The Computation of Discontinuous Solutions of Linear Hyperbolic Equations*

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Let L be a linear hyperbolic operator in any number of variables with C^∞ coefficients. As is well known, a solution u of $Lu=0$ which has C^∞ initial data is C^∞ for all time. Let L_h be a difference approximation to L that is stable, and accurate of order ν. Denote by U the solution of $L_hU=0$ whose initial values agree with those of u on the lattice points, h denoting the mesh width of the lattice. According to the basic theory of difference approximations, at all times t at which U is available, and in any fixed range $0 \leqq t \leqq T$,

$$|u(t)-U(t)|=O(h^\nu)\,. \tag{1}$$

Consider piecewise C^∞ initial data whose discontinuities occur across C^∞ surfaces. It is known that solutions u of $Lu=0$ with such initial data are themselves piecewise C^∞, and their discontinuities occur across characteristic surfaces issuing from the discontinuity surface of the initial data. What happens when such a solution is approximated by a solution of $L_hU=0$? Does U differ from u by $O(h^\nu)$ in those regions where u is C^∞, or have the discontinuities hopelessly polluted the approximate solutions even at smooth regions between discontinuities? In a recent paper [2], Majda and Osher have shown that, for second order accurate schemes applied to hyperbolic equations in one space variables and with constant coefficients, U differs from u in smooth regions by $O(h^2)$, provided that at lattice points of discontinuity the initial value of U is taken as the average of the values of u on the two sides of the discontinuity. In this note we show that, for a scheme of any order ν, the *moments* of U approximate those of u with accuracy $O(h^\nu)$, provided that the initial data of U are *pre-processed* appropriately near the discontinuities; this is true for equations in any number of variables, and with variable coefficients. We show further that we can, by *post-processing* the approximate solution, recover the exact solution, as well as its derivatives,

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with an error $O(h^{\nu-\delta})$, δ as small as we wish, at all points, no matter how close to the discontinuity. The idea of comparing the moments of a discontinuous solution with the moments of its approximation comes from [3]; there it is used also in studying discontinuous solutions of nonlinear conservation laws.

We need the following quadrature result, which goes back to the 18-th century, see [1]:

LEMMA 1. *Let f be any C^∞ function on R_+, with bounded support. Given any positive integer ν, there exists a quadrature formula accurate of order ν of the form*

$$\int_0^\infty f(x)\,dx = h\sum_0^\infty w_j f(jh) + O(h^\nu)\,, \tag{2}$$

where the weights w_j depend on ν, but

$$w_j = 1 \quad \textit{for} \quad j \geqq \nu\,. \tag{3}$$

EXAMPLE. For $\nu = 4$ we have

$$w_0 = \tfrac{3}{8}\,, \quad w_1 = \tfrac{7}{6}\,, \quad w_2 = \tfrac{23}{24}\,, \quad w_j = 1 \quad \text{for} \quad j > 2\,.$$

Using (2) twice we get this

COROLLARY. *Let f be a piecewise C^∞ function, the discontinuity occurring at $x = 0$. Let ν be any integer; then*

$$\int_{-\infty}^\infty f(x)\,dx = h\sum w_j' f(jh) + O(h^\nu)\,, \tag{2$'$}$$

where

$$w_0' = 2w_0\,, \quad w_j' = w_{|j|} \quad \textit{for} \quad j \neq 0\,, \tag{4}$$

and

$$f(0) = \frac{f(0-) + f(0+)}{2}\,.$$

Similar formulas can be obtained in case the discontinuity of f occurs between the mesh points, say at $x = h\theta$, $|\theta| < 1$, and for the integrals of discontinuous functions of several variables whose discontinuities occur along smooth surfaces.

Let L be a first-order hyperbolic matrix operator

$$L = \partial_t + \sum A_j \, \partial_j + B \,, \qquad \partial_j = \frac{\partial}{\partial x_j}; \tag{5}$$

the coefficients A_j and B are C^∞ functions of x and t; for simplicity we take them, and all solutions, to be real.

Denote the L_2 scalar product with respect to the x variables by

$$(u, v) = \int u(x) \cdot v(x) \, dx \,. \tag{6}$$

Denote by L^* the adjoint of L; suppose u and v satisfy

$$Lu = 0 \,, \qquad L^* v = 0 \,, \tag{7}$$

and one of them vanishes for $|x|$ large. Then, by Green's formula in the slab $0 \leqq t \leqq T$,

$$(u(T), v(T)) = (u(0), v(0)) \,. \tag{8}$$

Let L_h be a two-level, forward difference approximation:

$$L_h = D_t^+ + \sum S_j T^j \,, \qquad T^j \text{ translation by } jh \,, \tag{9}$$

Let U be a lattice function that satisfies $L_h U = 0$, i.e.,

$$U_k^{n+1} = \sum C_j U_{k-j}^n \,, \qquad C_j = C_j^n(k) \,. \tag{10}$$

Let V be another lattice function; multiplying (10) by V_k^{n+1} and summing gives

$$\begin{aligned} \sum_k U_k^{n+1} \cdot V_k^{n+1} &= \sum_{k,j} C_j U_{k-j}^n \cdot V_k^{n+1} \\ &= \sum_k U_k^n \cdot \sum_j C_j^*(k+j) V_{k+j}^{n+1} \,. \end{aligned} \tag{11}$$

Suppose V satisfies the two-level backward adjoint equation

$$L_h^* V = 0$$

defined to be

$$V_k^n = \sum C_j^*(k+j) V_{k+j}^{n+1} \,. \tag{10*}$$

Then (11) can be rewritten as

$$(U^{n+1}, V^{n+1})_h = (U^n, V^n)_h, \tag{12}$$

where $(\ ,\ )_h$ denotes the lattice scalar product

$$(U, V)_h = h \sum U_k \cdot V_k. \tag{13}$$

Note that, for u, v in C_0^∞,

$$(u, v)_h = (u, v) + O(h^\nu) \tag{14}$$

for any ν.

We draw two conclusions from (12):

(a) If L_h approximates L to order ν, then L_h^* approximates L^* also to order ν.

(b) For all N,

$$(U^N, V^N) = (U^0, V^0). \tag{15}$$

Proof: (a) L_h approximates L to order ν if

$$U^1 = u(h) + O(h^{\nu+1}), \tag{16}$$

where $u(t)$ is any C^∞ solution of $Lu = 0$ and U the solution $L_h U = 0$ with the same initial data as u:

$$U^0 = u(0). \tag{17}$$

Let $v(t)$ be any C^∞ solution of $L^* v = 0$, V a solution of $L_h^* V = 0$ which equals v at $t = h$:

$$V^1 = v(h). \tag{18}$$

Then by (10)* we can determine V^0:

$$V^0 = v(0) + h^{p+1} E + O(h^{p+2}), \tag{19}$$

where p is the order to which L_h^* approximates L^*, and E is the leading truncation error. It follows from (16) and (18) using (14), that

$$(U^1, V^1)_h = (u(h), v(h)) + O(h^{\nu+1}); \tag{20}$$

on the other hand, by (17) and (19),

$$(U^0, V^0)_h = (u(0), v(0)) + h^{p+1}(u(0), E(0)) + O(h^{p+2}). \tag{21}$$

By (12) the left sides of (20) and (21) are equal, and by (8) the first terms on the right are equal; from this we conclude that $p = \nu$.

(b) Relation (15) follows by summing (12) from 0 to $N-1$. It follows that if L_h is stable, so is L_h^*.

We turn now to discontinuous solutions; for the sake of simplicity we take the number of space dimensions to be 1. Suppose u is a piecewise C^∞ solution of $Lu = 0$ whose initial data contain a single discontinuity at, say, $x = 0$. Let U be a solution of $L_h U = 0$ whose initial data are related to those of u as follows:

$$\begin{aligned} U^0(jh) &= w_j' u(jh, 0), \qquad j \neq 0, \\ U^0(0) &= w_0'(u(0-, 0) + u(0+, 0))/2. \end{aligned} \tag{22}$$

where w_j' are the weights (4) entering formula (2)'. Let $\phi(x)$ be an arbitrary C_0^∞ function; denote by v the solution of

$$L^* v = 0, \qquad v(x, T) = \phi(x). \tag{23}$$

Let V be the solution of

$$L_h^* V = 0, \qquad V^N(jh) = \phi(jh), \tag{23$_h$}$$

where N is a time step corresponding to $t = T$. Since v is C^∞ and since L_h^* is accurate of order ν, it follows that if L_h^* is stable, then

$$V^0(jh) = v(jh, 0) + O(h^\nu). \tag{24}$$

We take, in the quadrature formula (2)', $f = u(x, 0)\, v(x, 0)$; this is a piecewise C^∞ function, with a discontinuity at $x = 0$, therefore

$$(u(0), v(0)) = \int f(x)\, dx = h \sum w_j' u(jh, 0)\, v(jh, 0) + O(h^\nu). \tag{25}$$

Using the definition (22) of $U_h^0(jh)$, the error estimate (24) and the definition (13) of scalar products for lattice functions, we see from (25) that

$$(u(0), v(0)) = (U^0, V^0)_h + O(h^\nu). \tag{26}$$

Now using (8) and (15) we deduce from (26) that

$$(u(T), v(T)) = (U^N, V^N)_h + O(h^\nu).$$

According to (23) and $(23)_h$, both $v(T)$ and V^N were chosen to be ϕ; therefore the relation above can be rewritten as

$$(u(T), \phi) = (U^N, \phi)_h + O(h^\nu). \tag{27}$$

We summarize:

THEOREM 1. *Let L_h be a stable two-level difference operator that approximates L to order ν. Let u be a solution of $Lu = 0$ whose initial data are piecewise C^∞ with a discontinuity at $x = 0$. Let U be the solution of $L_h U = 0$ whose initial data are related to those of U by formula (22). Then at any later time T the moments of u and U with any C_0^∞ function ϕ differ by $O(h^\nu)$.*

We show now how to use the weak error estimate (27) to deduce pointwise estimates. To this end we need to know the dependence of the error term in (27) on ϕ; since the error term comes from (24), it is of order of the size of the $(\nu+1)$-st derivative of v. Thus

$$(u(T), \phi) = (U^N, \phi)_h + ch^\nu, \tag{27'}$$

where

$$c = O(|\phi|_{\nu+1}), \tag{28}$$

the bars denoting the maximum norm of $\partial_x^{\nu+1}\phi$; for more space variables more derivatives of ϕ are needed in (28). Let $s(x)$ be an auxiliary function whose support is contained in the interval $(-1, 1)$, satisfying

$$\int s(x)\,dx = 1, \qquad \int x^l s(x)\,dx = 0, \qquad l = 1, \cdots, p-1, \tag{29}$$

p an arbitrary integer. We set

$$\phi(x) = \frac{1}{\varepsilon} s\left(\frac{x-y}{\varepsilon}\right). \tag{30}$$

For any function g that is C^∞ in $(y-\varepsilon, y+\varepsilon)$

$$g(y) = \int g(x)\phi(x)\,dx + O(\varepsilon^p).$$

So if the interval $(y-\varepsilon, y+\varepsilon)$ is free of discontinuities of u at time T,

$$\int u(x, T)\phi(x)\,dx = u(y, T) + O(\varepsilon^p).$$

Comparing this with (27), gives

$$u(y, T) = (U^N, \phi)_h + O(|\phi|_{\nu+1})h^\nu + O(\varepsilon^p). \tag{31}$$

From the definition of ϕ in (30) we see that

$$|\phi|_{\nu+1} = O\Big(\frac{1}{\varepsilon^{\nu+2}}\Big);$$

thus

$$u(y, T) = (U^N, \phi)_h + O\Big(\frac{h^\nu}{\varepsilon^{\nu+2}}\Big) + O(\varepsilon^p). \tag{31$'$}$$

We choose ε so that the two error terms are of the same order:

$$\frac{h^\nu}{\varepsilon^{\nu+2}} = \varepsilon^p, \qquad \varepsilon = h^{\nu/(\nu+2+p)}. \tag{32}$$

With this choice of ε,

$$u(y, T) = (U^N, \phi)_h + O(h^{\nu p/(\nu+p+2)}). \tag{33}$$

Taking p large enough we get

THEOREM 2. *Choose ϕ of the form* (30), (29); *formula* (33) *recovers $u(y, T)$ with accuracy as close to order ν as we wish if p is taken large enough.*

The same technique yields this

COROLLARY. *Any derivative of u can be recovered with an accuracy as close to order ν as desired.*

The points (y, T) at which u and its derivatives can be recovered by formula (33) and its analogue are subject to the restriction that the interval $(y-\varepsilon, y+\varepsilon)$ be free of discontinuities of u at time T. If, however, we choose the support of s to lie in $(0, 1)$ or $(-1, 0)$, this can be replaced by the one-sided restriction that u be free of discontinuities at time T in either $(y-\varepsilon, y)$ or $(y, y+\varepsilon)$. This allows one to get accurate pointwise estimates right up to the discontinuity.

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