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An unconditionally stable implicit method for hyperbolic conservation laws

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Summary

We construct a space-centered self-adjusting hybrid difference method for one-dimensional hyperbolic conservation laws. The method is linearly implicit and combines a newly developed minimum dispersion scheme of the first order with the recently developed second-order scheme of Lerat. The resulting method is unconditionally stable and unconditionally diagonally dominant in the linearized sense. The method has been developed for quasi-stationary problems, in which shocks play a dominant role. Numerical results for the unsteady Euler equations are presented. It is shown that the method is non-oscillatory, robust and accurate in several cases.

1. Introduction

We consider a one-dimensional quasi-linear hyperbolic system of m conservation laws

$$\frac{\partial w}{\partial t}=\frac{\partial f}{\partial x}, \quad f=f(w). \tag{1.1}$$

We construct a new space-centered linearly implicit scheme for the solution of (1.1). The scheme minimizes dispersion and is unconditionally stable and unconditionally diagonally dominant in the linearized sense for specific values of the parameters. We restrict our attention to these values of the parameters.

The scheme has been developed for quasi-steady problems, in which shocks play a dominant role. Implicit schemes have been found to be applicable for the computation of steady solutions by time-iteration at large Courant numbers, see, for example, several papers in [1] and [8]. Implicit methods are more difficult to apply in quasi-steady problems, because the schemes tend to develop strong oscillations in the neighbourhood of shocks ([9, page 186], [10], [11]). It is our aim to improve the shock resolution in this case.

In Section 2 we describe a general class of six-point difference methods. In Section 3 we introduce the USMD scheme. This scheme is first-order accurate. In Section 4 we apply the USMD scheme to the test problem of an isolated shock for the unsteady Euler equations. We discuss the choice of the parameters and we give some recommendations. We shall see that the USMD scheme has non-oscillatory and accurate shock profiles in several cases. As reference scheme we use the second-order scheme of Lerat, which has been constructed in [9, page 181], [12] and applied in [10].

In Section 5 we introduce a self-adjusting hybrid version of the USMD scheme, which we call the LUSMD scheme. The LUSMD scheme combines the USMD scheme with the above-mentioned scheme of Lerat, thus providing second-order accuracy in smooth parts of the flow. In Section 6 we finally present some numerical results for the LUSMD scheme. We consider two test problems for the unsteady Euler equations, with a maximum Courant number equal to 2 and 5. Both shocks and expansion fans occur. The choice of the parameters is discussed and recommendations are given. We shall see that the scheme is non-oscillatory, accurate and robust in several cases.

2. On a general class of six-point schemes

We consider the following linearly implicit difference method for (1.1):

$$\frac{w_j^{n+1}-w_j^n}{\Delta t}=\frac{1}{\Delta x}\left[h_{j+1/2}-h_{j-1/2}\right], \tag{2.1}$$

$$h_{j+1/2}=h\left(w_{j+1}^n, w_j^n, w_{j+1}^{n+1}, w_j^{n+1}\right), \tag{2.2}$$

$$h(u, v, \phi, \psi)=h_0(u, v)+h_1(u, v)\phi+h_2(u, v)\psi, \tag{2.3}$$

$$\begin{aligned} h_0(u, v)=\frac{1}{2\sigma}\Big\{&\delta(u-v)+\sigma[f(u)+f(v)]-\beta\sigma f_w\left(\frac{u+v}{2}\right)(u+v) \\ &+(1-\epsilon)\sigma^2 f_w\left(\frac{u+v}{2}\right)[f(u)-f(v)]\Big\}, \end{aligned} \tag{2.4}$$

$$h_1(u, v)=\frac{1}{2\sigma}\left\{\alpha+\beta\sigma f_w\left(\frac{u+v}{2}\right)+\gamma\sigma^2 f_w^2\left(\frac{u+v}{2}\right)\right\}, \tag{2.5}$$

$$h_2(u, v)=\frac{1}{2\sigma}\left\{-\alpha+\beta\sigma f_w\left(\frac{u+v}{2}\right)-\gamma\sigma^2 f_w^2\left(\frac{u+v}{2}\right)\right\}, \quad \text{where } \sigma=\frac{\Delta t}{\Delta x}. \tag{2.6}$$

The scheme is conservative, space-centered and at least first-order accurate. For $\delta=-\alpha$ and $\epsilon=2\beta+\gamma$ we obtain a scheme which is a slight modification of the second-order method considered in [9, page 174], [12]. Related schemes have been considered in [2], [6].

We may calculate the modified equation (of the second kind) in the case of a single conservation law (i.e. $m=1$) with the aid of theorem 3.5.1 in [14]. We obtain

$$\frac{\partial w}{\partial t}=\frac{\partial f}{\partial x}+\frac{1}{2}\Delta x\frac{\partial \mu_2}{\partial x}+\frac{1}{6}\Delta x^2\frac{\partial \mu_3}{\partial x}, \tag{2.7}$$

$$\mu_2=D(w)w_x, \quad \mu_3=E_1(w)w_{xx}+E_2(w)w_x^2, \tag{2.8}$$

$$D=\frac{1}{\sigma}\left(\chi_1+\chi_2\sigma^2 f_w^2\right), \quad E_2=\tfrac{1}{2}\left[\tau_1+\tau_2\sigma^2 f_w^2\right]f_{ww}, \tag{2.9}$$

$$\chi_1=\alpha+\delta, \quad \chi_2=2\beta+\gamma-\epsilon, \tag{2.10}$$

$$\tau_1 = 2 + 6\alpha - 3\chi_1, \quad \tau_2 = 3[(4\beta - 3)\chi_2 + 2(\epsilon - 1)], \tag{2.11}$$

$$E_1 = [1 + 3\alpha + (3\beta + 3\gamma - 1)\sigma^2 f_w^2 + 3(\beta - 1)\sigma D] f_w. \tag{2.12}$$

From (2.7), (2.8), (2.9) and (2.10) we see that the choice $\delta = -\alpha$ and $\epsilon = 2\beta + \gamma$ provides a second-order scheme, as already has been remarked. If we next also choose $\alpha = -1/3$ and $3\gamma = 1 - 3\beta$, then we see from (2.12) that $E_1 = 0$. The resulting compact nondispersive scheme is conditionally stable and has already been considered in Lerat [9, page 188].

We use the results obtained in [9, page 23, 174, 179], [12] to investigate the stability and diagonal dominance. For $f(w) = Aw$ with A a constant $m \times m$ matrix it follows, that the scheme is dissipative of order 2 in the sense of Kreiss if and only if

$$\chi_1 + \chi_2 c^2 > 0, \quad \text{and} \tag{2.13}$$

$$(\chi_1 + \chi_2 c^2 + (1 - 2\beta)c^2)(1 - (\delta - \alpha) - (1 - \gamma - \epsilon)c^2) > 0 \tag{2.14}$$

for all eigenvalues c of σA. For $f(w) = \lambda w$ with λ a real constant it follows that the scheme is unconditionally diagonally dominant if and only if

$$\alpha > -\tfrac{1}{2} \quad \text{and} \quad \left[\gamma > \frac{\beta^2}{4(1+\alpha)} \text{ or } \beta = \gamma = 0\right]. \tag{2.15}$$

It is difficult to verify the condition of diagonal dominance in general. It is common use to verify this condition only in the case of a single linear conservation law. We shall always take care that (2.15) is satisfied and we assume that direct LU-decomposition ([7, page 55], [13]) provides in general a stable solution of the block-tridiagonal systems.

3. The USMD scheme

The USMD scheme is obtained from the general scheme of the previous section by setting $E_1 = 0$, i.e. by minimizing dispersion in the modified equation. For a motivation of the minimum dispersion property the reader may consult [14], where numerical shock structures are examined in relation with modified equations. From (2.7), (2.8), (2.9), (2.10) and (2.12) we see that $E_1 = 0$ if

$$\delta = \frac{1}{3}\,\frac{1 + 3\alpha\beta}{1 - \beta}, \tag{3.1}$$

$$\epsilon = \frac{1}{3}\,\frac{1 + 3\beta - 6\beta^2 - 3\beta\gamma}{1 - \beta}, \quad \beta \neq 1. \tag{3.2}$$

If we substitute (3.1) and (3.2) into (2.4), (2.5) and (2.6), then we obtain a three-parameter scheme with parameters α, β and γ. This scheme is called the USMD scheme, because we restrict our attention to parameter values which imply an unconditionally stable scheme.

Using (2.10) we get from a simple substitution:

$$\delta - \alpha + (1 - \gamma - \epsilon)c^2 = -2(\alpha + (\beta + \gamma)c^2) + (\chi_1 + \chi_2 c^2) + c^2. \tag{3.3}$$

Table 3.1. Values of parameters.

α	β	γ	δ	ϵ	χ_1	χ_2	τ_1	τ_2
0	0	2/3	1/3	1/3	1/3	1/3	1	−7
0	0	1	1/3	1/3	1/3	2/3	1	−10
1/3	−1	14/9	0	−5/9	1/3	1/9	3	−35/3
−1/4	0	2/3	1/3	1/3	1/12	1/3	1/4	−7

Because $E_1 = 0$, it follows from (2.9) and (2.12) that

$$3(\alpha + (\beta + \gamma)c^2) = c^2 - 1 - 3(\beta - 1)(\chi_1 + \chi_2 c^2), \tag{3.4}$$

and hence by aid of (3.3),

$$\delta - \alpha + (1 - \gamma - \epsilon)c^2 = \tfrac{2}{3} + (2\beta - 1)(\chi_1 + \chi_2 c^2) + \tfrac{1}{3}c^2. \tag{3.5}$$

Finally, if we substitute (3.5) into (2.14), then it follows easily that the USMD scheme is unconditionally dissipative if and only if

$$\chi_1 > 0, \quad \beta < \tfrac{1}{2}, \quad \chi_2 \geqslant \frac{1}{3(1 - 2\beta)}. \tag{3.6}$$

We take always care that (3.6) is satisfied. As a consequence, the diffusion term D, given in (2.9), contains a term which grows with the Courant number.

In Section 4 we present the results of some numerical experiments. In Table 3.1 we summarize the chosen value of the parameters α, β, γ and the corresponding values of the other parameters.

We may easily verify (2.15) and (3.6), i.e. for these values of the parameters the USMD scheme is unconditionally stable and unconditionally diagonally dominant.

4. Numerical results for the USMD scheme

In the case of the unsteady Euler equations we have

$$w = \begin{pmatrix} \rho \\ \rho u \\ \rho E \end{pmatrix}, \quad f(w) = -\begin{pmatrix} \rho u \\ \rho u^2 + p \\ (\rho E + p)u \end{pmatrix}. \tag{4.1}$$

The functions f_w and f_w^2 are presented in the Appendix. The variables ρ, u, p and E denote the density, velocity, pressure and total energy. By definition $E = e + u^2/2$, where e denotes the internal energy. The gas is assumed to be perfect, i.e. $p = (\gamma - 1)\rho e$, where γ denotes the specific heat ratio. We choose $\gamma = 1.4$. It is very well known that (1.1), (4.1) is strictly hyperbolic with characteristic speeds $\lambda_{\pm 1} = u \pm a$ and $\lambda_0 = u$, where a denotes the speed of sound.

We consider a specific Riemann problem for (1.1), (4.1). We set $d_i = (p_i, \rho_i, u_i)^T$. The initial function reads

$$w(x,0)=\phi(x)=\begin{cases} d_1, & x<0 \\ \tfrac{1}{2}(d_1+d_2), & x=0 \\ d_2, & x>0. \end{cases} \tag{4.2}$$

We take $\Delta x = 1/250$. We set

$$\bar{\eta}=\bar{\lambda}\frac{\Delta t}{\Delta x},\ \bar{\lambda}=\max_{\substack{i=-1,0,1\\ x\in\mathbf{R},\, t=0}} |\lambda_i|. \tag{4.3}$$

The time step Δt is fixed and is such that $\bar{\eta}=2$ or $\bar{\eta}=5$. We restrict our attention to the computed values of the pressure in order to evaluate the experiments.

4.1. Shock resolution

Let s be an arbitrary number. In (4.2) we choose

$$u_1 = v_1 + s,\quad u_2 = v_2 + s, \tag{4.4}$$

$$\frac{\rho_2}{\rho_1}=\frac{v_1}{v_2}=\frac{(\gamma+1)M_1^2}{2+(\gamma-1)M_1^2}, \tag{4.5}$$

$$v_1^2 = a_1^2 M_1^2,\quad a_1^2=\gamma\frac{p_1}{\rho_1}, \tag{4.6}$$

$$M_1^2=\frac{\gamma+1}{2\gamma}\frac{p_2}{p_1}+\frac{\gamma-1}{2\gamma},\quad \text{where} \tag{4.7}$$

$$p_1=\rho_1=1,\quad p_2=4.5,\quad v_1>0. \tag{4.8}$$

Both the jump condition and the entropy condition are initially satisfied. The exact solution consists of an isolated shock wave, travelling along the line $x=st$, i.e. $w=\phi(x-st)$. This test problem has already been considered in [9, page 184].

In [9, page 184] it has already been noticed, that $s\Delta t/\Delta x$ is a fundamental parameter, relating the shock speed and the grid. Therefore we describe the experiments as a function of $\bar{\eta}$ and $s\Delta t/\Delta x$. We also present the results for the second-order method of Lerat, i.e. $\alpha=\beta=\delta=0$ and $\gamma=\epsilon=1/2$ in (2.4), (2.5), (2.6).

We choose $s\Delta t/\Delta t = 0,\ 1/4,\ \pm 1/2$ and we compute the solution over 100 time steps. In nearly all cases we have found that for large n the numerical solution converges to a travelling wave and in these cases the convergence has "settled" after 100 time steps, in the sense that

$$\max_{j\in\mathbf{Z}}\left|p_{j-j_0}^{n-40}-p_j^n\right|<10^{-3},\quad n=100,\quad j_0=40s\frac{\Delta t}{\Delta x}. \tag{4.9}$$

The exact shock position is denoted by x_s, i.e. $x_s = ns\Delta t/\Delta x$, $n=100$. In all cases x_s coincides with a grid point. In Figs. 4.1 and 4.2 we present two computed pressure

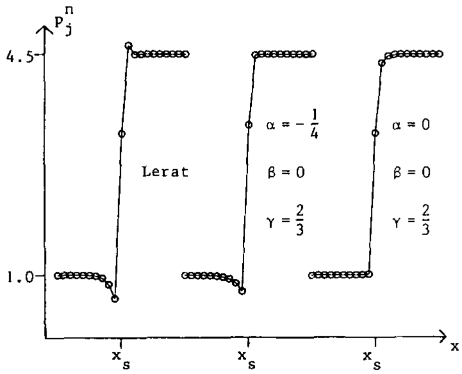

Figure 4.1. $\bar{\eta}=5$, $s\Delta t/\Delta x=0$.

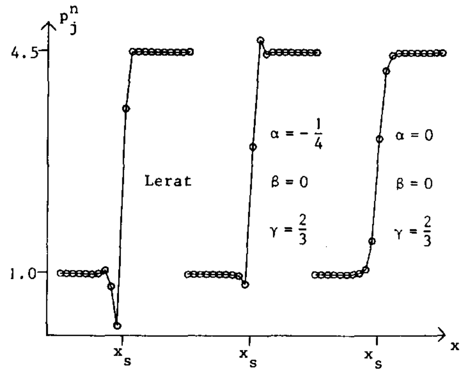

Figure 4.2. $\bar{\eta}=2$, $s\Delta t/\Delta x=\frac{1}{4}$.

profiles. From these figures and Table 3.1 we may draw the conclusion that it is important to introduce a significant constant diffusion term in the modified equation, besides the minimization of dispersion, if one searches for non-oscillatory profiles.

We set

$$E_j^n = p_{ex}(x_j, t_n) - p_j^n, \quad x_j = j\Delta x, \quad t_n = n\Delta t, \tag{4.10}$$

$$N = \sum_{j=j_0-10}^{j_0+10} |E_j^n|, \quad j_0 = \frac{x_s}{\Delta x}. \tag{4.11}$$

In Table 4.1 we present some computed values of N, where N has been given in (4.11). In cases with a triangle (Δ) we have found that condition (4.9) is not satisfied. In cases with an asterisk ($*$) we have found oscillations larger than one per cent of the shock strength. In Figs. 4.3 and 4.4 we finally illustrate the computed pressure profiles in two interesting cases.

From Table 4.1 it follows that a higher order of accuracy does not necessarily lead to a smaller error in the neighbourhood of shock waves, which in view of modified equations is not surprising ([14, page 68]). We also see that one may encounter (nonlinear) instabilities. At this time a theoretical explanation of these instabilities is not available, but it is clear

Table 4.1. Values of N, shock wave.

$\bar{\eta}$	$s\Delta t/\Delta x$	Lerat	$\alpha=0$ $\beta=0$ $\gamma=2/3$	$\alpha=0$ $\beta=0$ $\gamma=1$	$\alpha=1/3$ $\beta=-1$ $\gamma=14/9$
2	−0.5	1.84*	1.11	1.57	1.19
2	0	1.56*	1.21	1.31	1.72
2	0.5	2.37*	1.85	2.27	1.80
5	−0.5	1.77*	1.43*	1.33	2.16*
5	0	1.21*	0.70	0.94	1.83
5	0.25	1.58*Δ	0.81Δ	0.93	1.71Δ
5	0.5	unstable	unstable	0.85	1.36Δ

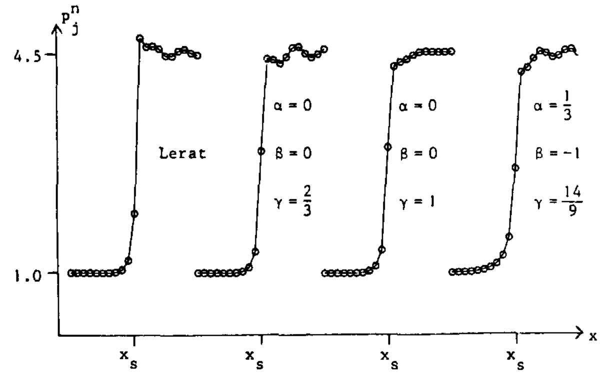

Figure 4.3. $\bar{\eta} = 5$, $s\Delta t/\Delta x = -0.5$.

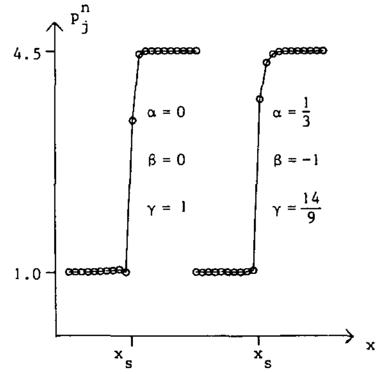

Figure 4.4. $\bar{\eta} = 5$, $s\Delta t/\Delta x = 0.5$.

that the occurrence depends on the speed of the shock. In fact we have succeeded in computing stationary shocks with the above schemes at maximum Courant numbers, which are much larger than 5.

The solution in the third column of Table 4.1 is more diffuse than the solution in the second column. This is in agreement with the modified equation (see Table 3.1 for the values of the parameters). To some extent it is surprising that the computed profiles are less diffuse for large value of $\bar{\eta}$, because the diffusion term in the modified equation of the USMD scheme contains a term which grows with the Courant number (see Section 3). In order to investigate this situation we have calculated by analytical means the shock thickness in the modified equation in the case that $m = 1$ and $f(w) = w^2/2$. We have found that in this model problem the shock thickness in the modified equation is a decreasing function of $\bar{\eta}$ as well. This means that it is the diminishing influence of the constant diffusion term in the modified equation which is more important for the balance between the two diffusion terms.

4.2. Conclusions

From Table 4.1 we first draw the conclusion that it is not very useful to introduce a nonzero value of the parameter β. The specific choice of β in Table 3.1 has been exaggerated a little in order to stretch out this fact clearly. A nonzero value of β leads to a significant increase of computer time. Furthermore, if $\beta = 0$, then from (2.4) we see that the explicit part is consistent, because $h_0(w, w) = f(w)$. We may therefore regard the scheme as a combination of an explicit predictor and an implicit corrector. This point of view has some advantages, in particular with respect to the possible formulation in curvilinear coordinates (compare with [10]).

Next the choice $\alpha = 0$ is attractive and (in view of the obtained results) very natural. We have also considered, but not reported, values of the parameters which extrapolate the sequence in Figs. 4.1 and 4.2, for example $\alpha = 1/6$, $\beta = 0$, $\gamma = 2/3$. Such a choice results in a larger constant diffusion term in the modified equation. The numerical experiments have indicated that very stable schemes may be obtained, but unfortunately with strongly diffuse profiles, in particular for low Courant numbers.

If we choose both $\alpha = 0$ and $\beta = 0$, then it follows from Table 4.1 that the scheme becomes more diffuse and stable if we let γ rise above the minimum value $\gamma = 2/3$. If stability is no serious problem, because the shock waves are slow, then we recommend of course the value $\gamma = 2/3$, because in this case the error takes a minimum value. If instead the shocks have a considerable speed, then we have to enlarge the value of γ.

5. Hybridizing the USMD scheme

We consider two six-point difference schemes DS_1 and DS_2 of the form (2.1) with numerical flux functions $h^{(1)}$ and $h^{(2)}$; DS_1 is a first and DS_2 a second-order scheme. Next we consider an eight-point molecule, consisting of five points on the old and three points on the new time level. We construct a new self-adjusting hybrid scheme ([4]) on this molecule with numerical flux function h given by

$$h_{j+1/2} = h\left(w_{j+2}^n, w_{j+1}^n, w_j^n, w_{j-1}^n, w_{j+1}^{n+1}, w_j^{n+1}\right), \tag{5.1}$$

$$\begin{aligned} h(\bar{u}, u, v, \bar{v}, \phi, \psi) &= h^{(2)}(u, v, \phi, \psi) \\ &\quad + \theta(\bar{u}, u, v, \bar{v})\left[h^{(1)}(u, v, \phi, \psi) - h^{(2)}(u, v, \phi, \psi)\right]. \end{aligned} \tag{5.2}$$

The function θ is called a switch function. Several switch functions θ have been investigated in literature. We choose the switch function used in [5]. Let $\sigma(w)$ denote a function to be specified later on. We set

$$\theta = \max(\hat{\theta}(\bar{u}, u, v), \hat{\theta}(u, v, \bar{v})) \tag{5.3}$$

where

$$\hat{\theta}(\bar{u}, u, v) = \begin{cases} \left|\dfrac{\alpha}{\beta}\right| & \text{if} \quad \beta > \epsilon \\ 0 & \text{if} \quad \beta \leqslant \epsilon \end{cases} \tag{5.4}$$

with

$$\alpha = |\sigma(\bar{u}) - \sigma(u)| - |\sigma(u) - \sigma(v)|, \tag{5.5}$$

$$\beta = |\sigma(\bar{u}) - \sigma(u)| + |\sigma(u) - \sigma(v)|. \tag{5.6}$$

We see that $0 \leqslant \theta \leqslant 1$. In the case of the Euler equations we choose ([5])

$$\sigma(w) = \rho, \tag{5.7}$$

where ρ denotes the density. The switch function measures variations in the solution. For example, near shocks we see that θ is close to one, leading to integration by a scheme which is close to DS_1. In smooth parts of the flow θ is close to zero, leading to integration by a scheme which is close to DS_2. The tolerance ϵ is a measure of insignificant variation.

For DS_1 we use the USMD scheme with $\alpha = \beta = 0$ and for DS_2 we use Lerat's scheme, i.e. $\alpha = \beta = \delta = 0$ and $\epsilon = \gamma$ in (2.4), (2.5), (2.6). In view of the recommendation of the previous section, it is natural to choose both $\alpha = 0$ and $\beta = 0$ (this choice has also been recommended by Lerat for his scheme). The resulting hybrid scheme depends on one parameter, viz. γ.

As usual, we freeze the value of the switch function θ in (5.2) ($0 \leqslant \theta \leqslant 1$) in order to investigate stability and diagonal dominance. We easily see that the resulting scheme is of the form (2.4), (2.5), (2.6) with parameter values: $\alpha = \beta = 0$, γ is free, $\delta = \theta/3$, $\epsilon = (1-\theta)\gamma + \theta/3$. Using [9, page 23, 174, 179] or [12], it next follows that the resulting scheme is unconditionally stable and unconditionally diagonally dominant for arbitrary values of θ, with $0 \leqslant \theta \leqslant 1$, if and only if $\gamma \geqslant 2/3$.

6. Numerical results for the LUSMD scheme

We apply the LUSMD scheme with $\gamma = 2/3$, 1, 3/2. For the tolerance ϵ of insignificant variation in (5.4) we choose $\epsilon = 10^{-4}$. We also choose $\Delta x = 1/250$.

6.1. Shock resolution

We consider the test problem described in Section 4.1. For more details the reader is referred to that section. In Table 6.1 we present several computed values of N. In cases

Table 6.1. Values of N, shock wave.

$\bar{\eta}$	$s\Delta t/\Delta x$	$\gamma = 2/3$	$\gamma = 1$	$\gamma = 3/2$
2	−0.5	0.55	0.67	0.84
2	0	0.82	0.87	0.95
2	0.5	1.19	1.37	1.65
5	−0.5	0.98*	1.02	1.29
5	0	0.60	0.77	0.98
5	0.5	1.18*	0.80*	0.91

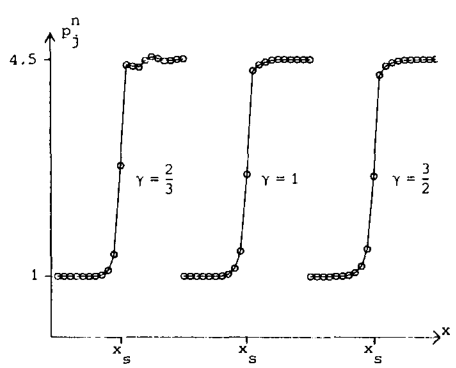

Figure 6.1. $\bar{\eta} = 5$, $s\Delta t/\Delta x = -0.5$.

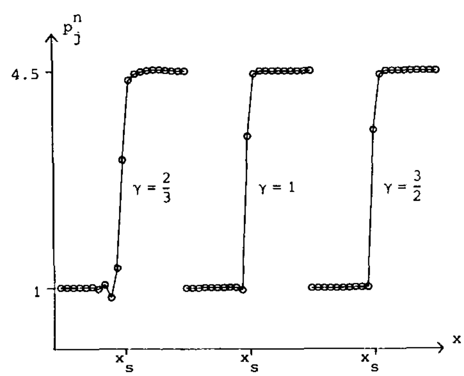

Figure 6.2. $\bar{\eta} = 5$, $s\Delta t/\Delta x = 0.5$.

with an asterisk (*) we have found oscillations larger than one per cent of the shock strength. In Figs. 6.1 and 6.2 we present the computed pressure profiles in two interesting cases.

If we compare Tables 4.1 and 6.1, then we see that in particular for low Courant numbers the accuracy for shock waves improves further.

6.2. Expansion fans

Let s be an arbitrary number with $0 \leqslant s \leqslant 1$. We consider the initial function (4.2) with

$$u_1 = -\tfrac{1}{2}(1+s), \quad u_2 = -\tfrac{1}{2}\left(1 - \frac{7-\gamma}{1+\gamma}s\right),$$

$$p_1 = \frac{1}{2}\left(\frac{a_1}{a_2}\right)^{2\gamma/(\gamma-1)}, \quad p_2 = \tfrac{1}{2},$$

$$\rho_1 = \gamma\frac{p_1}{a_1^2}, \quad \rho_2 = \frac{\gamma}{2a_2^2},$$

where

$$a_1 = \frac{1-s}{2}, \quad a_2 = \frac{1}{2}\left(1 - \frac{5-3\gamma}{1+\gamma}s\right), \quad \gamma = 1.4.$$

We have $\bar{\lambda} = 1$, $\forall s$, where $\bar{\lambda}$ has been given in (4.3). As can be seen from [3, page 94, 104], the exact solution consists of an isolated expansion fan in between the lines $x = \pm st$.

We choose $s = 1/5$ in order to obtain a weakly dynamic problem. The time step Δt is fixed and such that $\bar{\eta} = 2$ or $\bar{\eta} = 5$, where $\bar{\eta}$ has been given in (4.3). Because $\bar{\lambda} = 1$, we have $\eta = \Delta t/\Delta x$. We compute the solution over 60 time steps. The schemes have been found to be non-oscillatory in all computed cases. We set

$$N = \sum_{j=-80}^{80} |E_j^n|,$$

where E_j^n has been defined in (4.10). We present the computed values of N in Table 6.2. We give also the values for the USMD scheme with $\alpha = \beta = 0$ and $\gamma = 2/3$. In Fig. 6.3 we illustrate the computed error distribution for $\bar{\eta} = 5$.

From Table 6.2 we see that the hybrid scheme gives a good improvement of the corresponding first-order scheme, which is as expected. We also see that the accuracy in the hybrid scheme decreases if γ increases, which is also as expected in view of the

Table 6.2. Values of N, expansion fan.

$\bar{\eta}$	$\gamma = 2/3$	$\gamma = 1$	$\gamma = 3/2$	USMD
2	0.23	0.23	0.24	0.59
5	0.21	0.24	0.28	0.47

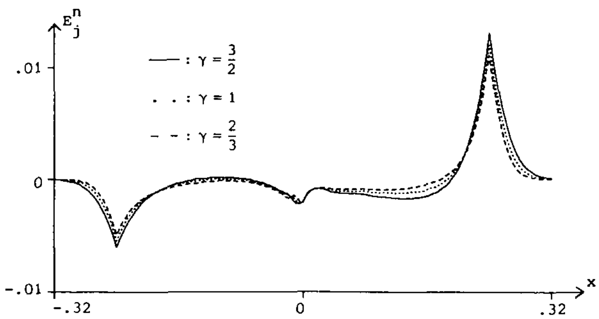

Figure 6.3. $\bar{\eta} = 5$.

increasing dissipation of the hybrid scheme. Finally, we remark that small expansion shocks have been found in the second-order method of Lerat. As can be seen from Fig. 6.3 near $x = 0$ this phenomenon has not completely been removed in the hybrid scheme. The magnitude of the expansion shock is very small in the hybrid scheme and further precautions are not necessary.

6.3. Conclusions

From the results of Section 6 it is clear that the LUSMD scheme is an attractive and robust scheme. The scheme is accurate, even in rather severe test problems. In view of the results in Table 6.1 and Figs. 6.1, 6.2, we recommend of course the value $\gamma = 3/2$. For this value we have found (nearby) monotonic profiles in all computed cases. If applicable without introducing severe oscillations, i.e. if the shock waves do not move too fast, then we of course suggest to decrease the value of γ in order to gain accuracy. If instead the shocks have a considerable speed, then we have to enlarge the value of γ. For the expansion fan some loss of accuracy results from enlarging γ, however, within acceptable bounds.

If one searches for an even more robust scheme, then it is suggested to include the parameter α in the LUSMD scheme. In the conclusion of Section 4 it has already been remarked that, for example, the choice $\alpha = 1/6$ leads to stabilized versions of the USMD scheme, because the constant diffusion term in the modified equation is increased. For the LUSMD scheme the same holds. Further tests have to be carried out, because the accuracy may become poor, in particular for low Courant numbers.

Appendix. The functions f_w and f_w^2 for the unsteady Euler equations

$$f_w = -\begin{pmatrix} 0 & 1 & 0 \\ -\dfrac{3-\gamma}{2}u^2 & (3-\gamma)u & \gamma - 1 \\ -\left(\dfrac{2-\gamma}{2}u^2 + \dfrac{a^2}{\gamma-1}\right)u & \dfrac{3-2\gamma}{2}u^2 + \dfrac{a^2}{\gamma-1} & \gamma u \end{pmatrix},$$

$$f_w^2 = -\begin{pmatrix} -\dfrac{3-\gamma}{2}u^2 & (3-\gamma)u & \gamma-1 \\ \left(\dfrac{3\gamma-7}{2}u^2 - a^2\right)u & 3(2-\gamma)u^2 + a^2 & 3(\gamma-1)u \\ \left(\dfrac{5\gamma-9}{4}u^2 - \dfrac{3+\gamma}{2(\gamma-1)}a^2\right)u^2 & \left(\dfrac{7-5\gamma}{2}u^2 + \dfrac{2}{\gamma-1}a^2\right)u & \dfrac{5\gamma-3}{2}u^2 + a^2 \end{pmatrix}.$$

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