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Author(s): Guang-Shan Jiang and Shih-Hsien Yu
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DISCRETE SHOCKS FOR FINITE DIFFERENCE APPROXIMATIONS TO SCALAR CONSERVATION LAWS*

GUANG-SHAN JIANG† AND SHIH-HSIEN YU†

Abstract. Numerical simulations often provide strong evidences for the existence and stability of discrete shocks for certain finite difference schemes approximating conservation laws. This paper presents a framework for converting such numerical observations to mathematical proofs. The framework is applicable to conservative schemes approximating stationary shocks of one-dimensional scalar conservation laws. The numerical flux function of the scheme is assumed to be twice differentiable but the scheme can be nonlinear and of any order of accuracy. To prove existence and stability, we show that it would suffice to verify some simple inequalities, which can usually be done using computers. As examples, we use the framework to give an unified proof of the existence of continuous discrete shock profiles for a modified first-order Lax–Friedrichs scheme and the second-order Lax–Wendroff scheme. We also show the existence and stability of discrete shocks for a third-order weighted essentially nonoscillatory (ENO) scheme.

Key words. conservation law, discrete shock, weighted ENO

AMS subject classifications. 65M06, 65M12, 65M15, 35L65

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1. Introduction.

In this paper, we provide a general framework for proving the existence and stability of continuous discrete shock profiles for conservative finite difference schemes which approximate scalar conservation laws

$$u_t + f(u)_x = 0 \qquad x \in \mathcal{R}. \tag{1.1}$$

We consider schemes of conservative form:

$$u_j^{n+1} = \mathcal{L}[u^n]_j \tag{1.2}$$

with

$$\mathcal{L}[u^n]_j = u_j^n - \lambda \left(g\left(u_{j-p+1}^n, u_{j-p+2}^n, \dots, u_{j+p}^n\right) - g\left(u_{j-p}^n, u_{j-p+1}^n, \dots, u_{j+p-1}^n\right)\right). \tag{1.3}$$

Here $g(.,.,\dots,.)$ is the numerical flux of the scheme which satisfies $g(u,u,\dots,u) = f(u)$ (consistency) and is twice continuously differentiable with respect to its arguments; u_j^n is an approximation to $u(j\Delta x, n\Delta t)$ ($\lambda = \Delta t/\Delta x = constant$); p is a constant integer such that $(2p+1)$ is the stencil width of the scheme. Schemes with such flux functions include the first-order Lax–Friedrichs scheme and some of its modified versions, the second-order Lax–Wendroff scheme, and a class of high-resolution weighted ENO schemes [4].

Let u_- and u_+ be two constants such that (1.1) with the initial data

$$u(x,0) = \begin{cases} u_- & \text{if } x < 0, \\ u_+ & \text{if } x > 0 \end{cases}$$

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†Department of Mathematics, UCLA, Los Angeles, CA 90095 (gsj@math.ucla.edu, shyu@math.ucla.edu).

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admits an entropy satisfying shock given by

$$u(x,t) = \begin{cases} u_- & \text{if } x < st, \\ u_+ & \text{if } x > st \end{cases}$$

where s is the shock speed and by the Rankine–Hugoniot condition,

$$s = \frac{f(u_+) - f(u_-)}{u_+ - u_-}.$$

In this paper, we will assume $s = 0$ or, equivalently, $f(u_+) = f(u_-)$. Introduce

$$\pi_j = \begin{cases} u_- & \text{if } j < 0, \\ (u_- + u_+)/2 & \text{if } j = 0, \\ u_+ & \text{if } j > 0. \end{cases} \tag{1.4}$$

Let us clarify the concepts we will use frequently in the paper by the following.

DEFINITION 1.1. *For $q \in [0,1]$, if $\varphi \equiv \{\varphi_j, j \in \mathcal{Z}\}$ satisfies*

1. $\sup_{j\in\mathcal{Z}} |\varphi_j|$ *is finite,*
2. $\lim_{j\to\pm\infty} \varphi_j = u_\pm$,
3. $\sum_{j\in\mathcal{Z}}(\varphi_j - \pi_j) = q(u_- - u_+)$,

we call φ an approximate stationary discrete shock with parameter q. *If, furthermore, φ also satisfies*

4. $\varphi_j = \mathcal{L}[\varphi]_j$ *for all $j \in \mathcal{Z}$,*

we call it an exact stationary discrete shock for scheme (1.2) with parameter q.

DEFINITION 1.2. *If a function $\varphi(x) (x \in \mathcal{R})$ is bounded, uniformly Lipschitz continuous in $\mathcal{R}$, and for any $q \in [0,1]$, $\{\varphi((j-q)\Delta x), j \in \mathcal{Z}\}$ is an exact stationary discrete shock for scheme* (1.2) *with parameter q, then φ is called* a continuous stationary discrete shock profile for scheme (1.2).

Remark. An approximate discrete shock is not related to the scheme (1.2). However, we will only be interested in approximate discrete shocks which are so accurate that condition 4 in Definition 1.1 is almost satisfied. In the following discussions, we will omit "stationary" when referring to stationary discrete shocks. We will often refer to exact discrete shocks plainly by "discrete shocks."

Existence and stability of discrete shocks are essential for the error analysis of difference schemes approximating (1.1). It is well known that solutions to (1.1) generally contain shocks and numerical schemes unavoidably commit $O(1)$ error around the shocks. It is important to understand whether this $O(1)$ error will destroy the accuracy of the scheme in smooth parts of the solution. For conservation laws, whose solutions are sufficiently smooth away from isolated shocks, Engquist and Yu [1] proved that the $O(1)$ error committed by a finite difference scheme around shocks will not pollute the accuracy of the scheme in smooth regions up to a certain time, *provided that,* (i) the scheme is linearly stable, and (ii) the scheme possesses stable discrete shocks.

This paper was motivated by the work of Liu and Yu [6]. Their approach was to linearize the scheme around some *constructed* approximate discrete shocks. Existence and stability of exact discrete shocks were then obtained by proving that this linearized scheme defines a contractive mapping for small perturbations on the approximated discrete shock and the original scheme behaves closely like the linearized scheme. We would like to mention that Goodman and Xin [2] utilized approximate solutions in their so-called matching method to obtain convergence to systems of conservation laws. Their approach was adopted by Teng and Zhang [12] in obtaining the optimal

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L^1 error estimate of monotone scheme in the case of piecewise constant solutions with shocks.

Our main observation is that, using computers, one can easily obtain approximate discrete shocks and most importantly, these approximate discrete shocks can be made as accurate as the machine limit allows. As we know, if a scheme possesses a discrete shock, it can often be observed from numerical experiments that the scheme converges quickly to a numerical discrete shock after a number of time iterations over an initial guess (this is often equivalently stated as "the residue quickly settles down to machine zero"). When it is linearized around such accurate approximate discrete shocks, the finite difference scheme can be expected to behave very closely like the linearized scheme (e.g., in terms of contractiveness of the induced mapping). The obvious advantage of using numerically computed approximate discrete shocks is that it can be applied to almost all schemes and all conservation laws with little efforts.

A brief review of the study of discrete shocks for finite difference schemes follows. The existence of a discrete shock was first studied by Jennings [3] for a monotone scheme by an L^1-contraction mapping and Brower's fixed point theorem. For a first-order system, Majda and Ralston [7] used a center manifold theory and proved the existence of a discrete shock. Yu [13] and Michaelson [8] followed the center manifold approach and showed the existence and stability of a discrete shock for the Lax–Wendroff scheme and a third-order scheme, respectively. In [11], Smylris and Yu studied the continuous dependence of the discrete shocks by extending the functional space for finite difference schemes from $L^\infty(\mathcal{Z})$ to $L^\infty(\mathcal{R})$ and a fixed point theorem. All existence theorems above require an artificial assumption on the shock speed. In [5], Liu and Xin proved the existence and stability of a stationary discrete shock for a modified Lax–Friedrichs scheme. For a first-order system, Liu and Yu [6] showed both the existence and stability of a discrete shock using a pointwise estimate and a fixed point theorem as well as the continuous dependence of the discrete shock on the end states (u_-, u_+).

Our paper is organized as follows.

First, we prove a basic fixed point theorem in section 2. Then we show in section 3 how the existence and stability problem can be formulated into a fixed point problem once an approximate discrete shock is available. In section 4, we derive sufficient conditions for the scheme (1.2) to possess a single stable discrete shock and in section 5, we derive sufficient conditions for scheme (1.2) to possess a continuous discrete shock profile. In section 6, we discuss how to obtain approximate discrete shocks and how to verify the conditions derived in earlier sections *by computers.* In section 7, we apply our framework to give a unified proof of the existence of continuous discrete shock profiles for a first-order modified Lax–Friedrichs scheme and the second-order Lax–Wendroff scheme. We will also show the proof of existence and stability of discrete shocks for a third-order weighted ENO scheme. Some remarks will be given in section 8.

2. A basic fixed point theorem. In this section, we prove a basic fixed point theorem. First, let us define a weighted l^2 norm. Suppose that $\alpha > 1$ and $\beta > 1$ are two constants. For any infinite dimensional vector $v \equiv \{v_j, j \in \mathcal{Z}\}$, we define

$$\|v\|_{\alpha,\beta} \equiv \left[\sum_{j\geq 0} v_j^2 \alpha^{2j} + \sum_{j<0} v_j^2 \beta^{-2j}\right]^{1/2}.$$

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If $\alpha = \beta = 1$, $\|\cdot\|_{\alpha,\beta}$ becomes the regular l^2 norm. We denote the corresponding space by

$$l^2_{\alpha,\beta} = \{v \quad | \quad \|v\|_{\alpha,\beta} < \infty\}.$$

Because $\alpha > 1$ and $\beta > 1$, it is easy to see that any vector $v \in l^2_{\alpha,\beta}$ satisfies

$$\lim_{j\to\pm\infty} v_j = 0 \qquad \text{and} \qquad \sup_{j\in\mathcal{Z}} |v_j| \le \|v\|_{\alpha,\beta}. \tag{2.1}$$

We denote a closed ball with radius r around a vector v_0 in $l^2_{\alpha,\beta}$ as $B_r(v_0)$, i.e.,

$$B_r(v_0) \equiv \{v \in l^2_{\alpha,\beta} \quad | \quad \|v - v_0\|_{\alpha,\beta} \le r\}.$$

Let $\mathcal{F}$ be a mapping from $l^2_{\alpha,\beta}$ to $l^2_{\alpha,\beta}$ and have the form

$$\mathcal{F}[v] = L[v] + N[v] + \mathcal{E} \tag{2.2}$$

where

1. L is a mapping which is linear, i.e., $L[av+bw] = aL[v]+bL[w]$ for any $a, b \in \mathcal{R}$ and $v, w \in l^2_{\alpha,\beta}$, and contractive, i.e.,

$$\|L\|_{\alpha,\beta} \equiv \sup_{\|v\|_{\alpha,\beta}=1} \|L[v]\|_{\alpha,\beta} < 1; \tag{2.3}$$

2. N is generally a nonlinear mapping and $N[\emptyset] = \emptyset$ where $\emptyset$ is the null vector. Moreover,

$$\|N\|^r_{\alpha,\beta} \equiv \sup_{v,w\in B_r(\emptyset)} \frac{\|N[v] - N[w]\|_{\alpha,\beta}}{\|v - w\|_{\alpha,\beta}} = O(r); \tag{2.4}$$

3. $\mathcal{E}$ is a constant vector in $l^2_{\alpha,\beta}$, i.e., independent of v.

We have the following basic fixed point theorem.

THEOREM 2.1 (a basic fixed point theorem). *If there exists $\sigma > 0$ such that*

$$\|\mathcal{E}\|_{\alpha,\beta} < \sigma \left\{1 - \|L\|_{\alpha,\beta} - \|N\|^\sigma_{\alpha,\beta}\right\}, \tag{2.5}$$

then the mapping $\mathcal{F}[v] = L[v] + N[v] + \mathcal{E}$

1. *is contractive in $B_\sigma(\emptyset)$ under the norm $\|\cdot\|_{\alpha,\beta}$;*
2. *has a unique fixed point, i.e., there exists only one $\bar{v} \in B_\sigma(\emptyset)$ such that $\bar{v} = \mathcal{F}[\bar{v}]$. Moreover,*

$$\|\bar{v}\|_{\alpha,\beta} \le \frac{\|\mathcal{E}\|_{\alpha,\beta}}{1 - \|L\|_{\alpha,\beta} - \|N\|^\sigma_{\alpha,\beta}}. \tag{2.6}$$

Proof. For any $v, w \in B_\sigma(\emptyset)$, we have

$$\begin{aligned}\|\mathcal{F}[v] - \mathcal{F}[w]\|_{\alpha,\beta} &= \|L[v] - L[w] + N[v] - N[w]\|_{\alpha,\beta} \\ &\le \|L[v - w]\|_{\alpha,\beta} + \|N[v] - N[w]\|_{\alpha,\beta} \\ &\le \|L\|_{\alpha,\beta}\|v - w\|_{\alpha,\beta} + \|N\|^\sigma_{\alpha,\beta}\|v - w\|_{\alpha,\beta}.\end{aligned}$$

Therefore,

$$\|\mathcal{F}[v] - \mathcal{F}[w]\|_{\alpha,\beta} \le (\|L\|_{\alpha,\beta} + \|N\|^\sigma_{\alpha,\beta})\|v - w\|_{\alpha,\beta}. \tag{2.7}$$

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Moreover, for any $v \in B_\sigma(\emptyset)$, since $\mathcal{F}[\emptyset] = L[\emptyset] + N[\emptyset] + \mathcal{E} = \mathcal{E}$, we have

$$\begin{aligned}\|\mathcal{F}[v]\|_{\alpha,\beta} &\leq \|\mathcal{F}[\emptyset]\|_{\alpha,\beta} + \|\mathcal{F}[v] - \mathcal{F}[\emptyset]\|_{\alpha,\beta} \\ &\leq \|\mathcal{E}\|_{\alpha,\beta} + (\|L\|_{\alpha,\beta} + \|N\|^\sigma_{\alpha,\beta})\|v\|_{\alpha,\beta} \\ &\leq \|\mathcal{E}\|_{\alpha,\beta} + \sigma(\|L\|_{\alpha,\beta} + \|N\|^\sigma_{\alpha,\beta}).\end{aligned}$$

From the condition (2.5), we obtain

$$\|\mathcal{F}[v]\|_{\alpha,\beta} \leq \sigma \qquad \text{for any } v \in B_\sigma(\emptyset). \tag{2.8}$$

Condition (2.5) implies that $\|L\|_{\alpha,\beta}+\|N\|^\sigma_{\alpha,\beta}$ is strictly less than 1. Therefore, (2.7) and (2.8) together imply that the mapping $\mathcal{F}$ maps $B_\sigma(\emptyset)$ into itself and is a contractive mapping. From Banach fixed point theorem, there exists a unique $\bar{v} \in B_\sigma(\emptyset)$ such that $\mathcal{F}[\bar{v}] = \bar{v}$. In addition, we have

$$\begin{aligned}\|\bar{v}\|_{\alpha,\beta} &= \|\mathcal{F}[\bar{v}]\|_{\alpha,\beta} \\ &\leq \|\mathcal{F}[\emptyset]\|_{\alpha,\beta} + \|\mathcal{F}[\bar{v}] - \mathcal{F}[\emptyset]\|_{\alpha,\beta} \\ &\leq \|\mathcal{E}\|_{\alpha,\beta} + (\|L\|_{\alpha,\beta} + \|N\|^\sigma_{\alpha,\beta})\|\bar{v}\|_{\alpha,\beta}\end{aligned}$$

from which (2.6) follows. $\square$

3. Formulation of a fixed point problem. For a fixed $q \in [0,1]$, assume $\bar{\varphi}$ is an approximate stationary discrete shock with parameter q and is given. By Definition 1.1, we know

$$\bar{k} \equiv \sup_{j\in\mathcal{Z}} |\bar{\varphi}_j| \quad < \quad \infty, \tag{3.1}$$

$$\lim_{j\to\pm\infty} \bar{\varphi}_j = u_\pm, \tag{3.2}$$

$$\sum_{j\in\mathcal{Z}} (\bar{\varphi}_j - \pi_j) = q(u_- - u_+). \tag{3.3}$$

Assume that there exists an exact discrete shock for scheme (1.2) with parameter q denoted as φ. If we set

$$\bar{u}_j = \varphi_j - \bar{\varphi}_j \qquad \text{for all } j \in \mathcal{Z},$$

then condition 2 in Definition 1.1 and (3.3), respectively, imply

$$\lim_{j\to\pm\infty} \bar{u}_j = 0 \qquad \text{and} \qquad \sum_{j\in\mathcal{Z}} \bar{u}_j = 0. \tag{3.4}$$

Condition 4 in Definition 1.1 gives

$$\bar{\varphi}_j + \bar{u}_j = \mathcal{L}[\bar{\varphi} + \bar{u}]_j \qquad \text{for all } j \in \mathcal{Z}. \tag{3.5}$$

Here $\bar{\varphi} + \bar{u}$ means a vector sum of $\bar{\varphi}$ and $\bar{u}$, i.e., $(\bar{\varphi} + \bar{u})_j = \bar{\varphi}_j + \bar{u}_j$ for all $j \in \mathcal{Z}$. Letting

$$\bar{v}_j \equiv \sum_{k\leq j} \bar{u}_k,$$

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we have $\bar{u}_j = \Delta_- \bar{v}_j \equiv \bar{v}_j - \bar{v}_{j-1}$. Equation (3.4) implies

$$\lim_{j \to \pm\infty} \bar{v}_j = 0. \tag{3.6}$$

Using (1.3) to write (3.5) in terms of $\bar{v}$, we have

$$\begin{aligned}
\bar{v}_j &= \sum_{k \leq j} \{\mathcal{L}[\bar{\varphi} + \Delta_- \bar{v}]_k - \bar{\varphi}_k\} \\
&= \sum_{k \leq j} \{\Delta_- \bar{v}_k - \lambda\, [g(\bar{\varphi}_{k-p+1} + \Delta_- \bar{v}_{k-p+1}, \ldots, \bar{\varphi}_{k+p} + \Delta_- \bar{v}_{k+p}) \\
&\qquad\qquad - g(\bar{\varphi}_{k-p} + \Delta_- \bar{v}_{k-p}, \ldots, \bar{\varphi}_{k+p-1} + \Delta_- \bar{v}_{k+p-1})]\} \\
&= \bar{v}_j - \lambda\, [g(\bar{\varphi}_{j-p+1} + \Delta_- \bar{v}_{j-p+1}, \ldots, \bar{\varphi}_{j+p} + \Delta_- \bar{v}_{j+p}) \\
&\qquad - \lim_{k \to -\infty} g(\bar{\varphi}_{k-p} + \Delta_- \bar{v}_{k-p}, \ldots, \bar{\varphi}_{k+p-1} + \Delta_- \bar{v}_{k+p-1})] \\
&= \bar{v}_j - \lambda\, [g(\bar{\varphi}_{j-p+1} + \Delta_- \bar{v}_{j-p+1}, \ldots, \bar{\varphi}_{j+p} + \Delta_- \bar{v}_{j+p}) - f(u_-)] \qquad (3.7)
\end{aligned}$$

where we have used

$$\lim_{k \to -\infty} g(\bar{\varphi}_{k-p} + \Delta_- \bar{v}_{k-p}, \ldots, \bar{\varphi}_{k+p-1} + \Delta_- \bar{v}_{k+p-1}) = f(u_-),$$

which is implied by (3.2), (3.6), and the consistency and continuity of the flux function g.

Let us define the right-hand side (RHS) of (3.7) as a mapping in some subspace of the space of infinite dimensional vectors, namely, let

$$\mathcal{F}[v]_j \equiv v_j - \lambda\, [g(\bar{\varphi}_{j-p+1} + \Delta_- v_{j-p+1}, \ldots, \bar{\varphi}_{j+p} + \Delta_- v_{j+p}) - f(u_-)] \tag{3.8}$$

with v be any vector satisfying (3.6), i.e., $\lim_{j \to \pm\infty} v_j = 0$. Then (3.7) gives

$$\bar{v} = \mathcal{F}[\bar{v}], \tag{3.9}$$

which means that $\bar{v}$ is a fixed point of $\mathcal{F}$.

If we reverse the above arguments, namely, assume that there exists an infinite dimensional vector $\bar{v}$ such that it satisfies (3.6) and is a fixed point of the mapping $\mathcal{F}$ defined in (3.8), then it is easy to see that $\varphi \equiv \bar{\varphi} + \Delta_- \bar{v}$ is an exact stationary discrete shock for scheme (1.2) with parameter q. So to prove the existence of a stationary discrete shock is equivalent to proving the existence of a fixed point for the mapping $\mathcal{F}$. We will restrict the space for the search of fixed points of $\mathcal{F}$ to $l^2_{\alpha,\beta}$ in our study, where α and β are some suitable constants.

We can rewrite the mapping $\mathcal{F}$ in (3.8) in the form of (2.2) with

$$L[v]_j \equiv v_j - \lambda \sum_{k=-p+1}^{p} g'_k(\bar{\varphi}_{j-p+1}, \cdots, \bar{\varphi}_{j+p}) \Delta_- v_{j+k}, \tag{3.10}$$

$$\begin{aligned}
N[v]_j \equiv -\lambda\, [&g(\bar{\varphi}_{j-p+1} + \Delta_- v_{j-p+1}, \cdots, \bar{\varphi}_{j+p} + \Delta_- v_{j+p}) \\
&-g(\bar{\varphi}_{j-p+1}, \cdots, \bar{\varphi}_{j+p}) - \sum_{k=-p+1}^{p} g'_k(\bar{\varphi}_{j-p+1}, \cdots, \bar{\varphi}_{j+p}) \Delta_- v_{j+k}],
\end{aligned} \tag{3.11}$$

$$\mathcal{E}_j \equiv -\lambda\, [g(\bar{\varphi}_{j-p+1}, \cdots, \bar{\varphi}_{j+p}) - f(u_-)]. \tag{3.12}$$

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Here, $g'_k(z_{-p+1},\ldots,z_p) \equiv \partial g/\partial z_k(z_{-p+1},\ldots,z_p)(k=-p+1,\ldots,p)$ are the first-order partial derivatives of the numerical flux function g.

It is easy to see that the mapping L is linear and the mapping N is generally nonlinear and satisfies $N[\emptyset]=\emptyset$. In addition, $\mathcal{E}$ is a vector depending on $\bar{\varphi}$ but independent of v. The linear mapping L is just the linearization of the mapping $\mathcal{F}$ around the approximate discrete shock $\bar{\varphi}$. If $\bar{\varphi}$ is accurate enough, we hope L becomes contractive under the weighted l^2 norm $\|\cdot\|_{\alpha,\beta}$ for some $\alpha>1$ and $\beta>1$. To see if the mapping N would satisfy (2.4), we write, for any $v,w\in B_r(\emptyset)$ and $j\in\mathcal{Z}$,

$$\begin{aligned}N[v]_j-N[w]_j = -\lambda[\, & g(\bar{\varphi}_{j-p+1}+\Delta_-v_{j-p+1},\ldots,\bar{\varphi}_{j+p}+\Delta_-v_{j+p})\\ & -g(\bar{\varphi}_{j-p+1}+\Delta_-w_{j-p+1},\ldots,\bar{\varphi}_{j+p}+\Delta_-w_{j+p})\\ & -\sum_{k=-p+1}^{p} g'_k(\bar{\varphi}_{j-p+1},\ldots,\bar{\varphi}_{j+p})(\Delta_-v_{j+k}-\Delta_-w_{j+k})]\end{aligned}$$

or

$$N[v]_j-N[w]_j=-\lambda\sum_{k,l=-p+1}^{p}\int_0^1\int_0^1 g''_{k,l}(\bar{\varphi}^*_{j-p+1},\ldots,\bar{\varphi}^*_{j+p})v^*_{j+l}\,d\eta\, d\xi\ (\Delta_-v_{j+k}-\Delta_-w_{j+k}) \tag{3.13}$$

where

$$\bar{\varphi}^*_i=\bar{\varphi}_i+\eta(\xi\Delta_-v_i+(1-\xi)\Delta_-w_i)\qquad v^*_i=\xi\Delta_-v_i+(1-\xi)\Delta_-w_i \tag{3.14}$$

and $g''_{k,l}(z_{-p+1},\ldots,z_p)\equiv\partial^2/\partial z_k\partial z_l(z_{-p+1},\ldots,z_p)(k,l=-p+1,\ldots,p)$ are the second-order derivatives of the flux function g. Notice that the summation on the RHS of (3.13) is a double summation over $k,l=-p+1,\ldots,p$. Introducing the shifting operator (or a mapping) E_k, which is defined as $E_k[v]_j=v_{j+k}$ for any vector v, we can rewrite (3.13) as

$$N[v]_j-N[w]_j=-\lambda\sum_{k,l=-p+1}^{p}\int_0^1\int_0^1 g''_{k,l}(\bar{\varphi}^*_{j-p+1},\ldots,\bar{\varphi}^*_{j+p})v^*_{j+l}\,d\eta\, d\xi\ (E_k-E_{k-1})[v_j-w_j]. \tag{3.15}$$

If we apply the norm $\|\cdot\|_{\alpha,\beta}$ on both sides of (3.15), due to the fact that v^* is $O(r)$, the mapping N would satisfy (2.4) provided that the second-order derivatives of g are bounded. We will give precise estimates of $\|N\|^r_{\alpha,\beta}$ in the next two sections for different choices of approximate discrete shocks $\bar{\varphi}$ and slightly different forms of the mapping N. Then we derive sufficient conditions on $\bar{\varphi}$ and the first and second derivatives of g, which will guarantee the existence and stability of exact discrete shocks.

Our estimates will be based upon the following two bounds on the first two derivatives of g:

$$\Gamma_1(r)\geq\sup\{|g'_k(z_{-p+1},\ldots,z_p)|\ \ |\ \ |z_l|\leq r,\quad k,l=-p+1,\ldots,p.\}. \tag{3.16}$$

$$\Gamma_2(r)\geq\sup\left\{|g''_{j,k}(z_{-p+1},\ldots,z_p)|\ \ |\ \ |z_l|\leq r,\quad j,k,l=-p+1,\ldots,p.\right\}. \tag{3.17}$$

The functions $\Gamma_1(r)$ and $\Gamma_2(r)$ can be obtained analytically from the given flux function g. Without loss of generality, we assume that both $\Gamma_1(r)$ and $\Gamma_2(r)$ are nondecreasing functions for $r\geq 0$.

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4. A single discrete shock profile. For any fixed $q \in [0,1]$, we estimate the upper bounds of $\|L\|_{\alpha,\beta}$ and $\|N\|^r_{\alpha,\beta}(r > 0)$ assuming that an approximate discrete shock $\bar{\varphi}$ with parameter q is known. We then give a sufficient condition which ensures the existence and stability of an exact discrete shock for scheme (1.2) with parameter q. The sense of stability will be made precise at the end of this section.

First, we estimate the upper bound of $\|L\|_{\alpha,\beta}$. Let us write the linear mapping L in the form of the matrix–vector product:

$$L[v] = Av,$$

thinking of v as a column vector with the jth row entry being $v_j (j \in \mathcal{Z})$. According to (3.10), the infinite dimensional matrix A is given by the following:

$$A_{ij} = \begin{cases} 0 & \text{if } |j-i| > p, \\ \lambda g'_{-p+1}(\bar{\varphi}_{i-p+1}, \cdots, \bar{\varphi}_{i+p}) & \text{if } j-i = -p, \\ -\lambda g'_p(\bar{\varphi}_{i-p+1}, \cdots, \bar{\varphi}_{i+p}) & \text{if } j-i = p, \\ \delta_{ij} + \lambda g'_{j-i+1}(\bar{\varphi}_{i-p+1}, \cdots, \bar{\varphi}_{i+p}) - \lambda g'_{j-i}(\bar{\varphi}_{i-p+1}, \cdots, \bar{\varphi}_{i+p}) & \text{if } |j-i| < p. \end{cases} \tag{4.1}$$

Here A_{ij} refers to the entry of A on ith row and jth column. $\delta_{ij} = 1$ if $i = j$ and $= 0$, otherwise. We define D to be an infinite dimensional diagonal matrix with the ith diagonal entry being

$$D_{ii} \equiv \begin{cases} \beta^{-i} & \text{if } i < 0, \\ \alpha^i & \text{if } i \geq 0. \end{cases} \tag{4.2}$$

Use $\|\cdot\|_2$ and $(\cdot,\cdot)_2$ to stand for the norm and the inner product in l^2. It is easy to see that, for any $v \in l^2_{\alpha,\beta}$, we have $\|v\|_{\alpha,\beta} = \|Dv\|_2$. Denote $\tilde{A} = DAD^{-1}$ (D^{-1} is the inverse of D), and we have

$$\|L[v]\|^2_{\alpha,\beta} = (DAv, DAv)_2 = (\tilde{A}Dv, \tilde{A}Dv)_2 = (\tilde{A}^T\tilde{A}Dv, Dv)_2,$$

$$\|L[v]\|^2_{\alpha,\beta} \leq \|\tilde{A}^T\tilde{A}\|_2 \|Dv\|_2^2 = \|\tilde{A}^T\tilde{A}\|_2 \|v\|^2_{\alpha,\beta}.$$

Since $\tilde{A}^T\tilde{A}$ is symmetric, its l^2 norm is just its spectral radius, $\rho(\tilde{A}^T\tilde{A})$. Using the Gerschgorin Circle theorem from matrix theory, we have

$$\rho(\tilde{A}^T\tilde{A}) \leq \sup_{i\in\mathcal{Z}} \sum_{j\in\mathcal{Z}} |(\tilde{A}^T\tilde{A})_{ij}|.$$

Notice that A is banded with bandwidth $2p+1$. Therefore, $\tilde{A}^T\tilde{A}$ is also banded with bandwidth not more than $4p+1$. We have the following upper bound for $\|L\|_{\alpha,\beta}$.

LEMMA 4.1.

$$\|L\|_{\alpha,\beta} \leq \sqrt{\sup_{i\in\mathcal{Z}} \sum_{|j-i|\leq 2p} \left|\left(\tilde{A}^T\tilde{A}\right)_{ij}\right|}. \tag{4.3}$$

For later use, we define

$$\delta = 1 - \sqrt{\sup_{i\in\mathcal{Z}} \sum_{|j-i|\leq 2p} \left|\left(\tilde{A}^T\tilde{A}\right)_{ij}\right|}. \tag{4.4}$$

Here $\tilde{A} = DAD^{-1}$ with A and D given by (4.1) and (4.2).

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Next we estimate the upper bound of $\|N\|^r_{\alpha,\beta}(r>0)$.

We start with a simple lemma on the norm of the shifting operator $E_k(k \in \mathcal{Z})$. Due to the nonunitary weight in the norm $\|\cdot\|_{\alpha,\beta}$, the shifting operator is not unitary.

LEMMA 4.2.

$$\|E_k\|_{\alpha,\beta} \leq \max(\alpha^{|k|}, \beta^{|k|}). \tag{4.5}$$

Proof. Assume $k>0$, for any $v \in l^2_{\alpha,\beta}$,

$$\begin{aligned}\|E_k[v]\|^2_{\alpha,\beta} &= \sum_{j\geq 0} v^2_{j+k}\alpha^{2j} + \sum_{j<0} v^2_{j+k}\beta^{-2j}\\ &= \sum_{j\geq k} v^2_j \alpha^{2(j-k)} + \sum_{j<k} v^2_j \beta^{-2(j-k)}\\ &= \alpha^{-2k}\sum_{j\geq k} v^2_j\alpha^{2j} + \beta^{2k}\sum_{0\leq j<k} v^2_j \alpha^{2j}(\alpha\beta)^{-2j} + \beta^{2k}\sum_{j<0} v^2_j\beta^{-2j}.\end{aligned}$$

Since $\alpha>1$ and $\beta>1$, the last equality implies

$$\|E_k[v]\|_{\alpha,\beta} \leq \beta^k \|v\|_{\alpha,\beta}.$$

Similarly, for $k<0$, we have

$$\|E_k[v]\|_{\alpha,\beta} \leq \alpha^{-k}\|v\|_{\alpha,\beta},$$

and (4.5) follows by combining the above two inequalities. □

Recall (3.14), we have for any $v, w \in B_r(\emptyset)$ and $0 \leq \xi, \eta \leq 1$,

$$\sup_{i\in\mathcal{Z}} |\bar{\varphi}^*_i| \leq \bar{k} + 2r \qquad \sup_{i\in\mathcal{Z}} |v^*_i| \leq 2r$$

where $\bar{k}$ is defined in (3.1). Thus, we have

$$\left|g''_{k,l}(\bar{\varphi}^*_{j-p+1},\ldots,\bar{\varphi}^*_{j+p})v^*_{j+l}\right| \leq 2r\Gamma_2(\bar{k}+2r) \quad \text{for all } j \in \mathcal{Z} \text{ and } k,l=-p+1,\ldots,p.$$

Here $\Gamma_2(\cdot)$ is the function defined in (3.17). From (3.15) and Lemma 4.2, we have

$$\begin{aligned}\|N[v]-N[w]\|_{\alpha,\beta} &\leq 4\lambda p r\Gamma_2(\bar{k}+2r)\sum_{k=-p}^{p}\|E_k - E_{k-1}\|_{\alpha,\beta}\|v-w\|_{\alpha,\beta}\\ &\leq 16\lambda p^2 r \max(\alpha^p,\beta^p)\Gamma_2(\bar{k}+2r)\|v-w\|_{\alpha,\beta}.\end{aligned}$$

Thus, we obtain the following upper bound for $\|N\|^r_{\alpha,\beta}$.

LEMMA 4.3.

$$\|N\|^r_{\alpha,\beta} \leq 16\lambda p^2 r\max(\alpha^p,\beta^p)\Gamma_2(\bar{k}+2r) \tag{4.6}$$

where $\bar{k}$ and $\Gamma_2(\cdot)$ are defined in (3.1) *and* (3.17), *respectively.*

Combining Theorem 2.1 and Lemma 4.1 and 4.3, we obtain the following.

THEOREM 4.4 (existence and stability of a single discrete shock). *If there exist $\alpha>1, \beta>1, \sigma>0$ and an approximate discrete shock $\bar{\varphi}$ with parameter q such that*

$$\|\mathcal{E}\|_{\alpha,\beta} < \frac{\sigma}{2}\left\{\delta - 16\lambda p^2\sigma\max(\alpha^p,\beta^p)\Gamma_2(\bar{k}+2\sigma)\right\} \tag{4.7}$$

where δ is defined in (4.4), *then*

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1. *The finite difference scheme* (1.2) *possesses an exact stationary discrete shock* φ *with parameter* q. *Moreover,* $k \equiv \sup_{j\in\mathcal{Z}} |\varphi_j| \leq \bar{k} + \sigma$;
2. φ *is stable in* $B_{\sigma/2}(\emptyset)$ *in the following sense: For any* $v \in B_{\sigma/2}(\emptyset)$, $\lim_{n\to\infty} \mathcal{L}^n[\varphi + \Delta_- v] = \varphi$ *under maximum norm. Here* $\mathcal{L}^n[\cdot]$ *means iterating the finite difference scheme* (1.2) n *times using its argument as the initial vector.*

Proof.

1. Based on the discussions in section 3, the existence of an exact discrete shock for scheme (1.2) is equivalent to the existence of a fixed point for the mapping $\mathcal{F}$ in (3.8). This mapping can be put in the form of $L[v] + N[v] + \mathcal{E}$ with $L, N, \mathcal{E}$ given by (3.10)–(3.12). By Lemma 4.1 and 4.3, condition (4.7) implies condition (2.5) in Theorem 2.1 (with an extra factor of 1/2); therefore the mapping (3.8) is contractive in $B_\sigma(\emptyset)$ and, as a result of this, possesses a fixed point $\bar{v} \in B_\sigma(\emptyset)$. Let $\varphi \equiv \bar{\varphi} + \Delta_- \bar{v}$. It is easy to see that φ is an exact stationary discrete shock for scheme (1.2) with parameter q. Moreover,
$$k \equiv \sup_{j\in\mathcal{Z}} |\varphi_j| \leq \sup_{j\in\mathcal{Z}} |\bar{\varphi}_j| + \sup_{j\in\mathcal{Z}} |v_j - v_{j-1}| \leq \bar{k} + 2\|\bar{v}\|_{\alpha,\beta}.$$
Due to the extra factor 1/2 in (4.7), by the second conclusion in Theorem 2.1, namely, (2.6), we have $\|\bar{v}\|_{\alpha,\beta} \leq \sigma/2$. Therefore, $k \leq \bar{k} + \sigma$.
2. For any $v \in B_{\sigma/2}(\emptyset)$, we write
$$\varphi + \Delta_- v = \bar{\varphi} + \Delta_-(\bar{v} + v).$$
Since $\|\bar{v}+v\|_{\alpha,\beta} \leq \sigma$ and the fact that the mapping $\mathcal{F}$ is contractive in $B_\sigma(\emptyset)$, we know that after applying the mapping $\mathcal{F}$ infinitely many times on $\bar{v} + v$ the mapping will converge to the fixed point $\bar{v}$. By the equivalence between the application of the mapping $\mathcal{F}$ on $\bar{v} + v$ and iteration of the scheme (1.2) with initial vector $\varphi + \Delta_- v$ established in section 3, we conclude that φ is stable in $B_{\sigma/2}(\emptyset)$.

5. Continuous discrete shock profiles. In this section, we derive sufficient conditions for the existence of a continuous discrete shock profile for the scheme (1.2).

For any fixed $q_0 \in [0, 1]$, assume now the conditions in Theorem 4.4 are satisfied. Namely, there exist $\alpha > 1, \beta > 1, \sigma_{q_0} > 0$ and an approximate discrete shock $\bar{\varphi}^{q_0}$ with parameter q_0 such that condition (4.7) is true. Then by Theorem 4.4, there exists an exact discrete shock φ^{q_0} with
$$k_{q_0} \equiv \sup_{j\in\mathcal{Z}} |\varphi_j^{q_0}| \leq \bar{k}_{q_0} + \sigma_{q_0}.$$
Here we have added superscripts and subscripts q_0 to indicate the dependence on parameter q_0. The constants α and β, however, will be chosen to be independent of q_0.

For proper choices of α and β, we are interested in finding the conditions on the approximate discrete shock $\bar{\varphi}^{q_0}$ and the first two derivatives of the numerical flux function g such that an exact discrete shock φ^q for the scheme (1.2) is guaranteed to exist for any q in a small neighborhood of q_0, say $[q_0 - 1/2M, q_0 + 1/2M]$ for some integer $M > 0$. Once such conditions are found and satisfied for a finite number of values of q_0, e.g.,
$$q_0 = \frac{i}{M}, \quad 0 \leq i \leq M,$$

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it becomes clear that for any $q \in [0,1]$, there exists an exact discrete shock for scheme (1.2). A continuous discrete shock profile is then obtained by properly arranging the family of exact discrete shocks which are parameterized by $q \in [0,1]$.

Let us take an approximate discrete shock with parameter $q \in [q_0 - 1/2M, q_0 + 1/2M]$ to be

$$\bar{\varphi}^q \equiv \varphi^{q_0} + (q - q_0)(u_- - u_+)e_0 \tag{5.1}$$

where $e_0 \equiv \{\delta_{j0}, j \in \mathcal{Z}\}$. It is easy to check that conditions 1, 2, 3 in Definition 1.1 are satisfied. We have

$$\bar{k}_q \equiv \sup_{j\in\mathcal{Z}} |\bar{\varphi}_j^q| \le \bar{k}_{q_0} + \sigma_{q_0} + \frac{|u_- - u_+|}{2M}, \tag{5.2}$$

$$\sup_{j\in\mathcal{Z}} |\bar{\varphi}_j^q - \bar{\varphi}_j^{q_0}| \le \sigma_{q_0} + \frac{|u_- - u_+|}{2M}. \tag{5.3}$$

Define a mapping based on the approximate discrete shock $\bar{\varphi}^q$ in (5.1)—for any $j \in \mathcal{Z}$,

$$\mathcal{F}[v]_j = v_j - \lambda\left[g(\bar{\varphi}^q_{j-p+1} + \Delta_- v_{j-p+1}, \ldots, \bar{\varphi}^q_{j+p} + \Delta_- v_{j+p}) - f(u_-)\right] \tag{5.4}$$

where v is any vector in $l^2_{\alpha,\beta}$. We can rewrite the mapping $\mathcal{F}$ in the form of (2.2) with

$$L[v]_j \equiv v_j - \lambda \sum_{k=-p+1}^{p} g'_k(\bar{\varphi}^{q_0}_{j-p+1}, \ldots, \bar{\varphi}^{q_0}_{j+p})\Delta_- v_{j+k}, \tag{5.5}$$

$$\begin{aligned} N[v]_j \equiv -\lambda \Big[&g(\bar{\varphi}^q_{j-p+1} + \Delta_- v_{j-p+1}, \ldots, \bar{\varphi}^q_{j+p} + \Delta_- v_{j+p}) \\ &- g(\bar{\varphi}^q_{j-p+1}, \ldots, \bar{\varphi}^q_{j+p}) - \sum_{k=-p+1}^{p} g'_k(\bar{\varphi}^{q_0}_{j-p+1}, \ldots, \bar{\varphi}^{q_0}_{j+p})\Delta_- v_{j+k}\Big]. \end{aligned} \tag{5.6}$$

$$\mathcal{E}_j \equiv -\lambda\left[g(\bar{\varphi}^q_{j-p+1}, \ldots, \bar{\varphi}^q_{j+p}) - f(u_-)\right]. \tag{5.7}$$

It is easy to see that L is a linear mapping and its norm can be estimated similarly as in Lemma 4.1, namely, we have

$$\|L\|_{\alpha,\beta} \le \sqrt{\sup_{i\in\mathcal{Z}} \sum_{|j-i|\le 2p} \left|\left(\tilde{A}^T_{q_0}\tilde{A}_{q_0}\right)_{ij}\right|} \tag{5.8}$$

and similar to (4.4), we define

$$\delta_{q_0} = 1 - \sqrt{\sup_{i\in\mathcal{Z}} \sum_{|j-i|\le 2p} \left|\left(\tilde{A}^T_{q_0}\tilde{A}_{q_0}\right)_{ij}\right|}. \tag{5.9}$$

Here $\tilde{A}_{q_0} = DA_{q_0}D^{-1}$, D is defined in (4.2), and A_{q_0} is given in (4.1) with $\bar{\varphi}$ replaced by $\bar{\varphi}^{q_0}$.

The mapping N satisfies $N[\emptyset] = \emptyset$ and its norm $\|N\|^r_{\alpha,\beta}$ can be estimated as follows. For any $v, w \in B_r(\emptyset)$, same as (3.15), we have

$$N[v]_j - N[w]_j = -\lambda \sum_{k,l=-p+1}^{p} \int_0^1\int_0^1 g''_{k,l}(\bar{\varphi}^*_{j-p+1}, \ldots, \bar{\varphi}^*_{j+p})v^*_{j+l}\,d\eta\, d\xi\ (E_k - E_{k-1})[v_j - w_j], \tag{5.10}$$

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but with (different from (5.11))

$$\bar{\varphi}_i^* = \eta\bar{\varphi}_i^q + (1-\eta)\bar{\varphi}_0^{q_0} + \eta(\xi\Delta_- v_i + (1-\xi)\Delta_- w_i),$$
$$v_i^* = \bar{\varphi}_i^q - \bar{\varphi}_i^{q_0} + \xi\Delta_- v_i + (1-\xi)\Delta_- w_i.$$

Using (5.2) and (5.3), we have

$$\sup_{j\in\mathcal{Z}} |\bar{\varphi}_j^*| \le \bar{k}_{q_0} + 2r + \sigma_{q_0} + \frac{|u_- - u_+|}{2M},$$

$$\sup_{j\in\mathcal{Z}} |v_j^*| \le 2r + \sigma_{q_0} + \frac{|u_- - u_+|}{2M}.$$

Thus similar to Lemma 4.3, we have

$$\|N\|_{\alpha,\beta}^r \le 8\lambda p^2 \max(\alpha^p,\beta^p)\left(2r + \sigma_{q_0} + \frac{|u_- - u_+|}{2M}\right)\Gamma_2\left(\bar{k}_{q_0} + 2r + \sigma_{q_0} + \frac{|u_- - u_+|}{2M}\right). \tag{5.11}$$

Now we estimate $\|\mathcal{E}\|_{\alpha,\beta}$. Since φ^{q_0} is an exact discrete shock profile, we have $\varphi^{q_0} = \mathcal{L}[\varphi^{q_0}]$, which implies

$$g(\varphi_{j-p+1}^{q_0},\dots,\varphi_{j+p}^{q_0}) = f(u_-) \qquad \text{for any } j \in \mathcal{Z}.$$

We can write

$$\begin{aligned}
\mathcal{E}_j &= -\lambda\left[g(\bar{\varphi}_{j-p+1}^q,\dots,\bar{\varphi}_{j+p}^q) - g(\varphi_{j-p+1}^{q_0},\dots,\varphi_{j+p}^{q_0})\right]\\
&= -\lambda\sum_{k=-p+1}^{p}\int_0^1 g_k'(\bar{\varphi}_{j-p+1}^*,\dots,\bar{\varphi}_{j+p}^*)d\xi\ (\bar{\varphi}_{j+k}^q - \varphi_{j+k}^{q_0})\\
&= -\lambda(q-q_0)(u_- - u_+)\sum_{k=-p+1}^{p}\int_0^1 g_k'(\bar{\varphi}_{j-p+1}^*,\dots,\bar{\varphi}_{j+p}^*)d\xi\ \delta_{j+k,0}\\
&= \begin{cases} -\lambda(q-q_0)(u_- - u_+)\int_0^1 g_{-j}'(\bar{\varphi}_{j-p+1}^*,\dots,\bar{\varphi}_{j+p}^*)d\xi & \text{if } -p \le j \le p-1,\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}$$

In the third equality above, we have used the definition of $\bar{\varphi}^q$ in (5.1), and in the second equality above, $\bar{\varphi}_j^* \equiv \xi\bar{\varphi}_j^q + (1-\xi)\varphi_j^{q_0}$ has the bound

$$\sup_{j\in\mathcal{Z}} |\bar{\varphi}_j^*| \le \bar{k}_{q_0} + \sigma_{q_0} + \frac{|u_- - u_+|}{2M}.$$

Recalling the function $\Gamma_1(\cdot)$ defined in (3.16), we get

$$\|\mathcal{E}\|_{\alpha,\beta} \le 2\lambda p\max(\alpha^p,\beta^p)|q-q_0||u_- - u_+|\Gamma_1\left(\bar{k}_{q_0} + \sigma_{q_0} + \frac{|u_- - u_+|}{2M}\right). \tag{5.12}$$

Using the fact that $|q - q_0| \le 1/2M$, we get a larger upper for $\|\mathcal{E}\|_{\alpha,\beta}$ which does not depend on q,

$$\|\mathcal{E}\|_{\alpha,\beta} \le \lambda p\max(\alpha^p,\beta^p)\frac{|u_- - u_+|}{M}\Gamma_1\left(\bar{k}_{q_0} + \sigma_{q_0} + \frac{|u_- - u_+|}{2M}\right). \tag{5.13}$$

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We want to note that the factor $|q - q_0|$ on the RHS of (5.12) will enable us to prove uniform Lipschitz continuity of a family of exact discrete shocks when they exist.

Based on the bounds found in (5.8), (5.11), and (5.13), we obtain the following sufficient condition for the existence of a continuous discrete shock profile for scheme (1.2).

THEOREM 5.1 (existence of a continuous discrete shock profile). *If there exist $\alpha > 1$, $\beta > 1$, and $M > 0$ (an integer) such that for each $q_0 = i/M (0 \le i \le M)$, there exist $\sigma_{q_0} > 0$ and an approximate discrete shock profile $\bar{\varphi}^{q_0}$ for which the following inequality is true:*

$$\max\left\{\|\mathcal{E}^{q_0}\|_{\alpha,\beta},\ \lambda p \max(\alpha^p, \beta^p)\frac{|u_- - u_+|}{M}\Gamma_1\left(\bar{k}_{q_0} + \sigma_{q_0} + \frac{|u_- - u_+|}{2M}\right)\right\}$$
$$< \frac{\sigma_{q_0}}{2}\left\{\delta_{q_0} - 8\lambda p^2 \max(\alpha^p, \beta^p)\left(2\sigma_{q_0} + \frac{|u_- - u_+|}{2M}\right)\Gamma_2\left(\bar{k}_{q_0} + 2\sigma_{q_0} + \frac{|u_- - u_+|}{2M}\right)\right\}. \tag{5.14}$$

Here $\mathcal{E}^{q_0}$ is given by

$$\mathcal{E}_j^{q_0} = -\lambda\left[g(\bar{\varphi}_{j-p+1}^{q_0}, \dots, \bar{\varphi}_{j+p}^{q_0}) - f(u_-)\right] \qquad j \in \mathcal{Z}; \tag{5.15}$$

$\bar{k}_{q_0} \equiv \sup_{j\in\mathcal{Z}} |\bar{\varphi}_j^{q_0}|$; δ_{q_0} is given in (5.9); $\Gamma_1(\cdot)$ and $\Gamma_2(\cdot)$ are two functions defined in (3.16) and (3.17). Then

1. *for any $q \in [0,1]$, there exists an exact discrete shock φ^q for scheme (1.2);*
2. *if for any $x \in \mathcal{R}$, we define $\varphi(x) \equiv \varphi_j^q$ where j and q are uniquely determined by $j - q = x$ with $j \in \mathcal{Z}$ and $q \in [0,1)$, then $\varphi(x) (x \in \mathcal{R})$ is a continuous discrete shock profile for scheme (1.2).*

Schematic proof.

1. Condition (5.14) clearly implies condition (4.7) in Theorem 4.4 for $q = q_0$. Therefore, there exists an exact discrete shock for scheme (1.2) with parameter q_0. For any $q \in [q_0 - 1/2M, q_0 + 1/2M]$, we can define an approximate discrete shock $\bar{\varphi}^q$ as in (5.1). According to the estimates (5.8), (5.11), and (5.13), condition (5.14) implies (2.5) for the mapping $\mathcal{F}$ in (5.4). By the same logic used in Theorem 4.4, we see that there exists an exact discrete shock φ^q for scheme (1.2) for any $q \in [q_0 - 1/2M, q_0 + 1/2M]$. Since condition (5.14) is true for all $q_0 = i/M, 0 \le i \le M$, conclusion 1 follows.
2. We only need to check that $\varphi(x)$ is bounded and uniformly Lipschitz continuous. For all $q_0 = i/M, 0 \le i \le M$, $\bar{\varphi}^{q_0}$ are uniformly bounded by the definition of approximate discrete shocks and the finiteness of M. Each φ^{q_0} differs from $\bar{\varphi}^{q_0}$ by a vector whose maximum can be bounded by σ_{q_0}, and so does φ^q differ from $\bar{\varphi}^q$ for $|q - q_0| \le 1/2M$. Due to choice of $\bar{\varphi}^q$ in (5.1), it is easy to see that φ^q is uniformly bounded for $q \in [0,1]$. To prove that $\varphi(x)$ is uniformly Lipschitz continuous, we first give the following two observations which can be shown easily.

   Observation 1. For any $q \in [0,1]$, there exists $q_0 = i/M$ for some integer i between 0 and M, such that

   $$\sup_{j\in\mathcal{Z}} |\varphi_j^q - \varphi_j^{q_0}| \le C|q - q_0|$$

   where the constant C are independent of q and q_0. This is a result of the estimate (5.12), the bound (2.6) in Theorem 2.1, and condition (5.14).

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Observation 2.

$$\varphi^0_{j-1} = \varphi^1_j \quad \text{for any } j \in \mathcal{Z}.$$

This is due to the way we parameterize the family of discrete shocks, namely the parameter q in Definition 1.1.
An easy generalization of Observation 1 is that for any $q_1, q_2 \in [0,1]$,

$$|\varphi^{q_1}_j - \varphi^{q_2}_j| \leq C|q_1 - q_2|$$

or

$$|\varphi(j - q_1) - \varphi(j - q_2)| \leq C|q_1 - q_2| \tag{5.16}$$

for any $j \in \mathcal{Z}$. Let $x_1 = j_1 - q_1$ and $x_2 = j_2 - q_2$ with $j_1, j_2 \in \mathcal{Z}$ and $q_1, q_2 \in [0,1)$. We only need to consider the case $j_1 < j_2$ for which we have

$$\begin{aligned}\varphi(x_1) - \varphi(x_2) &= \varphi(x_1) - \varphi(j_1) + \sum_{j=j_1+1}^{j_2-1} (\varphi(j-1) - \varphi(j)) + \varphi(j_2-1) - \varphi(x_2)\\ &= \varphi(j_1-q_1) - \varphi(j_1) + \sum_{j=j_1+1}^{j_2-1} (\varphi(j-1)-\varphi(j-0))\\ &\quad + \varphi(j_2-1) - \varphi(j_2-q_2).\end{aligned}$$

Because each term on the RHS of the last equality is of the form of (5.16), the uniform Lipschitz continuity of $\varphi(x)$ follows.

6. Algorithms for computer verification. In this section, we discuss how to use a computer to verify condition (4.7) in Theorem 4.4 to prove the existence and stability of an exact discrete shock or condition (5.14) in Theorem 5.1 to prove the existence of a continuous discrete shock profile for scheme (1.2).

6.1. Computing an approximate discrete shock. We start with providing a method of obtaining an accurate approximate discrete shock $\bar{\varphi}^q$ for any fixed $q \in [0,1]$ using scheme (1.2). Let J be an integer. Set

$$u^0_j = \pi_j + \tau_j \qquad \text{for all } |j| \leq J \tag{6.1}$$

where π_j is given by (1.4) and $\{\tau_j, |j| \leq J\}$ is chosen such that

$$\sum_{|j|\leq J} \tau_j = q(u_- - u_+).$$

For example, we can take

$$\tau_j = \frac{1}{2J} q(u_- - u_+)(1 + \cos\frac{\pi j}{J}) \qquad j = -J, -J+1, \ldots, J.$$

We then apply the finite difference scheme (1.2) to the initial data u^0 repeatedly for sufficiently many times. Note that when the scheme is applied to $u^n_j (n \geq 0, |j| \leq J)$, we need values of u^n_j, $J < |j| \leq J+p$ in order to compute values of u^{n+1}_j for all $|j| \leq J$. We can simply set $u_j = u_-$ for $-J-p \leq j < -J$ and $= u_+$ for $J < j \leq J+p$. Although this makes the scheme nonconservative in the bounded region (i.e., $|j| \leq J$), it actually

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does not make an error much bigger than the machine accuracy if J is taken to be large enough. This is because an exact discrete shock is generally believed to be converging to the two end states (u_-, u_+) exponentially fast. For the sake of rigorousness, we can modify the value of $u_0^{n_0}$ such that $\sum_{|j|\le J}(u_j^{n_0} - u_j^0) = 0$ to make the procedure conservative in the bounded region. Here n_0 is assumed to the number of iterations of the scheme on u^0. To determine how large n_0 should be, we can monitor $\sup_{|j|\le J} |\mathcal{E}_j|$ ($\mathcal{E}$ is defined in (3.12) with $\bar\varphi$ replaced by u^n) to see if it is small enough, say close to machine accuracy, for the purpose of our verification of the conditions in Theorem 4.4 or 5.1. Finally, we can set the approximate discrete shock to be

$$\bar\varphi_j^q = \begin{cases} u_- & \text{if } j < -J, \\ u_j^{n_0} & \text{if } |j| \le J, \\ u_+ & \text{if } j < J. \end{cases} \tag{6.2}$$

It is easy to check that $\bar\varphi^q$ satisfies the conditions 1 to 3 in Definition 1.1.

Remark. Theoretically, the larger the J is, the more accurate $\bar\varphi^q$ one can get. However, larger J means that it takes longer computer time to verify the conditions. If $|u_- - u_+| = O(1)$, $J = 10$ to 80 is believed to be good enough. If $|u_- - u_+|$ is very small, J needs to be very large and it may even exceed the computer power. In the latter case, the framework in this paper may be improper.

6.2. Choosing the constants α and β. Once we have an approximate discrete shock $\bar\varphi^q$, we can decide what to choose for α and β. The criterion for this is to make the norm $\|L\|_{\alpha,\beta}$ as much below 1 as possible, or in our estimates, to maximize δ in (4.4). The range of possible values of α can be obtained by studying the linearized scheme of (1.2) around u_+. Similarly, β can be obtained by studying the linearized scheme of (1.2) around u_-. See Smylris [10] for details. We can use one approximate discrete shock or a few such approximate discrete shocks (corresponding to different values of q in $[0,1]$) to choose the constants α and β. In the latter case, we should make the minimum of δ (over different values of q) above 0 as far as possible. Note that the matrix A in (4.4), which is given by (4.1) with $\bar\varphi$ replaced by $\bar\varphi^q$, is essentially finite due to constancy of $\bar\varphi_j^q$ for $|j| > J$ (see (6.2)). So we have a finite row sum to take a maximum of in (4.4).

6.3. Strategy for verification. We suggest the following strategy for verifying the condition (4.7) in Theorem 4.4. For a given $q \in [0,1]$:

1. Find the function $\Gamma_2(r)$. Make sure it is nondecreasing in r.
2. Compute an approximate discrete shock following the method described in subsection 6.1. Make it as accurate as possible should the condition (4.7) seem very demanding.
3. Find the range of the constants α and β from the linear analysis of the scheme and choose $\alpha > 1, \beta > 1$ such that δ in (4.4) is positive and maximized.
4. Find the largest possible value for σ for which condition (4.7) is true. Mostly, we can take

$$\sigma = \frac{\delta}{32\lambda p^2 \max(\alpha^p, \beta^p)\Gamma_2(k^*)}$$

where $k^* = \max(|u_-|, |u_+|)$.

If the above four steps are finished, we can conclude that scheme (1.2) does possess an exact discrete shock with parameter q and it is stable in the sense stated in the second part of Theorem 4.4.

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To verify condition (5.14), we suggest the following steps:

1. Find the functions $\Gamma_1(r)$ and $\Gamma_2(r)$.
2. For several q values, compute an approximate discrete shock $\bar{\varphi}^{q_0}$ for each q_0. Find the proper constants $\alpha > 1$ and $\beta > 1$ such that minimum of the values of δ_{q_0}, based on each $\bar{\varphi}^{q_0}$, is positive and maximized. We then set $M \approx \sigma_{q_0}^{-1}$ and find σ_{q_0} such that the RHS of (5.14) is maximized. Using this σ_{q_0} and replacing $\bar{k}_{q_0}$ by $\max(|u_-|, |u_+|)$, we can obtain an estimate of the size of M by requiring the second term on the LHS of (5.14) to be less than the RHS of (5.14). One can even replace M inside the functions $\Gamma_1(\cdot)$ and $\Gamma_2(\cdot)$ by $\sigma_{q_0}^{-1}$ as long as we eventually use an $M \geq \sigma_{q_0}^{-1}$ for all values of q_0 sampled (this is due to the monotonicity of $\Gamma_1(\cdot)$ and $\Gamma_2(\cdot)$). We suggest one always use a bigger M than necessary to attain a bigger margin of the RHS of (5.14) over the LHS. Usually, one can take
$$\sigma_{q_0} = \frac{\delta_{q_0}}{16\lambda p^2 \max(\alpha^p, \beta^p)(2 + \frac{|u_- - u_+|}{2})\Gamma_2(k^*)},$$
$$M > \frac{\max\{\sigma_{q_0}^{-1}, 4\lambda p \max(\alpha^p, \beta^p)|u_- - u_+|\Gamma_1(k^*)\}}{\sigma_{q_0}\delta_{q_0}}$$
where $k^* = \max(|u_-|, |u_+|)$.
3. For each $q_0 = i/M (0 \leq i \leq M)$, we compute an approximate discrete shock $\bar{\varphi}^{q_0}$ with parameter q_0 and check if for this $\bar{\varphi}^{q_0}$, there exists $\sigma_{q_0} > 0$ such that (5.14) is true.

If (5.14) is true for every $q_0 = i/M (0 \leq i \leq M)$, we can conclude that the finite difference scheme does have a continuous discrete shock profile.

7. Some examples. In this section, we apply Theorem 5.1 to give a unified proof of the existence of a continuous discrete shock profile for a modified Lax–Friedrichs scheme and the Lax–Wendroff scheme. For a third-order weighted essentially non-oscillatory (WENO) scheme, we apply Theorem 4.4 to prove the existence and stability of an exact discrete shock for some sample values of q in $[0, 1]$.

As an example, we take the conservation law to be the Burgers' equation:
$$u_t + \left(\frac{u^2}{2}\right)_x = 0$$
and take the end states $u_- = 1$ and $u_+ = -1$. It is well known that an entropy satisfying stationary shock exists for these two end states.

7.1. A modified Lax–Friedrichs scheme. The flux function for a modified first-order Lax–Friedrichs scheme is
$$g(z_0, z_1) = \frac{1}{2}\left(f(z_0) + f(z_1) - \frac{2}{3\lambda}(z_1 - z_0)\right). \tag{7.1}$$
It is clear that the stencil width constant $p = 1$. We take the upper bound of the first and second derivatives of this flux function to be $\Gamma_1(r) = 2/3 + r/2$ and $\Gamma_2(r) = 1/2$ (see Appendix A.1). For $\lambda = 0.5$, we use 41 points (i.e., $J = 20$) to compute approximate discrete shocks and choose $\alpha = \beta = 1.3$ to define the weighted l^2 norm. Roughly $\sigma_{q_0} = \delta_{q_0}/12\alpha$ maximizes the RHS of (5.14) and M can be estimated by $64\alpha^2/\delta_{q_0}^2$. We take $M = 10000$ and are able to verify condition (5.14) for all $q_0 =$

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$i/M, 0 \le i \le M$. Thus this modified Lax–Friedrichs scheme possesses a continuous stationary discrete shock profile for Burgers' equation with end states $u_- = 1$ and $u_+ = -1$. The constant σ_0, which represents the size of the stability region for the discrete shocks (see conclusion 2 in Theorem 4.4), is approximately 7×10^{-3}. We plot values of the LHS and RHS of (5.14) for 100 even-spaced samples of q_0 in $[0, 1]$ in Figure 7.1a. The discrete shock profile is plotted in Figure 7.1b.

7.2. The Lax–Wendroff scheme. The flux function for the second-order Lax–Wendroff scheme is

$$g(z_0, z_1) = \frac{1}{2}\left(f(z_0) + f(z_1) - \lambda f'(\frac{z_0 + z_1}{2})(f(z_1) - f(z_0))\right). \tag{7.2}$$

The stencil width constant $p = 1$. We take $\Gamma_1(r) = r/2 + 3r^2/8$ and $\Gamma_2(r) = 1 + r/2$ (see Appendix A.2). For $\lambda = 0.5$, we use 61 points (or $J = 30$) to compute approximate discrete shocks. For the weighted l^2 norm, we take $\alpha = \beta = 1.7$. Roughly $\sigma_{q_0} = \delta_{q_0}/24\alpha$ maximizes the RHS of (5.14) and M can be estimated by $96\alpha^2/\delta_{q_0}^2$. We take $M = 60000$ and are able to verify (5.14) for $q_0 = i/M, 0 \le i \le M$. Thus the Lax–Wendroff scheme possesses a continuous stationary discrete shock profile for Burgers' equation with end states $u_- = 1$ and $u_+ = -1$. The constant for the size of the stability region, namely, σ_0, is approximately 2.3×10^{-3}. We plot the LHS and RHS of (5.14) for 100 even-spaced samples of $q_0 \in [0, 1]$ in Figure 7.1c. The discrete shock profile is plotted in Figure 7.1d.

7.3. The third-order WENO scheme. The WENO schemes [4] are variations of the ENO schemes [9]. They both achieve essentially nonoscillatory property by favoring information from the smoother part of the stencil over that from the less smooth or discontinuous part. However, the numerical flux function of the ENO schemes is discontinuous while the numerical flux function of the WENO schemes are infinitely smooth (if one takes ϵ_w appearing below to be nonzero).

The numerical flux function for the third-order WENO scheme with global Lax–Friedrichs flux splitting is

$$\begin{aligned} g(z_{-1}, z_0, z_1, z_2) = \frac{1}{2}\,[f(z_0) + f(z_1) &- \psi(f^+(z_0) - f^+(z_{-1}), f^+(z_1) - f^+(z_0)) \\ &- \psi(f^-(z_1) - f^-(z_2), f^-(z_0) - f^-(z_1))] \end{aligned} \tag{7.3}$$

and

$$f^{\pm}(z) = \frac{1}{2}(f(z) \pm \Lambda z),$$

$$\psi(a, b) = \frac{b - a}{1 + 2(r(a, b))^2},$$

$$r(a, b) = \frac{\epsilon_w + a^2}{\epsilon_w + b^2}.$$

Here, ϵ_w is a small constant to avoid the denominator to be zero and is taken as $\epsilon_w = 10^{-6}$; Λ is a constant which is the maximum of $|f'(u)|$ over all possible values of u. In our case, we take Λ to be slightly above the maximum of the absolute value of the two end states, namely, we set $\Lambda = 1.1$. The first- and second-order partial derivatives of the numerical flux g are shown in the Appendix.

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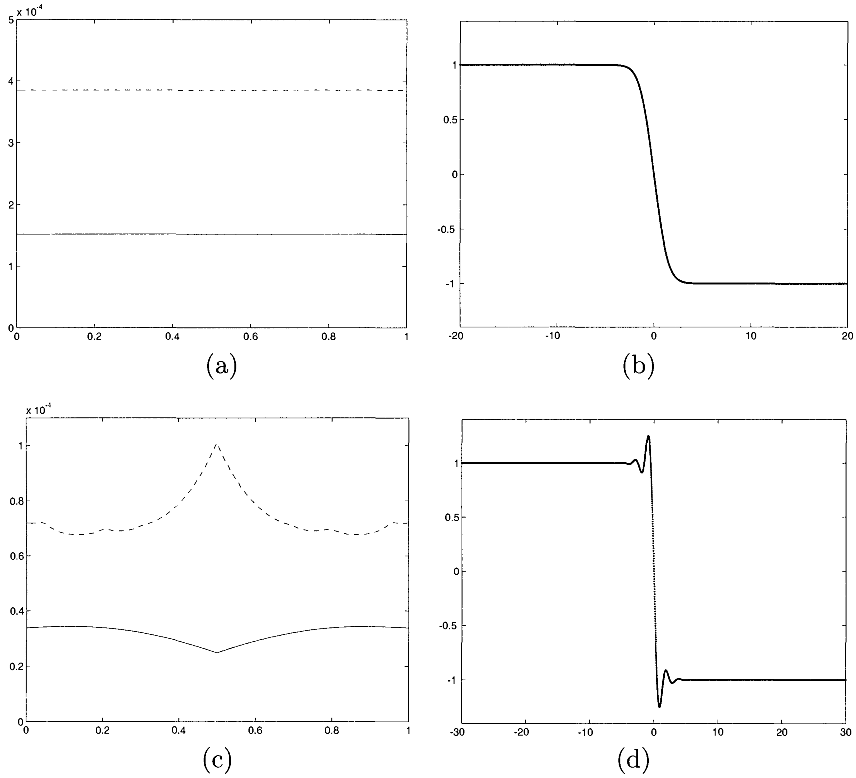

FIG. 7.1. (a) *The modified Lax–Friedrichs scheme. Solid line: the value of the LHS of* (5.14); *Dashed line: the value of the RHS of* (5.14). (b) *A discrete shock profile for the modified Lax–Friedrichs scheme.* (c) *Same as* (a) *but for the Lax–Wendroff scheme.* (d) *Same as* (b) *but for the Lax–Wendroff scheme.*

This scheme is third-order accurate in space where the solution is monotone and smooth. It degenerates to second order at smooth extrema. See [4] for details. If we use Euler forward in time, the scheme has the form

$$u_j^{n+1} = u_j^n - \lambda(g(u_{j-1}^n, u_j^n, u_{j+1}^n, u_{j+2}^n) - g(u_{j-2}^n, u_{j-1}^n, u_j^n, u_{j+1}^n)). \tag{7.4}$$

However, this scheme is linearly unstable for any constant $\lambda > 0$. Abbreviating the RHS of (7.4) as $E[u^n]_j$, we can express the scheme with the third-order Runge–Kutta scheme [9] in time as

$$\begin{aligned} u_j^{(1)} &= E[u^n]_j, \\ u_j^{(2)} &= \frac{3}{4}u_j^n + \frac{1}{4}E[u^{(1)}]_j, \\ u_j^{n+1} &= \frac{1}{3}u_j^n + \frac{2}{3}E[u^{(2)}]_j. \end{aligned} \tag{7.5}$$

We abbreviate (7.6) as

$$u_j^{n+1} = R[u^n]_j. \tag{7.6}$$

We have two observations: (i) If $\{\varphi_j, j \in \mathcal{Z}\}$ satisfies $\varphi = E[\varphi]$, then it also satisfies $\varphi = R[\varphi]$; (ii) Take $\mathcal{L}[\cdot] = E[\cdot]$ and define $\mathcal{F}[\cdot]$ as in (3.8); if this mapping is

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contractive under $\|\cdot\|_{\alpha,\beta}$ for some $\alpha > 1$ and $\beta > 1$, then the mapping derived from $\mathcal{L}[\cdot] = R[\cdot]$ is also contractive under the same norm.

The first observation is obvious. The second one is due to the fact that each stage in the third-order Runge–Kutta scheme (7.6) is a convex combination of u^n and $E[u]$ where u is u^n in stage one, $u^{(1)}$ in stage two, and $u^{(2)}$ in the final stage. See [9] for details.

Therefore, in order to prove the existence and stability of exact discrete shocks for (7.6), it suffices to prove the existence and stability of exact discrete shocks for (7.4). We have attempted to apply Theorem 5.1 to prove the existence of a continuous discrete shock profile for (7.4) and found that we need sample roughly 10^{19} even-spaced values of q_0 in $[0,1]$ which is far beyond the computer power. Nevertheless, we are able to use Theorem 4.4 to prove the existence and stability of exact discrete shocks for (7.4) for many sample values of $q_0 \in [0,1]$. Our computer verification strongly indicates that a continuous discrete shock profile does exist for this scheme.

Here are the details of the computer verification.

It is clear that the stencil width constant p equals 2. We take the upper bound for the second derivatives of the numerical flux function to be

$$\Gamma_2(r) = \frac{5}{4} + \frac{8r^2 + 16\Lambda r + 5\Lambda^2}{2\sqrt{\epsilon_w}} + \frac{55r(r+2\Lambda)(r+\Lambda)^2}{4\epsilon_w}.$$

Its derivation is detailed in Appendix A.3. We have taken $\lambda = 0.2$ and used 161 points (or $J = 80$) to compute the approximate discrete shocks. α and β are both taken to be 1.8. The condition in Theorem 4.4 is verified for $q_0 = i/1000, 0 \le i \le 1000$. This verification is done on CRAY C-90 using double precision.

We plot the LHS and RHS of (4.7) for the 1000 even-spaced samples of q_0 in Figure 7.2a. It can be seen that the curve for the LHS is properly below the curve for the RHS. We believe it is true for all $q_0 \in [0,1]$. The discrete shock profile with 40 even-spaced samples of q_0 is plotted in Figure 7.2b.

The discrete shock profile appears to be monotone. In Figure 7.2c and Figure 7.2d, we show the profile zoomed around $u_- = 1$ between the grid points indexed from -10 to 0 and around $u_+ = -1$ between points indexed from 0 to 10, respectively. Notice that the profile contains very small oscillations of magnitude around 10^{-4} on both sides of the shock. However, the profile looks very smooth, which leads us to believe that a continuous discrete shock profile does exist for this third-order WENO scheme. We have not been able to find a less stringent condition than that in Theorem 5.1 in order to prove this by computer.

8. Concluding remarks. We have provided sufficient conditions for a conservative scheme (1.2) to have a single discrete shock (Theorem 4.4) or a continuous discrete shock profile (Theorem 5.1) for a scalar conservation law in one dimension. These conditions can usually be verified by computers as demonstrated in the last section. The key idea here was to linearize the scheme around accurate numerical approximations of the discrete shocks and find suitable weighted l^2 norms for this linear part to define a contractive mapping. If we can find sufficiently accurate approximate discrete shocks, the original scheme behaves closely like the linearized scheme around this approximate discrete shock in terms of contractiveness of the induced mapping.

Several generalizations or implications of Theorem 4.4 or 5.1 are immediate. For example, we can find sufficient conditions, in the form of a single inequality, which assure the existence of discrete shocks or a continuous discrete shock profile for a *range* of the time-space ratio λ and the end states u_-, u_+. Using the result in [1],

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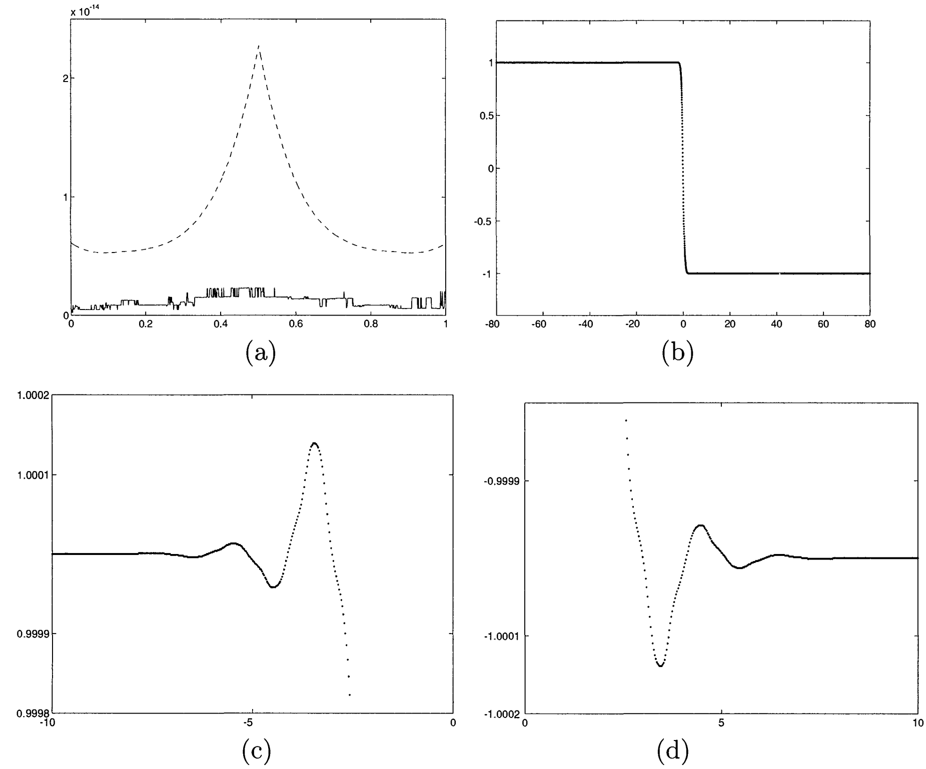

FIG. 7.2. *The third-order WENO scheme.* (a) *Solid line: the value of LHS of* (4.7); *Dashed line: the value of the RHS of* (4.7). (b) *The discrete shock profile.* (c) *Zoom of* (b) *around* $u_- = 1$. (d) *Zoom of* (b) *around* $u_+ = -1$.

we can generalize the theorems to nonstationary shocks with sufficiently small shock speed. Pointwise convergence rate estimates can also be obtained for schemes that possess stable discrete shocks and are linearly stable, when used to approximate scalar conservation laws whose solution is smooth except for some isolated shocks. However, we will not elaborate on such extensions.

Appendix A. $\Gamma_1(r)$ and $\Gamma_2(r)$ for several flux functions. Let

$$\tau_m(r) = \sup_{|z|\le r} \left| f^{(m)}(z) \right|$$

where f is the flux function of the conservation law (1.1). We derive the functions $\Gamma_1(r)$ and $\Gamma_2(r)$ for the numerical flux functions of the schemes discussed in section 7.

For simplicity, we use g'_k to denote the first-order derivative of $g(z_{-p+1}, \dots, z_p)$ with respect to z_k and use $g''_{k,l}$ to denote its second-order derivative with respect to z_k and z_l, where $k, l = -p+1, \dots, p$.

A.1. The modified Lax–Friedrichs scheme. The numerical flux function is given in (7.1). Its first-order derivatives are

$$g'_0 = \frac{1}{2}(\lambda_0 + \frac{2}{3\lambda}) \qquad\qquad g'_1 = \frac{1}{2}(\lambda_1 - \frac{2}{3\lambda})$$

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where $\lambda_k = f'(z_k), k = 0, 1$. Its second-order derivatives are

$$g''_{0,0} = \frac{1}{2}\mu_0 \qquad g''_{0,1} = g''_{1,0} = 0 \qquad g''_{1,1} = \frac{1}{2}\mu_1$$

where $\mu_k = f''(z_k)$. Therefore, we have

$$\Gamma_1(r) = \frac{1}{2}(\tau_1(r) + \frac{2}{3\lambda}), \tag{A.1}$$

$$\Gamma_2(r) = \frac{1}{2}\tau_2(r). \tag{A.2}$$

A.2. Lax–Wendroff scheme. The numerical flux function is given in (7.2). Its first-order derivatives are

$$g'_0 = \frac{1}{2}\left(\lambda_0 + \lambda\lambda_0\lambda_{\frac{1}{2}} - \frac{1}{2}\lambda\mu_{\frac{1}{2}}(f(z_1) - f(z_0))\right),$$

$$g'_1 = \frac{1}{2}\left(\lambda_1 - \lambda\lambda_1\lambda_{\frac{1}{2}} - \frac{1}{2}\lambda\mu_{\frac{1}{2}}(f(z_1) - f(z_0))\right)$$

where $\lambda_k = f'(z_k), k = 0, 1$ and $\lambda_{1/2} = f'(z_0 + z_1/2)$. Its second-order derivatives are

$$g''_{0,0} = \frac{1}{2}\left(\mu_0 + \lambda\lambda_{\frac{1}{2}}\mu_0 + \lambda\lambda_0\mu_{\frac{1}{2}} - \frac{1}{4}\lambda\gamma_{\frac{1}{2}}(f(z_1) - f(z_0))\right),$$

$$g''_{1,1} = \frac{1}{2}\left(\mu_1 - \lambda\lambda_{\frac{1}{2}}\mu_1 - \lambda\lambda_1\mu_{\frac{1}{2}} - \frac{1}{4}\lambda\gamma_{\frac{1}{2}}(f(z_1) - f(z_0))\right),$$

$$g''_{0,1} = \frac{1}{4}\left(\lambda\lambda_0\mu_{\frac{1}{2}} - \lambda\lambda_1\mu_{\frac{1}{2}} - \frac{1}{2}\lambda\gamma_{\frac{1}{2}}(f(z_1) - f(z_0))\right),$$

$$g''_{1,0} = g''_{0,1}$$

where $\mu_k = f''(z_k), k = 0, 1; \mu_{1/2} = f''(z_0 + z_1/2)$ and $\gamma_{1/2} = f'''(z_0 + z_1/2)$. Therefore, we have

$$\Gamma_1(r) = \frac{1}{2}\tau_1(r)(1 + \lambda\tau_1(r)) + \frac{\lambda}{2}\tau_0(r)\tau_2(r), \tag{A.3}$$

$$\Gamma_2(r) = \frac{1}{2}\tau_2(r)(1 + 2\lambda\tau_1(r)) + \frac{\lambda}{4}\tau_0(r)\tau_3(r). \tag{A.4}$$

A.3. The third-order WENO scheme. The flux function for the third-order WENO scheme is given in (7.3). To obtain $\Gamma_1(r)$ and $\Gamma_2(r)$, we first find the first two derivatives of the flux function.

The derivatives of the function $r = r(a, b) = \epsilon_w + a^2/\epsilon_w + b^2$ are

$$r'_a = \frac{2ra}{\epsilon_w + a^2},$$

$$r'_b = -\frac{2rb}{\epsilon_w + b^2},$$

$$r''_{aa} = \frac{2r}{\epsilon_w + a^2},$$

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$$r''_{bb} = -\frac{2r}{\epsilon_w + b^2} + \frac{8rb^2}{(\epsilon_w + b^2)^2},$$

$$r''_{ab} = -\frac{4abr}{(\epsilon_w + a^2)(\epsilon_w + b^2)}.$$

The derivatives of the function $\psi = \psi(a,b) = \dfrac{b-a}{1+2[r(a,b)]^2}$ are

$$\psi'_a = -\frac{1+4rr'_a\psi}{1+2r^2},$$

$$\psi'_b = -\frac{-1+4rr'_b\psi}{1+2r^2},$$

$$\psi''_{aa} = -\frac{4(2rr'_a\psi'_a + r'^2_a\psi + rr''_{aa}\psi)}{1+2r^2},$$

$$\psi''_{bb} = -\frac{4(2rr'_b\psi'_b + r'^2_b\psi + rr''_{bb}\psi)}{1+2r^2},$$

$$\psi''_{ab} = -\frac{4(rr'_a\psi'_b + rr'_b\psi'_a + r'_ar'_b\psi + rr''_{ab}\psi)}{1+2r^2}.$$

Let

$$\lambda_k = f'(z_k) \qquad \lambda_k^{\pm} = \frac{1}{2}(f'(z_k) \pm \Lambda) \qquad f^{\pm}(z_k) = \frac{1}{2}(f(z_k) \pm \Lambda z_k)$$

for $k = -1, 0, 1, 2$. We define

$$\psi^+ = \psi\left(f^+(z_0) - f^+(z_{-1}), f^+(z_1) - f^+(z_0)\right),$$

$$\psi^- = \psi\left(f^-(z_1) - f^-(z_2), f^-(z_0) - f^-(z_1)\right).$$

Similarly, we define by $\psi_a^{\pm}, \psi_b^{\pm}, \psi_{aa}^{\pm}\psi_{ab}^{\pm}, \psi_{bb}^{\pm}$ the first- and second-order derivatives of ψ with arguments, respectively, same as $\psi^{\pm}$. We drop $'$ and $''$ in the notation for derivatives of ψ for simplicity.

The first-order derivatives of the numerical flux function (7.3) are

$$g'_{-1} = \frac{1}{2}\lambda_{-1}^+\psi_a^+,$$

$$g'_0 = \frac{1}{2}(\lambda_0 - \lambda_0^+(\psi_a^+ - \psi_b^+) - \lambda_0^-\psi_b^-),$$

$$g'_1 = \frac{1}{2}(\lambda_1 - \lambda_1^-(\psi_a^- - \psi_b^-) - \lambda_1^+\psi_b^+),$$

$$g'_2 = \frac{1}{2}\lambda_2^-\psi_a^-.$$

Denote $\mu_k = f''(z_k), k = -1, 0, 1, 2$. The second-order derivatives are

$$g''_{-1,-1} = \frac{1}{4}\mu_{-1}\psi_a^+ - \frac{1}{2}(\lambda_{-1}^+)^2\psi_{aa}^+,$$

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$$g''_{-1,0} = \frac{1}{2}\lambda^+_{-1}\lambda^+_0(\psi^+_{aa} - \psi^+_{ab}),$$

$$g''_{-1,1} = \frac{1}{2}\lambda^+_{-1}\lambda^+_1\psi^+_{ab},$$

$$g''_{-1,2} = 0,$$

$$g''_{0,0} = \frac{1}{2}\left(\mu_0(1 - (\psi^+_a - \psi^+_b + \psi^-_b)/2) - (\lambda^+_0)^2(\psi^+_{aa} - 2\psi^+_{ab} + \psi^+_{bb}) + (\lambda^-_0)^2\psi^-_{bb}\right),$$

$$g''_{0,1} = -\frac{1}{2}(\lambda^+_0\lambda^+_1(\psi^+_{ab} - \psi^+_{bb}) + \lambda^-_0\lambda^-_1(\psi^-_{ab} - \psi^-_{bb})),$$

$$g''_{0,2} = \frac{1}{2}\lambda^-_0\lambda^-_2\psi^-_{ab},$$

$$g''_{1,1} = \frac{1}{2}\left(\mu_1(1 - (\psi^-_a - \psi^-_b + \psi^+_b)/2) - (\lambda^-_1)^2(\psi^-_{aa} - 2\psi^-_{ab} + \psi^-_{bb}) - (\lambda^+_1)^2\psi^+_{bb}\right),$$

$$g''_{1,2} = \frac{1}{2}\lambda^-_1\lambda^-_2(\psi^-_{aa} - \psi^-_{ab}),$$

$$g''_{2,2} = \frac{1}{4}\mu_2\psi^-_a - \frac{1}{2}(\lambda^-_2)^2\psi^-_{aa}.$$

Other second-order derivatives are known by symmetry, i.e., $g''_{k,l} = g''_{l,k}$.

We have the following simple observations:

$$|\lambda_k| \le \tau_1(r) \qquad |\lambda^\pm_k| \le \frac{1}{2}(\tau_1(r) + \Lambda) \qquad |\psi| \le 2(\tau_0(r) + \Lambda r),$$

$$|\psi'| \le \frac{1}{\sqrt{\epsilon_w}}(\sqrt{\epsilon_w} + 4(\tau_0(r) + \Lambda r)) \qquad |\psi''| \le \frac{1}{\epsilon_w}(\sqrt{\epsilon_w} + 11(\tau_0(r) + \Lambda r)),$$

$$\left|\frac{rr'}{1+2r^2}\right| \le \frac{1}{2\sqrt{\epsilon_w}} \qquad \left|\frac{(r')^2}{1+2r^2}\right| \le \frac{1}{2\epsilon_w} \qquad \left|\frac{rr''}{1+2r^2}\right| \le \frac{3}{\epsilon_w}$$

where r' can be r'_a or r'_b; r'' can be r''_{aa} or r''_{bb} or r''_{ab} and similarly, ψ' can be ψ'_a or ψ'_b; ψ'' can be ψ''_{aa} or ψ''_{bb} or ψ''_{ab}.

Based on the above observations, we have

$$\text{(A.5)} \qquad \Gamma_1(r) = (\tau_1(r) + \Lambda)\left(\frac{5}{4} + \frac{3(\Lambda r + \tau_0(r))}{\sqrt{\epsilon_w}}\right),$$

$$\text{(A.6)} \qquad \Gamma_2(r) = \frac{5\tau_2(r)}{4} + \frac{6\tau_2(r)(\Lambda r+\tau_0(r)) + 5(\Lambda+\tau_1(r))^2}{2\sqrt{\epsilon_w}} + \frac{55(\Lambda r+\tau_0(r))(\Lambda+\tau_1(r))^2}{2\epsilon_w}.$$

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