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ELLIPTIC RECONSTRUCTION AND A POSTERIORI ERROR ESTIMATES FOR PARABOLIC PROBLEMS*

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Abstract. It is known that the energy technique for a posteriori error analysis of finite element discretizations of parabolic problems yields suboptimal rates in the norm $L^\infty(0,T;L^2(\Omega))$. In this paper, we combine energy techniques with an appropriate pointwise representation of the error based on an elliptic reconstruction operator which restores the optimal order (and regularity for piecewise polynomials of degree higher than one). This technique may be regarded as the "dual a posteriori" counterpart of Wheeler's elliptic projection method in the a priori error analysis.

Key words. a posteriori error estimators, finite elements, semidiscrete parabolic problems, energy technique

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1. Introduction.

A posteriori error estimation and adaptivity are in many cases very successful tools for efficient numerical computations of linear as well as nonlinear PDEs. In particular, a posteriori error control provides a practical, as well as mathematically sound, means of detecting multiscale phenomena and doing reliable computations. Although the a posteriori error analysis of elliptic problems is now mature [2, 3, 6, 7, 18, 23], the time dependent case is still under development. Many papers have appeared for the discontinuous Galerkin method [9, 10, 11, 13, 14, 15, 20, 19] and other schemes [1, 4, 17, 21, 24, 25] mainly for linear parabolic problems.

One of the outstanding issues related to a posteriori estimation of (linear) time dependent problems is the known fact that the energy technique for a posteriori error analysis of finite element discretizations of parabolic problems yields suboptimal rates in the norm $L^\infty(0,T;L^2(\Omega))$. Since the energy method is the most elementary technique for estimating the error in the a priori analysis, the question of whether or not this method can be successfully applied in the a posteriori error analysis is very natural. In addition, we hope that examining this and related issues will enable us to increase our understanding on the important subject of error control for time dependent problems in general.

We will work with the following linear parabolic equation as a model:

$$
\begin{aligned}
&u_t + Au = f \quad \text{in } \Omega\times[0,T],\\
&u(\cdot,0) = u_0(\cdot) \quad \text{in } \Omega,\\
&u = 0 \quad \text{on } \partial\Omega\times[0,T].
\end{aligned}
\tag{1.1}
$$

Here A is a linear, symmetric, second order positive definite elliptic operator, and Ω is a bounded domain of $\mathbb{R}^d$ ($d \geq 1$) with sufficiently smooth boundary for our purposes.

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Let $H := L^2(\Omega)$, $V := H_0^1(\Omega)$, and $V^\star := H^{-1}(\Omega)$ be the dual of V. If $a(\cdot,\cdot)$ is the bilinear form that corresponds to A, our assumptions on A imply that

$$\|v\|_V := a(v,v)^{1/2}$$

defines a norm on V. We denote the norms on H and $V^\star$ by $\|\cdot\|_{V^\star}$ and $\|\cdot\|_H$, respectively, and we indicate with $\langle\cdot,\cdot\rangle$ the duality pairing in either H or $V^* - V$.

We assume that $f \in L^2(0,T;V^\star)$ and $u_0 \in H$ so that (1.1) admits a unique weak solution satisfying

$$\langle u_t(t), v\rangle + a(u(t),v) = \langle f, v\rangle \quad \text{for all } v \in V, \text{ a.e. } t \in [0,T]\,.$$

In this paper, we consider semidiscrete finite element discretizations of *arbitrary* degree. We combine energy techniques with an appropriate pointwise representation of the error based on a novel *elliptic reconstruction* operator which restores the optimal order in $L^\infty(0,T;L^2(\Omega))$. This technique may be regarded as the dual counterpart of Wheeler's elliptic projection method in the a priori error analysis [27]. In particular, for u_h as the finite element approximation, our estimates exhibit the following properties:

- the estimator is a computable quantity in terms of the approximate solution u_h and the data f, u_0, and Ω, but its actual form and quality depends only on the elliptic estimator at our disposal;
- the order is optimal in $L^\infty(0,T;L^2(\Omega))$ for any polynomial degree ≥ 1, and the regularity is the lowest compatible with (1.1) for polynomial degree > 1;
- the a posteriori estimates mimic completely the corresponding a priori estimates.

Hereafter, the use of "optimal order" and "optimal regularity" is consistent with classical terminology in approximation theory. Consequently, "optimal order" corresponds to the largest exponent r for which the error is $O(h^r)$, where h is the biggest element diameter of the partition. Likewise, "optimal regularity" refers to the lowest regularity which is compatible with (1.1) and an error of $O(h^r)$.

Finite element approximation. For $\mathcal{T}_h$, being a shape-regular partition of Ω, consider the finite element space

$$V_h = \{\chi \in H_0^1(\Omega) : \chi|_K \in \mathbb{P}_k(K) \quad \text{for all } K \in \mathcal{T}_h\},$$

where $\mathbb{P}_k(K)$ is the space of polynomials of degree $\leq k$ over K. The finite element approximation $u_h : [0,T] \to V_h$ of u is defined to satisfy the following linear ODE:

$$\begin{aligned} \langle u_{h,t}, \chi\rangle + a(u_h,\chi) &= \langle f, \chi\rangle \quad \text{for all } \chi \in V_h, \text{ a.e. } t \in [0,T], \\ u_h(\cdot,0) &= u_h^0 \in V_h\,. \end{aligned} \tag{1.2}$$

A posteriori error estimation. Residual-based a posteriori estimates are usually proved by estimating the linear functional $R \in V^\star$, so-called *residual*,

$$\begin{aligned} -\langle R, v\rangle &= \int_0^T \Big(\langle u_{h,t}, v\rangle + a(u_h,v) - \langle f,v\rangle\Big)dt \\ &= \int_0^T \Big(\langle u_{h,t}, v - I_hv\rangle + a(u_h, v - I_hv) - \langle f, v - I_hv\rangle\Big)dt, \end{aligned} \tag{1.3}$$

in appropriate norms. Here, in the second equality, we have used the definition of the semidiscrete scheme (1.2) and an interpolation operator $I_h : V \to V_h$ stable in V

(e.g., Clement's interpolant). Then, for $e = u - u_h$ as the error to be estimated, we have

$$\frac{1}{2}\|e\,(T)\|_H^2 + \int_0^T a(e,e)dt = \frac{1}{2}\|e\,(0)\|_H^2 + \langle R, e\rangle. \tag{1.4}$$

Due to the presence of $\int_0^T a(u_h, e - I_h e)dt$, which gives rise to the integral of an H^1 elliptic residual, the ensuing a posteriori estimate is of optimal order in $L^2(0,T;H_0^1(\Omega))$, as corresponds to an estimate of $\int_0^T a(e,e)dt$, but is *suboptimal* in $L^\infty(0,T;L^2(\Omega))$. It is well known that an analogous phenomenon occurs in the a priori analysis and that the use of an elliptic projection operator overcomes the difficulty [27]. This is now a standard tool in the finite element analysis.

In this paper, we introduce an *elliptic reconstruction* operator which restores the optimal order in the a posteriori error estimation in $L^\infty(0,T;L^2(\Omega))$. The key properties of the elliptic reconstruction U (cf. Definition 2.1) are (i) $u - U$ satisfies an appropriate pointwise equation (cf. (3.2)) that can be used to derive estimates in terms of $u_{h,t} - U_t$, and (ii) u_h is the finite element solution of an elliptic problem whose exact solution is U, and therefore $u_h - U$ (as well as $u_{h,t} - U_t$) can be estimated in various norms by any given a posteriori elliptic estimator. Note that a similar function U was introduced in [12] for a different purpose.

For clarity of exposition, we present the method in the simplest framework. The ideas of the present paper might be useful for linear problems of nondissipative character as well as for nonlinear dissipative problems. In this direction, they should be explored together with the recent a posteriori results of time discretization of nonlinear problems [17, 19]. The a posteriori analysis of [17, 19] is based on the same principles as those in the present paper, namely, an appropriate pointwise representation of the error and energy arguments.

Although it is possible to derive quasi-optimal order-regularity estimators in $L^\infty(0,T;L^2(\Omega))$ via *parabolic duality* [9, 22], this technique hinges on the parabolic regularizing effect which is not valid for estimates in $L^2(0,T;H_0^1(\Omega))$. For the latter, duality leads invariably to estimators similar to those obtained with the energy approach and which also bound the error in $L^\infty(0,T;L^2(\Omega))$ but with suboptimal order. In contrast, several contributions over the last few years are devoted to estimates that are based on the (forward) energy approach. Picasso [21] derives a posteriori error estimates of residual type that are optimal in $L^2(0,T;H_0^1(\Omega))$ for piecewise linear elements for space discretization and backward Euler for time discretization. Toward overcoming the barrier described above, Babuška, Feistauer, and Šolín [4] derive estimates in $L^2(0,T;L^2(\Omega))$ for (1.2) by a double integration in time; see also [1, 5]. In [24, 25], Verfürth proves a posteriori estimates in $L^r(0,T;L^\rho(\Omega))$, with $1 < r, \rho < \infty$, for fully discrete approximations of quasi-linear parabolic equations.

The paper is organized as follows. We introduce the elliptic reconstruction operator in section 2, and we derive abstract a posteriori error estimates in section 3. In particular, our estimator of Theorem 3.1 depends on an abstract *elliptic estimator function* for elliptic problems; any such estimator can be used. In section 4, we specify the form of the estimates for the classical residual-type elliptic estimators.

2. Elliptic reconstruction. We now introduce the elliptic reconstruction operator $\mathcal{R} : V_h \to V$. To this end, let $P_h^1 : V \to V_h$ be the elliptic projection operator, i.e.,

$$a(P_h^1 w, \chi) = a(w, \chi) \quad \text{for all } \chi \in V_h, \tag{2.1}$$

and let $P_h^0 : H \to V_h$ be the L^2-projection operator, i.e.,

$$(P_h^0 w, \chi) = \langle w, \chi\rangle \quad \text{for all } \chi \in V_h. \tag{2.2}$$

Let $w \in V$ satisfy the elliptic problem $Aw = g \in V^\star$ or, in weak form,

$$w \in V : \qquad a(w, v) = \langle g, v\rangle \quad \text{for all } v \in V. \tag{2.3}$$

Let $w_h \in V_h$ be the corresponding finite element solution

$$w_h \in V_h : \qquad a(w_h, \chi) = \langle g, \chi\rangle \quad \text{for all } \chi \in V_h; \tag{2.4}$$

hence $w_h = P_h^1 w$. We assume that we have at our disposal a posteriori estimators that control the error $\|w - w_h\|_X$ in the spaces $X = H, V$, or $V^\star$.

ASSUMPTION 2.1. *Let w and w_h be the exact solution and its finite element approximation given in* (2.3) *and* (2.4) *above. We assume that there exists an a posteriori estimator function $\mathcal{E} = \mathcal{E}(w_h, g; X)$, which depends on w_h, g and the space $X = H, V$, or $V^\star$ such that*

$$\|w - w_h\|_X \le \mathcal{E}(w_h, g; X). \tag{2.5}$$

Let $A_h : V_h \to V_h$ be the following discrete version of A:

$$\langle A_h v, \chi\rangle = a(v, \chi) \quad \text{for all } \chi \in V_h. \tag{2.6}$$

Then we have the following definition.

DEFINITION 2.1. *Let u_h be the finite element solution of* (1.2) *and $f_h := P_h^0 f$. We define the elliptic reconstruction $U = \mathcal{R}u_h \in H_0^1(\Omega)$ of u_h to be the solution of the elliptic problem in weak form*

$$a(U(t), v) = \langle g_h(t), v\rangle \quad \textit{for all } v \in H_0^1(\Omega), \textit{ a.e. } t \in [0, T], \tag{2.7}$$

where

$$g_h := A_h u_h - f_h + f. \tag{2.8}$$

We note that a similar function U was defined at the final time T in [12] in a different context, i.e., in postprocessing the Galerkin method at T with the aim of improving the order of convergence. We observe that U satisfies the strong form

$$AU = A_h u_h - f_h + f \tag{2.9}$$

as well as

$$a(U, \varphi) = a(u_h, \varphi) - \langle f_h - f, \varphi\rangle = a(u_h, \varphi) \quad \text{for all } \varphi \in V_h, \tag{2.10}$$

because $f_h = P_h^0 f$. This relation implies that u_h *is the finite element solution of the elliptic problem whose exact solution is the elliptic reconstruction* U, namely,

$$u_h = P_h^1 U. \tag{2.11}$$

Assume that $f \in H^1(0, T; V^*)$. Since $a(\cdot, \cdot)$ is independent of t, there holds $a(U_t, \varphi) = a(u_{h,t}, \varphi)$ for all $\varphi \in V_h$, or

$$u_{h,t} = P_h^1 U_t. \tag{2.12}$$

In addition,

$$a(U_t, v) = \langle g_{h,t}, v\rangle \quad \text{for all } v \in V. \tag{2.13}$$

3. Abstract a posteriori error analysis. In this section, we establish the improved a posteriori error estimate in H and make several comments about its optimality regarding both order and regularity.

THEOREM 3.1. *Assume that u is the solution of* (1.1) *and u_h is its finite element approximation* (1.2). *Let U be the elliptic reconstruction of u_h and $\mathcal{E}$ be as defined in Assumption* 2.1. *Then the following a posteriori error bounds hold for $0 < t \le T$:*

$$\|(u-U)(t)\|_H^2 + \int_0^t \|u-U\|_V^2 ds \le \|u(0)-U(0)\|_H^2 + \int_0^t \mathcal{E}(u_{h,t}, g_{h,t}; V^\star)^2 ds$$

and

$$\|(u-u_h)(t)\|_H \le \|u_0 - u_h^0\|_H + \left(\int_0^t \mathcal{E}(u_{h,t}, g_{h,t}; V^\star)^2 ds\right)^{1/2} + \mathcal{E}(u_h(0), g_h(0); H) + \mathcal{E}(u_h(t), g_h(t); H).$$

Proof. By virtue of definitions (1.2) and (2.9) of u_h and U, we have

$$u_{h,t} + AU = f,$$

whence U satisfies the following pointwise equation:

$$U_t + AU = f + (U - u_h)_t\,. \tag{3.1}$$

Thus the error equation for $u - U$ reads

$$(u-U)_t + A(u-U) = (u_h - U)_t\,. \tag{3.2}$$

Multiplying by $u - U$ and using standard energy arguments yield

$$\|(u-U)(t)\|_H^2 + \int_0^t \|(u-U)(s)\|_V^2 ds \le \|u(0)-U(0)\|_H^2 + \int_0^t \|(u_{h,t} - U_t)(s)\|_{V^\star}^2 ds\,. \tag{3.3}$$

Relations (2.12) and (2.13), in conjunction with Assumption 2.1, imply

$$\|u_{ht} - U_t\|_{V^\star} \le \mathcal{E}(u_{h,t}, g_{h,t}; V^\star)\,,$$

which in turn leads to the first assertion of Theorem 3.1. To show the second one, it suffices to note that (2.11) and Assumption 2.1 yield

$$\|(u_h - U)(t)\|_H \le \mathcal{E}(u_h(t), g_h(t); H) \quad \text{for all } 0 \le t \le T\,, \tag{3.4}$$

which, together with

$$\begin{aligned}\|u(0)-U(0)\|_H &\le \|u(0) - u_h(0)\|_H + \|P_h^1 U(0) - U(0)\|_H \\ &\le \|u_0 - u_h^0\|_H + \mathcal{E}(u_h(0), g_h(0); H),\end{aligned}$$

concludes the proof. $\square$

Remark 3.1 (L^2-based estimate). An alternative estimate that follows from the proof of Theorem 3.1 is

$$\begin{aligned}\max_{0\le t\le T}\|u-U\|_H^2 &\le \|u(0)-U(0)\|_H^2+\max_{0\le t\le T}\|u-U\|_H\int_0^T\|u_{h,t}-U_t\|_H dt\\ &\le \max_{0\le t\le T}\|u-U\|_H\left(\|u(0)-U(0)\|_H+\int_0^T\|u_{h,t}-U_t\|_H dt\right).\end{aligned}$$

Therefore, (2.5) and (3.4) imply

$$\max_{0\le t\le T}\|u-U\|_H \le \|u(0)-U(0)\|_H+\int_0^T\mathcal{E}(u_{h,t},g_{h,t};H)dt,$$

along with the corresponding a posteriori error bound

$$\max_{0\le t\le T}\|u-u_h\|_H\le\|u_0-u_h^0\|_H+\mathcal{E}(u_h(0),g_h(0);H)\;+2\int_0^T\mathcal{E}(u_{h,t},g_{h,t};H)dt.$$

Remark 3.2 (a priori vs. a posteriori bounds). Note that the elliptic reconstruction is an "a posteriori dual" to Wheeler's elliptic projection [22, 27]. Furthermore, the two results in Theorem 3.1 are indeed an *a posteriori dual* to the classical a priori estimate for semidiscrete linear parabolic problems [22, 27]

$$\begin{aligned}&\|(u_h-P_h^1u)(t)\|_H^2+\int_0^t\|u_h-P_h^1u\|_V^2ds\\ &\qquad\le\|u_h(0)-P_h^1u(0)\|_H^2+\int_0^t\|u_t-P_h^1u_t\|_{V^\star}^2ds\end{aligned}\tag{3.5}$$

and

$$\begin{aligned}\|(u-u_h)(t)\|_H\le{}&\|(u-P_h^1u)(t)\|_H\\ &+\left(\|u_h(0)-P_h^1u(0)\|_H^2+\int_0^t\|u_t-P_h^1u_t\|_{V^\star}^2dt\right)^{1/2}.\end{aligned}\tag{3.6}$$

Remark 3.3 (optimal regularity). The a priori bound in (3.5) (and therefore in (3.6)) is of optimal order. The regularity required is optimal only for polynomial degree $k\ge 2$. Indeed, by exploiting standard results on superconvergence in negative norms of elliptic finite element problems, we see that the following bound for the error of the elliptic projection holds [22, 26]:

$$\|v-P_h^1v\|_{V^\star}\le Ch^{(k+1)}\|v\|_k.\tag{3.7}$$

This estimate follows from the definition of the dual norm $\|w\|_{V^\star}=\sup_{\|z\|_V=1}\langle w,z\rangle$ and a standard duality argument. Using (3.7), we obtain

$$\int_0^T\|u_t-P_h^1u_t\|_{V^\star}^2dt\le C\int_0^T h^{2(k+1)}\|u_t\|_k^2dt\le Ch^{2(k+1)}\int_0^T\|u\|_{k+2}^2dt\,;$$

here $\|\cdot\|_s$ denotes the Sobolev norm of $H^s(\Omega)$, and for simplicity take $A=-\Delta$ and $f=0$. For an (optimal) convergence rate of order $O(h^{k+1})$ in $L^\infty(0,T;L^2(\Omega))$, the minimal regularity required by our finite element space is $u \in L^\infty(0,T;H^{k+1}(\Omega))$. However, it is a simple matter to check that for (1.1) both

$$\int_0^T \|u\|_{k+2}^2 dt \quad \text{and} \quad \max_{0\le t\le T} \|u\|_{k+1}^2$$

are bounded by the same constant depending on data. Thus the classical a priori estimate (3.6) is of optimal order and regularity for $k \ge 2$. The negative norm $\|\cdot\|_{V^*}$ appears in a complete similar fashion in the a posteriori error analysis of Theorem 3.1, and thus for polynomial degree $k \ge 2$ this indicates that the estimator is of *optimal order-regularity*.

4. Application: Residual-type error estimators. In this section, we derive the specific form of the estimates of section 3 in case we choose the classical residual-type estimators for (2.5) [6, 23]. Of course, any other choice, such as solving local problems [2, 7, 18, 23] or averaging techniques [3], is possible according to Theorem 3.1. For simplicity, we assume that $A = -\Delta$ and that Ω is sufficiently smooth in order for (4.2) below to be valid. However, Theorem 3.1 is general enough to allow for geometric singularities and corresponding elliptic estimators. We refer to [16] for *weighted* a posteriori estimators, which account for corner singularities in both H and $V^\star$ in an optimal fashion, as well as to [8], where an error estimator is derived for an elliptic problem with curved boundaries.

We first calculate $\mathcal{E}(u_{h,t}, g_{h,t}; V^\star)$ or, equivalently, estimate

$$\|\rho\|_{V^\star} = \sup_{\|\phi\|_V \le 1} \langle \rho, \phi\rangle, \quad \rho = (U - u_h)_t.$$

We accomplish this via standard duality arguments. Given $\phi \in V$, let $\psi \in V$ be defined by

$$a(\psi, v) = \langle \nabla\psi, \nabla v\rangle = \langle v, \phi\rangle \quad \text{for all } v \in V, \tag{4.1}$$

and suppose there exists a constant $C_\Omega > 0$, depending on the domain Ω, such that

$$\|\psi\|_{H^3(\Omega)} \le C_\Omega \|\phi\|_{H^1(\Omega)}. \tag{4.2}$$

If $\mathcal{T}_h = \{K\}$ is a shape-regular partition of Ω into finite elements K, then $\mathcal{S}_h = \{S\}$ denotes the set of internal interelement sides and $\mathcal{N}_h(E)$ stands for the union of all elements of $\mathcal{T}_h$ intersecting the *closed* set E ($= K$ or S). Then, assuming for the time being that the polynomial degree is $k \ge 2$ and recalling (2.12), we can write

$$\begin{aligned}\langle \rho, \phi\rangle &= a(\psi, \rho) = a(\psi - I_h\psi, \rho)\\ &\le \sum_{K\in\mathcal{T}_h} |(\psi - I_h\psi, \Delta\rho)_K| + \sum_{S\in\mathcal{S}_h} \int_S |\psi - I_h\psi||[\partial_n \rho]|\, ds\\ &\le C_I \sum_{K\in\mathcal{T}_h} h_K^3 |\psi|_{3,\mathcal{N}_h(K)} \|\Delta\rho\|_{L^2(K)}\\ &\quad + C_I \sum_{S\in\mathcal{S}_h} h_S^{5/2} |\psi|_{3,\mathcal{N}_h(S)} \|[\partial_n\rho]\|_{L^2(S)},\end{aligned} \tag{4.3}$$

where $C_I > 0$ is an interpolation constant associated with the local interpolation operator I_h. If we further set

$$\eta_{-1}(u_{h,t})^2 = \sum_{K\in\mathcal{T}_h} h_K^6 \|\Delta\rho\|_{L^2(K)}^2 + \sum_{S\in\mathcal{S}_h} h_S^5 \|[\partial_n u_{h,t}]\|_{L^2(S)}^2$$

and make use of (4.2), then we end up with the a posteriori error estimate

$$\mathcal{E}(u_{h,t}, g_{h,t}; V^\star) = \|\rho\|_{V^\star} \le C_I C_\Omega \eta_{-1}(u_{h,t}),$$

where C_I now contains an additional factor to account for the h-independent overlap of sets $\mathcal{N}_h(E)$ in (4.3).

The form of $\eta_{-1}(u_{h,t})$ can be further simplified upon using the definition of the elliptic reconstruction and the semidiscrete scheme:

$$\Delta\rho = \Delta U_t - \Delta u_{h,t} = -A_h u_{h,t} + f_{h,t} - f_t - \Delta u_{h,t}\,.$$

Since $u_{h,tt} + A_h u_{h,t} = f_{h,t}$, we have

$$\Delta\rho = -f_{h,t} + u_{h,tt} + f_{h,t} - f_t - \Delta u_{h,t} = (u_{h,t} - \Delta u_h - f)_t\,.$$

If we denote the element residuals as

$$r|_K := u_{h,t} - \Delta u_h - f \quad \text{for all } K \in \mathcal{T}_h, \qquad j|_S := [\partial_n u_h] \quad \text{for all } S \in \mathcal{S}_h,$$

we finally get

$$\eta_{-1}(u_{h,t})^2 = \sum_{K\in\mathcal{T}_h} h_K^6 \|r_t\|^2_{L^2(K)} + \sum_{S\in\mathcal{S}_h} h_S^5 \|j_t\|^2_{L^2(S)}, \tag{4.4}$$

and

$$\mathcal{E}(u_{h,t}, g_{h,t}; V^\star) \le C_I C_\Omega \eta_{-1}(u_{h,t}) \quad \text{if } k \ge 2.$$

Using similar arguments, we can derive

$$\mathcal{E}(u_h, g_h; H) \le C_I C_\Omega \eta_0(u_h) \quad \text{if } k \ge 2,$$

where

$$\eta_0(u_h)^2 = \sum_{K\in\mathcal{T}_h} h_K^4 \|r\|^2_{L^2(K)} + \sum_{S\in\mathcal{S}_h} h_S^3 \|j\|^2_{L^2(S)}. \tag{4.5}$$

Note that the constants C_I, C_Ω may have different values now. Finally, in the case $k = 1$, the use of negative norm does not give better results because of the lack of superconvergence in $V^\star$. Hence

$$\mathcal{E}(u_{h,t}, g_{h,t}; V^\star) \le \mathcal{E}(u_{h,t}, g_{h,t}; H) \le C_I C_\Omega \eta_0(u_{h,t})\,. \tag{4.6}$$

In summary, in view of Theorem 3.1, we have derived the following *explicit* error estimate.

THEOREM 4.1 (a posteriori estimators of residual type). *Assume that the domain Ω is sufficiently smooth, and let $t \in (0, T]$. If $k = 1$, then the following a posteriori estimate holds:*

$$\|(u - u_h)(t)\|_H \le \|u^0 - u_h^0\|_H + C_I C_\Omega \left\{ \eta_0(u_h(0)) + \eta_0(u_h(t)) + \left(\int_0^t \eta_0(u_{h,t}(s))^2 ds \right)^{1/2} \right\}.$$

In addition, for $k \geq 2$, we have

$$\|(u-u_h)(t)\|_H \leq \|u^0 - u_h^0\|_H + C_I C_\Omega \left\{ \eta_0(u_h(0)) + \eta_0(u_h(t)) + \left(\int_0^t \eta_{-1}(u_{h,t}(s))^2 ds \right)^{1/2} \right\},$$

where the estimators η_0 and η_{-1} are given by (4.5) *and* (4.4)*, respectively.*

Remark 4.1. The reasoning of Remark 3.3 applies and indicates that the estimator in Theorem 4.1 is of optimal order for polynomial degree $k \geq 1$ and of optimal regularity for $k \geq 2$. We do not actually show an a priori convergence rate for the a posteriori estimators $\eta_0(u_h)$ and $\eta_1(u_h)$; this is of interest but lies outside the scope of this paper.

REFERENCES

[1] S. Adjerid, J. E. Flaherty, and I. Babuška, *A posteriori error estimation for the finite element method-of-lines solution of parabolic problems*, Math. Models Methods Appl. Sci., 9 (1999), pp. 261–286.

[2] M. Ainsworth and J. Oden, *A unified approach to a posteriori error estimation using element residual methods*, Numer. Math., 65 (1993), pp. 23–50.

[3] M. Ainsworth, J. Z. Zhu, A. W. Craig, and O. C. Zienkiewicz, *Analysis of the Zienkiewicz-Zhu a posteriori error estimator in the finite element method*, Internat. J. Numer. Methods Engrg., 28 (1989), pp. 2161–2174.

[4] I. Babuška, M. Feistauer, and P. Šolín, *On one approach to a posteriori error estimates for evolution problems solved by the method-of-lines*, Numer. Math., 89 (2001), pp. 225–256.

[5] I. Babuška and S. Ohnimus, *A posteriori error estimation for the semidiscrete finite element method of parabolic differential equations*, Comput. Methods Appl. Mech. Engrg., 190 (2001), pp. 4691–4712.

[6] I. Babuška and W. C. Rheinboldt, *Error estimates for adaptive finite element computations*, SIAM J. Numer. Anal., 15 (1978), pp. 736–754.

[7] R. E. Bank and A. Weiser, *Some a posteriori error estimators for elliptic partial differential equations*, Math. Comp., 44 (1985), pp. 283–301.

[8] W. Dörfler and M. Rumpf, *An adaptive strategy for elliptic problems including a posteriori error controlled boundary approximation*, Math. Comp., 67 (1998), pp. 1361–1382.

[9] K. Eriksson and C. Johnson, *Adaptive finite element methods for parabolic problems.* I. *A linear model problem*, SIAM J. Numer. Anal., 28 (1991), pp. 43–77.

[10] K. Eriksson and C. Johnson, *Adaptive finite element methods for parabolic problems.* IV. *Nonlinear problems*, SIAM J. Numer. Anal., 32 (1995), pp. 1729–1749.

[11] K. Eriksson, C. Johnson, and S. Larsson, *Adaptive finite element methods for parabolic problems.* VI. *Analytic semigroups*, SIAM J. Numer. Anal., 35 (1998), pp. 1315–1325.

[12] B. García-Archilla and E. S. Titi, *Postprocessing the Galerkin method: The finite-element case*, SIAM J. Numer. Anal., 37 (2000), pp. 470–499.

[13] C. Johnson, *Error estimates and adaptive time-step control for a class of one-step methods for stiff ordinary differential equations*, SIAM J. Numer. Anal., 25 (1988), pp. 908–926.

[14] C. Johnson, *Discontinuous Galerkin finite element methods for second order hyperbolic problems*, Comput. Methods Appl. Mech. Engrg., 107 (1993), pp. 117–129.

[15] C. Johnson, Y. Y. Nie, and V. Thomée, *An a posteriori error estimate and adaptive timestep control for a backward Euler discretization of a parabolic problem*, SIAM J. Numer. Anal., 27 (1990), pp. 277–291.

[16] X. Liao and R. H. Nochetto, *Local a posteriori error estimates and adaptive control of pollution effects*, Numer. Methods Partial Differential Equations, 19 (2003), pp. 421–442.

[17] Ch. Makridakis and R. H. Nochetto, *A posteriori error analysis of a class of dissipative methods for nonlinear evolution problems*, submitted (2002).

[18] P. Morin, R. H. Nochetto, and K. G. Siebert, *Local problems on stars: A posteriori error estimators, convergence, and performance*, Math. Comp., 72 (2003), pp. 1067–1097.

[19] R. H. Nochetto, G. Savaré, and C. Verdi, *A posteriori error estimates for variable time-step discretizations of nonlinear evolution equations*, Comm. Pure Appl. Math., 53 (2000), pp. 525–589.

[20] R. H. Nochetto, A. Schmidt, and C. Verdi, *A posteriori error estimation and adaptivity for degenerate parabolic problems*, Math. Comp., 69 (2000), pp. 1–24.

[21] M. Picasso, *Adaptive finite elements for a linear parabolic problem*, Comput. Methods Appl. Mech. Engrg., 167 (1998), pp. 223–237.

[22] V. Thomée, *Galerkin Finite Element Methods for Parabolic Problems*, Springer-Verlag, Berlin, 1997.

[23] R. Verfürth, *A Posteriori Error Estimation and Adaptive Mesh Refinement Techniques*, Teubner, Stuttgart, 1995.

[24] R. Verfürth, *A posteriori error estimates for nonlinear problems.* $L^r(0,T;L^\rho(\omega))$*-error estimates for finite element discretizations of parabolic equations*, Math. Comp., 67 (1998), pp. 1335–1360.

[25] R. Verfürth, *A posteriori error estimates for nonlinear problems:* $L^r(0,T;W^{1,\rho}(\omega))$*-error estimates for finite element discretizations of parabolic equations*, Numer. Methods Partial Differential Equations, 14 (1998), pp. 487–518.

[26] L. B. Wahlbin, *Superconvergence in Galerkin Finite Element Methods*, Springer-Verlag, Berlin, 1995.

[27] M. F. Wheeler, *A priori* L_2 *error estimates for Galerkin approximations to parabolic partial differential equations*, SIAM J. Numer. Anal., 10 (1973), pp. 723–759.