Adaptive finite element methods for conservation laws

Chapter · November 2006

DOI: 10.1007/BFb0096354

4 authors, including:

Claes Johnson
KTH Royal Institute of Technology

Chi-Wang Shu
Brown University

Adaptive Finite Element Methods for Conservation Laws Based on a posteriori Error Estimates

CLAES JOHNSON

Chalmers University of Technology, Göteborg

ANDERS SZEPESSY

Royal Institute of Technology, Stockholm

Abstract

We prove a posteriori error estimates for a finite element method for systems of strictly hyperbolic conservation laws in one space dimension, and design corresponding adaptive methods. The proof of the a posteriori error estimates is based on a strong stability estimate for an associated dual problem, together with the Galerkin orthogonality of the finite element method. The strong stability estimate uses the entropy condition for the system in an essential way.

1. INTRODUCTION

A Basic Problem

One of the basic unsolved problems of computational fluid mechanics may be formulated as follows: Construct an algorithm for the numerical solution of the (compressible or incompressible) Navier-Stokes equations such that the error between the exact and computed solution in a given norm may be guaranteed to be below a given tolerance and such that (under suitable comparison) the computational work is nearly minimal. The problem clearly has two components, reliability and efficiency; the error should be guaranteed to be below the tolerance and the computational work should be nearly minimal. Solutions of the Navier-Stokes equations typically contain small scale features such as boundary layers and shocks, and thus efficiency requires adaptive methods, where the discretization (the mesh) is automatically adjusted to fit the nature of the exact solution with mesh refinement in regions with small scale features. For reliability, a posteriori error estimates are required where the error is estimated in terms of the computed solution. In this note we take a step towards the solution of the problem stated above: We prove a posteriori error estimates and design corresponding adaptive algorithms for the streamline diffusion finite ele-

Communications on Pure and Applied Mathematics, Vol. ???, 0–0 (???)
 CCC ???

ment method (SD-method below) for systems of conservation laws of the form: Find $u : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}^m$ such that

$$
\begin{array}{llcll}
(1.1\text{a}) & L(u) & \equiv & u_t + f(u)_x - \epsilon u_{xx} = 0 & x \in \mathbb{R}, t \in \mathbb{R}_+ \equiv \{t \in \mathbb{R} : t > 0\}, \\
(1.1\text{b}) & u(x,0) & = & u_0(x) \quad x \in \mathbb{R}, & \\
(1.1\text{c}) & u(x,t) & \to & 0 \quad \text{as} \quad |x| \to \infty, t \in \mathbb{R}_+ &
\end{array}
$$

where $f : \mathbb{R}^m \to \mathbb{R}^m$ is a given smooth strictly hyperbolic flux function, $u_0 : \mathbb{R} \to \mathbb{R}^m$ is a given bounded initial function with compact support, $\epsilon > 0$ is a constant (small) viscosity, m a positive integer and $v_t = \frac{\partial v}{\partial t}$, $v_x = \frac{\partial v}{\partial x}$. Our results appear, perhaps surprisingly so, to be the first to show that reliable and efficient adaptive error control (in L_2) with solid mathematical basis may be possible in the numerics of systems of conservation laws. Although the results of this note are restricted to one-dimensional problems as far as a more or less complete mathematical justification is concerned, the methodology for adaptive error control presented is general in nature and might be applicable also to conservation laws in several dimensions (for non-turbulent flows, cf. below). For numerical results for the compressible Euler equations in two dimensions supporting this belief, we refer to [12], [1], [28] and [30]. The results and techniques of this note extend to conservation laws, those of [3],[4], [20],[21], [18], concerning adaptive SD-methods for linear scalar convection-diffusion problems. As far as we know, the only previous results on a posteriori error control for numerical methods for conservation laws are those of Lucier [28], restricted to scalar one dimensional problems which may be solved exactly with piecewise polynomial initial data, and the recent results by Nessyahu and Tadmor [31] restricted to scalar one dimensional problems using a weak dual Lipschitz norm.

The SD-method

The SD-method is a modified Galerkin method for hyperbolic-type problems combining high accuracy with good stability, based on

- (1.2a) spacetime discretization with piecewise polynomials with the basis functions being continuous in space and discontinuous in time,
- (1.2b) a “streamline diffusion” modification of the test functions giving a weighted least sqauares control of the residual $R(U) = L(U)$ of the finite element solution U,
- (1.2c) modification of the given viscosity ϵ to an artificial viscosity $\hat{\epsilon}$ of the form $\hat{\epsilon} = \max(\epsilon, C_1 h|R(U)|/|\bigtriangledown U|, C_2 h^{3/2})$ or $\hat{\epsilon} = \max(\epsilon, C_1 h^2 |R(U)/|U|, C_2 h^{3/2})$, where h is the mesh size and the C_i are positive constants.

The design principles underlying the modifications (1.2b,c) may be phrased as: improvement of stability without loss of accuracy. Below, we shall in detail consider a simplified version of the SD-method with artificial viscosity $\hat{\epsilon}$ of the form $\hat{\epsilon} = \max(\epsilon, C_1 h)$, in which case the streamline modification may be omitted, and then more briefly indicate extensions to the case (1.2b,c) in Section 5. The space-time mesh may be oriented in space-time (x,t) in a given direction $(\sigma, 1)$, where σ is a (variable) "mesh tilting velocity". We shall consider the case of continuous piecewise linear approximation in space and discontinuous piecewise constant approximation in the direction $(\sigma, 1)$.

We now first give a brief sketch of the main contents of this note and then put our results and methods into perspective and discuss possible extensions and applications.

The a posteriori Error Estimate

The form of the a posteriori error estimate underlying the adaptive method is, for simplicity in the case of space discretization only, essentially as follows (or more generally a weighted norm analog thereof):

$$\|\hat{e}\|_Q \leqq C^s C^i \|\hat{\epsilon}^{-1} h^2 R(U)\|_Q, \tag{1.3a}$$

where $\hat{e} = \hat{u} - U$ with $\hat{u}$ the solution of (1.1) with ϵ replaced by the artificial viscosity $\hat{\epsilon}$, $h = h(x,t)$ is the mesh size in space, $Q = \mathbb{R} \times (0,T)$ with T a given final time and again $R(U) = L(U)$ is the residual (properly evaluated) of the finite element solution U. Here the total error $e = u - U$ is split into $e = \hat{e} + u - \hat{u}$, where $u - \hat{u}$ is a perturbation error caused by changing the viscosity in the continuous problem from ϵ to $\hat{\epsilon}$. Further, $\|\cdot\|_Q = \|\cdot\|_{L_2(Q)}$ denotes the $L_2(Q)$-norm, C^s is a stability constant and C^i an interpolation constant to be commented on below. Here, let us remark that C^i only depends on the degree of the polynomials used in the finite element discretization and the shape of the finite elements, while the stability constant C^s in general depends on both $\hat{u}$ and U. Thus, (1.3a) as stated is not a full a posteriori error estimate with only dependence on U and h. The problem of estimating C^s, analytically or computationally, is discussed below. Of course, the idea is that C^s should be of moderate size unless the given problem is inherently unstable. In the simplified case with $\hat{\epsilon} = Ch$, (1.3a) takes the form

$$\|\hat{e}\|_Q \leqq C^s C^i \|hR(U)\|_Q. \tag{1.3b}$$

The adaptive algorithm is based on (1.3a) and seeks to find a mesh with as few total degrees of freedom as possible such that

$$C^s C^i \|\hat{\epsilon}^{-1} h^2 R(U)\|_Q \leqq TOL, \tag{1.4}$$

where $TOL > 0$ is a given tolerance, from which $\hat{\epsilon}$ will be controlled via (1.3a):

$$\|\hat{\epsilon}\|_Q \leqq TOL. \tag{1.5}$$

The remaining part $u - \hat{u}$ may be controlled in different ways, e.g. by adaptively refining the mesh until $\hat{\epsilon} = \epsilon$ in which case $u = \hat{u}$, or by estimating $u = \hat{u}$ in terms of $\hat{\epsilon} - \epsilon$, and U via an estimate in terms of $\hat{u}$, see below. To approximately minimize the number of degrees of freedom of a mesh with mesh size $h(x,t)$ realizing (1.4), typically a simple iterative algorithm is used where a new mesh size h is computed through a principle of equidistribution (usually of element contributions) in the quantity $C^s C^i \|\hat{\epsilon}^{-1} h^2 R(U)\|_{L_2(\Omega)}$ with the values of $\hat{\epsilon}$ and $R(U)$ taken from the previous mesh, see Section 6 for more details.

The structure of the proof of the a posteriori error estimate (1.3) is as follows:

(1.6a) Representation of the error $\hat{e}$ in terms of the residual $R(U)$ and the solution φ of a linearized dual problem with $\hat{e}$ as right hand side.

(1.6b) Use of the Galerkin orthogonality to replace φ by $\varphi - \Phi$, where Φ is a finite element interpolant of φ.

(1.6c) Interpolation error estimates for $\varphi - \Phi$ in terms of certain derivatives $D\varphi$ of φ and the mesh size h.

(1.6d) Strong stability estimate for the dual solution φ estimating $D\varphi$ in terms of the data $\hat{e}$ of the dual problem.

To be more concrete, let us indicate the main steps (1.6a-d) in a simplified situation, which however contains the essentials of the argument. Thus, let $\hat{u}$ satisfy

$$\begin{aligned} \hat{u}_t + f(\hat{u})_x - \hat{\epsilon}\hat{u}_{xx} &= 0 \quad \text{in} \quad Q \equiv \mathbb{R} \times (0,T) && \text{(1.7a)} \\ \hat{u}(\cdot,0) &= u_0, && \text{(1.7b)} \end{aligned}$$

and suppose $U \in V_h$, where $V_h \subset L_2(Q)$ is a finite element space, is a Galerkin type approximate solution satisfying

$$\begin{aligned} U_t + f(U)_x - \hat{\epsilon}U_{xx} &\equiv R \quad \text{in} \quad Q, && \text{(1.8a)} \\ U(\cdot,0) &= u_0, && \text{(1.8b)} \end{aligned}$$

where the residual $R = L(U)$ satisfies the orthogonality relation

$$\int_Q R v \, dx dt = 0 \quad \forall v \in V_h. \tag{1.9}$$

Here (1.7) and (1.8) are complemented with a boundary condition corresponding to (1.1c). This condition is also imposed below when relevant without explicit notion. Subtracting (1.8) from (1.7) gives the following equation for the error $\hat{e} = \hat{u} - U$,

$$
\begin{aligned}
&(1.10a) \qquad & L\hat{e} \equiv \hat{e}_t + (A\hat{e})_x - \hat{\epsilon}\hat{e}_{xx} &= -R \quad \text{in} \quad Q,\\
&(1.10b) \qquad & \hat{e}(x,0) &= 0,
\end{aligned}
$$

where

$$
A \equiv \int_0^1 f'(s\hat{u} + (1-s)U)ds \tag{1.11}
$$

with f' the Jacobian of f, so that

$$
A\hat{e} = \int_0^1 \frac{d}{ds} f(s\hat{u} + (1-s)U)ds = f(\hat{u}) - f(U).
$$

Introducing now the linear dual problem

$$
\begin{aligned}
&(1.12a) \qquad & L^*\varphi \equiv -\varphi_t - A^*\varphi_x - \hat{\epsilon}\varphi_{xx} &= \hat{e} \quad \text{in} \quad Q,\\
&(1.12b) \qquad & \varphi(\cdot, T) &= 0,
\end{aligned}
$$

where A^* denotes the transpose of the matrix A, we get the following representation of the error $\hat{e}$:

$$
\begin{aligned}
(1.13) \qquad ||\hat{e}||_Q^2 &= \int_Q \hat{e} \cdot L^*\varphi dxdt = \int_Q L\hat{e} \cdot \varphi dxdt\\
&= -\int_Q R\varphi dxdt = -\int_Q R(\varphi - P_h\varphi)dxdt\\
&= -\int_Q (R - P_h R)\varphi - P_h\varphi)dxdt,
\end{aligned}
$$

where $P_h : L_2(Q) \rightarrow V_h$ is the $L_2(Q)$-projection. Note that with A defined by (1.11) with the dependence on both $\hat{u}$ and U, there is no linearization contribution in the error representation (1.13), which would occur with $A = f'(\hat{u})$ or $A = f'(U)$. Using now an interpolation estimate, for simplicity corresponding to space discretization only, of the form

$$
||\hat{\epsilon}h^{-2}(\varphi - P_h\varphi)||_Q \leqq C^i ||\hat{\epsilon}\varphi_{xx}||_Q, \tag{1.14}
$$

together with a strong stability estimate for the dual problem (1.12) of the form

$$
||\hat{\epsilon}\varphi_{xx}||_Q \leqq C^s ||\hat{e}||_Q, \tag{1.15}
$$

we get from (1.13)

$$\|\hat{e}\|_Q \leqq C^s C^i \|h^2 \hat{\epsilon}^{-1}(R - P_h R)\|_Q, \tag{1.16}$$

which is the proper analog of (1.3a) in the present context with a Galerkin method satisfying (1.9). Note that the Galerkin method for (1.1) to be studied below does not have exactly the form (1.9) with $R \in L_2(Q)$, and thus below the projection P_h will not enter in the same simple way as in (1.16), cf (1.3), where the projection is omitted.

The concrete form of (1.16) to be derived below will be essentially as follows if $m = 1$ and $\sigma = f'(U)$:

$$\|\hat{e}\|_Q \leqq C(\|h^2 D_1 U\|_Q + \|\hat{\epsilon}^{-1} h^4 f^{(3)}(U)|U_x|^3\|_Q + \|k\partial_t U\|_Q), \tag{1.17}$$

with also a (first order) time discretization term is included, where $D_1 U$ is a second difference quotient in $x, \partial_t U$ a first order difference quotient in the direction $(\sigma, 1)$ in space-time; k is a time step and $f' = \frac{df}{du}, f^{(3)} = \frac{d^3 f}{du^3}$. In the presence of rarefaction waves the norm $\|\cdot\|_Q$ is replaced by $\|\cdot\sqrt{T/t}\|_Q$ with the weight $\sqrt{T/t}$. In (1.17) the first and third terms naturally correspond to interpolation error terms $C\|h^2 u_{xx}\|_Q$ (if $\hat{\epsilon} = Ch$) and $C\|ku_t\|_Q$ related to piecewise linear approximation in space and piecewise constant approximation in time, while the middle (higher order) term results from the non-linearity of the problem. We note the simple form of the a posteriori error estimate (1.17) which also appears to be optimal indicating an $L_2(Q)$-estimate for $\hat{e}$ of order $O(h^{3/2} + k)$ in regions of smoothness with $\hat{\epsilon} = Ch^{3/2}$ (if $D_1 U, U_x$ and $\partial_t U$ are bounded), and order $O(\sqrt{h})$ in shock (and rarefaction) regions with $\hat{\epsilon} = O(h)$ and $k = O(h)$, see the discussion after Theorem 3.6. In the system case $m > 1$, we have analogous estimates with a reduction to order $O(h^{3/2} + kh^{-3/4})$ in regions of smoothness (with piecewise linears/constants in space/time). We recall that proofs of corresponding a priori error estimates are not known. The idea to base on adaptive method on interpolation error estimates was used in e.g. [1], [28], [30].

The estimates (1.3a) should be compared with the typical a posteriori error for elliptic problems derived in [2]:

$$\|e\|_{L_2} \leqq C\|h^2 R(U)\|_{L_2}, \tag{1.18}$$

which is a sharp estimate leading to reliable and efficient adaptive algorithms. Evidently (1.3a) may be viewed as a variant of (1.18) where the ellipticity introduced in the hyperbolic case through the artificial viscosity $\hat{\epsilon}^{-1}$. In the SD-method with $\hat{\epsilon} = \max(\epsilon, C_2 h R(U)/|\nabla U|, C_2 h^{3/2})$ we will have $\hat{\epsilon} \approx Ch$ close to shocks and $\hat{\epsilon} \sim C_2 h^{3/2}$ (with C_2 small) in smooth regions. Thus, $\hat{\epsilon}$ is a dynamic viscosity acting like an automatic "switch" adding considerable viscosity close to shocks and less in regions of smoothness of the exact solution. The dependence of $\hat{\epsilon}$ in the a posteriori error estimate (1.3a) or (1.17) gives new insight into the role played by artificial viscosity, and in particular indicates that the construction of $\hat{\epsilon}$ given above is reasonable. In particular, it follows that close to shocks $\hat{\epsilon}^{-1}h^2R(U) \sim O(1)$, which shows the optimal rate of convergence $O(\sqrt{h})$ in $L_2(Q)$ since the numerical shock width is $O(h)$, see [35]. Further, in regions of smoothness we will have $\hat{\epsilon}^{-1}h^2R(U) \sim h^{1/2}R(U) \sim O(h^{3/2})$ if as expected $R(U) \sim O(h)$ with piecewise linear approximation (cf. [3]). This is in conformity with the $O(h^{3/2})$ a priori error estimate in L_2 for linear convection-diffusion problems ([23]).

We have now outlined the main ingredients of the proof of the a posteriori error estimate (1.3) which clearly indicates the roles played by

1. Galerkin orthogonality,
2. Strong stability of the dual problem.

The strong stability (1.15) of the dual problem (1.12) is here the critical issue which will be a main focus of interest in the development to follow. We will be able to prove analytically (1.15) to be valid with C^s of moderate size in cases where the exact solution has weak noninteracting shocks and rarefaction waves of genuinely nonlinear fileds (cases including contact discontinuities will be considered in subsequent work). The proof uses the entropy condition in the shock case in an essential way and for rarefaction waves the presence of the weight $\sqrt{T/t}$ is essential. More precisely, the proof is based on certain qualitative properties of the operator A (depending on both $\hat{u}$ and U) which may be verified analytically in certain model cases. For more complicated situations with interacting shocks for which analytical techniques are not yet available, we anticipate to be able to check the validity of (1.15) and determine the constant C^s computationally, cf. Section 1.5. Note that for turbulent flows the constant C^s may be expected to be large as well as the residual r, in which case the a posteriori error bound of (1.3) will not be small unless h is extremely small.

Comparison with an Approach Based on General Perturbation Analysis

Let us now, in order to further stress some fundamental aspects, compare the approach to a posteriori error estimates for conservation laws presented in this note with an approach based on a direct perturbation analysis not using the Galerkin form of the approximation. With the latter approach we would also start from the equation (1.10) satisfied by the error $\hat{e}$. The hope would then be to estimate $\hat{e}$ in terms of R as follows,

$$\|\hat{e}\|_X \leqq C\|R\|_y, \tag{1.19}$$

corresponding to stability of (1.10) with respect to certain norms $\|\cdot\|_X$ and $\|\cdot\|_Y$. The question is now if we can choose X and Y so that (1.19) holds and $\|R\|_Y$

tends to zero as $h \to 0$ (reasonably fast). Let us first point out that choosing $X = Y = L_2(Q)$ is impossible since already in the case of a scalar conservation law, (1.19) may be invalid corresponding to the well-known phenomenon of instability in L_2 with respect to L_2-perturbations of initial data, cf. [32]. This is most easily seen e.g. for Burgers' equation with two shock solutions u_1 and u_2 with shock jumps $[u_1]$ and $[u_2]$ at $x = 0$ satisfying $[u_1] - [u_1] = \delta > 0$ at $t = 0$ corresponding to an L_2-perturbation of initial data $\|u_2(\cdot, 0) - u_1(\cdot, 0)\|_{L_2(-1,1)} \sim \delta$. Since the difference in shock speed is proportional to δ, we will have for $t \sim 1$, say, that $\|u_2(\cdot, t) - u_1(\cdot, t)\|_{L_2(-1,1)} = O(\sqrt{\delta})$ and instability follows since $O(\sqrt{\delta})/\delta \to \infty$ as $\delta \to 0$. For scalar conservatin laws it may be possible to choose $X = Y = L_1(Q)$ in (1.19), but it appears as if $\|R\|_{L_1}$ will tend to zero only very slowly with h. (Note that $\|R\|_{L_2}$ typically, e.g. in the presence of shocks, will increase with decreasing mesh size h until $h < \epsilon$). It would then remain to seek realizations of (1.19) with norms weaker than L_1. For scalar conservation laws it is possible (so far in the absence of rarefaction waves, see [31] and [38]) to choose $\|\cdot\|_X = \|\cdot\|_Y$ to be the weak dual norm of the Lipschitz (semi) norm

$$\|v\|_{Lip} = \text{ess sup}_{\substack{0<t<T \\ x \neq y}} |v(x, t) - v(y, t)|/|x - y|,$$

but it is not clear if the corresponding estimate extends to systems. The general perturbation estimate (1.19) with weak norms $X = Y$, if at all possible to derive, should be compared with the Galerkin based L_2-estimate (1.3), which can be proved to be valid also for systems in one dimension, where the additional factors $h^2 \hat{\epsilon}^{-1}$ or h make the right hand side converge to zero with decreasing mesh size at close to optimal rate.

Let us give some further remarks on the relation between a posteriori error estimates and stability or perturbation theory for conservation laws. First, an a posteriori error estimate for a numerical method may be viewed as a particular stability result estimating the effect on the solution resulting from a non-zero residual. Thus it appears conceivable that a posteriori error estimates could be viewed as special cases of a general stability theory, if available. However, no such theory for systems of conservation laws is yet available; basically only special results (see [5], [26], [37], [8], [39], [7]) are known concerning the stability with respect to perturbations of initial data for shock waves or rarefaction waves in one space dimension, and these results cannot directly be used to obtain a posteriori error estimates, since the perturbations in the finite element case have a different nature. Further, the perturbations in the finite element case satisfy certain orthogonality relations which, as indicated above, make a posteriori error estimates possible in cases where a general perturbation argument would fail. Thus, it appears that a posteriori error estimates for Galerkin methods for conservation laws have particular features and cannot be derived from the existing stability theory. However, there is in the technical details a remarkable simularity in the proof of the basic stability result for the dual problem underlying the a posteriori error estimate and the basic stability estimate in the proof of stability of shock waves. Shock wave stability is based on energy estimates of the integrated error and we use energy estimates of a differentiated dual solution, reflecting the duality of the two problems.

Let us sum up as follows: the full stability problem for conservation laws, concerned with the effect on the solution of a general perturbation, appears to be extremely difficult and only very limited results are available (or forseeable). On the other hand, in the approach to a posteriori error estimates for Galerkin methods presented, we avoid the full general stability problem by considering perturbations of a special form and by opening the possibility of direct computational evaluation of the stability of an associated linearized dual problem with coefficients depending on the computed solution only. The reliability of a corresponding adaptive method could in general be fully guaranteed only under certain qualitative assumptions concerning the error. Thus, we avoid the general stability problem and give instead mathematical support for a step from qualitative to quantitative error control for Galerkin methods.

The Strong Stability Estimate

As indicated, the crucial step in the proof of the a posteriori error estimate (1.3) is the strong stability (1.15) of the linearized dual problem. We prove by analytical techniques (1.15) to be valid with C^s bounded for strictly hyperbolic systems in one dimension allowing the presence of non-interacting shocks and rarefaction waves. For simplicity we first consider the scalar case with $f(u) = u^2/2$, i.e., Burgers' equation which is free from technical complications but yet has features of interest. We then pass to the case of systems using diagonalization and a weighted norm technique developed in [5],[6] to essentially reduce to the scalar case. The estimate (1.15) should be contrasted with the non-validity in general of a "weak stability" estimate for (1.12) of the form

$$||\varphi||_Q \leqq C||\hat{e}||_Q, \tag{1.20}$$

corresponding to the L_2-instability discussed above. We note that in (1.15) the derivative φ_{xx} of the dual solution φ is L_2-controlled (with the factor $\hat{\epsilon}$) in terms of $||\hat{e}||_{L_2}$, whereas L_2-control for φ itself as in (1.20) is not possible to achieve in general. In order to be able to use, in the proof of the a posteriori error estimate, the strong stability estimate (1.15) with derivative control only, we have to use Galerkin orthogonalities. Thus it appears that Galerkin based (finite element) methods have distinct advantages from an a posteriori error control point of view as compared to other methods for which only a general perturbation argument, which does not appear to work well, may be available.

Let us further emphasize one important aspect of presented framework related to the concept of strong stability as expressed by (1.15) and in particular to the definition (1.11) with the linear operator $A = A(\hat{u}, U)$ depending on both

$\hat{u}$ and U. The crucial question is how to determine approximately the stability constant C^s which is required to achieve quantitative error control through the a posteriori error estimate. There are basically two approaches to this problem, an analytical and a computational. With the analytical approach we would typically prove that $A(\hat{u}, U)$ has certain qualitative properties, such as monotonicity, and then prove again by analytical techniques the desired strong stability estimate together with an estimate of C^s. This approach would be restricted to certain model cases where the qualitative properties of $A(\hat{u}, U)$ could actually be verified and a reasonably realistic estimate of C^s could be given. It could also be natural to consider a variant of this type of argument based on assuming certain qualitative properties of $A(\hat{u}, U)$ and then proving by analytical techniques the boundedness of C^s. In this case we would get mathematical support for a step from qualitative to quantitative error control.

The possibility of estimating C^s through analytical techniques appears to be restricted to simple model cases. Nevertheless, such model cases would be important for the justification of the adaptive algorithm since they would indicate that the form of the a posteriori error estimate is correct. For more general problems, the only possibility to determine C^s would be through computations. In this case we would replace $\hat{u}$ by U in the definition of A and then seek to estimate the stability of the resulting linear dual problem by solving this problem numerically with suitably chosen data. The resulting adaptive method would not necessarily be fully reliable since we would (i) not account for the step from $A(\hat{u}, U)$ to $A(U, U)$, (ii) not in general determine C^s very accurately. The effect of the modification (i) may be expected to be small if U is sufficiently close to $\hat{u}$, i.e., we would again obtain quantitative error control under an assumption of qualitative error control. We sum up this discussion as follows: To determine the stability constant C^s in (1.15) we may use analytical techniques in certain model cases and probably computational techniques for more complex problems. For such problems, full reliability could be fully guaranteed only under the assumption that the error qualitatively is sufficiently small.

Summary of Results and Extensions

We give in this paper evidence that adaptive quantitative error control with solid mathematical basis is possible for a class of finite element methods for conservation laws in one space dimension. We prove a posteriori error estimates for strictly hyperbolic systems in cases, including non-interacting shocks and rarefaction waves under certain assumptions of qualitative nature, which may be verified in model cases. Extensions to include contact discontinuities seem to be possible and will be the subject of subsequent work. Of course the extension of the above results to systems in several dimensions is the real challenge. Formally, our methodology carries over to this case: The critical ingredient is again the stability of the associated dual problem, which is now more difficult to evaluate by analytical techniques (but could be approachable in certain model cases). It is probably impossible in situations with complicated flow to get hands analytically on the stability of the dual problem, which partially reflects the limited mathematical/analytical knowledge concerning systems of conservation laws in multi-dimensions. However, there seems to be a computational way out of this dilemma in the case of adaptive error control: Evaluate the stability properties of the dual problem linearized around a computed solution, by feeding in suitable data and computing the corresponding dual solution. Of course the question is then how to choose the data and how much work will have to be spent to evaluate the stability properties sufficiently accurately, which are questions we plan to consider in the future. As already indicated, error control for turbulent flows appears to be difficult to realize. In our framework, turbulence is probably reflected by the stability constant C^s and the residual $R(U)$ both being large. This situation may be changed back to the more favorable one with C^s bounded and $R(U)$ not large if a turbulence model is used, and we want to control the error in the numerical solution of the turbulence model. This seems to open possibilities of accurate resolution with quantitative error control of (non-turbulent) compressible flow with features of vastly different scales including shocks, boundary layers, reaction zones, et ce. For computational results for compressible flow in two dimensions with adaptive algorithms of the form presented in this note, we refer again to [12].

To sum up, the results of this note indicate that adaptive error control for compressible flow may be possible to realize, which seems to open interesting new perspectives in CFD.

By c and C we will denote positive constants not necessarily the same at each occurrence.

2. FINITE ELEMENT DISCRETIZATION

We shall discretize (1.1) by the SD-method with $\mathbb{P}_1 \times \mathbb{P}_0$ space-time elements oriented in space-time (cf. [19], [20], [4]). To define this method, let $0 = t_0 < t_1 < t_2 < \ldots < t_{N+1} = T$ be a sequence of discrete time levels with corresponding time intervals $I_n = (t_n, t_{n+1})$ and times steps $k_n = t_{n+1} - t_n$, and introduce the corresponding space-time "slabs" $S_n = \mathbb{R} \times I_N$. For each $n = 0, 1, 2, \ldots, N$, let $\{x_1^n\}_i$ be a mesh on $\mathbb{R}$, i.e., a partition of $\mathbb{R}$ into intervals $J_i^n = (x_i^n, x_{i+1}^n)$, and assume there is a mesh function $h^n \in C^1(\mathbb{R})$ such that for some positive constants c_1 and μ (independent of n and i)

$$\|h_x^n\|_{L^\infty(\mathbb{R})} \leqq \mu, \tag{2.1a}$$

$$c_1 h_i^n \leqq h^n(x) \leqq h_i^n \qquad x \in J_i^n, \tag{2.1b}$$

where $h_i^n = x_{i+1}^n - x_i^n$. We define the global mesh function $h(x,t)$ by $h(x,t) = h^n(x), x \in \mathbb{R}, t \in [t_n, t_{n+1})$. For simplicity we shall below first assume quasi uniformity of the meshes $\{x_i^n\}$, i.e., the function $h(x,t)$ may be chosen to be constant $h(x,t) = h, \forall x, t$. Further, let for each slab S_n a "mesh convection velocity" $\sigma_n \in \mathcal{C}(S_n)$ be given such that

$$\sigma_{n,t} + \sigma_n \sigma_{n,x} = 0 \qquad \text{in} \quad S_n, \tag{2.2}$$

and assume that for some sufficiently small constant c_2

$$|\sigma_{n,x}| \leqq c_2/k_n \qquad \text{on} \quad S_n. \tag{2.3}$$

We introduce the integral curves or characteristics $x^n(\bar{x},\bar{t})$, $\bar{t} \in I_n$, corresponding to σ_n, defined by

$$\frac{d}{dt}x^n(\bar{x},\bar{t}) \;=\; \sigma_n(x^n(\bar{x},\bar{t}),\bar{t}), \quad \bar{t} \in I_n, \tag{2.4a}$$

$$x^n(\bar{x},t_n) \;=\; \bar{x}. \tag{2.4b}$$

From (2.2) it follows that σ_n is constant along a characteristic $x^n(\bar{x},\cdot)$ so that each characteristic $x^n(\bar{x},\cdot)$ is a straight line $x^n(\bar{x},\bar{t}) = \bar{x} + (\bar{t} - t_n)\sigma_n(\bar{x},t_n)_+$, $\bar{t} \in I_n$. Here and below we use the notation: $v(t_n)_\pm = \lim_{s\to 0^\pm} v(t_n + s)$. Note that the condition (2.3) will ensure that different characteristics satisfying (2.4) do not cross in S_n.

Let us next introduce the 1-1 mapping $F_n : S_n \to S_n$ defined by

$$(x,t) = F_n(\bar{x},\bar{t}) = (x^n(\bar{x},\bar{t}),\bar{t}) = (\bar{x} + (\bar{t} - t_n)\sigma_n(\bar{x},t_n)_+,\bar{t}),\ \bar{x},\bar{t}) \in S_n. \tag{2.5}$$

For a given function $v(x,t)$ on S_n we now associate a function $\bar{v}(\bar{x},\bar{t})$ on S_n by setting

$$\bar{v}(\bar{x},\bar{t}) = v(x,t) \quad \text{if} \quad (x,t) = F_n(\bar{x},\bar{t}), \tag{2.6}$$

and vice versa. Let us for $\bar{t} \in I_n$ define $\bar{J}^n(\bar{x},\bar{t})$ to be the Jacobian of the mapping $\bar{x} \to x^n(\bar{x},\bar{t})$, i.e.,

$$\bar{J}^n(\bar{x},\bar{t}) = 1 + (\bar{t} - t_n)\sigma_n(\bar{x},t_n)_{+,\bar{x}},\ (\bar{x},\bar{t}) \in S_n.$$

We note that by (2.3) there are constants C and c such that

$$c \leqq \bar{J}^n(\bar{x},\bar{t}), \bar{J}^n(\bar{x},\bar{t})^{-1} \leqq C \quad \text{on} \quad S_n, n = 0,1,\ldots. \tag{2.7}$$

We shall below use the function $\bar{J}$ defined by $\bar{J}|_{S_n} = \bar{J}^n$ and the corresponding function $J(x,t)$ defined by (2.6).

Let now $\bar{T}^n = \{\bar{K}_i^n\}_i$ be the partition of S_n into elements $\bar{K}_i^n = J_i^n \times I_n$ induced by $\{x_i^n\}$ and introduce the corresponding finite element space $\bar{V}_n$

consisting of the continuous functions $\bar{v}(\bar{x},\bar{t})$ on S_n such that on each $\bar{K}_i^n$ the function $\bar{v}$ is linear in $\bar{x}$ and constant in $\bar{t}$, or in other words,

$$\bar{V}_n = \{\bar{v} \in \mathcal{C}(S_n)^m : \bar{v}(\bar{x},\bar{t}) = \bar{w}(\bar{x}) \quad \text{for} \quad (\bar{x},\bar{t}) \in S_n, \quad \text{where} \quad \bar{w} \in [X_n]^m\},$$

where X_n is the set of continuous piecewise linear functions on $\{x_i^n\}_i$. We introduce the corresponding space of functions $v(x,t)$

$$V_n = \{v \in \mathcal{C}(S_n)^m : v(x,t) = \bar{v}(\bar{x},\bar{t}), \bar{v} \in \bar{V}_n\},$$

and define finally

$$V_h^\sigma = \{v \in L_2(Q) : v|_{S_n} \in V_n,\ n = 0,1,\ldots,N\},$$

in which the approximate solution U will be sought. Below we will choose $\sigma_n(\cdot,t_n)_+ \in X_n$, in which case the functions $v(x,t)$ in V_h^σ are continuous and piecewise linear in x, constant along the characteristics $x^n(\bar{x},\bar{t}), \bar{t} \in S_n$, and in general are discontinuous across the discrete time levels t_n. In particular we will have $v_t + \sigma v_x = 0$ on S_n, where $\sigma|_{S_n} = \sigma_n$, if $v \in V_h^\sigma$.

We shall below use the following notation:

$$\begin{aligned}
\bar{W}_n &= \{\bar{v} \in L_2(S_n) : \bar{v}(\cdot,\bar{t}) \in [X_n]^m, \bar{t} \in I_n\},\\
W_n &= \{v \in L_2(S_n) : \bar{v} \in \bar{W}_n\}.
\end{aligned}$$

The SD-method for (1.1) can now be formulated as follows: Find $U \in V_h^\sigma$ such that for $n = 0,1,2,\ldots,N$

$$\begin{aligned}
&(U_t + f(U)_x, v + \delta(v_t + f'(U)^T v_x))_n \\
&\quad + (\hat{\epsilon} U_x, v_x)_n + ([U^n], v_+^n) = 0 \quad \forall v \in V_n^\sigma,
\end{aligned} \tag{2.8}$$

where $U(\cdot,0)_- = u_0$,

$$\begin{aligned}
\hat{\epsilon} &= \max(\epsilon, C_1 h R(U)|/|\bigtriangledown U|,\ C_2 h^{3/2}),\\
R(U) &= |U_t + f(U)_x| + |[U^n]|/k_n \quad \text{on} \quad S_n,\\
\delta^T &= \frac{1}{2} h (f'(U)^2)^{-\frac{1}{2}},\\
(v,w)_n &= \int_{S_n} v \cdot w\,dxdt, \quad [v^n] = v_+^n - v_-^n, \quad v_\pm^n = v(t_n)_\pm,
\end{aligned} \tag{2.9}$$

with $[U^n]$ extended as a constant to S_n. Here (with some abuse of notation) A^T denotes the transpose of the matrix A.

Below we shall consider the following simplified SD-method with $\hat{\epsilon} = Ch$ and $\delta = 0$: Find $U \in V_h^\sigma$ such that for $n = 0,1,2,\ldots,N$,

$$(U_t + f(U)_x, v)_n + (\hat{\epsilon} U_x, v_x)_n + ([U^n], v_+^n) = 0 \quad \forall v \in V_n^\sigma, \tag{2.10}$$

where $U(\cdot,0)_- = u_0$. Note that (2.10) with $\hat{\epsilon} = Ch$ corresponds to a first order accurate scheme with classical artificial viscosity $\hat{\epsilon} = Ch$. Note further that in this case we may take $\delta = 0$ since the $\hat{\epsilon}$-term dominates the δ-term. For simplicity we will further first assume that h is constant corresponding to quasi uniformity as indicated above. We shall below use the notation $(v,w)_Q = \sum_{n=0}^N (v,w)_n$, $\|v\|_Q = (v,v)_Q^{\frac{1}{2}}$.

3. THE SCALAR CASE: BURGERS' EQUATION

Introduction

In this section we shall consider the finite element method (2.10) for (1.1) in the simplest case of interest: $m = 1$, and $f(u) = u^2/2$, that is, Burgers' equation. In general the exact solution u of (1.1) will be smooth in certain regions of space-time and non-smooth in other regions typically containing shocks and rarefaction waves. In the development below, a critical role will be played by the quantity $u_x = (f'(u))_x$. In regions of smoothness, u_x will be bounded, while in regions of non-smoothness, u_x may be large. More precisely, in shocks, we will have u_x large negative as a consequence of the entropy condition stating that the velocity u should decrease through the shock. The sign of u_x in shocks corresponding to the entropy condition is the fundamental ingredient in the proof of the a posteriori error estimates. In rarefaction waves, u will typically have the form $u(x,t) = x/t$ so that there $u_x \leqq \frac{1}{t}$. The presence of a large positive u_x for t (small in the case of rarefactions) is counterbalanced by the use of a norm which is weighted in time.

The perturbed problem (1.7) takes the following form in the present case:

$$\hat{u}_t + f(\hat{u})_x - \hat{\epsilon}\hat{u}_{xx} = \quad \text{in} \quad \mathbb{R}\times\mathbb{R}_+, \tag{3.1a}$$

$$\hat{u}(\cdot,0) = u_0 \quad \text{on} \quad \mathbb{R}, \tag{3.1b}$$

where the constant $\hat{\epsilon}$ is the same as in the finite element method (2.10) for (1.1), i.e., $\hat{\epsilon} = Ch$ in the simplified case considered first. We shall no prove an a posteriori error estimate for $\hat{e} = \hat{u} - U$ in a weighted L_2-norm following the scheme outlined in (1.6).

Error Representation: The Dual Problem and Galerkin Orthogonality

The linearized continuous dual problem takes the form: Find φ such that

$$-\varphi_t - a\varphi_x - \hat{\epsilon}\varphi_{xx} = \hat{e}\psi^{-1} \quad \text{in} \quad Q, \tag{3.2a}$$

$$\varphi(\cdot,T) = 0 \quad \text{in} \quad \mathbb{R}, \tag{3.2b}$$

where ψ is a positive weight function to be specified below, and

$$a = (\hat{u}+U)/2, \tag{3.3}$$

The error representation is now obtained multiplying (3.3a) by $\hat{e}$, integrating over each S_n, integrating by parts, and summing over n, which gives since $a\hat{e} = (\hat{u}^2 - U^2)/2$,

$$\begin{aligned}
\|\hat{e}\psi^{-\frac{1}{2}}\|_Q^2 &= \sum_{n=0}^{N} (\hat{e}, -\varphi_t - a\varphi_x - \hat{\epsilon}_{xx})_n \\
&= \sum_{n=0}^{N} (\hat{e}_t + (a\hat{e})_x, \varphi)_n + (\hat{\epsilon}\hat{e}_x)_n - \sum_{n=0}^{N} ([U^n], \varphi^n) \\
&= \sum_{n=0}^{N} (\hat{u}_t + \hat{u}\hat{u}_x - \hat{\epsilon}_{xx}, \varphi)_n \\
&- \sum_{n=0}^{M} ((U_t + UU_x, \varphi)_n + (\hat{\epsilon}U_x, \varphi_x)_n) + \sum_{n=0}^{N} ([U^n], \varphi^n)
\end{aligned}$$

so that recalling (3.1a) and using (2.6) with $\Phi \in V_h^\sigma$,

$$\begin{aligned}
\|\hat{e}\psi^{-\frac{1}{2}}\|_Q^2 &= \sum_{n=0}^{N} (U_t + UU_x, \Phi - \varphi)_n + \sum_{n=0}^{N} (\hat{\epsilon}U_x, (\Phi - \varphi)_x)_n \\
&+ \sum_{n=0}^{N} ([U^n], (\Phi - \varphi)_+^n)\} \equiv I + II + III.
\end{aligned} \tag{3.4}$$

Below we shall choose Φ to be a suitable interpolant of φ. We note that in principle the error representation (3.4) has the form (1.13) with a proper definition of $R(U)$ to be given. As indicated, the idea is now to estimate $\varphi - \Phi$ in terms of $\hat{e}\psi^{-\frac{1}{2}}$ using a strong stability estimate for the solution φ of the dual problem (3.3).

Interpolation Estimates for the Dual Solution

Let $\bar{P}_n : L_2(S_n) \to \bar{W}_n$ be the L_2-projection defined by

$$(\bar{P}_n \bar{w}, \bar{v})_n = (\bar{w}, \bar{v})_n \qquad \forall \bar{v} \in \bar{W}_n. \tag{3.5}$$

Further, define $\bar{\pi}_n : L_2(S_n) \to L_2(S_n)$ to be the L_2-projection w.r.t. $\bar{t}$ onto functions constant on I_n, defined by

$$\bar{\pi}_n \bar{w}|_{S_n} = \frac{1}{k_n} \int_{I_n} \bar{w}(\cdot, \bar{t}) d\bar{t}. \tag{3.6}$$

Let us now define $P_n : L_2(S_n) \to W_n$ by

$$\overline{P_n \varphi} = \bar{P}_n \bar{\varphi}, \tag{3.7}$$

and $\pi_n : L_2(S_n) \to L_2(S_n)$, by

$$\overline{\pi_n \varphi} = \bar{\pi}_n \bar{\varphi}. \tag{3.8}$$

We finally define P and π by setting

$$(P\varphi)|_{S_n} = P_n(\varphi|_{S_n}), \tag{3.9}$$

$$(\pi\varphi)|_{S_n} = \pi_n(\varphi|_{S_n}). \tag{3.10}$$

We note that

$$P\pi = \pi P, \tag{3.11}$$

since $\bar{P}_n \bar{\pi}_n = \bar{\pi}_n \bar{P}_n$ for $n = 0, 1, \ldots,$.

We shall now choose $\Phi = P\pi\varphi = \pi P\varphi$ in (3.5), which is legitimate since for $n = 0, 1, 2, \ldots, \overline{\pi_n P_n \varphi} = \bar{\pi}_n \bar{P}_n \bar{\varphi} \in \bar{V}_n$ so that $\pi_n P_n \varphi \in V_n$ where $\varphi = \varphi|_{S_n}$. Below we shall use the following estimates for the interpolation error $\varphi - \Phi = \varphi - P\varphi + P(\varphi - \pi\varphi)$. For simplicity we first assume that k, h and $\hat{\epsilon}$ are constant and for $n = 0, 1, \ldots$ there are positive constants ψ_i such that $\psi_1 \leqq \psi \leqq \psi_2$ on S_n with ψ_2/ψ_1 bounded independently of n. We also recall that (2.7) is assumed to be valid. We use below the notation $\tilde{R} \equiv RJ$, with J the Jacobian defined above.

Lemma 3.1. *There is a constant C such that for $R \in L_2(Q)$*

$$|(R, \varphi - P\varphi)_Q| \leqq C\|\hat{\epsilon}^{-1} h^2 (I - P)\tilde{R}\psi^{-\frac{1}{2}}\|_Q \|\hat{\epsilon}\varphi_{xx}\psi^{\frac{1}{2}}\|_Q, \tag{3.12a}$$

$$|(\hat{\epsilon}U_x, (\varphi - P\varphi)_x)_Q| \leqq C\|h^2 D_1 U\psi^{-\frac{1}{2}}\|_Q \|\hat{\epsilon}\varphi_{xx}\psi^{\frac{1}{2}}\|_Q, \tag{3.12b}$$

where

$$\overline{D_1 U}(\bar{x}, \bar{t}) = \max_{\tau \in I_n} \max_{j = i, i+1} |[\bar{U}_{\bar{x}}(x_j^n, \tau)]/h_j^n, \bar{x} \in J_i^n = (x_i^n, x_{i+1}^n),$$

and

$$[v(x_j^n)] = \lim_{s \to 0^+} (v(x_j^n + s) - v(x_j^n - s)).$$

Proof: We have

$$(R, \varphi - P\varphi)_Q = \sum_{n=0}^{N} \int_{S_n} \bar{R}\bar{J}(\bar{\varphi} - \bar{P}_n\bar{\varphi}) d\bar{x} d\bar{t} \tag{3.13}$$

$$
\begin{aligned}
&= \sum_{n=0}^{N} \int_{S_n} (I-\bar{P}_n)\bar{R}\bar{J}(\bar{\varphi}-\bar{P}_n\bar{\varphi})d\bar{x}d\bar{t} \\
&\leqq \sum_{n=0}^{N} \|(I-\bar{P}_n)\bar{R}\bar{J}\|_{S_n}\|\bar{\varphi}-\bar{P}_n\bar{\varphi}\|_{S_n}.
\end{aligned}
$$

Next, we note that with $\bar{\varphi}_h \in \bar{W}_n$ the standard nodal interpolant we have

$$
\|\bar{\varphi}-\bar{P}_n\bar{\varphi}\|_{S_n} \leqq \|\bar{\varphi}-\bar{\varphi}_h\|_{S_n} \leqq C\|h^2\bar{\varphi}_{\bar{x}\bar{x}}\|^*_{S_n}
$$

where

$$
\|\bar{\varphi}_{\bar{x}\bar{x}}\|^*_{S_n} \equiv \Big(\sum_i \int_{J_i^n\times I_n} (\bar{\varphi}_{\bar{x}\bar{x}})^2 d\bar{x}d\bar{t}\Big)^{\frac{1}{2}} \leqq C\|\varphi_{xx}\|_{S_n}, \tag{3.14}
$$

where the last inequality follows from (2.3), the fact that $\sigma_{n,x}$ is piecewise linear (or smooth) and (2.7). The desired estimate (3.12a) in the case $\psi = 1$ now follows combining (2.13) and (3.14), and the extension to a variable ψ with the stated assumptions on ψ is direct.

To prove (3.12b), we first note that

$$
\begin{aligned}
(\hat{\epsilon}U_x, (\varphi-P\varphi)_x)_Q &= \sum_{n=0}^{N} \int_{S_n} \hat{\epsilon}\bar{U}_{\bar{x}}\bar{J}^{-1}(\bar{\varphi}-\bar{P}_n\varphi)_{\bar{x}}\bar{J}^{-1}\bar{J}d\bar{x}d\bar{t} \\
&= \sum_{n=0}^{N} \int_{I_n} \sum_i [\hat{\epsilon}\bar{U}_x\bar{J}^{-1}](x_i^n,\bar{t})(\bar{\varphi}-\bar{P}_n\bar{\varphi})(x_i^n,\bar{t})d\bar{t}.
\end{aligned}
$$

Now, by well-known estimates for the L_2-projection $\bar{P}_n : L_2(\mathbb{R}) \to X_n$ we have

$$
\Big(\sum_i (\bar{\varphi}-\bar{P}_n\bar{\varphi})(x_i^n,\bar{t}))^2 h\Big)^{\frac{1}{2}} \leqq C\|h^2\bar{\varphi}_{\bar{x}\bar{x}}(\cdot,\bar{t})\|_{\mathbb{R}},
$$

with again evaluation of $\bar{\varphi}_{\bar{x}\bar{x}}(\cdot,\bar{t})$ on each interval J_i^n separately, from which (3.12b) follows as above. □

In the proof of the next lemma, we need a bound on the variation of the function $\gamma|a_x|$, acting as a weight, from one element to the next. Such a bound may be assumed to be valid by replacing $\gamma|a_x|$ by a smoother (possibly smaller) function. For simplicity, we state and prove the lemma with $\gamma|a_x|$ as a weight assuming appropriate smoothness.

Lemma 3.2. *There is a constant C such that for $R \in L_2(Q)$ and $\gamma \geqq 0$ constant*

$$
|(R, P(\varphi-\pi\varphi))_Q| \leqq C\|k(I-\pi)\tilde{R}\psi^{-\frac{1}{2}}\|_Q\|(\varphi_t+a\varphi_x)\psi^{\frac{1}{2}}\|_Q \tag{3.15}
$$

$$+C\min\big[\|k\alpha\hat{\epsilon}^{-\frac{1}{2}}(I-\pi)\tilde{R}\psi^{-\frac{1}{2}}\|_{L_1(L_2)}\|\hat{\epsilon}^{\frac{1}{2}}\varphi_x\psi^{\frac{1}{2}}\|_{L_\infty(L_2)},$$
$$\|k\alpha(\hat{\epsilon}\gamma|a_x|)^{-\frac{1}{2}}(I-\pi)\tilde{R}\psi^{-\frac{1}{2}}\|_Q\|(\hat{\epsilon}\gamma|a_x|)^{\frac{1}{2}}\varphi_x\psi^{\frac{1}{2}}\|_Q\big],$$

where for $t\in I_n$

$$\bar{\alpha}(\bar{x},\bar{t})=\max_{s\in I_n}|\bar{\sigma}_n-\bar{a})(\bar{x},s)|.$$

Proof: Integrating the relation

$$\bar{w}(\bar{t})-\bar{w}(\bar{s})=\int_{\bar{s}}^{\bar{t}}\bar{w}'(\bar{\tau})d\bar{\tau},$$

where $\bar{t},\bar{s}\in I_n$, over I_n with respect to $\bar{s}$, we see that

$$|(\bar{w}-\bar{\pi}\bar{w})(\bar{t})|\leqq\int_{I_n}(\bar{w}'(\bar{\tau})|d\tau,\quad \bar{t}\in I_n,$$

so that by integration with respect to $\bar{t}$ and Cauchy's inequality

$$\|\bar{w}-\pi\bar{w}\|_{I_n}\leqq k_n\|\bar{w}'\|_{I_n}. \tag{3.16}$$

We have

$$\begin{aligned}(3.17)\quad (R,P\varphi-P\pi\varphi)_Q &= (R,P\varphi-\pi P\varphi)_Q\\ &= \sum_{n=0}^{N}\int_{S_n}\bar{R}\bar{J}(\bar{P}\bar{\varphi}-\bar{\pi}\bar{P}\bar{\varphi})d\bar{x}d\bar{t}\\ &= \sum_{n=0}^{N}\int_{S_n}(\bar{R}\bar{J}-\bar{\pi}(\bar{R}\bar{J}))(\bar{P}\bar{\varphi}-\bar{\pi}\bar{P}\bar{\varphi})d\bar{x}d\bar{t}\\ &\leqq \sum_{n=0}^{N}\|k_n(I-\bar{\pi})\bar{R}\bar{J}\|_{S_n}\|k_n^{-1}(\bar{P}\bar{\varphi}-\bar{\pi}\bar{P}\bar{\varphi})\|_{S_n}\\ &\leqq \sum_{n=0}^{N}\|k_n(I-\bar{\pi})\bar{R}\bar{J}\|_{S_n}\|k_n^{-1}(\bar{\varphi}-\bar{\pi}\bar{\varphi})\|_{S_n}\\ &\leqq C\sum_{n=0}^{N}\|k_n(I-\pi)\tilde{R}\|_{S_n}\|\frac{d\bar{\varphi}}{d\bar{t}}\|_{S_n}.\end{aligned}$$

where (3.16) was used in the last inequality.

Now,

$$\frac{d\bar{\varphi}}{d\bar{t}}=\bar{\varphi}_t+\overline{\sigma_n\varphi_x},$$

and thus

$$\begin{aligned}\|\frac{d\bar{\varphi}}{dt}\|_{S_n} &\leqq \|\overline{\varphi_t + a\varphi_x}\|_{S_n} + \|\overline{(a-\sigma_n)\varphi_x}\|_{S_n} \\ &\leqq C\|\varphi_t + a\varphi_x\|_{S_n} + \|\alpha\varphi_x\|_{S_n},\end{aligned}$$

which combined with (3.17) proves (3.15) in the case $\psi \equiv 1, \gamma|a_x| \equiv 1$. The extension to variable ψ and $\gamma|a_x|$ is direct with the stated conditions on ψ and with suitable conditions on the variation of $\gamma|a_x|$. □

We now turn to the stability of the continuous dual problem (3.3), which is the critical issue. Once this is settled we will be in position to complete the proof of the a posteriori error estimate.

Strong Stability of the Dual Problem

We shall use the following stability estimate.

LEMMA 3.3. *There is a constant C such that if $\hat{\epsilon}$ is constant and $\psi(x,t)$ is a positive weight function satisfying for some constants $\beta, \gamma \geqq 0$,*

$$-\psi_t - a\psi_x + a_x\psi + 2\hat{\epsilon}\psi_x^2\psi^{-1} \leqq -\beta\psi - \gamma|a_x|\psi \quad in \quad Q = \mathbb{R}\times(0,T), \tag{3.18}$$

then the solution φ of (3.2) satisfies

$$\begin{aligned}\|(\varphi_t + a\varphi_x)\psi^{\frac{1}{2}}\|_Q \quad &+ \quad \|\hat{\epsilon}^{\frac{1}{2}}\varphi_x\psi^{\frac{1}{2}}\|_{L_\infty(L_2)} + \|(\hat{\epsilon}\gamma|a_x|)^{\frac{1}{2}}\varphi_x\psi^{\frac{1}{2}}\|_Q \\ &+\|\hat{\epsilon}\varphi_{xx}\psi^{\frac{1}{2}}\|_Q \leqq C\|e\psi^{-\frac{1}{2}}\|_Q.\end{aligned}$$

Proof: Multiplying (3.2a) by $-(\varphi_t + a\varphi_x)\psi$ and integrating over $Q_\tau = \mathbb{R}\times(\tau,T)$ gives

$$\begin{aligned}-\int_{Q_\tau} e(\varphi_t + a\varphi_x)dxdt &= \int_{Q_\tau}(\varphi_t + a\varphi_x)^2\psi dxdt + \int_{Q_\tau}\hat{\epsilon}\varphi_{xx}\varphi_t\psi dxdt \\ &\quad + \int_{Q_\tau}\hat{\epsilon}\varphi_{xx}a\varphi_x\psi dxdt \\ &= \int_{Q_T}(\varphi_t + a\varphi_x)^2\psi dxdt - \int_{Q_\tau}\hat{\epsilon}\varphi_x\varphi_{xt}\psi dxdt \\ &\quad - \int_{Q_\tau}\hat{\epsilon}\varphi_x\varphi_t\psi_x dxdt + \int_{Q_\tau}\frac{\hat{\epsilon}}{2}(\varphi_x^2 a\psi)_x dxdt - \int_{Q_\tau}\frac{\hat{\epsilon}}{2}\varphi_x^2(a\psi)_x dxdt \\ &= \int_{Q_\tau}(\varphi_t + a\varphi_x)^2\psi dxdt - \int_{Q_\tau}\frac{\hat{\epsilon}}{2}(\varphi_x^2\psi)_t dxdt + \int_{Q_\tau}\frac{\hat{\epsilon}}{2}\varphi_x^2\psi_t dxdt\end{aligned}$$

$$
\begin{aligned}
&- \int_{Q_\tau} \hat{\epsilon}\varphi_x(\varphi_t + a\varphi_x)\psi_x dxdt + \int_{Q_\tau} \hat{\epsilon}\varphi_x^2 a\psi_x dxdt - \int_{Q_\tau} \frac{\hat{\epsilon}}{2}\varphi_x^2(a\psi)_x dxdt \\
&\geqq \int_{Q_\tau} (\varphi_t + a\varphi_x)^2\psi dxdt + \int_{\mathbb{R}} \frac{\hat{\epsilon}}{2}(\varphi_x^2\psi)(x,\tau)dx \\
&+ \int_{Q_\tau} \frac{\hat{\epsilon}}{2}\varphi_x^2\psi_t dxdt - \frac{1}{4}\int_{Q_\tau} (\varphi_t + a\varphi_x)^2\psi dxdt \\
&- \int_{Q_\tau} \hat{\epsilon}^2\varphi_x^2\psi_x^2\psi^{-1} dxdt - \int_{Q_\tau} \frac{\hat{\epsilon}}{2}\varphi_x^2(a_x\psi - a\psi_x)dxdt
\end{aligned}
$$

so that

$$
\begin{aligned}
&\int_{Q_\tau} (\varphi_t + a\varphi_x)^2\psi dxdt + \int_{\mathbb{R}} \frac{\hat{\epsilon}}{2}(\varphi_x^2\psi)(x,\tau)dx \\
\leqq &\int_{Q_\tau} \frac{\hat{\epsilon}}{2}\varphi_x^2(-\psi_t - a\psi_x + a_x\psi + 2\hat{\epsilon}\psi_x^2\psi^{-1}) \\
+ &\frac{1}{2}\int_{Q_\tau} (\varphi_t + a\varphi_x)^2\psi dxdt + \int_{Q_\tau} e^2\psi^{-1} dxdt,
\end{aligned}
$$

from which the lemma follows, using the equation (3.2a) to estimate $\hat{\epsilon}\varphi_{xx}\psi^{\frac{1}{2}}$. □

Corollary 3.4. *The assumption (3.18) of Lemma 3.3 is satisfied in the following basic cases*

1. $\psi \equiv 1$ *and* $a_x \leqq 0, \beta = 0, \gamma = 1$, *corresponding to a shock,*
2. $\psi(x,t) = \left(\frac{1}{T}\right)^{1+\gamma}, \max a_x(\cdot,t) \leqq \frac{1}{t}$, $\beta = 0$, $\gamma \geqq 0$, *corresponding to a rarefaction wave* $u(x,t) = \frac{x}{t}$, *and*
3. $\psi = \exp(3\beta(t-T))$ *and* $|a_x| \leqq \beta$, $\gamma = 1$ *corresponding to a regular solution.*

A posteriori Error Estimates

We shall now derive a posteriori error estimates by estimating the terms I - III in the error representation (3.4), using the strong stability estimate of Lemma 3.3 coupled with the interpolation estimates of Lemma 3.1-2. We shall below use the "discrete second derivative" $D_2 : W_n \rightarrow W_n$ defined by

$$(D_2 w, v)_n = -(w_x, v_x)_n \qquad \forall v \in W_n.$$

We have

$$I = (R_1, P\varphi - \varphi)_Q + (R_1, P(\pi\varphi - \varphi))_Q \equiv I_1 + I_2,$$

where
$$R_1 = U_t + UU_x = (U-\sigma)U_x.$$
First we note that $I_2 = 0$ since
$$I_2 = \sum_{n=0}^{N}\int_{S_n}(\bar U - \bar\sigma)\bar U_{\bar x}\frac{d\bar x}{dx}(\bar P\bar\varphi - \bar\pi\bar P\bar\varphi)\frac{dx}{d\bar x}d\bar x d\bar t = 0,$$
because $\bar U, \bar\sigma$ and $\bar U_{\bar x}$ are constant in $\bar t$ on each I_n.

Further, by Lemma 3.1 using the orthogonality related to the L_2-projection $\bar P_n$, we get
$$|I_1| \leqq C\|\hat\epsilon^{-1}h^2(I-P)\tilde R_1\psi^{-\frac12}\|_Q\|\hat\epsilon\varphi_{xx}\psi^{\frac12}\|_Q,$$
where, as indicated above, $\tilde R_1 = R_1 J$.

Next,
$$\begin{aligned} II &= (\hat\epsilon U_x, (P\varphi-\varphi)_x)_Q + (\hat\epsilon U_x, (P(\pi\varphi-\varphi)))_x)_Q \\ &= (\hat\epsilon U_x, (P\varphi-\varphi)_x)_Q - (\hat\epsilon D_2 U, P(\pi\varphi-\varphi))_Q = II_1 + II_2. \end{aligned}$$
By Lemma 3.1 we have
$$|II_1| \leqq C\|h^2 D_1 U\psi^{\frac12}\|_Q\|\hat\epsilon\varphi_{xx}\psi^{\frac12}\|_Q,$$
and by Lemma 3.2
$$\begin{aligned} |II_2| &\leqq C\|k\hat\epsilon(I-\pi)D_2U\psi^{-\frac12}\|_Q\|(\varphi_t + a\varphi_x)\psi^{\frac12}\|_Q \\ &+ C\min[\|ka\hat\epsilon^{\frac12}(I-\pi)D_2U\psi^{-\frac12}\|_{L_1(l_2)}\|\hat\epsilon^{\frac12}\varphi_x\psi^{\frac12}\|_{L_\infty(L_2)}, \\ &\qquad \|k\alpha|\hat\epsilon\gamma a_x|^{\frac12}(I-\pi)D_2U\psi^{-\frac12}\|_Q\|(\hat\epsilon\gamma|a_x|)^{\frac12}\varphi_x\psi^{\frac12}\|_Q]. \end{aligned}$$
For use below, we note that $II_2 = 0$ if $\bar J = 1$, corresponding to $\sigma^n(\bar x, t_n)_+$ begin constant in $\bar x$ (constant mesh convection). Finally, for the third term III in the error representation (3.4) we have
$$III = \sum_{n=0}^{N}([U^n], ((P\varphi-\varphi)_+^n) + \sum_{n=0}^{N}([U^n], (\pi P\varphi - P\varphi)_+^n) \equiv III_1 + III_2.$$
Considering first III_1, we have with $P_n : L_2(\mathbb{R}) \to [X_n]^m$, the L_2-projection
$$\begin{aligned} III_1 &= \sum_{n=0}^{N}(U_+^n - U_-^n, (P_n - I)\varphi_+^n) \\ &= \sum_{n=0}^{N}(P_nU_-^n - U_-^n, (P_n-I)\varphi_+^n) \\ &\equiv \sum_{n=0}^{N}(R_{31}(t_n)_+, (P_n-I)\varphi_+^n)k_n, \end{aligned} \tag{3.19}$$

where we define

$$R_{31}(t) \equiv (P_n - I)U_-^n/k_n \quad \text{for} \quad t \in I_n.$$

Now, to estimate $(P_n - I)\varphi_+^n$ we first note that

$$\varphi_+^n(\bar{x}) = \varphi(x,t) - \int_{t_n}^{t} \frac{d}{ds}\varphi(\bar{x} + \sigma^n(\bar{x},t_n)_+(s-t_n),s)ds,$$

where $x = \bar{x} + \sigma^n(\bar{x},t_n)_+(t-t_n)$ so that

$$\begin{aligned} k_n\varphi_+^n(\bar{x}) &= \int_{I_n} \varphi(\bar{x} + \sigma^n(\bar{x},t_n)_+(t-t_n),t)dt \\ &+ \int_{I_n}\int_{t}^{t_n} (\varphi_t + \sigma\varphi_x)(\bar{x} + \sigma^n(\bar{x},t_n)_+(s-t_n),s)dsdt. \end{aligned}$$

Inserting this representation into the right hand side of (3.19), using an estimate for $(P_n - I)$ analogous to (3.12a) together with the smoothness of σ, we get

$$\begin{aligned} |III_1| &\leqq C(\|\hat{\epsilon}^{-1}h^2R_{31}\psi^{-\frac{1}{2}}\|_Q\|\hat{\epsilon}\varphi_{xx}\psi^{\frac{1}{2}}\|_Q \\ &+ \|kR_{31}\|_Q\|(\varphi_t + a\varphi_x)\psi^{\frac{1}{2}}\|_Q \\ &+ \min(\|k\alpha\hat{\epsilon}^{-\frac{1}{2}}R_{31}\psi^{-\frac{1}{2}}\|_{L_1(L_2)}\|\hat{\epsilon}^{\frac{1}{2}}\varphi_x\psi^{\frac{1}{2}}\|_{L_\infty(L_2)}, \\ &\quad \|k\alpha(\hat{\epsilon}\gamma|a_x|)^{-\frac{1}{2}}R_{31}\psi^{-\frac{1}{2}}\|_Q\|(\hat{\epsilon}\gamma|a_x|)^{\frac{1}{2}}\varphi_x\psi^{\frac{1}{2}}\|_Q)). \end{aligned}$$

Finally, for III_2 we have with $R_{32} \equiv (U_+^n - U_-^n)/k_n$ on S_n,

$$\begin{aligned} |III_2| &\leqq C\|kR_{32}\psi^{-\frac{1}{2}}\|_Q\|(\varphi_t + a\varphi_x)\psi^{\frac{1}{2}}\|_Q \\ &+ \min(\|k\alpha\hat{\epsilon}^{-\frac{1}{2}}R_{32}\psi^{-\frac{1}{2}}\|_{L_1(L_2)}\|\hat{\epsilon}^{\frac{1}{2}}\varphi_x\psi^{\frac{1}{2}}\|_{L_\infty(L_2)}. \\ &\quad \|k\alpha(\hat{\epsilon}\gamma|a_x|)^{-\frac{1}{2}}R_{32}\psi^{-\frac{1}{2}}\|_Q\|(\hat{\epsilon}\gamma|a_x|)^{\frac{1}{2}}\varphi_x\psi^{\frac{1}{2}}\|_Q)). \end{aligned}$$

We have now proved the following a posteriori error estimate.

THEOREM 3.5. *Suppose, the assumptions of Lemma 3.1-3.3 are valid. Then there is a constant C such that if $\hat{u}$ is the solution of (3.1) and U that of (2.10), then*

$$\begin{aligned} \|(\hat{u} - U)\psi^{-\frac{1}{2}}\|_Q &\leqq C(\|\hat{\epsilon}^{-1}h^2(|(I-P)\tilde{R}_1| + |R_{21}| + |R_{31}|)\psi^{-\frac{1}{2}}\|_Q \\ &+ \|k(|I-\pi)\tilde{R}_{22}| + |R_{31}| + |R_{32}|)\psi^{-\frac{1}{2}}\|_Q \\ &+ \min(\|k\alpha\hat{\epsilon}^{-\frac{1}{2}}(|(I-\pi)\tilde{R}_{22}| + |R_{31}| + |R_{32}|)\psi^{-\frac{1}{2}}\|_{L_1(L_2)}, \\ &\quad \|k\alpha(\hat{\epsilon}\gamma|a_x|)^{-\frac{1}{2}}(|(I-\pi)\tilde{R}_{22} + +|R_{31}| + |R_{32}|\psi^{-\frac{1}{2}}\|_Q), \end{aligned}$$

where $\gamma \geqq 0$ is given in Lemma 3.3, and

$$\begin{aligned} R_1 &= U_t + f(U)_x = U_t + f'(U)U_x = (f'(U) - \sigma)U_x, \\ R_{21} &= \hat{\epsilon} D_1 U,\ R_{22} = \hat{\epsilon} D_2 U, \\ R_{31} &= (P_n - I)U_-^n / k_n \quad \text{on} \quad S_n, \\ R_{32} &= (U_+^n - U_-^n)/k_n \quad \text{on} \quad S_n. \end{aligned}$$

Theorem 3.5 may be stated in a more concrete form by estimating the terms $(I - P_n)U_-^n$ and $(I - P)\tilde{R}_1$ explicitly. In Theorem 3.6 below we will state such a concrete result. We than extend the setting to a general strictly convex flux function f. We have

$$\|(I - P_n)U_-^n\|_{S_n} \leqq C\|h^2 D_1 U\|_{S_{n-1}}. \tag{3.20}$$

Next, assuming for simplicity that $\sigma_n(\bar{x}, \bar{t})$ is constant in $\bar{x}$, we have

$$\begin{aligned} \|(I - P)\tilde{R}_1\|_{S_n} &= \|(I - P)(f'(U) - \sigma)U_x\|_{S_n} \\ &= \|(I - P)(f(U) - \sigma U)_x\|. \end{aligned} \tag{3.21}$$

To estimate the right hand side of (3.21), let now $\bar{U}$, with some abuse of notation, be a standard mollification of U on the length scale h, i.e.,

$$\bar{U}(x,t) \equiv \int_{-\infty}^{\infty} U(x - y, t)\omega_h(y)dy, \tag{3.22}$$

where $0 \leqq \omega \in \mathcal{C}_0^\infty(\mathbb{R})$ and

$$\int_{-\infty}^{\infty} \omega(x)dx = 1, \quad \omega_h(x) \equiv h^{-1}\omega(h^{-1}x).$$

Then, we have

$$\begin{aligned} \|(I - P)(f(\bar{U}) - \sigma\bar{U})_x\|_Q \ &\leqq \ C\|h^2(|f'(\bar{U}) - \sigma)\bar{U}_{xxx}| \\ &\quad +3|f'(\bar{U})_x \bar{U}_{xx}| + |f^{(3)}(\bar{U})\bar{U}_x^3|)\|_Q. \end{aligned} \tag{3.23}$$

It follows from (3.22) and an inverse estimate that

$$\|(I - P)[f(U) - f(\bar{U}) - \sigma(U - \bar{U})]_x\|_Q \leqq C\|h(\bar{\bar{f'}} - \sigma)D_1 U\|_Q, \tag{3.24}$$

where

$$\bar{\bar{f'}} \equiv \int_0^1 f'(\bar{U}s + U(1 - s))ds, \tag{3.25}$$

and

$$\|h^2(f'(\bar U)-\sigma)\bar U_{xxx}\|_Q+\|h^2 f'(\bar U)_x\bar U_{xx}\|_Q \leqq C\|h(f'(\bar U-\sigma)D_1U\|_Q,$$
$$\|h^2 f^{(3)}(\bar U)\bar U_x^3\|_Q \leqq C\|h^2 f^{(3)}(\bar U)U_x^3\|_Q.$$

We thus have the following particular case of Theorem 3.5, recalling that $(I-\pi)\tilde R_{22}=0$ if $\bar\sigma_n(\bar x,\bar t)$ is constant in $\bar x$.

THEOREM 3.6. *Let f be a scalar strictly convex flux function. Assume that $\max a_x(\cdot,t)\leqq \max(\frac{1}{t},\beta)$ where $a=\int_0^1 f'(s\hat u+(1-s)U)ds$, and that $\sigma_n(\bar x,\bar t)$ is constant in $\bar x$. Let $\hat u$ be the solution of (3.1) and U that of (2.10). Then there is a constant C such that*

$$\begin{aligned}(3.26)\qquad |||\hat u-U|||_Q \leqq\ & C(|||h^2(1+\hat\epsilon^{-1}h|f'-\sigma|)D_1U|||_Q\\ &+|||\hat\epsilon^{-1}h^4 f^{(3)}(\bar U)U_x^3|||_Q)+C|||k\partial_{\bar t}U|||_Q\\ &+C\min(|||\alpha\hat\epsilon^{-\frac12}h^2D_1U|||^{\#}_{L_1(L_2)},|||\alpha(\hat\epsilon\gamma|a_x|)^{-\frac12}h^2D_1U||_Q),\\ &+C\min(|||\alpha\hat\epsilon^{-\frac12}k\partial_{\bar t}U|||_{L_1(L_2)},|||\alpha(\hat\epsilon\gamma|a_x|)^{-\frac12}k\partial_{\bar t}U|||_Q),\end{aligned}$$

where

$$\begin{aligned}|f'-\sigma| &= \max(|\overset{=}{f'}-\sigma|,|f'(\bar U)-\sigma|),\\ |||\cdot|||. &= \|\cdot\psi^{-\frac12}\|.,\\ \psi &= \left(\frac{t}{T}\right)^{1+\gamma}\exp(3\beta(t-T)),\ 0\leqq\gamma<<1,\\ \partial_{\bar t}U &= (U_+^n-U_-^n)/k_n \quad on \quad S_n,\end{aligned}$$

and the terms marked with # only get contributions in case of remeshing in space, i.e., when $(I-P_n)U_-^n\neq 0$. Here $\overset{=}{f'}$ is the average of f' given by (3.25). If $\max(a_x(\cdot,t))\leqq 0$, then (3.26) holds with $\psi\equiv 1$.

We note in the estimate (3.26), the presence of the terms $\|h^2D_1U\|_Q$ (if $\hat\epsilon = Ch$) and $\|k\partial_tU\|_Q$ naturally corresponding to the terms $\|h^2u_{xx}\|_Q$ and $\|ku_{\bar t}\|_Q$, arising in pure interpolation with piecewise linears and constants, respectively. In addition, we have the term $\|\hat\epsilon^{-1}h^4\overset{=}{f}^{(3)}(\bar U)|U_x|^3\|_Q$ which is connected with the non-linearity of the problem and has no counterpart in pure interpolation. Finally, we have the $\hat\epsilon$-terms connected with h and k; in a region of smoothness of u (or a rarefaction wave, except close to $t=0$), we expect α to be small ($\alpha\leqq\hat\epsilon^{\frac12}$) in which case the $\hat\epsilon$-terms may be absorbed by the previous terms. Close to a shock, we expect to have $\hat\epsilon|a_x|\geqq c$, $\alpha\geqq c$, in which case again the $\hat\epsilon$-terms may be absorbed. To sum up, it appears that choosing $\sigma\sim U$, the a posteriori error estimate of Theorem 3.5 in the case $\hat{\epsilon} = Ch$ takes the simple form (1.17). In a shock we expect $\hat{\epsilon} \sim h$ and $h^3 u_x^3 \sim h^2 u_{xx} \sim 1$ while in a rarefaction wave $h^3 u_x^3 \sim (\frac{h}{t})^3$ and $h^2 u_{xx} \sim (\frac{h}{t})^2$.

Concerning the perturbation error $|||u - \hat{u}|||_Q$, we expect $O(|\hat{\epsilon} - \epsilon|^{\frac{1}{2}})$ contributions from shocks and rarefactions (see [8]) and a contribution of order $O(|\hat{\epsilon} - \epsilon|)$ from smooth regions of u, cf. also Section 4.3 and Remark 3.7 below.

Remark 3.7. To estimate the perturbation part of the error $u - \hat{u}$, we may again use a dual problem of the form (3.2) with now $a = (u + \hat{u})/2$ and $\hat{\epsilon} = \epsilon$ to obtain the following error representation (with $\psi = 1$ for simplicity):

$$\|u - \hat{u}\|_Q^2 = -\int_Q (\hat{\epsilon} - \epsilon)\hat{u}_{xx}\varphi dxdt = \int_Q (\hat{\epsilon} - \epsilon)\hat{u}_x \varphi_x dxdt, \tag{3.27}$$

where we integrated by parts assuming ϵ and $\hat{\epsilon}$ to be constant. Using an analog of Lemma 3.1, we obtain for instance the following estimate for $u - \hat{u}$ in the presence of a shock:

$$\|u - \hat{u}\|_Q \leqq C\|\min(\epsilon^{-\frac{1}{2}}, (\epsilon|a_x|)^{-\frac{1}{2}})|\hat{\epsilon} - \epsilon|\hat{u}_x\|_Q. \tag{3.28}$$

If $\hat{\epsilon}$ is comparable to ϵ, this would give the expected $O(|\hat{\epsilon} - \epsilon|^{\frac{1}{2}})$ estimate in L_2 in the shock case. In the adaptive algorithm, $\|u - \hat{u}\|_Q$ would be controlled by replacing $\hat{u}$ by U in relevant forms of (3.28), (cf. [3]). □

Remark 3.8. In practice the term $\|(I - P)\tilde{R}_1\|_{S_n}$ in the a posteriori error estimate of Theorem 3.5 may be estimated by direct computational evaluation. In particular, the tilting direction σ may be determined through minimization of $|f'(U) - \sigma)U_x|$ with U taken, e.g., from the previous time step or (see [11]).

□

4. SYSTEMS OF CONSERVATION LAWS

Introduction

We now return to the system of conservation laws (1.1): Find $u : \mathbb{R}\times\mathbb{R}_+ \rightarrow \mathbb{R}^m$ such that

$$u_t + f(u)_x - \epsilon u_{xx} = 0 \quad x \in \mathbb{R},\, t > 0, \tag{4.1a}$$

$$u(x,0) = u_0(x) \quad x \in \mathbb{R}, \tag{4.1b}$$

where $f : \mathbb{R}^m \to \mathbb{R}^m$ is a smooth flux and u_0 has compact support. Below we shall prove an a posteriori error estimate analogous to that of Theorem 3.5 for the error $\hat{e} = \hat{u} - U$, where U is the finite element solution and $\hat{u}$ is the solution of (4.1) with ϵ replaced by $\hat{\epsilon}$ with the artificial viscosity in (2.10). We assume that (4.1) is strictly hyperbolic and genuinely non-linear, i.e., for all $u \in \mathbb{R}^m$ the Jacobian $f'(u)$ of $f(u)$ has m distinct eigenvalues $\lambda_1(u) < \lambda_2(u) < \ldots < \lambda_m(u)$, satisfying $\nabla\lambda_i(u) \cdot r_i(u) \neq 0$ where $r_i(u)$ is a right eigenvector corresponding to the i:th eigenvalue $\lambda_i(u)$ of $f'(u)$. Denoting by $R(u) = (r_1(u), \ldots, r_m(u))$, the $m \times m$ matrix of right eigenvectors $r_i(u)$ of $f'(u)$ and by $\Lambda(u) = \mathrm{diag}(\lambda_i(u))$ the $m \times m$ diagonal matrix with diagonal elements $\lambda_i(u)$, we have

$$f'(u)R(u) = R(u)\Lambda(u). \tag{4.2}$$

Our aim is now to prove an a posteriori estimate for the SD-method (2.10) for (4.1) analogous to that stated in Theorem 3.5 for Burgers' equation above. We shall then assume that $Q = \mathbb{R} \times (0,T)$ may be split into a shock region S, a simple wave region E and a regular region R as specified below. In particular, contact discontinuities are excluded for the moment. We shall assume that all shock waves S_{ni} and simple waves E_{ni} in the i:th family, $n = 1, \ldots, i = 1, \ldots, m$, are weak and are separated by regions where the solution is smooth. We also assume that no new shock waves or simple waves are created which did not exist at $t = 0$, that is, wave collisions are not allowed.

We recall the associated dual problem

$$-\varphi_t - \bar{f}'^T\varphi_x - \hat{\epsilon}\varphi_{xx} = \hat{e}W^{-1} \quad \text{in} \quad Q, \tag{4.3a}$$

$$\varphi(\cdot, T) = 0. \tag{4.3b}$$

where

$$\bar{f}' = \int_0^1 f[(s\hat{u} + (1-s)U)ds, \tag{4.4}$$

and where W is a matrix-valued weight to be specified below. The main work will now be to prove an analog of the strong stability estimate of Lemma 3.3 for the system (4.3) using a weighted norm with weight Wconstructed through diagonalization of $\bar{f}'$.

We now state the assumptions to be used to construct the weight W with the desired properties. These assumptions concern both the exact solution $\hat{u}$ and the finite element solution. Verification of these assumptions is possible in model cases (see [35]), and thus the assumptions appear to be realistic. The assumptions on U only may also be verified a posteriori by a direct check. To check analytically the validity of the assumptions in a more general case is difficult, and we may have to rely on verifying instead a posteriori corresponding verifiable conditions with $\hat{u}$ replaced by U. First, we assume that

$$\|\hat{u}\|_{BV} + \|U\|_{BV} << 1, \tag{4.5}$$

where $\|\cdot\|_{BV}$ denotes the total variation in space-time, corresponding to the shocks and simple waves being weak. Further, we shall assume that there is a constant $c > 1$ such that

$$|(\hat{u} - U)_x| \leqq c|\bar{u}_x|, \tag{4.6}$$

where

$$\bar{u} = (\hat{u} + U)/2. \tag{4.7}$$

Next, we assume that the shock region S is the disjoint union $S = \cup_{\substack{n=1,\dots,N_p \\ p=1,\dots,m}} S_{np}$ of p-shock waves S_{np}:

$$S_{np} = \cup_{0<t<T}(x_-^{np}(t), x_+^{np}(t)) \times \{t\},$$

where $(x_-^{np}(t), x_+^{np}(t)) \neq \emptyset$ is an interval, and there is a smooth function $s = s(x,t)$ such that for a constant $0 < \delta << 1$

$$|\bar{u}_t + s\bar{u}_x| \leqq \delta|\bar{u}_x| \quad \text{in} \quad S_{np}, \tag{4.8a}$$

where

$$\lambda_p(\bar{u}(x_+^{np}(t), t)) < s < \lambda_p(\bar{u}(x_-^{np}(t), t), \; 0 < t < T, \tag{4.8b}$$

and

$$\lambda_p(\bar{u})_x \leqq 0 \quad \text{in} \quad S_{np}. \tag{4.9a}$$

$$|\bar{u}_t| + |\bar{u}_x| \leqq -C\lambda_p(\bar{u})_x \quad \text{in} \quad S_{np}. \tag{4.9b}$$

Similarly, we assume that the simple wave region $E = \cup_{\substack{n=1,\dots,N_p \\ p=1,\dots,m}} E_{np}$ is the disjoint union of p-simple waves

$$E_{np} = \cup_{0<t<T}(x_-^{np}(t), x_+^{np}(t)) \times \{t\},$$

where for a constant $0 < \delta << 1$,

$$\begin{aligned}
&(4.10\text{a}) & |\bar{u}_t + \lambda_p(\bar{u})\bar{u}_x| &\leqq \delta|\bar{u}_x|, \\
&(4.10\text{b}) & \frac{1-\delta}{t} < \lambda_p(\bar{u})_x &\leqq (1+\delta)/t, \\
&(4.10\text{c}) & |\bar{u}_t| + |\bar{u}_x| &\leqq C|\lambda_p(\bar{u})_x|.
\end{aligned}$$

We shall assume that $\bar{S} \cap \bar{E} = \emptyset$, corresponding to non-interacting waves, and finally we define the regular region $R = Q\backslash(\bar{S} \cup \bar{E})$ and we assume that for some positive constant β

$$|(\bar{u}_t, \bar{u}_x)| \leqq \beta \quad \text{in} \quad R. \tag{4.11}$$

Let us now note that we may assume that the matrix $\bar{f}'$ defined by (4.4), has m real and distinct eigenvalues $\bar{\lambda}_i$ with corresponding right eigenvectors $\bar{r}_i$, since

$$\bar{f}' = f'(\hat{u}) + O(\hat{e}), \tag{4.12}$$

and by (4.5) we have

$$||\hat{e}||_{L^\infty(Q)} \leqq ||\hat{e}||_{BV} << 1.$$

Thus, we have the factorization

$$\bar{f}'\bar{R} = \bar{R}\bar{\Lambda}, \tag{4.13}$$

where $\bar{\Lambda} = \text{diag}(\bar{\lambda}_i)$ and $\bar{R} = (\bar{r}_1, \ldots, \bar{r}_m)$. We shall use a matrix valued weight W below of the form

$$W = \bar{R}\Psi\bar{R}^T, \tag{4.14}$$

where $\Psi = \text{diag}(\psi_i)$ is a diagonal matrix with diagonal elements ψ_i defined as follows. For suitable constants $0 < c^* << 1$ and $C^* > 0$ to be specified below, let

$$\begin{aligned}
\tilde{\psi}_i(x,t) \equiv \exp[C^* &\sum_{\{np:S_{np}\subset S\}} a^i_{np}(\lambda_p(u^{np}) - \lambda_p(u^{np}_-)) \\
-C^* &\sum_{\{np:E_{np}\subset E\}} a^i_{np}(\lambda_p(u^{np}) - \lambda_p(u^{np}_-))],\ i = 1, \ldots, m,
\end{aligned} \tag{4.15a}$$

where

$$u^{np}(x,t) = \begin{cases} \bar{u}(x,t) & \text{if} \quad (x,t) \in S_{np}, \\ \bar{u}(x^{np}_\mp(t), t) & \text{if} \quad \pm(x^{np}_\mp(t) - st) > \pm(x - st), \end{cases}$$

$$\begin{aligned}
u^{np}_\mp(t) &= \bar{u}(x^{np}_\mp(t), t), \\
a^i_{np} &= \begin{cases} -1 & \text{if} \quad i < p, \\ 1 & \text{if} \quad i > p, \\ 0 & \text{if} \quad i = p, \end{cases}
\end{aligned} \tag{4.15b}$$

and let

$$\psi_0(t) = \begin{cases} 1 & \text{if} \quad E = \emptyset, \\ (t/T)^{1+c^*} & \text{if} \quad E \neq \emptyset. \end{cases} \tag{4.15c}$$

Then we define

$$\psi_i(x,t) = \tilde{\psi}_i(x,t)H(t), \quad i = 1, \ldots, m. \tag{4.15d}$$

where

$$H(t) = \exp(C^*\beta(T-t))\psi_0(t). \tag{4.15e}$$

Note that, if $(x,t) \in S_{ni}$, then $\psi_i(x,t)_x = 0$ since $a^i_{ni} = 0$.

Proposition 4.1. *We have*

$$\frac{1}{2} < \tilde{\psi}_i < 2 \quad i = 1, \ldots, m, \tag{4.16}$$

$$\psi_{i,c} = \begin{cases} C^* a^i_{np}\lambda_p(\bar{u})_x\tilde{\psi}_i & \textit{if} \quad (x,t) \in S_{np}, \\ -C^* a^i_{np}\lambda_p(\bar{u})_x\tilde{\psi}_i & \textit{if} \quad (x,t) \in E_{np}, \\ 0 & \textit{if} \quad (x,t) \in R. \end{cases} \tag{4.17}$$

Proof: By the definition of $\tilde{\psi}_i$ we have

$$\lim_{x \to -\infty} \tilde{\psi}_i(x,t) = 1.$$

Each time (x,t) cross a simple wave E_{np} or shock wave S_{np}, cf. fig. 4.1, the function $\psi_i(\cdot,t)$ changes monotonically with a factor bounded above and below respectively by

$$\exp[\pm C^*(\lambda_p(u^{np}_+(t)) - \lambda_p(u^{np}_-(t))].$$

From assumption (4.5) we conclude that (4.16) holds.

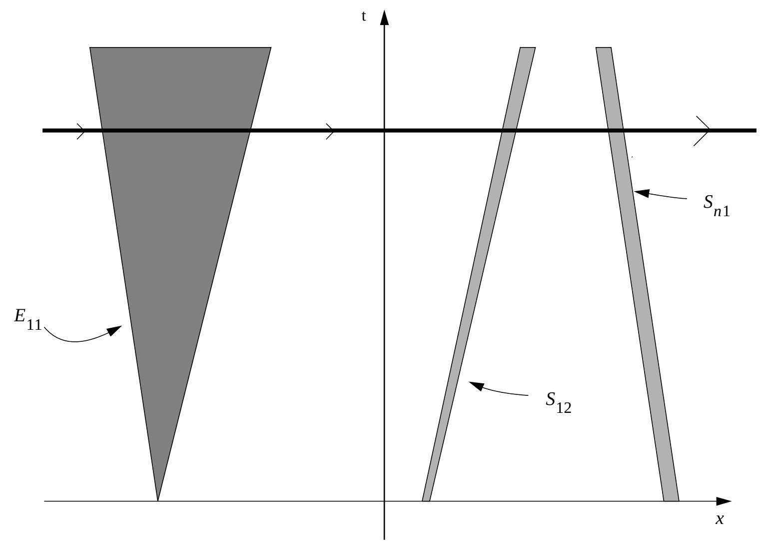

Fig. 4.1 Path of integration for shocks and rarefactions

The equality (4.17) follows by differentiating (4.15). □

A similar weight as (4.15a) was introduced by Goodman in [6] and was also used in [35] and [37]. The purpose of the weight functions $\tilde{\psi}_i(x)$ is essentially to introduce a quadratic term with the good sign with a weight $|\lambda_{p,x}|$ in S_{np}, $p \neq i$, cf. (4.31) below. We note the following properties of the weight W defined by (4.14).

Proposition 4.2. *The matrix W defined by (4.14) is positive-definite and symmetric and $W\bar{f}'^T$ is symmetric.*

Proof: The first statement concerning W is obvious. We have $W\bar{f}'^T = \bar{R}\Psi\bar{R}^T\bar{f}'^T = \bar{R}\Psi\bar{\Lambda}\bar{R}^T = \bar{R}\bar{\Lambda}\Psi\bar{R}^T = \bar{f}'\bar{R}\Psi\bar{R}^T = \bar{f}'W = (W\bar{f}'^T)^T$, which proves the symmetry of $W\bar{f}'^T$.

Proposition 4.3. *For $g(u) = r_i(u), l_i(u)$ or $\lambda_i(u)$, we have*

$$\bar{g} \equiv \int_{-\frac{1}{2}}^{\frac{1}{2}} g(\bar{u} + \hat{e}s)ds = g(\bar{u}) + O(|\hat{e}|^2), \tag{4.18}$$

$$\begin{aligned} \bar{g}_x &= g(\bar{u})_x + O(|\hat{e}|^2|\bar{u}_x| + |\hat{e}||\hat{e}_x|) \\ &= g(\bar{u})_x + O(|\hat{e}||\bar{u}_x|). \end{aligned} \tag{4.19}$$

Proof: The equality (4.18) follows directly by Taylor expansion. to prove (4.19), we note that

$$\bar{g}_x = \int_{-\frac{1}{2}}^{\frac{1}{2}} g'(\bar{u} + \hat{e}s)(\bar{u}_x + \hat{e}_x s)ds = g(\bar{u})_x + O(|\hat{e}|^2|\bar{u}_x| + |\hat{e}||\hat{e}_x|).$$

which by (4.6) prove (4.19). □

The Strong Stability Estimate

We define

$$\lambda^*(x,t) = \lambda_p(\bar{u}(x,t)) \quad \text{if} \quad (x,t) \in S_{np} \cup E_{np},$$

with λ^* varying linearly in x on each line segment $I(t) = \{(x,t) : a < x < b\}$ in the regular region R.

The main result of this section is

THEOREM 4.4. *Suppose that $\hat{\epsilon}$ is constant and that the flux f is strictly hyperbolic and genuinly nonlinear and that the conditions (4.5)-(4.10) are satisfied by the solutions $\hat{u}$ of (4.1) with ϵ replaced bt $\hat{\epsilon}$ and U of (2.10). Then there is a positive constant $\delta'' << 1$ such that the solution φ of (4.3) satisfies*

$$\begin{aligned} &\frac{1}{2}\|W^{\frac{1}{2}}(\varphi_t + \bar{f}'^T\varphi_x)\|_Q^2 + \frac{1}{2}\|W^{\frac{1}{2}}\hat{\epsilon}\varphi_{xx}\|_Q^2 + \|W^{-\frac{1}{2}}(\delta''\hat{\epsilon}|\lambda^*\lambda_x^*|)^{\frac{1}{2}}\varphi_x\|_Q^2 \\ &+\|W^{\frac{1}{2}}\hat{\epsilon}^{\frac{1}{2}}\varphi_x\|_{L^\infty(L^2)}^2 \leqq \|W^{-\frac{1}{2}}\hat{e}\|_Q^2. \end{aligned}$$

Proof: Using that (4.3) is uniformly parabolic and that $\bar{u}$ is uniformly Lipschitz continuous on $\mathbb{R} \times (0,T)$, local existence and uniqueness of a solution to the Cauchy problem (4.3) in H^2 follows by standard arguments. By the a priori estimates below combined with a standard continuation argument, we obtain a global solution in

$$C^0((0,T); L^2(R)) \cap H^1((0,T); L^2(\mathbb{R})) \cap L^2((0,T); H^2(\mathbb{R})).$$

Multiplying (4.3a) with $W(-\varphi_t - A^T\varphi_x)$ where $A = \bar{f}'$ integrating over $Q_\tau = \mathbb{R} \times (\tau, T)$, we get

$$
\begin{aligned}
(4.20)\quad & \|W^{\frac{1}{2}}(\varphi_t + A^T\varphi_x)\|^2_{Q_\tau} + \int_{Q_\tau} \hat{\epsilon}\varphi_{xx} \cdot W\varphi_t dxds \\
& + \int_{Q_\tau} \hat{\epsilon}\varphi_{xx} \cdot WA^T\varphi_x dxdt \\
& = \int_{Q_\tau} \hat{e}(-\varphi_t - A^T\varphi_x)dxdt \leqq \|W^{-\frac{1}{2}}\hat{e}\|^2_{Q_\tau} + \frac{1}{4}\|W^{\frac{1}{2}}(\varphi_t + A^T\varphi_x)\|^2_{Q_\tau}.
\end{aligned}
$$

Using that WA^T is symmetric we find integrating by parts that

$$
(4.21)\qquad \int_{Q_\tau} \hat{\epsilon}\varphi_{xx} \cdot WA^T\varphi_x dxds = -\int_{Q_\tau} \frac{\hat{\epsilon}}{2}\varphi_x \cdot (WA^T)_x\varphi_x dxdt \equiv I,
$$

and further

$$
\begin{aligned}
(4.22)\quad & \int_{Q_\tau} \hat{\epsilon}\varphi_{xx} \cdot W\varphi_t dxdt = -\int_{Q_\tau} \hat{\epsilon}\varphi_x \cdot W_x\varphi_t dxdt \\
& - \int_{Q_\tau} \hat{\epsilon}\varphi_x \cdot W\varphi_{xt} dxdt \\
& = \int_{Q_\tau} \hat{\epsilon}\varphi_x \cdot W_x(-\varphi_t - A^T\varphi_x)dxdt + \int_{Q_\tau} \hat{\epsilon}\varphi_x \cdot W_xA^T\varphi_x dxdt \\
& + \int_{Q_\tau} \frac{\hat{\epsilon}}{2}\varphi_x \cdot W_t\varphi_x dxdt + \int_{\mathbb{R}} \frac{\hat{\epsilon}}{2}\varphi_x \cdot W\varphi_x|_{t=\tau} dx \equiv II + III + IV + V.
\end{aligned}
$$

We have

$$
\begin{aligned}
(4.23)\qquad III &= 2\int_{Q_\tau} \frac{\hat{\epsilon}}{2}\varphi_x \cdot W_xA^T\varphi_x dxdt \\
&= 2\int_{Q_\tau} \frac{\hat{\epsilon}}{2}\varphi_x \cdot \lambda^* W_x\varphi_x dxdt \\
&\quad +2\int_{Q_\tau} \frac{\hat{\epsilon}}{2}\varphi_x \cdot W_x(A^T - \lambda^* I)\varphi_x dxdt \equiv III_1 + III_2.
\end{aligned}
$$

Writing, now recalling (4.15e), $W = \tilde{W}H$, where

$$
\tilde{W} = \bar{R}\tilde{\Psi}\bar{R}^T, \tilde{\Psi} = \mathrm{diag}(\tilde{\psi}_i),
$$

we have, since by (4.5) $|s - \lambda^*| << 1$, and using (4.8a), (4.10a), (4.11) and Proposition 4.3 that for $0 < \delta_0 << 1$

$$
\begin{aligned}
(4.24)\qquad III_1 &= -2\int_{Q_\tau} \frac{\hat{\epsilon}}{2}\varphi_x \cdot \tilde{W}_t\varphi_x H(t)dxdt \\
& \int_{Q_\tau} O(\delta_0 C^*|\lambda^*_x| + C\beta|\varphi_x|^2 H(t)dxdt \equiv III_{11} + III_{12}.
\end{aligned}
$$

Further, defining

$$w = \bar{R}^T \varphi_x, \bar{L} = (\bar{l}_1^T, \ldots, \bar{l}_m^T) = \bar{R}^{-1},$$

we have by Proposition 4.3, (4.21) and (4.22)

$$\begin{aligned} I + III_{11} + IV &= -\int_{Q_\tau} \frac{\hat{\epsilon}}{2}\varphi_x \cdot [\tilde{W}_t + (\tilde{W}A^T)_x]\varphi_x H(t)dxdt \\ &+ \int_{Q_\tau} \frac{\hat{\epsilon}}{2}(C^*\beta + \frac{1+c^*}{t})\varphi_x \cdot \tilde{W}\varphi_x H(t)dxdt \equiv IV_1 + IV_2. \end{aligned} \tag{4.25}$$

$$\begin{aligned} IV_1 &= -\int_{Q_\tau} \frac{\hat{\epsilon}}{2} w \cdot (\tilde{\Psi}_t + \bar{\Lambda}\tilde{\Psi}_x + \bar{\Lambda}_x \tilde{\Psi}) w dxdt \\ &- \int_{Q_\tau} \frac{\hat{\epsilon}}{2} w \cdot [\bar{L}(\bar{R}_t + \bar{R}_x\bar{\Lambda})\tilde{\Psi} + (L(\bar{R}_t + \bar{R}_x\bar{\Lambda})\tilde{\Psi})^T] w dxdt \\ &= -\int_{Q_\tau} \frac{\hat{\epsilon}}{2} w \cdot (\tilde{\Psi}_t + \Lambda(\bar{u})\tilde{\Psi}_x + \Lambda(\bar{u})_x \tilde{\Psi}) w H(t) dxdt \\ &- \int_{Q_\tau} \frac{\hat{\epsilon}}{2} w \cdot [L(\bar{u})(R(\bar{u})_t + R(\bar{u})_x \Lambda(\bar{u}))\tilde{\Psi} \\ &+ (L(\bar{u})(R(\bar{u})_t + R(\bar{u})_x \Lambda(\bar{u}))\tilde{\Psi})^T] w H(t) dxdt \\ &+ \int_{Q_\tau} \frac{\hat{\epsilon}}{2} O(|\lambda_x^*||\hat{e}|)|w|^2 H(t) dxdt \equiv VI + VII + VIII. \end{aligned}$$

From (4.8a), (4.10a) and (4.11) we conclude that

$$\begin{aligned} VII &= \int_{Q_\tau} \hat{\epsilon} \sum_{i,j} l_j(\bar{u}) \cdot (r_i(\bar{u})_t + \lambda_i(\bar{u}) r_i(\bar{u})_x) w_i w_j \tilde{\psi}_i H(t) dxdt \\ &\leqq \int_{Q_\tau} \hat{\epsilon} C \sum_{i,j} |\lambda_i(\bar{u}) - \lambda^*)||\lambda_x^*| w_i w_j H(t) dxdt. \end{aligned} \tag{4.26}$$

Similarly by (4.15)-(4.17)

$$\begin{aligned} III_2 &= \int_{Q_\tau} \hat{\epsilon} w \cdot (\bar{L}(\bar{R}\tilde{\Psi}\bar{R}^T)_x (A^T - \lambda^* I))\bar{L}^T w H(t) dxdt \\ &\leqq \int_{Q_\tau} \hat{\epsilon} C \sum_{i,j} |\lambda_x^*||\lambda_i(\bar{u}) - \lambda^*||w_i w_j| H(t) dxdt \\ &\quad + \int_{Q_\tau} \hat{\epsilon} C |\lambda_x^*||\hat{e}|w|^2 H(t) dxdt. \end{aligned} \tag{4.27}$$

To estimate the term II in (4.22) we first note that

$$|II| \leqq \frac{1}{4}||W^{\frac{1}{2}}(\varphi_t + A^T\varphi_x)||^2_{Q_\tau} + ||\hat{\epsilon} W^{-\frac{1}{2}} W_x^T \varphi_x||^2_{Q_\tau} \equiv II_1 + II_2. \tag{4.28}$$

Next, to estimate the term II_2, we note that by an inverse estimate and (4.5), we have

$$\hat{\epsilon}|\bar{u}_x| \leqq \|\hat{u}\|_{BV} + \|U\|_{BV} \equiv \delta' << 1.$$

Combining this with Proposition 4.3 yields

$$\begin{aligned} |II_2| &\leqq \int_{Q_\tau} C(\hat{\epsilon}|\bar{u}_x|\hat{\epsilon}|\lambda_x^*| + \hat{\epsilon}\beta)|\varphi_x|^2 H(t)dxdt \\ &\leqq \int_{Q_\tau} \hat{\epsilon}(C(C^*)^2\delta'|\lambda_x^*| + \beta)|\varphi_x|^2 H(t)dxdt. \end{aligned} \tag{4.29}$$

Here by assumming the error terms in (4.20-4.27), we have

$$\begin{aligned} &II_2 + III_2 + III_{12} + VII + VIII \\ &\leq \int_{Q_\tau} \hat{\epsilon}C\sum_{i,j}[\beta + |\lambda_x^*|[|\hat{e}| + |\bar{\lambda}_i(\bar{u}) - \lambda^*| + C^*\delta_0 + (C^*)^2\delta']]w_i w_j H(t)dxdt \end{aligned} \tag{4.30}$$

Proposition 4.1 combined with (4.8a), (4.10a) and the strict hyperbolicity of f' implies that there is a constant $\gamma > 0$ such that

$$-(\tilde{\psi}_{i,t} + \lambda_i(\bar{u})\tilde{\psi}_{i,c}) \geqq \begin{cases} \gamma C^*|\lambda_x^*|\tilde{\psi}_i & \text{for} \quad (x,t) \notin \cup_n(S_{ni} \cup E_{ni}) \\ 0 & \text{for} \quad (x,t) \in \cup_n(S_{ni} \cup E_{ni}) \end{cases} \tag{4.31a}$$

$$-\lambda_i(\bar{u})_x = |\lambda_x^*| \quad \text{for} \quad (x,t) \in \cup_n S_{ni}, \tag{4.31b}$$

$$\lambda_i(\bar{u})_x \leqq \begin{cases} (1+\delta^*)/t \qquad (x,t) \in \cup_n E_{ni}, \\ C\beta \qquad x \in R, \\ C|\lambda_x^*|(x,t) \notin R \cup (\cup_n(S_{ni} \cup E_{ni}). \end{cases} \tag{4.31c}$$

Noting that $\delta << 1$ and by choosing C^*, c^* in (4.15) so that $\gamma C^* > C((C^*)^2 + C^*\delta_0 + C)$, and using that $|\hat{e}| + |\lambda_i - \lambda^*| + \delta + \delta' << 1$, we conclude from (4.25), (4.30) and (4.31) that there is a constant $\delta'', 0 < \delta'' << 1$ such that

$$\begin{aligned} I + II + III + IV &\geqq \int_{Q_\tau} \delta''\hat{\epsilon}(|\lambda_x^*| + \beta|\varphi_x|^2)H(t)dxdt \\ &\quad -\frac{1}{4}\|W^{\frac{1}{2}}(\varphi_t + A^T\varphi_x)\|_{Q_\tau}^2. \end{aligned} \tag{4.32}$$

Combining this with (4.11) and (4.20-4.22) completes the proof of the theorem. □

The proof of the a posteriori error estimate of Theorem 3.5 directly carries over to the present system case since the stability estimates of Lemma 3.3 and Theorem 4.4 are fully analogous and the interpolation error estimates of Lemma 3.1 and Lemma 3.2 directly extend to the system case with obvious notation. Thus there is a direct analogue in the system case of Theorem 3.5 with the weight ψ replaced by W. Note that to have the variation of the weights ψ_i defined by (4.15d) bounded over each time interval I_n, as required in Lemma 3.1-3.2, the first time step should be excluded if $\psi_o \neq 1$, i.e., in the case of a rarefaction wave.

For definiteness, we state the following analogue of Theorem 3.6 using the notation of (3.22-25).

THEOREM 4.5. *If $\sigma_n(\bar{x},\bar{t})$ is constant in $\bar{x}$, and the assumptions of Theorem 4.1 are satisfied, then there is a constant C, independent of h and $\hat{\epsilon}$, such that*

$$\begin{aligned}(4.33)\quad |||(\hat{u}-U)|||_Q \leqq\ & C(|||(h^2(1+\hat{\epsilon}^{-1}h|f'-\sigma|)D_1U|||_Q + |||\hat{\epsilon}^{-1}h^4 f^{(3)}U_x^3|||_Q \\ & +C|||k\partial_{\bar{t}}U|||_Q + C\min(|||\alpha\hat{\epsilon}^{-\frac{1}{2}}h^2D_1U|||^{\#}_{L_1(L_2)}, (|||\alpha(\delta''\hat{\epsilon}|\lambda_x^*|)^{-\frac{1}{2}}h^2D_1U|||^{\#}_Q) \\ & +C\min(|||\alpha\hat{\epsilon}^{-\frac{1}{2}}k\partial_{\bar{t}}U|||_{L_1(L_2)}, |||\alpha(\hat{\epsilon}\delta''|\lambda_x^*|)^{-\frac{1}{2}}k\partial_{\bar{t}}U|||_Q),\end{aligned}$$

where

$$\begin{aligned} |f'-\sigma| &= \max(|\,\overset{=}{f}{}'-\sigma|, |f'(\bar{U})-\sigma|), \\ |||\cdot|||_{\cdot} &= |W^{-\frac{1}{2}}\cdot||_{\cdot}, \\ \overset{=}{f}{}^{(3)} &= \frac{\partial^3}{\partial s^3} f(\bar{U}+\frac{s}{|\bar{U}_x|}\bar{U}_x)|_{s=0}, \\ \bar{\alpha}(\bar{x},\bar{t}) &= \max_{s\in I_n}|\bar{\sigma}_n(\bar{x},\bar{s})I-\bar{f}'(\bar{x},\bar{s})|. \end{aligned}$$

with $\bar{U}$ and $\overset{=}{f}{}'$ defined in (3.22) and (3.24).

Let us now briefly discuss the estimate (4.33) of Theorem 4.5, concerning the quantity $\hat{u}-U$. We first note that in the present system case, we cannot expect α to be small since the mesh tilting speed σ_n cannot be close to all the eigenvalues of f'. This means that in regions of smoothness of $\hat{u}$ with $\hat{\epsilon}=Ch^{3/2}$, (4.33) indicates the order $O(h^{3/2}+kh^{-3/4})$. In shocks with $k=O(h)$ the order $O(h^{1/2})$ is indicated as above. The same rate in the weighted norm $|||\cdot||_Q$ is indicated for rarefactions (up to a logarithm). Concerning the remaining part of the error $u-\hat{u}$, we expect to have, if ϵ is very small, that $u-\hat{u}=O(\hat{\epsilon})$ in regions of smoothness, a contribution $\|u - \hat{u}\|_Q = O(\hat{\epsilon}^{\frac{1}{2}})$ from shocks and (see [8]), $|||u - \hat{u}||_Q = O(\hat{\epsilon}^{\frac{1}{2}})$ from rarefactions (up to a logarithm). Thus, with $\hat{\epsilon} = O(h)$ in shocks and rarefactions the two parts of the error $\hat{u} - U$ and $u - \hat{u}$ seem there to be of the same order, as seems to be the case in regions of smoothness with $\hat{\epsilon} = O(h^{3/2})$. Note further that in a typical case we will have a coarse space-mesh size H in the region of smoothness and a fine mesh h in shocks. Balancing the orders of H and h from $H^{3/2} = C\,\mathrm{TOL}$ and $h^{\frac{1}{2}} = C\,\mathrm{TOL}$ with TOL a given tolerance, we find $h = O(H^3)$. Typically, the time step k assumed to be uniform in space, will be of order $O(h)$ determined from the shock region, that is, we will have $k = O(H^3)$ in the region of smoothness so that there $kH^{-3/4} \leqq H^{3/2}$. Thus, effectively the time step will not be limited by the suboptimal form of the a posteriori error estimate as concerns the time discretization error in the region of smoothness. Higher order accuracy in time may be desirable in smooth regions and may be achieved by using, e.g., piecewise linear approximation in time $\bar{t}$, cf. Section 5 below.

5. Extensions

In this section we briefly comment on the extension of the above results to the case of variable space and time steps h and k, non-constant artificial viscosity $\hat{\epsilon}$, and higher order polynomial approximation in space and/or time. Basically, the indicated extension is possible under suitable smoothness assumptions. For instance, the relevant assumption on the variation of h is given by (2.1), which allows a very quick variation of h^n, corresponding to local quasi-uniformity in space: $h_i^n / h_{i\pm 1}^n \leqq \mu$. Note that no restriction on the variation of $h^n(x)$ or k_n with n is imposed. Concerning the variation of $\hat{\epsilon}$, the requirements are somewhat more stringent than a condition of the form $|\hat{\epsilon}_x| \leqq c$, corresponding to (2.1a)(see [18]). We postpone the details to forthcoming work. The desired regularity of $\hat{\epsilon}$ can always be obtained be suitable smoothing of $\hat{\epsilon}$ defined by (2.9). Such smoothing is also natural to perform to faciliate the numerical solution of the discrete system (2.8).

The finite element method (2.8) may be extended to the case with V_n defined by

$$\bar{V}_n = \{\bar{v} \in \mathcal{C}(S_n)^m : \bar{v}(\bar{x}, \bar{t}) = \sum_{j=0}^{r} (t - t_n)^j \bar{v}_j(\bar{x}), \quad \text{where} \quad \bar{v}_j \in [X_n]^m\},$$

where X_n is a space of piecewise polynomials on $\{x_i^n\}_i$. Thus, we may base (2.8) on piecewise polynomial approximation in $(\bar{x}, \bar{t})$ of any order. The a posteriori error estimates of Theorem 3.5, including the system analogue, remain valid with unchanged formulation also with such a more general polynomial approximation in time.

The general approach to a posteriori error estimates presented in principle may be applied to systems of conservation laws in several dimensions. The crucial step is again the quantitative evaluation of strong stability of the associated linearized dual problem. We may think of approaching this problem by considering related simplified problems for which the stability constant may be computed. For instance, when seeking to use analytical techniques, we may consider the corresponding dual problem linearized around the exact solution u and follow [29] to evaluate its stability properties in model cases containing, e.g., planar shocks. On the other hand, with a computational approach, we would linearize around the computed solution U and evaluate the stability of the corresponding dual problem through computational or analytical techniques (e.g., using a so-called normal mode analysis) or combinations thereof.

The problem of adaptive error control for conservation laws in several dimensions is very rich, and is one of the main challenges of CFD. Our hope is that the material presented in this paper may open some new possibilities of approaching this problem.

Acknowledgement. This work was in part supported by the Swedish Council for Engineering Sciences under grant TFR-92-691 and the Air Force Office of Scientific Research under grant AF OSR-91-0063

Bibliography

[1] M. Berger and P. Colella,*Local adaptive mesh refinement for shock hydrodynamics*, J. Comp. Phys. 82, 64-84.

[2] K. Eriksson and C. Johnson,*Adaptive finite element methods for parabolic problems I: A linear model problem*, SIAM J. Numer. Anal. 28 (1991), 43-77.

[3] K. Eriksson and C. Johnson, *Adaptive streamline diffusion finite element methods for stationary convection-diffusion problems*, Math. Comp., 60, 1993, 167-188.

[4] K. Eriksson and C. Johnson, *Adaptive streamline diffusion finite element methods for time-dependent convection-diffusion problems*, Math. Comp. (to appear).

[5] J. Goodman, *Nonlinear stability of viscous shock profiles for conservation laws*, Arch. Rat. Mech. Anal. 95 (1986), 325-344.

[6] J. Goodman, *Remarks on the stability of viscous shock waves; Viscous Profiles and Numerical Methods for Shock Waves* ed. M. Shearer, SIAM Philadelphia 1991, pp. 66-72.

[7] J. Goodman and Z. Xin, *Viscous limits for piecewise smooth solutions to systems of conservation laws*, Comm. Pure Appl. Math.

[8] E. Harabetian, *Rarefaction and large time behaviour for parabolic equations and monotone schemes*, Comm.Math. Phys. 114 (1988), 527-563.

[9] P. Hansbo, *The characteristic streamline diffusion method for convection-diffusion problems*, Comp. Meth., Appl. Mech. Engrg. 99 (1992), 171-186.

[10] P. Hansbo, *The characteristic streamline diffusion method for the time-dependent incompressible Navier-Stokes equations*, Comput. Methods Appl. Mech. Engrg., 99, 1992,

171-186.

[11] P. Hansbo, *Space-time oriented streamline diffusion methods for non-linear conservation laws in one dimension*, Preprint 1992-33, Dept. of Mathematics, Chalmers Univ. of Technology, 1992.

[12] P. Hansbo and C. Johnson, *Adaptive streamline diffusion methods for compressible flow using conservation variables*, Comp. Meth. Apl. Mech. Engrg. 87 (1991), 267-280.

[13] P. Hansbo and A. Szepessy, *A velocity pressure streamline diffusion finite element method for the incompressible Navier-Stokes equations*, Comp. Meth. Appl. Mech. Engrg. 84 (1990), 175-192.

[14] T.J.R. Hughes, L.P. Franca and M. Mallet, *A new finite element formulation for computational fluid dynamics: I Symmetric forms of the compressible Euler and Navier-Stokes equations and the second law of thermodynamics*, Comp. meth. Appl. Mech. and Eng. 54 (1986) 223-234.

[15] T.J.R. Hughes, M. Mallet and A. Mizukami,*A New Finite element formulation for Computational Fluid Dynamics II: Beyond SUPG*, Comp. Meth. Apl. Mech. Engrg. Vol.54 (1986), 341-355.

[16] T.J.R. Hughes, M. Mallet, *A new Finite Element Formulation for Computational Fluid Dynamics III: The General Streamline Operator for Multidimensional Advective-Diffusive Systems*, Comp. Meth. Appl. Mech. Engrg. Vol. 58 (1986), 329-336.

[17] T.J.R. Hughes and M. Mallet, *A New Finite Element Formulation for Computational Fluid Dynamics IV: A Discontinuity-Capturing Operator for Multidimensional Advective-Diffusive Systems*, Comp. Meth. Appl. Mech. Engrg., Vo. 58 (1986), 329-336.

[18] C. Johnson, *Adaptive finite element methods for diffusion and convection problems*, Comput. Meth., Appl. Mech. Engrg. 82 (1990), 301-322.

[19] C. Johnson,*The characteristic streamline diffusion finite element method,* Matematica Aplicada e Computacional, 10 (1992), Proc. from Conf. on Innovative Finite Element Methods, Rio de Janeiro, Dec., 1989.

[20] C. Johnson,*A new approach to algorithms for convection problems which are based on exact transport + projection,* Comp. Meth. Appl. Mech. Engrg.,100, 1992, 45-62.

[21] C. Johnson,*The streamline diffusion method for compressible and incompressible flow,* Proc. Finite elements in Fluids VII, Huntsvill 1989, Hemisphere 1992.

[22] C. Johnson and P. Hansbo,*Adaptive finite element methods in computational mechanics,* to appear in Comp. Meth. Appl. Mech. Engrg., Proc. from Workshop on Reliability in Computational Mechanics, Cracow, Oct., 1991.

[23] C. Johnson and U. Nävert and J. Pitkäranta,*Finite element methods for linear hyperbolic problems,* Comp. Meth. Appl. Mech. Engrg. 45 (1984) 427-312.

[24] C. Johnson and A. Szepessy,*On the convergence of a finite element method for a nonlinear hyperbolic conservation law,* Math. Comp. 49 (1987) 427-444.

[25] C. Johnson, A Szepessy and P. Hansbo,*On the convergence of shock-capturing streamline diffusion finite element methods for hyperbolic conservation laws,* Math. Comp. 54 (1990) 107-129.

[26] T. P. Liu,*Nonlinear stability of shock waves for viscous conservation laws,* AMS Memoars, Vol. 328 (986).

[27] Lucier, B.,*A moving mesh numerical method for hyperbolic conservation laws,* Math. Comp. 46, 1986, 56-69.

[28] R. Löhner,*Finite Element methods in CFD: Grid generation, Adaptivity and Parallelization,* AGARD report 787, AGARD 7 rue Ancelle, 92200 Neville sur Seine, France.

[29] A. Majda,*Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variables,* Springer, 1984.

[30] K. Morgan, J. Peraire and J. Peiro,*Unstructured grid methods for compressible flow,* AGARD report 787, AGARD 7 rue Ancelle, 92200 Neville sur Seine, France.

[31] H. Nessyahu and E. Tadmor,*The convergence rate of approximate solutions for nonlinear scalar conservation laws,* preprint, Tel-Aviv University, 1991.

[32] R. Pego,*Remarks on the stability of shock profiles for systems of conservation laws with*

dissipation, Trans. Ams. 291 (1985) 353-361.

[33] A. Szepessy, *Convergence of the streamline diffusion finite element method for conservation laws,* Thesis, Department of Mathematics, Chalmers University of Technology, 1989.

[34] A Szepessy, *Convergence of a shock-capturing streamline diffusion finite element method for a scalar conservation law in two space dimensions,* Math. Comp. 53 (1989) 27-545.

[35] A. Szepessy, *On the stability of finite element methods for shock waves,* Comm. Pure Appl. Math. Vol XLV (1992), 923-946.

[36] A. Szepessy, *Convergence of a streamline diffusion finite element method for scalar conservation laws with boundary conditions,* Mathematical Modeling and Numerical Analysis 25 (1991), 749-782.

[37] A. Szepessy and Z. Xin, *Nonlinear stability of viscous shock waves,* Arch. Rat. Mech. Anal., 122, 1993, 53-103.

[38] E. Tadmor, *Local error estimates for discontinuous solutions of nonlinear hyperbolic equations,* SIAM J. Num. Anal. 28 (1991), 891-906.

[39] Z. Xin, *Asymptotic stability of rarefaction waves for viscous hyperbolic conservation Laws,* J. Diff. eq. 73 (1988) 45-77.