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QUASIMONOTONE SCHEMES FOR SCALAR CONSERVATION LAWS. PART II*

BERNARDO COCKBURN†

Abstract. In this paper, the technique of construction and analysis of quasimonotone finite-difference numerical schemes for scalar conservation laws in one space dimension, developed in Part I [SIAM *J. Numer. Anal.*, 26 (1989), pp. 1325–1341], is extended to a wide class of Petrov–Galerkin finite-element methods. The resulting schemes are called the quasimonotone finite-element schemes. The approximate solution u_h is written as $\overline{u}_h + \tilde{u}_h$, where $\overline{u}_h$ is a piecewise-constant function. The Petrov–Galerkin methods are then considered to be a set of equations that defines "the parameter" $\tilde{u}_h$, plus a single equation, which is essentially a finite-difference scheme, that defines "the means" $\overline{u}_h$. All the results of the theory of quasimonotone finite-difference schemes can be carried over this finite-element framework by this simple point of view.

Key words. conservation laws, entropy schemes, finite elements

AMS(MOS) subject classifications. 65M60, 65N30, 35L65

1. Introduction. In this paper we introduce and analyze quasimonotone finite-element (QMFE) numerical methods for the scalar conservation law

$$\partial_t u + \partial_x f(u) = 0 \quad \text{in } (0,T)\times\mathbb{R}, \tag{1.1a}$$

$$u(t=0) = u_0 \quad \text{on } \mathbb{R}, \tag{1.1b}$$

where f is assumed to be C^1. These schemes are L^∞-stable, TVDM (total variation diminishing in the means), and, more important, they are entropy schemes; i.e., their approximate solution always converges to the entropy solution of (1.1). Moreover, assuming that the initial data u_0 belongs to $L^1(\mathbb{R}) \cap BV(\mathbb{R})$, where $BV(\mathbb{R})$ denotes the space of bounded total variation functions, error estimates in the $L^\infty(L^1_{\text{loc}})$-norm have been obtained. As far as we know, no other type of FE methods have been proved to have these stability and convergence properties. In the linear case many FE methods have been studied, and L^2-stability results, as well as L^2-type error estimates, have been obtained assuming enough regularity on the initial data u_0. Let us mention, for example, Lesaint and Raviart [9], in which the analysis is made by means of energy techniques and interpolation theory, and Johnson and Pitkaranta [7], which uses a mixture of FD and FE techniques. Unfortunately, these methods can hardly be extended to the nonlinear case. On the other hand, very recently Johnson and Szepessy [8] considered the stream-line diffusion Petrov–Galerkin FE method applied to (1.1), where $f(u) = u^2/2$, and proved convergence to the entropy solution assuming that the method is L^∞-stable. They used energy estimates, interpolation theory, and, most interesting, the theory of compensated compactness to obtain L^2-like error estimates. However, the latter theory can only be applied (... at least for the moment!) when the nonlinear function f is convex. Also, there are no results available for the d-dimensional case, $d > 1$. Our method of analysis does not make use of either energy estimates or interpolation theory, but employs the finite-difference techniques

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†Department of Mathematics, University of Chicago, Chicago, Illinois 60637. This work was supported by the Andrews Foundation. This work is contained in the author's Ph.D. dissertation, University of Chicago, 1986.

used to study monotone schemes; see [5] and the bibliography therein. We write the approximate solution as $u_h = \overline{u}_h + \tilde{u}_h$, where $\overline{u}_h$ is a piecewise-constant function obtained as a local average of u_h; the function $\tilde{u}_h$ is viewed as a parameter updated at each timestep. In this way, we can consider our Petrov–Galerkin method as a set of equations defining the parameter $\tilde{u}_h$ plus a single equation, that is essentially an FD scheme, defining "the means" $\overline{u}_h$. The theory of QMFD schemes is then applied to that "FD scheme" to get stability, convergence to the entropy solution, and error estimates—all of this only in terms of the means $\overline{u}_h$. If the approximate solution is discontinuous, it is possible to obtain similar results for the whole approximate solution u_h by means of nonconstraining local control of the growth of $\tilde{u}_h$ that does not destroy the order of accuracy of the method in smooth monotone regions. In a QMFE scheme, stability and convergence are obtained by controlling the means $\overline{u}_h$, whereas precision is achieved by an adequate choice of the function defined by the remaining degrees of freedom, $\tilde{u}_h$.

An outline of the paper is as follows. In §2 we display the class of Petrov–Galerkin methods that will be considered in this theory, and we establish a link between them and FD schemes. This will allow us to use for the QMFE schemes the same analysis techniques used for the QMFD schemes. In §3 we introduce and analyze QMFE schemes. In §4 we discuss in detail an example of an explicit first-order accurate QMFE scheme that produces less viscosity than the Godunov scheme, the G-$\frac{1}{2}$ scheme; numerical results are displayed. In §5 some extensions of QMFE are studied briefly. We end in §6 with some concluding remarks.

2. A certain class of Petrov–Galerkin FE methods. Let us first recall some notation. As usual, the sets $\{t^n\}_{n=1,\cdots,N}$ and $\{x_{i+1/2}\}_{i\in Z}$ are partitions of $[0,T]$ and $\mathbb{R}$, respectively. We set $\Delta t^n = t^{n+1} - t^n$ and $\Delta x_i = x_{i+1/2} - x_{i-1/2}$, and we denote by J^n and I_i the intervals (t^n, t^{n+1}) and $(x_{i-1/2}, x_{i+1/2})$, respectively. Finally, K_i^n will stand for the element $J^n \times I_i$, and δ for $\sup_{i,n}\{\Delta t^n, \Delta x_i\}$.

The Petrov–Galerkin methods we consider are defined as follows. First, multiply (1.1a) by a smooth function φ, integrate over the set K, and integrate by parts formally to obtain:

$$\begin{aligned}-\iint_K u\,\partial_t\varphi\;dt\,dx + \int_{\partial K} u\,\varphi\,n_t\;d\gamma\\ -\iint_K f(u)\,\partial_x\varphi\;dt\,dx + \int_{\partial K} f(u)\,\varphi\,n_x\;d\gamma = 0,\end{aligned}$$

where (n_x, n_t) is the unit outward normal of ∂K. The approximate solution $u_h \in V_h$ is determined as the unique solution of a discrete weak formulation that follows naturally from the preceding one, namely,

$$\begin{aligned}-\iint_K u_h\,\partial_t\varphi_h\;dt\,dx + \int_{\partial K} w_h\,\varphi_h\,n_t\;d\gamma\\ -\iint_K f(u_h)\,\partial_x\varphi_h\;dt\,dx + \int_{\partial K} f_h^{\mathrm{FE}}\,\varphi_h\,n_x\;d\gamma = 0 \quad \forall\varphi \in W_h,\end{aligned} \tag{2.1a}$$

where the sets K are, as is common in the finite-element method, the union of elements K_i^n containing the support of φ_h. For simplicity we shall assume that in such a union the time index n is held constant. To complete the scheme (2.1a) an initial data must be given. We shall denote it by $u_{0,h}$.

The fluxes w_h and f_h^{FE} are functions of the form

$$
\text{(2.1b)} \qquad \begin{aligned} w_h(t,x) &= w\big(\lim_{\epsilon\downarrow 0} u_h(t-\epsilon,x), \lim_{\epsilon\downarrow 0} u_h(t+\epsilon,x)\big),\\ f_h^{\mathrm{FE}}(t,x) &= f^{\mathrm{FE}}\big(\lim_{\epsilon\downarrow 0} u_h(t,x-\epsilon), \lim_{\epsilon\downarrow 0} u_h(t,x+\epsilon)\big), \end{aligned}
$$

that are consistent, i.e., $w(u,u) = u$ and $f^{\mathrm{FE}}(u,u) = f(u)$. As the entropy solution may develop strong discontinuities in finite time even if the initial data u_0 is very smooth, it is quite reasonable to allow u_h to have discontinuities across ∂K. In this case the *trace* of u_h on ∂K is not uniquely defined, hence the necessity of introducing the fluxes w and f^{FE} to give a meaning to the integrals over ∂K. The most natural choices for w are $w(u_1, u_2) = u_2$, which may give rise to explicit schemes, and $w(u1, u2) = u1$, which does give rise to implicit ones; see (3.2) below. For simplicity, we are going to consider only these two cases. On the other hand, the flux f^{FE} can be, for example, any two-point monotone flux, like the well-known Godunov, Enquist–Osher and Lax–Friedrichs fluxes.

In general, the *trial* function space V_h and the *test* space W_h are different; if they are not, the method is called simply a Galerkin method. We assume, of course, that the choice of the trial and test spaces ensures the existence and uniqueness of the approximate solution. Besides this, complete freedom in the choice of these spaces is allowed, except for the single crucial condition on the test space W_h:

$$
\text{(2.1c)} \qquad \chi_i^n \in W_h \qquad \forall\, i,\, n,
$$

where χ_i^n is the characteristic function of K_i^n. This condition allows us to establish an essential link between this class of Petrov–Galerkin methods and FD schemes. If in (2.1a) we take $\varphi_h = \chi_i^n$, we obtain

$$
\text{(2.2a)} \qquad (\overline{w}_i^{n+1} - \overline{w}_i^n)/\Delta t^n + (\overline{f}_{i+1/2}^{\mathrm{FE},n} - \overline{f}_{i-1/2}^{\mathrm{FE},n})/\Delta x_i = 0,
$$

where

$$
\text{(2.2b)} \qquad \overline{w}_i^n = (\Delta x_i)^{-1} \int_{I_i} w(t^n, x)\, dx,
$$

$$
\text{(2.2c)} \qquad \overline{f}_{i+1/2}^{\mathrm{FE},n} = (\Delta t^n)^{-1} \int_{J^n} f^{\mathrm{FE}}(t, x_{i+1/2})\, dt.
$$

If u_h is assumed to be piecewise-constant, say $u_h(t,x) = \overline{u}_h(t,x) = \overline{u}_i^n$ for $(t,x) \in K_i^n$, then

$$
\begin{aligned} \overline{w}_i^n &= w_h(\overline{u}_i^{n-1}, \overline{u}_i^n),\\ \overline{f}_{i+1/2}^{\mathrm{FE},n} &= f_h^{\mathrm{FE}}(\overline{u}_i^n, \overline{u}_{i+1}^n), \end{aligned}
$$

and the scheme (2.2) can be considered as an FD scheme defining the discrete approximate solution $\{\overline{u}_i^n\}$. In the general case we write u_h as $\overline{u}_h + \tilde{u}_h$, where $\overline{u}_h$ is piecewise-constant, and we consider the function $\tilde{u}_h$ as a parameter updated using (2.1a). Thus (2.1) can also be considered as an FD scheme defining $\{\overline{u}_h\}$. In this way, we can use it to construct QMFD schemes using the techniques introduced in [5]. Note that the algebraic equations (2.1a) can be rewritten as

$$
\begin{aligned} &-\iint_K u_h\, \partial_t \varphi_h \, dt\, dx + \int_{\partial K} w_h\, \varphi_h\, n_t \, d\gamma \\ &\quad - \iint_K f(u_h)\, \partial_x \varphi_h \, dt\, dx + \int_{\partial K} f_h^{\mathrm{FE}}\, (\varphi_h - \overline{\varphi}_h)\, n_x \, d\gamma \\ &\quad + \int_{\partial K} \overline{f}_h^{\mathrm{FE}}\, \overline{\varphi}_h\, n_x \, d\gamma = 0 \quad \forall \varphi \in W_h, \end{aligned}
$$

where $\overline{f}_h^{\text{FE}}$ is given by (2.2c), and $\overline{\varphi}_h$ is a function independent of t such that $\int_{\partial K}(\varphi_h - \overline{\varphi}_h)\, n_x \, d\gamma = 0$.

3. Quasimonotone finite-element schemes. If in the previous weak formulation we simply replace the flux $\overline{f}^{\text{FE}}$ by a QMFE flux, f^{QMFE},

$$
\begin{aligned}
&- \iint_K u_h \, \partial_t \, \varphi_h \, dt \, dx + \int_{\partial K} w_h \, \varphi_h \, n_t \, d\gamma \\
(3.1) \qquad &- \iint_K f(u_h) \, \partial_x \, \varphi_h \, dt \, dx + \int_{\partial K} f_h^{\text{FE}} \, (\varphi_h - \overline{\varphi}_h) \, n_x \, d\gamma \\
&+ \int_{\partial K} f_h^{\text{QMFE}} \, \overline{\varphi}_h \, n_x \, d\gamma = 0 \qquad \forall \varphi \in W_h,
\end{aligned}
$$

we obtain what we call a QMFE scheme. A QMFE flux is a QM flux as defined in [5, (2.4)-(2.5)] where the grid values u_i^n are replaced by *the values of the means of the approximate solution* u_h, $\overline{u}_i^n$. The function $\overline{u}_h(t,x)$ can be chosen in several ways, but the two most simple choices are:

$$\overline{u}_i^n = (\Delta x_i)^{-1} \int_{I_i} u_h(t^{n+0}, x) \, dx, \qquad \text{if } w(w1, w2) = w2, \tag{3.2a}$$

$$\overline{u}_i^n = (\Delta x_i)^{-1} \int_{I_i} u_h(t^{n+1-0}, x) \, dx, \qquad \text{if } w(w1, w2) = w1. \tag{3.2b}$$

Although this paper is concerned with the general features of the theory of QMFE schemes, a concrete application of the theory should give the reader a better idea of how the theory works. Thus, a detailed example of a formally first-order accurate explicit QMFE scheme, and numerical experiments showing the quality of its performance will be presented in the next section. Higher-order accurate schemes can be easily obtained by following the lines of the theory and will not be discussed here. Some of them can be be found in [4].

To analyze (3.1) is now a very simple matter, for if we take $\varphi = \chi_i^n$ in (3.1) we obtain

$$(\overline{u}_i^{n+1} - \overline{u}_i^n)/\Delta t^n + (f_{i+1/2}^{\text{QMFE},n} - f_{i-1/2}^{\text{QMFE},n})/\Delta x_i = 0 \quad \text{for the choice (3.2a) of } \overline{u}_h,$$
$$(\overline{u}_i^n - \overline{u}_i^{n-1})/\Delta t^n + (f_{i+1/2}^{\text{QMFE},n} - f_{i-1/2}^{\text{QMFE},n})/\Delta x_i = 0 \quad \text{for the choice (3.2b) of } \overline{u}_h.$$

If we assume that we know how to calculate the parameter function $\tilde{u}_h$ and we consider the scheme solely as a method to determine the means $\overline{u}_h$, it can be regarded as a QMFD scheme. As a consequence, we have the following result (we assume that f^{M} is the two-point monotone flux with which the corresponding QMFE flux f^{QMFE} is constructed).

THEOREM 3.1. *Suppose that the scheme*

$$(\overline{u}_i^{n+1} - \overline{u}_i^n)/\Delta t^n + (f_{i+1/2}^{\text{M},n} - f_{i-1/2}^{\text{M},n})/\Delta x_i = 0,$$

is monotone for CFL $\in [0, \text{CFL}_0]$. *Then, if* CFL $\in [0, \text{CFL}_0/2]$, *the* QMFE *scheme* (3.1)–(3.2a) *verifies the maximum principle*

$$\overline{u}_i^{n+1} \in I(\overline{u}_{i-1}^n, \overline{u}_i^n, \overline{u}_{i+1}^n) \tag{3.3a}$$

and is TVDM *(total variation diminishing in the means); i.e.,*

$$\|\overline{u}_h^{n+1}\|_{\text{TV}(\mathbb{R})} \le \|\overline{u}_h^n\|_{\text{TV}(\mathbb{R})}. \tag{3.4}$$

Moreover, the sequence $\{\overline{u}_h\}$ *converges to the entropy solution of* (1.1), *and verifies the estimates*

$$\int_{\mathbf{R}} J(u(T)-\overline{u}_h(T)) \le \int_{\mathbf{R}} J(u_0-\overline{u}_{0,h}) + \|u_0\|_{BV(\mathbf{R})}(C_1T^{1/2}\delta^{1/2}+C_2T\delta^{\alpha/2}), \tag{3.5a}$$

$$\begin{aligned}\|u(T)-\overline{u}_h(T)\|_{L^1(\Omega)} \le 2\|u_0-\overline{u}_{0,h}\|_{L^1(\mathbf{R})} + C_3T^{1/2}\|u_0\|_{BV(\mathbf{R})}\delta^{1/2}\\ + C_4T^{1/2}(\|u_0\|_{BV(\mathbf{R})}|\Omega|)^{1/2}\delta^{\alpha/2},\end{aligned} \tag{3.5b}$$

where J is an even nonnegative convex function with Lipschitz second derivative vanishing outside an interval of the form $[-c\,\delta^{\alpha/2}, c\,\delta^{\alpha/2}]$, *and Ω is an arbitrary compact subset of* $\mathbf{R}$.

The QMFE *scheme* (3.1)–(3.2b) *verifies the maximum principle*

$$\overline{u}_i^{n+1} \in I(\overline{u}_j^{n+1}, \overline{u}_{j+1}^n, \cdots, \overline{u}_i^n, \overline{u}_{k-1}^n, \overline{u}_k^{n+1}), \quad j<i<k, \tag{3.3b}$$

and the inequalities (3.4), (3.5).

This result is independent of the precise form of the flux f^{FE} and of the values of $\tilde{u}_h$. Since an appropriate choice of $\tilde{u}_h$ near discontinuities may reduce the error, QMFE schemes provide freedom to do this. Note also that the local accuracy can be changed without altering stability properties. In particular, if the approximate solution is piecewise-polynomial, its degree can vary in time and in space without affecting the stability. This could possibly give rise to a p-version of the QMFE schemes, but we shall not discuss the matter in this work.

Now, let us consider the problem of the convergence of the sequence $\{u_h\}$ to the entropy solution of (1.1). Since

$$\|u-u_h\|_{L^\infty(L^1)} \le \|u-\overline{u}_h\|_{L^\infty(L^1)} + \|\tilde{u}_h\|_{L^\infty(L^1)},$$

to obtain (3.5) with u_h instead of $\overline{u}_h$ it is enough to require that, as δ goes to zero, $\|\tilde{u}_h\|_{L^\infty(L^1)} = O(\delta^{\alpha/2})$. Note that $\tilde{u}_h(t)$ is essentially an approximation of $h\,\partial_x u(t)$, so that it is reasonable to expect that

$$\begin{aligned}\|\tilde{u}_h(t)\|_{L^\infty(L^1)} &\cong C\ h\|\partial_x u(t)\|_{L^\infty(L^1)}\\ &\cong C\ h\|u(t)\|_{L^\infty(BV)}\\ &\cong C\ h\|u_0\|_{BV}\end{aligned}$$

In fact, we expect $\tilde{u}_h$ to "behave well" in the regions in which the entropy solution is smooth (... if the initial Petrov–Galerkin method already has this property!). However, $\tilde{u}_h$ can have an exaggerated growth near discontinuities, and so must be modified. We shall denote by $\Lambda\Pi_h^n(\tilde{u}_h)$ this modification, and the only restriction we shall impose on the family of operators $\{\Lambda\Pi_h^n\}$ is the following

$$\|\Lambda\Pi_h^n(\tilde{u}_h)\|_{L^1(\mathbf{R})} \le C\delta. \tag{3.6}$$

Suppose, for the moment, that such a family exists for the scheme (3.1) under consideration. Then, we modify it as follows:

(3.7) Assuming u_h^n known, compute u_h^{n+1} as follows: First, calculate u_h^{n+1} using the QMFE scheme (3.1); then, compute $\Lambda\Pi_h^{n+1}(\tilde{u}_h^{n+1})$ and set $u_h^{n+1} = \overline{u}_h^{n+1} + \Lambda\Pi_h^{n+1}(\tilde{u}_h^{n+1})$.

We have the following immediate result.

COROLLARY 3.2. *Consider the* QMFE *scheme* (3.7). *Suppose that* (3.6) *and the assumptions of Theorem* 3.1 *are satisfied. Then the results of Theorem* 3.1 *hold, and the estimates* (3.5) *are verified with* $u_h(T)$ *replacing* $\overline{u}_h(T)$.

When the approximate solution u_h is discontinuous across ∂K_i^n the operators $\Lambda\Pi_h^n$ can be easily defined. Let us denote by $\widetilde{V}(K_i^n)$ the space to which the restriction of $\tilde{u}_h$ to the interior of K_i^n, $\tilde{u}_h\big|_{K_i^n}$, belongs. Now, denote by $\widetilde{V}'(K_i^n)$ the convex subset of $\widetilde{V}(K_i^n)$ for which we have

$$\text{(3.8a)} \qquad \left|\tilde{u}_h(t,x)\right| \le \Sigma_{m=r,\cdots,s} C_m \Theta^n_{i+m_1/2} \quad \text{for } (t,x) \text{ in } K_i^n.$$

$$\text{(3.8b)} \qquad \Theta^n_{i+1/2} = \left|\overline{u}^n_{i+1} - \overline{u}^n_i\right|,$$

$$\text{(3.8c)} \qquad C_m \ge 0,$$

where r and s (respectively, C_m) are arbitrary but fixed natural (respectively, real) numbers. Then, we require that

$$\text{(3.9a)} \qquad \Lambda\Pi_h^n : \widetilde{V}(K_i^n) \to \widetilde{V}'(K_i^n),$$

$$\text{(3.9b)} \qquad (\Lambda\Pi_h^n)^2 = \Lambda\Pi_h^n.$$

This means that we require $\Lambda\Pi_h^n$ to be a local projection. Note that as u_h is discontinuous across ∂K_i^n, the result of projecting $\tilde{u}_h\big|_{K_i^n}$ on $\widetilde{V}'(K_i^n)$ does not have any influence on the values of u_h outside of K_i^n. Also note that the very form of the projection has not been specified, and so there is absolute freedom in the choice of it. Finally, it is easy to verify that any family of operators verifying (3.8)–(3.9) satisfies the inequality (3.5), for we have

$$\begin{aligned} \|\Lambda\Pi_h^n(\tilde{u}_h(t))\|_{L^1(\mathbf{R})} &\le [\Sigma_{m=r,\cdots,s} C_m] \|\overline{u}_h^n\|_{BV(\mathbf{R})} h, \quad t \in J^n, \\ &\le [\Sigma_{m=r,\cdots,s} C_m] \|u_0\|_{BV(\mathbf{R})} \delta. \end{aligned}$$

If the initial Petrov–Galerkin method (2.1) "behaves well," the corresponding QMFE scheme (3.1) may coincide with it in the smooth monotone regions of the entropy solution, as it may happen in the FD case, see [5]. In this case, and if the constants C_m are not too small, we should have $\Lambda\Pi_h^n(\tilde{u}_h) = \tilde{u}_h$ in those regions, in which case the QMFE scheme (3.7) coincides with the initial method (2.1). Thus, the projection does not modify the already achieved accuracy in those regions and is really effective in a very small region. Ideally, the measure of this region should be $O(\delta)$ as δ goes to zero!

4. An explicit QMFE **scheme: the G-$\frac{1}{2}$ scheme.** This scheme was introduced by the author [3] as a modification of a scheme introduced by Chavent and Salzano [2]. The latter scheme will be called the P^0P^1 scheme. This method is a nonlinear explicit version of the discontinuous Galerkin finite-element schemes analyzed by LeSaint and Raviart [9].

The P^0P^1 is a Galerkin ($V_h = W_h$) method (2.1) for which

$$V_h = \{v_h \in L^\infty(0,T; L^1(\mathbf{R})) : v_h|_{K_i^n} \in P^0(J^n) \times P^1(I_i)\},$$

$w(u_1,u_2) = u_2$, and $f^{\mathrm{FE}} = f^{P^0P^1}$ is defined by

$$f_h^{P^0P^1}(t, x_{i+1/2}) = f^{\mathrm{G}}(u_h(t, x^-_{i+1/2}), u_h(t, x^+_{i+1/2})),$$

where f^{G} is the Godunov flux. Taking into account that the approximate solution is piecewise-constant in time, we can rewrite this scheme as follows:

(3.10a)
Set u_h^0 equal to the L^2-projection of u_o into $\mathcal{V}_h = \{v_h : v_h|_{I_i} \text{ is linear}, i \in \mathbb{Z}\}$;
(3.10b)
For $n = 0, \cdots, N-1$ compute u_h^{n+1} as the unique solution of:

$$\begin{aligned}&\Big[\int_{I_i} u_h^{n+1}\, \phi_h\, dx - \int_{I_i} u_h^n\, \phi_h\, dx\Big] / \Delta t^n \\ &\quad + [f_h^{P^0P^1}(t^n, x_{i+1/2})\, \phi_h(x_{i+1/2}) - f_h^{P^0P^1}(t^n, x_{i-1/2})\, \phi_h(x_{i-1/2})] \\ &\quad - \int_{I_i} f(u_h)\, \partial_x\, \phi_h\, dx = 0 \quad \forall \phi \in P^1(I_i).\end{aligned}$$

Note that in this case $\overline{f}^{P^0P^1} = f^{P^0P^1}$; see (2.1b). The P^0P^1-scheme is L^2-stable in the linear case if and only if CFL $= O(\sqrt{\Delta x})$, and may not converge to the entropy solution when the nonlinearity f is nonconvex; see [1]. It was in an effort to overcome these disadvantages that the G-$\frac{1}{2}$ scheme was found.

The G-$\frac{1}{2}$ scheme is a QMFE scheme (3.7). In order to define it we only have to define the QMFE flux $f^{\mathrm{G}-1/2}$, and the projections $\Lambda\Pi_h^n$. The flux $f^{\mathrm{G}\text{-}1/2}$ is constructed as in [5, Eqns. (2.4)–(2.6)] with $f^h = \overline{f}^{\mathrm{FE}}$ and $f^m = f^{\mathrm{G}}$; i.e.,

$$\begin{aligned}f_h^{\mathrm{G}-1/2}(t^n, x_{i+1/2}) &= f^{\mathrm{G}}(\overline{u}_i^n, \overline{u}_{i+1}^n) + a_{i+1/2}^n, \\ a_{i+1/2}^n &= \operatorname{sgn}(\overline{u}_{i+1}^n - \overline{u}_i^n) \cdot \max\{0, \Theta_{i+1/2}^n\}, \\ \Theta_{i+1/2}^n &= \min\{|f_{i+1/2}^{P^0P^1,n} - f^{\mathrm{G}}(\overline{u}_i^n, \overline{u}_{i+1}^n)|, \\ &\qquad |f^{\mathrm{G}}(\overline{u}_{i-1}^n, \overline{u}_i^n) - f^{\mathrm{G}}(\overline{u}_i^n, \overline{u}_{i+1}^n)|, \\ &\qquad \operatorname{sgn}((\overline{u}_{i-1}^n - \overline{u}_i^n)(\overline{u}_i^n - \overline{u}_{i+1}^n)), c_i \cdot (\Delta x_i)^\alpha, \\ &\qquad |f^{\mathrm{G}}(\overline{u}_{i+1}^n, \overline{u}_{i+2}^n) - f^{\mathrm{G}}(\overline{u}_i^n, \overline{u}_{i+1}^n)|, \\ &\qquad \operatorname{sgn}((\overline{u}_{i+1}^n - \overline{u}_i^n)(\overline{u}_{i+1}^n - \leq lu_{i+2}^n)), c_{i+1} \cdot (\Delta x_{i+1})^\alpha\}.\end{aligned}$$

If we define the degrees of freedom of $v_h \in V_h$ as

$$v_h(t,x) = \overline{v}_i^n + 2\frac{x - x_i}{\Delta x_i}\tilde{v}_i^n, \qquad (t,x) \in K_i^n,$$

then, the projections $\Lambda\Pi_h^n$ are defined as

$$\begin{aligned}\Lambda\Pi_h^n(\tilde{v}_i^n) = \operatorname{sgn}(\tilde{v}_i^n) \cdot \max\{0, \min\{&|\tilde{v}_i^n|, |\overline{u}_{i+1}^n - \overline{u}_i^n|, \operatorname{sgn}(\tilde{v}_i^n \cdot (\overline{u}_{i+1}^n - \overline{u}_i^n)), \\ &|\overline{u}_i^n - \overline{u}_{i-1}^n|, \operatorname{sgn}(\tilde{v}_i^n \cdot (\overline{u}_i^n - \overline{u}_{i-1}^n))\}\}.\end{aligned}$$

This is precisely one of the monotonicity-preserving projections Van Leer introduced [10] in an effort to enforce stability and convergence for his schemes. Note that in our framework the aim of these projections is not to ensure monotonicity, but to enforce the convergence of the whole sequence $\{u_h\}_{h\downarrow 0}$, since convergence of the means $\{\overline{u}_h\}_{h\downarrow 0}$ has already been obtained.

For the sake of completeness, let us write the G-$\frac{1}{2}$ scheme in terms of the degrees of freedom of u_h:

(3.11a) Compute u_h^0 as follows: compute the L^2-projection of u_o into $\mathcal{V}_h$, i.e.:

$$\overline{u}_i^0 = \int_{I_i} u_0(s)ds/\Delta x_i,$$
$$\tilde{u}_i^{0,\star} = 6\int_{I_i} (s - x_i)u_0(s)ds/\Delta {x_i}^2;$$

then set $\tilde{u}_i^0 = \Lambda\Pi_i^0(u_i^{0,\star})$;

(3.11b) For $n = 0, \cdots, N-1$ compute u_h^{n+1} as follows: compute $\overline{u}_h^{n+1}$, and $\tilde{u}_h^{n+1,\star}$ as the solution of:

$$(\overline{u}_i^{n+1} - \overline{u}_i^n)/\Delta t^n + (f_{i+1/2}^{\mathrm{G}-1/2,n} - f_{i-1/2}^{\mathrm{G}-1/2,n})/\Delta x_i = 0,$$
$$(\tilde{u}_i^{n+1,\star} - \tilde{u}_i^n)/\Delta t^n + 3(f_{i+1/2}^{\mathrm{G}-1/2,n} + f_{i-1/2}^{\mathrm{G}-12,n})/\Delta x_i$$
$$- 6\Big\{\int\int_{K_i^n} f(u_h(t,x))dtdx/(\Delta t^n \Delta x_i)\Big\}/\Delta x_i = 0.$$

Set $\tilde{u}_h^{n+1} = \Lambda\Pi_h(\tilde{u}_n^{n+1,\star})$.

Since the Godunov scheme is monotone for CFL $\in [0,1]$ Corollary (3.2) holds for the G-$\frac{1}{2}$ scheme for CFL$_o = 1$. Moreover, thanks to the definition of the projections $\Lambda\Pi_h^n$, the following maximum principle holds:

$$u_h(t^n, x) \in I(\overline{u}_{i-1}^n, \overline{u}_i^n, \overline{u}_{i+1}^n), \quad x \in I_i.$$

This implies that, if $u_{0,h}(x) \in [a,b]$ for $x \in \mathbb{R}$, the same is true for u_h. This is a very important property because in many physical problems only values inside a fixed interval have physical meaning.

Next we test numerically this scheme in the same test problems used for testing the QM leap-frog scheme in [5]. Numerical results are shown in Table 1 and Figs. 1 and 2. In Table 1 we show the $L^1(\Omega')$ and $L^\infty(\Omega')$ errors and their respective orders. The set Ω' is a set in which the entropy solution is smooth. It has been defined in [5, §2.7]. We recall that the entropy solutions of problems 1–3 present discontinuities, whereas the ones of problems 4–6 are very smooth. We see that the scheme displays a consistent first order of accuracy, even in the case of a contact discontinuity. In particular, its behavior in the Buckley–Leverett case is remarkable. Note how the shocks are captured in only two elements; see Figs. 1 and 2.

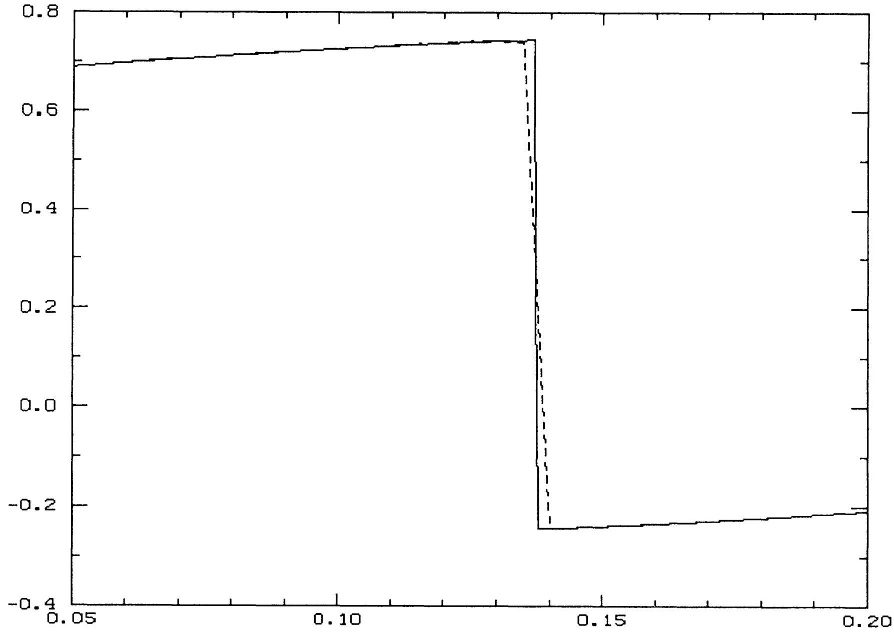

FIG. 1. *Detail of the approximation of the shock for the Burgers test problem* 2 : CFL = $\frac{1}{4}$ *and* $\Delta x = 1/200$. *The discontinuous line represents the approximate solution.*

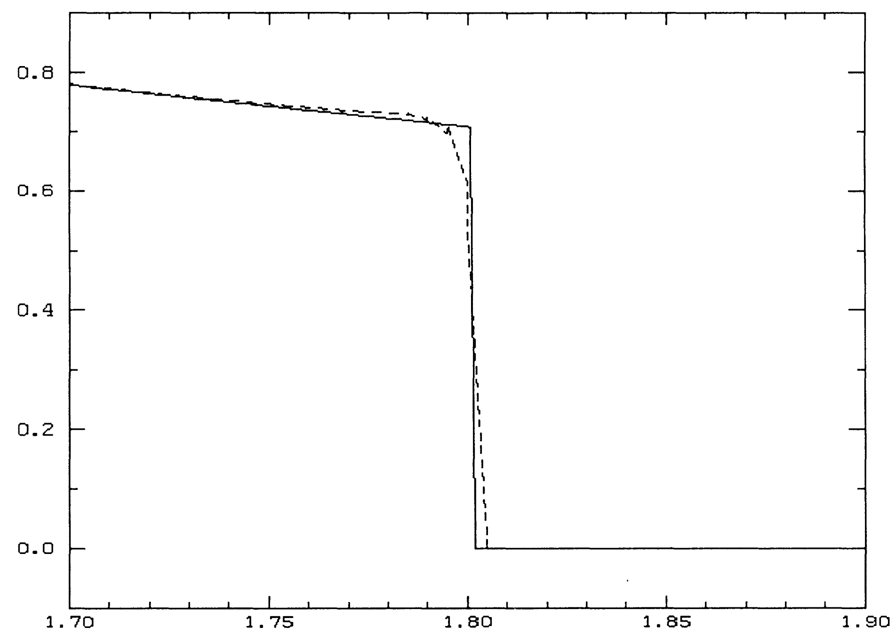

FIG. 2. *Detail of the approximation of the shock for the Buckley-Leverett test problem* 3 : CFL = $\frac{1}{4}$ *and* $\Delta x = 1/200$. *The discontinuous line represents the approximate solution.*

TABLE 1

$L^1(\Omega')$ and $L^\infty(\Omega')$ errors and their respective order of convergence: CFL $= \frac{1}{4}$ *and* $\Delta x = 1/200$.

problem	$10^4 \cdot L^1$-error	order	$10^4 \cdot L^\infty$-error	order
1	49.08	1.000	-	-
2	1.05	1.216	22.08	0.956
3	1.11	1.051	10.55	1.091
4	12.06	0.994	42.19	1.128
5	2.23	1.482	40.41	1.135
6	9.50	1.062	73.16	0.805

5. Extensions of implicit QMFE schemes. Throughout this section we assume that the choice (3.2b) has been taken for the time flux w. In §5.1 and 5.2 we shall take, accordingly,

$$\overline{u}_i^n = (\Delta x_i)^{-1} \int_{I_i} u_h(t^{n+1-0}, x)\, dx.$$

5.1. Time-dependent grid QMFE schemes. Time-dependent grid QMFE schemes can be defined in exactly the same way the corresponding QMFD schemes were defined. We only want to point out that if $u_h^{n,\mathrm{OLD}}(t^n)$ is the approximate solution defined on the mesh $\{x_{i+1/2}^n\}$, we define $u_h^{n,\mathrm{NEW}}(t^n)$ in the new mesh $\{x_{j+1/2}^n\}$ as

$$\int_{\mathbf{R}} u_h^{n,\mathrm{OLD}}(t^n)\varphi_h(t^n) = \int_{\mathbf{R}} u_h^{n,\mathrm{NEW}}(t^n)\varphi_h(t^n) \qquad \forall\, \varphi_h \in W_h.$$

Note that, thanks to condition (2.1c) the total mass is conserved. Theorem (3.1) is valid for these schemes. See [5], [6].

5.2. QMFE schemes with two time-level fluxes. We can extend the definition of a QM flux by saying that the flux f_s^{QMFE} is a QMFE flux if it is of the norm

$$f_s^{\mathrm{QMFE},n} = s\, f^{\mathrm{QMFE},n} + (1-s) f^{\mathrm{QMFE},n-1}, \qquad s \in (0,1], \tag{5.1}$$

where f^{QMFE} is a QMFE flux constructed as in §3. Note that in §3 we have already treated the case $s = 1$. The following result can now be proved immediately.

THEOREM 5.1. *Suppose that the scheme*

$$(\overline{u}_i^{n+1} - \overline{u}_i^n)/\Delta t^n + (f_{i+1/2}^{m,n} - f_{i-1/2}^{m,n})/\Delta x_i = 0,$$

is monotone for CFL $\in [0, \mathrm{CFL}_0]$. *Then, for* CFL $\in [0, \mathrm{CFL}_0/2(1-s)]$, *any* QMFE *scheme* (3.1)–(5.1) *is a* TVDM *scheme verifying the maximum principle*

$$\overline{u}_i^n \in I(\overline{u}_j^n, \overline{u}_j^{n-1}, \overline{u}_{j+1}^{n-1}, \cdots, \overline{u}_{k-1}^{n-1}, \overline{u}_k^{n-1}, \overline{u}_k^n), \quad j < i < k, \quad s \in (0,1),$$
$$\overline{u}_i^n \in I(\overline{u}_j^n, \overline{u}_{j+1}^{n-1}, \cdots, \overline{u}_{k-1}^{n-1}, \overline{u}_k^n), \quad j < i < k, \quad j < i < k, \quad s = 1.$$

Moreover, the sequence $\{\overline{u}_h\}$ converges to the entropy solution of (1.1), *and satisfies the error estimates* (3.5). *Also, any* QMFE *scheme* (3.7)–(5.1) *satisfies the error estimates* (3.5) *with u_h replacing $\overline{u}_h$.*

5.3. QMFE schemes with time-QMFE fluxes. Now, we define the equivalent of the implicit QMFD schemes of second type. To do this, take $w(u_1, u_2) = u_1$, define the means in a different way, namely:

$$\overline{u}_h(t,x) = \overline{u}_i^n = (\Delta x_i)^{-1} \int_I u_h(t^n + \rho\Delta t^n, x)\, dx \quad \text{for } (t,x) \in K_i^n \tag{5.2a}$$

where $\rho \in [0,1]$ is a fixed number, and rewrite (2.1a) as follows:

$$\begin{aligned} &-\iint_K u_h\, \partial_t \varphi_h \, dt\, dx + \int_{\partial K} w_h\, (\varphi_h - \tilde{\varphi}_h)\, n_t \, d\gamma + \int_{\partial K} \overline{w}_h\, \tilde{\varphi}_h\, n_t\, d\gamma \\ &-\iint_K f(u_h)\, \partial_x \varphi_h \, dt\, dx + \int_{\partial K} f_h^{\mathrm{FE}}\, (\varphi_h - \overline{\varphi}_h)\, n_x\, d\gamma \\ &+ \int_{\partial K} \overline{f}_h^{\mathrm{FE}}\, \overline{\varphi}_h\, n_x\, d\gamma = 0 \qquad \forall \varphi \in W_h, \end{aligned}$$

where $\overline{w}_h$, $\overline{f}^{\mathrm{FE}}$ are given by (2.2), and $\overline{\varphi}_h$ and $\tilde{\varphi}_h$ are functions independent of t and x, respectively, such that $\int_{\partial K} (\varphi_h - \overline{\varphi}_h)\, n_x\, d\gamma = \int_{\partial K} (\varphi_h - \tilde{\varphi}_h)\, n_t\, d\gamma = 0$. The scheme obtained by replacing $\overline{f}^{\mathrm{FE}}$ by the QMFE flux f_s^{QMFE}, and $\overline{w}_h$ by the QMFE time flux g^{QMFE} flux:

$$\begin{aligned} &-\iint_K u_h\, \partial_t \varphi_h \, dt\, dx + \int_{\partial K} w_h\, (\varphi_h - \tilde{\varphi}_h)\, n_t \, d\gamma \\ &+ \int_{\partial K} g_h^{\mathrm{QMFE}}\, \tilde{\varphi}_h\, n_t\, d\gamma \\ &-\iint_K f(u_h)\, \partial_x \varphi_h \, dt\, dx + \int_{\partial K} f_h^{\mathrm{FE}}\, (\varphi_h - \overline{\varphi}_h)\, n_x\, d\gamma \\ &+ \int_{\partial K} f_h^{\mathrm{QMFE}}\, \overline{\varphi}_h\, n_x\, d\gamma = 0 \qquad \forall \varphi \in W_h, \end{aligned} \tag{5.2b}$$

is also called a QMFE scheme. A QMFE time flux g^{QM} is a QM time flux constructed with the means $\overline{u}_h$ as in [5, Eqns. (3.4), (3.5)].

THEOREM 5.2. *Suppose the hypotheses of Theorem* 5.1 *are verified. Then, for* $\mathrm{CFL} \in [0, c_o \cdot \mathrm{CFL}_0/(1-s)]$, *any* QMFE *scheme* (5.2) *verifies the results of Theorem* 5.1.

6. Concluding remarks. In this paper the techniques used to define and analyze QMFD schemes have been used to define and analyze QMFE by following the simple idea that a Petrov–Galerkin method (2.1) can be also regarded as an FD scheme for the means of the approximate solution, $\overline{u}_h$. The remaining degrees of freedom defining the approximate solution u_h are considered parameters updated at each timestep. Stability and convergence to the entropy solution of the means are obtained via a suitable definition of the QMFE fluxes. When the approximate solution is discontinuous, convergence of the whole sequence can also be achieved by applying a simple nonrestrictive local projection. These schemes can be easily extended to the bounded domain case.

The Petrov–Galerkin formulation (2.1) leads to implicit schemes unless, of course, the form of u_h on $J^n \times \mathbf{R}$ depends only on $u_h(t^n)$. The theory of QMFE schemes allows a reasonable amount of freedom to establish such a dependence. Numerical experimentation remains to be done in this framework. On the other hand, implicit schemes are in general more difficult to code and to actually solve, but are useful if a time-dependent grid is to be used (we point out that even in the case of an implicit monotone scheme there are very few algorithms that allow us to solve them for a very large ratio $\Delta t/\Delta x$ in a satisfactory way!) There is much computational, as well as theoretical, work to be done for this type of scheme. The theory of QMFE schemes proposes one way of development.

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Note added in proof. The QMFE schemes can be rendered (formally) uniformly high-order accurate by using the TVB technique introduced by Shu [11]. In [5] we have indicated how to implement his technique in the framework of our quasi-monotone, finite-difference schemes. The extension of such implementation to QMFE schemes is straightforward.

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