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# QUASIMONOTONE SCHEMES FOR SCALAR CONSERVATION LAWS. PART III*

BERNARDO COCKBURN†

**Abstract.** In this paper the definition and analysis of the quasimonotone numerical schemes is extended to the general case $d > 1$, where $d$ is the number of space variables. These schemes are constructed using the simple but very important class of monotone schemes that are defined by two-point monotone fluxes. To enforce the compactness in $L^\infty(L^1_{\text{loc}})$ of the sequence of approximate solutions, the case of meshes that are a Cartesian product of one-dimensional partitions is addressed. It is proved that the main stability and convergence results for one-dimensional quasimonotone schemes (of the first type) also hold in the general case. As a by-product of this theory, the theory of relaxed, quasimonotone schemes is developed. These schemes are $L^\infty$-stable, and they can be more accurate than the quasimonotone schemes; unfortunately, the compactness in $L^\infty(L^1_{\text{loc}})$ is lost. Nevertheless, if they converge, they do so to the entropy solution.

**Key words.** conservation laws, entropy schemes, finite elements

**AMS(MOS) subject classifications.** 65M60, 65N30, 35L65

**1. Introduction.** In this paper we define and analyze the quasimonotone numerical schemes for the scalar conservation law

$$\partial_t u + \operatorname{div} \mathbf{f}(u) = 0, \qquad \text{in } (0,T) \times \mathbf{R}^d, \tag{1.1a}$$

$$u(t=0) = u_0, \qquad \text{on } \mathbf{R}^d, \tag{1.1b}$$

where $\mathbf{f}$ is assumed to be $C^1$. These entropy schemes are constructed using monotone schemes defined by means of two-point monotone fluxes. The monotone schemes of this type, [3], [5], are important not only because they can be defined employing a single one-dimensional flux, but because it is possible to prove they are entropy schemes in the case in which the mesh is a Cartesian product of one-dimensional partitions. In the case of an arbitrary triangulation these schemes are not TVD, as can be seen in simple examples, and there is no proof that they are TVB (total variation bounded). This is why in this paper we shall restrict ourselves to that type of grid, although quasimonotone schemes for arbitrarily-shaped triangulations can also be defined. This will be done in a forthcoming paper. Parallel to the theory of QM schemes we shall develop, as a by-product, the theory of relaxed quasimonotone (RQM) schemes. These schemes are $L^\infty$-stable, but they are not necessarily TVD. However, if they converge, they do converge to the entropy solution. Moreover, formal local high-order accuracy is easier to achieve than with QM schemes.

We shall restrict ourselves to the case $d = 2$, for the general case is a straightforward extension of it. An outline of the paper follows. Section 2 is devoted to setting the notation and stating and proving a new maximum principle for implicit monotone schemes. In §3 we define, analyze, and discuss QMFD schemes. The QMFE schemes are considered very briefly in §4. Also, some numerical results are displayed. We end with some concluding remarks in §5.

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†Department of Mathematics, University of Chicago, Chicago, Illinois 60637. This work was supported by the Andrews Foundation. This work is contained in the author's Ph.D. dissertation, University of Chicago, 1986.

**2. Preliminaries.** The notation of [1], [2] will be extended to this case in the obvious usual way. We set $Q_{i,j} = (x_{i-1/2}, x_{i+1/2}) \times (y_{j-1/2}, y_{j+1/2})$, $K^n_{i,j} = J^n \times Q_{i,j}$, and $\delta = \sup_{i,j,n}\{\Delta t^n, \Delta x_i, \Delta y_j\}$. The CFL number is taken to be CFL $= \max\{\mathrm{CFL}_x, \mathrm{CFL}_y\}$, where

$$\mathrm{CFL}_z = \sup_{l,n} \Delta t^n \Delta z_l \|f'_z\|_{L^\infty(a,b)} \quad \text{for } z = x, y,$$

and $a = \inf_x u_o(x), b = \sup_x u_o(x)$.

The monotone schemes that will be considered in this work are of the form

$$(u^n_{i,j} - u^{n-1}{}_{i,j})/\Delta t^n + (f_{xs}^{\mathrm{M},n}{}_{i+1/2,j} - f_{xs}^{\mathrm{M},n}{}_{i-1/2,j})/\Delta x_i + (f_{ys}^{\mathrm{M},n}{}_{i,j+1/2} - f_{ys}^{\mathrm{M},n}{}_{i,j-1/2})/\Delta y_j = 0, \tag{2.1a}$$

where

$$\begin{aligned} f_{xs}^{\mathrm{M},n} &= s\, f_x^{\mathrm{M},n} + (1-s) f_x^{\mathrm{M},n-1}, \\ f_{ys}^{\mathrm{M},n} &= s\, f_y^{\mathrm{M},n} + (1-s) f_y^{\mathrm{M},n-1} \quad \text{for } s \in [0,1], \end{aligned} \tag{2.1b}$$

and $f_x^{\mathrm{M}}$ and $f_y^{\mathrm{M}}$ are two-point monotone consistent fluxes; we shall use the notation $\mathbf{f}_s^{\mathrm{M}} = (f_{xs}^{\mathrm{M}}, f_{ys}^{\mathrm{M}})$. The results about the schemes (2.1) that we need can be found in [3], [5]. However, since the following stability result is new for $s \in (0,1]$ we include it here. Consider the union $U$ of the closed rectangle $\overline{Q}_{i,j}$ with a union of closed rectangles $\overline{Q}_{a,b}$ such that if $\overline{Q}_{ab} \subset U$, then there exists another $\overline{Q}_{l,m} \subset U$ such that $\overline{Q}_{a,b}$ and $\overline{Q}_{l,m}$ share a common edge. A generic set $U$ with this property will be denoted by $\Psi_{i,j}$. By $\partial\Psi_{i,j}$ we denote the union of all sets $\overline{Q}_{a,b}$ that do not belong to $\Psi_{i,j}$ and that share a common edge with some set $\overline{Q}_{l,m} \subset \Psi_{i,j}$.

PROPOSITION 2.1. *Suppose that the scheme* (2.1) *is monotone for* CFL $\in [0, \mathrm{CFL}_0/(1-s)]$. *Then, for* CFL $\in [0, \mathrm{CFL}_0/(1-s)]$,

$$u^n_{i,j} \in I(u^{n-1}_{i,j}, u^{n-1}_{i+1,j}, u^{n-1}_{i-1,j}, u^{n-1}_{i,j-1}) \quad \textit{if } s = 0, \tag{2.2a}$$

$$u^n_{i,j} \in I(u^{n-1}_{a,b}, u^n_{c,d} : \overline{Q}_{a,b} \subset \Psi_{i,j} \cup \partial\Psi_{i,j},\ \textit{and } \overline{Q}_{c,d} \subset \partial\Psi_{i,j}) \qquad \textit{if } s \in (0,1), \tag{2.2b}$$

$$u^n_{i,j} \in I(u^{n-1}_{a,b}, u^n_{c,d} : \overline{Q}_{a,b} \subset \Psi_{i,j},\ \textit{and } \overline{Q}_{c,d} \subset \partial\Psi_{i,j}) \qquad \textit{if } s = 1. \tag{2.2c}$$

*Moreover, the scheme is* TVD, *i.e.:*

$$\|u_h^{n+1}\|_{\mathrm{TV}(\mathbf{R}^2)} \le \|u_h^n\|_{\mathrm{TV}(\mathbf{R}^2)}. \tag{2.3}$$

A proof of (2.2a) and (2.3) can be found in Sanders [5]. Here we include the proof of (2.2b,c).

*Proof.* Let us take $s = 1$. The case $s \in (0,1)$ is done in a similar way. Set

$$\begin{aligned} \Theta^+_{x,a,b} &= -\Delta t^n (f^{\mathrm{M}}_{x,a+1/2,b} - f_x(u_{a,b}))/(\Delta x_a (u_{a+1,b} - u_{a,b})), \\ \Theta^-_{x,a,b} &= +\Delta t^n (f^{\mathrm{M}}_{x,a-1/2,b} - f_x(u_{a,b}))/(\Delta x_a (u_{a-1,b} - u_{a,b})), \\ \Theta^+_{y,a,b} &= -\Delta t^n (f^{\mathrm{M}}_{y,a,b+1/2} - f_y(u_{a,b}))/(\Delta y_b (u_{a,b+1} - u_{a,b})), \\ \Theta^-_{y,a,b} &= +\Delta t^n (f^{\mathrm{M}}_{y,a,b-1/2} - f_y(u_{a,b}))/(\Delta y_b (u_{a,b-1} - u_{a,b})), \end{aligned}$$

and note that these quantities are nonnegative because the fluxes $f_x^{\mathrm{M}}$ and $f_y^{\mathrm{M}}$ are monotone. Now, set

$$D_{a,b} = (1 + \Theta^+_{x,a,b} + \Theta^-_{x,a,b} + \Theta^+_{y,a,b} + \Theta^-_{y,a,b})^{-1},$$

and rewrite the fully implicit scheme (2.1) as follows:

$$(2.4a)\qquad \begin{aligned} u^n_{a,b} = D^n_{a,b}\{u^{n-1}_{a,b} &+ \Theta^+_{x,a,b}u^n_{a+1,b} + \Theta^-_{x,a,b}u^n_{a-1,b} \\ &+ \Theta^+_{y,a,b}u^n_{a,b+1} + \Theta^-_{y,a,b}u^n_{a,b-1}\}. \end{aligned}$$

To prove (2.2c) we proceed by contradiction. Assume then that (2.2c) is false; for example, assume that

$$(2.4b)\qquad u^n_{i,j} > v \quad \forall v \in I(u^{n-1}_{a,b}, u^n_{c,d} : \overline{Q}_{a,b} \subset \Psi_{i,j}, \text{ and } \overline{Q}_{c,d} \subset \partial\Psi_{i,j}).$$

The "less than or equal to" case is treated in the same way. Let us denote by $P_{i,j}$ the point $(x_i, y_j)$. We say that the oriented segment $P_{a,b} \to P_{c,d}$ is an elementary oriented segment if $|a-c| + |b-d| \leq 1$. A path $P_{i_0,j_0} \to P_{i_n,i_n}$ is an oriented path if it is of the form $P_{i_0,j_0} \to P_{i_1,j_1} \to P_{i_2,j_2} \to \cdots \to P_{i_{n-1},j_{n-1}} \to P_{i_n,j_n}$, where the path $P_{i_l,j_l} \to P_{i_{l+1},j_{l+1}}$ is an elementary oriented segment. Consider the set of those paths starting from $P_{i,j}$ with the property that if $P_{a,b} \to P_{c,d}$ then

$$(2.4c)\qquad u^n_{a,b} < u^n_{c,d},$$

and denote it by $\chi_{i,j} = \{\Gamma\}$. If $\Psi_{i,j} = \overline{Q}_{i,j}$ the assumption (2.4b) contradicts (2.4a), and hence (2.2c) is true. If $\Psi_{i,j}$ strictly contains $\overline{Q}_{i,j}$ the set $\chi_{i,j}$ is nonempty because by (2.4a) with $(a,b) = (i,j)$ and by (2.4b) one of the values of the set

$$\{u^n_{i+1,j}, u^n_{i-1,j}, u^n_{i,j+1}, u^n_{i,j-1}\}$$

must be strictly greater than $u^n_{i,j}$. By construction, any $\Gamma \in \chi_{i,j}$ advances in the direction in which $u^n$ increases strictly, and so it cannot intersect itself. Also, by (2.4b), $\Gamma$ cannot cross the boundary of $\Psi_{i,j}$. As a consequence, its ending point must then lie in the interior of that set. By (2.4c) and (2.4a) this is a contradiction, and the proof is complete. □

## 3. Quasimonotone, finite-difference schemes.

**3.1 Definition.** A FD numerical scheme

$$(3.1a)\qquad \begin{aligned} (u^n_{i,j} - u^{n-1}_{i,j})/\Delta t^n &+ (f^{h,n}_{x,i+1/2,j} - f^{h,n}_{x,i-1/2,j})/\Delta x_i \\ &+ (f^{h,n}_{y,i,j+1/2} - f^{h,n}_{y,i,j-1/2})/\Delta y_j = 0, \end{aligned}$$

is a QMFD scheme if $\mathbf{f}^h = (f^h_x, f^h_y)$ is a QMFD flux. Such a flux, $\mathbf{f}^{\mathrm{QM}}$, is a flux of the form

$$(3.1b)\qquad \mathbf{f}^{\mathrm{QM}} = \mathbf{f}^{\mathrm{M}} + \mathbf{a}^{\mathrm{QM}},$$

where $\mathbf{a}^{\mathrm{QM}} = (a^{\mathrm{QM}}_x, a^{\mathrm{QM}}_y)$, and

**(3.2a)** (*Stability in the $x$-direction.*) There exist two discrete functions

$$\nu_{xx} = (\nu^-_{xx}, \nu^+_{xx}) \text{ and } \nu_{xy} = (\nu^-_{xy}, \nu^+_{xy}) \text{ such that:}$$

$$\begin{aligned} \text{(i)}\quad a^{\mathrm{QM}}_{x,i+1/2,j} &= \nu^+_{xx,i+1/2,j}(f^{\mathrm{M}}_{x,i+3/2,j} - f^{\mathrm{M}}_{x,i+1/2,j}), \\ &= \nu^-_{xx,i+1/2,j}(f^{\mathrm{M}}_{x,i+1/2,j} - f^{\mathrm{M}}_{x,i-1/2,j}); \end{aligned}$$

$$\text{(ii)}\quad (1 + \nu^-_{xx,i+1/2,j} - \nu^+_{xx,i-1/2,j}) \in [0,2];$$

$$\text{(iii)}\quad a^{\mathrm{QM}}_{x,i+1/2,j} = \nu^{+}_{xy,i+1/2,j}(f^{\mathrm{M}}_{x,i+1/2,j+1} - f^{\mathrm{M}}_{x,i+1/2,j}),$$
$$= \nu^{-}_{xy,i+1/2,j}(f^{\mathrm{M}}_{x,i+1/2,j} - f^{\mathrm{M}}_{x,i+1/2,j-1});$$
$$\text{(iv)}\quad (1 + \nu^{-}_{xy,i+1/2,j+1} - \nu^{+}_{xy,i+1/2,j}) \in [0,2];$$

(3.2b) (*Stability in the y-direction.*) There exist two discrete functions

$\nu_{yy} = (\nu^{-}_{yy}, \nu^{+}_{yy})$ and $\nu_{yx} = (\nu^{-}_{yx}, \nu^{+}_{yx})$ such that:

$$\text{(i)}\quad a^{\mathrm{QM}}_{y,i,j+1/2} = \nu^{+}_{yy,i,j+1/2}(f^{\mathrm{M}}_{y,i,j+3/2} - f^{\mathrm{M}}_{y,i,j+1}),$$
$$= \nu^{-}_{yy,i,j+1/2}(f^{\mathrm{M}}_{y,i,j+1/2} - f^{\mathrm{M}}_{y,i,j-1/2});$$
$$\text{(ii)}\quad (1 + \nu^{-}_{yy,i,j+1/2} - \nu^{+}_{yy,i,j-1/2}) \in [0,2]:$$
$$\text{(iii)}\quad a^{\mathrm{QM}}_{y,i,j+12} = \nu^{+}_{yx,i,j1/2}(f^{\mathrm{M}}_{y,i+1} - f^{\mathrm{M}}_{y,i,j+1/2}),$$
$$= \nu^{-}_{xy,i,j+12}(f^{\mathrm{M}}_{y,i,j+1/2} - f^{\mathrm{M}}_{y,i-1,j+1/2});$$
$$\text{(iv)}\quad (1 + \nu^{-}_{yx,i+1,j+1/2} - \nu^{+}_{yx,i,j+1/2}) \in [0,2];$$

(3.2c) (*Entropy.*) $\mathbf{a}^{\mathrm{QM}} = O(h^{\alpha})$ for some $\alpha \in (0,1]$.

These conditions are a generalization of conditions (3.2) of [1] defining one-dimensional QM fluxes. It is not difficult to realize that conditions (3.2) state that each of the components of the QM flux $\mathbf{f}^{\mathrm{QM}}$ is itself a one-dimensional QM flux in the $x$-direction as well as in the $y$-direction! (If we relax the stability conditions (3.2a), and only require the $x$-component (respectively, $y$-component) of the flux to be a one-dimensional QM flux in the $x$-direction (respectively, $y$-direction) we obtain what we call a relaxed quasimonotone (RQM) flux, $\mathbf{f}^{\mathrm{RQM}} = \mathbf{f}^{\mathrm{M}} + \mathbf{a}^{\mathrm{RQM}}$.) A flux of the form

$$\text{(3.1c)}\qquad \mathbf{f}^{\mathrm{QM},n}_{s} = s\mathbf{f}^{\mathrm{QM},n} + (1-s)\mathbf{f}^{\mathrm{QM},n-1}, \quad s \in [0,1],$$

is also called a QM flux. (The RQM flux $\mathbf{f}^{\mathrm{RQM}}_{s}$ is then defined in a similar way.) We shall prove later that, as in the one-dimensional case, any QM scheme is TVD thanks to the stability conditions (3.2a,b). These conditions also ensure that the scheme verifies the same maximum principle as the corresponding monotone scheme. Finally, condition (3.2c) ensures convergence to the entropy solution. For RQM schemes the situation is different because they are not necessarily TVD schemes. Thus, by relaxing the stability conditions (3.2a,b) we have lost the compactness in $L^{\infty}(0,T;L^{1}_{\mathrm{loc}})$ of the sequence of approximate solutions. However, RQM schemes verify the same maximum principle as the corresponding monotone scheme, and, thanks to (3.2c), if they converge, they converge to the entropy solution. Moreover, formal local order of accuracy is easier to obtain with this type of scheme, as we shall see later.

The function $\mathbf{a}^{\mathrm{QM}}$ can be chosen using a two-point monotone flux $\mathbf{f}^{\mathrm{M}}$ and a high-order accurate one $\mathbf{f}^{h}$ generalizing the choices (3.3), (3.4) of [1] of the function in the case $d = 1$. For example:

$$\text{(3.3a)}\qquad a^{\mathrm{QM}}_{x,i+1/2,j} = \operatorname{sgn}(u_{i+1,j} - u_{i,j}) \max\{0, \Theta_{i+1/2,j}\},$$

$$\begin{aligned}
(3.3\text{b}) \qquad \Theta_{i+1/2,j} = \min\{&|f^h_{x,i+1/2,j} - f^{\mathrm{M}}_{x,i+12,j}|,\\
&|f^{\mathrm{M}}_{x,i+3/2,j} - f^{\mathrm{M}}_{x,i+1/2,j}|sx_{i+1,j}, cx_{i+1,j}(\Delta x_{i+1})^\alpha,\\
&|f^{\mathrm{M}}_{x,i+1/2,j} - f^{\mathrm{M}}_{x,i-1/2,j}|sx_{i,j}, cx_{i,j}(\Delta x_i)^\alpha,\\
&|f^{\mathrm{M}}_{x,i+1/2,j+1} - f^{\mathrm{M}}_{x,i+1/2,j}|sy_{i+1/2,j+1/2},\\
&|f^{\mathrm{M}}_{x,i+1/2,j} - f^{\mathrm{M}}_{x,i+1/2,j-1}|sy_{i+1/2,j-1/2}\},
\end{aligned}$$

$$(3.3\text{c}) \qquad sx_{i,j} = \operatorname{sgn}((u_{i+1,j} - u_{i,j})(u_{i,j} - u_{i-1,j})),$$

$$(3.3\text{d}) \qquad sy_{i+1/2,j+1/2} = \operatorname{sgn}((u_{i+1,j+1} - u_{i,j+1})(u_{i+1,j} - u_{i,j})),$$

$$(3.3\text{e}) \qquad c_{i,j} \in [0, K] \quad \text{for some fixed } k \in \mathbf{R}^+;$$

or

$$(3.4\text{a}) \qquad a^{\mathrm{QM}}_{x,i+1/2,j} = \operatorname{sgn}(u_{i+1,j} - u_{i,j}) \max\{0, \Theta_{i+1/2,j}\},$$

$$\begin{aligned}
(3.4\text{b}) \qquad \Theta_{i+1/2,j} = \min\{&|f^h_{x,i+1/2,j} - f^{\mathrm{M}}_{x,i+1/2,j}|,\\
&0.5|f^{\mathrm{M}}_{x,i+3/2,j} - f^{\mathrm{M}}_{x,i+1/2,j}|,\\
&0.5|f^{\mathrm{M}}_{x,i+1/2,j} - f^{\mathrm{M}}_{x,i+1/2,j}|,\\
&0.5|f^{\mathrm{M}}_{x,i+1/2,j+1} - f^{\mathrm{M}}_{x,i+1/2,j}|,\\
&0.5|f^{\mathrm{M}}_{x,i+1/2,j} - f^{\mathrm{M}}_{x,i+1/2,j-1}|\},
\end{aligned}$$

$$(3.4\text{c}) \qquad cx_{i,j} \in [0, K] \quad \text{for some fixed } K \in \mathbf{R}^+$$

If we want to define an RQM flux $\mathbf{f}^{\mathrm{RQM}} = \mathbf{f}^{\mathrm{M}} + \mathbf{a}^{\mathrm{RQM}}$, the function $\mathbf{a}$ can be defined in a similar way. For example, $a^{\mathrm{RQM}}_{x,i+1/2,j}$ can be obtained from the expressions (3.3) or (3.4) by simply dropping the terms containing either $f^{\mathrm{M}}_{x,i+1/2,j-1}$ or $f^{\mathrm{M}}_{x,i+1/2,j+1}$. In this case the TVD property of the QMFD schemes is lost.

**3.2. Analysis.** Theorem 3.1 is our main result.

THEOREM 3.1. *Suppose that the scheme* (2.1) *is monotone for* CFL $\in [0, \mathrm{CFL}_0/(1-s)]$. *Then, if* CFL $\in [0, \mathrm{CFL}_0/2(1-s)]$, *any* QM *scheme* (3.1) *verifies the maximum principle* (2.2) *and is* TVD. (*Under the same* CFL-*condition, any* RQM *scheme also verifies the maximum principle* (2.2)). *Moreover,*

$$(3.5) \qquad \begin{aligned}
\int_{\mathbf{R}} J(u(T) - u_h(T)) &\le \int_{\mathbf{R}} J(u_0 - u_{0,h}) + \|u_0\|_{BV(\mathbf{R})}(C_1 T^{1/2}\delta^{1/2} + C_2 T \delta^\alpha),\\
\|u(T) - u_h(T)\|_{L^1(\Omega)} &\le 2\|u_0 - u_{0,h}\|_{L^1(\mathbf{R})} + C_3 T^{1/2}\|u_0\|_{BV(\mathbf{R})}\delta^{1/2}\\
&\quad + C_4 T^{1/2}(\|u_0\|_{BV(\mathbf{R})}|\Omega|)^{1/2}\delta^{\alpha/2},
\end{aligned}$$

*where $J$ is an even nonnegative convex function with Lipschitz second derivative vanishing outside an interval of the form $[-c\,\delta^{\alpha/2}, c\,\delta^{\alpha/2}]$, and $\Omega$ is an arbitrary compact set.* (*Under the same* CFL-*condition, any* RQM *scheme that is* TVB *also verifies the error estimates* (3.5)).

The proof of the theorem is analogous to the one of the one-dimensional case; see [1]. However, the proof of its stability results contains some technicalities that it is necessary to display.

*Proof.* The proof of the maximum principles is straightforward and will be omitted. We shall prove that the explicit QM scheme is TVD; the proof for the implicit case is similar and will not be displayed. In this proof we shall use the following notation:

$$\Delta_{i+1/2}w_j^n = w_{i+1,j}^n - w_{i,j}^n,$$
$$\Delta_{j+1/2}w_i^n = w_{i,j+1}^n - w_{i,j}^n$$

Consider the following explicit QM scheme

$$\begin{aligned}(u_{i,j}^{n+1} - u_{i,j}^n)/\Delta t^n + (f_{x,i+1/2,j}^{M,n} - f_{x,i-1/2,j}^{M,n})/\Delta x_i \\ + (f_{y,i,j+12}^{M,n} - f_{y,i,j-1/2}^{M,n})/\Delta y_j = 0,\end{aligned}$$

and rewrite it as follows

$$u_{i,j}^{n+1} = u_{i,j}^n - \Delta t^n(\Delta x_i)^{-1}\Delta_i(f_{x,j}^{\mathrm{QM},n}) - \Delta t^n(\Delta y_j)^{-1}\Delta_j(f_{y,i}^{\mathrm{QM},n}).$$

Next, apply the difference operator $\Delta_{i+1/2}$ to this expression to obtain

$$\begin{aligned}\Delta_{i+1/2}(u_j^{n+1}) &= \Delta_{i+1/2}(u_j^n) - \Delta_{i+1/2}(\Delta t^n(\Delta x_i)^{-1}\Delta_i(f_{x,j}^{\mathrm{QM},n})) \\ &\quad - \Delta_{i+1/2}(\Delta t^n(\Delta y_j)^{-1}\Delta_j(f_y^{\mathrm{QM},n})) \\ &= \Delta_{i+1/2}(u_j^n) - \Delta_{i+1/2}(\Delta t^n(\Delta x_i)^{-1}\Delta_i(f_{x,j}^{\mathrm{QM},n})) \\ &\quad - \Delta t^n(\Delta y_j)^{-1}\Delta_{i+1/2}(f_y^{\mathrm{QM},n})).\end{aligned} \tag{3.6}$$

Notice that

$$\begin{aligned}\Delta_i(f_{x,j}^{\mathrm{QM}}) &= (1 + \nu_{xx,i+1/2,j}^- - \nu_{xx,i-1/2,j}^+)\Delta_i(f_{x,j}^{\mathrm{M}}), \\ \Delta_{i+1/2}(f_{y,j+1/2}^{\mathrm{QM}}) &= f_{y,i+1,j+1/2}^{\mathrm{QM}} - f_{y,i,j+1/2}^{\mathrm{QM}} \\ &= (1 + \nu_{yx,i+1,j+1/2}^- - \nu_{yx,i,j+1/2}^+)\Delta_{i+1/2}(f_{y,j+1/2}^{\mathrm{M}}),\end{aligned}$$

by (3.2a) (i) and (3.2b) (iii), respectively. Finally, set

$$\wedge_{x,i,j} = \Delta t^n\ \Delta y_j(1 + \nu_{xx,i+1/2,j}^- - \nu_{xx,i-1/2,j}^+)(\Delta x_i)^{-1},$$
$$\wedge_{y,i+1/2,j+1/2} = \Delta t^n(1 + \nu_{yxi_1,1/2}^- - \nu_{yx,i,j+1/2}^+),$$

and rewrite (3.6) times $\Delta y_j$ as follows

$$\begin{aligned}\Delta y_j\Delta_{i+1/2}(u_j^{n+1}) = \Delta y_j\Delta_{i+1/2}(u_j^n) - \Delta_{i+1/2}(\wedge_{x,j}\Delta_i(f_{x,j}^{M,n})) \\ - \Delta_j(\wedge_{y,i+1/2}\Delta_{i+1/2}(f_y^{M,n})).\end{aligned}$$

Now, we only have to follow word by word the argument used by Sanders [5] to obtain

$$\sum_{i,j}\ \Delta y_j|\Delta_{i+1/2}(u_j^{n+1})| \le \sum_{i,j}\ \Delta y_j\ |\Delta_{i+1/2}(u_j^n)|.$$

We have used properties (3.2a) (ii) and (3.2b) (iv) strongly, as well as the condition $\mathrm{CFL} \in [0, \mathrm{CFL}_0/2]$. Using an analogous procedure, we prove that

$$\sum_{i,j}\ \Delta x_i|\Delta_{j+1/2}(u_i^{n+1})| \le \sum_{i,j}\ \Delta x_i|\Delta_{j+1/2}(u_i^n)|,$$

and this completes the proof. □

**3.3. The conflict between accuracy and the TVD property.** In this section we illustrate with an example the conflict between accuracy and the TVD property pointed out by Goodman and LeVeque [4]. We also show under what formal conditions a QMFD scheme may still be more than first-order accurate. Although we work this out for a particular case, its generalization is straightforward.

Thus, consider the Crank-Nicholson scheme:

$$\begin{aligned} u_{ij}^n = u_{ij}^{n-1} &- \Delta t(f_{x,i+1/2,j}^{\mathrm{CN},n} - f_{x,i-1/2,j}^{\mathrm{CN},n})\Delta x \\ &- \Delta t(f_{y,i+1/2,j}^{\mathrm{CN},n} - f_{y,i-1/2,j}^{\mathrm{CN},n})/\Delta y, \end{aligned}$$

where

$$\begin{aligned} f_{x,i+1/2,j}^{\mathrm{CN},n} &= [(f_x(u_{i+1,j}^n) + f_x(u_{ij}^n))/2 + (f_x(u_{i+1,j}^{n-1}) + f_x(u_{ij}^{n-1}))/2]/2, \\ f_{y,i,j+1/2}^{\mathrm{CN},n} &= [(f_y(u_{i,j+1}^n) + f_y(u_{ij}^n))/2 + (f_y(u_{i,j+1}^{n-1}) + f_y(u_{ij}^{n-1}))/2]/2. \end{aligned}$$

This scheme is formally second-order accurate in space and time. We consider now the QM scheme obtained by combining the Crank–Nicholson and the Godunov fluxes:

$$\begin{aligned} f_{x,i+1/2,j}^{\mathrm{CNG},n} &= [f_{x,i+1/2,j}^{\mathrm{G},n+1} + f_{x,i+1/2,j}^{\mathrm{G},n}]2, \\ f_{y,i,j+1/2}^{\mathrm{CNG},n} &= [f_{y,i,j+1/2}^{\mathrm{G},n+1} + f_{y,i,j+1/2}^{\mathrm{G},n}]/2, \end{aligned}$$

where $\mathbf{f}^G$ is the Godunov flux. From the expressions

$$\begin{aligned} |f_{x,i+1/2,j}^{\mathrm{CNG},n} - f_{x,i+1/2,j}^{\mathrm{CN},n}| &= \Delta x|\partial_x f_x(u_{ij}^n)|/2 + 0(\delta^2), \\ |f_{x,i+3/2,j}^{\mathrm{CNG},n} - f_{x,i+1/2,j}^{\mathrm{CNG},n}| &= \Delta x|\partial_x f_x(u_{ij}^n)| + 0(\delta^2), \\ |f_{x,i+1/2,j}^{\mathrm{CNG},n} - f_{x,i-1/2,j}^{\mathrm{CNG},n}| &= \Delta x|\partial_x f_x(u_{ij}^n)| + 0(\delta^2), \end{aligned}$$

and

$$\begin{aligned} |f_{x,i+1/2,j+1}^{\mathrm{CNG},n} - f_{x,i+1/2,j}^{\mathrm{CNG},n}| &= \Delta y|\partial_y f_x(u_{ij}^n)| + 0(\delta^2), \\ |f_{x,i+1/2,j}^{\mathrm{CNG},n} - f_{x,i+1/2,j-1}^{\mathrm{CNG},n}| &= \Delta y|\partial_y f_x(u_{ij}^n)| + 0(\delta^2), \end{aligned}$$

it is clear that, if we want to have $f_x^{\mathrm{QM}} = f_x^{CN}$ in smooth monotone parts of the entropy solution, we must respect the condition

$$\Delta y|\partial_y f_x| > \Delta x|\partial_x f_x|/2,$$

or equivalently

$$\Delta y|\partial_y u| > \Delta x|\partial_x u|/2.$$

Roughly speaking, this means that the variation of the solution in the element $Q$ along the $y$-direction must be greater than the half of its variation along the $x$-direction. As a similar condition is obtained by analyzing the $f_y$-flux, we shall be able to recover the second-order accuracy of the Crank–Nicholson scheme only if

$$2\Delta x|\partial_x u| > \Delta y|\partial_y u| > \Delta x|\partial_x u|/2. \tag{3.7}$$

Thus, even if the solution is very smooth we may not recover the second-order accuracy of the CN scheme. This is in strong agreement with the result of Goodman and LeVeque [4].

Note, however, that we do not have this restriction if we consider only an RQM scheme, for in this case the second set of expressions is not taken into account. For example, for the RQM versions of the considered scheme, we can recover locally the second-order accuracy of the Crank–Nicholson scheme on smooth monotone parts of the entropy solution!

## 4. Quasimonotone, finite-element schemes.

**4.1. Definition and analysis.** Now that we have shown how to define and analyze QMFD schemes the definition and analysis of QMFE schemes is simply routine. However, we would like to briefly stress some points:

(1) The approximate solution $u_h \in V_h$ given by a QMFE scheme is determined via the weak formulation

$$
\begin{aligned}
&- \iint_K u_h\, \partial_t \varphi_h \; dt\, dx + \int_{\partial K} w_h\, \varphi_h\, n_t \; d\gamma \\
&- \iint_K \mathbf{f}(u_h)\, \partial_x \varphi_h \; dt\, dx + \int_{\partial K} \mathbf{f}_h^{fe}\, (\varphi_h - \overline{\varphi}_h)\, n_{\partial Q} \; d\gamma \\
&+ \int_{\partial K} \overline{\mathbf{f}}_h^{\mathrm{QMFE}}\, \overline{\varphi}_h\, n_{\partial Q} \; d\gamma = 0 \quad \forall \varphi \in W_h,
\end{aligned}
\tag{4.1}
$$

where $(\mathbf{n}_{\partial Q}, n_t)$ is the unit outward normal of $K$, the flux $\overline{\mathbf{f}}_h^{\mathrm{QMFE}}$ is a QMFE flux, and $\overline{\varphi}_h$ is a function independent of $t$ such that $\int_{\partial K} (\varphi_h - \overline{\varphi}_h)\, n_x \; d\gamma = 0$. Relaxed QMFE schemes are obtained if $\overline{\mathbf{f}}^{\mathrm{QMFE}}$ is replaced by an RQM flux $\overline{\mathbf{f}}^{\mathrm{RQMFE}}$

(2) The projections $\Lambda\Pi_h^n$ verifying the following inequality:

$$
\|\Lambda\Pi_h^n(\tilde{u}_h)\|_{L^1(\mathbb{R}^2)} \le C\,\delta
\tag{4.2}
$$

are used to enforce the convergence of the whole approximate solution to the entropy solution. When the approximate solution is discontinuous across the boundaries $\partial K_{i,j}^n$ a family of operators $\{\Lambda\Pi_h^n\}$ verifying (4.2) does exist. Let us denote by $\widetilde{V}(K_{i,j}^n)$ the space to which the restriction of $\tilde{u}_h$ to the interior of $K_{i,j}^n$, $\tilde{u}_h|_{K_{i,j}^n}$, belongs. Denote by $\widetilde{V}'(K_{i,j}^n)$ the convex subset of $\widetilde{V}(K_{i,j}^n)$ for which we have:

$$
\begin{aligned}
|\tilde{u}_h(t,x)| \le \sum \sum_{\ell, m = r, \cdots, s} & C x_{m,\ell}\; \Theta_{i+m+1/2, j+\ell}^n \\
+ \sum \sum_{\ell, m = r, \cdots, s} & C x_{m,\ell}\; \Theta_{i+m, j+\ell+1/2}^n \qquad \text{for } (t,x) \text{ in } K_{i,j'}^n,
\end{aligned}
\tag{4.3a}
$$

$$
\Theta_{i+1/2,j}^n = |\overline{u}_{i+1,j}^n - \overline{u}_{i,j}^n|,
\tag{4.3b}
$$

$$
\Theta_{i,j+1/2}^n = |\overline{u}_{i,j+1}^n - \overline{u}_{i,j}^n|,
\tag{4.3c}
$$

$$
C x_{m,\ell}, \;\; C y_{m,\ell} \ge 0,
\tag{4.3d}
$$

where $r$ and $s$ (respectively, $C_{m,\ell}$) are arbitrary but fixed natural (respectively, real) numbers. We impose the following conditions

$$
\Lambda\Pi_h^n : \widetilde{V}(K_i^n) \to \widetilde{V}'(K_i^n),
\tag{4.4a}
$$

$$
(\Lambda\Pi_h^n)^2 = \Lambda\Pi_h^n.
\tag{4.4b}
$$

For the QMFE schemes we have the following result.

THEOREM 4.1. *Theorem* 3.1 *is valid if we replace "a* (3.1) QM *scheme" by "a* (4.1) QMFE *scheme," the point-values $u_{i,j}$ by the means $\overline{u}_{i,j}$ in* (2.2), *the function $u_h$ by $\overline{u}_h$ in* (2.3) *and* (3.5), *"*TVD*" by "*TVDM.*" If the projections $\Lambda\Pi_h^n$ verifying* (4.2) *are used, as indicated in* (3.6) *of* [2], *then* (3.5) *also holds for $u_h$.*

We end this section by pointing out that time-dependent grid versions of the fully implicit QM scheme can be defined by a straightforward generalization of the procedure used in the one-dimensional case; see §4.3 of [2]. As in that case, the resulting scheme can be proven to be an entropy mass-conserving, positive (in the means!), TVDM scheme.

**4.2. An example.** Let us extend the definition of the G-$\frac{1}{2}$ scheme, see §4 of [2], to the $d = 2$ case. Set:

$$\begin{aligned}
\mathcal{V}_h &= \{v_h : v_h|_{Q_i^n} = \overline{v}_i^n + 2((x-x_i)/\Delta x_i)\tilde{v}^n_{x,ij} + 2((y-y_j)/\Delta y_j)\tilde{v}^n_{y,ij}\},\\
V_h &= \{v_h : [0,T] \to \mathcal{V}_h : v_h(t) = v(t^n), \text{ for } t \in (t^n, t^{n+1})\}.
\end{aligned}$$

Define the flux $\mathbf{f}^{P^0P^1}$ as

$$\begin{aligned}
f^{P^0P^1}_{x,h}(t, x_{i+1/2}, y) &= f^{\mathrm{G}}_x(u_h(t, x^-_{i+1/2}, y), u_h(t, x^+_{i+1/2}, y)),\\
f^{P^0P^1}_{y,h}(t, x, y_{j+1/2}) &= f^{\mathrm{G}}_y(u_h(t, x, y^-_{j+1/2}), u_h(t, x, y^+_{j+1/2})).
\end{aligned}$$

where $f^{\mathrm{G}}$ is the Godunov flux. Note that since these fluxes do not depend on time, we have $\overline{f}^{P^0P^1}_{x,h} = f^{P^0P^1}_{x,h}$ and $\overline{f}^{P^0P^1}_{y,h} = f^{P^0P^1}_{y,h}$. Construct with them the QM fluxes $f^{\mathrm{G}-1/2}_{x,h}$ and $f^{\mathrm{G}-1/2}_{y,h}$ as indicated by (3.1), and (3.3) or (3.4) with the Godunov flux as the monotone flux $\mathbf{f}^{\mathrm{M}}$ and $\mathbf{f}^{P^0P^1}$ as the flux $\mathbf{f}^h$. The weak formulation of the G-$\frac{1}{2}$ scheme is obtained when in (4.1) we take $V_h$ as above, $W_h = V_h$, $\mathbf{f}^{\mathrm{FE}} = \mathbf{f}^{P^0P^1}$, $\overline{\mathbf{f}}^{\mathrm{QMFE}} = \mathbf{f}^{\mathrm{QMFE}} = \mathbf{f}^{\mathrm{G}-1/2}$, *and* all the integrals are evaluated with the midpoint rule. We complete the scheme with a suitably chosen initial condition and the following definition of the projection $\Lambda\Pi^n_h$:

$$\Lambda\Pi^n_h(\tilde{v}_h)(x,y) = 2\frac{x-x_i}{\Delta x_i}\Lambda\Pi^n_{x,h}(\tilde{v}^n_{x,ij}) + 2\frac{y-y_j}{\Delta y_j}\Lambda\Pi^n_{y,h}(\tilde{v}^n_{y,ij}),$$

where

$$\begin{aligned}
\Lambda\Pi^n_{x,h}(\tilde{v}^n_{x,ij}) = \operatorname{sgn}(\tilde{v}^n_{x,ij}) \cdot \max\{0, \min\{&|\tilde{v}^n_{x,ij}|, |\overline{u}^n_{i+1,j} - \overline{u}^n_{i,j}|, \operatorname{sgn}(\tilde{v}^n_{x,ij} \cdot (\overline{u}^n_{i+1,j} - \overline{u}^n_{ij})),\\
&|\overline{u}^n_{ij} - \overline{u}^n_{i-1,j}|, \operatorname{sgn}(\tilde{v}^n_{x,ij} \cdot (\overline{u}^n_{i,j} - \overline{u}^n_{i-1,j}))\}\},\\
\Lambda\Pi^n_{y,h}(\tilde{v}^n_{y,ij}) = \operatorname{sgn}(\tilde{v}^n_{y,ij}) \cdot \max\{0, \min\{&|\tilde{v}^n_{y,ij}|, |\overline{u}^n_{i,j+1} - \overline{u}^n_{i,j}|, \operatorname{sgn}(\tilde{v}^n_{y,ij} \cdot (\overline{u}^n_{i,j+1} - \overline{u}^n_{ij})),\\
&|\overline{u}^n_{ij} - \overline{u}^n_{i,j-1}|, \operatorname{sgn}(\tilde{v}^n_{x,ij} \cdot (\overline{u}^n_{i,j} - \overline{u}^n_{i,j-1}))\}\}.
\end{aligned}$$

In terms of the degrees of freedom of the approximate solution, the scheme reads as follows:

(4.5a) Compute $u^0_h$ as follows: compute the $L^2$-projection of $u_o$ into $\mathcal{V}_h$, i.e.:

$$\begin{aligned}
\overline{u}^0_{ij} &= \int_{Q_{ij}} u_0(x,y)\,dx\,dy/(\Delta x_i \Delta y_j),\\
\tilde{u}^{0,\star}_{x,ij} &= 6\int_{Q_{ij}} (x-x_i)u_0(x,y)\,dx\,dy/(\Delta x_i{}^2 \Delta y_j),\\
\tilde{u}^{0,\star}_{y,ij} &= 6\int_{Q_{ij}} (y-y_j)u_0(x,y)\,dx\,dy/(\Delta x_i \Delta y_j)^2;
\end{aligned}$$

then set $\tilde{u}^0_{x,ij} = \Lambda\Pi^0_{x,h}(u^{0,\star}_{x,ij})$ and $\tilde{u}^0_{y,ij} = \Lambda\Pi^0_{y,h}(u^{0,\star}_{y,ij})$;

(4.5b) For $n = 0, \cdots, N-1$ compute $u_h^{n+1}$ as follows: compute $\overline{u}_h^{n+1}$, and $\tilde{u}_h^{n+1,\star}$ as the solution of:

$$
\begin{aligned}
&(\overline{u}_i^{n+1} - \overline{u}_i^n)/\Delta t^n + (f_{i+1/2}^{\mathrm{G}-1/2,n} - f_{i-1/2}^{\mathrm{G}-1/2,n})/\Delta x_i = 0,\\
&(\tilde{u}_i^{n+1,\star} - \tilde{u}_i^n)/\Delta t^n + 3(f_{i+1/2}^{\mathrm{G}-1/2,n} + f_{i-1/2}^{\mathrm{G}-1/2,n})/\Delta x_i\\
&\qquad\qquad - 6\left\{ \int\int_{K_i^n} f(u_h(t,x))dtdx/(\Delta t^n \Delta x_i) \right\}/\Delta x_i = 0.
\end{aligned}
$$

Set $\tilde{u}_h^{n+1} = \Lambda\Pi_h(\tilde{u}_n^{n+1,\star})$.

Since the Godunov scheme is monotone for CFL $\in [0, \frac{1}{2}]$, Theorem 3.2 holds for $\mathrm{CFL}_o = \frac{1}{2}$. Moreover, thanks to the definition of the projections $\Lambda\Pi_h^n$, the following maximum principle holds:

$$
u_h(t^n, x, y) \in I(\overline{u}_{i-1,j}^n, \overline{u}_{i,j-1}^n, \overline{u}_{i+1,j}^n, \overline{u}_{i,j+1}^n, \overline{u}_{i,j}^n), \quad (x,y) \in Q_{ij}.
$$

This implies that, if $u_{0,h}(x,y) \in [a,b]$ for $x, y \in \mathbb{R}$, the same is true for $u_h$. This is a very important property because in many physical problems only values inside a fixed interval have physical meaning.

Next, we present some numerical results. The test problems are defined on Table 1. Problems 1–4 have periodic boundary conditions, whereas the boundary condition for Problems 5–7 is the exact solution. Problem 1 is one of the most difficult, for it presents contact discontinuities. In Problems 2 and 3 the solution is smooth. Problems 2–4 are essentially one-dimensional problems along the diagonal of the domain. Problems 5–7 are truly two-dimensional Riemann problems solved by Wagner [6]. We have tested various choices of the fluxes for the scheme (4.5). Here we report the best results, which are obtained for the RQM version of the choice (3.4). We have to point out that the scheme (4.5) without projection and with the initial flux $f^{P^0P^1}$ is unconditionally unstable for CFL fixed; see [2]. So, these experiments show how the quasimonotone techniques can improve a very wild scheme. In Table 2 we show the $L^1$-errors and their corresponding order of convergence. We see that, except for problem 1, the scheme is first-order accurate even in the presence of discontinuities. In Figs. 1–5 we show how the discontinuities are approached. In Fig. 1 the approximate solution of problem 1 is shown. We see that there is no diffusion along the direction of the flux, but that there is an important amount of cross-wind diffusion which is responsible for the $\frac{3}{4}$ order of convergence. See in Fig. 2, in which we display the solution of problem 4, how the discontinuities are captured in a single element.

TABLE 1
*The test problems.*

| problem | $\Omega$ | $T$ | $\mathbf{f}(u)$ | $u_0(x,y)$ |
|---|---|---|---|---|
| 1 | $(-2,2)^2$ | 4 | $(u,u)$ | $\begin{cases} 1, & \text{if } x^2+y^2 \leq 1, \\ 0, & \text{otherwise.} \end{cases}$ |
| 2 | $(-2,2)^2$ | 4 | $(u,u)$ | $\frac{1}{2}+\sin(\pi(x+y)/2))$ |
| 3 | $(-2,2)^2$ | 0.1 | $(u^2/2,u^2/2)$ | $\frac{1}{2}+\sin(\pi(x+y)/2))$ |
| 4 | $(-2,2)^2$ | 1 | $(u^2/2,u^2/2)$ | $\frac{1}{2}+\sin(\pi(x+y)/2))$ |
| 5 | $(-1,1)^2$ | 0.5 | $(u^2/2,u^2/2)$ | $\begin{cases} -1.0, & \text{if } x>0,\ y>0, \\ 0.5, & \text{if } x<0,\ y>0, \\ -0.2, & \text{if } x<0,\ y<0, \\ 0.8, & \text{if } x>0,\ y<0. \end{cases}$ |
| 6 | $(-1,1)^2$ | 0.5 | $(u^2/2,u^2/2)$ | $\begin{cases} -1.0, & \text{if } x>0,\ y>0, \\ -0.2, & \text{if } x<0,\ y>0, \\ 0.8, & \text{if } x<0,\ y<0, \\ 0.5, & \text{if } x>0,\ y<0. \end{cases}$ |
| 7 | $(-1,1)^2$ | 0.5 | $(u^2/2,u^2/2)$ | $\begin{cases} 0.8, & \text{if } x>0,\ y>0, \\ -1.0, & \text{if } x<0,\ y>0, \\ 0.5, & \text{if } x<0,\ y<0, \\ -0.2, & \text{if } x>0,\ y<0. \end{cases}$ |

TABLE 2
*The $L^1$ - errors and their respective order of convergence for the* RQM (4.5) *scheme.*

| problem | $L^1(\Omega)$-error | order | $\Delta x(=\Delta y)$ | $\Delta t/\Delta x$ |
|---|---|---|---|---|
| 1 | 0.5637 | 0.7484 | 1/20 | 0.2 |
| 2 | 0.9926 | 1.2351 | 1/20 | 0.2 |
| 3 | 0.1174 | 1.0204 | 1/20 | 0.2 |
| 4 | 0.1855 | 0.9746 | 1/20 | 0.2 |
| 5 | 0.0378 | 1.0580 | 1/40 | 0.2 |
| 6 | 0.0273 | 1.1649 | 1/40 | 0.2 |
| 7 | 0.0383 | 1.1121 | 1/40 | 0.2 |

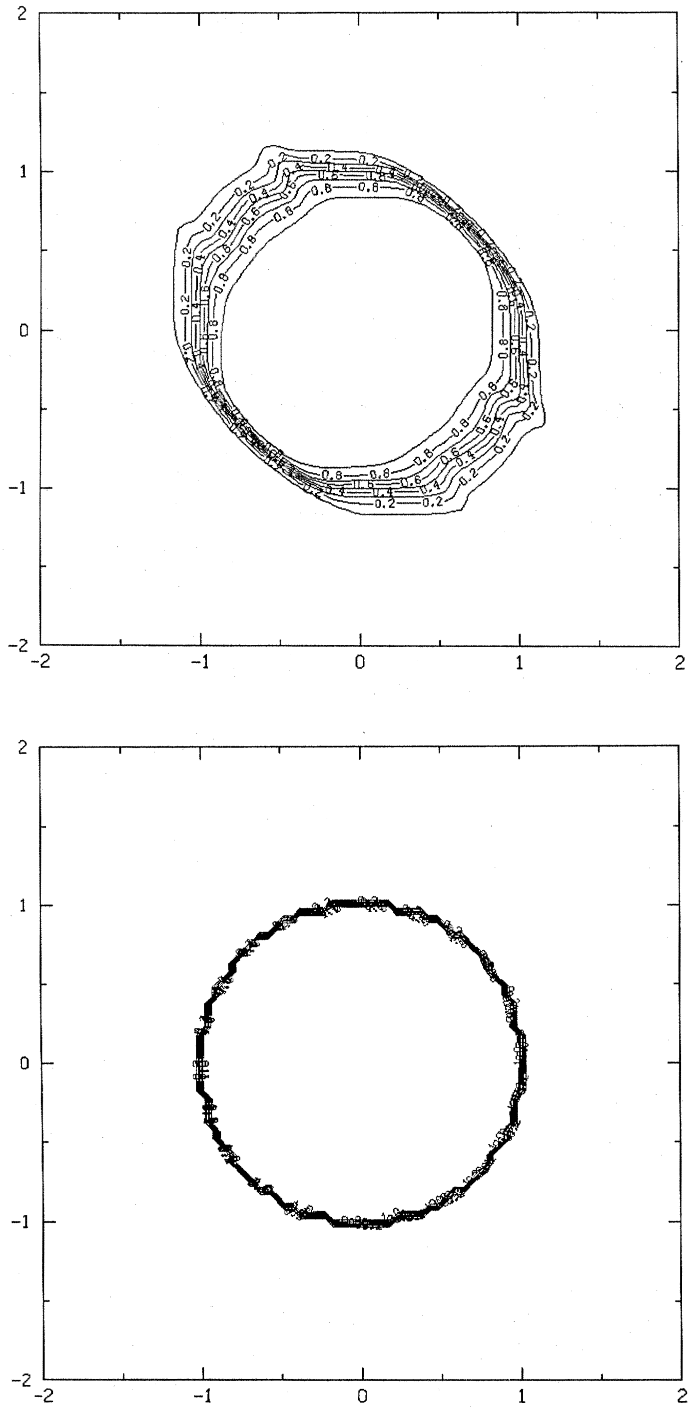


FIG. 1. *Level curves of the approximate solution (above) and the interpolate of the exact solution in the same mesh (below) for problem* 1.

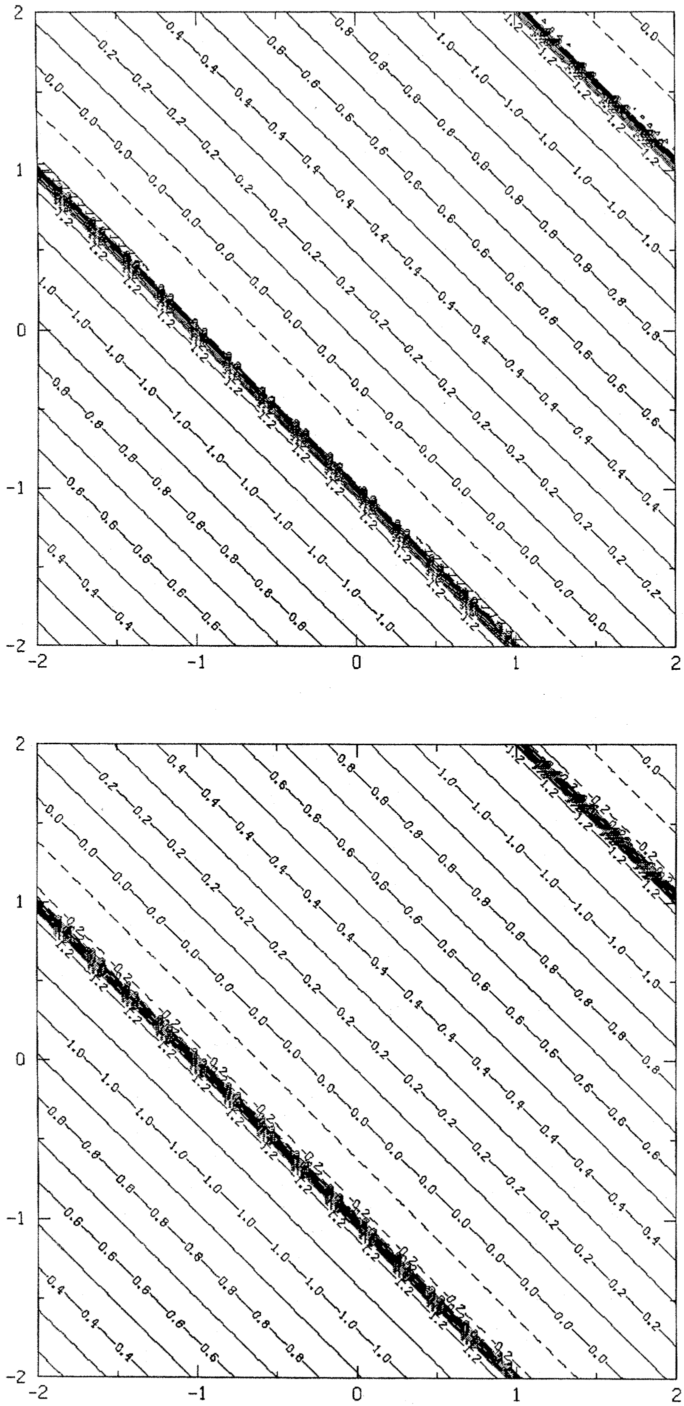

FIG. 2. *Level curves of the approximate solution (above) and the interpolate of the exact solution in the same mesh (below) for problem* 4.

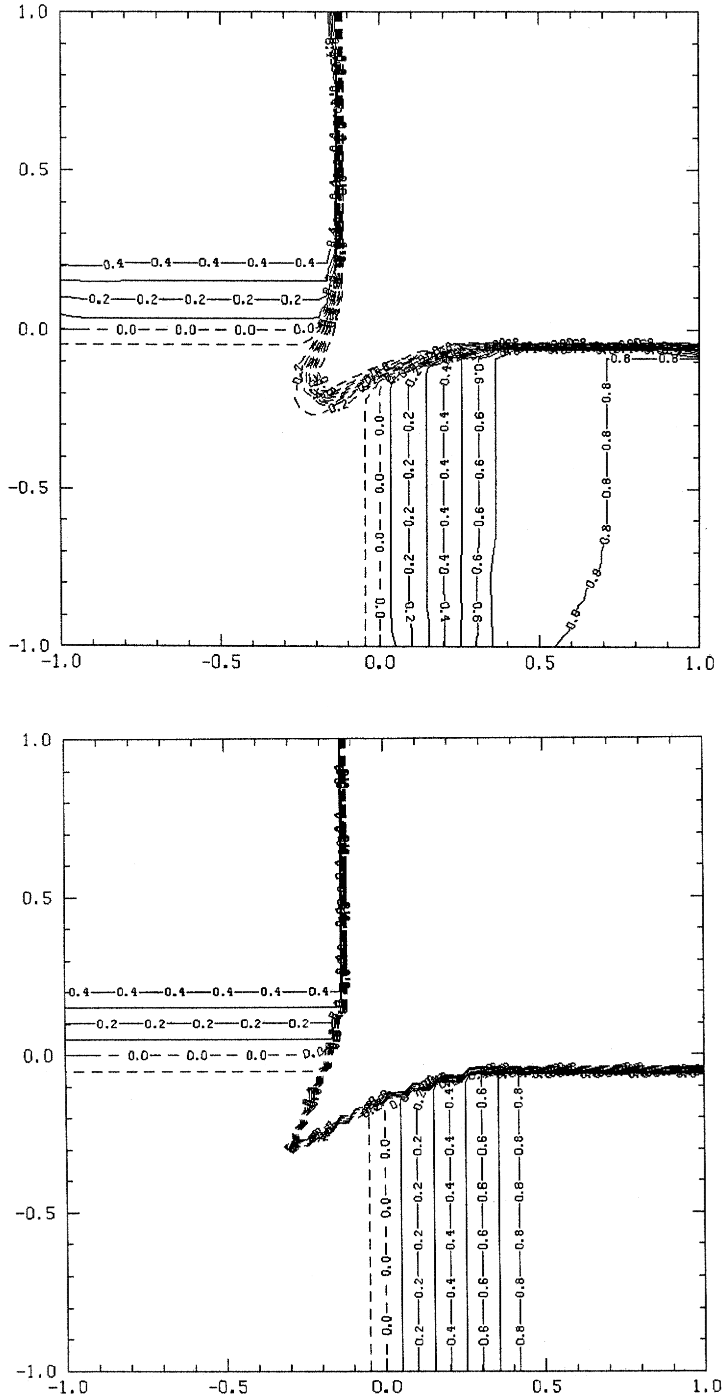


FIG. 3. *Level curves of the approximate solution (above) and the interpolate of the exact solution in the same mesh (below) for problem* 5.

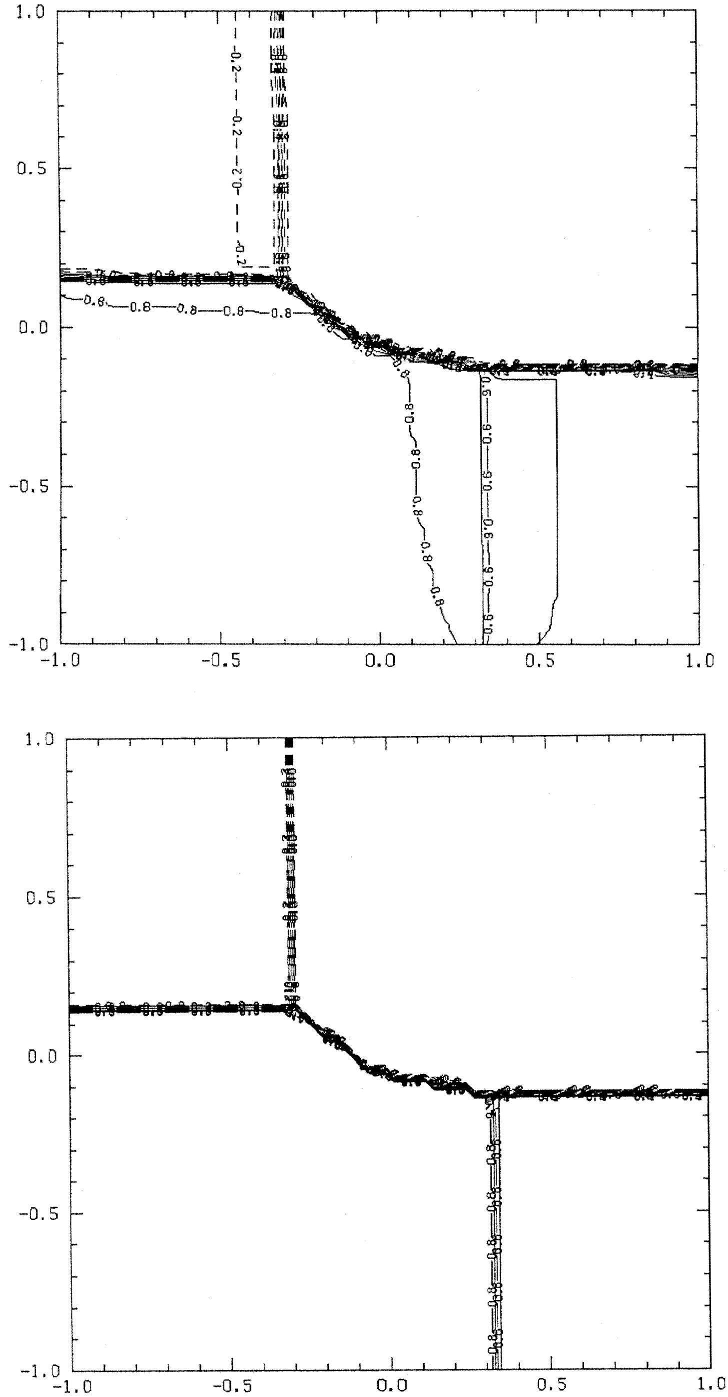


FIG. 4. *Level curves of the approximate solution (above) and the interpolate of the exact solution in the same mesh (below) for problem* 6.

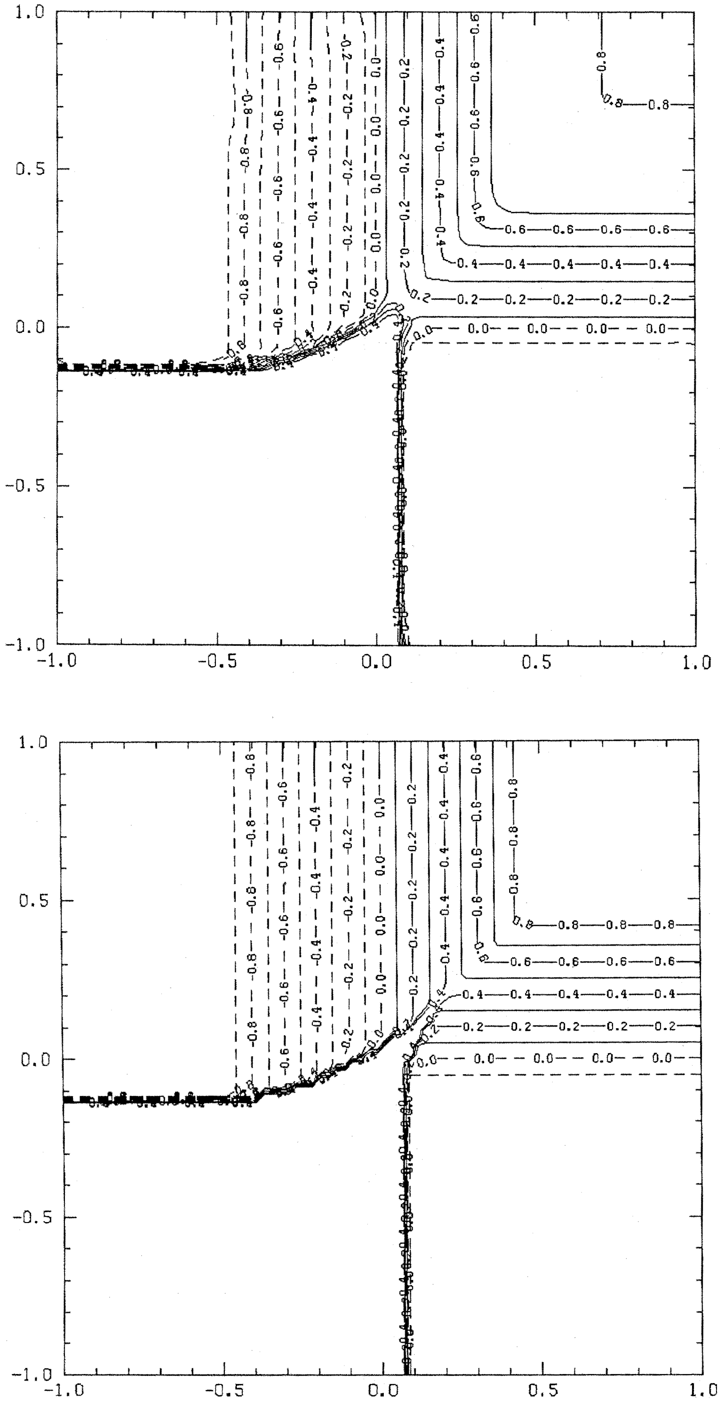


FIG. 5. *Level curves of the approximate solution (above) and the interpolate of the exact solution in the same mesh (below) for problem 7.*

**5. Concluding remarks.** In this paper we have extended to the case $d > 1$ the construction and analysis of the QM schemes devised in [1], [2] for the case $d = 1$. The main difference between this case and the one-dimensional one is the proof of the TVD property, and its influence on the formal accuracy of the scheme. The TVD property is obtained by rewriting the scheme in such a way that the proof of the TVD estimate for monotone schemes obtained by Sanders [5] can be used successfully. We show that under suitable conditions on the solution and the mesh, these schemes can be high-order accurate, even if they are TVD. Of course, these conditions are not verified in the general case, and so the QMFD schemes are at most first-order accurate, as expected; see Goodman and LeVeque [4].

In order to overcome this inconvenience, the RQM schemes were introduced. For them maximum principles have been proven and convergence to the entropy solution has also been proven to occur *provided* they do converge. No TVD property can be proven for them, but, in return, formal order of accuracy can be obtained. A property that implies compactness of the approximate solutions and that is weaker than the TVD property (like that well-known application of the theory of compensated compactness in the $d = 1$ case) seems to be lacking in the theory of schemes for (1.1), and constitutes an exciting topic of future research.

These schemes can be easily generalized to the case of a bounded domain, and also to systems. This paper was mainly concerned with the theoretical aspects of the QM schemes. However, as one of the main contributions of this work is the introduction of the QM finite-element schemes, we have included some numerical experiments displaying the performance of a QMFE to show the applicability of the ideas developed in this paper. Nevertheless, numerical experimentation is far from being completed and the search for particular QMFE schemes constitutes the subject of work in progress.

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**Note added in proof.** The RQM schemes can be rendered (formally) uniformly high-order accurate, by using the TVB technique introduced by Shu [7]. Since the fluxes of an RQM scheme are QM fluxes in either the $x$ or the $y$-direction, this can be done by applying this technique to each of these numerical fluxes as indicated in [1] (for quasimonotone finite-difference schemes) and in [2] (for quasimonotone finite-element schemes). We want to stress the fact that Shu's TVB technique cannot be applied successfully to the general QM schemes in multidimensions. This is because each of the numerical fluxes of a QM scheme is a QM flux in *both* the $x$ and the $y$-direction. This is an essential difficulty reflected in condition (3.7), which is a necessary condition for the (formal) uniform high-order accuracy of the QM schemes.

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