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CONVERGENCE OF UPWIND FINITE VOLUME SCHEMES FOR SCALAR CONSERVATION LAWS IN TWO DIMENSIONS*

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Abstract. This paper proves the convergence of a general class of monotone finite volume methods for numerical schemes of scalar conservation laws in two dimensions on unstructured meshes. There are convergence results for fractional step methods on cartesian grids and for finite element algorithms on unstructured grids. Even for finite volume methods there are some recent results concerning the Lax–Friedrichs and the Godunov finite volume method. The proof in this paper considers a general class including the Lax–Friedrichs and the Engquist–Osher finite volume schemes, and uses a completely different idea than in previous papers to control the entropy dissipation.

Key words. finite volume methods, conservation laws, measure-valued solution

AMS subject classifications. 35L60, 35L65, 35L67, 65M10, 76N15

1. Introduction. in this paper we shall consider finite volume discretizations for hyperbolic conservation laws in two space dimensions,

$$\partial_t u + \partial_x f_1(u) + \partial_y f_2(u) = 0 \quad \text{in} \quad \mathbf{R}^2 \times \mathbf{R}^+, \tag{1}$$

with initial values

$$u(x, y, 0) = u_0(x, y) \quad \text{in} \quad \mathbf{R}^2 \tag{2}$$

where $u : \mathbf{R}^2 \times \mathbf{R}^+ \to \mathbf{R}$, $f_1, f_2 : \mathbf{R} \to \mathbf{R}$, and $u_0 \in L^\infty(\mathbf{R}^2)$. Essentially, there are three different numerical methods that are generally used for solving (1) and (2). There are the finite difference methods on cartesian grids in the form of dimensional splitting (or fractional steps) [2]–[6], and the finite volume [1], [12]–[15], [20], [23], [24] and finite element methods [32] on arbitrary meshes.

For monotone finite difference methods Crandall and Majda [6] have proved convergence results for monotone dimensional splitting methods in several space dimensions. This type of scheme has two disadvantages. First, it is defined on structured meshes and therefore cannot be adapted to general geometries, and second, the convergence proof is based on a compactness argument for functions with bounded variations. The method of measure-valued solutions in the sense of DiPerna [10] assumes less regularity than is needed for BV estimates. This method was used by Szepessy [32] for the convergence proof of the streamline diffusion shock capturing method, and by Coquel and LeFloch [3], [4] for special dimensional splitting schemes of higher orders of accuracy. The streamline diffusion shock capturing method and the finite volume method are defined on unstructured grids and can be adapted to general geometries very easily.

For the proof in [3] and [4], Coquel and LeFloch analyze the local entropy dissipation in the single elementary waves of the Lax–Friedrichs and Godunov schemes in order to get an estimate which is weaker than the BV estimate. In a more recent

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paper [1], they also generalize these ideas to finite volume schemes under some special assumptions. They need a very restrictive condition either for the triangulation or for the numerical scheme. In the latter case, they must assume that the numerical viscosity is estimated from below.

In this paper we present a proof for the convergence of upwind finite volume schemes which does not need the strong assumptions of [4], and where we do not consider the entropy dissipation in the single elementary waves. Instead of this we choose suitable numerical entropy fluxes in order to get a global control of the entropy dissipation. These arguments can be used in particular for the Lax–Friedrichs and the Engquist–Osher finite volume schemes. Furthermore, they also hold for a more general class of finite volume schemes, which is characterized by the condition (54).

2. Notation, definitions, and assumptions. In this section we shall collect the most important notation, definitions, and assumptions.

DEFINITION 2.1. *Let a k-polygon be a closed, convex polygon with k vertices. The set*

$$T := \{T_i \mid T_i \textit{ is a k-polygon for } i \in I \subseteq \mathbf{N}\}$$

(where $I \subseteq \mathbf{N}$ is an index set) is called an unstructured grid of $\Omega \subset \mathbf{R^n}$ if the following two properties are satisfied:

(1) $\Omega = \cup_{i \in I} T_i$,

(2) *For two different T_i, T_j we have $T_j \cap T_i = \emptyset$ or $T_j \cap T_i =$ a common vertex of T_i, T_j or $T_j \cap T_i =$ a common edge of T_i, T_j.*

NOTATION 2.2. *Let $(T_j)_j$ denote an unstructured grid of the $\mathbf{R}^2$. In particular, we shall use the following notation:*

T_j: *the jth cell of the unstructured grid.*

$\mid T_j \mid$: *area of T_j.*

w_j: *center of gravity of T_j.*

$T_{jl}, l = 1, \ldots, k$: *neighboring cells of T_j.*

$\alpha(j,l)$: *global number of the lth-neighboring cell T_{jl} of T_j such that $T_{jl} = T_{\alpha(j,l)}$.*

$w_{jl}, l = 1, \ldots, k$: *center of gravity of T_{jl}, $w_{jl} = w_{\alpha(j,l)}$.*

u_j^n: *approximation of the exact solution u on T_j at time $n\Delta t$. The function u_j^n is assumed to be constant on T_j.*

u_{jl}^n: *approximation of the exact solution u on T_{jl} at time $n\Delta t$; $u_{jl}^n = u_{\alpha(j,l)}^n$.*

$u_j := u_j^n$.

S_{jl}: *lth edge of T_j.*

z_{jl}: *midpoint of the lth edge of the cell j.*

ν_{jl}: *outer normal to S_{jl} of length $\mid S_{jl} \mid$.*

$n_{jl} := \nu_{jl}/|\nu_{jl}|$. *By n_{jlx} and n_{jly} we denote the x- and y-coordinates of n_{jl}, respectively.*

$h := \sup_{j \in I} \operatorname{diam}(T_j)$.

$u^n(z) := u_j^n$ *if $z \in T_j$.*

$u_h(z,t) := u_j^n$ *if $z \in T_j$ and $t_n \leq t < t_{n+1}$, $n = 0, 1, \ldots$.*

Figure 1 illustrates the notation for unstructured grids of quadrangles ($k = 4$) and triangles ($k = 3$), respectively. Also, the dual cells of a cell-vertex mesh can be considered as an unstructured cell-centered grid. In Fig. 2 this situation is pointed out for hexagons as dual cells.

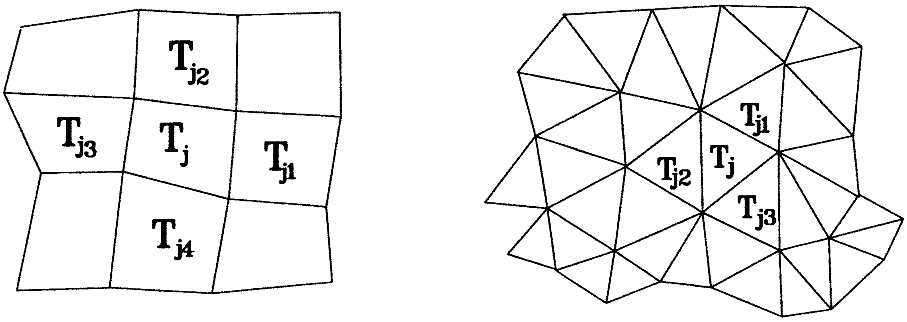

FIG. 1

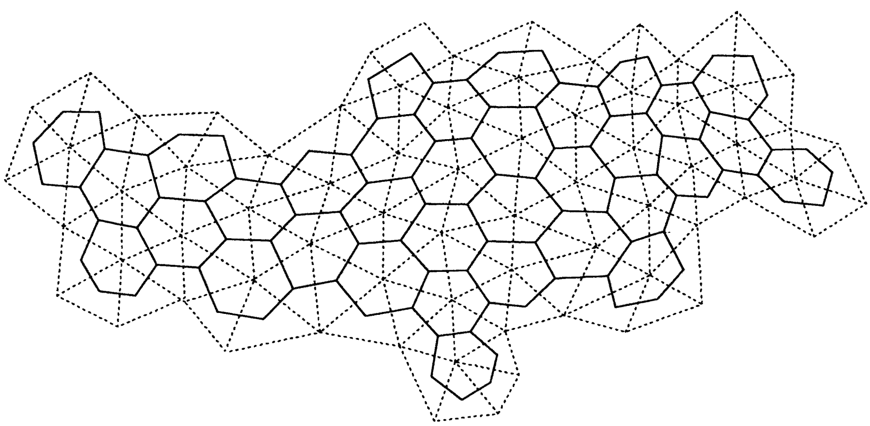

FIG. 2

ASSUMPTION 2.3. *For any j and l there exists a unique number m such that for $i := \alpha(j,l)$ we have*

$$\alpha(i,m) = j.$$

Since we use the notation $\nu_{j,\alpha(j,l)} := \nu_{jl}$, then $\nu_{im} = -\nu_{jl}$.

ASSUMPTION 2.4. *We assume that there are constants c_1, $c_2 \geq 0$ such that*

$$0 < c_1 \leq \frac{\Delta t}{h} \leq c_2 \tag{3}$$

if $\Delta t, h \to 0$. Moreover, we assume that there exists a constant $c_V > 0$ such that

$$\sup_j \frac{h^2}{|\, T_j \,|} \leq c_V. \tag{4}$$

DEFINITION 2.5. *For given initial values $u_0 \in L^\infty(\mathbf{R}^2)$ let u_j^n be defined by the following numerical scheme:*

$$u_j^0 := \frac{1}{|\, T_j \,|} \int_{T_j} u_0\,, \tag{5}$$

$$u_j^{n+1} := u_j^n - \frac{\Delta t}{| T_j |} \sum_{l=1}^{k} g_{jl}(u_j^n, u_{jl}^n) , \tag{6}$$

where for $g_{jl}, l = 1, \ldots, k$, *we assume that for any* $R > 0$ *and for all* $u, v, u', v' \in B_R(0)$ *we have*

$$| g_{jl}(u, v) - g_{jl}(u', v') | \leq c(R)\, h\, (| u - u' | + | v - v' |) , \tag{7}$$

$$g_{j,\alpha(j,l)}(u, v) = -g_{\alpha(j,l),j}(v, u) , \tag{8}$$

$$g_{j,\alpha(j,l)}(u, u) = \nu_{j,\alpha(j,l)} f(u) , \tag{9}$$

where

$$f(u) := \begin{pmatrix} f_1(u) \\ f_2(u) \end{pmatrix}$$

and where we have used the notation $g_{j,\alpha(j,l)} := g_{jl}$.

3. General results concerning measure-valued solutions. For the following notation we refer to [10]. By $K \subset\subset \mathbf{R}^2$ we denote a compact set in $\mathbf{R}^2$.

DEFINITION 3.1 (measure-valued solutions). *Let*

$$\mathrm{Prob}(\mathbf{R}) := \{\nu \ : \ \nu \textit{ is a probability measure on } \mathbf{R}\}.$$

Then a Young measure ν, *which is a measurable map*

$$\nu : \mathbf{R}^2 \times [0, T] \to \mathrm{Prob}(\mathbf{R}),$$

is a measure-valued solution to the conservation law (1) *if we have*

$$\partial_t \langle \nu, id \rangle + \partial_x \langle \nu, f_1 \rangle + \partial_y \langle \nu, f_2 \rangle = 0$$

in the distributional sense. Here we have used the notation

$$\langle \nu, f \rangle(x, y, t) := \int_{\mathbf{R}} f(\lambda) d\nu_{x,y,t}(\lambda).$$

DEFINITION 3.2 (consistency with the entropy). *Let* $U, F_1, F_2 \in C^1(\mathbf{R})$ *be defined such that* U *is convex and*

$$U'(s) f_1'(s) = F_1'(s), \qquad U'(s) f_2'(s) = F_2'(s)$$

for all $s \in \mathbf{R}$. *Then* (U, F_1, F_2) *is called an entropy for the conservation law* (1). *Now a Young measure is said to be consistent with the entropy* (U, F_1, F_2) *if we have*

$$\partial_t \langle \nu, U \rangle + \partial_x \langle \nu, F_1 \rangle + \partial_y \langle \nu, F_2 \rangle \leq 0 \tag{10}$$

in the distributional sense. If (10) *is satisfied for all entropies* (U, F_1, F_2) *and if* ν *is a measure-valued solution of* (1) *then* ν *is called the entropy measure-valued or admissible measure-valued solution.*

The most important tool for proving the convergence will be the following result of Diperna [10].

THEOREM 3.3. *Let us assume that* $u_0 \in L^1(\mathbf{R}^2) \cap L^\infty(\mathbf{R}^2)$ *and that there is a Young measure* ν *satisfying the following properties:*

(a) *The function* $(x, y, t) \to \langle \nu_{x,y,t}, | \, id \, | \rangle$ *is in* $L^\infty\left([0,T], L^1(\mathbf{R}^2)\right)$.

(b) ν *is a measure-valued solution to the conservation law* (1), (2).

(c) ν *is consistent with all entropies* (U, F_1, F_2).

(d) ν *assumes the initial values* u_0 *in the following sense:*

$$\lim_{t \to 0, t>0} \frac{1}{t} \int_0^t \int_{\mathbf{R}^2} \langle \nu_{x,y,s}, | \, id - u_0(x,y) \, | \rangle \, dx \, dy \, ds = 0.$$

(e) *Let us assume that the unique entropy solution* $u \in L^\infty(\mathbf{R}^2 \times [0,T])$ *of problem* (1), (2) *in the sense of Kruzkov* [21] *exists.*

Then the Young measure reduces to a Dirac measure, that is,

$$\nu_{x,y,t} = \delta_{u(x,y,t)}, \quad a.e. \, (x,y,t) \in \mathbf{R}^2 \times [0,T].$$

The following theorem provides a useful sufficient condition for property (d) of Theorem 3.3.

THEOREM 3.4. *Assume that* $u_0 \in L^\infty(\mathbf{R}^2) \cap L^1(\mathbf{R}^2)$ *and that* $\nu : \mathbf{R}^2 \times [0,T] \to \operatorname{Prob}(\mathbf{R})$ *is a Young measure. Furthermore, we assume that condition* (a) *of Theorem* 3.3 *is valid and that*

$$\lim_{t \to 0, t>0} \frac{1}{t} \int_0^t \int_{\mathbf{R}^2} \langle \nu_{x,y,s}, id \rangle \phi(x,y) \, dx \, dy \, ds = \int_{\mathbf{R}^2} u_0(x,y) \phi(x,y) \, dx \, dy \tag{11}$$

for all $\phi \in C_0^1(\mathbf{R}^2)$. *Additionally, we suppose*

$$\lim_{t \to 0, t>0} \frac{1}{t} \int_0^t \int_{\mathbf{R}^2} \langle \nu_{x,y,s}, U \rangle \, dx \, dy \, ds \le \int_{\mathbf{R}^2} U(u_0(x,y)) \, dx \, dy \tag{12}$$

for one *strictly convex continuous function* $U : \mathbf{R} \to \mathbf{R}$ *such that* $U(0) = 0$. *Then* ν *satisfies condition* (d) *of Theorem* 3.3, *that is,*

$$\lim_{t \to 0, t>0} \frac{1}{t} \int_0^t \int_{\mathbf{R}^2} \langle \nu_{x,y,s}, | \, id - u_0(x,y) \, | \rangle \, dx \, dy \, ds = 0.$$

For the proofs of Theorems 3.3 and 3.4, see [10].

Now we must show that the finite volume scheme (5), (6) will have a subsequence that converges to a measure-valued solution of (1) as $h \to 0$, and that this limit will satisfy the conditions (a)–(d) of Theorem 3.3. Let us start with the following proposition.

PROPOSITION 3.5. *For given initial values* $u_0 \in L^\infty(\mathbf{R}^2) \cap L^1(\mathbf{R}^2)$ *let* u_j^n *be defined by the numerical scheme* (5), (6) *such that* $c_1 \le \frac{\Delta t}{h} \le c_2$ *and*

$$\| \, u_h \, \|_{L^\infty(\mathbf{R}^2 \times \mathbf{R}^+)} \le M \qquad \text{uniformly in } h. \tag{13}$$

Furthermore, we assume that there exists $\beta \in [0,1[$, *such that for all* $K \subset\subset \mathbf{R}^2$

$$\Delta t^\beta h \sum_n \sum_{j, T_j \cap K = \emptyset} \sum_{l=1}^k | \, g_{jl}(u_j, u_{jl}) - \nu_{jl} f(u_j) \, | \le C_2(K) \tag{14}$$

uniformly in $\Delta t, h$. *Then there exists a subsequence* $(u_h)_h$ *and a Young measure* $\nu (= \nu_{x,y,t})$ *such that*

$$a(u_h) \to \int_{\mathbf{R}} a(\lambda) d\nu_{x,y,t}(\lambda) \quad \textit{weak}^* \textit{ in } L^\infty$$

for all $a \in C^0(\mathbf{R})$, *and that* $\nu_{x,y,t}$ *is a measure-valued solution of*

$$\partial_t u + \partial_x f_1(u) + \partial_y f_2(u) = 0 \quad in \quad \mathbf{R}^2 \times \mathbf{R}^+. \tag{15}$$

In particular, we have

$$\int_{\mathbf{R}^2 \times \mathbf{R}^+} u \partial_t \phi + \int_{\mathbf{R}^2 \times \mathbf{R}^+} \bar{f} \nabla \phi + \int_{\mathbf{R}^2} u_0 \phi(.,0) = 0 \tag{16}$$

for all $\phi \in C_0^\infty(\mathbf{R}^2 \times [0,T])$, *where* $\bar{f}(x,y,t) := \int_{\mathbf{R}} f(\lambda)\, d\nu_{x,y,t}(\lambda)$. *This means that* ν *satisfies condition* (b) *of Theorem* 3.3.

Remark 3.6. A sufficient criterion for (13) is given in Theorem 4.8 in [13]; see also Proposition 3.11 below.

Proof of Proposition 3.5. In this proof the summation with respect to j is always taken over $\{j, T_j \cap K \neq \emptyset\}$. To prove this proposition we have to multiply (6) by $\phi(w_j, t_n)$. We obtain

$$\sum_n \sum_j \mid T_j \mid (u_j^{n+1} - u_j^n) \phi(w_j, t_n) = -\Delta t \sum_n \sum_j \sum_l g_{jl}(u_j^n, u_{jl}^n) \phi(w_j, t_n) \tag{17}$$

where $\phi \in C_0^\infty(\mathbf{R^2} \times [0,T])$. As in Proposition 4.2 of [13], we get

$$\begin{aligned} &\sum_n \sum_j \mid T_j \mid (u_j^{n+1} - u_j^n) \phi(w_j, t_n) \\ &\quad = - \int_{\mathbf{R}^2 \times \mathbf{R}^+} u_h \partial_t \phi - \int_{\mathbf{R}^2} u_0 \phi(.,0) + O(h). \end{aligned} \tag{18}$$

For the right-hand side in (17) we obtain, using Lemmas 4.4 and 4.5 of [13] and the notation $\phi_{jl}^n := \phi(z_{jl}, t_n)$,

$$\begin{aligned} -\Delta t \sum_n \sum_j \sum_l g_{jl}(u_j^n, u_{jl}^n) \phi_j^n &= -\Delta t \sum_n \sum_j \sum_l g_{jl}(u_j^n, u_{jl}^n)(\phi_j^n - \phi_{jl}^n) \\ = -\Delta t \sum_n \sum_j \sum_l (g_{jl}(u_j^n, u_{jl}^n) &- \nu_{jl} f(u_j))(\phi_j^n - \phi_{jl}^n) \\ + \Delta t \sum_n \sum_j \sum_l \nu_{jl} f(u_j) \phi_{jl}^n &= -R + \int_{\mathbf{R}^2 \times \mathbf{R}^+} f(u_h) \nabla \phi + O(h), \end{aligned} \tag{19}$$

where

$$R := \Delta t \sum_n \sum_j \sum_l \left(g_{jl}(u_j^n, u_{jl}^n) - \nu_{jl} f(u_j) \right) (\phi_j^n - \phi_{jl}^n). \tag{20}$$

Since u_h is uniformly bounded there exists a $u \in L^\infty(\mathbf{R}^2 \times \mathbf{R}^+)$ such that

$$u_h \to u \quad \text{weak}^*,$$

and since f is continuous we obtain [34]

$$f(u_h) \to \bar{f} \quad \text{weak*},$$

where

$$\bar{f}(x,y,t) := \int_{\mathbf{R}} f(\lambda)\, d\nu_{x,y,t}(\lambda).$$

Therefore, if

$$R \to 0 \quad \text{for } h, \Delta t \to 0, \tag{21}$$

we obtain

$$\int_{\mathbf{R}^2\times\mathbf{R}^+} u\partial_t\phi + \int_{\mathbf{R}^2\times\mathbf{R}^+} \bar{f}\nabla\phi + \int_{\mathbf{R}^2} u_0\phi(.,0) = 0, \tag{22}$$

which proves Proposition 3.5. It remains to show (21). Using the assumption (14) we get

$$\begin{aligned} |\, R \,| &= \Delta t \left| \left(\sum_{n,j,l} (g_{jl} - \nu_{jl} f(u_j))\,(\phi_j^n - \phi_{jl}^n) \right) \right| \\ &\leq c(K)\Delta t^{1-\beta} h \Delta t^{\beta} \sum_{n,j,l} |\, g_{jl} - \nu_{jl} f(u_j) \,| \\ &\leq c(K,M)\, \Delta t^{1-\beta} \to 0 \quad \text{if } \beta < 1, \end{aligned} \tag{23}$$

where $\operatorname{supp}(\phi) \subset K$ and $M := \sup_{x,y} |u_0(x,y)|$. This completes the proof of Proposition 3.5.

Now let us verify condition (c) in Theorem 3.3. In particular, we shall show that the finite volume scheme (5), (6) will define a measure-valued solution of (1) as $h \to 0$ and that this limit is consistent with all entropies (U, F_1, F_2).

PROPOSITION 3.7. *Let us assume the conditions of Proposition* 3.5, *and additionally assume that there is an entropy* (U, F_1, F_2) *and a numerical entropy flux* $G_{jl} \in C^{0,1}(\mathbf{R}^2, \mathbf{R})$ *for* $j \in \mathbf{N}, l = 1, \ldots, k$ *such that* U *is convex and* G_{jl} *satisfies the following conditions:*

We assume that for any $R > 0$ *and for all* $u, v, u', v' \in B_R(0)$ *we have*

$$|\, G_{jl}(u,v) - G_{jl}(u',v') \,| \leq c(R)\, h\, (|\, u - u' \,| + |\, v - v' \,|)\,, \tag{24}$$

$$G_{j,\alpha(j,l)}(u,v) = -G_{\alpha(j,l),j}(v,u)\,, \tag{25}$$

$$G_{j,\alpha(j,l)}(u,u) = \nu_{j,\alpha(j,l)} F(u)\,, \tag{26}$$

where

$$F(u) := \begin{pmatrix} F_1(u) \\ F_2(u) \end{pmatrix}.$$

Furthermore, we assume

$$U(u_j^{n+1}) - U(u_j^n) \leq -\frac{\Delta t}{|\, T_j \,|} \sum_{l=1}^{k} G_{jl}(u_j^n, u_{jl}^n); \tag{27}$$

and we assume that there exists a $\beta \in [0,1[$ such that for all $K \subset\subset \mathbf{R}^2$ we have

$$\Delta t^\beta h \sum_n \sum_{j, T_j \cap K = \emptyset} \sum_{l=1}^{k} \mid G_{jl}(u_j, u_{jl}) - \nu_{jl} F(u_j) \mid \leq C_3(K) \tag{28}$$

uniformly in Δt, h. Then the measure-valued solution ν as obtained in Proposition 3.5 *is consistent with the entropy (U, F_1, F_2), that is,*

$$\partial_t \langle \nu, U \rangle + \partial_x \langle \nu, F_1 \rangle + \partial_y \langle \nu, F_2 \rangle \leq 0 \tag{29}$$

in the distributional sense.

COROLLARY 3.8. *Suppose the same assumptions as in Proposition* 3.7. *Additionally assume that there exists $\beta \in [0,1[$ such that* (27) *and* (28) *are valid for* all *entropies U and all corresponding entropy fluxes G_{jl}. Then condition* (c) *of Theorem* 3.3 *is satisfied.*

Proof of Proposition 3.7. In this proof the summation with respect to j is always taken over $\{j, T_j \cap K \neq \emptyset\}$. Let $\phi \in C_0^\infty(\mathbf{R}^2 \times [0,T])$, $\phi \geq 0$, $\text{supp}(\phi) \subset K$. Then we multiply (27) by $\frac{|T_j|}{\Delta t} \phi_j^n$ and sum over n and j. Then, using the same ideas as in the proof of Proposition 3.5, we obtain

$$- \int_{\mathbf{R}^2 \times \mathbf{R}^+} U(u_h) \partial_t \phi - \int_{\mathbf{R}^2 \times \mathbf{R}^+} F(u_h) \nabla \phi + R + o(1) \leq 0, \tag{30}$$

where

$$R := \Delta t \sum_n \sum_j \sum_l \left(G_{jl}(u_j^n, u_{jl}^n) - \nu_{jl} F(u_j) \right) (\phi_j^n - \phi_{jl}^n). \tag{31}$$

Now if $h \to 0$, then (30) and (28) imply

$$- \int_{\mathbf{R}^2 \times \mathbf{R}^+} \bar{U} \partial_t \phi - \int_{\mathbf{R}^2 \times \mathbf{R}^+} \bar{F}(u) \nabla \phi \leq 0, \tag{32}$$

where

$$\bar{U}(x,y,t) := \int_{\mathbf{R}} U(\lambda) \, d\nu_{x,y,t}(\lambda) = \langle \nu, U \rangle (x,y,t),$$

$$\bar{F}_1(x,y,t) := \int_{\mathbf{R}} F_1(\lambda) \, d\nu_{x,y,t}(\lambda) = \langle \nu, F_1 \rangle (x,y,t),$$

$$\bar{F}_2(x,y,t) := \int_{\mathbf{R}} F_2(\lambda) \, d\nu_{x,y,t}(\lambda) = \langle \nu, F_2 \rangle (x,y,t).$$

This completes the proof of Proposition 3.7.

Before we can verify property (d) in Theorem 3.3 we need some auxiliary results.

PROPOSITION 3.9. *Let the assumption of Proposition* 3.5 *be valid. Then the measure-valued solution ν of Proposition* 3.5 *satisfies* (11) *of Theorem* 3.4, *that is,*

$$\lim_{t \to 0, t > 0} \frac{1}{t} \int_0^t \int_{\mathbf{R}^2} \langle \nu_{x,y,s}, id \rangle \phi(x,y) \, dx \, dy \, ds = \int_{\mathbf{R}^2} u_0(x,y) \phi(x,y) \, dx \, dy \tag{33}$$

for all $\phi \in C_0^1(\mathbf{R}^2, \mathbf{R})$.

Proof. See [32] for the proof.

Now we shall recall the discrete maximum principle and the L^1 estimate for the scheme (6). To do this, we need the notion of monotonicity of the scheme (6).

DEFINITION 3.10 (monotonicity [13]). *The scheme* (6) *is called monotone if the function*

$$H(u_j, u_{j1}, \dots, u_{jk}) := u_j^n - \frac{\Delta t}{|T_j|} \sum_{l=1}^{k} g_{jl}(u_j, u_{jl}) \tag{34}$$

is monotonely nondecreasing in any argument $u_j, u_{j1}, \dots, u_{jk}$.

PROPOSITION 3.11. (a) *Let* $u_0 \in L^\infty(\mathbf{R}^2) \cap L^1(\mathbf{R}^2)$ *with* $\sup_{x,y} | u_0(x,y) | \leq M$. *Let* u_j^n *be defined by the monotone numerical scheme* (5), (6). *Then we have, for all* $n \in \mathbf{N}$,

$$\|u^n\|_{L^1(\mathbf{R}^2)} \leq \|u^0\|_{L^1(\mathbf{R}^2)}. \tag{35}$$

(b) *Instead of the monotonicity we assume that*

$$\sum_{l=1}^{k} \frac{g_{jl}(u_j, u_{jl}) - g_{jl}(u_j, u_j)}{u_j - u_{jl}} \geq 0. \tag{36}$$

(*In particular,* (36) *is automatically fulfilled for any monotone scheme.*) *Choose* Δt *such that*

$$\frac{\Delta t}{h} \leq \frac{1}{k \cdot c(M) \cdot c_V}, \tag{37}$$

where $c(M)$ *is the constant of Lipschitz continuity of* g_{jl} *and* G_{jl} (*see* (7) *and* (24)) *and* c_V *are defined as in* (4). *Then we have, for all* $n \in \mathbf{N}$,

$$\|u^n\|_{L^\infty(\mathbf{R}^2)} \leq \|u^0\|_{L^\infty(\mathbf{R}^2)}. \tag{38}$$

Proof. (a) The monotonicity of the scheme (6) and the property $H(0,0,\dots,0) = 0$ imply that

$$(H(u^n))^+ \leq H((u^n)^+) \quad \text{and} \quad (H(u^n))^- \geq H((u^n)^-).$$

Here we use the notation $a^+ := \max(a,0)$, $a^- := \min(a,0)$.

Then we have

$$\begin{aligned} |u_j^{n+1}| &= |H(u^n)| = (H(u^n))^+ - (H(u^n))^- \leq H((u^n)^+) - H((u^n)^-) \\ &= |u_j^n| - \frac{\Delta t}{|T_j|} \sum_{l=1}^{k} g_{jl}((u_j^n)^+, (u_{jl}^n)^+) + \frac{\Delta t}{|T_j|} \sum_{l=1}^{k} g_{jl}((u_j^n)^-, (u_{jl}^n)^-). \end{aligned} \tag{39}$$

Now the properties of g_{jl} imply that for $v = u^0$, $v = (u^0)^+$, $v = (u^0)^-$ we have

$$\begin{aligned} \sum_{l=1}^{k} |g_{jl}(v_j, v_{jl})| &= \sum_{l=1}^{k} |g_{jl}(v_j, v_{jl}) - g_{jl}(0,0)| \leq c(M)h\left(k|v_j| + \sum_{l=1}^{k} |v_{jl}|\right) \\ &\leq c(M)h\left(k|u_j^0| + \sum_{l=1}^{k} |u_{jl}^0|\right). \end{aligned}$$

So, since $u_0 \in L^1(\mathbf{R}^2)$, $\sum_j \sum_{l=1}^k g_{jl}(v_j, v_{jl})$ is absolutely convergent for $v = u^0$, $v = (u^0)^+$, or $v = (u^0)^-$. Moreover, we have by the conservative property (8) of g_{jl} that

$$\begin{aligned}\sum_j \sum_{l=1}^k g_{jl}(u_j^0, u_{jl}^0) &= \sum_j \sum_{l=1}^k g_{jl}((u_j^0)^+, (u_{jl}^0)^+) \\ &= \sum_j \sum_{l=1}^k g_{jl}((u_j^0)^-, (u_{jl}^0)^-) = 0.\end{aligned} \tag{40}$$

Then, multiplying (39) by $|T_j|$ and summing up over all j we get

$$\|u^1\|_{L^1(\mathbf{R}^2)} \le \|u^0\|_{L^1(\mathbf{R}^2)}, \tag{41}$$

and the proof is finished by induction.

(b) The definition of the algorithm (6) implies

$$\begin{aligned}u_j^{n+1} &= u_j^n - \frac{\Delta t}{|T_j|} \sum_{l=1}^k g_{jl}(u_j^n, u_{jl}^n) = u_j^n - \frac{\Delta t}{|T_j|} \sum_{l=1}^k (g_{jl}(u_j^n, u_{jl}^n) - g_{jl}(u_j^n, u_j^n)) \\ &= u_j^n - \frac{\Delta t}{|T_j|} \sum_{l=1}^k K_{jl}^n (u_j^n - u_{jl}^n),\end{aligned}$$

where

$$K_{jl}^n := \frac{g_{jl}(u_j, u_{jl}) - g_{jl}(u_j, u_j)}{u_j - u_{jl}} \quad \text{for } u_{jl} \ne u_j \tag{42}$$

and

$$K_{jl}^n := 0 \quad \text{for } u_{jl} = u_j. \tag{43}$$

Then we can continue with

$$u_j^{n+1} = u_j^n \left(1 - \frac{\Delta t}{|T_j|} \sum_{l=1}^k K_{jl}^n\right) + \frac{\Delta t}{|T_j|} \sum_{l=1}^k K_{jl}^n u_{jl}^n.$$

Let us assume $n = 0$. Then, by assumption (7),

$$|K_{jl}^0| \le c(M)h,$$

and therefore

$$\left|\frac{\Delta t}{|T_j|} \sum_{l=1}^k K_{jl}^0\right| \le k \frac{\Delta t}{\min_j |T_j|} c(M)h \le 1.$$

Altogether we get

$$\begin{aligned}|u_j^1| &\le |u_j^0| \left(1 - \frac{\Delta t}{|T_j|} \sum_{l=1}^k K_{jl}^0\right) + \frac{\Delta t}{|T_j|} \sum_{l=1}^k K_{jl}^0 |u_{jl}^0| \\ &\le \|u^0\|_{L^\infty(\mathbf{R}^2)} \left(1 - \frac{\Delta t}{|T_j|} \sum_{l=1}^k K_{jl}^0\right) + \frac{\Delta t}{|T_j|} \sum_{l=1}^k K_{jl}^0 \|u^0\|_{L^\infty(\mathbf{R}^2)} \\ &\le \|u^0\|_{L^\infty(\mathbf{R}^2)} \le M.\end{aligned} \tag{44}$$

Then we have $\|u^1\|_{L^\infty(\mathbf{R}^2)} \leq M$ and the proof can be completed by induction.

Remark 3.12. $\sum_j \sum_{l=1}^k g_{jl}(u_j^n, u_{jl}^n) = 0$ for all $n \in \mathbf{N}$ is a consequence of the proof of Proposition 3.11 and the same can be proved for G_{jl}.

PROPOSITION 3.13. *Let the assumptions of Propositions* 3.5 *and* 3.7 *be valid and let* $U(u) = u^2/2$. *Then the measure-valued solution* ν *of Proposition* 3.5 *satisfies* (12) *of Theorem* 3.4, *that is,*

$$\lim_{t\to 0, t>0} \frac{1}{t}\int_0^t \int_{\mathbf{R}^2} \langle \nu_{x,y,s}, U\rangle \, dx\, dy\, ds \leq \int_{\mathbf{R}^2} U(u_0(x,y))\, dx\, dy. \tag{45}$$

Proof. The properties of G_{jl} as defined in (24), (25), and (26) imply that we have, as in (40),

$$\sum_j \sum_l G_{jl}(u_j, u_{jl}) = 0.$$

Therefore, we obtain from (27),

$$\sum_j (U(u_j^{n+1}) - U(u_j^n))|T_j| \leq 0,$$

and summing this up with respect to n,

$$\sum_j (U(u_j^n) - U(u_j^0))|T_j| \leq 0.$$

Of course, the value of the sum $\sum_j U(u_j^n)$ is finite since we have, thanks to (35) and (38),

$$\|U(u^n)\|_{L^1} \leq \tfrac{1}{2}\|u^n\|_{L^1}\|u^n\|_{L^\infty} \leq \tfrac{1}{2}\|u^0\|_{L^1}\|u^0\|_{L^\infty} \leq \text{const.}$$

Let $t_1 := n_1 \Delta t$ and $t_2 = n_2 \Delta t$. Then we get

$$\frac{\Delta t}{t_2 - t_1} \sum_{n=n_1}^{n_2 - 1} \sum_j U(u_j^n)|T_j| \leq \sum_j U(u_j^0)|T_j|$$

or

$$\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} \int_{\mathbf{R}^2} U(u_h) \leq \int_{\mathbf{R}^2} U(u_0).$$

Now we proceed as in [32] to obtain that the function

$$t \mapsto \frac{1}{t_2 - t}\int_t^{t_2} \int_{\mathbf{R}^2} \langle \nu, U\rangle$$

is Lipschitz continuous and admits a trace at $t = 0$. So we can pass to the limit $t_1 \to 0$ to obtain

$$\frac{1}{t}\int_0^t \int_{\mathbf{R}^2} \langle \nu, U\rangle \leq \int_{\mathbf{R}^2} U(u_0).$$

Taking the limit $t \to 0$ proves the proposition.

Now let us summarize the results of this section in order to get a general convergence result for the finite volume scheme (5), (6).

THEOREM 3.14. *Let us assume that the conditions from §2 are satisfied. Let $u_0 \in L^\infty(\mathbf{R}^2) \cap L^1(\mathbf{R}^2)$ and let u_j^n be defined by the numerical scheme* (5), (6) *such that for g_{jl} the properties* (7), (8), *and* (9), *and for G_{jl} the properties* (24), (25), *and* (26), *are valid. Furthermore, let us assume that the following conditions are satisfied:*

$$0 < c_1 \le \frac{\Delta t}{h} \le c_2 \quad and \quad \sup_j \frac{h^2}{|T_j|} \le c_V, \tag{46}$$

$$|u_h| \le M \quad uniformly\ in\ h, \tag{47}$$

$$\|u_h(.,t)\|_{L^1(\mathbf{R}^2)} \le M \quad uniformly\ in\ h. \tag{48}$$

Moreover, let there exist $\beta \in [0,1[$ such that for all $K \subset\subset \mathbf{R}^2$,

$$\Delta t^\beta h \sum_n \sum_{j, T_j \cap K = \emptyset} \sum_{l=1}^k |g_{jl}(u_j, u_{jl}) - \nu_{jl} f(u_j)| \le C_2(K) \quad uniformly\ in\ \Delta t, h, \tag{49}$$

$$\Delta t^\beta h \sum_n \sum_{j, T_j \cap K = \emptyset} \sum_{l=1}^k |G_{jl}(u_j, u_{jl}) - \nu_{jl} F(u_j)| \le C_3(K) \quad uniformly\ in\ \Delta t, h, \tag{50}$$

$$U(u_j^{n+1}) - U(u_j^n) \le -\frac{\Delta t}{|T_j|} \sum_{l=1}^k G_{jl}(u_j^n, u_{jl}^n) \tag{51}$$

for all entropies (U, F_1, F_2) and G_{jl} satisfying the conditions of Proposition 3.7. *Then*

$$u_h \to u \quad weak^*, \qquad u \in L^\infty(\mathbf{R}^2 \times \mathbf{R}^+), \tag{52}$$

and u is the uniquely determined solution in the sense of Kruzkov of the initial value problem (1), (2).

Proof. We have to verify conditions (a)–(d) of Theorem 3.3. From (35) it follows that $\sup_{[0,T]} \int_{\mathbf{R}^2} |u_h(x,y,t)|\, dx\, dy \le \| u^0 \|_{L^1(\mathbf{R}^2)}$. Then $h \to 0$ gives us (a). Now (b) follows from Proposition 3.5 and (c) from Proposition 3.7 and Corollary 3.8. Propositions 3.13 and 3.9 imply that the assumptions of Theorem 3.4 are satisfied. Then from Theorem 3.4 we obtain property (d) of Theorem 3.3.

4. Convergence of upwind finite volume schemes. In this section we describe a class of upwind finite volume schemes for which the conditions of Theorem 3.14 are fulfilled. In particular, we show that the Engquist–Osher and Lax–Friedrichs schemes with suitably chosen numerical entropy fluxes belong to that class. This yields the convergence result for these schemes (see Theorem 4.11).

ASSUMPTION 4.1. *Using the notation and assumptions introduced in §2 we will assume in the sequel that the numerical flux $g_{jl} \in C^1(\mathbf{R}^2)$ satisfies the conditions* (7), (8), *and* (9) *and the numerical entropy flux $G_{jl} \in C^1(\mathbf{R}^2)$ satisfies the conditions* (24), (25), *and* (26). *Without loss of generality we also assume that $f_1(0) = f_2(0) = 0$. Additionally, we assume the monotonicity of the scheme* (6); *see Definition* 3.10.

Example 4.2 (Engquist–Osher and Lax–Friedrichs schemes). Let

$$c_{jl}(u) := n_{jl} f(u),$$

and define

$$c_{jl}^{+}(u) := \int_0^u \max\left(c_{jl}'(s), 0\right) ds, \qquad c_{jl}^{-}(u) := \int_0^u \min\left(c_{jl}'(s), 0\right) ds.$$

The Engquist–Osher numerical flux is then given by [11]

$$g_{jl}^{EO}(u,v) := |S_{jl}| \left(c_{jl}^{+}(u) + c_{jl}^{-}(v)\right).$$

Now, the Lax–Friedrichs numerical flux is given explicitly by (see, e.g., [1])

$$g_{jl}^{LF}(u,v) := \frac{1}{2}\left[\nu_{jl} f(u) + \nu_{jl} f(v)\right] - \frac{1}{2\lambda_{jl}}(v-u),$$

where λ_{jl} are arbitrarily chosen constants satisfying

$$\lambda_{jl} = \lambda_{lj} > \frac{\widetilde{c}}{h} > 0, \qquad \lambda_{jl} \sup_u \left(\nu_{jl} f'(u)\right) \leq 1.$$

Then g_{jl}^{EO}, g_{jl}^{LF} satisfy the conditions (7), (8), and (9). Moreover, if Δt is chosen in such a way that in the case of the Engquist–Osher scheme we have

$$\sup_j \frac{\Delta t}{|T_j|} \sum_{l=1}^{k} \max\{\nu_{jl} f'(u_j), 0\} \leq 1,$$

while in the case of the Lax–Friedrichs scheme we have

$$\sup_j \frac{\Delta t}{|T_j|} \sum_{l=1}^{k} \left(\frac{1}{2}\nu_{jl} f'(u_j) + \frac{1}{2\lambda_{jl}}\right) \leq 1,$$

then both schemes are monotone.

As we have seen in Proposition 3.11, the first three conditions, (46), (47), and (48), of Theorem 3.14 are satisfied for monotone schemes with properly chosen Δt. Next we shall show that condition (49) implies condition (50) for a suitably chosen numerical entropy flux.

We recall that Assumption 4.1 is supposed to be satisfied throughout this section.

PROPOSITION 4.3. *Let us assume the following conditions:*

(a) g_{jl} *is given and satisfies* (7), (8), *and* (9).

(b) *There exists* $\beta \in [0,1[$ *such that for all* $K \subset\subset \mathbf{R}^2$

$$\Delta t^{\beta} h \sum_n \sum_{j, T_j \cap K = \emptyset} \sum_{l=1}^{k} |g_{jl}(u_j, u_{jl}) - \nu_{jl} f(u_j)| \leq C_2(K) \quad \textit{uniformly in } \Delta t, h. \tag{53}$$

(c) *For given convex* $U \in C^2(\mathbf{R})$ *there exists a numerical entropy flux* G_{jl} *corresponding to the entropy* (U, F_1, F_2), *satisfying* (24), (25), *and* (26) *and the additional compatibility condition*

$$\frac{\partial G_{jl}}{\partial v}(u,v) = U'(v) \frac{\partial g_{jl}}{\partial v}(u,v) \tag{54}$$

for all $u,v \in \mathbf{R}$.

(d) *We suppose that*

$$g_{jl} \text{ is monotonely nonincreasing in the second variable} \tag{55}$$

(*which is fulfilled by the monotone schemes*).

Then for any $K \subset\subset \mathbf{R}^2$ *we have*

$$\Delta t^{\beta} h \sum_n \sum_{j, T_j \cap K = \emptyset} \sum_{l=1}^{k} |G_{jl}(u_j, u_{jl}) - \nu_{jl} F(u_j)| \leq C_2(K) \cdot c(U') \tag{56}$$

uniformly in $\Delta t, h$.

Proof. We have

$$|G_{jl}(u_j, u_{jl}) - \nu_{jl} F(u_j)| = \left| \int_{u_j}^{u_{jl}} \frac{\partial G_{jl}}{\partial s}(u_j, s)\, ds \right| = \left| \int_{u_j}^{u_{jl}} U'(s) \frac{\partial g_{jl}}{\partial s}(u_j, s)\, ds \right|$$

$$\leq c(U') \int_{\min(u_j, u_{jl})}^{\max(u_j, u_{jl})} \left(-\frac{\partial g_{jl}}{\partial s}(u_j, s) \right) ds = c(U') |g_{jl}(u_j, u_{jl}) - g_{jl}(u_j, u_j)|,$$

which proves our proposition.

Remark 4.4. The compatibility condition (54) can be viewed as an analogy of the entropy compatibility condition

$$F_j'(s) = U'(s) f_j'(s), \qquad j = 1, 2$$

(cf. Definition 3.2). Condition (54) turns out to be crucial in obtaining a global control of the entropy dissipation, as we will see later. In the following example we show that for the Engquist–Osher and Lax–Friedrichs schemes the numerical entropy flux can be defined in such a way that both the conditions, (24), (25), and (26) and the compatibility condition (54) are fulfilled.

Example 4.5 (entropy fluxes for the E–O and L–F schemes). Without loss of generality we assume that $F(0) = 0$. We choose the numerical entropy flux in the case of the Engquist–Osher scheme (see Example 4.2)

$$G_{jl}^{EO}(u, v) := |S_{jl}| \int_0^u U'(s) \max\left(c_{jl}'(s), 0\right) ds + |S_{jl}| \int_0^v U'(s) \min\left(c_{jl}'(s), 0\right) ds,$$

and in the case of the Lax–Friedrichs scheme

$$G_{jl}^{LF}(u, v) := \frac{1}{2}\left[\nu_{jl} F(u) + \nu_{jl} F(v)\right] - \frac{1}{2\lambda_{jl}} \left(U(v) - U(u)\right).$$

Then these numerical entropy fluxes satisfy the conditions (24), (25), and (26), and the compatibility condition (54).

Next we shall consider the condition (49) of Theorem 3.14.

PROPOSITION 4.6. *Let* $u_0 \in L^1(\mathbf{R}^2) \cap L^\infty(\mathbf{R}^2)$, $\|u_0\|_{L^\infty} \leq M$, *and let* u_j^n *be defined by the numerical scheme* (6) *with numerical flux* g_{jl}, *satisfying* (7), (8), (9), *and* (55). *Let* G_{jl} *be the numerical entropy flux corresponding to the entropy* $U(u) = u^2/2$ *satisfying* (24), (25), (26), *and* (54). *Moreover, choose* Δt *such that the Courant–Friedrichs–Levy* (CFL)*-like condition*

$$\frac{\Delta t}{h} \leq \frac{1}{(k+2) \cdot c(M) \cdot c_V} \tag{57}$$

holds. Here $c(M)$ *is the constant of Lipschitz continuity of* g_{jl} *and* G_{jl} *(see* (7) *and* (24)), *and* c_V *is defined as in* (4). *Then there exists* $\beta \in]\frac{1}{2}, 1[$ *such that for all* $K \subset\subset \mathbf{R}^2$ *we have*

$$\Delta t^\beta h \sum_n \sum_{j, T_j \cap K = \emptyset} \sum_{l=1}^{k} |g_{jl}(u_j, u_{jl}) - \nu_{jl} f(u_j)| \le C_2(K) \tag{58}$$

uniformly in $\Delta t, h$.

This proposition will now be proved in two lemmas.

LEMMA 4.7. *Under the assumptions of Proposition* 4.6 *we have*

$$\begin{aligned} &\frac{(u_j^{n+1})^2 - (u_j^n)^2}{2} + \frac{\Delta t}{|T_j|} \sum_{l=1}^{k} G_{jl}(u_j, u_{jl}) \\ &\qquad + \left(\frac{\Delta t}{|T_j|}\right)^2 \sum_{l=1}^{k} [g_{jl}(u_j, u_{jl}) - g_{jl}(u_j, u_j)]^2 \le 0, \end{aligned} \tag{59}$$

where G_{jl} *is the numerical entropy flux corresponding to the entropy* $U(u) = u^2/2$.

Proof. We set $\lambda := \Delta t / |T_j|$ and use the following notation: for all $t \in [-M, M]$,

$$\Delta_2 G(t) := G_{jl}(u_j, t) - G_{jl}(u_j, u_j),$$

$$\Delta_2 g(t) := g_{jl}(u_j, t) - g_{jl}(u_j, u_j),$$

$$\partial_2 G(t) := \frac{\partial G_{jl}}{\partial t}(u_j, t),$$

$$\partial_2 g(t) := \frac{\partial g_{jl}}{\partial t}(u_j, t).$$

Note that

$$\sum_{l=1}^{k} g_{jl}(u_j, u_{jl}) = \sum_{l=1}^{k} \Delta_2 g(u_{jl}),$$

since $\sum_{l=1}^{k} g_{jl}(u_j, u_j) = f(u_j) \sum_{l=1}^{k} \nu_{jl} = 0$. The same holds for G_{jl}. Then we can write, using (6),

$$\begin{aligned} \frac{(u_j^{n+1})^2 - (u_j^n)^2}{2} &= u_j^n (u_j^{n+1} - u_j^n) + \frac{1}{2} (u_j^{n+1} - u_j^n)^2 \\ &= u_j(-\lambda) \sum_{l=1}^{k} \Delta_2 g(u_{jl}) + \frac{1}{2} \lambda^2 \left(\sum_{l=1}^{k} \Delta_2 g(u_{jl}) \right)^2 \\ &\le -\lambda u_j \sum_{l=1}^{k} \Delta_2 g(u_{jl}) + \frac{1}{2} k \lambda^2 \sum_{l=1}^{k} (\Delta_2 g(u_{jl}))^2 . \end{aligned}$$

Hence, the left-hand side of (59) is less than or equal to

$$\lambda \sum_{l=1}^{k} \left[-u_j \Delta_2 g(u_{jl}) + \Delta_2 G(u_{jl}) + \lambda \left(\frac{k+2}{2} \right) (\Delta_2 g(u_{jl}))^2 \right]. \tag{60}$$

Setting $a := u_j$ and

$$p(t) := -a\Delta_2 g(t) + \Delta_2 G(t) + \lambda \left(\frac{k+2}{2}\right) (\Delta_2 g(t))^2,$$

the proof will be finished by showing that

$$p(t) \le 0 \quad \forall t \in \,]-M, M[. \tag{61}$$

Now,

$$p'(t) = -a\partial_2 g(t) + \partial_2 G(t) + \lambda(k+2)\left(\Delta_2 g(t)\right)\left(\partial_2 g(t)\right). \tag{62}$$

Using (54) with $U(u) = u^2/2$ and $\Delta_2 g(t) = \partial_2 g(\xi)(t-a)$ for ξ between a and t, we have

$$p'(t) = (t-a)\partial_2 g(t)\left(1 + \lambda(k+2)\partial_2 g(\xi)\right). \tag{63}$$

Now, since $\partial_2 g(t) \le 0$ for all $t \in \,]-M, M[$ (cf. (55)), we are finished, provided the third term in (63) is nonnegative. In fact, $p'(t) \ge 0$ for $t \le a$ and $p'(t) \le 0$ for $t \ge a$, which together with $p(a) = 0$ gives us (61).

But $\partial_2 g(\xi) \ge -c(M)h$ by Lipschitz continuity (24) and consequently

$$\begin{aligned} 1 + \lambda(k+2)\partial_2 g(\xi) &\ge 1 - \lambda(k+2)hc(M) \\ &\ge 1 - \frac{\Delta t}{h} c_V (k+2) c(M) \ge 0 \end{aligned} \tag{64}$$

because of (57).

LEMMA 4.8. *Under the assumptions of Proposition* 4.6 *we have*

$$\sum_n \sum_j \sum_{l=1}^{k} \left(g_{jl}(u_j, u_{jl}) - \nu_{jl} f(u_j)\right)^2 \le c(u_0). \tag{65}$$

Proof. Multiplying (59) by $|T_j|$ and summing up over j we get

$$\begin{aligned} &\frac{1}{2}\|u^{n+1}\|^2_{L^2(\mathbf{R}^2)} - \frac{1}{2}\|u^n\|^2_{L^2(\mathbf{R}^2)} \\ &\qquad + \sum_j \frac{(\Delta t)^2}{|T_j|} \sum_{l=1}^{k} \left(g_{jl}(u_j, u_{jl}) - \nu_{jl} f(u_j)\right)^2 \le 0, \end{aligned} \tag{66}$$

since $\sum_j \sum_{l=1}^k G_{jl}(u_j, u_{jl}) = 0$ (see Remark 3.12). Note that the L^2 norms on the left-hand side of (66) are finite since (see Proposition 3.11)

$$\|u^n\|^2_{L^2(\mathbf{R}^2)} \le \|u^n\|_{L^\infty(\mathbf{R}^2)} \cdot \|u^n\|_{L^1(\mathbf{R}^2)} \le M \cdot \|u_0\|_{L^1(\mathbf{R}^2)}.$$

Now, from the trivial property of unstructured grids $|T_j| \le h^2$, it follows that we can replace $|T_j|$ by h^2 in the third term of (66) with the inequality still holding true. Since the resulting coefficient is bounded,

$$0 < c_1^2 \le \left(\frac{\Delta t}{h}\right)^2 \le c_2^2$$

(see (46)), (66) yields, after summing over n from 0 to N,

$$c_1^2\sum_{n=0}^{N}\sum_j\sum_{l=1}^{k}\left(g_{jl}(u_j,u_{jl})-\nu_{jl}f(u_j)\right)^2 \le \frac{1}{2}\|u^0\|^2_{L^2(\mathbf{R}^2)}-\frac{1}{2}\|u^{N+1}\|^2_{L^2(\mathbf{R}^2)}$$
$$\le \frac{1}{2}\|u^0\|^2_{L^2(\mathbf{R}^2)}+\frac{1}{2}\|u^0\|_{L^\infty(\mathbf{R}^2)}\cdot\|u^0\|_{L^1(\mathbf{R}^2)} =: c(u_0).$$

This gives us the desired result, since the right-hand side does not depend on N.

Now we are ready to prove Proposition 4.6.

Proof of Proposition 4.6. In this proof the summation with respect to j is always taken over $\{j, T_j\cap K\neq\emptyset\}$. Using the Cauchy–Schwarz inequality and Lemma 4.8 we get

$$\Delta t^\beta h\sum_n\sum_j\sum_{l=1}^{k}|g_{jl}(u_j,u_{jl})-\nu_{jl}f(u_j)| = \sum_n\sum_j\sum_{l=1}^{k}|\Delta_2 g(u_{jl})|\Delta t^\beta h$$
$$\le\left(\sum_n\sum_j\sum_{l=1}^{k}(\Delta_2 g(u_{jl}))^2\right)^{\frac12}\cdot\left(\sum_n\sum_j\sum_{l=1}^{k}\Delta t^{2\beta}h^2\right)^{\frac12}$$
$$\le c(u_0,K)\cdot\left(\sum_n\sum_j\sum_{l=1}^{k}\Delta t^{2\beta}h^2\right)^{\frac12}\le c(u_0,K)\cdot\left(\widetilde{c}\cdot\Delta t^{2\beta-1}\right)^{\frac12}.$$

The last term is bounded for $\Delta t\to 0$, provided $\beta>\frac12$.

Next, we will focus on global control of the entropy dissipation (51) for all entropies (U,F_1,F_2) and corresponding G_{jl}. We will see that our compatibility condition (54) is again the crucial point in proving (51). The same condition as (51) was used by Tadmor in [33].

PROPOSITION 4.9. *Under the assumptions of Proposition* 4.6 *we have*

$$U(u_j^{n+1})-U(u_j^n)\le-\frac{\Delta t}{|T_j|}\sum_{l=1}^{k}G_{jl}(u_j^n,u_{jl}^n) \tag{67}$$

for all convex entropies $U\in C^2(\mathbf{R})$.

Proof. Throughout the proof we will use the same notation as in the proof of Lemma 4.7. We begin by rewriting the scheme (6):

$$u_j^{n+1}=\frac1k\left(\sum_{l=1}^{k}\left(u_j^n-\lambda k\Delta_2 g(u_{jl})\right)\right). \tag{68}$$

The main step in the proof is the following mean value property of convex functions:

$$U\left(\frac{a_1+\cdots+a_m}{m}\right)\le\frac1m\left(U(a_1)+\cdots+U(a_m)\right). \tag{69}$$

Applying this to (68) we obtain

$$U(u_j^{n+1})\le\frac1k\sum_{l=1}^{k}U\left(u_j^n-\lambda k\Delta_2 g(u_{jl})\right). \tag{70}$$

Hence, the proof will be completed if we prove the following inequality:

$$U\left(u_j^n - \lambda k \Delta_2 g(u_{jl})\right) \le U(u_j^n) - \lambda k \Delta_2 G(u_{jl}). \tag{71}$$

Now, setting $a := u_j^n$ and defining for all $t \in [-M, M]$

$$p(t) := U\left(a - \lambda k \Delta_2 g(t)\right) - U(a) + \lambda k \Delta_2 G(t),$$

we have (as in the proof of Lemma 4.7) $p(a) = 0$ and

$$\begin{aligned} p'(t) &= U'\left(a - \lambda k \Delta_2 g(t)\right) \cdot \left(-\lambda k \partial_2 g(t)\right) + \lambda k \partial_2 G(t) \\ &= \lambda k\, \partial_2 g(t) \big(U'(t) - U'\left(a - \lambda k \Delta_2 g(t)\right)\big) \\ &= \lambda\, k\, \partial_2 g(t)\, U''(\eta) \left(t - a + \lambda k \partial_2 g(\widetilde{\eta})(t-a)\right) \end{aligned}$$

for some η between t and $a - \lambda k \Delta_2 g(t)$ and $\widetilde{\eta}$ between a and t. Finally,

$$p'(t) = \lambda\, k\, \partial_2 g(t)\, U''(\eta)\, (t-a) \left(1 + \lambda k \partial_2 g(\widetilde{\eta})\right).$$

Since $\partial_2 g(t) \le 0$ for all $t \in]-M, M[$ (the monotonicity of g_{jl}; see (55)) and $U'' \ge 0$ (the convexity of U) the proof can be completed (as in Lemma 4.7) by showing $1 + \lambda k \partial_2 g(\widetilde{\eta}) \ge 0$. But this holds true because of the CFL-like condition (57) (cf. (64) and the end of the proof of Lemma 4.7).

Remark 4.10. Note that the CFL-like condition (57), which was used at the end of the proof, is a bit stronger than the CFL-like condition (37).

Now we can summarize the results of this paragraph in the following theorem.

THEOREM 4.11. *Let us assume the following conditions:*

(a) g_{jl} *is given and satisfies* (7), (8), *and* (9).

(b) *For any convex* $U \in C^2(\mathbf{R})$ *there exists a numerical entropy flux* G_{jl} *corresponding to the entropy* (U, F_1, F_2), *satisfying* (24), (25), *and* (26) *and the additional compatibility condition*

$$\frac{\partial G_{jl}}{\partial v}(u,v) = U'(v) \frac{\partial g_{jl}}{\partial v}(u,v) \tag{72}$$

for all $u,v \in \mathbf{R}$.

(c) *Let the scheme* (5), (6) *be monotone in the sense of Definition* 3.10.

(d) *Let the grid satisfy* (3), (4), *and* (57).

(e) *Let* $u_0 \in L^\infty(\mathbf{R}^2) \cap L^1(\mathbf{R}^2)$ *and let* u_j^n *be defined by the numerical scheme* (5), (6).

Then all the conditions of Theorem 3.14 *are satisfied and, consequently,*

$$u_h \to u \quad \textit{weak*}, \qquad u \in L^\infty(\mathbf{R}^2 \times \mathbf{R}^+), \tag{73}$$

and u *is the uniquely determined solution in the sense of Kruzkov of the initial value problem* (1), (2).

Proof. The proof follows immediately from Propositions 3.11, 4.3, 4.6, and 4.9.

Example 4.12 (Engquist–Osher and Lax–Friedrichs schemes). As an application of Theorem 4.11 we have the convergence proof for the Engquist–Osher and Lax–Friedrichs schemes with numerical entropy fluxes chosen as in Example 4.5. Note that the convergence result obtained in [1] can be applied to the Lax–Friedrichs scheme but not to the Engquist–Osher one, since the latter doesn't satisfy the condition (2.37) of Theorem 2.3 in [1].

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REFERENCES

[1] B. Cockburn, F. Coquel, Ph. LeFloch, and C. W. Shu, *Convergence of finite volume methods*, IMA Preprint #771, 1991, Univ. of Minnesota, IMA, Minneapolis, MN; SIAM J. Numer. Anal., submitted.

[2] P. Colella and P. R. Woodward, *The piecewise parabolic method for gas-dynamical simulation*, J. Comp. Phys., 54 (1984), pp. 174–201.

[3] F. Coquel and P. LeFloch, *Convergence of finite difference schemes for conservation laws in several space dimensions: The corrected antidiffusive flux approach*, Math. Comp., 57 (1991), pp. 169–210.

[4] ———, *Convergence of finite difference schemes for conservation laws in several space dimensions: General theory*, Preprint No. 207, Centre de Maths. Appl., Unité Associé au CNRS, Ecole Polytechnique, Palaiseau, France, 1989; SIAM J. Numer. Anal., 30 (1993), pp. 675–700.

[5] M. C. Crandall and A. Majda, *The method of fractional step for conservation laws*, Numer. Math., 34 (1980), pp. 285–314.

[6] ———, *Monotone difference approximation for scalar conservation laws*, Math. Comp., 34 (1980), pp. 1–21.

[7] H. Deconinck and R. Struijs, *Multidimensional upwind schemes for the Euler equations using fluctuation distribution on a grid consisting of triangles*, Preprint, Von Karman Inst. for Fluid Dynamics, Sint-Genesius-Rode, Belgium, 1989.

[8] ———, *A comparison of fluctuation splitting on triangular meshes for scalar equations and application to the Euler equations*, Preprint, Von Karman Inst. for Fluid Dynamics, Sint-Genesius-Rode, Belgium, 1989.

[9] H. Deconinck, P. L. Roe, and R. Struijs, *A conservative linearisation of the multidimensional Euler equations*, Preprint, Von Karman Inst. for Fluid Dynamics, Sint-Genesius-Rode, Belgium, 1991.

[10] R. J. DiPerna, *Measure valued solutions of conservation laws*, Arch. Rational Mech. Anal., 88 (1985), pp. 223–270.

[11] B. Engquist and S. Osher, *One sided difference approximations for nonlinear conservation laws*, Math. Comp., 36 (1981), pp. 321–351.

[12] L. Fezoui and B. Stoufflet, *A class of implicit upwind schemes for Euler simulations with unstructured meshes*, J. Comp. Phys., 84 (1989), pp. 174–206.

[13] M. Geiben, D. Kröner, and M. Rokyta, *A Lax–Wendroff type theorem for cell-centered, finite volume schemes in 2-D*, Preprint 278, Sonderforschungsbereich 256, Universität Bonn, 1993.

[14] W. Hackbusch, *On first and second order box schemes*, Computing, 41 (1989), pp. 277–296.

[15] B. Heinrich, *Finite difference methods on irregular networks*, International Series of Numerical Mathematics, Birkhaüser, Basel, 1987.

[16] C. Johnson, *Adaptive finite element methods for diffusion and convection problems*, Comput. Methods Appl. Mech. Engrg., 82 (1990), pp. 301–322.

[17] C. Johnson and J. Pitkäranta, *An analysis of the discontinuous Galerkin method for a scalar hyperbolic equation*, Math. Comp., 46 (1986), pp. 1–26.

[18] C. Johnson and J. Saranen, *Streamline diffusion methods for the incompressible Euler and Navier–Stokes equations*, Math. Comp., 47 (1986), pp. 1–18.

[19] C. Johnson, A. Szepessy, and P. Hansbo, *On the convergence of shock-capturing streamline diffusion finite element methods for hyperbolic conservation laws*, Math. Comp., 54 (1990), pp. 107–129.

[20] D. Kröner, *Directionally adapted upwind schemes in 2-D for the Euler equations*, Finite Approximations in Fluid Mechanics II. DFG Priority Research Program Results 1986–1988, Notes on Numerical Fluid Mechanics, E. H. Hirschel, ed., Vol. 25, 1989.

[21] S. N. Kruzkov, *First order quasilinear equations in several independent variables*, Mat. Sb., 81 (1970), pp. 228–255; Math. USSR-SB, 10 (1970), pp. 217–243.

[22] P. Lax and B. Wendroff, *Systems of conservation laws*, Comm. Pure Appl. Math., 13 (1960), pp. 217–237.

[23] R. J. LeVeque, *High resolution finite volume methods on arbitrary grids via wave propagation*, J. Comp. Phys., 78 (1988), pp. 36–63.

[24] K. W. Morton and M. F. Paisley, *A finite volume scheme with shock fitting for the Euler

equation, J. Comp. Phys., 80 (1989), pp. 168–203.

[25] K. G. Powell and B. van Leer, *A genuinely multi-dimensional upwind cell-vertex scheme for the Euler equations*, AIAA 89-0095, Reno, Nevada, January 9–12, 1989.

[26] T. H. Pulliam and D. S. Chaussee, *A diagonal form of an implicit approximate-factorization algorithm*, J. Comp. Phys., 39 (1981), pp. 347–363.

[27] J. Rauch, *BV estimates fail for most quasilinear hyperbolic systems in dimension greater than one*, Comm. Math. Phys., 106 (1986), pp. 481–484.

[28] P. L. Roe, *Linear advection schemes on triangular meshes*, Cranfield Inst. of Technology, CoA Report No. 8720, Cranfield, Bedford, UK, 1987.

[29] J. Smoller, *Shock Waves and Reaction-Diffusion Equations*, Springer-Verlag, New York, 1983.

[30] J. L. Steger and R. F. Warming, *Flux vector splitting of the inviscid gasdynamic equations with application to finite difference methods*, J. Comput. Phys., 40 (1981), pp. 263–293.

[31] G. Strang, *On the construction and comparison of difference schemes*, SIAM J. Numer. Anal., 5 (1968), pp. 506–517.

[32] A. Szepessy, *Convergence of a shock capturing streamline diffusion finite element method for a scalar conservation law in two space dimensions*, Math. Comp., 53 (1989), pp. 527–545.

[33] E. Tadmor, *Numerical viscosity and the entropy condition for conservative difference schemes*, Math. Comp., 43 (1984), pp. 369–381.

[34] L. Tartar, *The compensated compactness method applied to systems of conservation laws*, in Systems of Nonlinear PDE, J.M. Ball, ed., NATO ASI Series, 1983, pp. 263–285.

[35] G. Vijayasundaram, *Transonic flow simulations using an upstream centered scheme of Godunov in finite elements*, J. Comp. Phys., 63 (1986), pp. 416–433.