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A SUPRACONVERGENT SCHEME FOR NONLINEAR HYPERBOLIC SYSTEMS

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Abstract

Supraconvergent difference schemes are methods for which the local truncation error degrades when applied to an irregular mesh, yet the observed error in the solution is as good as if the mesh were uniform. For a version of the Lax-Wendroff scheme requiring the computation of the Jacobian we show that supraconvergence holds on an arbitrary grid (one-dimensional). For the standard version we present a computation which shows that second order accuracy is not always maintained, but the convergence is better than first-order.

Finite difference schemes for hyperbolic partial differential equations are typically designed and analyzed for a uniform spatial grid, but most often used on a nonuniform grid. It has been observed ([1], [2]) that some schemes are surprisingly accurate on quite irregular grids, the surprise being that since the truncation error is of lower order on such a grid, one would expect the solution error to be correspondingly worse. In [3] we gave an explanation of this phenomenon in some special cases, in particular, for edge-centered and cell-centered versions of the Lax-Wendroff (L-W) scheme applied to constant coefficient equations.

In this note, we will look at a cell-centered L-W scheme which is supraconvergent, that is, it is just as accurate in a sense to be defined on a completely arbitrary grid as on a uniform one for nonlinear hyperbolic equations. This method is not the usual L-W method and requires the computation of the Jacobian in the predictor portion of the method when written in two steps. However, there are indications that the standard L-W method fails to be second order in some cases in which the Jacobian predictor L-W method remains supraconvergent.

The grid and the centering of the unknowns is shown in Figure 1.

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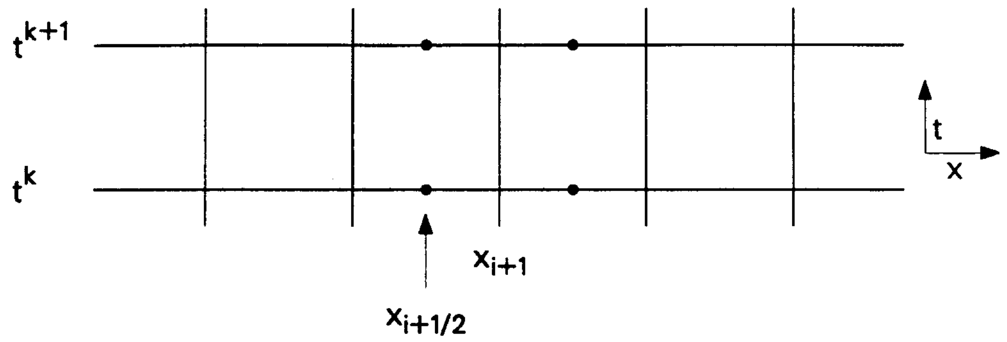

Fig. 1. Grid for difference scheme.

Let

$$\begin{aligned} h_{i+\frac{1}{2}} &= x_{i+1} - x_i \ , \\ \Delta t &= t^{k+1} - t^k \ , \\ h_i &= \frac{1}{2}(h_{i+\frac{1}{2}} + h_{i-\frac{1}{2}}) \ . \end{aligned}$$

The partial differential equation is

$$u_t + f_x(u) = 0 \ , \ u(x,0) \text{ given } . \tag{1}$$

The grid function $U^k_{i+\frac{1}{2}}$ lives on the large dots in Figure 1, and is thought of as approximately $u(x_{i+\frac{1}{2}}, t^k)$.

The difference equation is as follows. Let

$$U_i^k = \frac{h_{i+\frac{1}{2}} U^k_{i-\frac{1}{2}} + h_{i-\frac{1}{2}} U^k_{i+\frac{1}{2}}}{h_{i+\frac{1}{2}} + h_{i-\frac{1}{2}}} \ , \tag{2}$$

$$U_i^{k+\frac{1}{2}} = U_i^k - \frac{\Delta t}{2} f'(U_i^k) \frac{U^k_{i+\frac{1}{2}} - U^k_{i-\frac{1}{2}}}{h_i} \ , \tag{3}$$

and

$$U^{k+1}_{i+\frac{1}{2}} = U^k_{i+\frac{1}{2}} - \Delta t \frac{f(U^{k+\frac{1}{2}}_{i+1}) - f(U_i^{k+\frac{1}{2}})}{h_{i+\frac{1}{2}}} \ . \tag{4}$$

This is an explicit difference scheme relating U^{k+1} to U^k, which is in the form

$$U^{k+1} = \phi(U^k) \ . \tag{5}$$

The truncation error is obtained by replacing $U^k_{i+\frac{1}{2}}$ by $u(x_{i+\frac{1}{2}}, t^k)$, that is, if

$$u^k_{i+\frac{1}{2}} = u(x_{i+\frac{1}{2}}, t^k) \tag{6}$$

then the truncation error is T, where

$$T = u^{k+1} - \phi(u^k) \ . \tag{7}$$

Let

$$h = \max_i h_{i+\frac{1}{2}} \ .$$

Now, if the grid is *uniform* i.e., $h_{i+\frac{1}{2}} = h$ for all i, it is well known that

$$T = \Delta t O(h^2) \ .$$

Strang's theorem [4] provides a sufficient condition for the solution error to be $O(h^2)$; it says that if a certain linearization of the difference operator ϕ is L_2-stable, and if the differential equation and its solution are sufficiently smooth, then

$$|U^k_{i+\frac{1}{2}} - u^k_{i+\frac{1}{2}}| = O(h^2)$$

up to any finite time t_o, $k\Delta t \le t_o$. The L-W scheme is known to satisfy that stability condition.

For an arbitrary *irregular* grid it is only true that

$$T = \Delta t O(h) \ .$$

In this case Strang's theorem is not available, for its proof rests solidly on the assumptions that $h_{i+\frac{1}{2}} \equiv h$ and $\Delta t =$ constant * h. We could postulate a much stronger convergence condition, however. Let

$$E = U - u$$

so that

$$\begin{aligned} E^{k+1} &= \phi(U^k) - \phi(u^k) + \Delta t O(h) \\ &= \int_0^1 \phi'(u^k + sE^k) ds \cdot E^k + \Delta t O(h) \end{aligned}$$

where $\phi'(v)$ is the gradient of ϕ with respect to the grid function v.

A possible stability condition is that

$$\begin{aligned} &\| \int_o^1 \phi'(u(\cdot, t^n) + s\delta^n) ds \cdot \int_0^1 \phi'(u(\cdot, t^{n-1}) + s\delta^{n-1}) ds \cdots \int_0^1 \phi'(u(\cdot, t^1) + s\delta^1) ds \|_{L_2} \\ &\le M \ , \end{aligned} \tag{8}$$

for all sequences $0 < t^1 < \cdots < t^n \le t_o$, and all grid functions δ^k such that $\|\delta^k\| \le 1$, for then it would follow by induction that

$$\|E^{k+1}\|_{L_2} = O(h) \ , \ (k+1)\Delta t \le t_o \ .$$

Let us assume that (8) holds for L-W, even though we do not know if this is the case. Then the stage is set for the supraconvergence result. We are going to show that there is a grid function w such that

$$w^k_{i+\frac{1}{2}} - u^k_{i+\frac{1}{2}} = O(h^2) \ , \tag{9}$$

and

$$w^{k+1} = \phi(w^k) + \Delta t O(h^2) \ . \tag{10}$$

Now it follows that if

$$E^k = w^k - U^k$$

then, as above,

$$\|E^k\|_{L_2} = O(h^2)$$

and therefore

$$\|U^k - u^k\| = O(h^2) \ . \tag{11}$$

The demonstration of Eqs. (9) and (10) is an exercise in Taylor expansion, but we must be very careful about the remainders because ratios of unequal grid sizes appear.

We first look at (suppressing time)

$$\bar{u}_i = \frac{h_{i+\frac{1}{2}}u_{i-\frac{1}{2}} + h_{i-\frac{1}{2}}u_{i+\frac{1}{2}}}{2h_i} \ .$$

From

$$u_{i+\frac{1}{2}} = u_i + \frac{1}{2}h_{i+\frac{1}{2}}u_{i,x} + \frac{1}{8}(h_{i+\frac{1}{2}})^2 u_{i,xx} + h_{i+\frac{1}{2}}^3 O(1) \tag{12}$$

we get

$$\bar{u}_i = U_i + \frac{h_{i-\frac{1}{2}}}{2h_i}\frac{1}{8}(h_{i+\frac{1}{2}})^2 u_{i,xx} + \frac{h_{i+\frac{1}{2}}}{2h_i}\frac{1}{8}(h_{i-\frac{1}{2}})^2 u_{i,xx} + h_{i-\frac{1}{2}}h_{i+\frac{1}{2}}O(h) \ .$$

We can now define w, the correction to u, as

$$w_{i+\frac{1}{2}} = u_{i+\frac{1}{2}} - \frac{1}{8}(h_{i+\frac{1}{2}})^2 u_{i+\frac{1}{2},xx} \tag{13}$$

from which it follows readily that

$$\bar{w}_i = u_i + h_{i+\frac{1}{2}}h_{i-\frac{1}{2}}O(h) \ . \tag{14}$$

Also,

$$f'(\bar{w}_i) = f'(u_i) + h_{i-\frac{1}{2}}h_{i+\frac{1}{2}}O(h) \ .$$

We are now ready to look at the full predictor step, i.e., let

$$w_i^* = \bar{w}_i - f'(w_i)\frac{w_{i+\frac{1}{2}} - w_{i-\frac{1}{2}}}{h_i}\frac{\Delta t}{2} \ .$$

From Eq. (12) we see that

$$\frac{w_{i+\frac{1}{2}} - w_{i-\frac{1}{2}}}{h_i} = u_i' + h_{i+\frac{1}{2}}^2 O(1) + h_{i-\frac{1}{2}}^2 O(1)$$

and therefore

$$w_i^* = u_i - f'(u_i)u_i'\frac{\Delta t}{2} + h_{i-\frac{1}{2}}h_{i+\frac{1}{2}}O(h) + \Delta t[h_{i+\frac{1}{2}}^2 O(1) + h_{i-\frac{1}{2}}^2 O(1)] \ .$$

We expect that, at the very least, stability will require the assumption $\Delta t < h_{j+\frac{1}{2}}$ constant, for all j. With this we have

$$w_i^* = u_i - \frac{\Delta t}{2}f'(u_i)u_i' + h_{i-\frac{1}{2}}h_{i+\frac{1}{2}}O(h)$$

or

$$w_i^* = u_i^{k+\frac{1}{2}} + (\Delta t)^2 g(x_i) + h_{i-\frac{1}{2}} h_{i+\frac{1}{2}} O(h) \ ,$$

where $g(x)$ is a *smooth* function of x. The final truncation error will come from the corrector step, i.e., let

$$R = w_{i+\frac{1}{2}}^{k+1} - w_{i+\frac{1}{2}}^k + \frac{\Delta t}{h_{i+\frac{1}{2}}} \{f(w_{i+1}^*) - f(w_i^*)\} \ .$$

Clearly,

$$w_{i+\frac{1}{2}}^{k+1} - w_{i+\frac{1}{2}}^k = u_t^{k+\frac{1}{2}} \Delta t + O(\Delta t)^3 + h_{i+\frac{1}{2}}^2 O(\Delta t) \ ,$$

and

$$f(w_i^*) = f(u_i^{k+\frac{1}{2}}) + (\Delta t)^2 g_1(x_i) + h_{i+\frac{1}{2}} h_{i-\frac{1}{2}} O(h)$$

where $g_1(x)$ is *smooth*. Thus,

$$\frac{f(w_{i+1}^*) - f(w_i^*)}{h_{i+\frac{1}{2}}} = [f_{x,i+\frac{1}{2}}^{k+\frac{1}{2}} + O(h_{i+\frac{1}{2}}^2)] + O(h^2 + \Delta t^2)$$

or

$$R = \Delta t O(h^2)$$

as claimed.

We have shown that the Jacobian-predictor (J-p) scheme is supraconvergent with no grid restriction other than the usual stability condition that the time step is $O(h)$. The analysis we have used does not seem to work for the standard L-W scheme. We now present calculations which indicate that in some instances, the standard L-W yields answers which are less accurate than the Jacobian-predictor (J-p) scheme considered here. The standard L-W scheme is defined by the predictor step,

$$U_i^{k+\frac{1}{2}} = U_i^k - \frac{\Delta t}{2} \frac{f(U_{i+\frac{1}{2}}^k) - f(U_{i-\frac{1}{2}}^k)}{h_i} \tag{15}$$

together with (4). For comparison, the J-p Lax-Wendroff scheme is given by the predictor step

$$U_i^{k+\frac{1}{2}} = U_i^k - \frac{\Delta t}{2} f'(U_i^k) \frac{U_{i+\frac{1}{2}}^k - U_{i-\frac{1}{2}}^k}{h_i} \ ,$$

together with the same corrector step (4).

Although we believe that the standard scheme is supraconvergent in the scalar case when the spatial grid is quasiuniform, the time step is $O(h)$, and $f'(u)$ does not change sign, consider the effect of violating the last condition. The test problem is a simple one,

$$\frac{\partial u}{\partial t} + \frac{1}{2} \frac{\partial u^2}{\partial x^2} = R(x,t) \ ,$$

where $u = c + 2(x - \frac{1}{2})^2 - \frac{1}{4} \sin \pi x$. We will solve two problems with both schemes, $c = \frac{1}{2}$ and $c = -\frac{1}{4}$. In the first case, $f'(u) > 0$, while in the second case $f'(u)$ has two zeroes. In Figure 2, we plot the L_2 error for the L-W scheme [(15) and (4)] for ten different grids, each two periodic and note that there is a significant difference in convergence rate between the two problems. In Figure 3, we plot the same errors as in Figure 2, but for the J-p scheme. Here we see little difference. Had we plotted the L_∞ errors, the L-W scheme would clearly show a linear convergence rate for the problem with $f'(u)$ vanishing but the J-p L-W would show quadratic convergence.

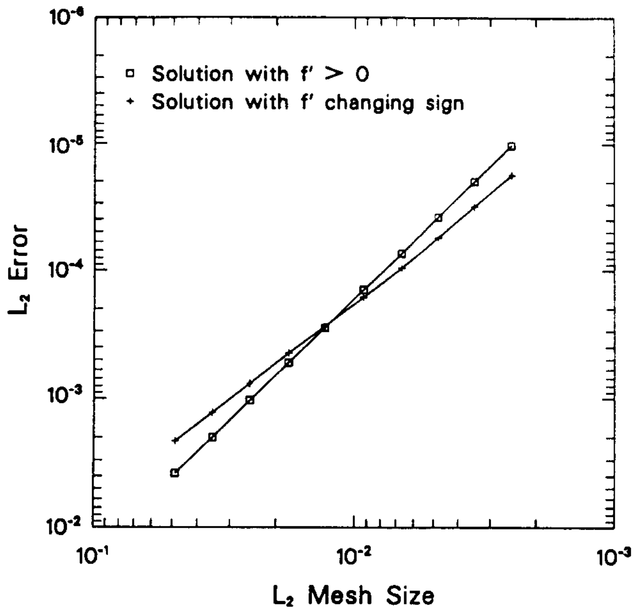

Fig. 2. Convergence rate for standard Lax-Wendroff.

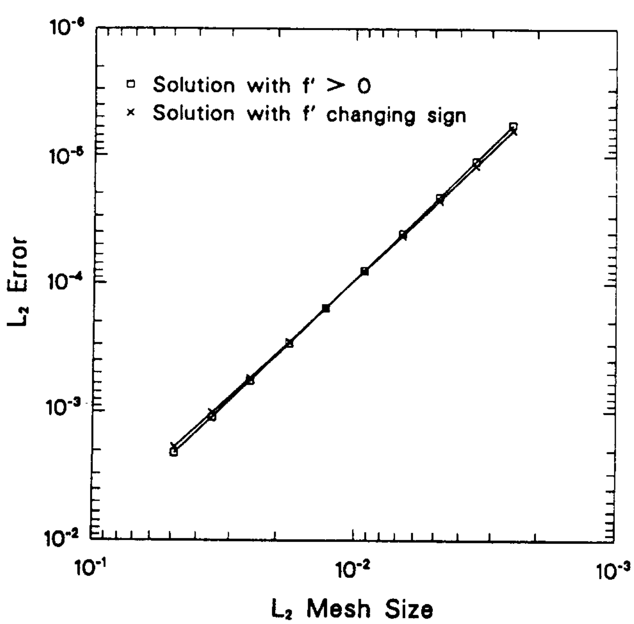

Fig. 3. Convergence rate for Jacobian-predictor Lax- Wendroff.

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