ACCURACY OF SOME APPROXIMATE METHODS FOR COMPUTING THE WEAK SOLUTIONS OF A FIRST-ORDER QUASI-LINEAR EQUATION*

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APPROXIMATE methods for solving the Cauchy problem for a quasi-linear equation in the class of measurable bounded functions are investigated. The convergence rate in $L_1(E_n)$ is estimated.

We estimate below the accuracy in $L_1(E_n)$ of some approximate methods for solving (in the class of discontinuous functions) the Cauchy problem for the quasi-linear equation

$$\frac{\partial v}{\partial t}+\sum\frac{\partial\varphi_s(v)}{\partial x_s}=0,\qquad v(0,x)=v^0(x). \tag{1}$$

We consider Lax's scheme (on a uniform mesh) and the smoothing method (see [1, 2]), and in addition we obtain a general estimate of the closeness of the solution v to the function u in terms of the "discrepancy". In particular, there easily follows from this estimate an estimate of the error of the method of vanishing viscosity.

The results obtained here are also automatically applicable to "running computation" difference schemes, i.e., to schemes of the type

$$u_j^{i+1}-u_j^i+\frac{\tau}{h}\left(\varphi(u_j^i)-\varphi(u_{j-1}^i)\right)=0,$$

since they reduce to Lax's scheme by a change of variables.

In the one-dimensional case (n = 1), under the extra assumption that the function is convex, the accuracy of Lax's scheme in the sense of weak convergence was investigated in [3, 4].

In Section 1 we give necessary information about the exact solutions of the problem (1). In Section 2 the basic estimate is obtained for the closeness of v to u in terms of the discrepancy of the function u, and we also estimate the accuracy of the smoothing method and the viscosity method. In section 3 we estimate the accuracy of Lax's scheme.

It may be mentioned that the present paper represents a development of the author's short paper [5].

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1

Let v^0 be a measurable function in Euclidean space E_n, let $|v^0(x)| \leqslant A$, and let the functions ϕ_s be continuously differentiable in the interval $|v| \leqslant A$, while

$$\max_s \max_{|v|\leqslant A} |\varphi_s'(v)| = B.$$

By a solution of the problem (1) in the domain $t > 0$ we mean a bounded measurable function v, which, given any smooth function g with compact support, satisfies the relation

$$\iint\limits_{\substack{t>0\\ x\in E_n}} \Big(g_t v + \sum_s g_{x_s} \varphi_s(v)\Big)\, dt\, dx + \int_{E_n} g(0,x)\, v^0(x)\, dx = 0$$

and in addition, is continuous with respect to t for $t \geqslant 0$ in the sense of $L_1^{\mathrm{loc}}(E_n)$, i.e.,

$$\int_\Omega |v(t+\Delta t, x) - v(t,x)|\, dx \to 0 \quad \text{as} \quad \Delta t \to 0,$$

and satisfies a stability condition. This means (see [6]) that, given any number a and any non-negative test function g,

$$\int \Big(g_t|v-a| + \sum g_{x_s} F_s(v,a)\Big) dt\, dx + \int g(0,x)\, |v^0(x) - a|\, dx \geqslant 0, \tag{2}$$

where $F_s(a, b) = \operatorname{sgn}(a-b)\,(\varphi_s(a) - \varphi_s(b))$. Inequality (2) ensures the uniqueness of the solution and its stability in the sense of $L_1^{\mathrm{loc}}(E_n)$.

We shall consider initial functions, and hence solutions, bounded in modulus by the constant A. Noting that the equation (1) is hyperbolic, it is sufficient to confine our future discussion to initial functions belonging to $L_1(E_n)$. The stability inequality, which follows from (2), has the form here

$$\|v_1(t) - v_2(t)\| \leqslant \|v_1(0) - v_2(0)\|, \tag{3}$$

where $\|\cdot\|$ is the norm in $L_1(E_n)$.

Estimates for the moduli of continuity (in the mean) of the solutions of problem (1) will play an important part later. For the element $z \in L_1(E_n)$ we put

$$\lambda_i(\delta, z) = \int |z(x+\delta e_i) - z(x)|\, dx, \qquad \bar\lambda_i(\delta, z) = \sup_{|\alpha|\leqslant\delta} \lambda_i(\alpha, z),$$

$$\lambda(\delta, z) = \sum_i \bar\lambda_i(\delta, z),$$

where e_i are the unit coordinate vectors in E_n. For the function $z = z(t)$ with values in $L_1(E_n)$, defined for $t \geqslant 0$, we put

$$\nu_t(\tau, z) = \|z(t+\tau) - z(t)\|, \qquad \tau > -t,$$

$$\bar\nu_t(\tau, z) = \sup_{|\tau'|\leqslant t,\, \tau' > -t} \nu_t(\tau', z).$$

If v is a solution of problem (1), then

$$\lambda_i(\delta, v(t)) \leqslant \lambda_i(\delta, v^0), \tag{4}$$

$$\nu_t(\tau, v) \leqslant \tilde{\lambda}(|\tau|, v^0), \tag{5}$$

where

$$\tilde{\lambda}(\tau, z) = \inf_{\varepsilon}\left(2\lambda(\varepsilon, z) + \tau B\frac{\lambda(\varepsilon, z)}{\varepsilon}\right) \leqslant C\lambda(\tau, z).$$

Inequality (4) follows immediately from (3). To obtain the inequality (5), we have to average Eq. (1) with respect to the space variables; we then obtain, for the mean functions $v^{\varepsilon}(t)$ and ϕ_s^{ε}.

$$\|v^{\varepsilon}(t+\tau) - v^{\varepsilon}(t)\| \leqslant \sum_s \int_t^{t+\tau} dt \int \left|\frac{\partial \varphi_s^{\varepsilon}}{\partial x_s}\right| dx \leqslant B|\tau| \frac{\lambda(\varepsilon, v^0)}{\varepsilon},$$

i.e.,

$$\|v(t+\tau) - v(t)\| \leqslant \|v^{\varepsilon}(t+\tau) - v(t+\tau)\| + \|v^{\varepsilon}(t) - v(t)\|$$
$$+ B|\tau| \frac{\lambda(\varepsilon, v^0)}{\varepsilon} \leqslant 2\lambda(\varepsilon, v^0) + B|\tau| \frac{\lambda(\varepsilon, v^0)}{\varepsilon}.$$

An important part will also be played by the solutions of problem (1) of bounded variation. Let $z \in L_1$ and let the functions

$$Z_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) = \operatorname*{var}_{x_i} z(x)$$

be (Lebesgue) integrable in E_{n-1}. Put

$$\operatorname{var}_i v = \inf_{z \infty v} \int_{E_{n-1}} Z_i(x)\, dx, \qquad \operatorname{var} v = \sum_i \operatorname{var}_i v,$$

where ∞ is the sign of equivalence.

We call var v the (total) variation of the function (in the Tonelli–Cesari sense), and if var $v < \infty$, we call the function v itself a function of bounded variation.

The class of functions of bounded variation is the same as the class of functions possessing a linear modulus of continuity $\lambda(\delta)$, i.e., functions which are Lipschitz continuous in the mean. This follows from the next Lemma 1, the proof of which we shall omit.

Lemma 1 (on functions of bounded variation)

Let $z \in L_1(E_n)$ and let the function $|\varepsilon|^{-1}\lambda_i(\varepsilon, z)$ be bounded. Then, $\operatorname{var}_i z < \infty$, and

$$\lim_{\varepsilon \to 0} |\varepsilon|^{-1}\lambda_i(\varepsilon, z) = \operatorname{var}_i z.$$

Conversely, if $\operatorname{var}_i z < \infty$, then $|\varepsilon|^{-1}\lambda_i(\varepsilon, z) \leqslant \operatorname{var}_i z$ and hence

$$\lim_{\varepsilon \to 0} |\varepsilon|^{-1}\lambda_i(\varepsilon, z) = \operatorname{var}_i z.$$

In fact we have the chain of inequalities

$$\frac{\lambda_t(\varepsilon, z)}{|\varepsilon|} \leqslant \operatorname{var}_t z \leqslant \varliminf_{\varepsilon\to 0} \frac{\lambda_t(\varepsilon, z)}{|\varepsilon|} \leqslant \varlimsup_{\varepsilon\to 0} \frac{\lambda_t(\varepsilon, z)}{|\varepsilon|} \leqslant \operatorname{var}_t z, \tag{6}$$

from which the lemma follows.

It follows from the inequality (4) that, if the variation of the initial function is bounded, then

$$\operatorname{var} v(t) \leqslant \operatorname{var} v^0. \tag{7}$$

The estimate of ν_t in this case has the form

$$\nu_t(\tau, v) \leqslant |\tau| B \operatorname{var} v^0. \tag{8}$$

2

Let the function $u = u(t)$ with values in $L_1(E_n)$ be defined for $t \geqslant 0$ and for every $t \geqslant 0$, let it have left- and right-hand limits. For clarity, assume that it is right continuous, and put $u(+0) = u(0) = u^0$.

For functions having the properties indicated, we shall put

$$\nu_t^{\pm}(\tau, u) = \sup_{0 \leqslant \tau' < \tau} \|u(t \pm \tau') - u(t \pm 0)\|.$$

Let $\omega = \omega(y)$ be a smooth non-negative function in E_1, $\omega(y) = \omega(-y)$, $\omega(y) = 0$ for $|y| \geqslant 1$, $\int \omega(y)\, dy = 1$. For $\varepsilon > 0$, $\varepsilon_0 > 0$ we put

$$\omega_\varepsilon(y) = \frac{1}{\varepsilon}\omega\left(\frac{y}{\varepsilon}\right), \qquad \Omega_\varepsilon(y_1, \ldots, y_n) = \omega_\varepsilon(y_1)\omega_\varepsilon(y_2)\ldots\omega_\varepsilon(y_n),$$
$$g(t', x', t'', x'') = \omega_{\varepsilon_0}(t' - t'')\Omega_\varepsilon(x' - x'').$$

We introduce the notation

$$\delta_t(h, u, a) = \iint\limits_{\substack{0<t'<t\\ x'\in E_n}} \left\{ \frac{\partial h(t', x')}{\partial t'} |u(t', x') - a| \right.$$
$$\left. + \sum_s \frac{\partial h(t', x')}{\partial x_s'} F_s(u(t', x'), a) \right\} dt'\, dx'$$
$$+ \int\limits_{E_n} h(0, x) |u^0(x') - a|\, dx' - \int\limits_{E_n} h(t, x') |u(t-0, x') - a|\, dx',$$
$$\delta_t^{\varepsilon_0, \varepsilon} = \delta_t^{\varepsilon_0, \varepsilon}(u, v) = \iint\limits_{\substack{0<t'<t\\ x'\in E_n}} \delta_t(g(t', x'), u, v(t', x'))\, dt'\, dx'.$$

where $g(t', x') = g(t', x', \dots)$.

The aim of the present section is to prove:

Lemma 2

For $0 < \varepsilon_0 < t$ and $\varepsilon > 0$ we have

$$\|u(t-0) - v(t)\| \leqslant \|v^0 - u^0\| + 2\lambda(\varepsilon, v^0) + \tilde{\lambda}(\varepsilon_0, v^0) + \frac{\bar{\nabla}_t^-(\varepsilon_0, u) + \bar{\nabla}_0^+(\varepsilon_0, u)}{2} - \delta_t^{\varepsilon_0,\varepsilon}. \tag{9}$$

Proof. In the inequality $\delta_t(g(t', x'), v, a) \geqslant 0$, which, by (2), holds for all t', x', we substitute $a = u(t', x')$, and integrate. We obtain

$$\int_0^t \omega_{\varepsilon_0}(t-t')\{\rho_\varepsilon(v(t'), u(t)) + \rho_\varepsilon(v(t), u(t'))\}dt'$$

$$\leqslant -\delta_t^{\varepsilon_0,\varepsilon} + \int_0 \omega_{\varepsilon_0}(t')\{\rho_\varepsilon(v(t'), u^0) + \rho_\varepsilon(v^0, u(t'))\}dt',$$

where $\rho_\varepsilon(z, w) = \int \Omega_\varepsilon(x-y)\,|z(x) - w(y)|\,dxdy$. By the triangle inequality,

$$\rho_\varepsilon(v(t'), u(t)) + \rho_\varepsilon(v(t), u(t')) \geqslant 2\|v(t) - u(t)\|$$
$$-2\rho_\varepsilon(v(t), v(t)) - \|u(t) - u(t')\| - \|v(t) - v(t')\|,$$
$$\rho_\varepsilon(v(t'), u^0) + \rho_\varepsilon(v^0, u(t')) \leqslant 2\|u^0 - v^0\| + 2\rho_\varepsilon(v^0, v^0)$$
$$+\|u(t') - u^0\| + \|v(t') - v^0\|.$$

Hence, noting that $\rho_\varepsilon(v, v) \leqslant \lambda(\varepsilon, v) \leqslant \lambda(\varepsilon, v^0)$, and also that

$$\int_0 \omega_{\varepsilon_0}(t-t')\,dt' = \int_0 \omega_{\varepsilon_0}(t')\,dt' = \frac{1}{2},$$

we get

$$\|u(t-0) - v(t)\| \leqslant \|u^0 - v^0\| + 2\lambda(\varepsilon, v^0) + \frac{\bar{\nabla}_t(\varepsilon_0, v) + \bar{\nabla}_0(\varepsilon_0, v)}{2}$$
$$+ \frac{\bar{\nabla}_t^-(\varepsilon_0, u) + \bar{\nabla}_0^+(\varepsilon_0, u)}{2} - \delta_t^{\varepsilon_0,\varepsilon},$$

and (9) follows from (5). The lemma is proved.

In the case when u is a solution of Eq. (1), $\delta_t^{\varepsilon_0,\varepsilon} \geqslant 0$, so that, on neglecting $\delta_t^{\varepsilon_0,\varepsilon}$ in (9) and letting ϵ_0 and ϵ tend to zero, we again obtain the estimate (3). In the general case, by using the inequality (9) we can estimate how close the function u is to the solution of problem (1), in terms of the discrepancy $\delta_t^{\varepsilon_0,\varepsilon}$ (it would be more precise to call $|\delta_t^{\varepsilon_0,\varepsilon}| - \delta_t^{\varepsilon_0,\varepsilon}$, the discrepancy (on the test function g), see [5]. If u^m is a sequence of functions, for which $\delta_t^{\varepsilon_0,\varepsilon}(u^m) \to 0$ for fixed ϵ_0 and ϵ, then, by making use of the arbitrariness in the choice of ϵ_0 and ϵ, we could in general estimate the rate of convergence to zero of $\|v - u^m\|$.

In one important case, inequality (9) can be greatly simplified.

Theorem 1

If $u = u(t)$ is an exact solution of Eq. (1) in the strips $i\tau<t<(i+1)\tau$, $i=0, 1, \ldots,$ then

$$\|u(t)-v(t)\|\leqslant\|u^0-v^0\|+2\lambda(\varepsilon, v^0)-\delta_t^\varepsilon, \tag{10}$$

where

$$\delta_t^\varepsilon=-\sum_{i=1}^{N(t,\tau)}\rho_\varepsilon(u(i\tau),v(i\tau))-\rho_\varepsilon(u(i\tau-0),v(i\tau)),$$

and $N(t, \tau)$ is the greatest integer less than t/τ.

Proof. Since u is an exact solution for $t_i<t<t_{i+1}$, $t_i=i\tau$, we have

$$\iint\limits_{t_i<t''<t_{i+1}}\left\{\frac{\partial g}{\partial t''}|u(t'',x'')-a|+\sum\frac{\partial g}{\partial x_s''}F_s(u(t'',x''),a)\right\}dt''\,dx''$$

$$+\int\limits_{E_n} g|_{t''=t_i}|u(t_i+0,x'')-a|dx''-\int g|_{t''=t_{i+1}}|u(t_i,x'')-a|dx''\geqslant0.$$

Summing these inequalities, we get

$$\delta_t(g(t',x'),u,v(t',x'))\geqslant-\sum_{i=1}^{N}\int g(t',x',t'',x'')\{|u(t_i,x'')$$

$$-v(t',x')|-|u(t_i-0,x'')-v(t',x')|\}dx''$$

and

$$\delta_t^{\varepsilon_0,\varepsilon}\geqslant-\sum_{i=1}^{N}\int_0^t\omega_{\varepsilon_0}(t'-t_i)\{\rho_\varepsilon(u(t_i),v(t'))-\rho_\varepsilon(u(t_i-0),v(t'))\}dt'.$$

Since

$$\varliminf_{\varepsilon_0\to0}\delta_t^{\varepsilon_0,\varepsilon}\geqslant\delta_t=-\sum_{i=1}^{N}\rho_\varepsilon(u(t_i),v(t'))-\rho_\varepsilon(u(t_i-0),v(t_i)),$$

we obtain the theorem on letting $\epsilon_0 \to 0$ in (9).

Let us apply Theorem 1 and Lemma 2 to estimate the error of the smoothing method and the viscosity method.

The approximate solution u_δ, obtained by the smoothing method, is the smooth solution of Eq. (1) in each strip $t_i<t<t_{i+1}$, $t_i=i\tau$, $i=0, 1, \ldots,$ and is defined by the conditions $u_\delta(0)=\Omega_\delta*v^0$, $u_\delta(t_i)=\Omega_\delta*u_\delta(t_i-0)$, where $\delta = k\tau$, and k is chosen in such a way that the smooth solution exists in the strips (for more details, see [1, 2]).

Theorem 2

If $\lambda_0(\varepsilon)=\lambda(\varepsilon, v^0)$, then the error of the smoothing method for solving problem (1) is given by the estimate

$$\|u_\delta(t)-v(t)\|\leqslant\lambda_0(\delta)+\inf_\varepsilon\left\{2\lambda_0(\varepsilon)+Dt\frac{\lambda_0(\delta)}{\varepsilon}\right\},$$

where D depends only on k and the kernel Ω. In particular, if the function v^0 has bounded variation, then

$$\|u_\delta(t)-v(t)\|\leqslant\{\delta+(8Dt\delta)^{1/2}\}\operatorname{var} v^0.$$

Proof. Theorem 1 can be applied to u_δ; hence,

$$-\delta_t^\varepsilon=\sum_{i=1}^{N(\tau,t)}\rho_\varepsilon(u_\delta(t_i),v(t_i))-\rho_\varepsilon(u_\delta(t_i-0),v(t_i)).$$

Since

$$\rho_\varepsilon(\Omega_\delta*u,v)-\rho_\varepsilon(u,v)\leqslant\int\Omega_\varepsilon(x-y)\Omega(z)\{|u(x+\delta z)-v(y)|$$

$$-|u(x)-v(y)|\}\,dx\,dy\,dz=\frac{1}{2}\int[\Omega_\varepsilon(x-y)-\Omega_\varepsilon(x+\delta z-y)]$$

$$\times\Omega(z)\{|u(x+\delta z)-v(y)|-|u(x)-v(y)|\}\,dx\,dy\,dz$$

$$\leqslant\frac{1}{2}\int|\Omega_\varepsilon(y+\delta z)-\Omega_\varepsilon(y)|\,\Omega(z)\,|u(x+\delta z)-u(x)|\,dx\,dy\,dz$$

$$\leqslant\omega(0)\operatorname{var}u\frac{\delta^2}{\varepsilon},$$

we have
$$-\delta_t^\varepsilon\leqslant N(t,\tau)\,\omega(0)\operatorname{var}u_\delta(0)\frac{\delta^2}{\varepsilon}\leqslant k\omega(0)t\delta\frac{1}{\varepsilon}\operatorname{var}u_\delta(0).$$

Putting $D = k\,\omega(0)$, and noting that

$$\|u_\delta(0)-v^0\|\leqslant\lambda_0(\delta),\qquad \operatorname{var}u_\delta(0)\leqslant\frac{\lambda_0(\delta)}{\delta},$$

we obtain the theorem.

The viscosity method consists in replacing problem (1) by the problem

$$\frac{\partial u_\delta}{\partial t}+\sum_s\frac{\partial\varphi_s(u_\delta)}{\partial x_s}=\delta\sum_s\frac{\partial^2u_\delta}{\partial x_s^2},\qquad u_\delta(0,x)=v^0(x).$$

See [7] concerning the solvability of this problem, and the properties of the function u_δ which we use below.

Theorem 3

If $v^0\in L_1(E_n)$ and $\lambda_0(\varepsilon)=\lambda(\varepsilon,v^0)$, then the error of the viscosity method is given by the estimate

$$\|u_\delta(t)-v(t)\|\leqslant\inf_\varepsilon\left(2\lambda_0(\delta)+(8t\delta)^{1/2}\frac{\lambda_0(\varepsilon)}{\varepsilon}\right)\leqslant C\lambda_0((t\delta)^{1/2}).$$

In particular, if $\operatorname{var} v^0<\infty$

$$\|u_\delta(t)-v(t)\|\leqslant(8t\delta)^{1/2}\operatorname{var}v^0. \tag{11}$$

Proof. Let var $v^0 < \infty$. Since $\operatorname{var} u_\delta(t) \leqslant \operatorname{var} v^0$, we have

$$-\delta_t^{\varepsilon_0,\varepsilon} \leqslant \delta \sum_s \iiint\limits_{\substack{0<t'<t\\ x\in E_n,\ y\in E_n}} \frac{\partial \Omega_\varepsilon(x-y)}{\partial x_s} \frac{\partial |u_\delta(t',y)-v(t',x)|}{\partial y_s}\, dx\, dy\, dt'$$

$$\leqslant \delta \sum_s \iiint \left| \frac{\partial \Omega_\varepsilon(x-y)}{\partial x_s} \right| \left| \frac{\partial u_\delta(t',y)}{\partial y_s} \right| dx\, dy\, dt' \leqslant 2\omega(0) \frac{t\delta}{\varepsilon} \operatorname{var} v^0.$$

Here, obviously, we can put $2\omega(0) = 1$. Hence, setting $\epsilon_0 = 0$ in inequality (9) and minimizing it with respect to ϵ, we obtain the estimate (11). Let var $v^0 = \infty$. We put $v_\varepsilon^0 = \Omega_\varepsilon * v^0$ and denote by $u_{\delta,\epsilon}$ the corresponding solution of the problem with viscosity. Since $\|u_{\delta,\varepsilon}(t) - u_\delta(t)\| \leqslant \|u_{\delta,\varepsilon}(0) - u_\delta(0)\| \leqslant \lambda_0(\varepsilon)$, and $\operatorname{var} v_\varepsilon^0 \leqslant \varepsilon^{-1}\lambda_0(\varepsilon)$, we have, by (11),

$$\|u_\delta(t) - v(t)\| \leqslant 2\lambda_0(\varepsilon) + (8t\delta)^{1/2} \frac{\lambda_0(\varepsilon)}{\varepsilon},$$

and this proves the theorem.

3

We shall obtain an estimate for the error of Lax's scheme. Let the half-space $E_n \times \{0 < t < \infty\}$ be divided by the planes $t = t_i = i\tau$ into the strips $S_i = \{t_i \leqslant t < t_{i+1}\}$, and let each strip S_i be divided by the planes $x_s = k_s h_s$, $h_s > 0$, where k_s runs over all integers of the same parity as the number i of the strip, into the parallelepipeds

$$P_m{}^i = \Pi_m{}^i \times \{t_i \leqslant t < t_{i+1}\},$$

where $\Pi_m{}^i = \{x \mid (m_s - 1)h_s < x_s < (m_s + 1)h_s\}$. Here, $m = (m_1, \ldots, m_n)$ is an integer-valued vector (multi-index).

We define the function u_τ as follows:

$$u_\tau(t, x) = u_m{}^i, \qquad (t, x) \in P_m{}^i,$$

where

$$u_m{}^0 = \frac{1}{2^n \cdot h_1 \ldots h_n} \int\limits_{\Pi_m{}^0} v^0(x)\, dx,$$

while the $u_m{}^i$, $i \geqslant 1$, are defined by the (Lax) difference scheme

$$u_m^{i+1} = 2^{-n} \left\{ \sum_{j\in I} u_{m+j}^i - \sum_{s=1}^n \sigma_s \sum_{j\in I} j_s \varphi_s(u_{m+j}) \right\}$$

(the summation is over the set I of multi-indices j with $j_k = \pm 1$; $\sigma_s = \tau/h_s$).

We can rewrite the scheme as

$$u_m^{i+1} = 2^{-n} \sum_{j\in I} \left\{ 1 - \sum_s j_s \sigma_s \beta_s \left(u^i_{m^{(s)}+j^{(s)},m_s+1}, u^i_{m^{(s)}+j^{(s)},m_s-1}\right) \right\} u^i_{m+j}, \tag{12}$$

where $\beta_s(a, b) = (a-b)^{-1}(\varphi_s(a) - \varphi_s(b))$, $m^{(s)} = (m_1, \ldots, m_{s-1}, m_{s+1}, \ldots, m_n)$.

Let
$$B_s = \sup_{|a| \leqslant A} |\varphi_s'(a)|$$

and let the mesh steps be subject to the stability condition

$$\sum_{s=1}^{n} \sigma_s B_s \leqslant 1. \tag{13}$$

Then, for $|u_m^i| \leqslant A$, all the expressions in the braces in (12) are non-negative. Since their sum is equal to 2^n, we have

$$\min_k u_k^i \leqslant u_m^{i+1} \leqslant \max_k u_k^i;$$

and in particular, $|u_m^i| \leqslant A$ for all $i=0, 1, \ldots$.

We put $u_\tau(t, \cdot) = u_\tau(t)$ and $u_\tau(t) = u^i(t)$ for $t_i \leqslant t < t_{i+1}$.

Lemma 3

If the stability condition (13) holds, the solutions of Lax's scheme have the following properties:

$$\|u_\tau(t) - \hat{u}_\tau(t)\| \leqslant \|u^0(0) - \hat{u}^0(0)\|, \tag{14}$$

$$\operatorname{var} u_\tau(t) \leqslant \operatorname{var} u^0, \tag{15}$$

$$\nu_t(\Delta, u_\tau) = \|u_\tau(t+\Delta) - u_\tau(t)\| \leqslant K_1(\tau + |\Delta|) \underset{\Delta > -t}{\operatorname{var}} u^0, \tag{16}$$

$$|u_m^{i+1} - a| \leqslant 2^{-n} \sum_{j\in I} \left\{ |u_{m+j}^i - a| - \sum_{s=1} \sigma_s j_s F_s(u_{m+j}^i, a) \right\}, \tag{17}$$

where the constant K_1 depends on the dimensionality of the space and on the constant B', representing an upper bound of the ratio $\tau^{-1} h_s$.

Proof. Let $\hat{u}_\tau$ be a solution of Lax's scheme, $|\hat{u}_m^i| \leqslant A$. Then,

$$|u_m^{i+1} - \hat{u}_m^{i+1}| \leqslant 2^{-n} \sum_{j\in I} \left\{ 1 - \sum_s \sigma_s j_s \beta_s (u_{m+j}^i, \hat{u}_{m+j}^i) \right\} |u_{m+j}^i - \hat{u}_{m+j}^i|, \tag{18}$$

so that

$$\sum_m |u_m^{i+1} - \hat{u}_m^{i+1}| \leqslant \sum_m |u_m^i - \hat{u}_m^i|,$$

and (14) is proved. Putting $\hat{u}_m^i = u_{m+f}^i$, $j \in I$, we obtain (15). Puting $u = a$, in (18), we get (17). To prove the estimate (16), we consider

$$J_m^i = \int_{\Pi_m^{i+1}} |u^{i+1}(x) - u^i(x)|\, dx = \int_{\Pi_m^{i+1}} |u_m^{i+1} - u^i(x)|\, dx.$$

Putting $\hat{\Pi}_{m+j}^i = \Pi_m^{i+1} \cap \Pi_{m+j}^i$ (this is the 2^{-n}-th part of the parallelepiped $\Pi_m{}^{i+1}$), we have

$$\begin{aligned} J_m^i &= \sum_{j\in I} \int_{\hat{\Pi}_{m+j}^i} |u_m^{i+1} - u^i(x)|\, dx = h_1 \ldots h_n \sum_{j\in I} |u_m^{i+1} - u_{m+j}^i| \\ &\leqslant 2^{-n} h_1 \ldots h_n \sum_{j,\, k\in I} \left\{1 - \sum_s \sigma_s k_s \beta_s (u_{m+j}^i, u_{m+k}^i)\right\} |u_{m+j}^i - u_{m+k}^i| \\ &\leqslant 2^{-n+1} h_1 \ldots h_n \sum_{j,\, k\in I} |u_{m+j}^i - u_{m+k}^i|, \end{aligned}$$

so that

$$\int_{E_n} |u^{i+1}(x)-u^i(x)|\,dx\leqslant 2^{-n+1}h_1\dots h_n \sum_{j,k\in I}\sum_m |u^i_{m+j}-u^i_{m+k}|$$

$$\leqslant K'\sum_{s=1}^n h_s \operatorname{var}_s u^i\leqslant B'K'\tau \operatorname{var} u^i\leqslant K_1\tau \operatorname{var} u^0,$$

and (16) is obtained by summation of these inequalities.

Lemma 4

Let $h_s \leqslant \tau B'$. Then,

$$-\delta_t^{\varepsilon_0,\varepsilon}(u_\tau, v)\leqslant\tau\left\{K_2+tK_3\left(\frac{1}{\varepsilon}+\frac{1}{\varepsilon_0}\right)\right\}\operatorname{var} u^0, \tag{19}$$

where the constants K_2 and K_3 depend on the function ω, the constants B and B', and the dimensionality of the space.

Proof. Let $f=f(t,x)$ be a smooth function and let $t_N\leqslant T\leqslant t_{N+1}$. We write $\Pi^i_{m(s)}$ for the projection of the parallelepiped $\Pi_m{}^i$ onto the space $x_s = 0$, so that $\Pi_m{}^i=\Pi^i_{m(s)}\times\{(m_s-1)h_s<x_s\ (m_s+1)h_s\}$. Introducing the set I_s of multi-indices $j^{(s)}=(j_1,\ \dots,\ j_{s-1},\ j_{s+1},\ \dots,\ j_n)$ with $j_k=\pm1$, we put $\hat\Pi^i_{m^{(s)}+j^{(s)}}=\Pi^{i+1}_{m^{(s)}}\cap\Pi^i_{m^{(s)}+j^{(s)}},\quad j^{(s)}\in I_s.$

Since the derivatives u_T are concentrated in the planes $t=t_i,\ x_s=m_sh_s$ the quantity $\delta_T(f,\ u_\tau,\ a)$ can easily be evaluated. Noting the inequality (17), we obtain from calculations:

$$\delta_T(f,u_\tau,a)\geqslant\sum_{i=0}^{N-1}(\delta_i^{(1)}+\delta_i^{(2)})+\delta_T^{(3)},$$

where

$$\delta_i^{(1)}(f,u_\tau,a)=-\sum_m\sum_{k,j\in I}2^{-n}\left\{\int_{\hat\Pi^i_{m+k}} f(t_{i+1},x)\,dx\right.$$
$$\left.-\int_{\hat\Pi^i_{m+j}} f(t_{i+1},x)\,dx\right\}\frac{|u^i_{m+j}-a|+|u^i_{m+k}-a|}{2},$$

$$\delta_i^{(2)}(f,u_\tau,a)=\sum_{s=1}^n\sum_m\sum_{j^{(s)}\in I_s}\left\{2^{-n}\sigma_s\int_{\Pi^{i+1}_m} f(t_{i+1},x)\,dx\right.$$
$$\left.-\int_{t_i}^{t_{i+1}}dt\int_{\hat\Pi^i_{m^{(s)}+j^{(s)}}} f(t,x^{(s)},m_sh_s)\,dx^{(s)}\right\}\{F_s(u^i_{m^{(s)}+j^{(s)},\,m_s+1},a)$$
$$-F_s(u^i_{m^{(s)}+j^{(s)},\,m_s-1},a)\},$$

$$\delta_T^{(3)}(f, u_\tau, a) = -\sum_{s=1}^{n}\sum_{m}\sum_{j^{(s)}\in I_s}\int_{t_N}^{T} dt \int_{\hat{\Pi}^N_{m^{(s)}+j^{(s)}}} f(t, x^{(s)}, m_s h_s)\, dx^{(s)} \qquad \text{(cont'd)}$$

$$\times \{F_s(u^N_{m^{(s)}+j^{(s)},\, m_s+1}, a) - F_s(u^N_{m^{(s)}+j^{(s)},\, m_s-1}, a)\};$$

here, $x^{(s)} = (x_1, \ldots, x_{s-1}, x_{s+1}, \ldots, x_n)$, $f(t, x^{(s)}, a)$ is the value of $f(t, x)$ for $x_s = a$.

Put

$$A^i_{m,\,k+j}(f) = \int_{\Pi^i_{m+k}} f(t_{i+1}, x)\, dx - \int_{\Pi^i_{m+j}} f(t_{i+1}, x)\, dx,$$

$$B^j_{m,\,m^{(s)}+j^{(s)}}(f) = 2^{-n}\sigma_s \int_{\Pi^{i+1}_m} f(t_{i+1}, x)\, dx$$

$$-\tau \int_{\hat{\Pi}^i_{m^{(s)}+j^{(s)}}} f(t_{i+1}, x^{(s)}, m_s h_s)\, dx^{(s)},$$

$$C^i_{m^{(s)}+j^{(s)}}(f) = \int_{t_i}^{t_{i+1}} dt \int_{\hat{\Pi}^i_{m^{(s)}+j^{(s)}}} [f(t_{i+1}, x^{(s)}, m_s h_s) - f(t, x^{(s)}, m_s h_s)]\, dx^{(s)},$$

$$D^T_{m^{(s)}+j^{(s)}}(f) = \int_{t_N}^{T} dt \int_{\hat{\Pi}^N_{m^{(s)}+j^{(s)}}} f(t, x^{(s)}, m_s h_s)\, dx^{(s)}.$$

Since $||b-a|-|c-a|| \leqslant |b-c|$ and $|F_s(b, a) - F_s(c, a)| \leqslant B|b-c|$, $|b| \leqslant A$, we have $|c| \leqslant A$,

$$|\delta_i^{(1)}(f, u_\tau, a)| \leqslant \sum_{m}\sum_{k,j\in I} 2^{-n-1}|A^i_{m,k,j}(f)|\,|u^i_{m+j} - u^i_{m+k}|,$$

$$|\delta_i^{(2)}(f, u_\tau, a)| \leqslant b \sum_{s}\sum_{m}\sum_{j^{(s)}\in I_s} (|B^i_{m,m^{(s)}+j^{(s)}}(f)|$$

$$+|C^i_{m^{(s)}+j^{(s)}}(f)|)\,|u^i_{m^{(s)}+j^{(s)},m_s+1} - u^i_{m^{(s)}+j^{(s)},m_s-1}|,$$

$$|\delta_T^{(3)}(f, u_\tau, a)| \leqslant$$

$$\leqslant b \sum_{s}\sum_{m}\sum_{j^{(s)}\in I_s} |D^T_{m^{(s)}+j^{(s)}}(f)|\,|u_{m^{(s)}+j^{(s)},m_s+1} - u_{m^{(s)}+j^{(s)},m_s-1}|.$$

Putting here $f(t, x) = g(t', y; t, x) = \omega_{\varepsilon_0}(t'-t)\,\Omega_\varepsilon(y-x)$ and $a = v(t', y)$ we get $0 < t' < T$, $y \in E_n$,

(cont'd)

$$\iint\limits_{0<t'<T,\ y\in E_n} |\delta_i^{(1)}(g(t',y),u_\tau,v(t',y))|\,dt'\,dy$$

$$\leqslant 2^{-n-1}A_\tau^T(\varepsilon_0,\varepsilon)\sum_m\sum_{k,j\in I}|\overset{i}{u}_{m+j}-\overset{i}{u}_{m+k}|h_1\ldots h_n\leqslant K_4|h|A_\tau^T(\varepsilon_0,\varepsilon)\operatorname{var}u^0,$$

$$\iint\limits_{0<t'<T,\ y\in E_n} |\delta_i^{(2)}(g(t',y),u_\tau,v(t',y))|\,dt'\,dy$$

$$\leqslant B\sum_s\sum_m\sum_{j^{(s)}}(B_\tau^T(\varepsilon_0,\varepsilon)+C_\tau^T(\varepsilon_0,\varepsilon))\,|\overset{i}{u}_{m^{(s)}+j^{(s)},m_s+1}$$

$$-\overset{i}{u}_{m^{(s)}+j^{(s)},m_s-1}|h_1\ldots h_{s-1}h_{s+1}\ldots h_n$$

$$\leqslant K_5\tau(B_\tau^T(\varepsilon_0,\varepsilon)+C_\tau^T(\varepsilon_0,\varepsilon))\operatorname{var}u^0,$$

$$\iint\limits_{0<t'<T,\ y\in E_n} |\delta_T^{(3)}(g(t',y),u_\tau,v(t',y))|\,dt'\,dy$$

$$\leqslant BD_\tau^T(\varepsilon_0,\varepsilon)\sum_s\sum_m\sum_{j^{(s)}}|\overset{i}{u}_{m^{(s)}+j^{(s)},m_s+1}$$

$$-\overset{i}{u}_{m^{(s)}+j^{(s)},m_s-1}|h_1\ldots h_{s-1}h_{s+1}\ldots h_n\leqslant K_2\tau D_\tau^T(\varepsilon_0,\varepsilon)\operatorname{var}u^0,$$

where K_4, K_5, K_2 are constants, dependent on the dimensionality of the space and on B, and

$$|h|=\max_i h_i,$$

$$A_\tau^T(\varepsilon_0,\varepsilon)=(h_1\ldots h_n)^{-1}\max_{i,m,k,j}\iint|\overset{i}{A}_{m,k,j}(g(t',y))|\,dt'\,dy,$$

$$B_\tau^T(\varepsilon_0,\varepsilon)=$$

$$=\tau^{-1}\max_{s,i,m,m^{(s)},j^{(s)}}\left\{\iint|\overset{i}{B}_{m,m^{(s)}+j^{(s)}}(g(t',y))|\,dt'\,dy\,(h_1\ldots h_{s-1}h_{s+1}\ldots h_n)^{-1}\right\},$$

$$C_\tau^T(\varepsilon_0,\varepsilon)=$$

$$=\tau^{-1}\max_{s,i,m^{(s)},j^{(s)}}\left\{\iint|\overset{i}{C}_{m^{(s)}+j^{(s)}}(g(t',y))|\,dt'\,dy\,(h_1\ldots h_{s-1}h_{s+1}\ldots h_n)^{-1}\right\},$$

$$D_\tau^T(\varepsilon_0,\varepsilon)=$$

$$=\tau^{-1}\max_{s,m^{(s)},j^{(s)}}\left\{\iint|\overset{T}{D}_{m^{(s)}+j^{(s)}}(g(t',y'))|\,dt'\,dy\,(h_1\ldots h_{s-1}h_{s+1}\ldots h_n)^{-1}\right\}.$$

In the last estimates we have utilized (13). It now remains to estimate the quantities $A_\tau^T,\ldots,D_\tau^T$. Denoting by Π_0 and Π'_0 respectively the $n-$ and $(n-1)$-dimensional unit cube $0\leqslant\xi_i\leqslant 1$, we have

$$\iint|\overset{i}{A}_{m,k,j}(g(t',y))|\,dy\,dt'\leqslant\int_0^T\omega_{\varepsilon_0}(t_{i+1}-t')\,dt'\int dy\left|\int\limits_{\Pi^i_{m+k}}\Omega_\varepsilon(x-y)\,dx\right.$$

(cont'd)

$$- \int\limits_{\Pi^{i}_{m+l}} \Omega_\varepsilon(x-y)\,dx \Bigg| \leqslant \int dy \int\limits_{\Pi_0} |\Omega_\varepsilon(y-\xi kh)-\Omega_\varepsilon(y-\xi_i jh)|\,d\xi\, h_1 \ldots h_n$$

$$\leqslant \alpha \frac{|h|}{\varepsilon} h_1 \ldots h_n, \qquad |h| = \max h_s,$$

$$\iint | B^{i}_{m,\, m^{(s)}+j^{(s)}}(g(t', y))|\, dt'\, dy \leqslant \frac{1}{2}\sigma_s \int\limits_{t_i}^{t_{i+1}} \omega_{\varepsilon_0}(t_{i+1}-t')dt'$$

$$\times \int dy \int\limits_{(m_s-1)h_s}^{(m_s+1)h_s} dx_s \Bigg| \frac{1}{2^{n-1}} \int\limits_{\Pi^{i+1}_{m^{(s)}}} \Omega_\varepsilon(x-y)\,dx^{(s)}$$

$$- \int\limits_{\hat{\Pi}^{i}_{m^{(s)}+j^{(s)}}} \Omega_\varepsilon(x^{(s)}-y^{(s)}, m_s h_s - y_s')\,dx^{(s)} \Bigg|$$

$$\leqslant \frac{1}{2^n}\sigma_s \int\limits_{(m_s-1)h_s}^{(m_s+1)h_s} dx_s \int dy \sum_{k^{(s)}\in I_s} \Bigg| \int\limits_{\hat{\Pi}^{i}_{m^{(s)}+k^{(s)}}} \Omega_\varepsilon(x-y)\,dx^{(s)}$$

$$- \int\limits_{\hat{\Pi}^{i}_{m^{(s)}+j^{(s)}}} \Omega_\varepsilon(x^{(s)}-y^{(s)}, m_s h_s - y_s)\,dx^{(s)} \Bigg|$$

$$\leqslant h_1 \ldots h_2 \frac{\sigma_s'}{2^n} \int\limits_{(m_s-1)h_s}^{(m_s+1)h_s} dx_s \sum_{k^{(s)}} \int\limits_{\Pi_0'} d\xi^{(s)} \int dy\, |\Omega_\varepsilon(y^{(s)} - \xi^{(s)}k^{(s)}h^{(s)}, y_s$$

$$- x_s + h_s m_s) - \Omega_\varepsilon(y^{(s)} - \xi^{(s)} j^{(s)} h^{(s)}, y_s)|$$

$$\leqslant \alpha \frac{|h|}{\varepsilon} h_1 \ldots h_{s-1} h_{s+1} \ldots h_n \frac{1}{2}\sigma_s 2h_s = \alpha\tau \frac{|h|}{\varepsilon} h_1 \ldots h_{s-1} h_{s+1}' \ldots h_n,$$

$$\iint | C^{i}_{m^{(s)}+j^{(s)}}(g(t', y))|\, dt'\, dy \leqslant \int\limits_{t_i}^{t_{i+1}} dt \int\limits_{0}^{T} dt'\, |\omega_{\varepsilon_0}(t_{i+1}-t')$$

$$- \omega_{\varepsilon_0}(t-t')| \int dy \int\limits_{\hat{\Pi}^{i}_{m^{(s)}+j^{(s)}}} \Omega_\varepsilon(x^{(s)}-y^{(s)}, m_s h_s - y_s)\,dx^{(s)}$$

$$\leqslant h_1 \ldots h_{s-1} h_{s+1} \ldots h_n \int\limits_{-\infty}^{\infty} dt' \int\limits_{0}^{\tau} |\omega_{\varepsilon_0}(t'+t) - \omega_{\varepsilon_0}(t')|\,dt$$

(cont'd)

$$\leqslant \alpha \frac{\tau^2}{\varepsilon_0} h_1 \ldots h_{s-1} h_{s+1} \ldots h_n,$$

$$\iint |D^T_{m^{(s)}+j^{(s)}}(g(t',y))|\, dt'\, dy \leqslant \iint dt'\, dy \int\limits_{t_N}^{T} \omega_\varepsilon(t'-t)\, dt$$

$$\times \int\limits_{\Pi^N_{m^{(s)}+j^{(s)}}} \Omega_\varepsilon(x^{(s)}-y^{(s)}, m_s h_s' - y_s)\, dx^{(s)} \leqslant \tau h_1 \ldots h_{s-1} h_{s+1} \ldots h_n,$$

where the constant α depends only on the estimate of the derivative of ω.

In short, for $A_\tau^T, \ldots, D_\tau^T$ we obtain the estimates $A_\tau^T(\varepsilon_0, \varepsilon) \leqslant \alpha\varepsilon^{-1}|h|$, $B_\tau^T(\varepsilon_0, \varepsilon) \leqslant \alpha\varepsilon^{-1}|h|$, $C_\tau^T(\varepsilon_0, \varepsilon) \leqslant \alpha\varepsilon_0^{-1}\tau$, $D_\tau^T(\varepsilon_0, \varepsilon) \leqslant 1$. Since $N \leqslant \tau^{-1}T$, we get

$$\delta_T^{\varepsilon_0,\varepsilon} = -\iint\limits_{\substack{0<t'<T \\ y\in E_n}} \delta_T(g(t',y), u_\tau, v(t',y))\, dt'\, dy$$

$$\leqslant \left\{\frac{T}{\tau}[K_4|h|A_\tau^T + K_5\tau(B_\tau^T + C_\tau^T)] + K_2\tau D_\tau^T\right\} \operatorname{var} u^0$$

$$\leqslant \tau\left\{K_2 + K_3 T\left(\frac{1}{\varepsilon} + \frac{1}{\varepsilon_0}\right)\right\} \operatorname{var} u^0,$$

and the lemma is proved.

Theorem 4

Let $v^0 \in L_1(E_n)$ and let it have bounded variation; let the mesh steps satisfy the stability condition (13) and the condition $\tau^{-1}h_s \leqslant B'$. Then the estimate of the error of Lax's scheme is

$$\|u_\tau(t) - v(t)\| \leqslant [L_1\tau + L(t\tau)^{1/2}] \operatorname{var} v^0.$$

Proof. Since $\operatorname{var} u_\tau(t) \leqslant \operatorname{var} u^0 \leqslant \operatorname{var} v^0$ and $\|v^0 - u^0\| \leqslant h \operatorname{var} v^0 \leqslant B'\tau \operatorname{var} v^0$, we find, on substituting the estimates (16) and (19) into (9), and then minimizing it with respect to ϵ and ϵ_0, that

$$(\operatorname{var} v^0)^{-1}\|u_\tau(t) - v(t)\| \leqslant (B' + K_1 + K_2)\tau + \inf_{\varepsilon>0}\left(2\varepsilon + K_3\frac{t\tau}{\varepsilon}\right)$$

$$+ \inf_{\varepsilon_0>0}\left(B\varepsilon_0 + K_1\varepsilon_0 + K_3\frac{t\tau}{\varepsilon_0}\right) = (B' + K_1 + K_2)\tau + (2K_3 t\tau)^{1/2}$$
$$+ [(B + K_1)K_3 t\tau]^{1/2} \leqslant L(t\tau)^{1/2} + L_1\tau,$$

and the theorem is proved.

Theorem 5

Assume that $v^0 \in L_1(E_n)$ and that it is bounded; let $\lambda(\varepsilon, v^0) = \lambda_0(\varepsilon)$ and let the mesh steps satisfy the conditions stated in Theorem 4. Then, the error of Lax's scheme has the estimate

$$\|u_\tau(t)-v(t)\| \leqslant \inf_{\varepsilon>0} \left\{ 2\lambda_0(\varepsilon) + \left[L_1\tau + L(t\tau)^{1/2} \frac{\lambda_0(\varepsilon)}{\varepsilon} \right] \right\}$$
$$\leqslant (L'+t^{1/2}L)\lambda_0(\tau^{1/2}).$$

Proof. Assume that $v_\varepsilon^0=\Omega_\varepsilon * v^0$, v^ε is the solution of Eq. (1) with the initial function v_ϵ^0, and u_τ^ϵ is the corresponding solution of Lax's scheme. Then, by Theorem 4, we have

$$\|u_\tau^\varepsilon(t)-v^\varepsilon(t)\| \leqslant [L(t\tau)^{1/2}+L_1\tau]\operatorname{var} v_\varepsilon^0 \leqslant [L(t\tau)^{1/2}+L_1\tau]\frac{\lambda_0(\varepsilon)}{\varepsilon}.$$

By the estimate (14),

$$\|u_\tau^\varepsilon(t)-u_\tau(t)\| \leqslant \|u_\tau^\varepsilon(0)-u_\tau(0)\| \leqslant \|v^\varepsilon(0)-v(0)\|,$$

so that

$$\|u_\tau(t)-v(t)\| \leqslant 2\|v_\varepsilon^0-v^0\| + [L(t\tau)^{1/2}+L_1\tau]\frac{\lambda_0(\varepsilon)}{\varepsilon}$$
$$\leqslant 2\lambda_0(\varepsilon) + [L(t\tau)^{1/2}+L_1\tau]\frac{\lambda_0(\varepsilon)}{\varepsilon},$$

and the theorem is proved.

Notice that, in the case when the moduli of continuity λ_0 are not too poor, an estimate can be obtained directly from the inequality (9), by minimizing it with respect to ϵ_0 and ϵ. It can easily be seen, however, that this estimate is worse than the estimate of Theorem 5, except in the case of a linear modulus of continuity.

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