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003

# CONVENIENT TOTAL VARIATION DIMINISHING CONDITIONS FOR NONLINEAR DIFFERENCE SCHEMES*

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**Abstract.** Convenient conditions for nonlinear difference schemes to be total variation diminishing (TVD) are derived. It is shown that such schemes share the TVD property, provided their numerical fluxes meet a certain positivity condition at local extreme values but can be arbitrary otherwise. Local TVD conditions are invariant under different incremental representations of the nonlinear schemes, and thus provide a simplified generalization of the global TVD conditions established by previous work.

**Key words.** nonlinear difference schemes, total variation, extreme values

**AMS(MOS) subject classifications.** 35L65, 65M10

## 1. Introduction.

We consider discrete approximations to the scalar conservation law

$$\frac{\partial}{\partial t}[u(x,t)]+\frac{\partial}{\partial x}[f(u(x,t))]=0, \qquad (x,t)\in R\times[0,\infty). \tag{1.1}$$

Let $v(t)=\{v_\nu(t)\}$ be the approximate solution, and denote by

$$TV[v(t)]=\sum_\nu |\Delta v_{\nu+1/2}|, \qquad \Delta v_{\nu+1/2}\equiv v_{\nu+1}(t)-v_\nu(t), \tag{1.2}$$

its total variation at time level $t$. A desirable property for such an approximate solution to share with the exact one, is that its total-variation should decrease in time.[1] Difference schemes which give rise to such total variation diminishing solutions—called TVD schemes after Harten [3]—are the subject of this paper.

TVD schemes prevent spurious oscillations in their solutions, and unlike monotone schemes, they can still allow for high accuracy in most of the computational domain. Consequently, the TVD schemes can offer a substantial gain in computational efficiency as indeed was verified in a wide range of applications (e.g., [12], [13] and the references therein).

Sufficient TVD criteria for explicit and implicit fully discrete schemes were given by Harten in [3], [4], and analogously for semidiscrete schemes in [14], [8]. Necessity for three-point schemes was proved in [16, Lemma 2.2] and a general TVD characterization for multipoint stencils was provided in [5], [10]. Roughly speaking, these criteria assert that a given scheme has the TVD property, provided it can be written in an appropriate incremental form which meets a certain positivity condition, augmented with a CFL restriction in the explicit case. A difference approximation of (1.1) can be equally represented by a variety of different incremental forms, yet the positivity

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[1] We use the notion of order in its weak sense; thus, decrease means nonincrease, positive refers to nonnegative, etc.

condition mentioned above is not invariant under such different representations. Hence, the key step in seeking the TVD property according to the above criteria, requires us to find a particular incremental in which this positivity condition holds (e.g., [10, Part I]).

In this paper, we provide alternative, more convenient TVD characterizations, in the sense that they are uniformly valid for the various different incremental representations of a given scheme. To this end, we first note that the total variation of a grid function depends solely on its extreme values (see (1.2)). It is therefore plausible to assert that in order for a difference scheme to share the TVD property, its incremental coefficients should be controlled only at critical neighborhoods where the approximate grid solution attains extreme values. Indeed, our sufficient TVD conditions have the flavor of this assertion, namely, they place a positivity restriction only on those incremental coefficients which are associated with such critical neighborhoods. Moreover, we are also able to express these local TVD conditions solely in terms of the numerical fluxes of the nonlinear schemes, rather than invoking any of their special incremental decompositions. Putting it in different words, we show that the TVD property holds for difference schemes whose numerical viscosity corresponds to upwind differencing at extreme values but can be arbitrary otherwise. Thus, in contrast to the more restrictive *global* positivity conditions mentioned earlier, our TVD criteria are *localized* to extreme values, and consequently can be equally applied to different incremental representations. In fact, the essentially local nature of our criteria enables one to achieve the TVD property, by a simple local modification of quite arbitrary schemes whether stable or not; in Examples 2.7 and 2.8 below, this point is demonstrated with central differencing of arbitrary order of accuracy.

We begin by discussing the semidiscrete case in § 2. Fully discrete implicit and explicit schemes are treated in § 3. To utilize our TVD criteria in the latter cases, the extreme values at the *next* time level are to be known in advance. This necessitates additional ingredients in the fully discrete case, whose purpose is to provide us with such a priori control on the behavior of extreme values at the next time level. In particular, standard recipes of constructing TVD schemes, which make use of anti-diffusive correctors and limiters, are all shown to naturally follow in light of our above arguments.

**2. Semidiscrete schemes.** We consider semidiscrete schemes in the conservative form

$$\frac{d}{dt} v_\nu(t) = -\frac{1}{\Delta x_\nu}[h_{\nu+1/2} - h_{\nu-1/2}], \tag{2.1}$$

with $\Delta x_\nu \equiv \frac{1}{2}(x_{\nu+1} - x_{\nu-1})$ being the variable meshsize and $h_{\nu+1/2}$ denotes the Lipschitz continuous numerical flux which is consistent with the differential one

$$h_{\nu+1/2} = h(v_{\nu-p+1}, \cdots, v_{\nu+p}), \qquad h(u, u, \cdots, u) = f(u). \tag{2.2}$$

To study the TVD property of these schemes, we forward difference (2.1),

$$\frac{d}{dt} \Delta v_{\nu+1/2} = -\frac{1}{\Delta x_{\nu+1}}[h_{\nu+3/2} - h_{\nu+1/2}] + \frac{1}{\Delta x_\nu}[h_{\nu+1/2} - h_{\nu-1/2}], \tag{2.3}$$

multiply by $s_{\nu+1/2} \equiv s_{\nu+1/2}(t) = \operatorname{sgn}[\Delta v_{\nu+1/2}(t)]$, and sum by parts, obtaining

$$\frac{d}{dt} TV[v(t)] = \sum_\nu \frac{1}{\Delta x_\nu}[s_{\nu+1/2} - s_{\nu-1/2}] \cdot [h_{\nu+1/2} - h_{\nu-1/2}]. \tag{2.4}$$

The only contributions to the sum on the right came from extreme values where $s_{\nu+1/2} \neq s_{\nu-1/2}$, and the requirement of these contributions to be negative yields the following result.

LEMMA 2.1. *The semidiscrete scheme* (2.1) *is* TVD, *if we have*

$$h_{\nu+1/2} \geqq h_{\nu-1/2} \quad \text{at maximum values } v_\nu(t), \tag{2.5a}$$

$$h_{\nu+1/2} \leqq h_{\nu-1/2} \quad \text{at minimum values } v_\nu(t). \tag{2.5b}$$

In other words, Lemma 2.1 requires maximum values to decrease in time and minimum values to increase in time. Moreover, if the distance between such extreme values exceeds the stencil width of $2p+1$ cells, then the corresponding terms inside the summation on the right of (2.4) are independent and consequently (2.5) is also *necessary* for TVD in this case.

We now turn to discuss the relation between the TVD criteria in Lemma 2.1 and a different kind of TVD conditions due to Harten [3], [4] and Osher [8]; see also [5], [14]. In order to implement the latter, we should start with nonlinear semidiscrete schemes which assume the incremental form

$$\frac{d}{dt} v_\nu(t) = C^+_{\nu+1/2}\Delta v_{\nu+1/2} - C^-_{\nu-1/2}\Delta v_{\nu-1/2}, \qquad \Delta v_{\nu+1/2} \equiv v_{\nu+1}(t) - v_\nu(t). \tag{2.6}$$

The nonlinearity is reflected here by the possible dependence of the coefficients $C^\pm_{\nu+1/2}$ on $v_{\nu-p+1}, \cdots, v_{\nu+p}$. Forward differencing of (2.6) gives

$$\frac{d}{dt}\Delta v_{\nu+1/2} = (C^+_{\nu+3/2}\Delta v_{\nu+3/2} - C^+_{\nu+1/2}\Delta v_{\nu+1/2}) - (C^-_{\nu+1/2}\Delta v_{\nu+1/2} - C^-_{\nu-1/2}\Delta v_{\nu-1/2}). \tag{2.7}$$

Multiplying (2.7) by $s_{\nu+1/2}$ and summing by parts we find

$$\begin{aligned} \frac{d}{dt}\sum_\nu |\Delta v_{\nu+1/2}| &= \sum_\nu s_{\nu+1/2} \cdot \frac{d}{dt}\Delta v_{\nu+1/2} \\ &= -\sum_\nu [(s_{\nu+1/2} - s_{\nu-1/2}) \cdot C^+_{\nu+1/2} + (s_{\nu+1/2} - s_{\nu+3/2}) \cdot C^-_{\nu+1/2}]\Delta v_{\nu+1/2}, \end{aligned} \tag{2.8}$$

and using the fact that[2] $\Delta v_{\nu+1/2} = s_{\nu+1/2} \cdot |\Delta v_{\nu+1/2}|$ where $s^2_{\nu+1/2} \equiv 1$, we end up with

$$\frac{d}{dt} TV[v(t)] = -\sum_\nu [(1 - s_{\nu-1/2}s_{\nu+1/2}) \cdot C^+_{\nu+1/2} + (1 - s_{\nu+1/2}s_{\nu+3/2})C^-_{\nu+1/2}] \cdot |\Delta v_{\nu+1/2}|. \tag{2.9}$$

The quantities inside the two round brackets on the right are equal either to 0 or 2. Hence, the summation on the right is positive and consequently the scheme (2.6) is TVD, provided the incremental coefficients, $C^\pm_{\nu+1/2}$, are positive

$$C^+_{\nu+1/2} \geqq 0, \qquad C^-_{\nu+1/2} \geqq 0. \tag{2.10}$$

The positivity requirement (2.10) is the usual condition which characterizes the TVD schemes (2.6), e.g., [3], [5], [8], [14]. Given a semidiscrete conservative approximation of (1.1), it can be equally represented in a variety of different incremental forms. The positivity condition is not invariant, however, under such different representations. Thus, a key step in seeking the TVD property for a given scheme requires us to find a particular incremental form in which this positivity condition, (2.10), holds (e.g., [11]).

[2] The signum function at zero is defined to be $\pm 1$, so that its square equals 1.

Lemma 2.1 provides us with a local TVD criterion which makes no reference to the incremental representation of the scheme (2.1). How does this compare with the global positivity condition placed on the incremental coefficients in (2.10)? A second glance at (2.9) shows that whenever the grid values $v_\nu$ and $v_{\nu+1}$ are located in a monotone profile, i.e., when both $1-s_{\nu-1/2}s_{\nu+1/2}$ and $1-s_{\nu+1/2}s_{\nu+3/2}$ vanish, then the corresponding term in the summation on the right of (2.9) also vanishes *independently* of the incremental coefficients $C^{\pm}_{\nu+1/2}$. This tells us, therefore, that the positivity condition (2.10) can be localized to extreme values, bearing a close similarity to the local nature of the TVD criterion (2.5). To be more precise, let us abbreviate

$$\chi_\nu = 1 - s_{\nu-1/2}s_{\nu+1/2}; \tag{2.11}$$

then (2.9) reads

$$\frac{d}{dt} TV[v(t)] = -\sum_\nu [\chi_\nu C^+_{\nu+1/2} + \chi_{\nu+1} C^-_{\nu+1/2}] \cdot |\Delta v_{\nu+1/2}|, \tag{2.12}$$

and we are led to the following.

LEMMA 2.2. *The semidiscrete scheme* (2.6) *is* TVD, *if we have*

$$\chi_\nu C^+_{\nu+1/2} + \chi_{\nu+1} C^-_{\nu+1/2} \geqq 0. \tag{2.13}$$

For smooth grid functions, we have almost everywhere (i.e., with the exception of critical neighborhoods), $\chi_\nu = \chi_{\nu+1} = 0$; hence the TVD condition (2.13) is automatically fulfilled in these cases.

Once the positivity condition (2.10) is localized to those incremental coefficients associated with extreme values (2.13), we can go one step further and complete the comparison with Lemma 2.1, dealing with the numerical fluxes instead.

To this end, the scheme (2.1) is rewritten in its canonical incremental representation (2.6) where (see [16, § 2])

$$C^+_{\nu+1/2} = \frac{1}{\Delta x_\nu} \cdot \frac{f(v_\nu) - h_{\nu+1/2}}{\Delta v_{\nu+1/2}}, \quad C^-_{\nu-1/2} = \frac{1}{\Delta x_\nu} \cdot \frac{f(v_\nu) - h_{\nu-1/2}}{\Delta v_{\nu-1/2}}, \qquad \Delta v_{\nu+1/2} \neq 0. \tag{2.14}$$

Applying Lemma 2.2 to these coefficients, then (2.13) reads

$$\frac{\chi_\nu}{\Delta x_\nu} \cdot \frac{f(v_\nu) - h_{\nu+1/2}}{\Delta v_{\nu+1/2}} + \frac{\chi_{\nu+1}}{\Delta x_{\nu+1}} \cdot \frac{f(v_{\nu+1}) - h_{\nu+1/2}}{\Delta v_{\nu+1/2}} \geqq 0, \tag{2.15a}$$

or, equivalently,

$$\frac{\chi_\nu}{\Delta x_\nu} \cdot s_{\nu+1/2}(h_{\nu+1/2} - f(v_\nu)) + \frac{\chi_{\nu+1}}{\Delta x_{\nu+1}} \cdot s_{\nu+1/2}(h_{\nu+1/2} - f(v_{\nu+1})) \leqq 0. \tag{2.15b}$$

Hence, in case of an isolated extrema value where $\chi_\nu = 2$, $\chi_{\nu\pm1} = 0$, the inequality (2.15) is fulfilled if and only if

$$s_{\nu+1/2}(h_{\nu+1/2} - f(v_\nu)) \leqq 0, \tag{2.16a}$$

$$s_{\nu-1/2}(h_{\nu-1/2} - f(v_\nu)) \leqq 0. \tag{2.16b}$$

In fact, (2.16) covers the general case of extreme values whether isolated or not. For, if $\chi_\nu = \chi_{\nu+1} = 2$, then, since $v_{\nu+1}$ is also an extreme value, we have in view of (2.16b)

$$s_{\nu+1/2}(h_{\nu+1/2} - f(v_{\nu+1})) \leqq 0, \tag{2.17}$$

and a weighted average of (2.16a), (2.17) yields (2.15).

We summarize this by stating the following corollary.

COROLLARY 2.3. *The semidiscrete scheme* (2.1) *is* TVD, *if we have*

(2.18a) $$h_{\nu+1/2} \geqq f(v_\nu) \geqq h_{\nu-1/2} \quad \textit{at maximum values } v_\nu(t),$$

(2.18b) $$h_{\nu+1/2} \leqq f(v_\nu) \leqq h_{\nu-1/2} \quad \textit{at minimum values } v_\nu(t).$$

The TVD condition (2.18), which was derived on the basis of the incremental decomposition (2.14), is somewhat more stringent than our TVD criterion (2.5) in that the former requires $f(v_\nu)$ to separate between the numerical fluxes on both sides of extreme values. Incremental decompositions of (2.1) other than (2.14) may lead to slightly different local TVD conditions; yet, they all share a similar kind of a separation requirement at extreme values, which in view of the consistency relation (2.2) is a generic property of the TVD numerical fluxes.

Lemma 2.1 and Corollary 2.3 enable one to verify the TVD property of first as well as higher order accurate semidiscrete schemes, without making reference to any of their special incremental representations. To demonstrate this point, we turn to our first example.

*Example* 2.4. Consider the class of generalized MUSCL schemes [9], where

(2.19) $$h_{\nu+1/2} = h^E\left(v_\nu + \frac{\Delta x}{2} d_\nu,\ v_{\nu+1} - \frac{\Delta x}{2} d_{\nu+1}\right).$$

Here, $\Delta x_\nu \equiv \Delta x$ is the uniform mesh spacing, $h^E(\cdot, \cdot)$ stands for any $E$-flux [8], satisfying

(2.20) $$\operatorname{sgn}(w_{\nu+1} - w_\nu) \cdot (h^E(w_\nu, w_{\nu+1}) - f(w)) \leqq 0$$

for all $w$ between $w_\nu$ and $w_{\nu+1}$, and $d_\nu$ is an approximate derivative at $x_\nu$ which guarantees second-order accuracy if chosen so that

(2.21) $$\Delta d_{\nu\pm1/2} = \frac{1}{\Delta x} \cdot [\Delta v_{\nu+1/2} - \Delta v_{\nu-1/2}] + O(\Delta x)^2.$$

In [9, Lemma 2.3], Osher introduces a special incremental decomposition of these schemes in order to show that they meet the positivity condition (2.10) and hence share the TVD property, provided for each $\nu$ we have

(2.22) $$0 \leqq \frac{\Delta x}{\Delta v_{\nu\pm1/2}} \cdot d_\nu \leqq 1.$$

Note that in the particular case of $v_\nu$ being an extreme value, (2.22) implies that $d_\nu$ must vanish and, consequently, that accuracy degenerates to first order at these points.

In contrast to the special positivity arguments made above, Lemma 2.1 suggests a straightforward TVD derivation in this case. Localized at extreme values, we set $d_\nu = 0$, so that in view of the $E$-condition (2.20), TVD is guaranteed if in addition we have

(2.23a) $$\operatorname{sgn}(v_{\nu+1} - v_\nu) = \operatorname{sgn}\left(v_{\nu+1} - v_\nu - \frac{\Delta x}{2} d_{\nu+1}\right),$$

(2.23b) $$\operatorname{sgn}(v_\nu - v_{\nu-1}) = \operatorname{sgn}\left(v_\nu - v_{\nu-1} - \frac{\Delta x}{2} d_{\nu-1}\right),$$

i.e., if the neighboring discrete derivatives of extreme values satisfy

(2.24) $$\frac{1}{2}\left|\frac{\Delta x}{\Delta v_{\nu-1/2}} d_{\nu-1}\right| \leqq 1, \qquad \frac{1}{2}\left|\frac{\Delta x}{\Delta v_{\nu+1/2}} d_{\nu+1}\right| \leqq 1.$$

One possible choice for such discrete derivatives, $d_\nu$, could be

$$d_\nu = \frac{s_\nu}{\Delta x} \cdot \text{Min}\,[|\Delta v_{\nu-1/2}|, |\Delta v_{\nu+1/2}|], \qquad s_\nu \equiv \frac{1}{2}[s_{\nu-1/2} + s_{\nu+1/2}]. \tag{2.25}$$

The above TVD analysis relies on the conservative form of nonlinear difference schemes. We now turn to another representation which is useful for utilizing TVD criteria for such schemes, making use of this viscosity form. To this end, recall the definition of the incremental coefficients, $C^{\pm}_{\nu+1/2}$, in (2.14). The identity

$$\Delta x_{\nu+1} C^{-}_{\nu+1/2} - \Delta x_\nu C^{+}_{\nu+1/2} = \frac{\Delta f_{\nu+1/2}}{\Delta v_{\nu+1/2}}, \qquad \Delta f_{\nu+1/2} \equiv f(v_{\nu+1}) - f(v_\nu), \tag{2.26}$$

shows that between these two incremental coefficients, $C^{\pm}_{\nu+1/2}$, there is only one degree of freedom, which could be expressed in terms of $Q_{\nu+1/2}$,

$$Q_{\nu+1/2} = \Delta x_{\nu+1} C^{-}_{\nu+1/2} + \Delta x_\nu C^{+}_{\nu+1/2}. \tag{2.27}$$

Eliminating $C^{\pm}_{\nu+1/2}$ from (2.26) and (2.27) we find

$$C^{-}_{\nu+1/2} = \frac{1}{2\Delta x_{\nu+1}}\left(Q_{\nu+1/2} + \frac{\Delta f_{\nu+1/2}}{\Delta v_{\nu+1/2}}\right), \qquad C^{+}_{\nu+1/2} = \frac{1}{2\Delta x_\nu}\left(Q_{\nu+1/2} - \frac{\Delta f_{\nu+1/2}}{\Delta v_{\nu+1/2}}\right). \tag{2.28}$$

Our scheme (2.1) is then recast into the viscosity form [17]

$$\frac{d}{dt} v_\nu(t) = -\frac{1}{2\Delta x_\nu}[f(v_{\nu+1}) - f(v_{\nu-1}) + \frac{1}{2\Delta x_\nu}[Q_{\nu+1/2}\Delta v_{\nu+1/2} - Q_{\nu-1/2}\Delta v_{\nu-1/2}], \tag{2.29}$$

thus revealing the role $Q$ plays as the numerical viscosity coefficient. Applying (2.13) to $C^{\pm}_{\nu+1/2}$ given in (2.28), Lemma 2.2 gives us the following lemma.

LEMMA 2.5. *The scheme* (2.29) *is* TVD, *if its viscosity satisfies*

$$(\chi_\nu \Delta x_{\nu+1} + \chi_{\nu+1}\Delta x_\nu) Q_{\nu+1/2} \geqq (\chi_\nu \Delta x_{\nu+1} - \chi_{\nu+1}\Delta x_\nu)\frac{\Delta f_{\nu+1/2}}{\Delta v_{\nu+1/2}}. \tag{2.30}$$

In the case of equally spaced meshpoints, $\Delta x_\nu \equiv \Delta x$, we conclude that the scheme

$$\frac{d}{dt} v_\nu(t) = -\frac{1}{2\Delta x}[f(v_{\nu+1}) - f(v_{\nu-1})] + \frac{1}{2\Delta x}[Q_{\nu+1/2}\Delta v_{\nu+1/2} - Q_{\nu-1/2}\Delta v_{\nu-1/2}], \tag{2.31}$$

is TVD, provided the following simple inequality is fulfilled at the neighborhood of extreme values

$$(\chi_\nu + \chi_{\nu+1}) Q_{\nu+1/2} \geqq (\chi_\nu - \chi_{\nu+1}) \frac{\Delta f_{\nu+1/2}}{\Delta v_{\nu+1/2}}. \tag{2.32}$$

*Example* 2.6. Consider a first-order accurate TVD scheme of the form

$$\begin{aligned} \frac{d}{dt} v_\nu(t) &= -\frac{1}{\Delta x_\nu}[h_{\nu+1/2} - h_{\nu-1/2}] \\ &= \frac{1}{2\Delta x_\nu}[f(v_{\nu+1}) - f(v_{\nu-1})] + \frac{1}{2\Delta x_\nu}[Q_{\nu+1/2}\Delta v_{\nu+1/2} - Q_{\nu-1/2}\Delta v_{\nu-1/2}]. \end{aligned} \tag{2.33}$$

In order to convert it into a second-order accurate TVD scheme, we add to it an antidiffusive conservative difference

$$\frac{d}{dt} v_\nu(t) = -\frac{1}{\Delta x_\nu}[h_{\nu+1/2} - h_{\nu-1/2}] - \frac{1}{\Delta x_\nu}[\tilde{h}_{\nu+1/2} - \tilde{h}_{\nu-1/2}]. \tag{2.34a}$$

The numerical flux correction, $\tilde{h}_{\nu+1/2}$, is chosen to be of the form [10, Cor. 4.9]

$$\tilde{h}_{\nu+1/2}=\tfrac{1}{2}[\tilde{g}_\nu+\tilde{g}_{\nu+1}-s_{\nu+1/2}|\tilde{g}_{\nu+1}-\tilde{g}_\nu|], \tag{2.34b}$$

where the so-called modified flux correction, $\tilde{g}_\nu$, should satisfy

(i) $\operatorname{sgn}(\tilde{g}_\nu)=s_{\nu-1/2}=s_{\nu+1/2}$ at nonextreme values $v_\nu(t)$,

(ii) $\tilde{g}_\nu=0$ at extreme values $v_\nu(t)$.

Now, if $v_\nu(t)$ or $v_{\nu+1}(t)$ is an extreme value, then by (ii) we have that $\tilde{g}_\nu=0$ or $\tilde{g}_{\nu+1}=0$, and consequently $\tilde{h}_{\nu+1/2}$ vanishes in both cases since by (i)

$$\tilde{h}_{\nu+1/2}=\tfrac{1}{2}[\tilde{g}_{\nu+1}-s_{\nu+1/2}|\tilde{g}_{\nu+1}|]=0 \quad \text{or} \quad h_{\nu+1/2}=\tfrac{1}{2}[\tilde{g}_\nu-s_{\nu+1/2}|\tilde{g}_\nu|]=0.$$

Hence, $\tilde{h}_{\nu+1/2}=\tilde{h}_{\nu-1/2}=0$ at extreme values $v_\nu(t)$, and in view of Lemma 2.1, the modified scheme (2.34) inherits the TVD property of (2.33). Next we observe that the modification of (2.33) into (2.34) has the net effect of *decreasing* the original first-order viscosity, $Q_{\nu+1/2}$, into $Q_{\nu+1/2}-2(\tilde{h}_{\nu+1/2}/\Delta v_{\nu+1/2})$; for second-order accuracy [18, Lemma 4.4], the latter should be a Lipschitz continuous grid function of order $O(|\Delta v_{\nu+1/2}|)$, i.e.,

$$\tilde{h}_{\nu+1/2}=\tfrac{1}{2}Q_{\nu+1/2}\Delta v_{\nu+1/2}+O(\Delta v_{\nu+1/2})^2.$$

To this end, one could choose the modified flux correction, $\tilde{g}_\nu$, as

$$\tilde{g}_\nu=\frac{s_\nu}{2}\cdot \operatorname{Min}\,[|Q_{\nu+1/2}\Delta v_{\nu+1/2}|, |Q_{\nu-1/2}\Delta v_{\nu-1/2}|], \qquad s_\nu\equiv\frac{1}{2}[s_{\nu-1/2}+s_{\nu+1/2}]. \tag{2.34c}$$

In this way, second-order accuracy is achieved away from extreme values, noting that $\tilde{h}_{\nu+1/2}$ takes the value $\tilde{g}_\nu$ or $\tilde{g}_{\nu+1}$ whose modulo quadratic error terms are equal to $\tfrac{1}{2}Q_{\nu+1/2}\Delta v_{\nu+1/2}$.

*Example* 2.7. A simple recipe suggested by (2.32), for constructing a TVD scheme with second-order accuracy away from extreme values, is to set the numerical viscosity $Q$ to be

$$Q_{\nu+1/2}=(\chi_\nu-\chi_{\nu+1})\frac{\Delta f_{\nu+1/2}}{\Delta v_{\nu+1/2}}. \tag{2.35}$$

The resulting scheme (2.31), (2.35), amounts to the usual second-order central differencing augmented with first-order conservative correction at extreme values

$$\frac{d}{dt}v_\nu(t)=-\frac{1}{2\Delta x}[f(v_{\nu+1})-f(v_{\nu-1})]-\frac{1}{2\Delta x}[\Delta\chi_{\nu+1/2}\Delta f_{\nu+1/2}-\Delta\chi_{\nu-1/2}\Delta f_{\nu-1/2}]. \tag{2.36}$$

The last two examples dealt with TVD schemes which are second-order accurate away from extreme values. In our final example for this section, we demonstrate a simple recipe of enforcing the TVD property on *arbitrary* conservative schemes while maintaining their high accuracy away from extreme values (see also [10]).

*Example* 2.8. Let $h_{\nu+1/2}$ be any highly accurate consistent flux. For example, the $2p$th order accurate central differencing is identified with the numerical flux (e.g., [19])

$$h_{\nu+1/2}=\sum_{k=1}^{p} d_{kp}[f(v_{\nu+k})+\cdots+f(v_{\nu-k+1})],$$

where

$$d_{11}=\tfrac{1}{2}, \quad d_{12}=\tfrac{2}{3}, \quad d_{22}=-\tfrac{1}{12}$$

for second- and fourth-order accuracy, or

$$d_{kN}=-\frac{(-1)^k\Delta x}{2\sin(k\Delta x/2)},\qquad \Delta x\equiv\frac{2\pi}{2N+1}$$

for spectral accuracy occupying periodic stencils of $2N+1$ meshpoints.

Next, we denote by

$$h^U_{\nu+1/2}=\tfrac{1}{2}[f(v_\nu)+f(v_{\nu+1})-s_{\nu+1/2}\cdot|f(v_{\nu+1})-f(v_\nu)|] \tag{2.37}$$

the usual first-order accurate upwind numerical flux, and let us consider the semidiscrete conservative scheme

$$\frac{d}{dt}v_\nu(t)=-\frac{1}{\Delta x_\nu}[H_{\nu+1/2}-H_{\nu-1/2}], \tag{2.38a}$$

where $H_{\nu+1/2}$ is defined as follows:

$$H_{\nu+1/2}=|s_\nu s_{\nu+1}|\cdot h_{\nu+1/2}+(1-|s_\nu s_{\nu+1}|)h^U_{\nu+1/2}, \tag{2.38b}$$

$$s_\nu\equiv\tfrac{1}{2}(s_{\nu-1/2}+s_{\nu+1/2}).$$

Away from extreme values $|s_\nu|=|s_{\nu\pm1}|=1$, and the original high accuracy of $H_{\nu+1/2}=h_{\nu+1/2}$ is retained in those regions. At extreme values $s_\nu=0$, hence $H_{\nu+1/2}$ coincides with the upwind flux $h^U_{\nu+1/2}$, satisfying

$$h^U_{\nu+1/2}-h^U_{\nu-1/2}=\mathrm{Max}\,[f(v_\nu),f(v_{\nu+1})]-\mathrm{Min}\,[f(v_{\nu-1}),f(v_\nu)]\geqq 0,$$

at maximum values, and the inverse inequality

$$h^U_{\nu+1/2}-h^U_{\nu-1/2}=\mathrm{Min}\,[f(v_\nu),f(v_{\nu+1})]-\mathrm{Max}\,[f(v_{\nu-1}),f(v_\nu)]\leqq 0,$$

at minimum values. Consequently, the scheme (2.38) is TVD by Lemma 2.1.

We note that the numerical flux $H_{\nu+1/2}$ in (2.38b) is in general not smooth, except for the second-order case, $p=1$, where the scheme (2.38) coincides with the previous example (2.36) and its global second-order accuracy is maintained (e.g., [3], [4], [10]). The highly accurate stencils, $p>1$, require further numerical and analytical investigation with regard to their accumulated accuracy in extrema free regions.

*Remarks.* (i) It is instructive to see why the necessary and sufficient conservative TVD criterion in Lemma 2.1 is reduced to the sufficient incremental TVD conditions derived from (2.12). To this end, let us insert the incremental coefficients (2.14) into (2.12) obtaining

$$\frac{d}{dt}\mathrm{TV}[v(t)]=-\sum_\nu\left[\frac{\chi_\nu}{\Delta x_\nu}\cdot\frac{f(v_\nu)-h_{\nu+1/2}}{\Delta v_{\nu+1/2}}+\frac{\chi_{\nu+1}}{\Delta x_{\nu+1}}\cdot\frac{f(v_{\nu+1})-h_{\nu+1/2}}{\Delta v_{\nu+1/2}}\right]\cdot|\Delta v_{\nu+1/2}|.$$

Now, Lemma 2.2 and Corollary 2.3 were derived by requiring a *termwise* positivity of the brackets inside the summation on the right (see (2.15)). Instead, if we first reindex this summation writing it as

$$-\sum_\nu\frac{\chi_\nu}{\Delta x_\nu}\cdot[s_{\nu+1/2}(f(v_\nu)-h_{\nu+1/2})+s_{\nu-1/2}(f(v_\nu)-h_{\nu-1/2})],$$

we then end up with the necessary and sufficient TVD criterion (2.5). This makes apparent the difference between the two derivations due to the nonlinearity of the schemes.

(ii) The inequality (2.32) shows that the scheme (2.31) has the TVD property with an arbitrary amount of viscosity, except for intervals containing isolated extreme values where we need at least

$$Q_{\nu+1/2} \geqq \left|\frac{\Delta f_{\nu+1/2}}{\Delta v_{\nu+1/2}}\right|. \tag{2.39}$$

The quantity on the right corresponds to upwind differencing, and is responsible for the familiar first-order "clipping" phenomenon at the extreme of TVD schemes (e.g., [3], [8], [11]).

(iii) A classical argument which involves Helly's theorem, Lipschitz continuity of $|v(\cdot, t)|_{L^1}$, and the diagonal process implies the convergence of TVD schemes to a weak solution of (1.1) (e.g., [4]). In particular, this is true for central differencing, $Q_{\nu+1/2}=0$, augmented with extreme upwind differencing (2.36). However, the limit solution may still be a physically irrelevant one (e.g., [9]). To avoid the latter, say in the convex case where $f''(u)>0$, it is enough to have viscosity at the amount which exceeds [18]

$$Q_{\nu+1/2} \geqq \tfrac{1}{6}\cdot f''(v_{\nu+1})\cdot \Delta v^+_{\nu+1/2}, \qquad \Delta v^+_{\nu+1/2}=\tfrac{1}{2}(\Delta v_{\nu+1/2}+|\Delta v_{\nu+1/2}|). \tag{2.40}$$

Thus, central differencing will do along the monotone decreasing profiles, and an additional $O(|\Delta v_{\nu+1/2}|)$ amount of viscosity is required along the monotone increasing ones.

**3. Fully discrete schemes.** We consider two-level fully discrete explicit or implicit schemes in the conservative form

$$v_\nu(t+\Delta t)=v_\nu(t)-\lambda_\nu[h_{\nu+1/2}-h_{\nu-1/2}]. \tag{3.1}$$

Here, $\Delta x_\nu \equiv \frac{1}{2}(x_{\nu+1}-x_{\nu-1})$ and $\Delta t$ are the variable meshsize and timestep such that $\lambda_\nu \equiv \Delta t/\Delta x_\nu$, and $h_{\nu+1/2}$ is the consistent Lipschitz continuous numerical flux which depends on $2p+1$ neighboring gridvalues from both time-levels, $t$ and $t+\Delta t$.

To study the TVD properties of these schemes, we forward difference (3.1)

$$\Delta v_{\nu+1/2}(t+\Delta t)=\Delta v_{\nu+1/2}(t)-\lambda_{\nu+1}[h_{\nu+3/2}-h_{\nu+1/2}]+\lambda_\nu[h_{\nu+1/2}-h_{\nu-1/2}], \tag{3.2}$$

multiply by $s_{\nu+1/2}(t+\Delta t)\equiv \operatorname{sgn}[\Delta v_{\nu+1/2}(t+\Delta t)]$ and sum by parts, obtaining

$$\begin{aligned} \mathrm{TV}[v(t+\Delta t)] &= \sum_\nu s_{\nu+1/2}(t+\Delta t)\cdot \Delta v_{\nu+1/2}(t) \\ &\quad +\sum_\nu \lambda_\nu\cdot[s_{\nu+1/2}(t+\Delta t)-s_{\nu-1/2}(t+\Delta t)]\cdot[h_{\nu+1/2}-h_{\nu-1/2}]. \end{aligned} \tag{3.3}$$

The first summation on the right does not exceed $\mathrm{TV}[v(t)]$, and the requirement for the second one to be negative yields the following result.

LEMMA 3.1. *The fully discrete scheme* (3.1) *is* TVD, *if we have*

$$h_{\nu+1/2}\geqq h_{\nu-1/2} \quad \textit{at maximum values } v_\nu(t+\Delta t), \tag{3.4a}$$

$$h_{\nu+1/2}\leqq h_{\nu-1/2} \quad \textit{at minimum values } v_\nu(t+\Delta t). \tag{3.4b}$$

Lemma 3.1 is a manifestation of our previous assertion, namely, that the TVD properties of conservative schemes are determined solely by the behavior of their numerical fluxes at extreme values. Yet, unlike the semidiscrete case we had previously, here there is the additional difficulty of tracing these unknown extreme values at the next time-level, $t+\Delta t$.

A similar situation occurs with the incremental representations of fully discrete nonlinear schemes. Consider, for example, two-level explicit schemes in the incremental form

$$v_\nu(t+\Delta t)=v_\nu(t)+C^+_{\nu+1/2}\Delta v_{\nu+1/2}(t)-C^-_{\nu-1/2}\Delta v_{\nu-1/2}(t), \tag{3.5}$$

with $C^{\pm}_{\nu+1/2} \equiv C^{\pm}_{\nu+1/2}(v_{\nu-p+1}(t), \cdots, v_{\nu+p}(t))$. To study the TVD properties of such schemes, we forward difference (3.5), multiply by $s_{\nu+1/2}(t+\Delta t)$, and sum by parts, obtaining

$$\begin{aligned} \text{TV}[v(t+\Delta t)] &= \sum_\nu s_{\nu+1/2}(t+\Delta t) \cdot \Delta v_{\nu+1/2}(t) \\ &\quad - \sum_\nu s_{\nu+1/2}(t+\Delta t) s_{\nu+1/2}(t) [\chi_\nu(t+\Delta t) C^+_{\nu+1/2} + \chi_{\nu+1}(t+\Delta t) C^-_{\nu+1/2}] \cdot |\Delta v_{\nu+1/2}(t)|. \end{aligned} \tag{3.6}$$

Since the first summation on the right does not exceed $\text{TV}[v(t)]$, we arrive at the following sufficient TVD condition.

LEMMA 3.2. *The explicit scheme* (3.5) *is* TVD, *if we have*

$$\chi_\nu(t+\Delta t) C^+_{\nu+1/2} + \chi_{\nu+1}(t+\Delta t) C^-_{\nu+1/2} \geqq 0 \quad \textit{when } s_{\nu+1/2}(t+\Delta t) = s_{\nu+1/2}(t), \tag{3.7a}$$

$$\chi_\nu(t+\Delta t) C^+_{\nu+1/2} + \chi_{\nu+1}(t+\Delta t) C^-_{\nu+1/2} \leqq 2 \quad \textit{when } s_{\nu+1/2}(t+\Delta t) \neq s_{\nu+1/2}(t). \tag{3.7b}$$

Applying the last result to the incremental coefficients (compare (2.28)),

$$C^{\pm}_{\nu+1/2} = \frac{1}{2}(Q_{\nu+1/2} \pm \lambda a_{\nu+1/2}), \qquad a_{\nu+1/2} \equiv \frac{\Delta f_{\nu+1/2}}{\Delta v_{\nu+1/2}}, \tag{3.8}$$

we find that for equally spaced explicit schemes given in the viscosity form

$$v_\nu(t+\Delta t) = v_\nu(t) - \frac{\lambda}{2}[f(v_{\nu+1}) - f(v_{\nu-1})] + \frac{1}{2}[Q_{\nu+1/2}\Delta v_{\nu+1/2} - Q_{\nu-1/2}\Delta v_{\nu+1/2}], \qquad \lambda_\nu \equiv \lambda, \tag{3.9}$$

the following TVD characterization holds.

LEMMA 3.3. *The explicit scheme* (3.9) *is* TVD *if its viscosity coefficient satisfies*

$$[\chi_\nu(t+\Delta t) + \chi_{\nu+1}(t+\Delta t)] \cdot Q_{\nu+1/2} \geqq [\chi_\nu(t+\Delta t) - \chi_{\nu+1}(t+\Delta t)] \cdot \lambda a_{\nu+1/2}, \tag{3.10a}$$

*and the following* CFL-*like condition is fulfilled*

$$\text{Max}\,[Q_{\nu+1/2}, \lambda |a_{\nu+1/2}|] \leqq 1. \tag{3.10b}$$

Thus, we conclude that the TVD property of either scheme, (3.5) or (3.9), is determined by the behavior of their incremental and viscosity coefficients at extreme values, but, as before, the difficulty lies in obtaining a priori knowledge about these values at time level $t+\Delta t$.

The inequality $0 \leqq \chi \leqq 2$ suggests one way of avoiding this difficulty, namely, to replace (3.7) by the simpler positivity condition

$$C^+_{\nu+1/2} \geqq 0, \qquad C^-_{\nu+1/2} \geqq 0, \tag{3.11a}$$

together with a CFL restriction

$$C^+_{\nu+1/2} + C^-_{\nu+1/2} \leqq 1. \tag{3.11b}$$

Yet, the simplicity of this sufficient TVD condition, which is originally due to Harten [3], [5], [9], [15], is obtained at the expense of its *global* dependence on the special incremental form being used.

Another attractive approach to circumvent the difficulty of tracing the next time-level extreme values is to view the scheme (3.1) just as a first predictor step. Then, the resulting spatial variation at time-level $t+\Delta t$ can be made the basis for an augmenting corrector step which will preserve the monotonicity of the predictor step and which will comply with (3.4). Such an argument was used in connection with the FCT algorithm [1] and the ACM method [2]. In the following example, borrowed from

[11, Cor. 4.9], we work out another corrective-type recipe of this kind, which highlights the essential features distinguishing the fully discrete explicit case from the semidiscrete one.

*Example* 3.4. Consider an explicit first-order accurate TVD scheme of the form

$$\begin{aligned} v_\nu^*(t+\Delta t) &= v_\nu(t) - \lambda[h_{\nu+1/2} - h_{\nu-1/2}] \\ &= v_\nu(t) - \frac{\lambda}{2}[f(v_{\nu+1}) - f(v_{\nu-1})] + \frac{1}{2}[Q_{\nu+1/2}\Delta v_{\nu+1/2} - Q_{\nu-1/2}\Delta v_{\nu-1/2}]. \end{aligned} \tag{3.12a}$$

The asterisk indicates the predicted values at time-level $t+\Delta t$. In order to convert this scheme into a second-value accurate TVD one, these values are corrected to second-order accuracy, by augmenting an antidiffusive corrector step of the form

$$v_\nu(t+\Delta t) = v_\nu^*(t+\Delta t) - [\tilde{h}_{\nu+1/2} - \tilde{h}_{\nu-1/2}]. \tag{3.12b}$$

The numerical flux correction, $\tilde{h}_{\nu+1/2}$, is chosen to be (compare Example 2.6),

$$\tilde{h}_{\nu+1/2} = \frac{1}{2}[\tilde{g}_\nu + \tilde{g}_{\nu+1} - s_{\nu+1/2}^*|\tilde{g}_{\nu+1} - \tilde{g}_\nu|], \qquad s_{\nu+1/2}^* \equiv \operatorname{sgn}[\Delta v_{\nu+1/2}^*(t+\Delta t)], \tag{3.12c}$$

where $\tilde{g}_\nu$ should satisfy the two properties

(i) $\operatorname{sgn}(\tilde{g}_\nu) = s_{\nu-1/2}^* = s_{\nu+1/2}^*$ at nonextremum values $v_\nu^*(t+\Delta t)$,

(ii) $\tilde{g}_\nu = 0$ at extremum values $v_\nu^*(t+\Delta t)$.

In addition, we require that the predicted monotonicity should be preserved, i.e.,

(iii) $s_{\nu+1/2}(t+\Delta t) = s_{\nu+1/2}^*(t+\Delta t)$,

so that by the usual summation by parts we have

$$\begin{aligned} \mathrm{TV}[v(t+\Delta t)] &= \sum_\nu s_{\nu+1/2}^*(t+\Delta t)\Delta v_{\nu+1/2}^*(t+\Delta t) \\ &\quad + \sum_\nu [s_{\nu+1/2}^*(t+\Delta t) - s_{\nu-1/2}^*(t+\Delta t)] \cdot [\tilde{h}_{\nu+1/2} - \tilde{h}_{\nu-1/2}]. \end{aligned}$$

Since the first summation on the right equals $\mathrm{TV}[v^*(t+\Delta t)] \leqq \mathrm{TV}[v(t)]$, while the second is nonnegative, consult Example 2.6, the scheme (3.12) is TVD. Next, its second-order accuracy is achieved if

$$\tilde{h}_{\nu+1/2} = \tfrac{1}{2}[Q_{\nu+1/2} - \lambda^2 a_{\nu+1/2}^2] \cdot \Delta v_{\nu+1/2}^*(t+\Delta t) + O[\Delta v_{\nu+1/2}^*(t+\Delta t)]^2.$$

To satisfy this (away from extreme values), and the first two properties listed above, we choose

$$\tilde{g}_\nu = \frac{s_\nu^*}{2} B[(Q_{\nu+1/2} - \lambda^2 a_{\nu+1/2}^2) \cdot |\Delta v_{\nu+1/2}^*|, (Q_{\nu-1/2} - \lambda^2 a_{\nu-1/2}^2) \cdot |\Delta v_{\nu-1/2}^*|] \tag{3.12d}$$

where the Lipschitz continuous form, $B[\cdot, \cdot]$, is yet to be determined so that the third property of monotonicity preserving will be satisfied. To this end, we note that

$$|\tilde{h}_{\nu+1/2}| \leqq \operatorname{Min}[|\tilde{g}_\nu|, |\tilde{g}_{\nu+1}|]$$

and therefore, since $\tilde{h}_{\nu+1/2}$ and $\tilde{h}_{\nu-1/2}$ must agree in sign,

$$|\Delta v_{\nu+1/2}(t+\Delta t) - \Delta v_{\nu+1/2}^*(t+\Delta t)| = |\tilde{h}_{\nu+3/2} - 2\tilde{h}_{\nu+1/2} + \tilde{h}_{\nu-1/2}| \leqq |\tilde{g}_\nu| + |\tilde{g}_{\nu+1}|.$$

Hence, the sum on the right does not exceed $|\Delta v_{\nu+1/2}^*(t+\Delta t)|$, and consequently the monotonicity preserving property holds, provided $B[\cdot, \cdot]$ is chosen as the bilinear limiter form

$$B[w_1, w_2] = \operatorname{Min}[|w_1|, |w_2|]. \tag{3.12e}$$

The last example demonstrates the typical situation with explicit schemes, where the TVD property necessitates one kind or another of a Minmod limiter in order to prevent new extrema values other than those which propagate from time-level $t$.

An implicit version of the above corrective procedure is given in the following example.

*Example* 3.5. Consider an implicit first-order accurate TVD scheme of the form

$$v_\nu^*(t+\Delta t)+\frac{\lambda}{2}[f(v_{\nu+1}^*)-f(v_{\nu-1}^*)]+\frac{1}{2}[Q_{\nu+1/2}\Delta v_{\nu+1/2}^*-Q_{\nu-1/2}\Delta v_{\nu-1/2}^*]=v_\nu(t). \tag{3.13a}$$

We augment it with an antidiffusive fully implicit corrector step of the form

$$v_\nu(t+\Delta t)+[\tilde{h}_{\nu+1/2}-\tilde{h}_{\nu-1/2}]=v_\nu^*(t+\Delta t), \tag{3.13b}$$

where

$$\tilde{h}_{\nu+1/2}=\tfrac{1}{2}[\tilde{g}_\nu+\tilde{g}_{\nu+1}-s_{\nu+1/2}(t+\Delta t)\cdot|\tilde{g}_{\nu+1}-\tilde{g}_\nu|]. \tag{3.13c}$$

Then (3.13b) serves as a second-order accurate solvable correction, if we set

$$\tilde{g}_\nu=\tfrac{1}{2}\cdot s_\nu(t+\Delta t)\cdot \operatorname{Min}\,[(Q_{\nu\pm1/2}+\lambda^2a_{\nu\pm1/2}^2(t+\Delta t))\cdot|\Delta v_{\nu\pm1/2}(t+\Delta t)|]. \tag{3.13d}$$

The resulting scheme (3.13) is TVD under the original (possibly unlimited) CFL condition. Indeed, we have

$$\begin{aligned}\mathrm{TV}[v(t+\Delta t)]&=\sum_\nu s_{\nu+1/2}(t+\Delta t)\Delta v_{\nu+1/2}^*(t+\Delta t)\\&\quad+\sum_\nu[s_{\nu+1/2}(t+\Delta t)-s_{\nu-1/2}(t+\Delta t)]\cdot[\tilde{h}_{\nu+1/2}-\tilde{h}_{\nu-1/2}];\end{aligned}$$

the first summation on the right does not exceed $\mathrm{TV}[v^*(t+\Delta t)]\leqq \mathrm{TV}[v(t)]$, while the second vanishes since $\tilde{h}_{\nu\pm1/2}$ do at extreme values where $s_{\nu+1/2}(t+\Delta t)\neq s_{\nu-1/2}(t+\Delta t)$.

Other recipes for constructing implicit TVD schemes which are second-order accurate away from extreme values are suggested by the following analogue of Lemma 3.3.

LEMMA 3.6. *The implicit scheme given in the viscosity form*

$$\begin{aligned}&v_\nu(t+\Delta t)+\frac{\lambda}{2}[f(v_{\nu+1}(t+\Delta t))-f(v_{\nu-1}(t+\Delta t))]\\&\quad-\frac{1}{2}[Q_{\nu+1/2}\Delta v_{\nu+1/2}(t+\Delta t)-Q_{\nu-1/2}\Delta v_{\nu-1/2}(t+\Delta t)]=v_\nu(t)\end{aligned} \tag{3.14}$$

*is* TVD, *if we have*

$$[\chi_\nu(t+\Delta t)+\chi_{\nu+1}(t+\Delta t)]Q_{\nu+1/2}\geqq[\chi_\nu(t+\Delta t)-\chi_{\nu+1}(t+\Delta t)]\cdot\lambda a_{\nu+1/2}(t+\Delta t). \tag{3.15}$$

We omit the proof and turn to our final example.

*Example* 3.7. The viscosity of the second-order accurate implicit Lax–Wendroff scheme is modified at extreme values, by setting

$$Q_{\nu+1/2}=-\lambda^2a_{\nu+1/2}^2+\frac{1}{2}(\chi_\nu+\chi_{\nu+1})\cdot[\lambda|a_{\nu+1/2}|+\lambda^2a_{\nu+1/2}^2], \tag{3.16}$$

where the quantities on the right are evaluated at time level $t+\Delta t$. The resulting scheme (3.14), (3.16) can be easily checked to satisfy (3.15) and hence is TVD. However, the linearized implicit LW scheme is unconditionally unstable—the amplification factors of its nonconstant modes all lie outside the unit disc [7]. Consequently, the TVD property of (3.16) is achieved by switching to upwind differencing at the extreme of these unstable oscillatory modes, at the expense of lowering the effective overall accuracy.

*Remarks.* We note that the TVD characterization in Lemma 3.1 does not assume the CFL condition; it enters, indirectly, through the requirement of controlling extrema values at the next time-level. Substitution of the canonical incremental decomposition (2.14) into (3.7) reveals that the same is true with respect to the TVD conditions in Lemma 3.2 and 3.3, where the CFL limitation is implicitly contained already in (3.7a) and (3.10a).

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