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CONVERGENCE OF THE FINITE VOLUME METHOD FOR MULTIDIMENSIONAL CONSERVATION LAWS*

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Abstract. We establish the convergence of the finite volume method applied to multidimensional hyperbolic conservation laws and based on monotone numerical flux-functions. Our technique applies with a fairly unrestrictive assumption on the triangulations ("flat elements" are allowed) and to Lipschitz continuous flux-functions. We treat the initial and boundary value problem and obtain the strong convergence of the scheme to the unique entropy discontinuous solution in the sense of Kruzkov. The proof of convergence is based on a convergence framework [Coquel and LeFloch, *Math. Comp.*, 57 (1991), pp. 169–210 and *J. Numer. Anal.*, 30 (1993), pp. 675–700]. From a convex decomposition of the scheme, we derive a new estimate for the rate of entropy dissipation and a new formulation of the discrete entropy inequalities. These estimates are shown to be sufficient for the passage to the limit in the discrete equation. Convergence follows from DiPerna's uniqueness result in the class of entropy measure-valued solutions.

Key words. conservation law, measure-valued solution, finite volume method, entropy dissipation

AMS subject classifications. primary, 65M12; secondary, 35L65

1. Introduction.

This paper considers the finite volume method for the approximation of the initial and boundary value problem associated with a multidimensional conservation law:

$$\partial_t u + \operatorname{div} f(u) = 0, \qquad u(t,x) \in \boldsymbol{R}, \quad t > 0, \quad x \in \Omega \subset \boldsymbol{R}^d, \tag{1.1}$$

$$u(0,x) = u_0(x), \qquad x \in \Omega, \tag{1.2}$$

and, for all convex entropy pairs (U, F) and almost all (t, x) in $\partial\Omega$ (with respect to the $(d-1)$-dimensional Haussdorf measure),

$$N(t,x)\big\{F(u(t,x)) - F(u_1(t,x)) - \nabla U(u_1(t,x))\big(f(u(t,x)) - f(u_1(t,x))\big)\big\} \geq 0. \tag{1.3}$$

The flux-function $f : \boldsymbol{R} \to \boldsymbol{R}^d$ is assumed to be locally Lipschitz continuous, and Ω is an open (and not necessarily bounded) subset of $\boldsymbol{R}^d$ that has a polygonal boundary $\partial\Omega$. We denote by $N(t,x)$ the outward unit normal along $\partial\Omega$. The initial data u_0 is assumed to belong to $L^1(\Omega) \cap L^\infty(\Omega)$, and the boundary data u_1 belongs to $L^1\big(\boldsymbol{R}_+, L^1(\partial\Omega)\big) \cap L^\infty\big(\boldsymbol{R}_+, L^\infty(\partial\Omega)\big)$. A Lipschitz continuous function $(U,F) : \boldsymbol{R} \to \boldsymbol{R} \times \boldsymbol{R}^d$ is said to be a convex entropy pair if U is a convex function and

$$\frac{dF}{du} = \nabla U \, \frac{df}{du}. \tag{1.4}$$

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The formulation (1.3) is actually a weak form of the standard boundary condition $u(t,x) = u_1(t,x)$, $(t,x) \in \partial\Omega$, which would not lead to a well-posed problem, since (1.1) is a nonlinear hyperbolic equation.

In view of the results by Kruzkov [21], [22] and others [1], [25], [36], [42], it is not difficult to see that there exists a unique weak solution in $L^\infty(\boldsymbol{R}_+ \times \Omega)$ to the problem that satisfies the entropy inequalities:

$$\partial_t U(u) + \operatorname{div} F(u) \leq 0 \tag{1.5}$$

for all convex entropy pairs (cf. also Lax [23], [24] for background on hyperbolic equations).

The finite volume method is based on the local conservation property satisfied by the solutions to a conservation law and can be defined on general triangulations. The method is widely used in computational fluid dynamics. The question of the convergence hence arises naturally and has indeed received important attention in the last ten years. The case of cartesian triangulations (made of a cartesian product of one-dimensional partitions) with monotone numerical flux-functions was completely solved after the works by Conway–Smoller, Crandall–Majda, Lucier, Osher, Sanders, and others [6], [12], [22], [28], [29], [32], [33]. The case of cartesian triangulations is special because, when the mesh is translation invariant, a uniform estimate in the bounded variation norm (BV) can be derived from the L^1 contraction property. Compactness for the scheme then follows from the BV estimate and Helly's theorem. These arguments do not work with the finite volume method on arbitrary triangulations, as was observed by Sanders [34].

Following the work by Szepessy [35], where the focus was the streamline diffusion finite element method, a technique for proving convergence of numerical schemes for multidimensional equations was introduced by Coquel and LeFloch in [7], [8], [9], [26]. In this work, convergence is established without appealing to a uniform BV estimate but based on a result by DiPerna [14], which, under some circumstances, ensures the uniqueness of the measure-valued solution to (1.1). We also recall that a Young measure can be constructed from any sequence of approximate solutions, say u^h, in order to represent all its composite weak-star limits $a(u^h)$ for $a \in C^0(\boldsymbol{R})$. A Young measure that is consistent with all the entropy inequalities (i.e., by definition, an entropy measure-valued solution) and reduces to a Dirac mass $\delta_{u_0(x)}$ at $t = 0$ is a Dirac mass $\delta_{u(t,x)}$ for all times. Moreover, u is the unique entropy solution to the problem with initial data u_0 (cf. also Szepessy [36]). DiPerna applied this technique to check the convergence of the vanishing viscosity method using the L^∞ stability only.

As a matter of fact, the main step in applying DiPerna's uniqueness theorem is the passage to the limit in the sense of Young measures in the approximate entropy inequalities. For approximate solutions obtained by a difference scheme or the finite volume method, it is not difficult to derive discrete entropy inequalities. These inequalities involve numerical flux-functions that do not have the form $a(u^h)$ with $a \in C^0(\boldsymbol{R})$ but instead introduce a coupling between the values of u^h at several space locations (typically two points in three point schemes). For this reason, the passage to the limit, in the sense of Young measures, necessitates an a priori estimate that controls in the weak sense the oscillations in u^h.

Following previous work by DiPerna [13] based on the compensated compactness approach, Coquel and LeFloch first showed in [8] how to derive such an a priori estimate from the rate of entropy dissipation in a finite difference scheme. They pointed out that this estimate is weaker than the classical BV estimate (a counterexample to strong convergence is constructed in [8]) but nevertheless sufficient with which to work in the setting of Young measures. The technique was applied in [9] to the control of the entropy production in the high-order accurate antidiffusive-flux methods. The result in [9] was the first result of convergence for multidimensional problems.

In the present paper, we follow the approach in [8], [9] and prove a fairly optimal result of convergence for the finite volume method constructed from entropy satisfying numerical flux-functions. Our technique of proof allows us to treat arbitrary triangulations (with virtually no restriction on the shape of the polyhedra that can become "flat" at the limit) and arbitrary flux-functions and applies to the initial and boundary value problems. The method can also be partially carried out with systems of conservation laws [11]. Two new ingredients in this paper are essential to the analysis: an estimate of the rate of entropy dissipation and a new formulation of the discrete entropy inequalities. They not only provide us with an optimal result of convergence but also make the proof remarkably simple.

Specifically, we prove in this paper that the rate of entropy dissipation can be estimated from the classical cell entropy inequalities. As opposed to the estimates in [8], [9], we need not study the detailed properties of the numerical flux-functions. Our estimate is weaker than that in [8]. As a matter of fact, we avoid the direct passage to the limit in the numerical flux-functions and entropy flux-functions. Our method in the setting of measure-valued solutions shows that L^∞ stability plus discrete consistency with all the entropy inequalities imply convergence and extends the theory based on compensated compactness arguments developed by Tadmor [39] for one-dimensional equations to multidimensional equations.

Estimates of the rate of entropy dissipation play a crucial role in this paper. We refer to §5 for further comments.

The authors know of several recent contributions that have come to complete their results. The present work was first published in the IMA Preprint series in February 1991 in collaboration with Chi-Wang Shu [4]. The convergence of the strictly monotone methods was established therein. This result was extended to arbitrary monotone methods by Coquel–LeFloch [10] in December 1991. The present article gathers together the results in these two preprints. After the work [4], Champier–Gallouet–Herbin [2] obtained a proof of convergence for a special flux-function, $f_0(u)V$, where V is a fixed vector in $\boldsymbol{R}^d$ and $f_0 : \boldsymbol{R} \to \boldsymbol{R}$ a nondecreasing function. Kröner and Rokyta [20] proved the convergence of the Engquist–Osher method, improving upon the idea in [4]. Very recently, Vila has addressed in [41] the question of deriving an entropy estimate under a strengthened CFL condition, improving upon the original estimates in [9]. Another work by Noelle [30] gives an a priori entropy dissipation estimate that extends the idea by Kröner–Rokyta [20] to arbitrary fluxes. Independently of the present work, Noelle introduces in [30] an unrestrictive condition on the triangulations, which is analogous to our condition (2.1) below. We emphasize that the estimates derived in the present paper are quite different from those in the above references.

The convergence of the discontinuous Galerkin method based on strictly monotone numerical flux-functions was recently established by Jaffré–Johnson–Szepessy [19]. The uniqueness theorem by DiPerna was also used by Chen–Du–Tadmor [3] for the spectral methods. We also observe that the setting of measure-valued solutions can be bypassed by extending the well-known Kuznetsov's approach [5]. The proof therein is more technical than in the present paper but provides an error estimate for the (high-order accurate) finite volume methods in the L^1-norm.

An outline of the paper follows. In §2, we introduce the finite volume method and state the main result of this paper, Theorem 2.1. The proof is contained in §§3 and 4. We derive first the estimate of the rate of entropy dissipation, Proposition 3.1, and then the new formulation of the entropy inequalities, Proposition 3.2. Section 4 treats the passage to the limit in the setting of the measure-valued solutions. Finally §5 presents several examples and comments.

2. Main result of convergence. Let $\mathcal{T}^h$ be a (locally finite) triangulation of the domain $\Omega \subset \boldsymbol{R}^d$ into nonoverlapping, nonempty, and open polyhedra : $\bigcup_{K \in \mathcal{T}^h} \overline{K} = \overline{\Omega}$. We assume that, for two distinct polyhedra K and K' in $\mathcal{T}^h$, the intersection $\overline{K} \cap \overline{K}'$ is either a face of both K and K' or a set with Haussdorf dimension less than or equal to $d-2$. The set of faces of a polyhedron K is denoted by ∂K; and for each face on K, $\nu_{e,K} \in \boldsymbol{R}^d$ represents the outward unit normal vector to the face e. Given a face e of K, K_e is the unique polyhedron that shares the same face e with K. The volume of K and the $(d-1)$-measure of e are denoted by $|K|$ and $|e|$, respectively. By $\partial\mathcal{T}^h$ we denote the set of edges of elements in $\mathcal{T}^h$ that belong to $\partial\Omega$. When $K \in \mathcal{T}^h$ admits an edge $e \in \partial\mathcal{T}^h \cap \partial K$, we use the notation K_e to denote an artificial element located along the boundary of Ω. This polyhedron will be used to support the boundary data.

We consider without loss of generality that

$$h = \sup_{K \in \mathcal{T}^h} h_K < +\infty,$$

where h_K is the exterior diameter of a polyhedron. The perimeter of K is defined by $p_K = \sum_{e \in \partial K} |e|$. The time increment, denoted by τ, is assumed to satisfy the condition, as $h \to 0$,

$$\tau \to 0 \quad \text{and} \quad \frac{h^2}{\tau} \to 0. \tag{2.1}$$

We shall use the notation $t_n = n\tau$. We introduce a family of locally Lipschitz continuous functions $g_{e,K}\colon \boldsymbol{R} \times \boldsymbol{R} \to \boldsymbol{R}$, associated with each face e of each polyhedron K of $\mathcal{T}^h$, that satisfy the conservation property

$$g_{e,K}(u,v) = -g_{e,K_e}(v,u), \tag{2.2}$$

the consistency property

$$g_{e,K}(u,u) = f(u) \cdot \nu_{e,K}, \tag{2.3}$$

valid for all real numbers u and v, and the monotonicity property

$$\frac{\partial g}{\partial u} \geq 0, \quad \frac{\partial g}{\partial v} \leq 0. \tag{2.4}$$

For instance, the Godunov scheme and the Lax–Friedrichs scheme possess numerical flux-functions that satisfy (2.2)–(2.4). It is easy to check that $\|\frac{df}{du}\|_{L^\infty}$ is a Lipschitz constant for the functions $g_{e,K}$. The main results of this paper are easily extended to the E-scheme in the sense of Osher [31].

The finite volume approximation is defined from the above family of numerical flux-functions by the following scheme:

$$u_K^{n+1} = u_K^n - \sum_{e \in \partial K} \frac{\tau|e|}{|K|}\, g_{e,K}(u_K^n, u_{K_e}^n) \tag{2.5}$$

for all polyhedra K and all integers n. The initial condition u_0 provides us with

$$u_K^0 = \frac{1}{|K|} \int_K u_0(x)\, dx \tag{2.6}$$

for all polyhedra K. Following Dubois and LeFloch [15], [16], we set

$$u_{K_e}^n = \frac{1}{\tau |e|} \int_{t_n}^{t_n+1} \int_e u_1(t,x) dt d\Gamma, \quad e \in \partial \mathcal{T}^h. \tag{2.7}$$

It will be convenient to define the piecewise constant function $u^h : \boldsymbol{R}_+ \times \Omega \to \boldsymbol{R}$ by

$$u^h(t,x) = u_K^n, \quad n\tau \le t < (n+1)\tau, \quad x \in K. \tag{2.8}$$

The finite volume method is an explicit scheme, so it is natural to impose a restriction on the time increment τ (cf. (2.9) below).

The main result of this paper is stated in the following theorem.

THEOREM 2.1. *Consider the initial and boundary value problem* (1.1)–(1.5) *and the approximate solutions* $\{u^h\}$ *defined by the finite volume method* (2.5)–(2.8). *Assume that the triangulation and the numerical flux-functions satisfy the conditions* (2.1)–(2.4) *and for all polyhedra* K, *the* CFL *condition*:

$$\frac{\tau p_K}{|K|} \sup_{u \in [a,b]} \Big| \frac{df}{du}(u) \Big| \le 1, \tag{2.9}$$

where $a = \inf\big(\inf_\Omega u_0, \inf_{\boldsymbol{R}_+ \times \partial\Omega} u_1\big)$ *and* $b = \sup\big(\sup_\Omega u_0, \sup_{\boldsymbol{R}_+ \times \partial\Omega} u_1\big)$.

Then the method is uniformly stable in all the L^p *norms,* $1 \le p \le \infty$, *and converges in the* L^p *norm strongly,* $1 \le p < \infty$, *to the unique entropy solution* u *to the problem* (1.1)–(1.5): *for all times* $T \ge 0$ *and* $1 \le p < \infty$,

$$\begin{aligned} \|u^h(T)\|_{L^p(\Omega)} \le{}& \|u_0\|_{L^p(\Omega)} + O(1)\, \|u_1\|^{1/p}_{L^1(L^1(\partial\Omega))} \|u_1\|^{1-1/p}_{L^\infty(L^\infty(\partial\Omega))} \\ &+ O(1) \left(\|u_0\|^{1/p}_{L^\infty(\Omega)} + \|u_1\|^{1/p}_{L^\infty(L^\infty(\partial\Omega))} \right) \|u_1\|^{1-1/p}_{L^1(L^{p-1}(\partial\Omega))}, \end{aligned} \tag{2.10}$$

and

$$\|u^h - u\|_{L^p((0,T)\times\Omega)} \le o(h) \quad \textit{for} \quad 1 \le p < \infty \quad \textit{and all times} \quad T > 0, \tag{2.11}$$

where $o(h)$ *is a function tending to zero with* h.

The proof of Theorem 2.1 is given in §§3 and 4.

3. Entropy dissipation estimate and entropy inequalities. Throughout this section, we assume that the assumptions made in §2 hold. First, in Proposition 3.4, we derive an a priori estimate for the entropy dissipation in the approximate solutions. Second, we give a new form of the discrete entropy inequalities (cf. Proposition 3.7). The passage to the limit in the entropy inequalities is the subject of §4.

We recall first several elementary properties of the method.

LEMMA 3.1. *The finite volume method satisfies the local maximum principle*:

$$\min(u_K^n, \min_{e \in \partial K} u_{K_e}^n) \le u_K^{n+1} \le \max(u_K^n, \max_{e \in \partial K} u_{K_e}^n), \tag{3.1}$$

valid for all polyhedra K and all integers n, and the L^1 contraction property, that is, let $u^h \equiv \{u_K^n\}$ and $v^h \equiv \{v_K^n\}$ be computed by the finite volume method using the initial and boundary data u_0, u_1 and v_0, v_1, respectively; *then we have*

$$(3.2)\qquad \begin{aligned} &||u^h(t+\tau) - v^h(t+\tau)||_{L^1(\Omega)} \\ &\quad \leq ||u^h(t) - v^h(t)||_{L^1(\Omega)} + \left\|\frac{df}{du}\right\|_{L^\infty} \int_t^{t+\tau} ||u_1(s) - v_1(s)||_{L^1(\partial\Omega)} ds. \end{aligned}$$

Proof of Lemma 3.1. The proof follows from the assumptions (2.2)–(2.4) and (2.9) and the properties of the boundary condition (1.3). We consider the decomposition

$$(3.3)\qquad \begin{aligned} u_K^{n+1} &= \Big\{1 + \frac{\tau}{|K|} \sum_{e\in\partial K} \frac{g_{e,K}(u_K^n, u_{K_e}^n) - g_{e,K}(u_K^n, u_K^n)}{u_{K_e}^n - u_K^n} \mid e \mid\Big\} u_K^n \\ &\quad + \frac{\tau}{|K|} \sum_{e\in\partial K} \Big\{ -\frac{g_{e,K}(u_K^n, u_{K_e}^n) - g_{e,K}(u_K^n, u_K^n)}{u_{K_e}^n - u_K^n} \Big\} \mid e \mid u_{K_e}^n, \end{aligned}$$

which is a convex combination of u_K^n and $(u_{K_e}^n)_{e\in\partial K}$. Namely, (2.4) and (2.9) imply

$$-\frac{\tau}{|K|} \frac{g_{e,K}(u_K^n, u_{K_e}^n) - g_{e,K}(u_K^n, u_K^n)}{u_{K_e}^n - u_K^n} \geq 0$$

and

$$\begin{aligned} \sum_{e\in\partial K} -\frac{\tau \mid e \mid}{|K|} \frac{g_{e,K}(u_K^n, u_{K_e}^n) - g_{e,K}(u_K^n, u_K^n)}{u_{K_e}^n - u_K^n} &\leq \sum_{e\in\partial K} \frac{\tau \mid e \mid}{|K|} \left\|\frac{df}{du}\right\|_{L^\infty} \\ &\leq \frac{\tau p_K}{|K|} \left\|\frac{df}{du}\right\|_{L^\infty} \leq 1. \end{aligned}$$

The local maximum principle (3.1) follows immediately from (3.3).

We now turn to the inequality (3.2). Let us introduce the notation

$$u^n = (u_K^n)_{K\in\mathcal{T}^h}, \quad u^{n+1} = (H_K(u^n))_{K\in\mathcal{T}^h} = H(u^n),$$

and a similar notation with u^n replaced by v^n. In view of (2.4) and (2.9), H is a monotone operator, in the sense that

$$(3.4)\qquad \left(u_K^n \leq v_K^n,\ K \in \mathcal{T}^h\right) \Rightarrow \left(H_K(u^n) \leq H_K(v^n),\ K \in \mathcal{T}^h\right).$$

We set $a \vee b = \max(a, b)$ and $a \wedge b = \min(a, b)$. Following Crandall and Majda [12] and using (3.4) and (2.2), we get

$$\begin{aligned} \sum_{K\in\mathcal{T}^h} \mid u_K^{n+1} - v_K^{n+1} \mid |K| &\leq \sum_{K\in\mathcal{T}^h} \mid u_K^n - v_K^n \mid |K| \\ &\quad - \tau \sum_{e\in\partial\mathcal{T}^h} \Big(g_{e,K}(u_K^n \vee v_K^n, u_{K_e}^n \vee v_{K_e}^n) - g_{e,K}(u_K^n \wedge v_K^n, u_{K_e}^n \wedge v_{K_e}^n)\Big)|e|. \end{aligned}$$

For each $e \in \partial\mathcal{T}^h$, the monotonicity condition (2.4) ensures that the following inequality holds:

$$g_{e,K}(u_K^n \wedge v_K^n, u_{K_e}^n \wedge v_{K_e}^n) - g_{e,K}(u_K^n \vee v_K^n, u_{K_e}^n \wedge v_{K_e}^n) \leq 0.$$

So we have

$$g_{e,K}(u_K^n \wedge v_K^n, u_{K_e}^n \wedge v_{K_e}^n) - g_{e,K}(u_K^n \vee v_K^n, u_{K_e}^n \vee v_{K_e}^n)$$
$$\leq g_{e,K}(u_K^n \vee v_K^n, u_{K_e}^n \wedge v_{K_e}^n) - g_{e,K}(u_K^n \vee v_K^n, u_{K_e}^n \vee v_{K_e}^n).$$

Multiplying the latter inequality by $\tau|e|$ and summing it over all faces $e \in \mathcal{T}^h$ yield

$$\begin{aligned}
&-\tau \sum_{e\in\partial\mathcal{T}^h} \Big(g_{e,K}(u_K^n \vee v_K^n, u_{K_e}^n \vee v_{K_e}^n) - g_{e,K}(u_K^n \wedge v_K^n, u_{K_e}^n \wedge v_{K_e}^n)\Big)|e| \\
&\leq \sum_{e\in\partial\mathcal{T}^h} \Big(g_{e,K}(u_K^n \vee v_K^n, u_{K_e}^n \wedge v_{K_e}^n) - g_{e,K}(u_K^n \vee v_K^n, u_{K_e}^n \vee v_{K_e}^n)\Big)\tau|e| \\
&\leq \left\|\frac{df}{du}\right\|_{L^\infty} \sum_{e\in\partial\mathcal{T}^h} \mid u_{K_e}^n \wedge v_{K_e}^n - u_{K_e}^n \vee v_{K_e}^n \mid \tau|e| \\
&= \left\|\frac{df}{du}\right\|_{L^\infty} \sum_{e\in\partial\mathcal{T}^h} \mid u_{K_e}^n - v_{K_e}^n \mid \tau|e| \\
&\leq \left\|\frac{df}{du}\right\|_{L^\infty} \sum_{e\in\partial\mathcal{T}^h} \int_{t_n}^{t_{n+1}} \int_e \mid u_1 - v_1 \mid d\Gamma dt.
\end{aligned}$$

The desired conclusion follows. □

The following *convex decomposition* for u_K^{n+1} will be central in our analysis:

$$u_K^{n+1} = \frac{1}{p_K} \sum_{e\in\partial K} |e|\, u_{K,e}^{n+1}, \tag{3.5}$$

with $u_{K,e}^{n+1}$ defined by

$$u_{K,e}^{n+1} = u_K^n - \frac{\tau p_K}{|K|}\big\{g_{e,K}(u_K^n, u_{K_e}^n) - g_{e,K}(u_K^n, u_K^n)\big\}. \tag{3.6}$$

It is easy to deduce (3.5) from (2.2) and (2.3). This decomposition was introduced by Tadmor [38] for one-dimensional problems and used by Coquel–LeFloch [9] for multidimensional equations.

A function of two variables $G = G(u,v)$ is said to be *consistent* with a function of a single variable $F = F(u)$ if it satisfies $G(u,u) = F(u)$ for all u. The following result is a consequence of (2.2)–(2.4) (cf. for instance [31], [33]).

LEMMA 3.2. *Let $(U,F)\colon \boldsymbol{R} \to \boldsymbol{R}\times\boldsymbol{R}^d$ be a convex entropy pair. Then there exists a family of numerical entropy flux-functions $G_{e,K}\colon \boldsymbol{R}^2 \to \boldsymbol{R}$ that are consistent with $\nu_{e,K}\cdot F$ and satisfy the conservation property*

$$G_{e,K}(u,v) = -G_{e,K_e}(v,u) \quad \textit{for all } u \textit{ and } v \tag{3.7}$$

and the discrete entropy inequalities

$$U(u_{K,e}^{n+1}) - U(u_K^n) + \frac{\tau p_K}{|K|}\big\{G_{e,K}(u_K^n, u_{K_e}^n) - G_{e,K}(u_K^n, u_K^n)\big\} \leq 0 \tag{3.8}$$

for all faces $e \in \partial K$, all polyhedra K, and all integers n.

As an immediate consequence of Lemma 3.2, we now derive a discrete formulation of the boundary condition (1.3).

LEMMA 3.3. *For all convex entropy pairs (U, F), the following discrete boundary condition holds*:

$$\begin{aligned} &G_{e,K}(u_K^n, u_{K_e}^n) - F(u_{K_e}^n)\cdot\nu_{e,K} \\ &\quad -\nabla U(u_{K_e}^n)\Big\{g_{e,K}(u_K^n, u_{K_e}^n) - f(u_{K_e}^n)\cdot\nu_{e,K}\Big\} \geq 0 \end{aligned} \tag{3.9}$$

for each $e \in \partial\mathcal{T}^h$ and all integers n.

Proof of Lemma 3.3. Using the notation in Lemma 3.2, one can check that, similarly to Lemma 3.2, $\overline{u}$ defined by ($\lambda > 0$ small enough)

$$\overline{u} = u_{K_e}^n + \lambda\{g_{e,K}(u_K^n, u_{K_e}^n) - f(u_{K_e}^n)\cdot\nu_{e,K}\} \tag{3.10}$$

satisfies the following discrete entropy inequality

$$U(\overline{u}) - U(u_{K_e}^n) - \lambda\{G_{e,K}(u_K^n, u_{K_e}^n) - F(u_{K_e}^n)\cdot\nu_{e,K}\} \leq 0 \tag{3.11}$$

for each $e \in \partial\mathcal{T}^h$ and each integer n. On the other hand, by the convexity property of U, (3.10) yields

$$\begin{aligned} U(\overline{u}) - U(u_{K_e}^n) &\geq \nabla U(u_{K_e}^n)\{\overline{u} - u_{K_e}^n\} \\ &\geq \lambda\nabla U(u_{K_e}^n)\{g_{e,K}(u_K^n, u_{K_e}^n) - f(u_{K_e}^n)\cdot\nu_{e,K}\}. \end{aligned} \tag{3.12}$$

The conclusion follows from (3.11) and (3.12). □

We now derive our main a priori estimate, which gives us a weak control of the entropy dissipated in the approximate solution.

PROPOSITION 3.4. *Let $U\colon \boldsymbol{R} \to \boldsymbol{R}$ be a strictly convex function of class $\mathcal{C}^2$. Then for each integer n, we have*

$$\begin{aligned} &\sum_{K\in\mathcal{T}^h} U(u_K^{n+1})|K| + \frac{\alpha}{2}\sum_{\substack{K\in\mathcal{T}^h\\ e\in\partial K}} \frac{|e|\,|K|}{p_K}|u_{K,e}^{n+1} - u_K^{n+1}|^2 \\ &\quad\leq \sum_{K\in\mathcal{T}^h} U(u_K^n)|K| + \|\nabla U(u_1)\|_{L^\infty}\left\|\frac{df}{du}\right\|_{L^\infty}\|u_1\|_{L^1\left([t_n,t_{n+1}),L^1\right)} \\ &\qquad + 2\max\big(\|u_0\|_{L^\infty}, \|u_1\|_{L^\infty}\big)\left\|\frac{df}{du}\right\|_{L^\infty}\|\nabla U(u_1)\|_{L^1\left([t_n,t_{n+1}),L^1\right)}, \end{aligned} \tag{3.13}$$

where α denotes the modulus of convexity of U.

Estimate (3.13) gives not only a uniform bound for the total entropy, i.e.,

$$\sum_{K\in\mathcal{T}^h} U(u_K^n)|K| \leq O(1)$$

(and so (2.10) with the choice $U(u) = |u|^p$) but also a *uniform estimate* for the entropy dissipation. Setting $U(u) = u^2/2$ in (3.13) and using (2.6) lead to the uniform bound

$$\begin{aligned} &\sum_{n=1}^{\infty}\sum_{\substack{K\in\mathcal{T}^h\\ e\in\partial K}} \frac{|e|\,|K|}{p_K}\left|u_{K,e}^n - u_K^n\right|^2 \\ &\quad\leq \|u_0\|_{L^2(\Omega)}^2 + 6\max(\|u_0\|_{L^\infty}, \|u_1\|_{L^\infty})\left\|\frac{df}{du}\right\|_{L^\infty}\|u_1\|_{L^1(L^1)}. \end{aligned} \tag{3.14}$$

Proposition 3.4 is a direct consequence of the local entropy inequalities (3.8). In this respect, the present approach simplifies the technique in Coquel–LeFloch [9]. For convenience, we first state the following lemma whose proof is immediate.

LEMMA 3.5. *Let U be a strictly convex function of class C^2. Let $v = \sum_{j=1}^q a_j v_j$ be a convex combination of the real numbers $v_1, v_2, \dots, v_q$; i.e., $a_j \in [0,1]$ and $\sum_{j=1}^q a_j = 1$. Then one has*

$$U(v) + \frac{\beta}{2}\sum_{j=1}^{q} a_j |v_j - v|^2 \le \sum_{j=1}^{q} a_j U(v_j), \tag{3.15}$$

where

$$\beta = \inf\{U''(w) \,/\, w \in \mathrm{Co}\{v_1, v_2, \dots, v_q\}\},$$

where Co *denotes the convex closure of a set.*

Proof of Proposition 3.4. Applying Lemma 3.5 with the choice $v = u_K^{n+1} = \sum_{e\in\partial K} \frac{|e|}{p_K} u_{K,e}^{n+1}$ gives

$$U(u_K^{n+1}) + \frac{\alpha}{2}\sum_{e\in\partial K} \frac{|e|}{p_K} \mid u_{K,e}^{n+1} - u_K^{n+1} \mid^2 \le \sum_{e\in\partial K} \frac{|e|}{p_K} U(u_{K,e}^{n+1}),$$

which, after summation with respect to K, becomes

$$\sum_{K\in\mathcal{T}^h} U(u_K^{n+1})|K| + \frac{\alpha}{2}\sum_{\substack{K\in\mathcal{T}^h\\ e\in\partial K}} \frac{|e|\,|K|}{p_K} \mid u_{K,e}^{n+1} - u_K^{n+1} \mid^2 \le \sum_{\substack{K\in\mathcal{T}^h\\ e\in\partial K}} \frac{|e|\,|K|}{p_K} U(u_{K,e}^{n+1}). \tag{3.16}$$

On the other hand, the local entropy inequalities (3.8) yield (after summation again)

$$\sum_{\substack{K\in\mathcal{T}^h\\ e\in\partial K}} \frac{|e||K|}{p_K} U(u_{K,e}^{n+1}) - \sum_{K\in\mathcal{T}^h} U(u_K^n)|K| \le -\tau \sum_{e\in\partial\mathcal{T}^h} G_{e,K}(u_K^n, u_{K_e}^n)|e|. \tag{3.17}$$

Here and except along $\partial\mathcal{T}^h$, the flux terms cancel out because of the consistency and conservation properties of the $G_{e,K}$'s. Now, the inequality (3.9) in Lemma 3.3 yields the bound

$$\begin{aligned} &- \sum_{e\in\partial\mathcal{T}^h} G_{e,K}(u_K^n, u_{K_e}^n)|e| \\ &\qquad \le - \sum_{e\in\partial\mathcal{T}^h} \Big\{ F(u_{K_e}^n)\cdot\nu_{e,K} + \nabla U(u_{K_e}^n)\big(g_{e,K}(u_K^n, u_{K_e}^n) - f(u_{K_e}^n)\cdot\nu_{e,K}\big)\Big\}\tau|e|, \end{aligned}$$

so (3.17) becomes

$$\begin{aligned} &\sum_{\substack{K\in\mathcal{T}^h\\ e\in\partial K}} \frac{|e||K|}{p_K} U(u_{K,e}^{n+1}) - \sum_{K\in\mathcal{T}^h} U(u_K^n)|K| \\ &\qquad \le - \sum_{e\in\partial\mathcal{T}^h} \Big\{ F(u_{K_e}^n)\cdot\nu_{e,K} + \nabla U(u_{K_e}^n)\big(g_{e,K}(u_K^n, u_{K_e}^n) - f(u_{K_e}^n)\cdot\nu_{e,K}\big)\Big\}|e|. \end{aligned} \tag{3.18}$$

On one hand, we have

$$-\sum_{e\in\partial\mathcal{T}^h} F(u^n_{K_e})\cdot\nu_{e,K}\tau|e| \le \|\nabla U(u_1)\|_{L^\infty}\left\|\frac{df}{du}\right\|_{L^\infty}\|u_1\|_{L^1\left([t_n,t_{n+1}),L^1(\partial\Omega)\right)};$$

and on the other hand,

$$\begin{aligned}-&\sum_{e\in\partial\mathcal{T}^h}\nabla U(u^n_{K_e})\big(g_{e,K}(u^n_K,u^n_{K_e})-f(u^n_{K_e})\cdot\nu_{e,K}\big)\tau|e|\\ &\le\left\|\frac{df}{du}\right\|_{L^\infty}2\max(\|u_0\|_{L^\infty},\|u_1\|_{L^\infty})\|\nabla U(u_1)\|_{L^1\left([t_n,t_{n+1}),L^1(\partial\Omega)\right)}.\end{aligned}$$

Combining (3.17) and (3.18) and using the two latter bounds give (3.13). □

Estimate (3.14) will be used to establish the consistency of the scheme. Note that (3.14) gives us a control of the entropy dissipation without providing us with an explicit form for it. In particular, (3.14) *does not* necessarily give a control for

$$\sum_{n=0}^{+\infty}\sum_{\substack{K\in\mathcal{T}^h\\ e\in\partial K}}|u^n_{K_e}-u^n_K|^2\,|e|\,\tau,$$

which would be a natural "quadratic" extension to the total variation:

$$\sum_{n=0}^{+\infty}\sum_{\substack{K\in\mathcal{T}^h\\ e\in\partial K}}|u^n_{K_e}-u^n_K|\,|e|\,\tau.$$

Lemma 3.6 and Proposition 3.7 below give (local and then global) entropy inequalities that involve functions of *only one variable* (i.e., $U(u)$ and $F(u)$, but not the numerical flux-functions). This plays a central role in our analysis.

LEMMA 3.6. *Let (U,F) be a convex entropy pair. Then for all polyhedra K and all faces $e\in\partial K\setminus\partial\mathcal{T}^h$, we have*

$$\begin{aligned}&\frac{|K|}{p_K}U(u^{n+1}_{K,e})+\frac{|K_e|}{p_{K_e}}U(u^{n+1}_{K_e,e})-\frac{|K|}{p_K}U(u^n_K)-\frac{|K_e|}{p_{K_e}}U(u^n_{K_e})\\ &\qquad+\tau\big\{F(u^n_{K_e})-F(u^n_K)\big\}\cdot\nu_{e,K}\le 0.\end{aligned}\tag{3.19}$$

In the case of an edge $e\in\partial\mathcal{T}^h$, we have

$$\begin{aligned}&\frac{|K|}{p_K}\Big(U(u^{n+1}_{K,e})-U(u^n_K)\Big)+\nabla U(u^n_{K_e})\big(g_{e,K}(u^n_K,u^n_{K_e})-f(u^n_{K_e})\cdot\nu_{e,K}\big)\\ &\qquad+\tau\big\{F(u^n_{K_e})-F(u^n_K)\big\}\cdot\nu_{e,K}\le 0.\end{aligned}\tag{3.20}$$

Proof of Lemma 3.6. Inequality (3.19) is a consequence of the local entropy inequalities (3.8). Let K be a polyhedron and e be a face in $\partial K\setminus\partial\mathcal{T}^h$. Note that (3.8) is valid in the polyhedron K but also in K_e, so we have

$$U(u^{n+1}_{K_e,e})-U(u^n_{K_e})-\frac{\tau\,p_{K_e}}{|K_e|}\big\{G_{e,K}(u^n_K,u^n_{K_e})-G_{e,K}(u^n_{K_e},u^n_{K_e})\big\}\le 0,\tag{3.21}$$

where we have used the conservation property (3.7). We then multiply (3.8) by $|K|/p_K$ and (3.21) by $|K_e|/p_{K_e}$, and we sum the resulting inequalities. Observe that the terms $G_{e,K}(u_K^n, u_{K_e}^n)$ indeed cancel together. Thus we obtain

$$\frac{|K|}{p_K}\{U(u_{K,e}^{n+1}) - U(u_K^n)\} + \frac{|K_e|}{p_{K_e}}\{U(u_{K_e,e}^{n+1}) - U(u_{K_e}^n)\}$$
$$+ \tau\big\{G_{e,K}(u_{K_e}^n, u_{K_e}^n) - G_{e,K}(u_K^n, u_K^n)\big\} \le 0,$$

which gives (3.19) since $G_{e,K}(u,u) = F(u)\cdot\nu_{e,K}$. Now turning to an edge $e \in \partial\mathcal{T}^h$, we sum the entropy inequality (3.8), valid for $K \in \mathcal{T}^h$ with $e \in \partial\mathcal{T}^h$, and the discrete boundary condition (3.9) in order to deduce (3.20). The proof of Lemma 3.6 is complete. □

PROPOSITION 3.7. *Let (U,F) be a convex entropy pair and $\phi = \phi(t,x)$ be a nonnegative continuous function with compact support in $[0,\infty)\times\overline{\Omega}$. Let K be a polyhedron and e be a face of K, and set*

$$\phi_e^n = \frac{1}{\tau|e|}\int_{t_n}^{t_{n+1}}\int_e \phi(t,x)\,dt\,d\Gamma(x), \qquad \hat\phi_K^n = \sum_{e\in\partial K}\frac{|e|}{p_K}\phi_e^n, \tag{3.22}$$

and

$$\widehat{\partial_t\phi}_K^n = \frac{1}{\tau}(\hat\phi_K^n - \hat\phi_K^{n-1}). \tag{3.23}$$

Then the following inequality is satisfied:

$$\begin{aligned}&-\sum_{n=0}^{\infty}\sum_{K\in\mathcal{T}^h}\int_{t_n}^{t_{n+1}}\int_K\{U(u_K^n)\,\widehat{\partial_t\phi}_K^n + F(u_K^n)\cdot\nabla\phi(t,x)\}\,dt\,dx\\&+\sum_{K\in\mathcal{T}^h}\int_K U(u_K^0)\phi_K^0\,dx + \sum_{n=0}^{\infty}\sum_{e\in\partial\mathcal{T}^h}\int_{t_n}^{t_{n+1}}\int_e\Big\{F(u_{K_e}^n)\cdot\nu_{e,K}\\&+\nabla U(u_{K_e}^n)\big(g_{e,K}(u_K^n,u_{K_e}^n) - f(u_{K_e}^n)\cdot\nu_{e,K}\big)\Big\}\phi(t,x)\,dtd\Gamma(x) \le E^h(\phi),\end{aligned} \tag{3.24}$$

with

$$E^h(\phi) = \sum_{n=0}^{\infty}\sum_{\substack{K\in\mathcal{T}^h\\ e\in\partial K}}\int_{\substack{K\in\mathcal{T}^h\\ e\in\partial K}}\frac{|e|\,|K|}{p_K}U(u_{K,e}^{n+1})\,(\hat\phi_K^n - \phi_e^n). \tag{3.25}$$

Proof of Proposition 3.7. We multiply inequality (3.19) by $\phi_e^n|e|$ and sum over all polyhedra and all faces e, $e \notin \partial\mathcal{T}^h$. In view of the identities

$$\sum_{\substack{K\in\mathcal{T}^h\\ e\in\partial K\setminus\partial\mathcal{T}^h}}\frac{|K_e|\,|e|}{p_{K_e}}U(u_{K_e,e}^{n+1})\phi_e^n = \sum_{\substack{K\in\mathcal{T}^h\\ e\in\partial K\setminus\partial\mathcal{T}^h}}\frac{|K|\,|e|}{p_K}U(u_{K,e}^{n+1})\,\phi_e^n$$

and

$$\sum_{\substack{K\in\mathcal{T}^h\\ e\in\partial K\setminus\partial\mathcal{T}^h}}\frac{|K_e|\,|e|}{p_{K_e}}U(u_{K_e}^n)\phi_e^n = \sum_{\substack{K\in\mathcal{T}^h\\ e\in\partial K\setminus\partial\mathcal{T}^h}}\frac{|K|\,|e|}{p_K}U(u_K^n)\,\phi_e^n,$$

we obtain

$$\sum_{\substack{K\in\mathcal{T}^h\\ e\in\partial K\setminus\partial\mathcal{T}^h}} 2\frac{|K|\,|e|}{p_K}\left\{U(u_{K,e}^{n+1})-U(u_K^n)\right\}\phi_e^n$$
$$+\sum_{\substack{K\in\mathcal{T}^h\\ e\in\partial K\setminus\partial\mathcal{T}^h}} \tau|e|\left\{F(u_{K_e}^n)-F(u_K^n)\right\}\cdot\nu_{e,K}\,\phi_e^n\le 0.$$

We also observe that

$$\sum_{\substack{K\in\mathcal{T}^h\\ e\in\partial K\setminus\partial\mathcal{T}^h}} \tau|e|F(u_{K_e}^n)\cdot\nu_{e,K}\phi_e^n=-\sum_{\substack{K\in\mathcal{T}^h\\ e\in\partial K\setminus\partial\mathcal{T}^h}} \tau|e|F(u_K^n)\cdot\nu_{e,K}\phi_e^n.$$

It thus follows that

$$\sum_{\substack{K\in\mathcal{T}^h\\ e\in\partial K\setminus\partial\mathcal{T}^h}} \frac{|K||e|}{p_K}\Big\{U(u_{K,e}^{n+1})-U(u_K^n)\Big\}\phi_e^n-\tau\sum_{\substack{K\in\mathcal{T}^h\\ e\in\partial K\setminus\partial\mathcal{T}^h}} F(u_K^n)\cdot\nu_{e,K}\phi_e^n|e|\le 0. \tag{3.26}$$

Now considering the edges $e\in\partial\mathcal{T}^h$, after multiplication by $\phi_e^n|e|$ inequalities (3.20) yield

$$\begin{aligned}&\sum_{e\in\partial\mathcal{T}^h}\Big\{\frac{|K|}{p_K}\big(U(u_{K,e}^{n+1})-U(u_K^n)\big)-F(u_K^n)\cdot\nu_{e,K}\tau\Big\}\phi_e^n|e|\\ &+\sum_{e\in\partial\mathcal{T}^h}\Big\{F(u_{K_e}^n)\cdot\nu_{e,K}+\nabla U(u_{K_e}^n)\big(g_{e,K}(u_K^n,u_{K_e}^n)-f(u_{K_e}^n)\cdot\nu_{e,K}\big)\Big\}\phi_e^n\tau|e|\le 0.\end{aligned} \tag{3.27}$$

Summing inequalities (3.26) and (3.27) provides us with

$$\begin{aligned}&\sum_{\substack{K\in\mathcal{T}^h\\ e\in\partial K}}\frac{|K||e|}{p_K}U(u_{K,e}^{n+1})\phi_e^n-\sum_{K\in\mathcal{T}^h}U(u_K^n)\hat\phi_K^n|K|-\sum_{\substack{K\in\mathcal{T}^h\\ e\in\partial K}}F(u_K^n)\cdot\nu_{e,K}\phi_e^n\tau|e|\\ &+\sum_{e\in\partial\mathcal{T}^h}\Big\{F(u_{K_e}^n)\cdot\nu_{e,K}+\nabla U(u_{K_e}^n)\big(g_{e,K}(u_K^n,u_{K_e}^n)-f(u_{K_e}^n)\cdot\nu_{e,K}\big)\Big\}\phi_e^n\tau|e|\le 0,\end{aligned} \tag{3.28}$$

where from the definition of the ϕ_e^n and the Green's theorem one has for each polyhedron K and each integer n exactly

$$\sum_{e\in\partial K}F(u_K^n)\cdot\nu_{e,K}\phi_e^n\tau|e|=\int_{t_n}^{t_{n+1}}\int_K F(u_K^n)\cdot\nabla\phi(t,x)\,dt\,dx.$$

Finally, from Jensen's inequality applied to the convex combination (3.5) we deduce that

$$U(u_K^{n+1})\le\frac{1}{p_K}\sum_{e\in\partial K}|e|U(u_{K,e}^{n+1}),$$

which implies by summation over K

$$\sum_{K\in\mathcal{T}^h}|K|\,U(u_K^{n+1})\hat\phi_K^n\le\sum_{\substack{K\in\mathcal{T}^h\\ e\in\partial K}}\frac{|K||e|}{p_K}U(u_{K,e}^{n+1})\hat\phi_K^n. \tag{3.29}$$

Comparing (3.28) and (3.29) and summing over all n yield immediately

$$\sum_{n=0}^{\infty}\sum_{K\in\mathcal{T}^h}|K|\{U(u_K^{n+1})-U(u_K^n)\}\hat{\phi}_K^n-\sum_{n=0}^{\infty}\sum_{K\in\mathcal{T}^h}\int_{t_n}^{t_{n+1}}\int_K F(u_K^n)\cdot\nabla\phi\,dt\,dx$$

$$+\sum_{n=0}^{\infty}\sum_{e\in\partial\mathcal{T}^h}\Big\{F(u_{K_e}^n)\cdot\nu_{e,K}+\nabla U(u_{K_e}^n)\big(g_{e,K}(u_K^n,u_{K_e}^n)-f(u_{K_e}^n)\cdot\nu_{e,K}\big)\Big\}\phi_e^n\tau|e|\le E^h,$$

where E^h is defined by (3.25). Inequality (3.24) then follows after discrete integration by parts in the first term above. The proof of Proposition 3.7 is complete. □

For the passage to the limit in the inequality (3.24)–(3.25), we need the following approximation result. The proof is elementary.

LEMMA 3.8. *Let $\phi=\phi(t,x)$ be a function of class $\mathcal{C}^2$ having compact support and ϕ_e^n, $\hat{\phi}_K^n$, $\widehat{\partial_t\phi}_K^n$ be defined by formulas* (3.22)–(3.23). *Then, for all integers n and all polyhedra K, one has*

$$\sup_{\substack{t_n\le t\le t_{n+1}\\ x\in K}}|\partial_t\phi(t,x)-\widehat{\partial_t\phi}_K^n|\le(\tau+h_K)\|\phi\|_{\mathcal{C}^2([t_n,t_{n+1}]\times K)}\tag{3.30}$$

and, if e is a face of K,

$$|\phi_e^n-\hat{\phi}_K^n|\le(\tau+h_K)\|\phi\|_{\mathcal{C}^1([t_n,t_{n+1}]\times K)}.\tag{3.31}$$

4. Convergence via measure-valued solutions. Finally, we use the entropy rate estimate for the passage to the limit in the entropy inequalities. We prove that the Young measure, describing all the weak-star limits of u^h, is an entropy measure-valued solution in the sense of DiPerna. Applying DiPerna's uniqueness theorem yields the strong convergence of the finite volume method. Let us consider the Young measure ν: $\boldsymbol{R}_+\times\Omega\to\mathrm{Prob}(\boldsymbol{R})$ associated with the sequence u^h (or a subsequence of it). Here $\mathrm{Prob}(\boldsymbol{R})$ denotes the set of all measures of probability. It is known that, in view of the L^∞ stability property satisfied by u^h (cf. (3.1)), such a Young measure exists. It is characterized by the following property: for every continuous function a: $\boldsymbol{R}\to\boldsymbol{R}$, one has

$$a(u^h)\rightharpoonup\langle\nu,a\rangle\quad\text{as}\quad h\to 0\tag{4.1}$$

in the L^∞ weak-star topology. We quote the paper by Tartar [40], for instance, as a general reference on Young measures.

Our goal is to deduce properties satisfied by this Young measure ν from what is known regarding the sequence of functions u^h. For that purpose, we mainly use the entropy inequalities (3.19), (3.20), and more precisely their consequence (3.24)–(3.25). We emphasize again that (3.15) involves functions of only *one variable.* The passage to the limit in the left-hand side of (3.24)–(3.25) is immediate in view of (4.1). The right-hand side of (3.24)–(3.25) is treated based on the a priori estimate (3.14).

We obtain that ν is consistent with all the entropy inequalities, the initial condition, and the boundary condition.

LEMMA 4.1. *There exists a function g in $L^\infty(\boldsymbol{R}_+\times\partial\Omega)$ with the following property. For all convex entropy pairs (U,F) and for any nonnegative test-function ϕ:$\boldsymbol{R}_+\times\overline{\Omega}\to\boldsymbol{R}_+$, which vanishes for t large enough, one has*

$$-\int_{\boldsymbol{R}_+\times\Omega}\Big(\langle\nu,U\rangle\partial_t\phi+\langle\nu,F\rangle\cdot\mathrm{grad}\;\phi\Big)\,dtdx+\int_\Omega U\big(u_0(x)\big)\phi(0,x)\,dx$$

$$+\int_{\boldsymbol{R}_+\times\partial\Omega}\Big\{F(u_1)\cdot N+\nabla U(u_1)\big(g-f(u_1)\cdot N\big)\Big\}\phi\,d\Gamma dt\leq 0. \tag{4.2}$$

Proof of Lemma 4.1. Using the error estimate in Lemma 3.8, it is clear that the first term in the left-hand side of inequality (3.24) converges to

$$-\int_{\boldsymbol{R}_+}\int_\Omega\{\langle\nu,U\rangle\partial_t\phi+\langle\nu,F\rangle\text{ grad }\phi\}\,dt\,dx+\int_\Omega U\big(u_0(x)\big)\phi(0,x)\,dx. \tag{4.3}$$

Let $g\in L^\infty\big(\boldsymbol{R}_+\times\partial\Omega\big)$ be the weak limit of the sequence $\Big\{g_{e,K}(u_K^n,u_{K_e}^n)\Big\}$. This means that for all compactly supported test-function $\phi\colon\boldsymbol{R}_+\times\overline{\Omega}\to\boldsymbol{R}$,

$$\sum_{n\in N}\sum_{e\in\partial\mathcal{T}^h}g_{e,K}(u_K^n,u_{K_e}^n)\phi_e^n\tau|e|\to\int_{\boldsymbol{R}_+\times\partial\Omega}g\phi\,d\Gamma dt.$$

Since $\{u_{K_e}^n\}\to u_1$ strongly and $\{g_{e,K}(u_K^n,u_{K_e}^n)\}\rightharpoonup g$ weakly, the second term of the left-hand side of (3.24) converges to

$$\int_{\boldsymbol{R}_+\times\partial\Omega}\Big\{F(u_1)\cdot N+\nabla U(u_1)\big(g-f(u_1)\cdot N\big)\Big\}\phi\,d\Gamma dt. \tag{4.4}$$

Finally, we treat the term $E^h(\phi)$ in (3.25). In view of (3.22), we have

$$\sum_{n=0}^{\infty}\sum_{\substack{K\in\mathcal{T}^h\\ e\in\partial K}}\frac{|e|\,|K|}{p_K}U(u_K^{n+1})(\hat\phi_K^n-\phi_e^n)=0.$$

Using the latter identity and Cauchy–Schwartz inequality in the expression of $E^h(\phi)$, we get

$$|E^h(\phi)|\leq\left\{\sum_{n=0}^{\infty}\sum_{\substack{K\in\mathcal{T}^h\\ e\in\partial K}}\frac{|e|\,|K|}{p_K}(u_{K,e}^{n+1}-u_K^{n+1})^2\right\}^{1/2}\cdot\left\{\sum_{n=0}^{\infty}\sum_{\substack{K\in\mathcal{T}^h\\ e\in\partial K}}\frac{|e|\,|K|}{p_K}\big(\hat\phi_K^n-\phi_e^n\big)^2\right\}^{1/2}.$$

In view of the a priori estimate (3.14) and the error estimate (3.31), we get

$$|E^h(\phi)|\leq O(1)\|\phi\|_{C^1(\boldsymbol{R}_+\times\Omega)}\left(\sum_{(n,K)\in A}|K|(\tau+h_K)^2\right)^{1/2}.$$

The last sum is over a set A of pairs (n,K) determined in an obvious manner from the (compact) support of the function ϕ. Using (2.1), it follows that

$$|E^h(\phi)|\leq O(1)\|\phi\|_{C^1(\boldsymbol{R}_+\times\Omega)}\left(\tau^{1/2}+\frac{h}{\tau^{1/2}}\right)\to 0. \tag{4.5}$$

Combining the results in (4.3)–(4.5) gives (4.2). □

It can be checked that (4.2) implies

$$\partial_t\langle\nu,U\rangle+\text{ div }\langle\nu,F\rangle\leq 0, \tag{4.6}$$

in the sense of distributions in Ω, and

$$\lim_{\substack{t\to 0\\ t>0}} \frac{1}{t}\int_0^t \int_\Omega \langle \nu_{s,x}, id - u_0(x)\rangle\, \phi(x)\, ds\, dx = 0. \tag{4.7}$$

From (4.6), it follows that

$$\lim_{t\to 0} \frac{1}{t}\int_0^t \int_\Omega \langle \nu_{s,x}, \lambda^2\rangle\, \phi(x)\, ds\, dx \le \int_\Omega u_0^2 \phi\, dx \tag{4.8}$$

for all nonnegative test-functions $\phi: \Omega \to \boldsymbol{R}_+$ with compact support. The statements (4.7) and (4.8) actually imply that ν assumes its initial data in a strong sense [14]:

$$\lim_{\substack{t\to 0\\ t>0}} \frac{1}{t}\int_0^t \int_\Omega \langle \nu_{s,x}, |id - u_0(x)|\rangle\, ds\, dx = 0. \tag{4.9}$$

We now treat the boundary condition.

LEMMA 4.2. *The normal trace of $\langle \nu, F\rangle$ along the boundary $\partial\Omega$ exists in the weak L^1 sense. In particular, the trace of $\langle \nu, f \cdot N\rangle$ coincides with the function g.*

For a proof, see LeFloch–Shearer [27]. In view of Lemma 4.2, we deduce from the formulation (4.2) that

$$\langle \nu, F\cdot N\rangle - F(u_1)\cdot N - \nabla U(u_1)\big(\langle \nu, f\cdot N\rangle - f(u_1)\cdot N\big) \ge 0$$

almost everywhere on $\boldsymbol{R}_+ \times \partial\Omega$.

THEOREM 4.3. *An entropy measure-valued solution ν to* (1.1)–(1.5), *that is, a Young measure satisfying* (4.2) *for all convex entropy pairs* (U, F), *is Dirac mass, i.e.,*

$$\nu_{t,x} = \delta_{u(t,x)},$$

where u is the Kruzkov's solution to (1.1)–(1.5).

For the proof, see DiPerna [14] and Szepessy [36].

The proof of Theorem 2.1 is then an immediate corollary of Lemma 4.1 and Theorem 4.3. We deduce that $\nu_{t,x} = \delta_{u(t,x)}$, which means that

$$u^h \to u \quad L^p \text{ strong}\ ,\ 1 \le p < \infty,$$

where u is the unique Kruzkov's solution to (1.1)–(1.5).

5. Examples and remarks. This section contains several examples of monotone finite volume methods, which we divide into two main families depending upon whether or not the direction splitting technique is being used. We complete this section with several comments and an application related to the entropy dissipation estimates.

Let e be a face e of a given polyhedron $K \in \mathcal{T}^h$. The flux term $g_{e,K}$ can be defined from a monotone one-dimensional numerical flux, such as the Godunov, Lax–Friedrichs, or Engquist–Osher ones. However, this can be done in two different ways. The most natural definition consists of defining $g_{e,K}$ as a one-dimensional monotone numerical flux consistent with the following one-dimensional equation:

$$\partial_t w + \partial_y\big(f(w)\cdot \nu_{e,K}\big) = 0, \quad t > 0,\ y \in \boldsymbol{R}. \tag{5.1}$$

A second approach is based on the direction splitting technique. In that situation $g_{e,K}$ is obtained in the form

$$g_{e,K} = \sum_{i=1,\dots,d} |\nu^i_{e,K}|\, g^i_{e,K}, \tag{5.2}$$

where, for each function $g^i_{e,K}$ is a monotone flux consistent with

$$\partial_t w + \partial_y\big(\text{ sgn } (\nu^i_{e,K}) f_i(w)\big) = 0, \quad t > 0,\, y \in \boldsymbol{R}, \tag{5.3}$$

where sgn $(a) = 1$ if $a \geq 0$, -1 if $a < 0$. Observe that the two above techniques can be combined together as well.

We shall refer to the first approach (respectively the second one) as the finite volume method without direction splitting (respectively with direction splitting). We begin by providing more details on the constrution of the methods based on direction splitting. This approach seems to have remained unnoticed in the literature.

Finite volume methods based on direction splitting. For each $i = 1, \dots, d$, we consider a monotone numerical flux-function $g_i: \boldsymbol{R}^2 \to \boldsymbol{R}$ such that $g_i(u,u) = f_i(u)$, $u \in \boldsymbol{R}$. Let us define the functions $L^i_{e,K}$ and $R^i_{e,K} : \boldsymbol{R}^2 \to \boldsymbol{R}$ by, for all $u, v \in \boldsymbol{R}$,

$$\begin{aligned} &L^i_{e,K}(u,v) = u \text{ if } \nu^i_{e,K} > 0, \quad v \text{ if } \nu^i_{e,K} \leq 0,\\ &R^i_{e,K}(u,v) = L^i_{e,K}(v,u). \end{aligned}$$

The formula

$$g^i_{e,K}(u,v) = \text{ sgn } (\nu^i_{e,K}) g_i(L^i_{e,K}(u,v), R^i_{e,K}(u,v)), \quad u, v \in \boldsymbol{R},$$

defines a monotone numerical flux $g^i_{e,K}$ consistent with (5.3). In view of (5.2) $g_{e,K}$ is given by

$$g_{e,K}(u,v) = \sum_{i=1,\dots,d} |\nu^i_{e,K}|\, g_i(L^i_{e,K}(u,v), R^i_{e,K}(u,v)), \quad u, v \in \boldsymbol{R}. \tag{5.4}$$

Formula (5.4) produces a flux consistent with $f \cdot \nu_{e,K}$.

As an illustration, we apply this construction to the Godunov and Lax–Friedrichs schemes.

Godunov method based on direction splitting. For each $i = 1, \dots, d$, the function g_i is defined by

$$g_i(u,v) = f_i(w(0^+; u, v; f_i)), \quad u, v \in \boldsymbol{R}. \tag{5.5}$$

Here the function $w = w(.; u, v; f_i)$ is the solution to the Riemann problem

$$\begin{aligned} &\partial_t w + \partial_y f_i(w) = 0, \quad t > 0,\, y \in \boldsymbol{R},\\ &w(0,y) = u \text{ if } y < 0, \quad v \text{ if } y > 0. \end{aligned}$$

The Godunov method with direction splitting is defined by (5.4), (5.5).

Modified Lax–Friedrichs method based on direction splitting. Here g_i is chosen to be

$$g_i(u,v) = \frac{1}{2}\big(f_i(u) + f_i(v)\big) - \frac{1}{2\lambda^i_{e,K}}(v-u), \quad u,\, v \in \boldsymbol{R}, \tag{5.6}$$

where, for $i = 1, \dots, d$, $\lambda^i_{e,K}$ is a given positive number. One can check that (2.2)–(2.4) hold if

$$\lambda^i_{e,K} = \lambda^i_{e,K_e} > 0, \quad i = 1, \dots, d,$$

and

$$\lambda^i_{e,K} \| \frac{df_i}{dw} \|_{L^\infty} \leq 1.$$

Formulas (5.4) and (5.6) yield the modified Lax–Friedrichs method based on direction splitting.

Finite volume methods without direction splitting.

Godunov method without direction splitting. In that case, the flux $g_{e,K}$ is defined by

$$g_{e,K} = \nu_{e,K} \cdot f\big(w(0^+; u, v; f \cdot \nu_{e,K})\big), \quad u, v \in \boldsymbol{R}, \tag{5.7}$$

where the function $w = w(.; u, v; f \cdot \nu_{e,K})$ is the corresponding Riemann solution (cf. the notation introduced above). We emphasize that (5.7) in general is not identical with the formula (5.4)–(5.5). We also note that (5.4)–(5.5) requires somewhat less computational effort than (5.7) does. When the conservation law (1.1) is invariant under rotation in space (this is the case for many systems in physics!), the two methods coincide.

Modified Lax–Friedrichs method without direction splitting. For each face e of $K \in \mathcal{T}^h$, the numerical flux is defined by

$$g_{e,K}(u, v) = \frac{1}{2}\big(\nu_{e,K} \cdot f(u) + \nu_{e,K} \cdot f(v)\big) - \frac{1}{2\lambda_{e,K}}(v - u), \quad u, v \in \boldsymbol{R}. \tag{5.8}$$

Here the coefficient $\lambda_{e,K}$ is assumed to be independent of (u, v) and satisfy

$$\lambda_{e,K} = \lambda_{e,K_e} > 0$$

and

$$\lambda_{e,K} \left\| \frac{d(f \cdot \nu_{e,K})}{du} \right\|_{L^\infty} \leq 1.$$

Formula (5.8) yields a numerical flux that, generally speaking, differs from (5.4), (5.6). However, the two formulas coincide when the coefficients $\lambda^i_{e,K}$ in (5.8) are chosen such that

$$\lambda^i_{e,K} = d\, \lambda_{e,K}, \quad i = 1, \dots, d.$$

The proof of convergence given in §4 applies to any monotone finite volume method and, in particular, to both the methods based on direction splitting and those defined without direction splitting. These methods converge strongly to the unique solution. Indeed, as was proved in §4, the entropy dissipation estimate (3.14) is sufficient to ensure that the Young measure ν is consistent with all the entropy inequalities. We emphasize that our estimate, however, does not imply an a priori estimate of the form ($p > 0$)

$$\sum_{n \in N} \sum_{\substack{K \in \mathcal{T}^h \\ e \in \partial K}} | G^n_{e,K} - F(u^n_K) \cdot \nu_{e,K} |^p \, |e| \tau \leq O(1), \tag{5.9}$$

where $G_{e,K}$ denotes the numerical entropy-flux associated with a given entropy flux F. Put in other words, we cannot a priori pass to the limit in any given numerical flux-function. This precisely explains why the discrete entropy inequalities (3.19) and (3.20) are essential to our analysis: they yield global discrete entropy inequalities free of numerical flux-functions. Consistency in the sense of (4.2) is achieved at a discrete level up to an error term E^h, which is controlled without appealing to an estimate like (5.9).

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