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QUASIMONOTONE SCHEMES FOR SCALAR CONSERVATION LAWS. PART I*

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This paper is dedicated to Jim Douglas, Jr., on the occasion of his 60*th birthday.*

Abstract. In this work, the quasimonotone schemes for scalar conservation laws are introduced. These new schemes share with monotone schemes both maximum principles and convergence to the entropy solution. However, they are not necessarily first-order accurate. They include both finite-difference schemes (that are total variation diminishing (TVD)) and finite-element ones (that are not TVD), they can be either explicit or implicit; and they can be used with time-dependent grids. They can be defined for d-space variables in the case of a grid that is a Cartesian product of one-dimensional partitions. Error estimates are provided. Part I covers the quasimonotone finite-difference schemes in one dimension. Part II is devoted to quasimonotone finite-element schemes, also in one dimension. Finally, Part III discusses the general case.

Key words. conservation laws, entropy schemes

AMS(MOS) subject classifications. 65M60, 65N30, 35L65

1. Introduction.

In this work, quasimonotone (QM) numerical schemes for the scalar conservation law [12], [14], [26]

$$\partial_t u + \operatorname{div} \mathbf{f}(u) = 0 \qquad \text{in } (0,T)\times\mathbb{R}^d, \tag{1.1a}$$

$$u(t=0) = u_0 \qquad \text{on } \mathbb{R}^d, \tag{1.1b}$$

where $\mathbf{f}$ is a C^1 function, are introduced and studied. These schemes are obtained by "combining" in a systematic way an arbitrary high-order accurate scheme with an arbitrary three-point monotone scheme, and they share with monotone schemes L^∞ and total-variation-stability properties. More importantly, these schemes are entropy schemes, i.e., they always converge to the entropy solution of (1.1), the only physically relevant weak solution. They behave like monotone schemes near discontinuities and like high-order accurate ones away from them. Thus, they can be considered as high-order accurate versions of monotone schemes. Some of their main features are:

(i) The formal local order of accuracy of a QM scheme can be made arbitrarily high in smooth monotone regions of the entropy solution. It can also vary in space without altering stability properties. In particular, the Courant–Friedrichs–Lewy (CFL) condition for explicit QM schemes is independent of the formal local accuracy;
(ii) QM schemes can be either explicit or implicit;
(iii) QM schemes include both finite-difference (FD) and finite-element (FE) schemes;
(iv) They can be used with time-varying meshes;
(v) They can be defined for d-space variables.

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Quasimonotone schemes will be obtained as a further development of the ideas contained in Book and Boris [1], LeRoux [16], and Cockburn [5]. In [1] the Flux-Corrected Transport (FTC) technique is introduced to ensure spatial monotonicity. At each time step, FTC schemes consist of a prediction phase of transport followed by diffusion, and a correction phase of antidiffusion. In the correction phase, an adequately chosen part of the viscosity introduced in the first phase is removed. See also [2], [3]. In [16], the antidiffusion technique is used to modify the Godunov scheme by that of Lax–Wendroff away from regions in which the approximate solution loses its spatial monotonicity. This procedure amounts to subtracting part of the difference of viscosities produced by the Lax–Wendroff and Godunov fluxes from the viscosity produced by the Godunov flux. The scheme that results is a single-phase entropy scheme that is close to having second-order accuracy. When this technique is used to modify an arbitrary monotone scheme by an arbitrary high-order one, a quasimonotone finite-difference (QMFD) scheme is obtained. By following the simple idea that an FE scheme for which the approximate solution is piecewise constant can also be considered an FD scheme, we used this procedure to define a family of FE entropy schemes called the $G - k/2$ schemes [5]. A generalization of this procedure allows us to define and analyze the quasimonotone finite-element (QMFE) schemes.

Finite-difference monotone schemes form the most important class of schemes for the scalar conservation law (1.1). Stability properties and convergence to the entropy solution of explicit monotone schemes have been established by Crandall and Majda [7] using uniform grids; see also the work of Harten, Hyman, and Lax [10] for partial results in the one-dimensional case. Sanders [22] generalized these results to the case in which the grid is a Cartesian product of nonuniform one-dimensional partitions, and the schemes are either explicit or implicit three-point schemes. He also obtained an $L^\infty(L^1)$-error estimate, extending the work of Kuznetsov [13]. Error estimates have also been obtained by Lucier [17]–[19]. As the theory of monotone three-point schemes is fairly complete, only this type of monotone scheme will be taken into consideration to construct our QM schemes. Quasimonotone schemes produce much less viscosity globally than monotone schemes do. However, their stability properties, as well as the fact that they are entropy schemes, are essentially a consequence of the corresponding properties of monotone schemes. Note that this is true for QMFD and QMFE schemes. It is very important, nevertheless, to point out that there is a deep qualitative difference between QMFE schemes: QMFE schemes are total variation diminishing (TVD), whereas FEQM schemes are not in general (they are only TVDM, i.e., TVD "in the means"; see Part II). This suggests that with QMFE schemes the shocks could be approximated in a better way. Moreover, since QMFE schemes are not TVD, better accuracy possibly can be obtained since in one-space dimension a three-point TVD scheme is at most first-order accurate [25], and in two-space dimensions any TVD scheme, in particular the QMFD schemes, are at most first-order accurate [9].

Quasimonotone finite-difference schemes can be considered schemes using so-called flux limiters. Flux limiters have been used by Van Leer [24], Osher and Chakravarthy [21], and Harten [11], among others. An important class of such schemes has been analyzed by Sweby [23]. Also, Vila [25] has constructed a family of entropy schemes of quasi-second-order accuracy following ideas similar to those that led to our QM schemes. His schemes are two-phase entropy finite-difference schemes that apply to systems of conservation laws in one space dimension. The second phase of these schemes implements the FTC technique. This makes the construction of his schemes distinctly different in nature and philosophy from the construction of QM schemes. Although some improvements have been made for QMFD schemes and error estimates have been obtained, the main contribution of this work is the introduction of the QMFE schemes and the technique to analyze them.

In §2 we study the explicit QMFD schemes. Section 3 is devoted to the implicit QMFD schemes. In §4 we give a proof of one of our main results, Theorem 2.2. We end with some concluding remarks in §5.

2. Explicit QMFD schemes.

2.1. The key idea. In this section we introduce and analyze the explicit quasi-monotone finite-difference schemes. The standard finite-difference notations are used; we set $\delta = \sup_{n,i}\{\Delta t^n, \Delta x_i\}$, $\lambda = \sup_{n,i} \Delta t^n/\Delta x_i$, and CFL $= \lambda\|f'\|_{L^\infty(a,b)}$, where $a = \inf_x u_0(x)$ and b$= \sup_x u_0(x)$. Finally, we denote by $I(u_1, \cdots, u_k)$ the interval $[\min\{u_1, \cdots, u_k\}, \max\{u_1, \cdots, u_k\}]$.

We consider FD schemes of the form

$$(u_i^{n+1} - u_i^n)/\Delta t^n + (f_{i+1/2}^{h,n} - f_{i-1/2}^{h,n})/\Delta x_i = 0, \tag{2.1}$$

where $f_{i+1/2}^{h,n} = f^h(u_{i-k}^n, \cdots, u_{i+m}^n)$ is a consistent numerical flux, i.e., $f^h(u, \cdots, u) = f(u)$. The scheme

$$(u_i^{n+1} - u_i^n)/\Delta t^n + (f_{i+1/2}^{\mathrm{QM},n} - f_{i-1/2}^{\mathrm{QM},n})/\Delta x_i = 0 \tag{2.2}$$

is an QMFD scheme if the flux f^{QM} is a QM flux. Such a flux is constructed by combining in a suitable way a two-point monotone flux f^{M} [7] and an arbitrary flux f^h for which the scheme (2.1) is formally "high-order accurate". This is done in such a way that: (i) the flux f^{QM} becomes equal to f^h in smooth monotone regions of the entropy solution, thus recovering the formal high-order accuracy of scheme (2.1); and (ii) the resulting scheme (2.2) can be rewritten as

$$(u_i^{n+1} - u_i^n)/\Delta t^n + \Theta_i^n \cdot (f_{i+1/2}^{\mathrm{M},n} - f_{i-1/2}^{\mathrm{M},n})/\Delta x_i = 0, \tag{2.3}$$

where $\Theta_i^n \in [0, 2]$. It is very well known that if $\Theta_i^n \equiv 1$, then (2.3) is a monotone scheme for λ small enough, say $\lambda \leq \lambda_0$. Moreover, the stability of such a scheme depends *only* on the size of λ. Thus, similar stability results can be obtained for the QMFD scheme (2.3) for $\lambda \leq \lambda_0/2$. This is the key idea of the construction of the QM schemes.

After giving the precise definition of a QM flux, we analyze the stability and convergence properties of QMFD schemes. Then, we extend this construction to three time-level explicit schemes by considering QM versions of the well-known leap-frog scheme. Some numerical results are displayed.

2.2. Definition of QM fluxes. A flux f^{QM} is a QM numerical flux if

$$f^{\mathrm{QM}} = f^{\mathrm{M}} + a, \tag{2.4}$$

where

(2.5a) (Stability) There exist two discrete functions ν^- and ν^+ such that

$$\text{(i)} \qquad a_{i+1/2} = \nu_{i+1/2}^+(f_{i+3/2}^{\mathrm{M}} - f_{i+1/2}^{\mathrm{M}}) = \nu_{i+1/2}^-(f_{i+1/2}^{\mathrm{M}} - f_{i-1/2}^{\mathrm{M}}),$$

$$\text{(ii)} \qquad (1 + \nu_{i+1/2}^- - \nu_{i-1/2}^+) \in [0, 2],$$

(2.5b) (Entropy) $a = O((\Delta x)^\alpha)$ for some $\alpha \in (0, 1]$.

As will be proved later, the condition (2.5a) implies that the QMFD (2.3) scheme is TVD, and so it converges to a weak solution of (1.1) [11]. The condition (2.5b) ensures the convergence to the entropy solution.

Note that the viscosity [15], [16], [4] produced by the QM flux is given by

$$\begin{aligned} v^{\mathrm{QM}}_{i+1/2} &= (f(u_{i+1}) - 2f^{\mathrm{QM}}_{i+1/2} + f(u_i))/(u_{i+1} - u_i) \\ &= v^{M}_{i+1/2} - 2\, a_{i+1/2}/(u_{i+1} - u_i). \end{aligned}$$

It is then clear that the role of a is to control the amount of viscosity that is added or subtracted from the viscosity already produced by the monotone numerical flux. Thanks to conditions (2.5), near discontinuities it will make the viscosity v^{QM} large enough to prevent oscillations and to smooth out nonentropy shocks.

Note that if $\mathrm{sgn}(a_{i+1/2}) = -\mathrm{sgn}(u_{i+1} - u_i)$, then $v^{\mathrm{QM}} \geq v^M$ and the QM scheme is at most first-order accurate. So, if the QM scheme is required to be locally at least second-order accurate, we must have $\mathrm{sgn}(a_{i+1/2}) = \mathrm{sgn}(u_{i+1} - u_i)$. One way to recover locally the order of accuracy of the scheme associated to the flux f^h is to have $f^{\mathrm{QM}} = f^h$ in regions in which the entropy solution is "smooth" and monotone. As this is equivalent to having $a = f^h - f^{\mathrm{M}}$, the equality

$$\mathrm{sgn}(u_{i+1} - u_i) = \mathrm{sgn}(f^h_{i+1/2} - f^{\mathrm{M}}_{i+1/2})$$

must then be satisfied in these regions. We will show in §2.4 that this is indeed possible to achieve.

2.3. Some choices of a. We present in this subsection two choices of a. The first one is the following:

$$a_{i+1/2} = \mathrm{sgn}(u_{i+1} - u_i)\max\{0, \Theta_{i+1/2}\}, \tag{2.6a}$$

$$\begin{aligned} \Theta_{i+1/2} = \min\{&|f^h_{i+1/2} - f^{\mathrm{M}}_{i+1/2}|, \\ &|f^{\mathrm{M}}_{i+3/2} - f^{\mathrm{M}}_{i+1/2}| s_{i+1}, c_{i+1}(\Delta x_{i+1})^\alpha, \\ &|f^{\mathrm{M}}_{i-1/2} - f^{\mathrm{M}}_{i+1/2}| s_i, c_i(\Delta x_i)^\alpha\}, \end{aligned} \tag{2.6b}$$

$$s_i = \mathrm{sgn}((u_{i+1} - u_i)(u_i - u_{i-1})), \tag{2.6c}$$

$$c_i \in [0, K] \text{ for some fixed } K \in \mathbb{R}^+. \tag{2.6d}$$

Let us show that a defined by (2.6) verifies conditions (2.5). First, note that both $|\nu^+_{i+1/2}|$ and $|\nu^-_{i+1/2}| \in [0, 1]$ and that $\nu^-_{i+1/2}$ and $\nu^+_{i-1/2}$ have the same sign; this is due to (2.6c). This implies that the stability condition (2.5a) is verified. The entropy condition (2.5b) is verified trivially. Note also that, when u_h has a local extremum, we have $a = 0$ and so, the scheme is locally of order one.

A second choice of a is the following:

$$a_{i+1/2} = \mathrm{sgn}(f^h_{i+1/2} - f^{\mathrm{M}}_{i+1/2})\Theta_{i+1/2}, \tag{2.7a}$$

$$\begin{aligned} \Theta_{i+1/2} = \min\{&|f^h_{i+1/2} - f^{\mathrm{M}}_{i+1/2}|, \\ &0.5|f^{\mathrm{M}}_{i+3/2} - f^{\mathrm{M}}_{i+1/2}|, c_{i+1}(\Delta x_{i+1})^\alpha, \\ &0.5|f^{\mathrm{M}}_{i-1/2} - f^{\mathrm{M}}_{i+1/2}|, c_i(\Delta x_i)^\alpha\}, \end{aligned} \tag{2.7b}$$

$$c_i \in [0, K] \text{ for some fixed } K \in \mathbb{R}^+. \tag{2.7c}$$

The analysis of (2.7) is analogous to that of (2.6). Note that in this case the sign of $\nu^-_{i+1/2}$ and the one of $\nu^+_{i-1/2}$ are not necessarily equal, this is why we must have $|\nu^+_{i+1/2}|$, $|\nu^-_{i+1/2}| \in [0,1/2]$. Also, $a_{i+1/2}$ is not necessarily equal to zero when either u_{i+1} or u_i is a local extremum of u_h, as in the preceding case.

2.4. Recovering formal high-order accuracy. For both the two preceding choices of a it is possible to have $a = f^h - f^{\mathrm{M}}$ formally in smooth monotone regions of the solution. Let us show this via a simple example. Assume, for simplicity that $\Delta x_i \equiv \Delta x$, and that $\Delta t^n \equiv \Delta t$. Take as f^{M} the Godunov flux f^{G} ([15], [20], [4]), and as f^h the Lax–Wendroff flux

$$f^{\mathrm{LW}}_{i+1/2} = (f(u_{i+1}) + f(u_i))/2 - \lambda f'_{i+1/2}(f(u_{i+1}) - f(u_i))/2,$$

where $f'_{i+1/2} = f'((u_{i+1} + u_i)/2)$. Since

$$f^{\mathrm{LW}}_{i+1/2} - f^{\mathrm{G}}_{i+1/2} = 0.5[v^{\mathrm{G}}_{i+1/2} - \lambda f'_{i+1/2}(f(u_{i+1}) - f(u_i))/(u_{i+1} - u_i)](u_{i+1} - u_i),$$

it is clear that for CFL in the interval $[0,1]$ the expression in brackets is positive and $\mathrm{sgn}(f^{\mathrm{LW}}_{i+1/2} - f^{\mathrm{G}}_{i+1/2}) = \mathrm{sgn}(u_{i+1} - u_i)$, as expected. This is due to the fact that

$$v^{\mathrm{G}}_{i+1/2} \geq |(f(u_{i+1}) - f(u_i))/(u_{i+1} - u_i)|.$$

The viscosity of any other two-point monotone flux is not smaller than v^{G}, so this result is valid in general. Moreover, if the solution is smooth, we obtain formally that

$$\begin{aligned}
|f^{\mathrm{LW}}_{i+1/2} - f^{\mathrm{G}}_{i+1/2}| &= 0.5[1 - \lambda|f'(u(x))|]|\partial_x f(u(x))| + O(\Delta^2 x),\\
|f^{\mathrm{G}}_{i+3/2} - f^{\mathrm{G}}_{i+1/2}| &= |\partial_x f(u(x))|\Delta x + O(\Delta^2 x),\\
|f^{\mathrm{G}}_{i+1/2} - f^{\mathrm{G}}_{i-1/2}| &= |\partial_x f(u(x))|\Delta x + O(\Delta^2 x),
\end{aligned}$$

where $x - x_{i+1/2} = O(\Delta x)$. Thus, if $\partial_x f(u(x)) \neq 0$ and CFL < 1, it is clear that if a is chosen as in (2.6) or (2.7) we have $a = f^{\mathrm{LW}} - f^{\mathrm{G}}$, as claimed. Notice that only the two first terms of the Taylor expansion of the fluxes f^{LW} and f^{G} are involved in this argument. This result is then true for fluxes f^h, f^{M} whose Taylor expansion coincide, up to the second term, with those of the Lax–Wendroff and the Godunov fluxes, respectively.

2.5. Stability and convergence analysis. In what follows the function u_h denotes the following piecewise constant extension of the grid function $\{u^n_i\}$:

$$u_h(t,x) = u^n_i \quad \text{for } (t,x) \in [t^n, t^{n+1}) \times (x_{i-1/2}, x_{i+1/2}).$$

Let us begin by proving the following stability result.

LEMMA 2.1. *Suppose that the scheme*

$$(u^{n+1}_i - u^n_i)/\Delta t^n + (f^{\mathrm{M},n}_{i+1/2} - f^{\mathrm{M},n}_{i-1/2})/\Delta x_i = 0,$$

is monotone for CFL $\in [0, \mathrm{CFL}_0]$. *Then, if* CFL $\in [0, \mathrm{CFL}_0/2]$, *any* QM *scheme* (2.3) *verifies the maximum principle*

$$u^{n+1}_i \in I(u^n_{i-1}, u^n_i, u^n_{i+1}), \tag{2.8a}$$

and is TVD, *i.e.,*

$$\|u^{n+1}_h\|_{\mathrm{TV}} = \sum_i |u^{n+1}_{i+1} - u^{n+1}_i| \leq \|u^n_h\|_{\mathrm{TV}}. \tag{2.8b}$$

Proof. Consider the scheme

$$u_i^{n+1} = u_i^n - \lambda_i^n (f_{i+1/2}^{\mathrm{M},n} - f_{i-1/2}^{\mathrm{M},n}),$$

where $\lambda_i^n = \Delta t^n / \Delta x_i$. By hypothesis, this scheme is monotone under the CFL condition

$$(*) \qquad \lambda_i^n \|f'\|_{L^\infty(a,b)} \in [0, \mathrm{CFL}_0].$$

It is well known that the scheme verifies (2.8) and is TVD [22]. This result depends only on condition (*). Now, use (2.5) to rewrite the QM scheme (2.3)

$$u_i^{n+1} = u_i^n - \lambda_i^n (f_{i+1/2}^{\mathrm{QM},n} - f_{i-1/2}^{\mathrm{QM},n})$$

in the following way:

$$u_i^{n+1} = u_i^n - \Lambda_i^n (f_{i+1/2}^{\mathrm{M},n} - f_{i-1/2}^{\mathrm{M},n}),$$

where $\Lambda_i^n = \lambda_i^n (1 + \nu_{i+1/2}^{-,n} - \nu_{i-1/2}^{+,n})$. It is clear that if

$$\Lambda_i^n \|f'\|_{L^\infty(a,b)} \in [0, \mathrm{CFL}_0],$$

then the QM scheme verifies (2.8) and is TVD. But by (2.5a) this condition is achieved if in (*) CFL_0 is replaced by $\mathrm{CFL}_0/2$. □

Lemma 2.1 implies that there exists a subsequence $\{u_{h'}\}$ that converges in $L^\infty(0,T;L^1(\mathbb{R}))$ to a weak solution of (1.1). Next we show that this weak solution is in fact the entropy solution. Let us denote by J an even nonnegative convex function such that J'' is Lipschitz and $J''(\omega) = 0$ for $|\omega| \geq c\delta^{\alpha/2}$, and by Ω an arbitrary compact subset of $\mathbb{R}$. With this notation we have the following result.

THEOREM 2.2. *Suppose that the scheme*

$$(u_i^{n+1} - u_i^n)/\Delta t^n + (f_{i+1/2}^{\mathrm{M},n} - f_{i-1/2}^{\mathrm{M},n})/\Delta x_i = 0$$

is a monotone scheme for CFL *in* $[0, \mathrm{CFL}_0]$. *Then, if* CFL *is in* $[0, \mathrm{CFL}_0/2]$, *any explicit* QM *scheme is an entropy scheme. Moreover,*

$$(2.9\text{a}) \qquad \int_{\mathbb{R}} J(u(T) - u_h(T)) \leq \int_{\mathbb{R}} J(u_0 - u_{0,h}) + \|u_0\|_{\mathrm{BV}(\mathbb{R})} (C_1 T^{1/2} \delta^{1/2} + C_2 T \delta^\alpha),$$

and

$$(2.9\text{b}) \qquad \begin{aligned} \|u(T) - u_h(T)\|_{L^1(\Omega)} \leq{} & 2\|u_0 - u_{0,h}\|_{L^1(\mathbb{R})} + C_3 T^{1/2} \|u_0\|_{\mathrm{BV}(\mathbb{R})} \delta^{1/2} \\ & + C_4 T^{1/2} (\|u_0\|_{\mathrm{BV}(\mathbb{R})} |\Omega|)^{1/2} \delta^{\alpha/2}. \end{aligned}$$

A proof will be given in §5.

2.6. Quasimonotone versions of the leap-frog scheme. The leap-frog scheme

$$(u_i^{n+1} - u_i^{n-1})/2\Delta t + (f_{i+1/2}^{\mathrm{LF},n} + f_{i-1/2}^{\mathrm{LF},n})/\Delta x = 0,$$

where $f_{i+1/2}^{\mathrm{LF}} = (f(u_{i+1}) + f(u_i))/2$, is a well-known, formally second-order accurate scheme. In the linear case a necessary condition for its L^2-stability is that $\mathrm{CFL} \in [0,1]$. This condition is also sufficient if the way of choosing $u_h(\Delta t)$ is adequate. In the nonlinear case, convergence to the entropy solution is not guaranteed.

Even if this scheme is not of the form (2.1) we can use it to construct QM schemes. We present two ways of doing this.

(a) Set $w_i^{n+1} = (u_i^{n+1} + u_i^n)/2$, and $p_i^n = u_i^n$. Now, rewrite the leap-frog scheme in terms of the new variables

$$(w_i^{n+1} - w_i^n)/\Delta t + (\tilde{f}_{i+1/2}^{\mathrm{LF},n} - \tilde{f}_{i-1/2}^{\mathrm{LF},n})/\Delta x = 0,$$
$$p_i^{n+1} = 2w_i^{n+1} - p_i^n,$$

where $\tilde{f}_{i+1/2}^{\mathrm{LF},n} = (f(p_{i+1}^n) + f(p_i^n))/2$. If we think of w_h as the approximate solution, and of $\{p_i^n\}$ as parameters updated at each time step, we can view the preceding scheme as a scheme of the form (2.1). Now we can define QM fluxes as indicated in (2.6) or (2.7) with $f^h = \tilde{f}^{\mathrm{LF}}$ in order to obtain the QM scheme

$$(w_i^{n+1} - w_i^n)/\Delta t + (f_{i+1/2}^{\mathrm{QMLF},n} - f_{i-1/2}^{\mathrm{QMLF},n})/\Delta x = 0, \tag{2.10a}$$
$$p_i^{n+1} = 2w_i^{n+1} - p_i^n. \tag{2.10b}$$

We start the computations by taking $f^{\mathrm{QMLF},0} = f^{\mathrm{M},0}$, and $p_i^0 = w_i^0$.

(b) We also can simply take a QM flux constructed as in (2.6) or (2.7), where the flux $f^h = f^{\mathrm{LF}}(u_h^n)$ is evaluted at time $t = t^n$, and the monotone flux $f^{\mathrm{M}} = f^{\mathrm{M}}(u^{n-1})$ is evaluated at the previous time step. With this flux we construct the scheme

$$(u_i^{n+1} - u_i^{n-1})/2\Delta t + (f_{i+1/2}^{\mathrm{QMLF},n-1} + f_{i-1/2}^{\mathrm{QMLF},n-1})/\Delta x = 0, \tag{2.11}$$

which we also consider to be a QM scheme. The function u_h^1 is computed using the corresponding monotone scheme.

Lemma 2.1 and Theorem 2.2 apply directly to scheme (2.10) with u_h replaced by w_h. Note that these results do not tell us anything about the "parameters" $\{p_i^n\}$. Lemma 2.1 and Theorem 2.2 are also valid for the scheme (2.11) if in (2.8) we replace u^n by u^{n-1}. Note that the CFL condition for the QM scheme (2.10) is twice less restrictive than the one for the QM scheme (2.11) when the same monotone fluxes are used.

Extensions of the procedures described above can be easily obtained for multi-time-level numerical schemes; see [6] for some examples. We shall not develop this point further in this paper.

2.7. Some numerical results. We tested numerically the QM leap-frog schemes (2.10) and (2.11) on the problems (1.1) described in Table 1. The entropy solution of the first three problems does have discontinuities, so these problems are used to test the convergence to the entropy solution of the schemes; the last three problems have a very smooth solution, and are used to test if the QM schemes can indeed recover the second-order formal accuracy of the leap-frog scheme. Also, note that the function f in problems 1 and 4 is linear, in problems 2 and 5 is strictly convex, and in problems 3 and 6 is nonconvex. The f^{QMLF} flux was constructed as indicated above with the Godunov flux. We report the best results, which were obtained for the scheme (2.11) with the choice (2.6) for a with $s_i \equiv 1$, $c_i \equiv \infty$, and $\alpha = 0$. Although this choice is a crime from the point of view of the theory, it is perfectly reasonable from the practical point of view.

The numerical results are shown in Figs. 1 and 2 and in Table 2. Table 2 contains the $L^1(\Omega')$ and $L^\infty(\Omega')$ errors and their respective orders of convergence. The set Ω' is defined as follows: $\Omega' = \Omega$, given in Table 1, for problems 1, 4, 5, and 6. For problem 2 we take $\Omega' = [0, 0.05] \cup [0.2, 1]$, because there is a shock near $x = 0.14$. For

TABLE 1
The test problems.

Problem	Ω	T	$f(u)$	$u_0(x)$
1	(0,1)	0.50	u	$\begin{cases} 1 & \text{if } x \in (0.4, 0.6), \\ 0 & \text{otherwise.} \end{cases}$
2	(0,1)	0.55	$u^2/2$	$\frac{1}{2}(\frac{1}{2} + \sin(2\pi x))$
3	(0,2)	0.50	$\frac{1}{2}\frac{u^2}{u^2+(1-u)^2}$	$\begin{cases} 1 & \text{if } x \in (0.5, 1.5), \\ 0 & \text{otherwise.} \end{cases}$
4	(0,1)	0.15	u	$\frac{1}{2}(1 + \frac{1}{2}\sin(4\pi x))$
5	(0,1)	0.15	$u^2/2$	$\frac{1}{2}(\frac{1}{2} + \sin(2\pi x))$
6	(0,1)	0.10	$\frac{1}{2}\frac{u^2}{u^2+(1-u)^2}$	$\frac{1}{2}(1 + \frac{1}{2}\sin(4\pi x))$

problem 3 we take $\Omega' = [.6, .7] \cup [1.6, 1.7]$, because this is a region in which the entropy solution, which has two shocks near $x = 0.8$ and $x = 1.8$, is smooth and not constant. Very good results are obtained.

(1) Second-order accuracy almost everywhere is achieved for the problems in which the solution is smooth;

(2) Convergence to the entropy solution is obtained even in the nonconvex case;

(3) Second-order accuracy in L^1 and 1.5-order accuracy in L^∞ is achieved for the Burgers test problem 2 away from discontinuities;

(4) First-order accuracy away from discontinuities is obtained in problem 3 in which f is nonconvex. This is the maximum attainable accuracy since the initial condition is discontinuous;

(5) The discontinuities are very well captured.

3. Implicit QM schemes.

3.1. Implicit QM schemes of first type. When dealing with implicit schemes, it is useful to generalize the definition of a QM flux as follows: we say that the flux f_s^{QM} is a QM flux if it is of the form

$$f_s^{\mathrm{QM},n} = s f^{\mathrm{QM},n} + (1-s) f^{\mathrm{QM},n-1}, \quad s \in (0,1], \tag{3.1a}$$

where f^{QM} is a QM flux defined by (2.4)–(2.5). A scheme

$$(u_i^n - u_i^{n-1})/\Delta t^n + (f_{s,i+1/2}^{\mathrm{QM},n} - f_{s,i-1/2}^{\mathrm{QM},n})/\Delta x_i = 0 \tag{3.1b}$$

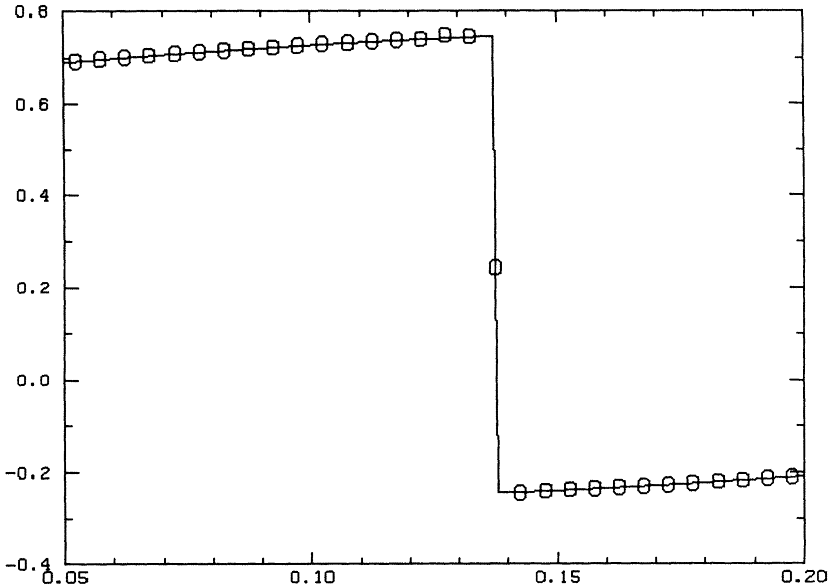

FIG. 1. *Detail of the approximation of the shock for the Burgers test problem* 2*:* CFL $= 1/4$ *and* $\Delta x = 1/200$.

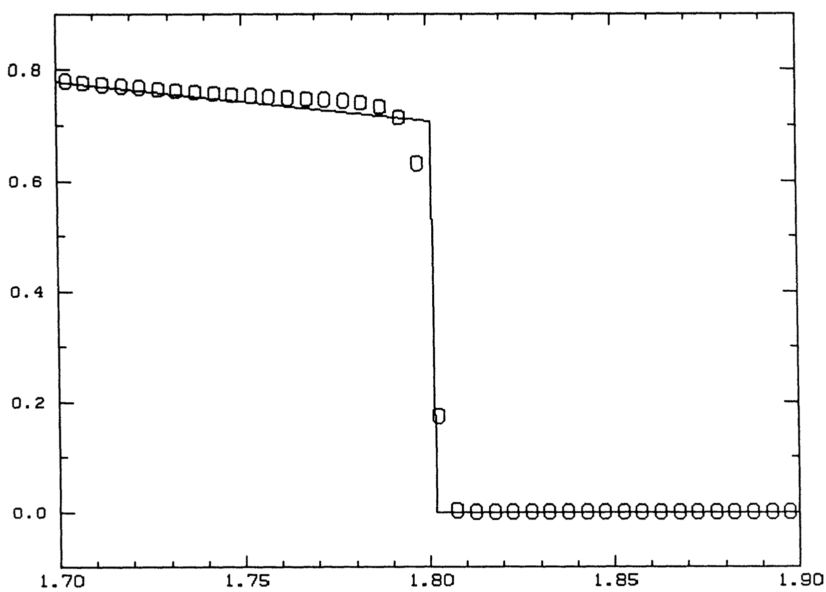

FIG. 2. *Detail of the approximation of the shock for the Buckley–Leverett test problem* 3*:* CFL $=$ $1/4$ *and* $\Delta x = 1/200$.

TABLE 2
$L^1(\Omega')$ and $L^\infty(\Omega')$ errors and their respective orders of convergence: CFL $= 1/4$ *and* $\Delta x = 1/200$.

Problem	$10^4 \cdot L^1$ – error	α_1	$10^4 \cdot L^\infty$ – error	α_∞
1	229.50	0.661	-	-
2	0.65	2.105	11.10	1.497
3	1.73	0.932	22.52	0.781
4	4.11	1.873	19.21	1.115
5	1.07	1.967	9.34	1.089
6	5.63	1.841	64.13	1.120

is called an implicit QMFD scheme of first type. The following result is an extension of Lemma 2.1 and Theorem 2.2.

THEOREM 3.1. *Suppose that the scheme*

$$(u_i^{n+1} - u_i^n)/\Delta t^n + (f_{i+1/2}^{\mathrm{M},n} - f_{i-1/2}^{\mathrm{M},n})/\Delta x_i = 0,$$

is monotone for CFL $\in [0, \mathrm{CFL}_0]$. *Then, if* CFL $\in [0, \mathrm{CFL}_0/2(1-s)]$, *any implicit* QMFD *scheme of first type* (3.1) *is* TVD, *and verifies the maximum principle*

$$\begin{aligned} &u_i^n \in I(u_j^n, u_j^{n-1}, u_{j+1}^{n-1}, \ldots, u_{k-1}^{n-1}, u_k^{n-1}, u_k^n), \quad j < i < k, \quad s \in (0,1), \\ &u_i^n \in I(u_j^n, u_{j+1}^{n-1}, \ldots, u_{k-1}^{n-1}, u_k^n), \quad j < i < k, \quad s = 1. \end{aligned} \tag{3.2}$$

Moreover, the approximate solution verifies the estimates (2.9).

This result is a particular case of Proposition 2.1 in Part III of this paper. Note that for $s = 1$ no bound on the CFL number is necessary in order to achieve convergence. However, if this number is too big, only first-order accuracy may be achieved. To illustrate this point, let us assume that $\Delta t^n \equiv \Delta t$ and that $\Delta x_i \equiv \Delta x$. Consider the implicit QMFD scheme (3.1) for which the QM flux f_s^{QM}, with $s = 1$, is constructed as indicated in (2.6) or (2.7) with the Godunov flux, f^G, as monotone flux and the Lax–Wendroff implicit flux,

$$f_{i+1/2}^{\mathrm{LW}} = (f(u_{i+1}) + f(u_i))/2 + \lambda\, f'_{i+1/2}(f(u_{i+1}) - f(u_i))/2,$$

where $f'_{i+1/2} = f'((u_{i+1} + u_i)/2)$, as the flux f^h (Theorem 3.1 holds in this case with $\mathrm{CFL}_0 = \infty$). Proceeding as in §2.4 we obtain

$$\begin{aligned} &|f_{i+1/2}^{\mathrm{LW}} - f_{i+1/2}^{\mathrm{G}}| = 0.5(1 + \lambda|f'(u(x))|)|\partial_x f(u(x))|\Delta x + O(\Delta^2 x), \\ &|f_{i+3/2}^{\mathrm{G}} - f_{i+1/2}^{\mathrm{G}}| = |\partial_x f(u(x))|\Delta x + O(\Delta^2 x), \\ &|f_{i+1/2}^{\mathrm{G}} - f_{i-1/2}^{\mathrm{G}}| = |\partial_x f(u(x))|\Delta x + O(\Delta^2 x). \end{aligned}$$

It is then clear that second-order or higher-order accuracy can be achieved only if CFL < 1 for the choice (2.6). If the choice (2.7) is taken, only first-order accuracy can be obtained! We overcome this difficulty by introducing the QMFD schemes of second type.

3.2. Implicit QM schemes of second type. A scheme of the form

$$(w_i^{\mathrm{QM},n} - w_i^{\mathrm{QM},n-1})/\Delta t^n + (f_{s,i+1/2}^{\mathrm{QM},n} - f_{s,i-1/2}^{\mathrm{QM},n})/\Delta x_j = 0, \tag{3.3}$$

where $s \in (0,1]$, and w^{QM} and f_s^{QM} are QM fluxes, is called an implicit QM scheme of the second type. A time-numerical flux is a QM flux if

$$w^{\mathrm{QM},n} = u^n + b^n, \tag{3.4}$$

where

(3.5a) (Stability) There exist two discrete functions η^- and η^+ such that

$$\begin{aligned}&\text{(i)} \quad b^n = \eta^{+,n}(u^{n+1} - u^n)\\ &\qquad\quad = \eta^{-,n}(u^n - u^{n-1}),\\ &\text{(ii)} \ (1+\eta^{-,n+1} - \eta^{+,n}) \in [c_0, \infty), \qquad c_0 > 0,\end{aligned}$$

(3.5b) (Entropy) $b = O(\Delta t^\alpha)$ for some $\alpha \in (0,1]$.

The role of b is analogous to that of a. The function b controls the amount of viscosity that has to be added to or subtracted from the viscosity already produced by the time flux $w^n = u^n$. Note that the viscosity of $w^{\mathrm{QM},n}$ is

$$v^{\mathrm{QM},n} = v^n - 2b^n/(u^{n+1} - u^n),$$

where $v^n \equiv 1$; see §2.2. If the approximation to the time derivative is to be locally second-order accurate, the sign of b must then be equal to $\mathrm{sgn}(u^{n+1} - u^n)$. Explicit expressions for b can be easily obtained taking choices analogous to those of (2.6), (2.7), i.e.,

$$b^n = \mathrm{sgn}(u^n - u^{n-1}) \max\{0, \Theta^{n+1/2}\}, \tag{3.6a}$$

$$\begin{aligned}\Theta^{n+1/2} = \min\{&|w^{h,n} - u^n|,\\ &(1-c_0)|u^{n+1} - u^n| s^n, c^{n+1}(\Delta t^n)^\alpha,\\ &(1-c_0)|u^n - u^{n-1}| s^{n-1}, c^n(\Delta t^{n-1})^\alpha\},\end{aligned} \tag{3.6b}$$

$$s^n = \mathrm{sgn}((u^{n+1} - u^n)(u^n - u^{n-1})), \tag{3.6c}$$

$$c^n \in [0, K] \text{ for some fixed } K \in \mathbb{R}^+, \tag{3.6d}$$

and

$$b^n = \mathrm{sgn}(w^{h,n} - u^n)\Theta^{n+1/2}, \tag{3.7a}$$

$$\begin{aligned}\Theta^{n+1/2} = \min\{&|w^n - u^n|,\\ &0.5(1-c_0)|u^{n+1} - u^n|, c^{n+1}(\Delta t^n)^\alpha,\\ &0.5(1-c_0)|u^n - u^{n-1}|, c^n(\Delta t^{n-1})^\alpha\},\end{aligned} \tag{3.7b}$$

$$c^n \in [0, K] \text{ for some fixed } K \in \mathbb{R}^+, \tag{3.7c}$$

where $w^{h,n}$ represents the time flux of the high-order accurate scheme. It is useful to think of $w^{h,n}$ as an approximation of $u(t^n - \rho\Delta t^n)$, where ρ is some fixed number usually in $(0,1)$. For these schemes we have the following result.

THEOREM 3.2. *Theorem* 3.1 *is also valid for* QMFD *schemes of second type* (3.2).

The proof of Theorem 3.2 is a straightforward extension of the proof of Theorem 3.1. Let us now show that we can obtain an implicit QMFD second type version of the implicit Lax–Wendroff scheme for which second-order accuracy can be recovered for any CFL. Consider the Lax–Wendroff implicit scheme

$$(v_i^n - v_i^{n-1})/\Delta t + (f_{i+1/2}^{\mathrm{LW},n} - f_{i-1/2}^{\mathrm{LW},n})/\Delta x = 0,$$

where $f^{\mathrm{LW}} = f^{\mathrm{LW}}(v_h)$. Let us introduce the new function u_h: $u^n = 2v^n - u^{n-1}$ or simply $v^n = (u^n + u^{n-1})/2$. Define the time flux $w^{h,n} = (u^n + u^{n-1})/2$. Note that in this case $w^{h,n}$ represents an approximation to $u(t^{n-1/2} = t^n - \Delta t/2)$; thus, we have $\rho = 1/2$. Construct with it a QM time flux $w^{\mathrm{QM},n}$ as indicated above. Finally, construct a QM space flux f^{QM} as in (2.6) or (2.7) with $f^{\mathrm{M},n} = f^{\mathrm{M}}(u_h^n)$ and $f^{h,n} = f^{\mathrm{LW}}(v_h^n = (u_h^n + u_h^{n-1})/2)$. With these fluxes, a QMFD scheme of second type (3.2) can be obtained (Theorem 3.2 is valid for the resulting scheme with $\mathrm{CFL}_0 = \infty$). Using Taylor expansions we obtain

$$\begin{aligned}
|w^{h,n} - u^n| &= 0.5|\partial_t u|\Delta t + O(\Delta t^2),\\
|u^{n+1} - u^n| &= |\partial_t u|\Delta t + O(\Delta t^2),\\
|u^n - u^{n-1}| &= |\partial_t u|\Delta t + O(\Delta t^2),
\end{aligned}$$

and

$$\begin{aligned}
|f_{i+1/2}^{\mathrm{LW},n}(v_h) - f_{i+1/2}^{\mathrm{G},n}(u_h)| &= 0.5|\partial_x f(u(x))|\Delta x + O(\Delta^2 x + \Delta^2 t),\\
|f_{i+3/2}^{\mathrm{G},n} - f_{i+1/2}^{\mathrm{G},n}| &= |\partial_x f(u(x))|\Delta x + O(\Delta^2 x),\\
|f_{i+1/2}^{\mathrm{G},n} - f_{i-1/2}^{\mathrm{G},n}| &= |\partial_x f(u(x))|\Delta x + O(\Delta^2 x).
\end{aligned}$$

We see from the preceding expressions and (2.6)–(2.7) and (3.6)–(3.7) that we can recover the local second-order of the Lax–Wendroff scheme independently of the value of the CFL number.

This seems to be optimal, but, as usual, we have to pay a price. Rewrite the general QMFD scheme of second type (3.2) as follows:

$$(u_i^n + b_i^n - u^{n-1} - b_i^{n-1})/\Delta t^n + (f_{i+1/2}^{\mathrm{QM},n} - f_{i-1/2}^{\mathrm{QM},n})/\Delta x_i = 0.$$

Note that, by (3.4a), the function b^n depends also on u^{n+1}, which will be known only in the next time step. This means that we are forced to use some kind of iterative procedure to compute b^n in the correct way. If we want to avoid this difficulty we could lose either stability or convergence. Consider the scheme

$$(u_i^n + \tilde{b}_i^n - u^{n-1} - b_i^{n-1})/\Delta t^n + (f_{i+1/2}^{\mathrm{QM},n} - f_{i-1/2}^{\mathrm{QM},n})/\Delta x_i = 0, \tag{3.8a}$$

where $\tilde{b}$ is obtained from the corresponding expression of b by simple elimination of the expressions involving u^{n+1}. This modification gives a scheme that is TVD and verifies the maximum principle (2.9). However, it is not conservative, and so it may converge to something that is not the solution of problem (1.1). Thus, we can instead consider the scheme

$$(u_i^n + \tilde{b}_i^n - u^{n-1} - \tilde{b}_i^{n-1})/\Delta t^n + (f_{i+1/2}^{\mathrm{QM},n} - f_{i-1/2}^{\mathrm{QM},n})/\Delta x_i = 0, \tag{3.8b}$$

which is conservative. If this scheme is TVB, it converges to the entropy solution and verifies the error estimates (2.9). However, its stability is not ensured!

3.3. Time-dependent grid QMFD schemes. In this subsection we define a time-dependent grid version of fully implicit (i.e., $s = 1$) QMFD schemes of first type. To do this we follow the ideas introduced in [8]. Let $P = \{x_{i+1/2}\}_{i \in Z}$ denote a partition of $\mathbb{R}$. Set $I_i = (x_{i-1/2}, x_{i+1/2})$. Extend the grid function $\{u_i\}_{i\in Z}$ to $\mathbb{R}$ via the extension operator E_P:

$$u_h(x) = E_P(\{u_i\}_{i\in Z})(x) = u_i \quad \text{for } x \in I_i.$$

Denote by Π_P the L^2-projection into the space of functions that are constant on each of the intervals I_i:

$$\Pi_P(v)(x) = \int_{I_j} u_h(x)\,dx/|I_i| \quad \text{for } x \in I_i.$$

Now suppose that u_h^{n-1} is known, and we want to compute u_h^n. Denote by P_h^{n-2} the partition of $\mathbb{R}$ used during the time interval (t^{n-2}, t^{n-1}); note that u_h^{n-1} is constant on each interval I_i associated to the partition P_h^{n-1}. We proceed as follows:

(1) First, find the partition P_h^{n-1} that will be used during the time step (t^{n-1}, t^n). It can be obtained using, for example, the method of characteristics [8];

(2) Project u_h^{n-1} in the new partition, i.e., set $\overline{u}_h^{n-1} = \Pi_{P_h^{n-1}}(u_h^{n-1})$;

(3) Finally, solve for $\{u_i^n\}$:

$$(u_i^n - \overline{u}_i^{n-1})/\Delta t^n + (f_{i+1/2}^{\mathrm{QM},n} - f_{i-1/2}^{\mathrm{QM},n})/\Delta x_i = 0.$$

Note that if $\overline{u}_h^{n-1} = \Pi_{P_h^{n-1}}(u_h^{n-1})$ we have

$$\int_{\mathbb{R}} \overline{u}_h^{n-1} = \int_{\mathbb{R}} u_h^{n-1},$$
$$(\inf)\sup\{\overline{u}_h^{n-1}\}(\geq) \leq (\inf)\sup\{u_h^{n-1}\},$$
$$\|\overline{u}_h^{n-1}\|_{\mathrm{TV}(\mathbb{R})} \leq \|u_h^{n-1}\|_{\mathrm{TV}(\mathbb{R})},$$

i.e., mass is conserved and the scheme remains positive and TVD. Moreover, it is not difficult to show that it converges to the entropy solution of (1.1); see [8].

4. Proof of Theorem 2.2. In this section we prove Theorem 2.2. We proceed in several steps.

4.1. A discrete entropy inequality for QM schemes. Rewrite the QM scheme under consideration as follows:

$$\tilde{u}_i^{n+1} = u_i^n - \lambda_i^n(f_{i+1/2}^{\mathrm{M},n} - f_{i-1/2}^{\mathrm{M},n}),$$

where

$$\tilde{u}_i^{n+1} = u_i^{n+1} + \lambda_i^n(a_{i+1/2}^n - a_{i-1/2}^n).$$

It is well known that

$$(J_c(u_i^{n+1}) - J_c(u_i^n))/\Delta t^n + (G_{c,i+1/2}^n - G_{c,i-1/2}^n)/\Delta x_i \leq (J_c(u_i^{n+1}) - J_c(\tilde{u}_i^{n+1}))/\Delta t^n,$$

where $J_c(u) = J(u-c)$ is a functon as described in §2.5 and G_c a discrete entropy flux associated to J_c. Note that the second number is not necessarily nonpositive. For $u \neq w$, set $L_c(u,w) = (J_c(u) - J_c(w))/(u-w)$ and rewrite the previous inequality as

$$\begin{aligned}(4.1)\qquad &(J_c(u_i^{n+1}) - J_c(u_i^n))/\Delta t^n + (G_{c,i+1/2}^n - G_{c,i-1/2}^n)/\Delta x_i \\ &\leq L_{c,i}^{n+1}(a_{i+1/2}^n - a_{i-1/2}^n)/\Delta x_i.\end{aligned}$$

4.2. An estimate for the rate of entropy. We shall need an estimate for the rate of entropy defined by

$$\text{(4.2a)}\qquad \begin{aligned}\mathrm{RE} = \int_Q \sum_{i,n} \{ (J_c(u_i^{n+1}) - J_c(u_i^n))\Delta x_i \\ + (G^n_{c,i+1/2} - G^n_{c,i-1/2})\Delta t^n \}\varphi(t' - \tau^n, x' - x_{i+1/2})\, dt'dx',\end{aligned}$$

where $\tau^n \in (t^n, t^{n+1})$ and $c = c(t', x')$ is an arbitrary function in $L^\infty(0,T;L^1(\mathbb{R})) \cap L^\infty(Q)$. Note that for monotone schemes RE ≤ 0, however, this is not true in the general case, and we are forced to estimate it. By (4.1), this expression can be bounded by

$$\begin{aligned}\Theta &= \int_Q \sum_{i,n} L^{n+1}_{c,i}(a^n_{i+1/2} - a^n_{i-1/2})\Delta t^n \varphi(t' - \tau^n, x' - x_{i+1/2})\, dt'dx' \\ &= \Theta_1 + \Theta_2,\end{aligned}$$

where

$$\begin{aligned}\Theta_1 &= \int_Q \sum_{i,n} L^{n+1}_{c,i} a^n_{i+1/2} \Delta t^n \\ &\qquad \cdot (\varphi(t' - \tau^n, x' - x_{i+1/2}) - \varphi(t' - \tau^n, x' - x_{i+3/2}))\, dt'dx', \\ \Theta_2 &= \int_Q \sum_{i,n} (L^{n+1}_{c,i-1} - L^{n+1}_{c,i}) a^n_{i+1/2} - \Delta t^n \varphi(t' - \tau^n, x' - x_{i+1/2})\, dt'dx'.\end{aligned}$$

If we assume that $J \in C^1$ we can estimate Θ_1 as follows:

$$\text{(4.2b)}\qquad \Theta_1 \leq CTL \|\partial_x \varphi\|_{L^1(Q)} \|u_Q\|_{\mathrm{BV}(\mathbf{R})} \delta,$$

where $L = \|J(u_0 - c)\|_{L^\infty(a,b)}$, where $[a,b]$ includes the range of $u_0(t,x) - c(t',x')$. If $J \in C^2$,

$$\text{(4.2c)}\qquad \Theta_2 \leq CTM \|\varphi\|_{L^1(Q)} \|u_0\|_{\mathrm{BV}(\mathbf{R})} \delta^\alpha,$$

where $M = \|J''(u_0 - c)\|_{L^\infty(a,b)}$.

4.3. An error estimate for J in C^2. In [22], Sanders obtains error estimates, extending the work of Kuznetsov [13] to the case of a variable space step size. However, in that paper some important boundary terms are missing. A correct treatment of those can be found in [19]. Using the estimates (4.2b)–(4.2c) of the rate of entropy RE (4.2a) we follow [22], [19] and obtain

$$\text{(4.3)}\qquad \begin{aligned}\int_{\mathbf{R}} J(u(T) - u_h(T)) \leq \int_{\mathbf{R}} J(u_0 - u_{o,h}) + C_1 L T^{1/2} \|u_0\|_{\mathrm{BV}(\mathbf{R})} \delta^{1/2} \\ + C_2 M T \|u_0\|_{\mathrm{BV}(\mathbf{R})} \delta^\alpha.\end{aligned}$$

This proves (2.9a). In this case we can easily obtain an $L^\infty(L^1)$-error estimate, since via a classical density argument, (4.3) can be proved valid for $J(w) = |w|$; see [13], [22], [19]. We have been forced to consider functionals J with Lipschitz second derivatives because of the presence of the second member in (4.1). Nevertheless, it is still possible to get a local $L^\infty(L^1)$-error estimate, as we show in the next subsection.

4.4. The $L^\infty(L^1_{\text{loc}})$ estimate. The inequality (4.3) is valid for the following convex function, if $0 < \mu \le c\delta^{\alpha/2}$:

$$J_\mu(e) = \begin{cases} |e| - \mu/2 & \text{for } |e| \ge \mu, \\ e^2/2\mu & \text{otherwise,} \end{cases} \tag{4.4}$$

for which we have

$$L = 1, \quad M = \mu^{-1}, \quad \int_{\mathbb{R}} J_\mu(e) \le \|e\|_{L^1(\mathbb{R})}. \tag{4.5}$$

Set $w = \|u(T) - u_h(T)\|_{L^1(\Omega)}|\Omega|^{-1}$. By Jensen's inequality,

$$J_\mu(w) \le \left[\int_\Omega J_\mu(u(T) - u_h(T))\right]|\Omega|^{-1} \le \Theta, \tag{4.6}$$

where, by (4.3), (4.4), and (4.5), Θ can be taken as follows:

$$\Theta = [\|u_0 - u_{o,h}\|_{L^1(\mathbb{R})} + C_1T^{1/2}\|u_0\|_{\mathrm{BV}(\mathbb{R})}\delta^{1/2} + C_2T\mu^{-1}\|u_0\|_{\mathrm{BV}(\mathbb{R})}\delta^\alpha]|\Omega|^{-1}.$$

Choose μ equal to 2Θ. If $w > \mu$, then $w = J_\mu(w) + \mu/2 \le \Theta + \mu/2 = \mu$, and this is a contradiction. So, $w \le \mu$, and consequently $J_\mu(w) = w^2/2\mu \le \mu/2 = \Theta$. In other words, (4.6) is always satisfied for the choice $\mu = 2\Theta$. Solving for μ we obtain

$$\mu = [b + (b^2 + c)^{1/2}]|\Omega|^{-1} \le \mu_0 = [2b + c^{1/2}]|\Omega|^{-1},$$

where

$$b = \|u_0 - u_{0,h}\|_{L^1(\mathbb{R})} + C_1T^{1/2}\|u_0\|_{\mathrm{BV}(\mathbb{R})}\delta^{1/2},$$
$$c = 2C_2T|\Omega|\|u_0\|_{\mathrm{BV}(\mathbb{R})}\delta^\alpha.$$

The estimate (2.9b) follows from the definition of w and the fact that $w \le \mu \le \mu_0$. □

5. Concluding remarks. In this paper, quasimonotone finite-difference (QMFD) schemes for the scalar conservation law (1.1) have been introduced and analyzed. The schemes are constructed in such a way that they inherit the stability and convergence properties of the monotone schemes with which they are built. The quasimonotone schemes are constructed in a systematic way from already known three-point monotone schemes and formal high-order accurate ones. In this respect, this approach, taken from LeRoux [16], is different from thie approach used by Harten [11] and others; see also Sweby [23]. Quasimonotone schemes can also be constructed using multi-time-level schemes. Although we have not developed this point to its full extent, we have indicated how to obtain quasimonotone versions of the well-known (three-time-level) leap-frog scheme. Finally, we point out that these schemes can be easily defined in the case where the space domain is bounded, and where the nonlinearity f depends also on (t, x) and the second member of (1.1) is nonzero. Results similar to the ones displayed in this paper can be obtained in those cases.

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Note added in proof. Since QMFD schemes are TVD, their accuracy degenerates to first-order at nonsonic ($f'(u) \ne 0$) critical points. However, by a simple application of the total variation bounded (TVB) modification technique introduced by Shu [27], the QM schemes can be rendered (formally) uniformly high-order accurate. In our framework, this technique amounts to adding to $a_{i+1/2}$ the term

$$(f^h_{i+1/2} - f^{\mathrm{M}}_{i+1/2} - a_{i+1/2}) \cdot \begin{cases} 1, & \text{if } |f^h_{i+1/2} - f^{\mathrm{M}}_{i+1/2}| \leq M \cdot (\Delta x)^2, \\ 0, & \text{otherwise.} \end{cases}$$

(The constant M is essentially an upper bound for the second derivative of the initial condition u_0.). Theorem (2.2) remains valid for the modified schemes. Similar remarks are valid for the implicit QMFD.

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