq t < T\}$$

and the set Ω_j defined by (5.1) is a Lipschitz centered j-rarefaction wave. It is not difficult to prove that a solution w with the above structure exists in a small strip $\mathscr{S}(T)$ provided one is given initial data of the form

$$w_0(x) = \begin{Bmatrix} w_\ell(x) & \text{if} \quad x < 0 \\ w_r(x) & \text{if} \quad x > 0 \end{Bmatrix}$$

where w_ℓ and w_r are Lipschitz continuous functions whose limiting values at the origin, $w_\ell(0-)$ and $w_r(0+)$ have the same j-Riemann invariants.

Theorem 5.1. *For every $\bar{u} \in R_2$ there exists a constant $\delta > 0$ depending only on $\bar{u}$ and f with the following property. If $u \in K\{\mathscr{S}(T)\}$, $|u - \bar{u}|_\infty < \delta$, $|w - \bar{u}|_\infty < \delta$ and $u(x, 0) = w(x, 0)$ for almost all x then $u = w$ for almost all (x, t) in $\mathscr{S}(T)$.*

Theorem 5.2. *If w is a solution to the quasilinear wave equation (1.5) with $p' < 0$ and $p'' < 0$ and if $u \in K\{\mathscr{S}(T)\}$ with $u(x, 0) = w(x, 0)$ for almost all x then $u = w$ for almost all (x, t) in $\mathscr{S}(T)$.*

We shall begin by studying the structure of the measure $\gamma(u, v)$ on centered rarefaction waves in the base solution v for general systems of equations.

Lemma 5.1. *Let u and v be arbitrary weak solutions. Suppose that Ω_j is a centered Lipschitz j-rarefaction wave in v. If $E \subset \Omega_j$ then*

$$\gamma(E) = \theta_u(E) - \iint_E \sum_{k=1}^{n} \alpha_k \ell_k(v)\, Qf(u, v)dxdt \tag{5.3}$$

where $\ell_k(v) \equiv r_k^t(v)\nabla^2\eta(v)$ and Qf is the quadratic part of f at v, i.e.,

$$Qf = f(u) - \nabla f(v)(u - v).$$

Proof. As we observed in Lemma 3.2 the right eigenvectors of ∇f are biorthogonal with respect to $\nabla^2\eta$. Thus ℓ_k is a left eigenvector of ∇f and the lemma follows from the decomposition (3.4) and the representation (5.2).

The dominant coefficient in (5.3), $\alpha_j\ell_jQ$, reflects the increased coupling between u and v near the center (0, 0) of the wave. The remaining coefficients are tame since α_k lies in L^∞ if $k \neq j$. The stability of a j-rarefaction wave is a consequence of the geometry of j-characteristics; the influence of waves crossing transversally, *i.e.* k-waves, $k \neq j$, is subordinate to the spreading of j-characteristics. We shall presently show that the only unfavorable coupling occurs in the directions complementary to r_j. For the purpose of formulating this result, let us restrict our attention to a small neighborhood $N(\tilde{v})$ of a fixed state $\tilde{v}$ in R^n and choose a coordinate system of functions $\phi_j = \phi_j(u), j = 1, 2, \cdots, n$ which are defined in $N(\tilde{v})$ and satisfy

$$r_j(u) \cdot \nabla\phi_j(u) = 1 \quad \text{and} \quad r_j(\tilde{v}) \cdot \nabla\phi_k(\tilde{v}) = 0 \qquad \text{if} \quad k \neq j.$$

The functions ϕ_j provide a convenient way to estimate the coordinates of a given vector in the basis of eigenvectors: if u and v lie in $N(\tilde{v})$ then

$$u - v = \sum_{k=1}^{n} c_k\, r_k(v)$$

where

$$c_k = \phi_k(u) - \phi_k(v) + 0(|u - v|^2).$$

Lemma 5.2. *There exist positive constants c_1, c_2 and δ with the following property. If u and v lie in $N(\tilde{v})$ and $|u - v| < \delta$ then*

$$\ell_j(v)\, Qf(u, v) \geq c_1\{\phi_j(u) - \phi_j(v)\}^2 - c_2 \sum_{k \neq j} \{\phi_k(u) - \phi_k(v)\}^2.$$

Proof. For the moment let r_k and ℓ_k be arbitrary right and left eigenvectors of ∇f. Differentiating the eigenvalue equation $\nabla f\, r_k = \lambda_k\, r_k$ in the direction r_k yields

$$\nabla f\, \nabla r_k \cdot r_k + \nabla^2 f\,(r_k, r_k) = \lambda_k \nabla r_k \cdot r_k + r_k \cdot \nabla\lambda_k.$$

Taking the inner product with ℓ_k yields the identity

$$\ell_k \nabla f^2 (r_k, r_k) = (\ell_k \cdot r_k)(r_k \cdot \nabla\lambda_k).$$

Thus, under the normalization $r_j \cdot \nabla\lambda_j = 1$, the vector $\ell_j \equiv r_j^t \nabla^2\eta$ satisfies

$$\ell_j \nabla^2 f (r_j, r_j) \equiv r_j^t \nabla^2\eta\, r_j > 0.$$

Letting $u - v = \sum c_k r_k(v)$ we obtain

$$\ell_j(v)\, Qf(u, v) = \ell_j \nabla^2 f(u - v)^2 + 0(u - v)^3$$

$$= c_j^2 \ell_j \nabla^2 f(r_j(v), r_j(v)) + 0\Big(\sum_{k \neq j} c^\ell c_k\Big) + 0(u - v)^3.$$

This completes the proof of the lemma.

Corollary 5.1. *Suppose that Ω_j is a centered Lipschitz j-rarefaction wave in a solution v and suppose that u is an arbitrary weak solution. If $|u - v|_\infty < \delta$ and $E \subset \Omega$ then*

(5.4) $$\gamma(E) \leq \theta_u(E) + \iint_E \frac{\text{const.}}{t} \sum_{k \neq j} \{\phi_k(u) - \phi_k(v)\}^2 + \text{const.}\ |u - v|^2 dxdt,$$

where δ depends only on $|v|_\infty$ and f.

Remark. The constants which appear in (5.4) depend only on the bounds for the coefficients α_k. The estimate (5.4) may be sharpened by taking advantage of the favorable sign of the coefficient $\{\phi_j(u) - \phi_j(v)\}$. We shall not make use of this refinement.

Henceforth, we shall restrict our attention to genuinely nonlinear systems of two equations and solutions with small oscillation. An extension to solutions with large oscillation will be discussed below. The following lemma shows that the coupling between u and w on a j-rarefaction wave in w and in L^2 of the complementary direction r_k, $k \neq j$ is bounded by a small fraction of the dissipation in u plus a quantity on the order of the L^2-deviation between u and w in space-time. Suppose that $j = 1$ and let

$$S_M^1(t) = \{(x, s) : x \leq x_M^1(s)\ ;\ 0 \leq s < t\}$$

where x_M^1 is the maximal 1-characteristic in w passing through the origin.

Lemma 5.3. *For every $\epsilon > 0$ and $M > 0$ there exists a constant $\delta(\epsilon, M)$ with the following property. If u is a solution in $K\{\mathcal{S}(T)\}$ whose initial data coincide with $w(x, 0)$ at almost all x and if*

$$|u|_\infty + |w|_\infty \leq M$$

$$\text{osc}\ u + \text{osc}\ w \leq \delta$$

then for $0 \leq t < T$

(5.5) $$\int_{-\infty}^{x_M^1(t)} \{\phi_2(u(x, t)) - \phi_2(w(x, t))\}^2 dx \leq \text{const.}\ \epsilon |\theta_u\{S_M^1(t)\}|$$

$$+ \text{ const. } \int_0^t \int_{-\infty}^{x_M^1(t)} |u(x, t) - w(x, t)|^2 dx dt.$$

Remarks. The constants appearing in (5.5) depend only on f and M. A similar result holds in the case $j = 2$ for the region S_m^2 to the right of the minimal 2-characteristic x_m^2 passing through the origin:

$$(5.6) \qquad \int_{x_m^2(t)}^{\infty} \{\phi_1(u(x, t)) - \phi_1(w(x, t))\}^2 dx \le \text{const. } \epsilon|\theta_u\{S_m^2(t)\}|$$
$$+ \text{ const } \int_0^t \int_{x_m^2(t)}^{\infty} |u(x, t) - w(x, t)|^2 dx dt,$$

if $u(x, 0)$ and $w(x, 0)$ coincide for almost all $x > 0$.

In order to prove Lemma 5.3 we shall construct two one-parameter families of non-convex entropies. We may assume without loss of generality that $\{\phi_j\}$ forms a coordinate system of Riemann invariants, *i.e.*,

$$r_j(u) \cdot \nabla\phi_k(u) \equiv \delta_{jk}.$$

For each $\bar{u} \in R^2$ we shall construct two families of entropies $\eta_j = \eta_j(u, \phi)$ and corresponding fluxes $q_j = q_j(u, \phi)$, $j = 1, 2$, which are defined and twice continuously differentiable on a set of the (convenient) form $B(\bar{u}) \times I_j(\bar{u})$ where

$$B(\bar{u}) = \{u \in R^2 : |\phi_j(u) - \phi_j(\bar{u})| \le \epsilon, j = 1, 2\}$$
$$I_j(\bar{u}) = \{\phi \in R : |\phi - \phi_k(\bar{u})| \le \epsilon, k \ne j\}.$$

We regard η_j and q_j as parametrized by ϕ. In addition η_j and q_j will have the following three properties. If u and v lie in $B(\bar{u})$ and $\phi_k(v)$ lies in $I_j(\bar{u})$ then

P_1: $\text{const.}\{\phi_k(u) - \phi_k(v)\}^2 \le \eta_j(u, \phi_k(v)) \le \text{const.}\{\phi_k(u) - \phi_k(v)\}^2, \; k \ne j.$

P_2: $\nabla_u^2 \eta_j(\bar{u}, \phi_k(\bar{u})) \ge 0.$

P_3: $\lambda_1(v)\eta_1(u, \phi_2(v)) - q_1(u, \phi_2(v)) \le 0$ and

$$\lambda_2(v)\eta_2(u, \phi_1(v)) - q_2(u, \phi_1(v)) \ge 0$$

where the constants in P_1 are positive.

Roughly speaking η_j is nearly convex in the minor characteristic direction r_k and nearly flat in the major characteristic direction r_j. In order to construct η_j and q_j let us take the curl of the compatibility equation $\nabla\eta\nabla f = \nabla q$ to obtain a smooth strictly hyperbolic second order linear equation for η:

$$(5.7) \qquad \operatorname{curl}\{\nabla\eta(u) \, \nabla f(u)\} = 0.$$

The characteristic directions for (5.7) are given by the right eigenvectors r_j of ∇f: taking the inner product of the compatibility equation with r_j yields

$$\lambda_j \nabla\eta_j \cdot r_j - \nabla q_j \cdot r_j = 0.$$

Thus, the level curves of the Riemann invariants ϕ_j are precisely the characteristics of (5.7) and one may deal equally well with the canonical form of (5.4):

$$\eta_{\phi_1\phi_2} + a\eta_{\phi_1} + b\eta_{\phi_2} + c\eta = 0.$$

In constructing η_j and q_j it is convenient to employ the rarefaction wave curves and the compression wave curves through a specified point u_0:

$$R_j(u_0) = \{u : \phi_j(u) \geq \phi_j(u_0),\ \phi_k(u) = \phi_k(u_0),\ k \neq j\}$$

$$C_j(u_0) = \{u : \phi_j(u) \leq \phi_j(u_0),\ \phi_k(u) = \phi_k(u_0),\ k \neq j\}.$$

First, we shall construct the family of entropies $\eta_1(u, \phi)$. Fix a point $\tilde{u} \in R^2$ and consider a nearby point $\tilde{v} \neq \tilde{u}$ which lies on $R_1(\tilde{u})$. Let $\bar{v}$ denote a typical point on the wave curve $W_2(\tilde{v}) = R_2(\tilde{v}) \cup C_2(\tilde{v})$. We shall associate with such $\bar{v}$ the following two local Goursat problems for the equation (5.7), *cf.* Figure 2.

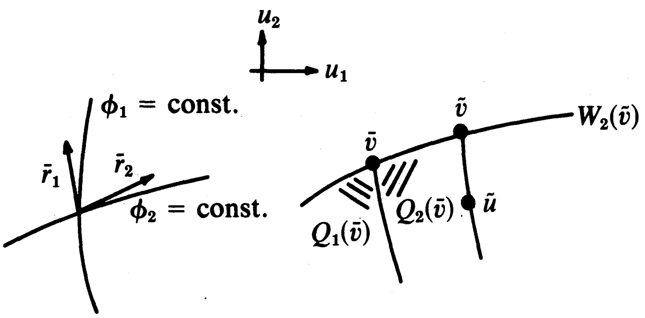

Figure 2

(I): Solve (5.7) in the quadrant $Q_1(\bar{v}) = \{u : \phi_1(u) \leq \phi_1(\bar{v}),\ \phi_2(u) \leq \phi_2(\bar{v})\}$ with data

$$\eta_1 = 0 \quad \text{on} \quad C_1(\bar{v}) \quad \text{and} \quad \eta_1 = \{\phi_2(u) - \phi_2(\bar{v})\}^2 \quad \text{on} \quad C_2(\bar{v}).$$

(II): Solve (5.7) in the quadrant $Q_2(\bar{v}) = \{u : \phi_1(u) \leq \phi_1(\bar{v}),\ \phi_2(u) \geq \phi_2(\bar{v})\}$ with data

$$\eta_1 = 0 \quad \text{on} \quad C_1(\bar{v}) \quad \text{and} \quad \eta_1 = \{\phi_2(u) - \phi_2(\bar{v})\}^2 \quad \text{on} \quad R_2(\bar{v}).$$

The entropy $\eta_1 = \eta_1(u, \phi)$ is defined as follows. Let $\bar{v}(\phi)$ denote the point of $W_2(\tilde{v})$ such that

$$\phi_2(\bar{v}(\phi)) = \phi.$$

In the quadrant $Q_1(\bar{v}(\phi))$ define $\eta_1(u, \phi)$ to be the solution of problem (I) and in $Q_2(\bar{v}(\phi))$ define $\eta_1(u, \phi)$ to be the solution of problem (II). It follows directly from the classical representation formula for the solution of the Goursat problem that η_1 is defined and twice continuously differentiable in a neighborhood $N(\bar{v})$ of $(\bar{v}, \phi_2(\bar{v}))$ of the form

$$N(\bar{v}) = \{u \in R^2 : |\phi_j(u) - \phi_j(\bar{v})| < \bar{\epsilon}, j = 1, 2\} \times \{\phi : |\phi - \phi_2(\bar{v})| < \bar{\epsilon}\},$$

and satisfies there property P_1. Furthermore, we may guarantee that $\bar{\epsilon}$ is uniformly bounded away from zero if $\bar{v}$ is confined to a compact set. Thus, one may associate with each vector $\bar{u}$ in a given compact subset of R^2 an auxiliary vector $\bar{v}$ such that

$$N(\bar{v}) \supset B(\bar{u}) \times I_1(\bar{u})$$

and ϵ is bounded away from zero. We observe that property P_2 is a corollary of property P_1 : P_1 implies that

$$r_2^t(\bar{u})\ \nabla_u^2\, \eta_1(\bar{u}, \phi_2(\bar{u}))\ r_2(\bar{u}) > 0$$

and

$$r_1^t(\bar{u})\ \nabla_u^2\, \eta_1(\bar{u}, \phi_2(\bar{u}))\ r_2(\bar{u}) = 0$$

since η_1 vanishes along $C_1(\bar{v})$.

Lastly, we shall show that property P_3 holds in $B(\bar{u}) \times I_1(\bar{u})$ with a possibly smaller choice of ϵ. We normalize the flux $q_1(u, \phi)$ by the requirement that it vanish at $\bar{v}(\phi_2)$:

$$q_1(u, \phi) = \int_{\bar{v}(\phi)}^{u} \nabla_u \eta(u, \phi)\ \nabla f(u).$$

Suppose now that u and v lie in $B(\bar{u})$ and that $\phi_2(v)$ lies in $I_1(\bar{u})$. Let us fix v and examine the function

$$\pi(u) \equiv \lambda_1(v)\, \eta_1(u, \phi_2(v)) - q_1(u, \phi_2(v)).$$

Our goal is to show that π is non-positive. We first observe that π vanishes on that portion of the wave curve $W_1(v) = R_1(v) \cup C_1(v)$ which lies in $B(\bar{u})$: η_1 vanishes by definition and q_1 vanishes since

$$r_1(u) \cdot \nabla_u q_1(u, \phi_2(v)) = \lambda_1(u)\ r_1(u) \cdot \nabla_u \eta_1(u, \phi_2(v)) = 0$$

if u lies on $W_1(v)$. Similarly it can be seen that the derivative of π in the direction r_2 vanishes on W_1 : if $u \in W_1(v)$ then

$$\begin{aligned} r_2(u) \cdot \nabla_u \pi &= \lambda_1(v)\ r_2(u)\ \nabla_u \eta_1(u, \phi_2(v)) - r_2(u)\ \nabla_u q_1(u, \phi_2(v)) \\ &= \{\lambda_1(v) - \lambda_2(v)\} r_2(u) \cdot \nabla_u\, \eta_1(u, \phi_2(v)) = 0 \end{aligned}$$

by property P_1. Next, we shall expand π in a finite Taylor series. Fix a point u near v and let z denote the point of intersection of $W_1(v)$ and $W_2(u)$. Then

$$\begin{aligned} \eta_1(u, \phi_2(v)) &= \frac{1}{2}\, r_2^t(z)\ \nabla_u^2 \eta_1(z, \phi_2(v))\ r_2(z)\{\phi_2(u) - \phi_2(v)\}^2 + o\{\phi_2(u) - \phi_2(v)\}^2 \\ q_1(u, \phi_2(v)) &= \frac{1}{2}\, r_2^t(z)\ \nabla_u^2 q_1(z, \phi_2(v))\ r_2(z)\{\phi_2(u) - \phi_2(v)\}^2 + o\{\phi_2(u) - \phi_2(v)\}^2. \end{aligned} \tag{5.9}$$

Differentiating the compatibility equation twice in the direction r_2 shows that the leading terms in (5.9) differ by a factor of $\lambda_2(z)$. Therefore

$$\lambda_2(z)\,\eta_1(u, \phi_2(v)) - q_1(u, \phi_2(v)) = o\{\phi_2(u) - \phi_2(v)\}^2.$$

We conclude that

$$\begin{aligned}\pi &= \{\lambda_1(v) - \lambda_2(z)\}\eta_1(u, \phi_2(v)) + \lambda_2(z)\eta_1(u, \phi_2(v)) - q_1(u, \phi_2(v))\\ &= \{\lambda_1(v) - \lambda_2(z)\}\eta_1(u, \phi_2(v)) + o\{\phi_2(u) - \phi_2(v)\}^2 \leq 0,\end{aligned}$$

if u and v are sufficiently close. This completes the verifications of properties $P_1 - P_3$ for η_1 and q_1. The construction of η_2 and q_2 is similar.

Proof of Lemma 5.3. For concreteness we shall consider the case where $w(x, t)$ contains a centered 1-rarefaction wave. Applying Green's theorem to the measure

$$\mu \equiv \eta_1(u, \phi_2(w))_t + q_1(u, \phi_2(w))_x$$

we obtain

$$\mu\{S_M^1(t)\} = \int_{-\infty}^{x_M^1(t)} \eta_1\{u(x, t), \phi_2(w(x, t))\}dx + \int \nu_t\eta_1 + \nu_x q_1 ds$$

where the second integral is taken over $\{(x, s) : x = x_M^{(1)}(s);\ 0 \leq s < t\}$. Since the second integral is non-negative by virtue of property P_3 we need only estimate the μ-measure of $S_M^1(t)$. To this end, let us partition the domain into jump points of u and points of approximate continuity of u:

$$S_M^1(t) = J \cup A$$

where $J = J(t) \equiv S_M^1(t) \cap \Gamma(u)$ and $A = A(t) = S_M^1(t) \cap \Gamma^c(u)$. We claim that

$$\mu(J) \leq \text{const.}\ \epsilon|\theta_u(J)| \tag{5.10}$$

if the range of u and w is contained in a sufficiently small set. It follows from a theorem of Lax [26] that if $\tilde{\eta}(u)$ is an arbitrary C^2 entropy and $\tilde{q}(u)$ is its associated flux and if u_ℓ and u_r are connected by a j-shock with speed τ and magnitude ω then the associated rate of dissipation is third order in ω:

$$\tau[\tilde{\eta}] - [\tilde{q}] = \frac{1}{2}\, r_j^t(u_\ell)\, \nabla^2\tilde{\eta}(u_\ell)\, r_j(u_\ell)\, \omega^3 + o(\omega^3).$$

Therefore, if u and v are two nearby vectors in R^2, property P_2 implies that

$$\nabla_u^2\eta(u, \phi_2(v)) \geq -\epsilon$$

and we conclude that

$$\begin{aligned}&\tau\{\eta_1(u_\ell, \phi_2(w)) - \eta_1(u_r, \phi_2(w))\} - q_1(u_\ell, \phi_2(w)) + q_1(u_r, \phi_2(w))\\ &\qquad \leq \text{const.}\ \epsilon|\omega|^3 \leq \text{const.}\ \epsilon|\tau[\eta] - [q]|\end{aligned}$$

if u_ℓ and u_r are connected by a weak admissible j-shock. This completes the proof of (5.10). Next, we shall estimate the μ-measure of A. We claim that if $E \subset A$ then

(5.11) $$|\mu(E)| \leq \iint_E \text{const. } |u - w|^2 \frac{\partial}{\partial x} \phi_2 dx dt$$

where $\phi_2 = \phi_2(w(x, t))$ and the constant depends only on f. Now, by the chain rule we have on the set A

$$\begin{aligned} \mu &= \eta(u, \phi)_t + q(u, \phi)_x \\ &= \{q_u(u, \phi) - \eta_u(u, \phi) \nabla f(u)\}u_x + \eta_\phi(u, \phi)\phi_t + q_\phi(u, \phi)\phi_x \end{aligned}$$

where we have suppressed the subscripts for simplicity in printing. We observe that the coefficient of u_x vanishes and that $\phi = \phi(w(x, t))$ satisfies the characteristic equation

$$\phi_t + \lambda_2(w)\phi_x = 0.$$

Thus on A, $\mu = p(u(x, t), w(x, t), \phi(x, t))\phi_x(x, t)$ where

$$p(u, w, \phi) \equiv q_\phi(u, \phi) - \lambda_2(w)\, \eta_\phi(u, \phi).$$

In order to prove (5.11) it is sufficient to show that

(5.12) $$|p(u, w, \phi(w))| \leq \text{const. } |u - w|^2.$$

If we write $\eta(u, \phi)$ in the form $\eta(u, \phi) = a(u, \phi)(\phi(u) - \phi)^2$ and then differentiate with respect to ϕ we obtain

$$\begin{aligned} \eta_\phi(u, \phi) &= -2a\{\phi(u) - \phi\} + a_\phi\{\phi(u) - \phi\}^2 \\ &= -2\eta(u, \phi)/\{\phi(u) - \phi\} + 0\{\phi(u) - \phi\}^2. \end{aligned}$$

The boundedness of a_ϕ follows from the integral representation of $\eta(u, \phi)$. In a similar way we obtain

$$q_\phi(u, \phi) = -2q(u, \phi)/\{\phi(u) - \phi\} + 0\{\phi(u) - \phi\}^2.$$

Therefore,

$$p(u, w, \phi(w)) = \tilde{p}(u, w, \phi(w))/\{\phi(u) - \phi(v)\} + 0(u - v)^2.$$

We shall complete the proof of (5.11) by showing that

(5.13) $$|\tilde{p}(u, w, \phi(w))| \leq \text{const. } |u - v|\{\phi(u) - \phi(v)\}^2.$$

Fix the states u and v and let u' denote the point of intersection of $W_1(u)$ and $W_2(v)$. We observe that

$$\begin{aligned} \lambda_2(v)\eta(u', \phi(v)) - q(u', \phi(v)) &= 0\{\phi(u') - \phi(v)\}^3 \\ \eta(u, \phi(v)) - \eta(u', \phi(v)) &= 0\{|r_1 \cdot \eta_u|\, |u - u'|\} \\ r_1 \cdot \eta_u &= 0\{\phi(u) - \phi(v)\}^2. \end{aligned}$$

These facts together with corresponding estimates for q establish (3.12). This completes the proof of Lemma 5.3.

Proof of Theorem 5.1. Suppose u lies in K and $u_0(x) = w_0(x)$ for almost all x. Then

$$\int_{-\infty}^{\infty} \alpha\{u(x, t), w(x, t)\}dx = \gamma\{\mathscr{S}(t)\}.$$

Let us assume for concreteness that w contains a centered Lipschitz 1-rarefaction wave Ω_1. Since w is Lipschitz on Ω_1^c,

$$\gamma\{\mathscr{S}(t) \cap \Omega_1^c\} \leq \theta_u\{\mathscr{S}(t) \cap \Omega_1^c\} + \iint_{\Omega_1^c} \text{const. } |u - w|^2 dxdt,$$

while Corollary 5.1 implies that

$$\gamma\{\mathscr{S}(t) \cap \Omega_1\} \leq \theta_u\{\mathscr{S}(t) \cap \Omega_1\} + \iint_{\Omega_1} \frac{\text{const.}}{t} \{\phi_2(u) - \phi_2(w)\}^2 + \text{const. } |u - w|^2 dxdt.$$

Therefore,

$$\int_{-\infty}^{\infty} \alpha dx \leq \theta_u\{\mathscr{S}(t)\} + \iint_{\Omega_1} \frac{\text{const.}}{t} \{\phi_2(u) - \phi_2(w)\}^2 dxdt + \int_0^t \int_{-\infty}^{\infty} \text{const. } |u - w|^2 dxdt.$$

It follows from Lemma 5.3 that

$$\theta_u\{S_M^1(t)\} \leq -\int_{-\infty}^{x_M^1(t)} \frac{\text{const.}}{\epsilon} \{\phi_2(u) - \phi_2(w)\}^2 dx + \int_0^t \int_{-\infty}^{x_M^1} \frac{\text{const.}}{\epsilon} |u - w|^2 dxdt$$

if the oscillation of u and w is sufficiently small. Therefore,

$$(5.13) \quad \int_{-\infty}^{x_M^1(t)} \text{const.}(1 + 1/\epsilon)\{\phi_2(u) - \phi_2(w)\}^2 dx + \int_{x_M^1(t)}^{\infty} \text{const.}\{\phi_2(u) - \phi_2(w)\} + \int_{-\infty}^{\infty} \text{const.}\{\phi_1(u) - \phi_1(w)\}^2 dx \leq \int_0^t \int_{-\infty}^{\infty} \frac{\text{const.}}{t} \{\phi_2(u) - \phi_2(w)\}^2 dxdt + \int_0^t \int_{-\infty}^{\infty} \frac{\text{const.}}{\epsilon} |u - w|^2 dxdt.$$

If we let $g(t)$ denote the right hand side of (5.13) then

$$0 \leq g(t) \leq \int_0^t \left\{ \frac{\text{const.}}{\epsilon} + \frac{\text{const. } \epsilon}{t} \right\} g(s)ds$$

which in general implies that $g \equiv 0$ if $g = o(t^\epsilon)$ as t approaches zero. However the latter condition is satisfied since we may take ϵ small and

$$\int_{-\infty}^{\infty} |u(x, t) - v(x, t)|^2 dx \leq \text{const. } t$$

by virtue of finite propagation speed. This completes the proof of Theorem 5.1.

Proof of Theorem 5.2. A straightforward calculation shows that

$$\ell_j(v)Qf(u, v) \geq 0$$

for both $j = 1$ and $j = 2$ if $p' < 0$ and $p'' > 0$. We conclude from Lemma 5.1 that in the case of quasilinear wave equation (1.5)

$$\gamma(u, w)\{E\} \leq \theta_u(E) + \iint_E \text{const. } |u - w|^2 dxdt$$

if E is contained in a centered Lipschitz j-rarefaction wave, $j = 1$ or $j = 2$. Theorem 5.2 follows immediately by applying γ to the strip $\mathscr{S}(T)$.

6. General solutions. We may now combine the results of the previous sections to obtain the uniqueness of piecewise Lipschitz solutions.

Theorem 6.1. *Consider a genuinely nonlinear system of two conservation laws of the form (1.1). For every $\bar{u} \in R^2$ there exists a constant $\delta > 0$ depending only on $\bar{u}$ and f with the following property. If $u \in K\{\mathscr{S}(T)\}$ $w \in PL\{\mathscr{S}(T)\}$, $|u - \bar{u}|_\infty < \delta$, $|w - \bar{u}|_\infty < \delta$ and $u(x, 0) = w(x, 0)$ for almost all x then $u = w$ for almost all (x, t) in $\mathscr{S}(T)$.*

Theorem 6.2. *If $u \in K\{\mathscr{S}(T)\}$ and $u \in PL\{\mathscr{S}(T)\}$ are solutions to the quasilinear wave equation (1.5) with $p' < 0$ and $p'' > 0$ and if $u(x, 0) = w(x, 0)$ for almost all x then $u = v$ for almost all (x, t) in $\mathscr{S}(T)$.*

Suppose that $u \in K\{\mathscr{S}(T)\}$ and $w \in PL\{\mathscr{S}(T)\}$. Let $x_m^1(t)$ and $y_m^1(t)$ denote the minimal 1-characteristics in u and w passing through the point $(b, 0)$. Let $x_M^2(t)$ and $y_M^2(t)$ denote the maximal 2-characteristics in u and w passing through the point $(a, 0)$, $a < b$. Put

$$\phi^1(t) = \max\{x_m^1(t), y_m^1(t)\} \quad \text{and} \quad \phi^2(t) = \min\{x_M^1(t), y_M^1(t)\}.$$

Theorem 6.3. *Suppose that $u(x, 0) = w(x, 0)$ for almost all $x \in (a, b)$. Under the hypotheses of Theorem 6.1 and Theorem 6.2 respectively, $u(x, t) = w(x, t)$ for almost all (x, t) satisfying*

$$\phi^1(t) < x < \phi^2(t), \qquad 0 \leq t < T.$$

We also obtain uniqueness of classical solutions to the Riemann problem for system with degenerate eigenvalues. We recall that the classical solution of the Riemann problem is a similarity solution $w = w(x/t)$ which consists of constant states separated by admissible shock waves, contact discontinuities and centered rarefaction waves.

Theorem 6.4. *Consider a system of two conservation laws whose eigenvalues λ_j are either genuinely nonlinear or linearly degenerate in the sense of Lax. For every $\bar{u} \in R^2$ there exists a constant $\delta > 0$ depending only on $\bar{u}$ and f*

with the following property. If w is a classical solution to the Riemann problem and if $u \in K$ with $|w - \bar{u}|_\infty < \delta$, $|u - \bar{u}|_\infty < \delta$ and $u(x, 0) = w(x, 0)$ almost everywhere then $u = w$ almost everywhere.

Sketch of proof. We observe that if λ_j is linearly degenerate, *i.e.*, $r_j \cdot \nabla\lambda_j \equiv 0$, then the dissipation function

$$d(\tau, w_\ell, w_r) = \tau\{\eta(w_\ell) - \eta(w_r)\} - q(w_\ell) + q(w_r)$$

vanishes if w_ℓ and w_r are connected by a j-contact with speed τ:

$$\tau(w_\ell - w_r) = f(w_\ell) - f(w_r), \qquad \tau = \lambda_j(w_\ell) = \lambda_j(w_r). \tag{6.1}$$

Therefore, the function $D(\tau, u_\ell, u_r, v_\ell, v_r)$ vanishes if both pairs (u_ℓ, u_r) and (v_ℓ, v_r) are connected by j-contacts such that either $u_\ell = v_\ell$ or $u_r = v_r$. Let us assume for simplicity that $w = w(x/t)$ consists of a 1-contact separating two constant states:

$$w = \begin{Bmatrix} w_\ell & \text{if } x/t < \tau \\ w_r & \text{if } x/t > \tau \end{Bmatrix}$$

where w_ℓ, w_r and τ satisfy (6.1) with $j = 1$. Let S_m^1 denote the minimal 1-characteristic in u passing through the origin and consider the comparison function $\tilde{w}$ defined by

$$\tilde{w} = \begin{Bmatrix} w_\ell & \text{if } x < x(t) \\ w_r & \text{if } x > x(t) \end{Bmatrix}.$$

We observe that

$$\gamma(u, \tilde{w})(S_m^1) = 0$$

since both of the limiting states of u and $\tilde{w}$ on the right side of S_m^1 lie on the 1-contact curve through w_ℓ. In addition, by (3.4) $\gamma(u, \tilde{w})$ is non-positive on the complement of S_m^1 since $\tilde{w}$ is identically constant there. We conclude that u and $\tilde{w}$ coincide almost everywhere. Hence S_m^1 is a straight line propagating with speed $\lambda_1(w_\ell)$ and $w \equiv \tilde{w}$. This completes the proof of Theorem 6.3.

In a similar way one may obtain uniqueness theorems in the large for the Riemann problem for systems of two equations such that one characteristic field is linearly degenerate while the other satisfies $\ell_j \cdot Qf \geq 0$.

7. The entropy rate criterion. In this section we shall show that the entropy rate criterion of Dafermos [4] is satisfied by Lipschitz continuous solutions to general systems of n equations and by PL solutions to the quasilinear wave equation. For simplicity we shall formulate the entropy rate criterion for weak solutions defined on the strip $\mathcal{S}(T)$. A weak solution u defined on $\mathcal{S}(T)$ is said to satisfy the entropy rate criterion with respect to a class G of weak solutions if for every a and b there exists a null set $N(a, b)$ of $[0, T)$ with the following property: Fix $\tau \in N^c(a, b)$ and suppose that u is a weak solution whose domain of definition Ω is open and contains

$$\{(x, t) : a \leq x \leq b,\ t = \tau\}.$$

If

$$u(x, \tau) = w(x, \tau)$$

for almost all x in $\Omega \cap \{(x, t) : t = \tau\}$, then

$$D\,\theta_w\{R(\tau)\} \leq \underline{D}^+\,\theta_u\{R(\tau)\} \tag{7.1}$$

where $\underline{D}^+$ denotes the lower right hand derivative and

$$R(\tau) = \{(x, t) : a < x < b,\ 0 \leq t < \tau\}.$$

We note that the derivative

$$D\,\theta_w\{R(\tau)\} \equiv \frac{d}{d\tau}\,\theta_w\{R(\tau)\}$$

necessarily exists for almost all τ since $\theta_w\{R(\tau)\}$ is an absolutely continuous function of τ. (As always we assume that $u = \bar{u}$.) Indeed, if u is an arbitrary weak solution then

$$\theta_u\{R(t, \tau)\} = \int_a^b \eta(u(x, t)) - \eta(u(x, \tau))dx + \int_\tau^t q(u(b - 0, s) - q(u(a + 0, s))ds \tag{7.2}$$

where

$$R(t, \tau) = \{(x, s) : a < x < b,\ t < s < \tau\}.$$

The terms on the right hand side of (7.2) are absolutely continuous functions. Using (7.2) the entropy rate criterion can be stated in terms of the rate of decay of local entropy: for almost all τ, $D\theta_u$ equals the time rate of change of total entropy in (a, b) modulo the flux of q at the boundary points a and b, *i.e.*

$$D\,\theta_u(R(\tau)) = D\int_a^b \eta(u(x, \tau)dx + q(u(b - 0, \tau)) - q(u(a + 0, \tau)).$$

One may, of course, formulate slight variants of (7.1).

We shall first show that Lipschitz continuous solutions satisfy (7.1) with respect to the class G_1 of weak solutions u with the following properties: u is defined on an open domain $\Omega \subset \mathcal{S}(T)$ and if

$$\text{ess}\lim\{u(x, t_0) : x \to x_0\}$$

exists where $(x_0, t_0) \in \Omega$ then

$$\text{ess lim}\{u(x, t) : (x, t) \to (x_0, t_0), t > t_0\}$$

exists and they are equal. We recall that the existence of the essential limit means that the function can be modified on a set of zero Lebesgue measure in such a way that the corresponding pointwise limit exists. The above condition may be regarded as a very mild form of finite propagation speed for waves.

Theorem 7.1. *Suppose that (1.1) is a system of n conservation laws with a smooth entropy-entropy flux pair (η, q) where η is strictly convex. If w is a Lipschitz continuous solution defined on $\mathcal{S}(T)$ then w satisfies (7.1) with respect to the class G_1.*

Proof. Fix a, b and τ with $0 < \tau < T$. Consider the trapezoidal regions of the form

$$\Omega(\tau, t) = \{(x, s) : x(s) < x < y(s), \tau \leq s < t\}$$

where

$$x(s) = a + c(s - \tau), \qquad y(s) = b - c(s - \tau)$$

and $t - \tau$ is small. Suppose that u is a solution in G_1 where the domain of definition contains

$$\{(x, t) : a < x < b, t = \tau\}$$

and such that $u(x, \tau) = w(x, \tau)$ for almost all x. It follows from the proof of Theorem 3.1 that, for a sufficiently large constant,

$$g(t) \leq \theta_u\{\Omega(\tau, t)\} + \text{const.} \int_\tau^t g(s)ds$$

where

$$g(t) = \int_{x(t)}^{y(t)} \alpha\{u(x, t), w(x, t)\}dx.$$

Thus,

$$g(t) \leq \theta_u\{\Omega(\tau, t)\} + \text{const.} \int_\tau^t e^{(t-s)}\theta_u\{\Omega(\tau, s)\}ds$$

and we conclude that

$$0 \leq \theta_u\{\Omega(\tau, t)\} + o(t - \tau) \tag{7.3}$$

since $\theta_u\{\Omega(t, \tau)\}$ is continuous and vanishes at $t = \tau$. We shall complete the proof by showing that

$$\theta_u\{R(\tau, t)\} = \theta_u\{\Omega(\tau, t)\} + o(t - \tau). \tag{7.4}$$

Combining (7.3) and (7.4) yields

$$D\theta_w\{R(\tau)\} = 0 \leq \underline{D}^+\theta_u\{R(\tau)\}.$$

Now, a straightforward calculation shows that

$$\theta_u\{R(\tau, t)\} = \int_\tau^t \left\{\sum \sigma[\eta] - [q]\right\}(s)ds \tag{7.5}$$

where, for almost all s, the summation is taken over all points of discontinuity $\{x_j(s)\}$ in the restriction $u(\cdot, s)$ of u to the line $t = s$:

$$\begin{aligned}\sum \sigma[\eta] - [q] = \sum_j \sigma\{\eta(u(x_j - 0, s)) - \eta(u(x_j + 0, s))\}\\ -q(u(x_j - 0, s)) + q(u(x_j + 0, s)).\end{aligned}$$

Relation (7.4) follows from (7.5) and the fact that $u(x, \tau)$ coincides with $w(x, \tau)$ for almost all x. Indeed, since

$$\sigma[\eta] - [q] = 0([u]^3).$$

The difference between $\theta_u\{R(t, \tau)\}$ and $\theta_u\{\Omega(t, \tau)\}$ is $o(t - \tau)$ by virtue of the fact that $u(\cdot, \tau)$ is approximately continuous at $x = a$ and at $x = b$. This completes the proof of Theorem 7.1.

Next, we shall show that PL solutions to the quasilinear wave equation (1.5) with $p' < 0$ and $p'' > 0$ satisfy the entropy rate criterion with respect to a large class G_2 of solutions. For simplicity in the formulation, we shall call a measurable function Lipschitz continuous if it is equal almost everywhere to a function which is Lipschitz continuous in the standard sense. By G_2 we shall mean the class of weak solutions u with the following three properties. First, u is defined on an open domain $\Omega \subset \mathcal{S}(T)$. Second, if $u(\cdot, t_0)$ is Lipschitz continuous on (a, b) then $u(x, \tau)$ is Lipschitz continuous in the region between the maximal forward 2-characteristic through (a, t_0) and the minimal forward 1-characteristic through (b, t_0) intersect a small strip of the form

$$\{(x, t) : t_0 \leq t \leq t_0 + \epsilon\}.$$

Third, if $u(\cdot, t_0)$ is piecewise Lipschitz on (a, b) with a jump at c, $a < c < b$, *i.e.* if

$$u(x, t_0) = \begin{Bmatrix} u_\ell(x) & a < x < c \\ u_r(x) & c < x < b \end{Bmatrix}$$

where $u_\ell(x)$ and $u_r(x)$ are Lipschitz, then the speeds of propagation of the minimal forward 1-characteristic through (c, t_0) and the maximal forward 2-characteristic through (c, t_0) approach the speeds of propagation, of the corresponding extreme characteristics in the classical solution to the Riemann problem with initial data $u_\ell(c - 0)$ and $u_r(c + 0)$ as t approaches t_0, $t > t_0$.

Theorem 7.2. *Consider the quasilinear wave equation (1.5) with $p' < 0$ and $p'' > 0$ and suppose that (η, q) is a smooth entropy-entropy flux pair with η*

strictly convex. If $w \in PL\{\mathscr{S}(T)\}$ then w satisfies (7.1) with respect to the class G_2.

Remark. The purpose of presenting Theorem 7.2 is simply to indicate that the entropy rate criterion is satisfied by PL solutions relative to a fairly broad class of comparison solution G_2. We expect that G_2 can be substannally enlarged.

Proof of Theorem 7.2. For simplicity, let us assume that w is Lipschitz continuous on either side of an admissible 1-shock

$$S_w = \{(y(t), t) : 0 \leq t < T\}$$

passing through the origin. Let u be a solution in G_2 whose initial data coincides with w_0 and consider the approximate solution $\tilde{w}$ defined in terms of u and w by (4.11). It follows from Green's theorem that

$$\int_{-\infty}^{\infty} \alpha(u(x, t), \tilde{w}(x, t))dx = \gamma\{\mathscr{S}(t)\}. \tag{7.6}$$

Let $S_u(t)$ denote the restriction of the minimal forward 1-characteristic in u to the interval $[0, t)$:

$$S_u(t) = \{(x(s), s) : 0 \leq s < t\}.$$

Then

$$\gamma\{S_u(t)\} = \int_0^t D(\tau, u_\ell, u_r, \tilde{w}_\ell, \tilde{w}_r)dt$$

where $\tau = x'(t)$ and the subscripts denote the limiting values of u and $\tilde{w}$ along the edges of S_u:

$$u_\ell = u\{x(t) - 0, t\}, \qquad u_r = u\{x(t) + 0, t\}, \text{ etc.}$$

Lemma 4.3 implies that

$$\begin{aligned} D(\tau, u_\ell, u_r, \tilde{w}_\ell, \tilde{w}_r) &= D(\tau, u_\ell, u_r, w_\ell, w_r) + 0(|w_\ell - \tilde{w}_\ell| + |w_r - \tilde{w}_r|) \\ &= d(\tau, u_\ell, u_r) - d(\sigma, w_\ell, w_r) - \alpha(w_\ell, w_r)(\tau - \sigma) \\ &\quad + 0(|u_\ell - \tilde{w}_\ell| + |w_\ell - \tilde{w}_\ell| + |w_r - \tilde{w}_r|) \end{aligned}$$

where $\sigma = y^1(t)$. It follows from the definitions of G_2 and $\tilde{w}$ that

$$\tau \to \sigma, \qquad \tilde{w}_\ell \to w_\ell, \quad \text{and} \quad \tilde{w}_r \to w_r$$

as t approaches zero where

$$w_\ell = w\{y(t) - 0, t\} \quad \text{and} \quad w_r = w\{y(t) + 0, t\}.$$

Then

$$\gamma\{S_u(t)\} = \theta_u\{S_u(t)\} - \theta_w\{S_w(t)\} + o(t) \tag{7.7}$$

where $S_w(t) = S_w \cap [0, t)$. Since

$$\gamma\{S_{\tilde{U}}^c(t)\} \le \theta_u\{S_u^c(t)\} + \text{const.} \int_0^t \int_{-\infty}^{\infty} |u(x, t) - \tilde{w}(x, t)|^2 dxdt,$$

we deduce from (7.6) and (7.7) that

$$g(t) \le \theta_u\{\mathcal{S}(t)\} - \theta_w\{\mathcal{S}(t)\} + o(t) + \int_0^t \text{const. } g(s)ds$$

where $g(t) = \int_{-\infty}^{\infty} |u(x, t) - w(x, t)|^2 dx$. Therefore,

$$0 \le \theta_u\{\mathcal{S}(t)\} - \theta_w\{\mathcal{S}(t)\} + o(t)$$

and the theorem follows in the case of PL solutions w with the indicated structure.

We note that the proof did not appeal to the special structure of the quasilinear wave equation: indeed, the result above is valued for such solutions w to systems in the Smoller-Johnson class. Unfortunately, in order to treat the general solution in PL we must appeal to the following special property of the quasilinear wave equation:

$$\ell_j(v)\, Qf(u, v) \ge 0 \tag{7.8}$$

for both $j = 1$ and $j = 2$, *cf.* proof of Theorem 5.2. If (7.8) holds then the restriction of γ to Lipschitz centered rarefaction waves in w is bounded from above by a measure which is absolutely continuous with respect to 2-dimensional Lebesgue measure and we may use virtually the same proof given above for the case of general solutions in PL. We omit the details.

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