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FIRST ORDER QUASILINEAR EQUATIONS IN SEVERAL INDEPENDENT VARIABLES

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Abstract. In this paper we construct a theory of generalized solutions in the large of Cauchy's problem for the equations

$$u_t + \sum_{i=1}^{n} \frac{d}{dx_i} \varphi_i(t, x, u) + \psi(t, x, u) = 0$$

in the class of bounded measurable functions. We define the generalized solution and prove existence, uniqueness and stability theorems for this solution. To prove the existence theorem we apply the "vanishing viscosity method"; in this connection, we first study Cauchy's problem for the corresponding parabolic equation, and we derive a priori estimates of the modulus of continuity in L_1 of the solution of this problem which do not depend on small viscosity.

Bibliography: 22 items.

§1. Introduction

The central problem of the theory of generalized (discontinuous) solutions of the quasilinear equations

$$u_t + \sum_{i=1}^{n} \frac{d}{dx_i} \varphi_i(t, x, u) + \psi(t, x, u) = 0,$$

$$\frac{d}{dx_i} \varphi(t, x, u) \equiv \varphi_{x_i} + \varphi_u u_{x_i}, \quad x = (x_1, \ldots, x_n) \in E_n, \tag{1.1}$$

is to describe the existence and uniqueness classes of the solution in the large (with respect to t) of Cauchy's problem with the initial condition

$$u\,|_{t=0} = u_0(x) \tag{1.2}$$

at $t = 0$. Several papers have been devoted to studying this problem under different assumptions about the initial function $u_0(x)$ and about the structure of equation (1.1). Ever since the first fundamental paper [1] was published on the theory of generalized solutions of quasilinear equations, the basic method for investigating these equations has remained the "vanishing viscosity method," which is based on the idea of passing to the limit as $\epsilon \to +0$ in the parabolic equation[1)]

$$u_t + \frac{d}{dx_i} \varphi_i(t, x, u) + \psi(t, x, u) = \varepsilon \Delta u, \quad \varepsilon > 0, \tag{1.3}$$

1) Some equations which are model equations for gas dynamics have the form (1.1) and (1.3); the parameter ϵ in (1.3) corresponds to the gas-dynamic notion of viscosity.

where Δ is the Laplace operator over the space variables $x_1, \cdots, x_n$ (here and below, if two of the indices i, j, k are equal in a monomial, then summation is taken from 1 to n). This method, which has deep physical meaning, not only allows us to prove the existence of a generalized solution of problem (1.1), (1.2) in the sense of the corresponding integral identity, but also makes it possible to show those additional conditions on the generalized solutions which characterize the uniqueness class (concerning the necessity of these conditions in the nonlocal theory of Cauchy's problem and the physical significance of these conditions, see, for example, [2] or [3]).

Up to now, the case $n = 1$ with the function $\phi_1(t, x, u)$ in equation (1.1) convex in u is the one that has been studied most thoroughly; in this case a theory of generalized solutions of problem (1.1), (1.2) has been constructed for an arbitrary bounded measurable initial function $u_0(x)$ (see [4]–[6], survey article [2], and others; various methods for constructing generalized solutions with estimates of speed of convergence are given in [7]). Several results concerning the case of a function $\phi_1(t, x, u)$ which is not convex in u are obtained in [8]–[11] and elsewhere. In particular, [8] (see also [9]) contains a uniqueness condition for a generalized solution of Cauchy's problem in the class of piecewise smooth functions; however, as is well known, it is impossible to construct a nonlocal theory of generalized solutions in this class.

The class BV of functions with bounded Tonelli-Cesàro variation is a natural generalization of the class of piecewise smooth functions (at least for the theory of quasilinear equations); one of the necessary and sufficient conditions for a bounded function $w(x)$ to belong to the class $BV(E_n)$ is that, for any compact Ω and any vector $\Delta x \in E_n$,

$$\int_{\Omega} |w(x+\Delta x) - w(x)|\, dx \leqslant \text{const} \cdot |\Delta x|, \tag{1.4}$$

where the constant does not depend on Δx. Article [12] contains a proof of the existence of a generalized solution $u(t, x) \in BV(E_{n+1})$ of Cauchy's problem in the large for the equation

$$u_t + (\varphi_i(u))_{x_i} = 0 \tag{1.5}$$

with an arbitrary bounded initial function $u_0(x)$ in $BV(E_n)$; on the cross-sections $t = \text{const}$ the function $u(t, x)$ also belongs to $BV(E_n)$, so that the class $BV(E_n)$ has an invariance property. It was shown in [13] that, for any function $u(t, x) \in BV(E_{n+1})$, at every point of discontinuity of this function, with the possible exception of the points of a set of n-dimensional Hausdorff measure zero, there is a first order discontinuity and there exists a normal to the set of points of discontinuity (one-sided limits are understood in the approximate sense), where the uniqueness condition for the generalized solution of Cauchy's problem in the class $BV(E_{n+1})$ is written, in principle, in the same way as in the class of piecewise smooth functions (see inequality (1.3) in §2; this condition can be easily derived for solutions of equation (1.1) in the class of piecewise smooth functions using the results and methods of [8] and [9]). Article [13] establishes the existence and uniqueness of a generalized solution of problem (1.5), (1.2) in the case when $u_0(x) \in BV(E_n)$. We note that in this proof of uniqueness we take into account the behavior of the generalized solutions on sets of dimension n; this procedure is connected with using a local (pointwise) uniqueness condition and requires us to take into account rather delicate and complicated results from the theory of BV function classes (it follows from the results in §3 of this paper that to prove uniqueness it is sufficient to know the generalized solutions on certain $(n+1)$-dimensional sets of full Lebesgue measure). The vanishing viscosity method was justified in [13] only for the case of a sufficiently smooth finite initial function $u_0(x)$.

The purpose of this paper is to construct a nonlocal theory of generalized solutions of Cauchy's problem (1.1), (1.2) in the class of bounded measurable functions. This very broad class of functions is the most natural class for constructing such a theory (especially when we are interested in questions of uniqueness and stability of generalized solutions and the question of justifying the vanishing viscosity method). We note that in the sense of "visibility" the solutions in the class of bounded measurable functions are practically equivalent to solutions in the class $BV(E_{n+1})$, since any function in these classes either is piecewise smooth (to within certain visible singularities) or else has an essential "pathology."

In §2 we formulate a definition of a generalized solution of problem (1.1), (1.2) and make some preliminary observations.

In §3 we prove uniqueness and stability theorems for the generalized solutions relative to changes in the initial data; in proving these theorems, from the theory of functions of a real variable we only apply Lebesgue's theorem on passing to the limit under the integral sign, the concept of a Lebesgue point and the result that almost all points of the open domain of an integrable function are Lebesgue points of this function (see [14]).

In §4 we use the vanishing viscosity method to prove an existence theorem for a generalized solution of problem (1.1), (1.2); we first consider Cauchy's problem for the parabolic equation (1.3). In the vanishing viscosity method convergence is proved for any bounded measurable initial function $u_0(x)$.

The author stated the result on existence of a generalized solution of problem (1.5), (1.2) in the sense of the definition in §2 at the International Congress of Mathematicians in Moscow in August, 1966 in discussing a related report by A. I. Vol'pert; the proof of this result was published in [15], where the author also announced the uniqueness theorem for the generalized solution of this problem.

Existence theorems for generalized solutions of problem (1.1), (1.2) in the sense of the integral identity

$$\int_0^T \int_{-\infty}^{+\infty} [uf_t + \varphi_i(t, x, u) f_{x_i} - \psi(t, x, u) f]\, dx\, dt = 0, \tag{1.6}$$

which is valid for any smooth finite function $f(t, x)$ (without determining uniqueness conditions) are established in [16].

The fundamental results of this paper were published in our note [17].

§5 contains some remarks and addenda concerning the questions considered in §§2–4. The arguments in subsection 7° occupy a special place here, where we discuss the problem of a generalized solution of Cauchy's problem for the quasilinear hyperbolic system

$$\frac{\partial \varphi_0(u)}{\partial t} + \frac{\partial \varphi_i(u)}{\partial x_i} = 0, \tag{1.7}$$

with

$$u = (u^1, \ldots, u^N), \quad \varphi_i(u) = (\varphi_i^1(u), \ldots, \varphi_i^N(u)).$$

§2. Statement of Cauchy's problem (1.1), (1.2); some notation and preliminary observations

We let π_T denote the band $\{(t, x)\} \equiv [0, T] \times E_n$. We shall assume that the functions $\phi_i(t, x, u)$ and $\psi(t, x, u)$ are defined and are continuously differentiable for $(t, x) \in \pi_T$ and $-\infty < u < +\infty$ (the assumptions concerning the properties of these functions will be refined in each section).

Let $u_0(x)$ be an arbitrary bounded function which is measurable in E_n: $|u_0(x)| \leq M_0$.

Definition 1. A bounded measurable function $u(t, x)$ is called a *generalized solution of problem* (1.1), (1.2) *in the band* π_T if:

1) for any constant k and any smooth function $f(t, x) \geq 0$ which is finite in π_T (the support of f is strictly contained inside π_T), the following inequality holds:

$$\iint_{\pi_T} \{ |u(t, x) - k| f_t + \operatorname{sign}(u(t, x) - k)[\varphi_i(t, x, u(t, x)) - \varphi_i(t, x, k)] f_{x_i} \\ - \operatorname{sign}(u(t, x) - k)[\varphi_{ix_i}(t, x, k) + \psi(t, x, u(t, x))] f\} \, dx\, dt \geqslant 0; \tag{2.1}$$

2) there exists a set $\mathcal{E}$ of zero measure on $[0, T]$ such that for $t \in [0, T] \setminus \mathcal{E}$ the function $u(t, x)$ is defined almost everywhere in E_n, and for any ball $K_r = \{|x| \leq r\} \subset E_n$

$$\lim_{\substack{t \to 0 \\ t \in [0, T] \setminus \mathcal{E}}} \int_{K_r} |u(t, x) - u_0(x)| \, dx = 0. \tag{2.2}$$

Since the smooth function $f \geq 0$ is arbitrary, it is obvious that inequality (2.1) for $k = \pm \sup |u(t, x)|$ implies that the generalized solution $u(t, x)$ of problem (1.1), (1.2) satisfies integral identity (1.6). But Definition 1 also contains a condition which characterizes the permissible discontinuities of the solutions. This condition is especially easy to visualize when the generalized solution is a piecewise smooth function in some neighborhood of the point of discontinuity; in this case, using integration by parts and the fact that f was chosen arbitrarily, we easily obtain from inequality (2.1) that, for any constant k along the surface of discontinuity,

$$|u^+ - k| \cos(\nu, t) + \operatorname{sign}(u^+ - k)[\varphi_i(t, x, u^+) - \varphi_i(t, x, k)] \cos(\nu, x_i) \\ \leqslant |u^- - k| \cos(\nu, t) + \operatorname{sign}(u^- - k)[\varphi_i(t, x, u^-) - \varphi_i(t, x, k)] \cos(\nu, x_i), \tag{2.3}$$

where ν is the normal vector to the surface of discontinuity at the point (t, x), and u^+ and u^- are the one-sided limits of the generalized solution at the point (t, x) from the positive and negative side of the surface of discontinuity, respectively. It is easily seen that for $n = 1$ inequality (2.3) is equivalent to condition E in [8] (we note that in the case $n \geq 2$ inequality (2.3) can be derived from condition E if the desired solution is approximated by a plane wave in a neighborhood of the point of discontinuity).

Before proceeding to the proofs of the uniqueness and existence theorems for a generalized solution of problem (1.1), (1.2) in the sense of Definition 1, we introduce some notation and make some elementary preliminary observations.

We let $\delta(\sigma)$ designate a function which is infinitely differentiable on $(-\infty, +\infty)$ such that $\delta(\sigma) \geq 0$, $\delta(\sigma) \equiv 0$ for $|\sigma| \geq 1$, and

$$\int_{-\infty}^{+\infty} \delta(\sigma)\, d\sigma = 1.$$

For any number $h > 0$ we set

$$\delta_h(\sigma) \equiv h^{-1}\delta(h^{-1}\sigma). \tag{2.4}$$

It is obvious that $\delta_h(\sigma) \in C^{\infty}(-\infty, +\infty)$ and

$$\delta_h(\sigma) \geqslant 0,\ \delta_h(\sigma) \equiv 0 \ \text{ for } |\sigma| \geqslant h, \quad |\delta_h(\sigma)| \leqslant \frac{\text{const}}{h}, \quad \int_{-\infty}^{+\infty} \delta_h(\sigma)\, d\sigma = 1 \tag{2.5}$$

(for $h \to +0$ the sequence $\{\delta_h(\sigma)\}$ is a delta-shaped sequence at the point $\sigma = 0$).

Let the function $v(x)$ be defined and locally integrable in E_n (we shall assume a function defined only in some region $\Omega \subset E_n$ to be continued by zero on $E_n \setminus \Omega$); we agree to let $v^h(x)$ denote the *mean functions* [1)]

$$v^h(x) \equiv \int_{E_n} \frac{1}{h^n} \lambda\left(\frac{x-y}{h}\right) v(y)\, dy, \quad h > 0, \tag{2.6}$$

with averaging kernel

$$\lambda(x) \equiv \prod_{i=1}^{n} \delta(x_i) \geqslant 0, \quad \int_{E_n} \lambda(x)\, dx = 1. \tag{2.7}$$

We call x_0 a *Lebesgue point* of the function $v(x)$ if

$$\lim_{h\to 0} \frac{1}{h^n} \int_{|x-x_0|\leqslant h} |v(x) - v(x_0)|\, dx = 0.$$

It is easily seen that at any Lebesgue point x_0 of the function $v(x)$

$$\lim_{h\to 0} v^h(x_0) = v(x_0).$$

Since the set of points which are not Lebesgue points of $v(x)$ has measure zero (see, for example, [14], Russian p. 396), it follows that $v^h(x) \to v(x)$ as $h \to 0$ almost everywhere.

We let $\omega(\sigma)$ designate *modulus of continuity type functions*. These functions are defined and continuous for $\sigma \geq 0$, are nondecreasing, and take on zero values at $\sigma = 0$.

Lemma 1. *Let the function $v(x)$ be integrable in the ball $K_{r+2\rho} = \{|x| \leq r + 2\rho\}$, $r > 0$, $\rho > 0$, where*

$$J_s(v, \Delta x) \equiv \int_{K_s} |v(x + \Delta x) - v(x)|\, dx \leqslant \omega_s(|\Delta x|) \tag{2.8}$$

for $|\Delta x| \leq \rho$ and $s \in [0, r + \rho]$. Then for $h \leq \rho$

$$J_r(v^h, \Delta x) \leqslant \omega_{r+h}(|\Delta x|), \tag{2.9}$$

$$\int_{K_r} \big|\, |v| - v(\operatorname{sign} v)^h \big|\, dx \leqslant 2\omega_r(h). \tag{2.10}$$

Estimate (2.9) follows from the obvious inequality

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1) Concerning mean functions, see [17].

$$J_r(v^h, \Delta x) \leqslant \int_{E_n} \lambda(z) \int_{K_r} |v(x + \Delta x - hz) - v(x - hz)|\, dx\, dz.$$

To prove estimate (2.10), it suffices to note that

$$\big|\, |v(x)| - v(x) \operatorname{sign} v(y) \big| = \big|\, |v(x)| - |v(y)| - [v(x) - v(y)] \operatorname{sign} v(y) \big| \leqslant 2\, |v(x) - v(y)|$$

and consequently

$$\int_{K_r} \big|\, |v| - v (\operatorname{sign} v)^h \big|\, dx$$

$$\leqslant \int_{K_r} \int_{E_n} h^{-n} \lambda\left(\frac{x-y}{h}\right) \big|\, |u(x)| - u(x) \operatorname{sign} v(y) \big|\, dy\, dx$$

$$\leqslant 2 \int_{E_n} \lambda(z) \int_{K_r} |u(x) - u(x - hz)|\, dx\, dz \leqslant 2\omega_r(h).$$

Lemma 2. *Let the function $v(t, x)$ be bounded and measurable in some cylinder $Q = [0, T] \times K_r$. If for some $\rho \in (0, \min[r, T])$ and any number $h \in (0, \rho)$ we set*

$$V_h = \frac{1}{h^{n+1}} \iiiint\limits_{\substack{\left|\frac{t-\tau}{2}\right| \leqslant h,\ \rho \leqslant \frac{t+\tau}{2} \leqslant T-\rho, \\ \left|\frac{x-y}{2}\right| \leqslant h,\ \left|\frac{x+y}{2}\right| \leqslant r-\rho}} |v(t, x) - v(\tau, y)|\, dx\, dt\, dy\, d\tau, \tag{2.11}$$

then $\lim_{h \to 0} V_h = 0$.

Proof. After substituting

$$\frac{t+\tau}{2} = \alpha, \quad \frac{t-\tau}{2} = \beta, \quad \frac{x+y}{2} = \eta, \quad \frac{x-y}{2} = \xi$$

we have

$$V_h = 2^{n+1} \iint\limits_{\substack{\rho \leqslant \alpha \leqslant T-\rho \\ |\eta| \leqslant r-\rho}} G_h(\alpha, \eta)\, d\eta\, d\alpha,$$

$$G_h(\alpha, \eta) = \frac{1}{h^{n+1}} \iint\limits_{\substack{|\beta| \leqslant h \\ |\xi| \leqslant h}} |v(\alpha + \beta, \eta + \xi) - v(\alpha - \beta, \eta - \xi)|\, d\xi\, d\beta.$$

Since almost all points (α, η) of the cylinder $Q_\rho = [\rho, T - \rho] \times K_{r-\rho}$ are Lebesgue points of the function $v(\alpha, \eta)$, and since

$$|v(\alpha+\beta, \eta+\xi) - v(\alpha-\beta, \eta-\xi)| \leqslant |v(\alpha+\beta, \eta+\xi) - v(\alpha, \eta)| + |v(\alpha, \eta) - v(\alpha-\beta, \eta-\xi)|,$$

it follows that $G_h(\alpha, \eta) \to 0$ as $h \to 0$ almost everywhere in Q_ρ. It remains to note that $|G_h(\alpha, \eta)| \leq c(n) \sup |v|$ and that the assertion of the lemma follows from Lebesgue's theorem on passing to the limit under the integral sign ([14], Russian p. 139).

Lemma 3. *If the function $F(u)$ satisfies a Lipschitz condition on the interval $[-M, M]$ with constant L, then the function $H(u, v) \equiv \operatorname{sign}(u - v)[F(u) - F(v)]$ also satisfies the Lipschitz condition in u and v with the constant L.*

To prove this, it suffices to take into account that $H_u(u, v) = F'(u)\operatorname{sign}(u - v)$ for fixed $v \in [-M, M]$ and almost all $u \in [-M, M]$, and that $H_v(u, v) = F'(v)\operatorname{sign}(v - u)$ for fixed $u \in [-M, M]$ and almost all $v \in [-M, M]$.

Finally, we introduce notation connected with the concept of a *characteristic cone*. For any $R>0$ and $M > 0$ we set

$$N = N_M(R) = \max_{\substack{(t,x)\in[0,T]\times K_R \\ |u|\leqslant M}} \left[\sum_{i=1}^{n} \varphi_{iu}^2(t, x, u)\right]^{1/2} \tag{2.12}$$

and let K designate the cone $\{(t, x): |x| \le R - Nt,\ 0 \le t \le T_0 = \min(T, RN^{-1})\}$; we let S_τ designate the cross-section of the cone K by the plane $t = \tau$, $\tau \in [0, T_0]$.

§3. Uniqueness of the generalized solution of problem (1.1), (1.2); stability with respect to the initial condition

In this section we shall assume that the functions $\phi_i(t, x, u)$ and $\psi(t, x, u)$ are continuously differentiable in the region $\{(t, x) \in \pi_T, -\infty < u < +\infty\}$, while the functions $\phi_{ix_j}(t, x, u)$ and $\phi_{it}(t, x, u)$ satisfy the Lipschitz condition in u on any compact set.

Uniqueness of the generalized solution of problem (1.1), (1.2) follows from the following proposition concerning stability of the solutions relative to changes in the initial data in the norm of the space L_1.

Theorem 1. *Let the functions $u(t, x)$ and $v(t, x)$ be generalized solutions of problem* (1.1), (1.2) *with initial functions $u_0(x)$ and $v_0(x)$, respectively, where $|u(t, x)| \le M$ and $|v(t, x)| \le M$ almost everywhere in the cylinder $[0, T] \times K_R$; let $\gamma = \max[-\psi_u(t, x, u)]$ in the region $\{(t, x) \in K, |u| \le M\}$. Then for almost all $t \in [0, T_0]$*

$$\int_{S_t} |u(t, x) - v(t, x)|\, dx \leqslant e^{\gamma t} \int_{S_0} |u_0(x) - v_0(x)|\, dx. \tag{3.1}$$

Proof. Let the smooth function $g(t, x; \tau, y) \ge 0$ be finite in $\pi_T \times \pi_T$. In inequality (2.1) we set $k = v(\tau, y)$ and $f = g(t, x; \tau, y)$ for a fixed point (τ, y) (we note that the function $v(\tau, y)$ is defined almost everywhere in π_T), and we then integrate over π_T (in the variables (τ, y)):

$$\begin{aligned}\iint\iint_{\pi_T\times\pi_T} \{ |u(t, x) - v(\tau, y)|\, g_t + \operatorname{sign}(u(t, x) - v(\tau, y))[\varphi_i(t, x, u(t, x)) \\ - \varphi_i(t, x, v(\tau, y))]\, g_{x_i} - \operatorname{sign}(u(t, x) - v(\tau, y))[\varphi_{ix_i}(t, x, v(\tau, y)) \\ + \psi(t, x, u(t, x))]\, g\}\, dx\, dt\, dy\, d\tau \geqslant 0.\end{aligned} \tag{3.2}$$

In exactly the same way, starting from integral inequality (2.1) for the function $v(\tau, y)$ written in the variables (τ, y), for $k = u(t, x)$ and $f = g(t, x; \tau, y)$ we integrate over π_T (in the variables (t, x)) to obtain the inequality

$$\begin{aligned}\iint\iint_{\pi_T\times\pi_T} \{ |v(\tau, y) - u(t, x)|\, g_\tau + \operatorname{sign}(v(\tau, y) - u(t, x))[\varphi_i(\tau, y, v(\tau, y)) \\ - \varphi_i(\tau, y, u(t, x))]\, g_{y_i} - \operatorname{sign}(v(\tau, y) - u(t, x))[\varphi_{iy_i}(\tau, y, u(t, x)) \\ + \psi(\tau, y, v(\tau, y))]\, g\}\, dy\, d\tau\, dx\, dt \geqslant 0.\end{aligned} \tag{3.3}$$

Combining (3.2) and (3.3) and making some elementary identity transformations in the integrand (which consist of adding and subtracting identical functions and arranging terms), we find that for any smooth function $g(t, x; \tau, y) \geq 0$ which is finite in $\pi_T \times \pi_T$ the following inequality is fulfilled:

$$\begin{aligned}\iint\iint_{\pi_T\times\pi_T}\{ & |u(t, x)-v(\tau, y)|(g_t+g_\tau)\\ & + \operatorname{sign}(u(t, x)-v(\tau, y))[\varphi_i(t, x, u(t, x))-\varphi_i(\tau, y, v(\tau, y))](g_{x_i}+g_{y_i})\\ & + \operatorname{sign}(u(t, x)-v(\tau, y))([\varphi_i(\tau, y, v(\tau, y))-\varphi_i(t, x, v(\tau, y))]g_{x_i}\\ & - \varphi_{ix_i}(t, x, v(\tau, y))g+[\varphi_i(\tau, y, u(t, x))\\ & - \varphi_i(t, x, u(t, x))]g_{y_i}+\varphi_{iy_i}(\tau, y, u(t, x))g)\\ & + \operatorname{sign}(u(t, x)-v(\tau, y))[\psi(\tau, y, v(\tau, y))-\psi(t, x, u(t, x))]g\}\,dx\,dt\,dy\,d\tau\\ & \equiv \iint\iint_{\pi_T\times\pi_T}\{I_1+I_2+I_3+I_4\}\,dx\,dt\,dy\,d\tau \geqslant 0.\end{aligned} \tag{3.4}$$

We first go through the later part of the proof for the case of equation (1.5) (then $I_3 \equiv 0$, $I_4 = 0$), so that, when we consider the general case, our attention can be focused on the additional difficulties of a technical character which result when the functions ϕ_i depend on t and x. In the case of equation (1.5) inequality (3.4) takes the form

$$\begin{aligned}\iint\iint_{\pi_T\times\pi_T}\{ & |u(t, x)-v(\tau, y)|(g_t+g_\tau)\\ & +\operatorname{sign}(u(t, x)-v(\tau, y))[\varphi_i(u(t, x))-\varphi_i(v(\tau, y))](g_{x_i}+g_{y_i})\}\,dx\,dt\,dy\,d\tau \geqslant 0.\end{aligned} \tag{3.5}$$

Let $f(t, x)$ be an arbitrary test function from Definition 1; we may assume that $f(t, x) \equiv 0$ outside some cylinder

$$\{(t, x)\} = [\rho, T-2\rho]\times K_{r-2\rho}, \quad 2\rho \leqslant \min(T, r).$$

In (3.5) we set

$$g = f\left(\frac{t+\tau}{2}, \frac{x+y}{2}\right)\delta_h\left(\frac{t-\tau}{2}\right)\prod_{i=1}^{n}\delta_h\left(\frac{x_i-y_i}{2}\right) \equiv f(\ldots)\,\lambda_h(\,:\,),\ h \leqslant \rho, \tag{3.6}$$

where

$$(\ldots) \equiv \left(\frac{t+\tau}{2}, \frac{x+y}{2}\right), \quad (\,:\,) \equiv \left(\frac{t-\tau}{2}, \frac{x-y}{2}\right)$$

and the function $\delta_h(\sigma)$ was defined in (2.4); noting that

$$g_t+g_\tau = f_t(\ldots)\,\lambda_h, \quad g_{x_i}+g_{y_i} = f_{x_i}(\ldots)\,\lambda_h,$$

we let h approach zero. We show that as $h \to 0$, (3.5) implies the inequality

$$\begin{aligned}\iint_{\pi_T}\{ & |u(t, x)-v(t, x)|f_t(t, x)\\ & +\operatorname{sign}(u(t, x)-v(t, x))[\varphi_i(u(t, x)-\varphi_i(v(t, x))]f_{x_i}(t, x)\}\,dx\,dt \geqslant 0.\end{aligned} \tag{3.7}$$

In fact, for this choice of g each of the two terms in the integrand of (3.5) can be represented in the form

$$P_h(t, x;\ \tau, y) \equiv F(t, x, \tau, y, u(t, x), v(\tau, y))\,\lambda_h(\,:\,), \tag{3.8}$$

where the function F satisfies a Lipschitz condition in all its variables (here we use Lemma 3), $P_h \equiv 0$ outside the region

$$\{(t, x;\ \tau, y)\} = \left\{\rho \leqslant \frac{t+\tau}{2} \leqslant T - 2\rho, \frac{|t-\tau|}{2} \leqslant h, \frac{|x+y|}{2} \leqslant r - 2\rho, \frac{|x_i - y_i|}{2} \leqslant h\right\}$$

and

$$\iint_{\pi_T \times \pi_T} \iint P_h \, dx\, dt\, dy\, d\tau = \iint_{\pi_T \times \pi_T} \iint [F(t, x, \tau, y, u(t, x), v(\tau, y))$$

$$- F(t, x, t, x, u(t, x), v(t, x))] \lambda_h(\,\vdots\,)\, dx\, dt\, dy\, d\tau$$

$$+ \iint_{\pi_T \times \pi_T} \iint F(t, x, t, x, u(t, x), v(t, x)) \lambda_h(\,\vdots\,)\, dx\, dt\, dy\, d\tau \equiv J_1(h) + J_2.$$

Taking into account the obvious estimate $|\lambda_h(\,\vdots\,)| \leq \text{const} \cdot h^{-(n+1)}$ and the above properties of the function F, we find that

$$|J_1(h)| \leqslant C\left[h + \frac{1}{h^{n+1}} \int\int\int\int_{\substack{\left|\frac{t-\tau}{2}\right| \leqslant h,\ \rho \leqslant \frac{t+\tau}{2} \leqslant T-\rho \\ \left|\frac{x_i - y_i}{2}\right| \leqslant h,\ \left|\frac{x+y}{2}\right| \leqslant r-\rho}} |v(t, x) - v(\tau, y)|\, dx\, dt\, dy\, d\tau,\right.$$

where the constant C does not depend on h. By Lemma 2, $J_1(h) \to 0$ as $h \to 0$. The integral J_2 does not depend on h; in fact, after substituting $t = \alpha$, $(t-\tau)/2 = \beta$, $x = \eta$, $(x-y)/2 = \xi$ and taking into account the obvious equation

$$\int_{-h}^{h} \int_{E_n} \lambda_h(\beta, \xi)\, d\xi\, d\beta = 1$$

we find that

$$J_2 = 2^{n+1} \iint_{\pi_T} \left\{ F(\alpha, \eta, \alpha, \eta, u(\alpha, \eta), v(\alpha, \eta)) \int_{-h}^{h} \int_{E_n} \lambda_h(\beta, \xi)\, d\xi\, d\beta \right\} d\eta\, d\alpha$$

$$= 2^{n+1} \iint_{\pi_T} F(t, x, t, x, u(t, x), v(t, x))\, dx\, dt.$$

Hence

$$\lim_{h \to 0} \iint_{\pi_T \times \pi_T} \iint P_h\, dx\, dt\, dy\, d\tau = 2^{n+1} \iint_{\pi_T} F(t, x, t, x, u(t, x), v(t, x))\, dx\, dt.$$

Thus (3.5) implies (3.7).

Let K be a characteristic cone, and let $\mathcal{E}_u$ and $\mathcal{E}_v$ be the sets of measure zero on $[0, T]$ in the definition of a generalized solution (see requirement 2) for the functions u and v, respectively. We let $\mathcal{E}_\mu$ designate the set of points on $[0, T]$ which are not Lebesgue points of the bounded measurable function

$$\mu(t) = \int_{S_t} |u(t, x) - v(t, x)|\, dx. \tag{3.9}$$

Let $\mathcal{E}_0 = \mathcal{E}_u \cup \mathcal{E}_v \cup \mathcal{E}_\mu$; it is clear that $\operatorname{mes} \mathcal{E}_0 = 0$. We define

$$\alpha_h(\sigma) = \int_{-\infty}^{\sigma} \delta_h(\sigma)\, d\sigma \qquad (\alpha_h'(\sigma) = \delta_h(\sigma) \geqslant 0)$$

and take two numbers ρ and $\tau \in (0, T_0) \setminus \mathcal{E}_0$, $\rho < \tau$. In (3.7) we set

$$f = [\alpha_h(t-\rho) - \alpha_h(t-\tau)]\, \chi(t, x), \quad h < \min(\rho, T_0 - \tau),$$

where[1]

$$\chi = \chi_\varepsilon(t, x) \equiv 1 - \alpha_\varepsilon(|x| + Nt - R + \varepsilon), \quad \varepsilon > 0,$$

and we note that $\chi(t, x) \equiv 0$ outside the cone K, while for $(t, x) \in K$ we have the relations

$$0 \equiv \chi_t + N|\chi_x| \geqslant \chi_t + \frac{\varphi_i(u) - \varphi_i(v)}{u - v} \chi_{x_i}.$$

From (3.7) we obtain the inequality

$$\iint_{\pi_{T_0}} [\delta_h(t-\rho) - \delta_h(t-\tau)]\, \chi_\varepsilon(t, x) |u(t, x) - v(t, x)|\, dx\, dt \geqslant 0. \tag{3.10}$$

Letting ϵ approach zero in (3.10), we find that

$$\int_0^{T_0} \left\{ [\delta_h(t-\rho) - \delta_h(t-\tau)] \int_{S_t} |u(t, x) - v(t, x)|\, dx \right\} dt \geqslant 0.$$

Since ρ and τ are Lebesgue points of the function $\mu(t)$ (see (3.9)), it follows that as $h \to 0$

$$\mu(\tau) = \int_{S_\tau} |u(\tau, x) - v(\tau, x)|\, dx \leqslant \int_{S_\rho} |u(\rho, x) - v(\rho, x)|\, dx = \mu(\rho) \tag{3.11}$$

(for example, by properties (2.5) of the functions $\delta_h(\sigma)$ for $h \le \min(\rho, T_0 - \rho)$ we have for the point $t = \rho$:

$$\left| \int_0^{T_0} \delta_h(t-\rho)\, \mu(t)\, dt - \mu(\rho) \right| = \left| \int_0^{T_0} \delta_h(t-\rho)\, [\mu(t) - \mu(\rho)]\, dt \right|$$
$$\leqslant \mathrm{const} \cdot h^{-1} \int_{\rho-h}^{\rho+h} |\mu(t) - \mu(\rho)|\, dt,$$

where the constant does not depend on h). Taking into account that

$$|u(\rho, x) - v(\rho, x)| \leqslant |u(\rho, x) - u_0(x)| + |v(\rho, x) - u_0(x)| + |u_0(x) - v_0(x)|_c$$

and letting ρ approach zero over a sequence of points in $\mathcal{E}_0$, we obtain estimate (3.1) from (3.11) in the case under consideration.

We now proceed to the general case, where we shall follow the same scheme of proof. We show that, after substituting the function g defined in (3.6) into (3.4), we have (3.4) in the limit as $h \to 0$ implying the following inequality, which is analogous to inequality (3.7):

1) It is easily seen that the function f defined in this way is a permissible test function.

$$\iint_{\pi_T} \{ |u(t, x) - v(t, x)| f_t$$
$$+ \operatorname{sign}(u(t, x) - v(t, x)) [\varphi_i(t, x, u(t, x)) - \varphi_i(t, x, v(t, x)) f_{x_i}$$
$$- \operatorname{sign}(u(t, x) - v(t, x)) [\psi(t, x, u(t, x)) - \psi(t, x, v(t, x))] f \} \, dx \, dt \geqslant 0. \tag{3.12}$$

We first note that as $h \to 0$ the integrals

$$\iint_{\pi_T \times \pi_T} \iint [I_1 + I_2 + I_4] \, dx \, dt \, dy \, d\tau$$

approach the integral in the left side of inequality (3.12) multiplied by 2^{n+2}, since I_1, I_2 and I_4 have the form (3.8), and the corresponding functions P_h and F have all the properties needed above to establish the limit as $h \to 0$ of the integrals of expressions of the form (3.8). Thus it suffices to prove that the integrals of I_3 in (3.4) approach zero as $h \to 0$; moreover, since the coefficients of g_{x_i} and g_{y_i} in I_3 vanish for $|t - \tau| + |x - y| = 0$, it follows by the concrete form of the function

$$I_h = \iint_{\pi_T \times \pi_T} \iint f(\ldots) \operatorname{sign}(u(t, x) - v(\tau, y)) \{ [\varphi_i(\tau, y, v(\tau, y))$$
$$- \varphi_i(t, x, v(\tau, y))] (\lambda_h)_{x_i} - \varphi_{ix_i}(t, x, v(\tau, y)) \lambda_h$$
$$+ [\varphi_i(\tau, y, u(t, x)) - \varphi_i(t, x, u(t, x))] (\lambda_h)_{y_i} + \varphi_{iy_i}(\tau, y, u(t, x)) \lambda_h \} \, dx \, dt \, dy \, d\tau. \tag{3.13}$$

Since the first derivatives of the functions $\phi_i(t, x, u)$ are uniformly continuous on any compact region,[1] we have the following relations (the index h of the function λ will be omitted in the computations; here δ_{ij} is the Kronecker symbol):

$$[\varphi_i(\tau, y, v(\tau, y)) - \varphi_i(t, x, v(\tau, y))] \lambda_{x_i} - \varphi_{ix_i}(t, x, v(\tau, y)) \lambda$$
$$= \varphi_{i\tau}(\tau, y, v(\tau, y)) (\tau - t) \lambda_{x_i} + \varphi_{iy_j}(\tau, y, v(\tau, y)) [(y_j - x_j) \lambda_{x_i} - \delta_{ij} \lambda]$$
$$+ \varepsilon_i \lambda_{x_i} + \varepsilon_0 \lambda \equiv \varphi_{i\tau}(\tau, y, v(\tau, y)) ((\tau - t) \lambda)_{x_i}$$
$$+ \varphi_{iy_j}(\tau, y, v(\tau, y)) ((y_j - x_j) \lambda)_{x_i} + \varepsilon_i \lambda_{x_i} + \varepsilon_0 \lambda;$$

similarly, taking into account the identity $\lambda_{y_i} \equiv - \lambda_{x_i}$, we obtain that

$$[\varphi_i(\tau, y, u(t, x)) - \varphi_i(t, x, u(t, x))] \lambda_{y_i}$$
$$+ \varphi_{iy_i}(\tau, y, u(t, x)) \lambda = \varphi_{i\tau}(\tau, y, u(t, x)) (\tau - t) \lambda_{y_i}$$
$$+ \varphi_{iy_j}(\tau, y, u(t, x)) [(y_j - x_j) \lambda_{y_i} + \delta_{ij} \lambda] + \beta_i \lambda_{y_i}$$
$$\equiv \varphi_{i\tau}(\tau, y, u(t, x)) ((t - \tau) \lambda)_{x_i} - \varphi_{iy_j}(\tau, y, u(t, x)) ((y_j - x_j) \lambda)_{x_i} + \beta_i \lambda_{y_i},$$

where

$$|\varepsilon_0| + \sum_{i=1}^{n} (|\varepsilon_i| + |\beta_i|) \leqslant d\varepsilon(d), \quad d = |t - \tau| + |x - y|,$$

and $\epsilon(d) \to 0$ as $d \to 0$. Since $\lambda = \lambda_h \equiv 0$ for $|t - \tau| \geq 2h$ or $|x_i - y_i| \geq 2h$, and

$$|\lambda_{x_i}| + |\lambda_{y_i}| \leqslant \text{const} \cdot h^{-(n+2)}, \quad |f(\ldots) - f(\tau, y)| \leqslant \text{const} \cdot (|t - \tau| + |x - y|),$$

1) We easily see that the integrand in I_h equals zero outside the region $\{0 \leq t \leq T,\ 0 \leq \tau \leq T,\ |x| \leq nr,\ |y| \leq nr\}$.

it follows that

$$I_h = \iint\limits_{\pi_T \times \pi_T}\iint f(\tau, y)\,\mathrm{sign}\,(u(t, x) - v(\tau, y))\,\{[\varphi_{i\tau}(\tau, y, v(\tau, y))$$
$$- \varphi_{i\tau}(\tau, y, u(t, x))]((\tau - t)\lambda)_{x_i} + [\varphi_{iy_j}(\tau, y, v(\tau, y))$$
$$- \varphi_{iy_j}(\tau, y, u(t, x))]((y_j - x_j)\lambda)_{x_i}\}\,dx\,dt\,dy\,d\tau + \beta(h), \tag{3.14}$$

where $\beta(h) \to 0$ as $h \to 0$. We designate the integrand in (3.14) by B_h; obviously B_h has a representation in the form

$$B_h = F_i(\tau, y, u(t, x), v(\tau, y))((t - \tau)\lambda_h f(\tau, y))_{x_i}$$
$$+ G_{ij}(\tau, y, u(t, x), v(\tau, y))((y_j - x_j)\lambda_h f(\tau, y))_{x_i},$$

where, by Lemma 3, the functions F_i and G_{ij} satisfy a Lipschitz condition in u (here we take into account the assumptions in the beginning of this section concerning ϕ_{it} and ϕ_{ix_j}). Since the function $\lambda_h f(\tau, y)$ is finite in $\pi_T \times \pi_T$, we have

$$\iint\limits_{\pi_T \times \pi_T}\iint \{F_i\ (\tau, y, u(\tau, y), v(\tau, y))((\tau - t)\lambda_h f(\tau, y))_{x_i}$$
$$+ G_{ij}(\tau, y, u(\tau, y), v(\tau, y))((y_j - x_j)\lambda_h f(\tau, y))_{x_i}\}\,dx\,dt\,dy\,d\tau = 0$$

and consequently (after subtracting the last equation in (3.14))

$$|I_h - \beta(h)| = \Big|\iint\limits_{\pi_T \times \pi_T}\iint B_h\,dx\,dt\,dy\,d\tau\Big|$$
$$\leqslant \mathrm{const}\cdot\iint\limits_{\pi_T \times \pi_T}\iint f(\tau, y)[\lambda_h + (|t - \tau| + |x - y|)|(\lambda_h)_x|]\cdot$$
$$\cdot\,|u(t, x) - u(\tau, y)|\,dx\,dt\,dy\,d\tau$$
$$\leqslant \frac{\mathrm{const}}{h^{n+1}}\iiiint\limits_{\substack{\left|\frac{t-\tau}{2}\right| \leqslant h,\ \left|\frac{t+\tau}{2}\right| \leqslant T-\rho, \\ \left|\frac{x_i - y_i}{2}\right| \leqslant h,\ \left|\frac{x+y}{2}\right| \leqslant r-\rho}} |u(t, x) - u(\tau, y)|\,dx\,dt\,dy\,d\tau,$$

which, by Lemma 2, implies that $I_h - \beta(h) \to 0$ as $h \to 0$ (and hence also $I_h \to 0$). Inequality (3.12) is thereby proved.

Further, choosing numbers ρ and $r \in \mathcal{E}_0$, $0 < \rho < r < T_0$, and substituting the same function f in (3.12) as in the proof for the case of equation (1.5), we obtain the following analogs of inequalities (3.10) and (3.11):

$$\iint\limits_{\pi_{T_0}} \{[\delta_h(t - \rho) - \delta_h(t - \tau)]\,\chi_\varepsilon(t, x)\,|u(t, x) - v(t, x)|$$
$$+ \gamma\chi_\varepsilon(t, x)\,|u(t, x) - v(t, x)|\}\,dx\,dt \geqslant 0$$

and

$$\mu(\tau) = \int\limits_{S_\tau} |u(\tau, x) - v(\tau, x)|\,dx \leqslant$$

$$\leqslant \int\limits_{S_\rho} |u(\rho, x) - v(\rho, x)|\, dx + \gamma \int\limits_{\rho}^{\tau} \int\limits_{S_t} |u(t, x) - v(t, x)|\, dx\, dt.$$

Letting ρ approach zero over the set $\mathcal{E}_0$, we find that for $\tau \in \mathcal{E}_0$

$$\mu(\tau) \leqslant \mu(0) + \gamma \int\limits_0^{\tau} \mu(t)\, dt,$$

from which estimate (3.1) follows in an obvious way. Theorem 1 is proved.

To prove the uniqueness theorem for the generalized solution of problem (1.1), (1.2), it is necessary to make certain assumptions concerning the growth of the functions $\phi_{iu}(t, x, u)$ as $|x| \to \infty$. Here we give one of the simplest conditions. Let K be the characteristic cone with base radius R for $|u| \leq M$ (see the end of §1), and let $N = N_M(R)$ be the number defined in (2.12). We shall assume that

$$R^{-1} N_M(R) \to 0 \quad \text{as} \quad R \to \infty \tag{3.15}$$

(for any $M > 0$). It is clear that, when this condition is fulfilled for any point $(t, x) \in \pi_T$, we can find a characteristic cone containing the point (for any $M > 0$), and so Theorem 1 implies

Theorem 2. *The generalized solution of problem* (1.1), (1.2) *in the band* π_T *is unique.*

We have the following proposition concerning monotonic dependence of the generalized solutions of problem (1.1), (1.2) on the initial data.

Theorem 3. *Let the functions* $u(t, x)$ *and* $v(t, x)$ *be the generalized solutions of problem* (1.1), (1.2) *with initial functions* $u_0(x)$ *and* $v_0(x)$, *respectively. Let* $u_0(x) \leq v_0(x)$ *almost everywhere in* E_n. *Then* $u(t, x) \leq v(t, x)$ *almost everywhere in* π_T.

It obviously suffices to show that the following analog of estimate (3.1) holds for the solutions $u(t, x)$ and $v(t, x)$:

$$\int\limits_{S_t} \Phi(u(t, x) - v(t, x))\, dx \leqslant e^{\gamma t} \int\limits_{S_0} \Phi(u_0(x) - v_0(x))\, dx, \tag{3.16}$$

where $\Phi(\sigma) \equiv \sigma + |\sigma|$.

Taking inequality (3.4) into account, we note that, since each of the functions $u(t, x)$ and $v(t, x)$ satisfies integral identity (1.6), the following identity for the functions $g(t, x; \tau, y)$ follow from inequality (3.4):

$$\begin{aligned} \iint\limits_{\pi_T \times \pi_T}\iint &\{[u(t, x) - v(\tau, y)](g_t + g_\tau) \\ &+ [\varphi_i(t, x, u(t, x)) - \varphi_i(\tau, y, v(\tau, y))](g_{x_i} + g_{y_i}) \\ &- [\psi(t, x, u(t, x)) - \psi(\tau, y, v(\tau, y))]g\}\, dx\, dt\, dy\, d\tau = 0. \end{aligned} \tag{3.17}$$

Adding the integrals (3.4) and (3.17), we obtain the inequality

$$\iint\limits_{\pi_T \times \pi_T}\iint \{I_1' + I_2' + I_3' + I_4'\}\, dx\, dt\, dy\, d\tau \geqslant 0, \tag{3.18}$$

where the integrand I_3' coincides with I_3 in (3.4), and the expressions I_1', I_2' and I_4' are obtained from the corresponding expressions I_1, I_2 and I_4 in (3.4) by replacing $|u(t, x) - v(\tau, y)|$

and $\operatorname{sign}(u(t, x) - v(\tau, y))$ by $\Phi(u(t, x) - v(\tau, y))$ and $\Phi'(u(t, x) - v(\tau, y))$, respectively. Further, taking into account that $\sigma\Phi'(\sigma) \equiv \Phi(\sigma)$, we derive inequality (3.16) from (3.18) in exactly the same way as estimate (3.1) was obtained from (3.4) in the proof of Theorem 1.

A proof of Theorem 3 based on Theorem 2 and a method of constructing generalized solutions will be given at the end of §4 for the case of equation (1.5).

§4. Existence of the generalized solution of problem (1.1), (1.2)

The fundamental result on the existence of a generalized solution of problem (1.1), (1.2) will be proved in this section under the following assumptions:

1) The functions $\phi_i(t, x, u)$ are three times continuously differentiable.

2) The functions $\phi_{iu}(t, x, u)$ are uniformly bounded for $(t, x, u) \in D_M = \pi_T \times [-M, M]$ (the numbers $N = N_M(R)$ in (2.12) are bounded by a constant $\overline{N}$ which does not depend on R).

3) The function $\Psi(t, x, u) \equiv \phi_{ix_i}(t, x, u) + \psi(t, x, u)$ is twice continuously differentiable and uniformly bounded in D_M, where

$$\sup_{(t,x)\in\pi_T} |\Psi(t, x, 0)| \leqslant c_0 = \text{const}, \tag{4.1}$$

$$\sup_{\substack{(t,x)\in\pi_T \\ -\infty<u<+\infty}} [-\Psi_u(t, x, u)] \leqslant c_1 = \text{const}. \tag{4.2}$$

4) $u_0(x)$ is an arbitrary bounded measurable function in E_n ($|u_0(x)| \leq M_0$).

The assumptions concerning smoothness of the functions $\phi_i(t, x, u)$ and $\psi(t, x, u)$ in conditions 1) and 3) were made without taking into account the "inequivalence" of the arguments t, x_j and u. Hence, in the context of the methods of this section, conditions 1) and 3) can be refined and weakened (see subsection 4 in §5); for example, in the case of equation (1.5) it is sufficient to require continuity of only the first derivatives of the functions $\phi_i(u)$. Undoubtably, assumptions (4.1) and (4.2) in condition 3), which ensure the a priori estimate of the maximum modulus of the generalized solution of problem (1.1), (1.2), can be replaced by other well-known assumptions of the same type.

To construct the generalized solution of problem (1.1), (1.2), we apply the vanishing viscosity method. We first investigate Cauchy's problem for the parabolic equation (1.3) with initial condition (1.2), where the main object here is to obtain an a priori estimate of the modulus of continuity in L_1 of the solution $u^\epsilon(t, x)$ of problem (1.3), (1.2) which ensures compactness of the family $\{u^\epsilon(t, x)\}$ in the L_1-norm, where this estimate does not depend on small viscosity ϵ. This estimate is established using similar methods separately in the following two cases A and B:

A. The initial function $u_0(x)$ is an arbitrary bounded function in E_n, but then (in addition to conditions 1)–3) in the beginning of the section) the functions ϕ_i do not depend on x, and the functions $\phi_{iut}(t, x, u)$, $\psi_u(\cdots)$, $\psi_{x_j}(\cdots)$ and $\psi_t(\cdots)$ are bounded in D_M.

B. The initial function $u_0(x)$ is bounded in E_n and satisfies a Lipschitz condition in the $L_1(K_R)$-norm for any $R > 0$:

$$\int_{K_R} |u_0(x + \Delta x) - u_0(x)|\, dx \leqslant c\,(R^\mu + 1)\,|\Delta x|, \;\; c = \text{const} \geqslant 1, \;\; \mu = \text{const} > 0; \tag{4.3}$$

the functions ϕ_i can now depend on x, while (in addition to conditions 1)–3)) the derivatives

$\phi_{ix_ix_j}(t, x, u)$, $\phi_{iux_j}(\cdots)$, $\phi_{itx_j}(\cdots)$, $\phi_{iut}(\cdots)$, and also $\psi_u(\cdots)$, $\psi_{x_j}(\cdots)$, $\psi_t(\cdots)$, are bounded in D_M.

Case A is singled out largely for considerations of method, since in this technically simple but nevertheless typical case (which essentially corresponds to equation (1.5)) we can emphasize the fundamental ideas of the proof with special clarity.

The estimate of the modulus of continuity in case B, which is also of independent interest, plays the role of a preliminary result for obtaining the desired estimate in the *general case* O. We let the *general case* O be characterized by the following conditions: $u_0(x)$ is an arbitrary bounded measurable function, while the functions ϕ_i and ψ satisfy the same assumptions as in case B. The fundamental result used to justify the vanishing viscosity method will be formulated under conditions O (concerning the possibility of weakening these conditions, see subsection 4 in §5).

1. **Cauchy's problem for the parabolic equation (1.3).** We first note that, by well-known results from the theory of second order quasilinear parabolic equations (see, for example, [19] or [20]), under our assumptions about the functions ϕ_i and ψ problem (1.3), (1.2) has a unique classical solution $u^\epsilon(t, x)$ if the initial function $u_0(x)$ is bounded in E_n along with its derivatives through the third order, inclusive; here the solution $u^\epsilon(t, x)$ is bounded in π_T and has bounded and uniformly Hölder continuous derivatives in equation (1.3).

We first prove several a priori estimates for the classical solution of problem (1.3), (1.2), but we shall take care that these estimates depend only on the above properties of the functions ϕ_i and ψ, on M_0, and on the function $\omega_R(\sigma)$ such that (see (2.8))

$$J_R(u_0(x), \Delta x) \leqslant \omega_R(|\Delta x|) \ \forall R > 0 \tag{4.4}$$

(for $\omega_R(\sigma)$ we can take the modulus of continuity of the function $u_0(x)$ in $L_1(K_R)$; in case B by (4.3) we have $\omega_R(\sigma) \equiv c(R^\mu + 1)\sigma$). We agree to let const designate different constants which depend on the "data" of problem (1.3), (1.2), but not on $\epsilon \in (0, 1]$.

Equation (1.3) can be written in the form

$$u_t + \varphi_{iu} u_{x_i} + \Psi(t, x, u) = \varepsilon \Delta u. \tag{4.5}$$

Since $\Psi(t, x, u) = \Psi(t, x, 0) + \Psi_u(t, x, \tilde{u})u$, we have by (4.1), (4.2) and the maximum principle that

$$|u^\varepsilon(t, x)| \leqslant \mathrm{const} = (M_0 + c_0 T)e^{c_1 T} = M. \tag{4.6}$$

We now prove an estimate of the modulus of continuity in L_1 for the solution $u^\epsilon(t, x)$ in case A. We take a vector $z \in E_n$ and set $w(t, x) \equiv u^\epsilon(t, x+z) - u^\epsilon(t, x)$; it is clear that the function $w(t, x)$ satisfies the equation

$$w_t + (a_i w)_{x_i} + cw + e_i z_i = \varepsilon \Delta w, \tag{4.7}$$

where

$$a_i(t, x) = \int_0^1 \varphi_{iu}(t, \alpha u^\varepsilon(t, x+z) + (1-\alpha)u^\varepsilon(t, x))\, d\alpha,$$

$$c(t, x) = \int_0^1 \psi_u(t, \alpha(x+z) + (1-\alpha)x, \alpha u^\varepsilon(t, x+z) + (1-\alpha)u^\varepsilon(t, x))\, d\alpha \equiv \int_0^1 \psi_u(\ldots)\, d\alpha,$$

$$e_i(t, x) = \int_0^1 \psi_{x_i}(\ldots)\, d\alpha, \quad \sum_{i=1}^{n} (|a_i| + |e_i|) + |c| \leqslant \text{const},$$

and all the functions a_i, c and e_i satisfy a Lipschitz condition on any compact set in π_T. We multiply equation (4.7) by a function $g(t, x)$ which is finite in x in the band $\pi_\tau \subset \pi_T$ and has continuous derivatives g_t, g_{x_i}, $g_{x_i x_j}$, and we integrate over π_τ; integrating by parts, we find that

$$\int_{E_n} wg|_{t=\tau}\, dx - \iint_{\pi_\tau} \mathscr{L}(g)\, w\, dx\, dt = \int_{E_n} wg|_{t=0}\, dx - \iint_{\pi_\tau} z_i e_i g\, dx\, dt, \tag{4.8}$$

where

$$\mathscr{L}(g) = g_t + a_i g_{x_i} - cg + \varepsilon \Delta g. \tag{4.9}$$

Lemma 4. *Let the function $q(t, x)$ be continuous in π_τ and satisfy the inequality $\mathscr{L}(q) \geq 0$; let $|q(t, x)| \leq q^0$ and $q(\tau, x) \equiv 0$ for $|x| \geq r$ (q^0 and r are constants). Then for*

$$(t, x) \in \Omega = \{(t, x) : |x| \geqslant r + H(\tau - t),\ 0 \leqslant t \leqslant \tau\}, \quad \textit{where} \quad H = 1 + \sup_{(t,x)\in\pi_\tau} \left(\sum_{i=1}^{n} a_i^2\right)^{1/2},$$

the following estimate is fulfilled:

$$q(t, x) \leqslant q^0 \exp[\varepsilon^{-1}(H(\tau - t) + r - |x|) + \tau \sup_{\pi_\tau} |c| + (t - \tau) \inf_{\pi_\tau} c] \equiv Q_\varepsilon(t, x).$$

Proof. It is easily verified that $\mathscr{L}(Q_\epsilon) \leq 0$ in Ω, and that

$$Q_\varepsilon|_{|x|=r+H(\tau-t)} \geqslant q^0, \quad Q_\varepsilon\Big|_{\substack{t=\tau \\ |x|\geqslant r}} \geqslant 0.$$

Hence, by the maximum principle, $q(t, x) \leq Q_\epsilon(t, x)$ everywhere in Ω.

We fix a number $r > 1$ and define the function $q_h(t, x)$ as the solution of Cauchy's problem for the equation $\mathscr{L}(q_h) = 0$ in π_τ with the initial condition $q_h(\tau, x) = \beta^h(x)$, where $\beta(x) = \operatorname{sign} w(\tau, x)$ for $|x| \leq r - h$, $\beta(x) \equiv 0$ for $|x| > r - h$. Obviously, by the maximum principle, $|q_h(t, x)| \leq$ const. In (4.8) we set

$$g = q_h(t, x)\, \eta_m(|x|), \quad \eta_m(\sigma) = 1 - \int_{-\infty}^{\sigma} \delta(\sigma - m)\, d\sigma, \tag{4.10}$$

where m is a natural number. Transferring the derivatives in x_i from the function q_h in the integral of $2\epsilon w(q_h)_{x_i}(\eta_m)_{x_i}$, we find that

$$\begin{aligned} \int_{E_n} wq_h \eta_m|_{t=\tau}\, dx = &- \iint_{\pi_\tau} \Big[a_i \frac{x_i}{|x|} \delta(|x| - m)\, w \\ &- 2\varepsilon w_{x_i} \frac{x_i}{|x|} \delta(|x| - m) + \varepsilon w \Delta \eta_m \Big] q_h\, dx\, dt \\ &- \iint_{\pi_\tau} z_i e_i q_h \eta_m\, dx\, dt + \int_{E_n} wq_h \eta_m|_{t=0}\, dx. \end{aligned} \tag{4.11}$$

We note that, by Lemma 4, for $\epsilon \in [0, 1]$

$$|q_h(t, x)| \leqslant \text{const} \cdot \exp\left(-\frac{|x|}{2}\right)$$

and for $R \geq \bar{r} = r + (1 + \bar{N}) T > 1$

$$\int_{E_n \setminus K_R} |q_h(0, x)| \, dx \leqslant \text{const} \cdot R^{n-1} \cdot \exp[\varepsilon^{-1}(\bar{r} - R)]$$

(here Lemma 4 is applied to the functions $\pm q_h(t, x)$). First letting m approach $+\infty$ in (4.11), and then letting h approach zero, we find that

$$\int_{K_r} |w(\tau, x)| \, dx \leqslant \text{const} \cdot \{|z| + \omega_R(|z|)$$

$$+ R^{n-1} \exp[\varepsilon^{-1}(\bar{r} - R)]\} = \lambda_R^\varepsilon(|z|).$$

Consequently for $0 \leq t \leq T$

$$J_r(u^\varepsilon, \Delta x) \leqslant \min_{R \geqslant \bar{r}} \lambda_R^\varepsilon(|\Delta x|) = \omega_r^x(|\Delta x|), \tag{4.12}$$

where the function $\omega_r^x(\sigma)$ does not depend on ϵ.

To estimate the modulus of continuity in t, we use the following interpolation theorem.

Lemma 5. *Let the function $u(t, x)$ be measurable in the cylinder $\{(t, x)\} = [0, T] \times K_{r+\rho}$ $(0 < 2\rho \leq r)$ and $|u(t, x)| \leq M = \text{const}$; for $0 \leq t \leq T$, $|\Delta x| \leq \rho$ let*

$$J_r(u(t, x), \Delta x) \leqslant \omega_r^x(|\Delta x|)$$

and for any t, $t + \Delta t \in [0, T]$, $\Delta t > 0$, and any twice smooth function $g(x)$ which is finite in K_r let

$$\left| \int_{K_r} g(x) [u(t + \Delta t, x) - u(t, x)] \, dx \right|$$

$$\leqslant c_r \Delta t \max_{x \in K_r} \left[|g| + |g_x| + \sum_{i,j=1}^n |g_{x_i x_j}| \right]. \tag{4.13}$$

Then for $0 \leq t \leq t + \Delta t \leq T$

$$I_r(u(t, x), \Delta t) \equiv \int_{K_r} |u(t + \Delta t, x) - u(t, x)| \, dx$$

$$\leqslant \text{const} \cdot \min_{0 < h \leqslant \rho} \left[h + \omega_r^x(h) + \frac{\Delta t}{h^2} \right], \tag{4.14}$$

where the constant depends only on c_r, M, r and n.

Proof. In (4.13) we set $g(x) = \beta^h(x)$, where $\beta(x) = \operatorname{sign}(u(t + \Delta t, x) - u(t, x))$ for $|x| \leq r - h$, $\beta(x) \equiv 0$ for $|x| > r - h$ and $h \leq \rho$. Noting that $|g(x)| \leq 1$, $|g_x| \leq \text{const} \cdot h^{-1}$, $|g_{x_i x_j}| \leq \text{const} \cdot h^{-2}$, we obtain the following estimate for the function $w(x) \equiv u(t + \Delta t, x) - u(t, x)$:

$$\left| \int_{K_{r-2h}} w(x) (\operatorname{sign} w)^h \, dx \right| \leqslant \left| \int_{K_r} w(x) \beta^h(x) \, dx \right| + \text{const} \cdot h$$

$$\leqslant \text{const} \cdot [h + (\Delta t) h^{-2}].$$

Applying Lemma 1 to the function $w(x)$ in K_{2r-h} (see (2.10)), we further find that

$$I_r(u(t, x), \Delta t) \leqslant \text{const} \cdot \left[h + \omega_r^x(h) + \frac{\Delta t}{h^2}\right]$$

for any $h \in (0, \rho)$, and this is equivalent to estimate (4.14).

Lemma 5 allows us to estimate the modulus of continuity in L_1 with respect to t for the solution $u^\epsilon(t, x)$ of equation (1.3) in terms of the modulus of continuity $\omega_r^x(\sigma)$ with respect to the space variables. In fact, it easily follows directly from equation (1.3) for $0 < \epsilon < 1$ that estimate (4.13) holds for the function $u^\epsilon(t, x)$ with constant $c_r = \text{const} \cdot r^n$ (we may assume that $r \geq 2$ and $\rho = 1$). Thus

$$I_r(u^\varepsilon(t, x), \Delta t) \leqslant \omega_r^t(\Delta t) = \text{const} \cdot \min_{0<h\leqslant 1}\left[h + \omega_r^x(h) + \frac{\Delta t}{h^2}\right]. \tag{4.15}$$

We now prove the analogs of estimates (4.12) and (4.15) in case B. To do this we note that in the case of a smooth initial function $u_0(x)$ inequality (4.3) implies the estimate

$$\int_{K_R} |u_{0x}|\, dx \leqslant \sqrt{n}\, c\,(R^\mu + 1) \tag{4.16}$$

and that the functions $v^k(t, x) \equiv u^\epsilon_{x_k}(t, x)$ satisfy the parabolic system[1)]

$$\begin{aligned} v_t^k + \frac{d}{dx_i}[\varphi_{iu}(t, x, u^\varepsilon) v^k] + \varphi_{iux_k}(\ldots) v^i + \varphi_{ix_kx_i}(\ldots) \\ + \psi_u(\ldots) v^k + \psi_{x_k}(\ldots) = \varepsilon \Delta v^k, \quad k = 1, \ldots, n. \end{aligned} \tag{4.17}$$

We multiply the kth equation in (4.17) by a sufficiently smooth function $g^k(t, x)$ which is finite in x in the band π_τ, integrate over π_τ, and then sum over k from 1 to n; integrating by parts, we find that

$$\begin{aligned} &\int_{E_n} v^k g^k|_{t=\tau}\, dx - \iint_{\pi_\tau} \mathcal{L}_k(g) v^k\, dx\, dt \\ &= \int_{E_n} v^k g^k|_{t=0}\, dx - \iint_{\pi_\tau} (\varphi_{ix_kx_i} + \psi_{x_k}) g^k\, dx\, dt, \end{aligned} \tag{4.18}$$

where

$$g = (g^1, \ldots, g^n),$$
$$\mathcal{L}_k(g) = g_t^k + a_i g_{x_i}^k - [\varphi_{kux_i} + \delta_{ik}\psi_u] g^i + \varepsilon \Delta g_k,$$
$$a_i = \varphi_{iu}(t, x, u^\varepsilon), \quad k = 1, \ldots, n.$$

We fix a number $r > 0$ and let $q_h^k(t, x)$, $k = 1, \cdots, n$, designate the solution of Cauchy's problem for the parabolic system $\mathcal{L}_k(q_h) = 0$ in π_τ with the initial condition $q_h^k(\tau, x) = (\beta_k(x))^h$, where $\beta_k(x) = \operatorname{sign} v^k(\tau, x)$ for $|x| \leq r - h$, $\beta_k(x) \equiv 0$ for $|x| > r - h$ (see [21]). Since

$$0 = 2\mathcal{L}_k(q_h) q_h^k \leqslant (q_h^2)_t + a_i (q_h^2)_{x_i} + \text{const} \cdot q_h^2 + \varepsilon \Delta q_h^2 \equiv \mathcal{L}(q_h^2), \quad q_h^2 = q_h^k q_h^k,$$

it follows by the maximum principle that $|q_h^k(t, x)| \leq q^0 = \text{const}$, and, by Lemma 4, for $\epsilon \in (0, 1]$

$$|q_h^k(t, x)| \leqslant \text{const} \cdot \exp\left(-\frac{|x|}{2}\right).$$

1) Under our smoothness assumptions for the functions ϕ_i and ψ the possibility of differentiating equation (1.3) with respect to x_k follows from well-known results for linear equations (see, for example, [19], Chapter 3, §5).

Substituting $g^k = q_h^k \eta_m(|x|)$ in (4.18) (see (4.10)) and, as in case A, first letting m approach ∞ and then letting h approach zero, we obtain the estimate

$$\int_{K_r} \sum_{k=1}^{n} |v^k|\, dx = \text{const} \cdot \left(1 + \int_{E_n} e^{-\frac{|x|}{2}} |u_{0x}(x)|\, dx\right).$$

Taking (4.16) into account, we find that

$$\int_{E_n} e^{-\frac{|x|}{2}} |u_{0x}|\, dx = \int_{K_1} + \sum_{m=1}^{\infty} \int_{K_{m+1}\setminus K_m}$$

$$\leqslant 2\sqrt{n}\; c + \sqrt{n}\; c \sum_{m=1}^{\infty} e^{-\frac{m}{2}} [1 + (m+1)^{\mu}] = \text{const}.$$

Consequently in case B we have the estimates

$$J_r(u^{\varepsilon}, \Delta x) \leqslant \text{const} \cdot |\Delta x| = \omega_r^x(|\Delta x|),$$

$$I_r(u^{\varepsilon}, \Delta t) \leqslant \text{const} \cdot |\Delta t|^{1/2} = \omega_r^t(|\Delta t|). \tag{4.19}$$

To derive estimates (4.12) and (4.15) in the general case O we note that the constant c in (4.3) and (4.16) is a factor in const in estimates (4.19). We let $u_h^{\epsilon}(t, x)$ designate the solution of Cauchy's problem for equation (1.3) with the initial condition $u_h^{\epsilon}(0, x) = u_0^h(x)$, $0 < h \leq 1$; since $|(u_0^h)_x| \leq M_0 h^{-1}$, and consequently

$$\int_{K_R} |u_0^h(x + \Delta x) - u_0^h(x)|\, dx \leqslant \text{const} \cdot h^{-1} R^n |\Delta x|,$$

it follows by the above remark that

$$J_r(u_h^{\varepsilon}, \Delta x) \leqslant \frac{\text{const}}{h} |\Delta x|, \quad I_r(u_h^{\varepsilon}, \Delta t) \leqslant \frac{\text{const}}{h} |\Delta t|^{1/2}. \tag{4.20}$$

The function $w = u_h^{\epsilon}(t, x) - u^{\epsilon}(t, x)$ satisfies an equation of the form (4.7), where $e_i \equiv 0$ $(i = 1, \cdots, n)$, and

$$a_i(t, x) = \int_0^1 \varphi_{iu}(t, x, \alpha u_h^{\varepsilon} + (1 - \alpha) u^{\varepsilon})\, d\alpha,$$

$$c(t, x) = \int_0^1 \psi_u(t, x, \alpha u_h^{\varepsilon} + (1 - \alpha) u^{\varepsilon})\, d\alpha.$$

Estimating the norm of the function $w(t, x)$ for $t = r$ in $L_1(K_r)$ in exactly the same way as the norm of the function w satisfying equation (4.7), we obtain that for $0 \leq t \leq T$

$$\int_{K_r} |u_h^{\varepsilon}(t, x) - u^{\varepsilon}(t, x)|\, dx \leqslant \text{const} \cdot \int_{E_n} e^{\frac{-|x|}{2}} |u_0^h(x) - u_0(x)|\, dx.$$

It is well known that for any $R \geq 1$

$$\int_{K_R} |u_0^h(x) - u_0(x)|\,dx \leqslant \omega_R(h),$$

where $\omega_R(\sigma)$ is the function in inequality (4.4) (for example, the modulus of continuity of the function $u_0(x)$ in $L_1(K_R)$). Consequently

$$\int_{K_r} |u_h^\varepsilon(t,x) - u^\varepsilon(t,x)|\,dx \leqslant \text{const} \cdot \left[\omega_R(h) + R^{n-1}\exp\left(-\frac{R}{2}\right)\right] \quad \forall R \geqslant 1. \tag{4.21}$$

From (4.20) and (4.21) we conclude that for $0 \le t \le T$

$$J_r(u^\varepsilon, \Delta x) \leqslant \text{const} \cdot \min_{\substack{0<h\leqslant 1\\ 1\leqslant R<+\infty}} \left[\omega_R(h) + R^{n-1}\exp\left(-\frac{R}{2}\right) + |\Delta x|\,h^{-1}\right] = \omega_r^x(|\Delta x|),$$

$$I_r(u^\varepsilon, \Delta t) \leqslant \text{const} \cdot \min_{\substack{0<h\leqslant 1\\ 1\leqslant R<+\infty}} \left[\omega_R(h) + R^{n-1}\exp\left(-\frac{R}{2}\right) + |\Delta t|^{1/3}\,h^{-1}\right] = \omega_r^t(|\Delta t|).$$

Thus in each of the cases A, B and O we can find functions $\omega_r^x(\sigma)$ and $\omega_r^t(\sigma)$ which do not depend on ϵ such that for $0 \le t \le T$

$$J_r(u^\varepsilon, \Delta x) + I_r(u^\varepsilon, \Delta t) \leqslant \omega_r^x(|\Delta x|) + \omega_r^t(|\Delta t|) \tag{4.22}$$

(however, this estimate was obtained under an additional assumption concerning sufficient smoothness of the function $u_0(x)$).

Let $\Phi(u)$ be an arbitrary twice smooth convex downward function on the line $-\infty < u < +\infty$. We multiply equation (1.3) by the function $\Phi'(u)f(t,x)$, where $f(t,x) \ge 0$ is a twice smooth function which is finite in π_T, and we integrate over π_T. Transferring the derivatives with respect to t and x_i to the test function f and taking into account that $\Phi''(u)u_{x_i}u_{x_i}f \ge 0$, we obtain the inequality

$$\iint_{\pi_T} \Big\{\Phi(u)f_t + \int_k^u \Phi'(u)\varphi_{iu}(t,x,u)\,du\, f_{x_i} - \Phi'(u)\varphi_{ix_i}(\dots)f$$
$$+ \Big[\int_k^u \Phi'(u)\varphi_{iux_i}(\dots)\,du - \Phi'(u)\psi(\dots)\Big]f + \varepsilon\Phi(u)\Delta f\Big\}\,dx\,dt \geqslant 0,$$

where k is a constant. Hence (using an approximation of the function $|u-k|$ by twice smooth convex functions $\Phi(u)$) we conclude that this inequality also holds for $\Phi = |u-k|$:

$$\iint_{\pi_T} \{|u-k|(f_t + \varepsilon\Delta f) + \operatorname{sign}(u-k)[\varphi_i(t,x,u) - \varphi_i(t,x,k)]f_{x_i}$$
$$- \operatorname{sign}(u-k)[\varphi_{ix_i}(t,x,k) + \psi(t,x,u)]f\}\,dx\,dt \geqslant 0. \tag{4.23}$$

To free ourselves from the requirement that the function $u_0(x)$ be sufficiently smooth, we make the following observations, which are based on elementary considerations of approximation and compactness. We approximate the bounded measurable function $u_0(x)$ by the mean functions $u_0^h(x)$ and note that the moduli of continuity in L_1 of the functions $u_0^h(x)$ are estimated in terms of the modulus of continuity of the function $u_0(x)$ (see (2.9)). Hence, for the classical solutions $u_h^\epsilon(t,x)$ of Cauchy's problem for equation (1.3) with initial functions $u_0^h(x)$, estimates (4.6) and (4.22) hold uniformly for $h \in (0, 1]$ and $\epsilon \in (0, 1]$. On the other hand, inner estimates of Schauder type (see [19], Chapter 7, 3 and 4) hold for the solutions $u_h^\epsilon(t, x)$ with fixed $\epsilon > 0$ as a result of our smoothness assumptions for the functions ϕ_i and ψ. Using these estimates, we can find a subsequence $u_{h_m}^\epsilon(t, x)$ which converges uniformly to the function $u^\epsilon(t, x)$ in any cylinder $\{(t, x)\} = [\rho, T] \times K_R$, $\rho > 0$ along with the derivatives in equation (1.3). Obviously for $t > 0$ the twice smooth function $u^\epsilon(t, x)$ satisfies equation (1.3) in the usual sense, estimates (4.6) and (4.22) hold for it, and for any $r > 0$ and $\rho \in [0, T]$

$$\int_{K_r} | u^\varepsilon (\rho, x) - u_0 (x) | \, dx \leqslant \omega_r^t (\rho). \tag{4.24}$$

It is also clear that the function $u^\epsilon(t, x)$ satisfies inequality (4.23). We shall henceforth understand the functions $u^\epsilon(t, x)$ to be the solutions of problem (1.3), (1.2) constructed in this way.

2. **Justification of the vanishing viscosity method. Existence theorem for a generalized solution of problem (1.1), (1.2).**

Theorem 4. *Let the assumptions of the general case* O *be fulfilled. Then the solutions* $u^\epsilon(t, x)$ *of problem* (1.3), (1.2) *converge as* $\epsilon \to 0$ *almost everywhere in* π_T *to a function* $u(t, x)$ *which is a generalized solution of problem* (1.1), (1.2).

Proof. By the estimates in subsection 1 of this section, the family $\{u^\epsilon(t, x)\}$ is compact in the L_1-norm in any cylinder $[0, T] \times K_r$, $r = 1, 2, 3, \cdots$. Using the diagonal process, we can find a subsequence $u^{\epsilon_m}(t, x)$ which converges almost everywhere in π_T to a bounded function $u(t, x)$. Passing to the limit as $\epsilon_m \to 0$ in inequality (4.23), where $u = u^{\epsilon_m}$, we find that the function $u(t, x)$ satisfies requirement 1) of the definition of a generalized solution of problem (1.1), (1.2) (here we take into account that only the first derivatives of the function f appear in the integrand in inequality (2.1) and that the smooth finite function $f(t, x) \geq 0$ can be uniformly approximated along with its first derivatives using twice smooth finite nonnegative functions). We can obviously find a set $\mathcal{E}$ of measure zero on $[0, T]$ such that if $t \in [0, T] \setminus \mathcal{E}$, then the sequence $u^{\epsilon_m}(t, x)$ converges to $u(t, x)$ almost everywhere in E_n. Passing to the limit as $\epsilon = \epsilon_m \to 0$ in inequality (4.24), where $\rho \in [0, T] \setminus \mathcal{E}$, we conclude that the function $u(t, x)$ satisfies requirement 2) of the definition of a generalized solution of problem (1.1), (1.2).

The function $u(t, x)$ is hence a generalized solution of problem (1.1), (1.2). By the uniqueness theorem for the generalized solution of this problem that was proved in §3, the sequence $u^\epsilon(t, x)$ converges to the function $u(t, x)$ as ϵ approaches zero in any way.

Theorem 5. *A generalized solution of problem* (1.1), (1.2) *exists if conditions* 1)–4) *in the beginning of this section are fulfilled.*

Proof. In case O, the existence of a generalized solution was proved in Theorem 4. Using the finiteness property of the domain of dependence of the generalized solution on the initial condition, we discard superfluous assumptions concerning boundedness of certain derivatives of ϕ_i and ψ (see condition B). Along with equation (1.1) we consider the sequence of equations

$$u_t + \frac{d}{dx_i} [\eta_m (|x|) \varphi_i (t, x, u)] - (\eta_m)_{x_i} \varphi_i (t, x, u) + \eta_m \psi (t, x, u) = 0,$$

$$\eta_m (\sigma) = 1 - \int_{-\infty}^{\sigma} \delta (\sigma - m) \, d\sigma, \quad \eta_m = \eta_m (|x|).$$

Since for the mth equation the corresponding function $\Psi_m = \eta_m \Psi(t, x, u)$ and the corresponding functions ϕ_{im} and ψ_m are finite in x, this equation satisfies all the requirements of case O. We let $u_m(t, x)$ designate the generalized solution of Cauchy's problem for the mth equation with initial condition (1.2). Noting that $|(\phi_{im})_u| \le |\phi_{iu}|$, we fix a number $r > 0$. By Theorem 1, all the functions $u_m(t, x)$ will coincide almost everywhere in the cylinder $[0, T] \times K_r$ for $m \ge \overline{r} + 1 = r + \overline{N}T + 1$ (we note that $\eta_m(|x|) \equiv 1$ for $|x| \le m - 1$). Hence the sequence $u_m(t, x)$ converges almost everywhere in π_T to a bounded measurable function $u(t, x)$; since in any cylinder $[0, T] \times K_r$ the function $u(t, x)$ coincides with the solution $u_{m_r}(t, x)$ where $m_r = 2 + [\overline{r}]$, it follows that the function $u(t, x)$ is a generalized solution of problem (1.1), (1.2).

3. Proof of Theorem 3 for the case of equation (1.5). By Theorem 4, any generalized solution of problem (1.5), (1.2) can be obtained as the limit as $\epsilon \to 0$ of solutions $u^\epsilon(t, x)$ of Cauchy's problem for the parabolic equation

$$u_t + (\varphi_i(u))_{x_i} = \varepsilon \, \Delta u \tag{4.25}$$

with initial condition (1.2). Since for any classical solutions $u_1(t, x)$ and $u_2(t, x)$ of equation (4.25), where $u_1(0, x) \ge u_2(0, x)$, the maximum principle implies that the inequality $u_1(t, x) \ge u_2(t, x)$ holds everywhere in π_T, it follows from the construction of the functions $u^\epsilon(t, x)$ and $v^\epsilon(t, x)$ which approximate the functions $u(t, x)$ and $v(t, x)$ considered in Theorem 3 that $u^\epsilon(t, x) \ge v^\epsilon(t, x)$ in π_T for any $\epsilon \in (0, 1]$. Consequently $u(t, x) \ge v(t, x)$ almost everywhere in π_T.

§5. Remarks and additions

1°. All the results of this paper can easily be carried over to the case of the following equation, which is more general than (1.1):

$$\frac{d}{dt} \varphi_0(t, x, u) + \frac{d}{dx_i} \varphi_i(t, x, u) + \psi(t, x, u) = 0, \tag{5.1}$$

$$\varphi_{0u}(t, x, u) \neq 0.$$

In particular, the corresponding results concerning stability and uniqueness of the generalized solution of problem (5.1), (1.2) are valid under the same conditions on the functions $\phi_i(t, x, u)$, $i = 0, 1, \cdots, n$ and $\psi(t, x, u)$ as in the beginning of §3. However, in the case $\phi_0(t, x, u) \equiv u$, considered in §3, we can use a slight modification of the proof of Theorem 1 to weaken the assumptions concerning smoothness of these functions in t.

2°. The requirement that the generalized solution of problem (1.1), (1.2) be bounded in π_T can be replaced by a boundedness condition on any compact set; a uniqueness theorem holds for such a solution, for example in the class of functions $u(t, x)$ such that as $R \to \infty$

$$\sup_{\substack{(t, x) \in [0, T] \times K_R \\ |v| \leqslant \sup\limits_{(t, x) \in [0, T] \times K_R} |u(t, x)|}} \Big(\sum_{i=1}^{n} \varphi_{iv}^2(t, x, v) \Big)^{1/2} = o(R).$$

3°. From Theorem 1 we can obviously derive a proposition on compactness of the family of generalized solutions of problem (1.1), (1.2) in the L_1-norm, assuming that the corresponding initial functions are uniformly bounded in C and are equicontinuous in L_1 on any compact set.

4°. The smoothness requirements on the functions $\phi_i(t, x, u)$ and $\psi(t, x, u)$ under which the existence of a generalized solution was proved (see the beginning of §4) are certainly excessive even with the methods of §4. But it is not hard to discard the superfluous requirements. In fact, it follows from the proof of Theorem 4 that to construct a generalized solution of problem (1.1), (1.2) using the vanishing viscosity method it suffices to prove the existence of a solution $u^\epsilon(t, x)$ of problem (1.3), (1.2) which is continuous for $t > 0$ and satisfies inequality (4.23) as well as estimates (4.6) and (4.22) which do not depend on $\epsilon \in (0, 1]$. Estimate (4.6) is ensured by assumptions (4.1) and (4.2); a quick analysis of the derivation of estimate (4.22) in case O shows that this estimate depends (if we are interested in the smoothness assumption for the functions ϕ_i and ψ) only on the least upper bound of the moduli of the derivatives $\phi_{iu}, \phi_{iux_j}, \phi_{ix_ix_j}, \psi_u, \psi_{x_k}$ in D_M. Using elementary methods for approximating the functions ϕ_i and ψ by sufficiently smooth functions, making use of estimates of Schauder type for fixed ϵ (see [19], Chapter 7, 3, 4), and taking into account the method of proof of Theorem B, we conclude that a generalized solution of problem (1.1), (1.2) exists when the following requirements on the functions ϕ_i and ψ are fulfilled: these functions are continuous, they have continuous derivatives $\phi_{iu}, \phi_{iux_j}, \phi_{ix_ix_j}, \psi_u, \psi_{x_k}$, and the functions $\phi_{iu}(t, x, u)$ and $\Psi(t, x, u)$ are bounded in the regions D_M; inequalities (4.1) and (4.2) are fulfilled. In particular, in the case of equation (1.5) we only need continuous differentiability of the functions $\phi_i(u)$ (see also [15]).

5°. It is easily seen that the derivation of the estimates of the moduli of continuity in case B is still suitable when the following inequality is fulfilled instead of (4.3):

$$\int_{K_R} |u_0(x + \Delta x) - u_0(x)|\, dx \leqslant c \cdot \exp(\text{const } R)\, |\Delta x|.$$

6°. The method of obtaining the norm estimate for the function $w = u^\epsilon(t, x + \Delta x) - u^\epsilon(t, x)$ in L_1 in case A (see §4, subsection 1) is also applicable to prove uniqueness and stability in L_1 of the bounded solutions $u^\epsilon(t, x)$ of problem (1.3), (1.2) constructed at the end of subsection 1 of §4 in the sense of the integral identity

$$\iint_{\pi_T} \{uf_t + \varphi_i(t, x, u) f_{x_i} - \psi(t, x, u) f + \varepsilon u \Delta f\}\, dx\, dt + \int_{E_n} f(0, x) u_0(x)\, dx = 0.$$

An analogous investigation of the difference of the two solutions $u^\epsilon(t, x)$ and $v^\epsilon(t, x)$ of this problem with initial functions $u_0(x)$ and $v_0(x)$, respectively, leads to the following estimate (for $0 < \epsilon \leq 1$):

$$\int_{K_r} |u^\varepsilon(t, x) - v^\varepsilon(t, x)|\, dx \leqslant \text{const} \cdot \int_{E_n} e^{-|x|} |u_0(x) - v_0(x)|\, dx.$$

We note that these results (like the L_1 estimates in §4) are based on the elementary fact (see Lemma 4) of the decrease as $|x| \to \infty$ of the solutions of Cauchy's problem with finite initial functions for the equation ($\mathfrak{L}(g) = 0$), which is conjugate to the variation of the nonlinear parabolic equation under consideration.

7°. Cauchy's problem for quasilinear hyperbolic systems. The approach to defining a generalized solution of equation (1.1) used in this article permits a natural generalization to the case of quasilinear hyperbolic systems (here we only consider systems of the form (1.7)). We first note that requirement 1) of the definition of a generalized solution of problem (1.1), (1.2) (see §2, Definition 1) generalizes to the case of a system in the following (equivalent) form: for an arbitrary convex downward function $\Phi(u)$ and any smooth function $f(t, x) \geq 0$ which is finite in π_T we have the inequality

$$\iint_{\pi_T} \Big\{ \Phi(u) f_t + \int_0^u \Phi'(u) \varphi_{iu}(t, x, u)\, du f_{x_i} - \Phi'(u) \varphi_{ix_i}(\ldots) f + \Big[\int_0^u \Phi'(u) \varphi_{iux_i}(\ldots)\, du - \Phi'(u) \psi(\ldots) \Big] f \Big\}\, dx\, dt \geqslant 0. \tag{5.2}$$

For $\Phi \equiv |u - k|$ inequality (5.2) coincides with (2.1). We easily see that, conversely, inequality (2.1) (for any k!) implies (5.2). In fact, as we noted in §2, if the function $u(t, x)$ satisfies inequality (2.1), then it also satisfies identity (1.6), and hence inequality (5.2) with the function $\Phi_k(u) = \max(u - k, 0)$; it remains to note that any function $\Phi(u)$ which is convex downward on $[-M, M]$ can be approximated by "inscribed broken lines", i.e. functions of the form $\Phi(-M) + \Phi'(-M)(u + M) + \Sigma_{i=1}^m \alpha_l \Phi_{k_l}(u)$, where $\alpha_l = \text{const} \geq 0$, $-M < k_l < k_{l+1} < M$.

We now consider the quasilinear hyperbolic system

$$\frac{\partial \varphi_0(u)}{\partial t} + \frac{\partial \varphi_i(u)}{\partial x_i} = 0, \tag{5.3}$$

where $u = (u^1, \cdots, u^N)$, $N \geq 2$, $\phi_i(u) = (\phi_i^1(u), \cdots, \phi_i^N(u))$, $i = 0, 1, \cdots, n$. We introduce the simple viscosity $\epsilon\Delta u$, $\epsilon = \text{const} > 0$ in system (5.3) and assume that the generalized solution which interests us of Cauchy's problem for the system (5.3) with the initial condition

$$u|_{t=0} = u_0(x) \tag{5.4}$$

can be obtained as the limit as $\epsilon \to +0$ (for example, in L_1) of solutions $u^\epsilon(t, x)$ of Cauchy's problem for the system

$$\frac{\partial \varphi_0}{\partial t} + \frac{\partial \varphi_i(u)}{\partial x_i} = \varepsilon \Delta u \tag{5.5}$$

with the initial condition (5.4), where

$$\sup_{\pi_T} |u^\varepsilon| + \int_0^T \int_{K_r} |u^\varepsilon_{x_i}|\, dx\, dt \leqslant \text{const}$$

uniformly in ϵ (the integral estimate assumption can be weakened, and in many cases can be entirely removed). Let the components $H^k(u)$ of the vector function $H(u) = (H^1(u), \cdots, H^N(u))$ be smooth functions, and let the matrix $H'(u) = \|H^k_{u^l}\|$ be nonnegative. For any real vector $\xi = (\xi^1, \cdots, \xi^N)$

$$(H'(u)\xi, \xi) \geqslant 0 \tag{5.6}$$

(for any values of u under consideration). We multiply the system (5.5) by the vector $H(u)$ using scalar multiplication, and we require that the expressions $(H(u), \phi_0' u_t)$ and $(H(u), \phi_i' u_{x_i})$ $(\phi_i' = \|\phi^k_{iu^l}\|)$ be total derivatives with respect to t and x^i of certain functions $\Phi(u)$ and $\Psi_i(u)$ respectively; the latter requirement means that $H(u)$ must satisfy the following system of linear equations, which is generally overdetermined:

$$\operatorname{rot}(\varphi_i'^*(u)H(u)) = 0, \quad i=0,1,\ldots,n, \tag{5.7}$$

(this system is not overdetermined only if $n = 1$, $N = 2$; here $\phi_i'^*$ is the transpose matrix of ϕ_i'). Taking (5.6) into account, we have

$$\begin{aligned}\frac{\partial \Phi(u)}{\partial t} + \frac{\partial \Psi_i(u)}{\partial x_i} &= \varepsilon \frac{\partial}{\partial x_i}\left(H(u), \frac{\partial u}{\partial x_i}\right) \\ - \varepsilon\,(H'(u)\,u_{x_i},\ u_{x_i}) &\leqslant \varepsilon \frac{\partial}{\partial x_i}\left(H(u), \frac{\partial u}{\partial x_i}\right).\end{aligned} \tag{5.8}$$

Multiplying inequality (5.8) by the test function $f(t, x) \geq 0$, integrating over π_T (interchanging the first derivatives with respect to t and x_i using integration by parts on f) and passing to the limit as $\epsilon \to +0$, we find that the limit function $u(t, x)$ satisfies the inequality

$$\iint_{\pi_T} [\Phi(u) f_t + \Psi_i(u) f_{x_i}]\, dx\, dt \geqslant 0 \tag{5.9}$$

for any smooth finite function $f \geq 0$.

Thus we arrive at the following notion of a generalized solution.

Definition 2. A bounded measurable vector function $u(t, x)$ is called a *generalized solution of problem* (5.3), (5.4) *in the band* π_T if the following conditions are satisfied.

1) Any smooth function $f(t, x) \geq 0$ which is finite in π_T satisfies inequality (5.9), where $\Phi(u)$ and $\Psi_i(u)$ are the functions constructed as above for an arbitrary solution $H(u)$ of system (5.7) so as to satisfy condition (5.6).

2) Requirement 2) of Definition 1 in §2 is fulfilled.

We note that the functions $H \equiv \pm(\underbrace{0, \cdots, 0}_{k-1}, 1, 0, \cdots, 0)$, which clearly satisfy condition (5.6), correspond to the functions $\Phi = \pm\phi_0^k(u)$ and $\Psi_i = \pm\phi_i^k(u)$; hence our generalized solution is also a generalized solution in the sense of the usual integral identity

$$\iint_{\pi_T} [\varphi_0^k(u) f_t + \varphi_i^k(u) f_{x_i}]\, dx\, dt = 0.$$

But the arbitrariness in the choice of the function $H(u)$ (and hence in the choice of Φ and Ψ_i) assumed in requirement 1) of the definition of a generalized solution certainly also takes into account the "entropy" relations at the discontinuities.

Here we have considered the simplest situation connected with an implicit "equivalence" relation for all the equations of system (5.3) (this is reflected in the choice of a viscosity of the form $\epsilon\Delta u$). However, an analogous approach is applicable in more general situations, in particular, for gas dynamic systems.

We conclude by noting that the problem of a generalized solution in the theory of quasilinear equations and in gas dynamics is discussed in [22].

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