# Stability Theorem and Truncation Error Analysis for the Glimm Scheme and for a Front Tracking Method for Flows with Strong Discontinuities

I-LIANG CHERN
*Courant Institute and Academia Sinica*

**Abstract**

Typical nonlinear wave interaction problems involve strong waves moving through a background of weak disturbance. Previous existence theorems and error analysis for computations are usually restricted to more idealized situations such as small data or single equations. We consider here the problem of a single strong discontinuity interacting with a weak background for general hyperbolic systems of conservation laws. We obtain the stability, consistency theorems and upper bounds of the truncation errors for the Glimm scheme and for a front tracking method. The major error in the Glimm scheme is the error generated by the strong discontinuity. This error is reduced when a front tracking method is applied to follow the location of the strong discontinuity. This demonstrates an advantage of front tracking methods in one-space dimension.

## 1. Introduction

Typical nonlinear wave interaction problems involve strong waves moving through a background of a weak disturbance. Commonly, the existence theorems and error analysis for computations were previously restricted to more idealized situations such as small initial data (see [5]) or single equations (see [8], [9]). We consider here the problem of a single strong discontinuity interacting with a weak background for general hyperbolic systems of conservation laws. More precisely, we consider the $N \times N$ hyperbolic system

$$u_t + f(u)_x = 0, \qquad t > 0, -\infty < x < \infty, u \in \mathbb{R}^N, \tag{1.1}$$

with the initial data

$$u(x,0) = \begin{cases} u_l(x) & \text{if} \quad x < 0, \\ u_r(x) & \text{if} \quad x > 0, \end{cases} \tag{1.2}$$

where $u_l(x), u_r(x)$ are close to the constant states $u_l^0, u_r^0$, respectively, and $(u_l^0, u_r^0)$ forms either a strong shock or a strong contact discontinuity with *arbitrary* strength.

In the first part of this paper, we shall prove that the deterministic version of the Glimm scheme [5] (known as the random choice method) is stable and consistent for this problem. The main idea in the proof of the stability theorem is

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the following. The approximate solution is interpreted as a composite of discrete waves emitted from the grid points. In the time evolution, the growth of the total strengths of these waves comes from two parts: the interactions of the strong discontinuity with incident small waves and the interactions among all small waves. The first kind of interaction produces the scattered waves, which have the same order of magnitude as the incident small waves. We design a functional—the potential wave production—which consists of the sum of all incident small waves, to control the production of these scattered waves. The second kind of interaction generates second-order small waves. These waves can be controlled by the potential wave interaction among the original small waves and the scattered waves.

In the proof of the consistency theorem, we adopt the wave tracing technique of Liu [11] to analyze the truncation error of the approximate solution. The discrete waves of the approximate solution will be partitioned. Except for a set of quadratic small waves, each partitioned wave can be traced back to the initial time or to the boundary of the strong discontinuity. These waves move in a zig-zag way in order to align with the grid. This effect causes a diffusion in the locations, but not in the strength, of these waves. The truncation error thus generated by a single wave $a$ moving without interaction is $|a|\Delta t|\log \Delta t|$. Here, $|a|$ indicates the magnitude of wave $a$. However, for flows with strong discontinuities, due to a technical difficulty arising from the nonlinear wave interactions, we obtain $\Sigma|a|(\Delta t)^{1/2}|\log \Delta t|$ as an upper bound. Here, the summation is over all waves on the initial time. In contrast, previous estimates on the upper bound of the truncation error of the Glimm scheme were restricted to small initial data. The upper bound was $O(\Sigma|a|(\Delta t)^{1/2}|\log \Delta t|)$ for single equations (see [7]) and $O(\Sigma|a|(\Delta t)^{1/3}|\log \Delta t|)$ for systems (see [12]). However, it is believed that the true truncation error should be $O(\Sigma|a|\Delta t|\log \Delta t|)$ for the Glimm scheme and $O(\Sigma|a|\Delta t)$ for first-order schemes. Applied to flows with strong discontinuities, the major error terms obtained either from our analysis or from the conjecture are the errors at the strong discontinuities.

In the second part of the paper, we shall introduce a simple front tracking method to reduce the error from the strong discontinuity. The scheme is a hybridization of front propagation to follow the location of the strong discontinuity and the random choice method for the rest of the flow. This scheme is also stable and consistent for the above problem. The truncation error released from the strong discontinuity is reduced to $O(\Sigma|a|\Delta t)$—an error caused by the scattered waves near the strong discontinuity. Here, $\Sigma|a|$ is a summation over all incident small waves and is less than the total amount of the background disturbance. This error is relatively small even compared with the conjectured error $O(|S|\Delta t|\log \Delta t|)$ at the strong discontinuity in the Glimm scheme. Here, $|S|$ is the magnitude of the strong discontinuity. For computational purposes, we propose and will publish in the near future a second-order front tracking method. The method consists of (i) a second-order scheme for the off-front flow, (ii) a second-order front propagation, and (iii) a second-order coupling between the front propagation and the off-front flow calculation. We expect that the errors from the strong discontinuities in this method will be reduced to $O(\Sigma|a|(\Delta t)^2)$. Here, the summation is over all incident small waves.

Other front tracking methods that have been introduced for computational purposes are first-order. We refer the reader to [1], [2], [6] and references therein.

This paper is organized as follows. Section 2 contains assumptions and notation. Section 3 is the stability theorem. The crucial lemma is the estimate of the interaction between the strong discontinuity and the small waves. Section 4 is devoted to the truncation error analysis and the consistency theorem. The wave tracing technique is adopted here. Section 5 demonstrates the stability, consistency and truncation error of the simple front tracking method.

## 2. Assumption and Notation

**2.1. Assumptions.** The basic assumptions for problem (1.1), (1.2) are the following: (i) the system (1.1) is assumed to be strictly hyperbolic, i.e., $f'(u)$ has only real and distinct eigenvalues $\lambda_1(u) < \cdots < \lambda_N(u)$ associated with $N$ independent right eigenvectors $r_1(u), \cdots, r_N(u)$; (ii) each characteristic field is either genuinely nonlinear or linearly degenerate; (iii) the unperturbed strong discontinuity $(u_l^0, u_r^0)$ is assumed to be either an entropy shock or a contact discontinuity with *arbitrary* strength. Without loss of generality, we shall assume that it is an entropy $k$-shock for some $k$, $1 \leqq k \leqq N$. For this unperturbed $k$-shock, we require that it satisfies the following stability condition:

A. *There exist $\eta$-neighborhoods $U_l$ and $U_r$ of $u_l^0$ and $u_r^0$, respectively, which have the following properties*:

(i) *For any states $u_l, u_l' \in U_l$ (respectively $u_r, u_r' \in U_r$), the states separating the waves generated by the Riemann problem $(u_l, u_l')$ (respectively $(u_r, u_r')$) lie in $U_l$ (respectively $U_r$).*

(ii) *For any $u_l \in U_l$, $u_r \in U_r$, the Riemann problem $(u_l, u_r)$ can be solved uniquely by a composite of $N$ elementary waves. The states separating these waves lie in $U_l \cup U_r$ and depend on $u_l$ and $u_r$ smoothly. Moreover, the corresponding $k$-wave is an entropy shock.*

We remark that this assumption is valid for the equations of gas dynamics with a convex constitutive equation of state (see [3], page 5).

**2.2. The Glimm scheme.** In the first part of the paper, we shall solve (1.1), (1.2) by the Glimm scheme [5]. To define such a scheme, let $\Delta x$ and $\Delta t$, respectively, denote the spatial and temporal mesh sizes satisfying the Courant-Friedrichs-Lewy condition

$$\text{(C-F-L)} \qquad \frac{\Delta x}{2\,\Delta t} \geqq \max_{i,u} |\lambda_i(u)|.$$

Given an equidistributed sequence $\{\theta^n\}_{n=0}^{\infty}$ on the interval $(-\frac{1}{2}, \frac{1}{2})$, the approximate solution $u_{\Delta x}$ of the Glimm scheme is defined inductively as follows. Suppose $u_{\Delta x}(\cdot, t)$ has been defined for $t < n\,\Delta t$. Then at $t = n\,\Delta t+$, $j\Delta x < x < (j+1)\,\Delta x$, define

$$(2.1)\quad u_{\Delta x}(x, n\,\Delta t+) = \begin{cases} u_{\Delta x}((j+1+\theta^n)\,\Delta x, n\,\Delta t-) & \text{if } \theta^n < 0, \\ u_{\Delta x}((j+\theta^n)\,\Delta x, n\,\Delta t-) & \text{if } \theta^n \geqq 0. \end{cases}$$

For $n\,\Delta t < t < (n+1)\,\Delta t$, $u_{\Delta x}$ is defined to be the solutions to the Riemann problems issued from $(j\Delta x, n\,\Delta t)$, $j \in Z$.

**2.3. Notation.** We recall some basic notation from the theory of hyperbolic conservation laws.

1. *The nonlinear waves*; see [10], [13]. For $1 \leqq i \leqq N$, let $R_i(u_0)$, $S_i(u_0)$ represent, respectively, the rarefaction curve and the shock curve of the $i$-th characteristic field passing through $u_0$. Let $R_i^+(u_0) \equiv \{u \in R_i(u_0) | \lambda_i(u) \geqq \lambda_i(u_0)\}$, $S_i^-(u_0) \equiv \{u \in S_i(u_0) \,|\, \lambda_i(u) \leqq \lambda_i(u_0)\}$, $T_i(u_0) \equiv R_i^+(u_0) \cup S_i^-(u_0)$. Suppose the $i$-th characteristic field is genuinely nonlinear. Then, for $u_1 \in T_i(u_0)$, $(u_0, u_1)$ forms an elementary $i$-wave which is either an entropy shock or a rarefaction wave, depending on whether $u_1 \in S_i^-(u_0)$ or $u_1 \in R_i^+(u_0)$. Its wave strength is measured by $\lambda_i(u_1) - \lambda_i(u_0)$. Its speed, in the case of a shock, will be denoted by $\sigma_i(u_{i-1}, u_i)$. If the $i$-th field is linearly degenerate, then $T_i(u_0) \equiv R_i(u_0) \equiv S_i(u_0)$. As $u_1 \in T_i(u_0)$, the $i$-wave $(u_0, u_1)$ forms a contact discontinuity. Its wave strength is measured by $\tau_i(u_1) - \tau_i(u_0)$, where $\tau_i$ is some monotonic parameter on $T_i(u_0)$. In this paper, an elementary wave $(u_0, u_1)$ will be denoted by a Roman character (for instance, $a, b, c, \cdots$), *its state difference $u_1 - u_0$ will also be denoted by the same Roman character without confusion*. The associated wave strength will be denoted by the same letter with an over bar (such as $\bar{a}, \bar{b}, \bar{c}, \cdots$).

2. *The Glimm scheme*; see [5]. We shall define the dependent grid points of a grid point $(j\Delta x, n\,\Delta t)$ at time level $(n-1)$ to be $\{((j-1)\,\Delta x, (n-1)\,\Delta t), (j\Delta x, (n-1)\,\Delta t)\}$ if $\theta^n \geqq 0$ and to be $\{(j\Delta x, (n-1)\,\Delta t), ((j+1)\,\Delta x, (n-1)\,\Delta t)\}$ if $\theta^n < 0$. This definition is natural because waves issued from $(j\Delta x, n\,\Delta t)$ only depend on the waves issued from the dependent grid points of $(j\Delta x, n\,\Delta t)$. The grid domain of dependence of $(j\Delta x, n\,\Delta t)$ can therefore be defined inductively. A collection of grid points is said to be space-like if none of them lies in the grid domain of dependence of others. An $i$-wave $a$ and a $j$-wave $b$ are said to approach each other if (i) they are issued from two space-like grid points, respectively, and (ii) either $a$ lies to the left of $b$ and $i > j$, or $a$ lies to the right of $b$ and $i < j$, or $i = j$ and one of them is a shock. The interaction among a sequence of waves $\{a_{i,p}\}_{i=1}^N$, $p = 1, \cdots, s$, is defined as follows:

$$(2.2)\qquad Q\big(\{a_{i,1}\}_{i=1}^N, \cdots, \{a_{i,s}\}_{i=1}^N\big) = \sum{}' |\bar{a}_{i,p}|\,|\bar{a}_{j,q}|.$$

Here $\Sigma'$ denotes the summation over all approaching pairs. We can connect the sample point $((j + \theta^n)\,\Delta x, n\,\Delta t)$ by two line segments to the two sample points on $t = (n-1)\,\Delta t$ which are on the two sides of the grid point $(j\Delta x, (n-1)\,\Delta t)$ and next to it. An interaction diamond is a quadrilateral with edges being these segments. An "S-curve" is a piecewise linear curve which is composed of these segments; moreover, the waves passing through this curve are emitted from space-like grid points. Here, "S" stands for "space-like". Given two S-curves $J_1$ and $J_2$, the notation $J_1 \leqq J_2$ stands for $J_2$ lying in the future of $J_1$. Let $J$ be an S-curve. $S_k(J)$ represents the portion of the strong shock passing through $J$. $W(J)$ denotes the collection of all small waves passing through $J$. $W_a(J)$ is the subset of $W(J)$ whose elements are approaching the strong shock $S_k(J)$. Here, a rarefaction wave divided by a sample point will be treated as two different rarefaction waves[1]. This convention is natural because $J$ is composed of segments which connect sample points.

3. *The initial perturbation*. We shall denote

$$\mathrm{TV} \equiv \underset{x<0}{\mathrm{TV}}\left|u_l(x) - u_l^0\right| + \underset{x>0}{\mathrm{TV}}\left|u_r(x) - u_r^0\right|,$$

$$S_k^0 \equiv |u_r^0 - u_l^0|.$$

## 3. Stability Theorem

The approximate solutions $u_{\Delta x}$ of subsection 1.2 are in $L^\infty(\mathbb{R}^2) \cap \mathrm{BV}(\mathbb{R}^2)$. By Assumption A, they are uniformly bounded in $L^\infty(\mathbb{R}^2)$. A scheme is considered to be stable in the sense that these approximate solutions are uniformly bounded in the total variation norm in $\mathrm{BV}(\mathbb{R} \times [0, T])$ for any fixed $T > 0$. If so, we can extract a convergent subsequence of $\{u_{\Delta x}\}$ in $L^1_{\mathrm{loc}}(\mathbb{R}^2)$ (see [4]), and then, by a consistency theorem, which will be proved in the next section, this subsequence converges to a weak solution of (1.1), (1.2).

Following the definition of Tonelli [4], to say that the functions $\{u_{\Delta x}\}$ are uniformly bounded in $\mathrm{BV}(\mathbb{R} \times [0, T])$ means that the generalized derivatives $D_t u_{\Delta x}$, $D_x u_{\Delta x}$ are Radon measures and their total measures on $\mathbb{R} \times [0, T]$ are uniformly bounded. We shall demonstrate this property by showing:

(i) $\mathrm{TV}_x u_{\Delta x}(x, t)$ are uniformly bounded for all $t > 0$.
(ii) The total measures of $D_x u_{\Delta x}$, $D_x f(u_{\Delta x})$ on $\mathbb{R} \times [0, T]$ are uniformly bounded.
(iii) The total measures of the truncation errors $\nu_{\Delta x} \equiv D_t u_{\Delta x} + D_x f(u_{\Delta x})$ on $\mathbb{R} \times [0, T]$ are uniformly bounded.

[1] The author thanks Professor Steve Schochet for a helpful communication.

From (ii) (iii), we can conclude that the total measures of $D_t u_{\Delta x} = \nu_{\Delta x} - D_x f(u_{\Delta x})$ on $\mathbb{R} \times [0, T]$ are also uniformly bounded. Statement (i) is the crucial step, and is a direct consequence of Theorem 3.1 below; (ii) and (iii) are corollaries of (i), and will be proved at the end of this section.

Since $u_{\Delta x}$ is self-similar in every mesh zone, to estimate its total variation in $x$ on any fixed time level it is sufficient to estimate it on any S-curve. Let $J$ be an S-curve. Let us denote $L(J) \equiv \Sigma_{a \in W(J)} |\bar{a}|$, and let $\bar{S}_k(J)$ be the strength of the strong shock passing through $J$. Since the strength of a wave is equivalent to the norm of its state difference, $L(J)$ and $|\bar{S}_k(J)|$ together measure the total variation of $u_{\Delta x}$ on $J$. The following theorem says that $L(J)$ and $\bar{S}_k(J)$ are uniformly bounded, provided the initial perturbation is small.

THEOREM 3.1. *Consider the perturbation problem* (1.1), (1.2) *with the initial perturbation*

$$\mathrm{TV} \equiv \underset{x<0}{\mathrm{TV}} |u_l(x) - u_l^0| + \underset{x>0}{\mathrm{TV}} |u_r(x) - u_r^0|. \tag{3.1}$$

*Under Assumption* A *with $\eta$ sufficiently small, there exist positive constants $C$ and $\varepsilon_0$, which depend on $f$, $u_l^0$, $u_r^0$ and $\eta$ only, such that, for any S-curve $J$,*

$$|L(J) - L(0)| \leqq C\,\mathrm{TV}, \tag{3.2}$$

$$|\bar{S}_k(J) - \bar{S}_k(0)| \leqq C\,\mathrm{TV}, \tag{3.3}$$

*provided* $\mathrm{TV} \leqq \varepsilon_0$.

The proof of this theorem will be broken below into several lemmas. The total variation of $u_{\Delta x}$ will be estimated inductively on the S-curves with each step advancing the S-curve by one diamond. The next two lemmas are devoted to the estimates of the wave interaction of two Riemann problems in a diamond. In Lemma 3.2 (originally due to Glimm), we consider the case when $S_k$ is not in the diamond (see Figure 3.1). Let $u_l$, $u_m$, $u_r$ be constant states which are either all in $U_l$ or all in $U_r$. Let $a_i$, $b_i$, $c_i$ denote, respectively, the state differences of the $i$-th waves of the Riemann problems $(u_l, u_m)$, $(u_m, u_r)$ and $(u_l, u_r)$.

LEMMA 3.2 (see [5], [11]). *Consider Figure* 3.1. *Under Assumption* A *with $\eta$ sufficiently small, we have*

$$c_i = a_i + b_i + O(1)Q(\Delta), \tag{3.4}$$

$$\bar{c}_i = \bar{a}_i + \bar{b}_i + O(1)Q(\Delta). \tag{3.4$'$}$$

*Here, $Q(\Delta) = Q(\{a_i\}_{i=1}^N, \{b_i\}_{i=1}^N)$, the function $Q$ is defined by* (2.2), *and $O(1)$ is bounded by a constant $K_1(f, u_l^0, u_r^0, \eta)$.*

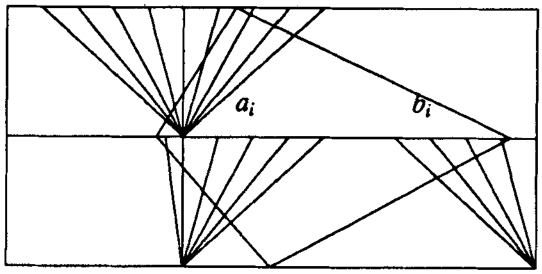


Figure 3.1

Thus, the interactions among small waves only generate second-order small waves. The next lemma considers the case when $S_k$ is in the diamond (see Figure 3.2). In this case, the small waves interacting with the strong shock will produce waves with the same order of magnitude. We only consider the case in which $S_k$ is in the left family of the incoming waves. The other case can be treated similarly. Let $u_l$, $u_m$ and $u_r$ be constant states such that $u_l, u_m \in U_l$ and $u_r \in U_r$. Further, let $\{a_1,\cdots, a_N\}$, $\{b_1,\cdots, S_k,\cdots, b_N\}$ and $\{c_1,\cdots, S_k',\cdots, c_N\}$ denote the state differences of the waves associated with the Riemann problems $(u_l, u_m)$, $(u_m, u_r)$ and $(u_l, u_r)$, respectively. Define

$$A(\Delta) = \sum_{l \geqq k} |\bar{a}_l|, \tag{3.5}$$

$$Q(\Delta) = \sum_{i,\, j<k}' |\bar{a}_i|\,|\bar{b}_j| + \sum_{\substack{l \geqq k \\ i \neq l}} |\bar{a}_i|\,|\bar{a}_l| + \sum_{\substack{l \geqq k \\ j \neq k}} |\bar{a}_l|\,|\bar{b}_j|. \tag{3.6}$$

LEMMA 3.3. *Consider Figure 3.2. Under Assumption* **A** *with $\eta$ sufficiently small, for each $a_l$, $l \geq k$, there exists a set of waves $\{a_{l,i} \mid i = 1,\cdots, N,\ i \neq k\}$*

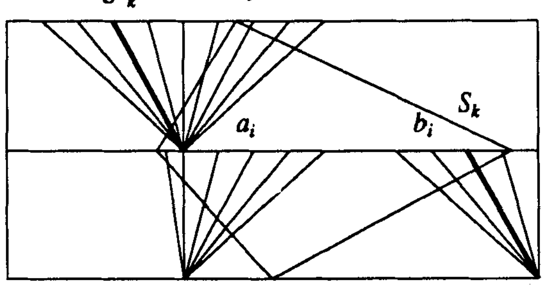


Figure 3.2

*such that*

$$|\bar{a}_{l,i}| = O(1)|\bar{a}_l| + O(1)Q(\Delta). \tag{3.7}$$

*The waves $c_i$, $S_k'$ satisfy*

$$c_i = a_i + b_i + \sum_{l\geq k} a_{l,i} + O(1)Q(\Delta), \qquad i < k, \tag{3.8}$$

$$c_i = b_i + \sum_{l\geq k} a_{l,i} + O(1)Q(\Delta), \qquad i > k, \tag{3.9}$$

$$|S_k' - S_k| = O(1)(A(\Delta) + Q(\Delta)), \tag{3.10}$$

*and*

$$\bar{c}_i = \bar{a}_i + \bar{b}_i + \sum_{l\geq k} \bar{a}_{l,i} + O(1)Q(\Delta), \qquad i < k, \tag{3.8$'$}$$

$$\bar{c}_i = \bar{b}_i + \sum_{l\geq k} \bar{a}_{l,i} + O(1)Q(\Delta), \qquad i > k, \tag{3.9$'$}$$

$$|\bar{S}_k' - \bar{S}_k| = O(1)(A(\Delta) + Q(\Delta)). \tag{3.10$'$}$$

*Here, $A(\Delta), Q(\Delta)$ are defined by* (3.5) *and* (3.6), *and all $O(1)$ functions are bounded by a constant $K_2(f, u_l^0, u_r^0, \eta)$.*

*Remark.* We shall call $a_l$, for $l \geq k$, the incident waves, and $a_{l,i}$, for $l \geq k$, $i \neq k$, the scattered waves.

Proof: The interaction process can be broken into a sequence of elementary interactions. Each of them is either between two sets of small waves or between the strong shock and a set of incident waves so that Lemma 3.2 and Assumption A can apply to these interactions. The idea is to lump all slow waves ($i < k$) from a single Riemann problem into one packet and the fast waves ($i \geq k$) into another. In Figure 3.3, each box represents such an elementary interaction. The figure suggests a possible space-time location for such interactions. However, these locations are only schematic or conceptual devices for reducing bounds on multiple wave interactions. For example, in order for wave $a_N$ to interact with $S_k$, $a_N$ has to first interact with $\{b_1, \cdots, b_{k-1}\}$. This yields the waves $\{a_{N+1,i}^N\}_{i=1}^N$. By Lemma 3.2,

$$a_{N+1,i}^N = b_i + O(1)Q^N, \qquad i < k,$$

$$a_{N+1,i}^N = O(1)Q^N, \qquad k \leq i < N,$$

$$a_{N+1,N}^N = a_N + O(1)Q^N,$$

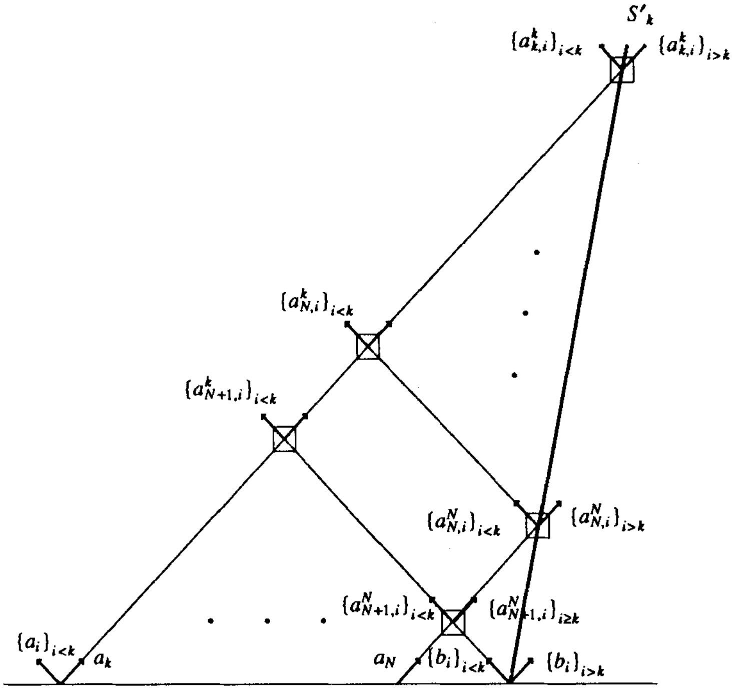


Figure 3.3

where $Q^N = Q(\{a_N\}, \{b_i\}_{i<k})$. Next, $\{a^N_{N+1,i}\}_{i\geqq k}$ will interact with $S_k$. The resulting waves are $\{a^N_{N,i}\}_{i\neq k}$ and the strong shock $S^N_k$. By Assumption **A**,

$$\begin{aligned} |a^N_{N,i}| &= O(1)|\overline{a^N_{N+1,N}}| + O(1)Q^N \\ &= O(1)|\bar{a}_N| + O(1)Q^N, \qquad\qquad i \neq k, \\ |S^N_k - S_k| &= O(1)|\bar{a}_N| + O(1)Q^N. \end{aligned}$$

These processes are repeated until all waves $a_l$, $l \geqq k$, have interacted with the strong shock. The resulting waves can be classified into the following three families:

$$\text{The left family: } \{a_i\}_{i<k}, \{a^k_{l,i}\}_{i<k}, \qquad k \leqq l \leqq N+1.$$

$$\text{The right family: } \{a^l_{l,i}\}_{i>k}, \qquad k \leqq l \leqq N.$$

$$\text{The strong shock: } S^k_k.$$

The estimates on these waves can be obtained by applying Lemma 3.2 and

Assumption A to each elementary interaction in Figure 3.3:

$$a_{N+1,i}^{k} = b_i + O(1)Q^k, \qquad i < k,$$

$$|\overline{a_{l,i}^{k}}| = O(1)\big(|\bar{a}_l| + Q^k\big), \qquad k \leqq l \leqq N,\ i < k,$$

$$|\overline{a_{l,i}^{l}}| = O(1)\big(|\bar{a}_l| + Q^k\big), \qquad k \leqq l \leqq N,\ i > k,$$

$$|S_k^k - S_k| = O(1)\Big(\sum_{l \geqq k} |\bar{a}_l| + Q^k\Big).$$

Here,

$$Q^k = \sum_{\substack{l \geqq k \\ i < k}} |\bar{a}_l|\,|\bar{b}_i| + \sum_{\substack{l,\, m \geqq k \\ l \neq m}} |\bar{a}_l|\,|\bar{a}_m|,$$

and all $O(1)$ functions are bounded by the constant $K_1$ of Lemma 3.2 times some constant $C(N)$. We then define the scattered waves $a_{l,i}$ to be $a_{l,i}^k$ for $i < k$ and to be $a_{l,i}^l$ for $i > k$. Next, by Lemma 3.2, the interaction among the left family waves yields the waves $a_i + b_i + \sum_{l \geqq k} a_{l,i} + O(1)Q^L$ for $i < k$, and $O(1)Q^L$ for $i \geqq k$, where

$$Q^L = Q^k + \sum_{\substack{l \geqq k \\ i < k}} |\bar{a}_i|\,|\bar{a}_l| + \sum_{\substack{i < k \\ j < k}}{}' |\bar{a}_i|\,|\bar{b}_j|.$$

Similarly, the interaction among the right family waves yields $b_i + \sum_{l \geqq k} a_{l,i} + Q^R$ for $i > k$, and $O(1)Q^R$ for $i \leqq k$, with

$$Q^R = Q^k + \sum_{\substack{l \geqq k \\ i > k}} |\bar{b}_i|\,|\bar{a}_l|.$$

Finally, the interaction of the strong shock $S_k^k$ with the resulting waves of the above two interactions yields the outgoing waves $c_1, \cdots, S_k', \cdots, c_N$. Estimates (3.8)–(3.10) then follow by Assumption A. The above proof is also applicable to the wave strength estimations (3.8)′–(3.10)′. This is because the wave strength is measured through a monotonic parameter on $T_i$ and this parameter is smooth.

Proof of Theorem 3.1: We shall estimate the differences of $L$ and $S_k$ on any two S-curves $J_1$ and $J_2$ with $J_1 \leqq J_2$. Let us denote the region between $J_1$ and $J_2$ by $\Lambda_{J_1, J_2}$. By applying Lemmas 3.2, 3.3 successively, we obtain

$$|L(J_2) - L(J_1)| \leqq K\big[A(J_1, J_2) + Q(\Lambda_{J_1, J_2})\big], \tag{3.11}$$

$$|S_k(J_2) - S_k(J_1)| \leqq K\big[A(J_1, J_2) + Q(\Lambda_{J_1, J_2})\big], \tag{3.12}$$

where

$$K \equiv \max\{1, NK_1, NK_2\},$$

$$(3.13)\quad A(J_1, J_2) \equiv \sum \{A(\Delta) \mid \Delta \subset \Lambda_{J_1, J_2} \text{ and } S_k \text{ passes through } \Delta\},$$

$$Q(\Lambda_{J_1, J_2}) \equiv \sum \{Q(\Delta) \mid \Delta \subset \Lambda_{J_1, J_2}\}.$$

$A(J_1, J_2)$ and $Q(\Lambda_{J_1, J_2})$ measure, respectively, the total amounts of the scattered waves and the second-order small waves generated in $\Lambda_{J_1, J_2}$. We shall introduce two generating functionals, the potential wave production $\tilde{A}$ and the potential wave interaction $\tilde{Q}$, to bound them. Given any S-curve $J$, we define

$$(3.14)\qquad \tilde{A}(J) \equiv \sum_{a \in W_a(J)} |\bar{a}|,$$

$$(3.15)\qquad \tilde{Q}(J) \equiv \tilde{Q}_{oo}(J) + K\tilde{Q}_{op}(J) + K^2\tilde{Q}_{pp}(J),$$

with

$$\tilde{Q}_{oo} \equiv \sum{}' \{|\bar{a}|\,|\bar{b}| \,|\, a, b \in W(J),\ a, b \text{ are on the same side of } S_k(J)\},$$

$$\tilde{Q}_{op} \equiv \sum \{|\bar{a}|\,|\bar{b}| \,|\, a \in W_a(J),\ b \in W(J),\ a \neq b\} + \frac{1}{2}\sum\{|\bar{a}|^2 \,|\, a \in W_a(J)\},$$

$$\tilde{Q}_{pp} \equiv \sum \{|\bar{a}|\,|\bar{b}| \,|\, a, b \in W_a(J),\ a \neq b\} + \frac{1}{2}\sum\{|\bar{a}|^2 \,|\, a \in W_a(J)\}.$$

We claim that

$$(3.16)\qquad A(J_1, J_2) \leqq \tilde{A}(J_1) - \tilde{A}(J_2) + KQ(\Lambda_{J_1, J_2}),$$

$$(3.17)\quad Q(\Lambda_{J_1, J_2}) \leqq 2(\tilde{Q}(J_1) - \tilde{Q}(J_2)) \quad \text{provided} \quad L(J_1) \leqq \frac{1}{32K^4}.$$

Inequality (3.16) follows directly from the definitions of $A$, $\tilde{A}$ and Lemma 3.2; (3.17) can be proved easily by an induction process on $J_2$ through (3.11), (3.16) and the help of the following lemma.

LEMMA 3.4. *Suppose $J$, $J'$ are two consecutive S-curves with $J$ preceding $J'$ and let $\Delta$ be the diamond between them. Under Assumption* A *with $\eta$ sufficiently small, we have*

$$(3.18)\qquad Q(\Delta) \leqq 2(\tilde{Q}(J) - \tilde{Q}(J')),$$

*provided $L(J) \leqq 1/8K^3$.*

To complete the proof of Theorem 3.1, we choose $\varepsilon_0 = 1/32K^4$, $C = 4K^2$. Then (3.2)–(3.3) follow from (3.11)–(3.12) and (3.16)–(3.17) with $J_1 = 0$, $J_2 = J$. This completes the proof of Theorem 3.1.

Proof of Lemma 3.4: We recall that a rarefaction wave divided by a sample point is treated as two different waves on an S-curve $J$. Thus, if a rarefaction wave $c_i$ is split into $c_i^1$ and $c_i^2$ by a sample point, we have

$$|\bar{c}_i| = |\overline{c_i^1}| + |\overline{c_i^2}|.$$

In the case when a wave $c_i$ is not split by a sample point, this formula is still valid if we define $c_i^1 = c_i$ and $c_i^2 = 0$. With this convention, let us apply Lemmas 3.2–3.3 to estimate $\tilde{Q}(J') - \tilde{Q}(J)$ as follows.

*Case 1. $S_k$ is not in $\Delta$* (see Figure 3.1). Without loss of generality, we may assume that $\Delta$ lies on the left-hand side of $S_k$. From (3.15) and (3.4)′ we obtain

$$\begin{aligned}\tilde{Q}_{oo}(J') - \tilde{Q}_{oo}(J) &= -\sum{}' |\bar{a}_i|\,|\bar{b}_j| + \sum{}' |\bar{d}|\big(|\bar{c}_i| - |\bar{a}_i| - |\bar{b}_i|\big)\\ &\leqq -Q(\Delta) + KL(J)Q(\Delta).\end{aligned}$$

Here, $d \in W(J)$, $d$ is not in $\Delta$, and $d$ is on the left-hand side of $S_k$. We also obtain

$$\begin{aligned}&\tilde{Q}_{op}(J') - \tilde{Q}_{op}(J)\\ &\quad= \sum_{\substack{l \geqq k\\ i \neq l}} \big(|\bar{c}_l|\,|\bar{c}_i| - |\bar{a}_l|\,|\bar{a}_i| - |\bar{b}_l|\,|\bar{b}_i|\big) - \sum_{\substack{l \geqq k\\ i \neq l}} \big(|\bar{b}_l|\,|\bar{a}_i| + |\bar{a}_l|\,|\bar{b}_i|\big)\\ &\qquad+ \sum_{l \geqq k} \Big(\frac{1}{2}\big(|\overline{c_l^1}|^2 + |\overline{c_l^2}|^2\big) + |\overline{c_l^1}|\,|\overline{c_l^2}| - \frac{1}{2}\big(|\bar{a}_l|^2 + |\bar{b}_l|^2\big) - |\bar{a}_l|\,|\bar{b}_l|\Big)\\ &\qquad+ \sum_{l \geqq k} \big(|\bar{c}_l| - |\bar{a}_l| - |\bar{b}_l|\big)|\bar{d}| + \sum |\bar{e}|\big(|\bar{c}_i| - |\bar{a}_i| - |\bar{b}_i|\big).\end{aligned}$$

Here, $d \in W(J)$, $e \in W_a(J)$, $d, e$ are not in $\Delta$. The third term on the right-hand side is equal to

$$\sum_{l \geqq k} \frac{1}{2}\Big(|\bar{c}_l|^2 - \big(|\bar{a}_l| + |\bar{b}_l|\big)^2\Big) \leqq \frac{1}{2} \sum_{l \geqq k} \big(|\bar{c}_l| - |\bar{a}_l| - |\bar{b}_l|\big)\Big(2\big(|\bar{a}_l| + |\bar{b}_l|\big) + KQ(\Delta)\Big).$$

Thus, by (3.4)′, $\tilde{Q}_{op}(J') - \tilde{Q}_{op}(J) \leqq 2KL(J)Q(\Delta)$. From (3.15), (3.4)′ we further have

$$\begin{aligned}&\tilde{Q}_{pp}(J') - \tilde{Q}_{pp}(J)\\ &\quad= \sum_{\substack{l, m \geqq k\\ l \neq m}} \big(|\bar{c}_l|\,|\bar{c}_m| - |\bar{a}_l|\,|\bar{a}_m| - |\bar{b}_l|\,|\bar{b}_m|\big) - \sum_{\substack{l, m \geqq k\\ l \neq m}} |\bar{a}_l|\,|\bar{b}_m|\\ &\qquad+ \sum_{l \geqq k} \Big(\frac{1}{2}\big(|\overline{c_l^1}|^2 + |\overline{c_l^2}|^2\big) + |\overline{c_l^1}|\,|\overline{c_l^2}| - \frac{1}{2}\big(|\bar{a}_l|^2 + |\bar{b}_l|^2\big) - |\bar{a}_l|\,|\bar{b}_l|\Big)\\ &\qquad+ \sum_{l \geqq k} \big(|\bar{c}_l| - |\bar{a}_l| - |\bar{b}_l|\big)|\bar{e}|\\ &\quad\leqq KQ(\Delta)L(J).\end{aligned}$$

Here, $e \in W_a(J)$, and $e$ is not in $\Delta$. Combining these three inequalities, we obtain

$$\tilde{Q}(J') - \tilde{Q}(J) \leqq -Q(\Delta) + (K + 2K^2 + K^3)L(J)Q(\Delta). \tag{3.19}$$

*Case 2. $S_k$ is in $\Delta$* (see Figure 3.2). Let $J_l$ (respectively $J_r$) denotes the portion of $J$ lying to the left (respectively right) of $\Delta$. By Lemma 3.3 and (3.15), we have

$$\begin{aligned}
\tilde{Q}_{oo}(J') - \tilde{Q}_{oo}(J) &= -\sum_{\substack{j<k\\ i}}{}' |\bar{a}_i||\bar{b}_j| + \sum_{\substack{i<k\\ d\in W(J_l)}}{}' |\bar{d}|\big(|\bar{c}_i| - |\bar{a}_i| - |\bar{b}_i|\big)\\
&\quad - \sum_{\substack{l\geqq k\\ d\in W(J_l)}}{}' |\bar{d}||\bar{a}_l| + \sum_{\substack{j>k\\ e\in W(J_r)}}{}' \big(|\bar{c}_j| - |\bar{b}_j|\big)|\bar{e}|\\
&\leqq -\sum_{\substack{j<k\\ i}}{}' |\bar{a}_i||\bar{b}_j| + \sum_{d\in W(J_l\cup J_r)} |\bar{d}|\big(KA(\Delta) + KQ(\Delta)\big),
\end{aligned}$$

$$\begin{aligned}
\tilde{Q}_{op}(J') - \tilde{Q}_{op}(J) &= -\sum_{\substack{l\geqq k\\ i\neq l}} |\bar{a}_l||\bar{a}_i| - \frac{1}{2}\sum_{l\geqq k} |\bar{a}_l|^2\\
&\quad - \sum_{\substack{l\geqq k\\ j\neq k}} |\bar{a}_l||\bar{b}_j| - \sum_{\substack{l\geqq k\\ d\in W(J_l\cup J_r)}} |\bar{a}_l||\bar{d}|\\
&\quad + \sum_{e\in W_a(J_l\cup J_r)} |\bar{e}|\Big\{\sum_{i\neq k} \big(|\bar{c}_i| - |\bar{a}_i| - |\bar{b}_i|\big) - |\bar{a}_k|\Big\}\\
&\leqq -\sum_{\substack{l\geqq k\\ i\neq l}} |\bar{a}_l||\bar{a}_i| - \sum_{\substack{l\geqq k\\ j\neq k}} |\bar{a}_l||\bar{b}_j| - \sum_{d\in W(J_l\cup J_r)} |\bar{d}|A(\Delta)\\
&\quad + \sum_{e\in W_a(J_l\cup J_r)} |\bar{e}|\big(KA(\Delta) + KQ(\Delta)\big),
\end{aligned}$$

$$\begin{aligned}
\tilde{Q}_{pp}(J') - \tilde{Q}_{pp}(J) &= -\sum_{\substack{l,m\geqq k\\ l\neq m}} |\bar{a}_l||\bar{a}_m| - \frac{1}{2}\sum_{l\geqq k} |\bar{a}_l|^2 - \sum_{\substack{l\geqq k\\ e\in W_a(J_l\cup J_r)}} |\bar{a}_l||\bar{e}|\\
&\leqq -\sum_{\substack{l,m\geqq k\\ l\neq m}} |\bar{a}_l||\bar{a}_m| - \sum_{e\in W_a(J_l\cup J_r)} |\bar{e}|A(\Delta).
\end{aligned}$$

Thus, by (3.15),

$$\tilde{Q}(J') - \tilde{Q}(J) \leqq (-1 + KL(J) + K^2L(J))Q(\Delta). \tag{3.20}$$

From (3.19)–(3.20), we obtain (3.18), provided $L(J) \leqq 1/2(K + 2K^2 + K^3)$. Since $K \geqq 1$, $L(J) \leqq 1/8K^3$ is enough to conclude that (3.18) holds.

We summarize this section by the following stability theorem.

THEOREM 3.5 (Stability Theorem). *Under the assumptions of Theorem* 3.1, *the Radon measures* $|D_x u_{\Delta x}|$, $|D_t u_{\Delta x}|$ *of the approximate solutions* $u_{\Delta x}$ *are uniformly bounded on* $\mathbb{R} \times [0, T]$ *for any fixed* $T > 0$.

Proof: We shall proved (i), (ii) and (iii) mentioned in the beginning of this section.

(i). From (3.2), (3.3),

$$\operatorname*{TV}_x u_{\Delta x}(x, t) \leqq S_k^0 + 2C\,\mathrm{TV},$$

where $S_k^0 = |S_k(0)|$.

(ii).

$$|D_x u_{\Delta x}|(\mathbb{R} \times [0, T]) = \int_0^T \operatorname*{TV}_x u_{\Delta x}(x, t)\,dt \leqq (S_k^0 + 2C\,\mathrm{TV})T,$$

$$|D_x f(u_{\Delta x})|(\mathbb{R} \times [0, T]) \leqq \max_u \left|\frac{\partial f}{\partial u}(u)\right| |D_x u_{\Delta x}|(\mathbb{R} \times [0, T]).$$

Notice that we have used a chain rule theorem for the BV-functions (see [14]) in the last formula.

(iii). Let $m = [T/\Delta t] + 1$. Then

$$\begin{aligned} |\nu_{\Delta x}|(\mathbb{R} \times [0, T]) &= \sum_{n=0}^{m} \int |u_{\Delta x}(x, n\,\Delta t+) - u_{\Delta x}(x, n\,\Delta t-)|\,dx \\ &\leqq \sum_{n=0}^{m} \operatorname*{TV}_x u_{\Delta x}(x, n\,\Delta t-)N\Delta x \\ &\leqq (S_k^0 + 2C\,\mathrm{TV})N\frac{\Delta x}{\Delta t}T. \end{aligned}$$

## 4. Truncation Error Analysis and Consistency Theorem

The truncation error of the approximate solution $u_{\Delta x}$ is defined to be the Radon measure $\nu_{\Delta x} \equiv D_t u_{\Delta x} + D_x f(u_{\Delta x})$. The goal of this section is to prove the consistency theorem, i.e., that $\nu_{\Delta x} \to 0$ weakly in the space of Radon measures on $\mathbb{R} \times [0, \infty)$ as $\Delta x \to 0$, provided the sequence $\{\theta^n\}$ is equidistributed on $(-\frac{1}{2}, \frac{1}{2})$, where $\{\theta^n\}$ is the sequence used in Glimm's construction of the approximate solutions. We recall that $|\nu_{\Delta x}|$ are uniformly bounded on $\mathbb{R} \times [0, T]$ for any $T > 0$. Therefore, it is enough to show that $\langle \nu_{\Delta x}, \phi \rangle \to 0$ as $\Delta x \to 0$ for any test function $\phi \in C^1(\mathbb{R} \times [0, \infty))$ with compact support. We shall use the wave tracing technique of Liu [11] to evaluate $\langle \nu_{\Delta x}, \phi \rangle$. The technique will first be demonstrated through partition of waves in diamonds. Then, it will be applied to a region bounded by two S-curves.

**4.1. Wave tracing.** We recall that the notation "$a$" stands for both the wave itself and its state difference.

DEFINITION 4.1 (see [11]). A partition of an $i$-wave $a_i = (u_{i-1}, u_i)$ is a collection of "waves" $\{a_i^h\}_{h=1}^p$ associated with speeds $\{\lambda_i^h\}_{h=1}^p$ with

$$a_i^h = y^h - y^{h-1},$$

$$\lambda_i^h = \begin{cases} \sigma_i(u_{i-1}, u_i) & \text{if } a_i \text{ is a shock,} \\ \lambda_i(y^{h-1}) & \text{otherwise,} \end{cases}$$

where $(y^0, \cdots, y^p)$ is a partition of the interval $(u_{i-1}, u_i)$ on the wave curve $T_i(u_{i-1})$ in $\mathbb{R}^N$ with $y^0 = u_{i-1}$, $y^p = u_i$.

DEFINITION 4.2. Two waves $a, b$ of the same family are said to satisfy the correspondence relation with bound $B$ if they have equal strength and satisfy the following condition:

Any partition on wave $a$, say $\{a^h\}_h$ associated with speeds $\{\lambda^h\}_h$, corresponds to a partition on wave $b$, say $\{b^h\}_h$ associated with speeds $\{\mu^h\}_h$, such that

$$\sum_h |b^h - a^h| \leqq B,$$

$$\sum_h |\lambda^h - \mu^h|(|a^h| + |b^h|) \leqq B.$$

We now study the wave partition in diamonds.

*Case 1. $S_k$ is not in $\Delta$* (see Figure 3.1). Let us denote, in Figure 3.1, the side states of the wave $a_i, b_i, c_i$ by $(u_{i-1}, u_i), (v_{i-1}, v_i), (w_{i-1}, w_i)$, respectively. The interaction formula (3.4)′ induces a natural partition on wave curves $T_i(u_{i-1})$, $T_i(v_{i-1})$, and $T_i(w_{i-1})$ through the monotonic parameter on $T_i$. It also induces a natural wave correspondence in the sense of equal strength between partitioned incoming waves and partitioned outgoing waves. More precisely, we have the following lemma.

LEMMA 4.3 (see [11]). *Consider Figure* 3.1; *the interaction formula* (3.4)′ *induces a partition on waves* $a_i$, $b_i$, $c_i$, *which has the following properties*:

*The partitioned incoming waves* $\{a_i\}_{i=1}^N, \{b_i\}_{i=1}^N$ *can be classified into translated waves, cancelled waves and quadratic waves. The partitioned outgoing waves* $\{c_i\}_{i=1}^N$ *can be classified into translated waves and quadratic waves. These partitioned waves satisfy the properties below.*

(i) *The total amount of quadratic waves is bounded by* $K_1Q(\Delta)$.

(ii) *There is a one-to-one correspondence between the translated incoming waves and the translated outgoing waves, i.e.,* $d \leftrightarrow e$ *with strength of* $d$ *equal to strength of* $e$. *Moreover, these correspondences satisfy the correspondence relation with bound* $K_3Q(\Delta)$ *for some positive constant* $K_3$ *depending on* $f$, $u_l^0$, $u_r^0$, *and* $\eta$.

*Case* 2. $S_k$ *is in* $\Delta$ (see Figure 3.2).

LEMMA 4.4. *Consider Figure* 3.2; *the interaction formulae* (3.8)′–(3.9)′ *induce partitions on the incoming waves* $\{a_i\}_{i<k}, \{b_i\}_{i\neq k}$, *the scattered waves* $\{a_{l,i}\}_{l\geqq k,\, i\neq k}$ *and the outgoing waves* $\{c_i\}_{i\neq k}$, *which have the following properties.*

*The partitioned incoming waves and scattered waves can be classified into translated waves, cancelled waves and quadratic waves. The partitioned outgoing waves can be classified into translated waves and quadratic waves. These partitioned waves satisfy the properties below.*

(i) *The total amount of the quadratic waves is bounded by* $K_2Q(\Delta)$.

(ii) *There are one-to-one wave correspondences between the translated outgoing waves and the union of the translated incoming waves and the translated scattered waves. Moreover, the correspondences satisfy the correspondence relation with bound* $K_4Q(\Delta)$ *for some positive constant* $K_4$ *depending on* $f$, $u_l^0$, $u_r^0$, *and* $\eta$.

Proof: We shall demonstrate how to carry out the partition and to build up the correspondence relation (ii) for waves on the left-hand side of the strong shock. Waves on the other side can be handled similarly. Following the proof of Lemma 3.3, let us denote the constant states separating the waves generated from the Riemann problems $(u_l, u_m), (u_m, u_r), (u_l, u_r)$ by $\{u_i\}_{i=0}^N$, $\{v_i\}_{i=0}^N$, and $\{w_i\}_{i=0}^N$, respectively. We recall that the outgoing waves $c_i$, $i<k$, are, with a possible second-order error, the resulting waves of the interactions among the waves $\{a_i\}_{i<k}, \{a_{n+1,i}^k\}_{i<k}, \{a_{l,i}\}_{i<k,\, l\geqq k}$. These interactions satisfy

$$\bar{c}_i = \bar{a}_i + \overline{a_{N+1,i}^k} + \sum_{l\geqq k} \bar{a}_{l,i} + O(1)Q(\Delta) \qquad i<k.$$

We can then find the constant states $\tilde{w}_\alpha, \tilde{w}_{N+1}, \cdots, \tilde{w}_k$ on $T_i(w_{i-1})$ in $\mathbb{R}^N$ (see Figure 4.1) such that

$$\lambda_i(\tilde{w}_\alpha) - \lambda_i(w_{i-1}) = \bar{a}_i,$$

$$\lambda_i(\tilde{w}_{N+1}) - \lambda_i(\tilde{w}_\alpha) = \overline{a_{N+1,i}^k},$$

$$\lambda_i(\tilde{w}_l) - \lambda_i(\tilde{w}_{l+1}) = \bar{a}_{l,i}, \qquad k \leqq l \leqq N.$$

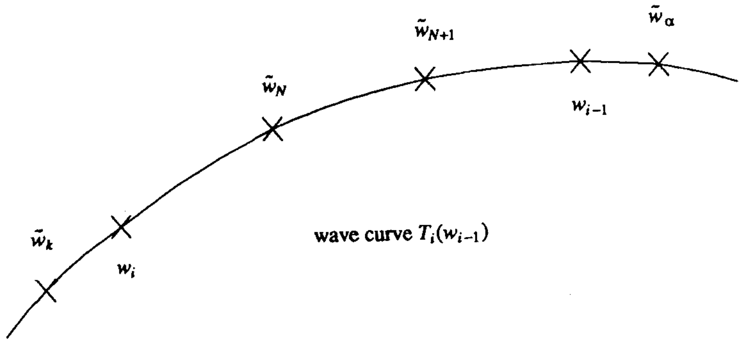


Figure 4.1

By applying Lemma 4.3 repeatedly on the pairwise interactions among $\{a_i\}_{i<k}, \{a^k_{N+1,i}\}_{i<k}, \{a_{l,i}\}_{i<k,l\geqq k}$, we see that the correspondences $(w_{i-1}, \tilde{w}_\alpha) \leftrightarrow a_i, (\tilde{w}_\alpha, \tilde{w}_{N+1}) \leftrightarrow a^k_{N+1,i}, \cdots, (\tilde{w}_{k+1}, \tilde{w}_k) \leftrightarrow a^k_{k,i}$ satisfy the correspondence relation with bound $K_4 Q(\Delta)$ for some positive constant $K_4$. The path $(w_{i-1}, \tilde{w}_\alpha, \tilde{w}_{N+1}, \cdots, \tilde{w}_k)$ on $T_i(w_{i-1})$ determines a partition on the wave curve $T_i(w_{i-1})$. This partition also induces partitions on wave curves $T_i(u_{i-1}), T_i(u_{l,i-1})$, $l = k, \cdots, N+1$, according to the above wave correspondence, where $u_{l,i-1}$ is the left state of wave $a_{l,i}$. The classification of these partitioned waves is determined by assigning: (a) the overlapping parts of the path to the cancelled waves, (b) the net parts—the summing-over of the path—to the translated waves, and (c) the rest to the quadratic waves. Finally, by Lemma 4.3, with the possible exception of quadratic waves, $a^k_{N+1,i}$ and $b_i$ satisfy a correspondence relation with bound $K_3 Q(\Delta)$. Therefore, the partition and classification on $a^k_{N+1,i}$ induced by the interaction (4.2) determine a partition and classification on $b_i$. This completes the build-up of the correspondence relations (ii) for waves on the left-hand side of $S_k$.

Let us study the wave tracing technics in a region bounded by two S-curves now. Suppose $J_1$, $J_2$ are two S-curves, $J_1 \leqq J_2$. Further, let

$$K \equiv \max\{1, NK_1, NK_2, NK_3, NK_4\}.$$

THEOREM 4.5. *There exist wave partitions for the waves in* $\Lambda_{J_1, J_2}$ *other than the strong shock* $S_k$ *and for the scattered waves generated from the strong shock boundary between* $J_1$ *and* $J_2$. *These partitioned waves have the following classification. Suppose* $J$ *is any S-curve between* $J_1$ *and* $J_2$. *The partitioned waves on* $J$ *are classified into* $W_t(J; J_1, J_2)$*—the set of translated waves, and* $W_q(J; J_1, J_2)$*—the set of quadratic waves. The partitioned scattered waves between* $J_1$ *and* $J_2$ *are* *classified into* $V_t(J_1, J_2)$*—the translated scattered waves, and* $V_q(J_1, J_2)$*—the quadratic scattered waves.* $W_t, W_q, V_t, V_q$ *have the following properties*:

(i) *for any* $J$, $J_1 \leqq J \leqq J_2$,

$$\sum_{a \in W_q(J;\, J_1, J_2)} |a| \leqq KQ(\Lambda_{J_1, J_2}), \tag{4.1}$$

$$\sum_{a \in V_q(J_1, J_2)} |a| \leqq KQ(\Lambda_{J_1, J_2}). \tag{4.2}$$

(ii) *Each* $a \in W_t(J_1;\, J_1, J_2) \cup V_t(J_1, J_2)$ *corresponds to a trace of waves in* $\Lambda_{J_1, J_2}$, *namely, there exists a sequence of waves* $\tilde{a}(n, a)$ *for time step* $n$ *between two positive numbers* $n_1(a), n_2(a)$ *with* $\tilde{a}(n_1(a), a) = a$. *Moreover, for any* $J$ *with* $J_1 \leqq J \leqq J_2$, $W_t(J;\, J_1, J_2)$ *is the disjoint union of*

$$\left\{\tilde{a}(n, a) \,|\, a \in W_t(J_1;\, J_1, J_2) \cup V_t(J_1, J_2),\ \tilde{a}(n, a) \text{ passes through } J\right\}. \tag{4.3}$$

*If we denote the state difference of the wave* $\tilde{a}(n, a)$ *by the same notation* $\tilde{a}(n, a)$, *its wave speed by* $\tilde{\lambda}(n, a)$ *and its x-mesh index by* $\tilde{j}(n, a)$, *then*

$$\sum_{a \in W_t(J_1;\, J_1, J_2) \cup V_t(J_1, J_2)} \left\{|\tilde{a}(n, a) - a| \,\big|\, \tilde{a} \text{ is on } J\right\} \leqq KQ(\Lambda_{J_1, J}), \tag{4.4}$$

$$\sum_{a \in W_t(J_1;\, J_1, J_2) \cup V_t(J_1, J_2)} \left\{|\tilde{\lambda}(n, a) - \tilde{\lambda}(n_1(a), a)|\,|a| \,\big|\, \tilde{a} \text{ is on } J\right\} \leqq KQ(\Lambda_{J_1, J}), \tag{4.5}$$

$$|\tilde{j}(n, a) - \tilde{j}(m, a)| \leqq m - n \quad \text{for} \quad n < m. \tag{4.6}$$

*Remark.* $n_1(a), n_2(a)$ will be called the starting and terminating time steps of the wave $a$, respectively. (4.3) means that all waves in $W_t(J;\, J_1, J_2)$ can be traced back to a wave on $J_1$ or to a scattered wave on the strong shock boundary. However, while tracing a wave forward in time the wave may terminate before leaving $\Lambda_{J_1, J_2}$. This is due to the fact that it may be cancelled by another wave or hit the strong shock.

Proof: The proof is an induction procees on $J_2$. In the initial induction step, we put all waves on $J_1$ into $W_t(J_1;\, J_1, J_1)$, and all scattered waves issued from the boundary of $S_k(J_1)$ into $V_t(J_1, J_1)$. The starting time steps of these waves are set to be the time step where they are issued. Now suppose all waves below $J_2$, $J_2 \geqq J_1$, are partitioned, classified, and satisfy (4.1)–(4.6). Suppose $J_2'$ is an immediate successor of $J_2$. Suppose $\Delta$ is the diamond between them. Let us first consider the case when $S_k$ is not in $\Delta$.

*Case 1.* $S_k$ *is not in* $\Delta$. The wave partition of Lemma 4.3 in $\Delta$ induces further partitions on those waves which are in $W_t(J_2;\, J_1, J_2)$ and are the incoming waves of $\Delta$. These partitions also induce a sequence of further partitions on the backward traces of these waves. The reclassification of them is determined as follows. Those which correspond to quadratic incoming waves of $\Delta$ are reclassified as quadratic waves. The rest stays as translated waves. This defines $W_t(J; J_1, J_2')$, $W_q(J; J_1, J_2')$, $V_t(J_1, J_2')$, $V_q(J_1, J_2')$ for $J_1 \leqq J \leqq J_2$. Furthermore, we define the terminating time step of those waves which correspond to a cancelled incoming wave of $\Delta$ to be the time step of the incoming waves of $\Delta$. We also define the terminating time step of those waves which correspond to a translated incoming wave of $\Delta$ to be the time step of the outgoing wave of $\Delta$. The outgoing waves of $\Delta$ will also be further partitioned and reclassified. Quadratic outgoing waves are classified into $W_t(J_2'; J_1, J_2')$. Translated outgoing waves are further partitioned and reclassified according to the partition and classification of their corresponding incoming waves in $W_t(J; J_1, J_2')$ and $W_q(J_2; J_1, J_2')$. This completes the partition and classification for waves below $J_2'$.

From the above construction, (4.3) and (4.6) are obvious. To show (4.1), from (i) of Lemma 4.3, for $J_1 \leqq J \leqq J_2$, we have

$$\begin{aligned}
\sum\left\{|a| \,|\, a \in W_q(J; J_1, J_2')\right\} &= \sum\left\{|a| \,|\, a \in W_q(J; J_1, J_2)\right\} \\
&\quad + \sum\{|a| \,|\, a \text{ is partitioned from a wave in} \\
&\qquad W_t(J; J_1, J_2) \text{ and } a \text{ corresponds to a} \\
&\qquad \text{quadratic incoming wave of } \Delta\} \\
&\leqq KQ(\Lambda_{J_1, J_2}) + KQ(\Delta) = KQ(\Lambda_{J_1, J_2'}),
\end{aligned}$$

$$\begin{aligned}
\sum\left\{|a| \,|\, a \in W_q(J_2'; J_1, J_2')\right\} &= \sum\left\{|a| \,|\, a \in W_q(J_2; J_1, J_2)\right\} \\
&\quad + \sum\{\text{quadratic outgoing waves of } \Delta\} \\
&\leqq KQ(\Lambda_{J_1, J_2'}).
\end{aligned}$$

To show (4.2), we have, from (i) of Lemma 4.4,

$$\begin{aligned}
\sum\left\{|a| \,|\, a \in V_q(J_1, J_2')\right\} &= \sum\left\{|a| \,|\, a \in V_q(J_1, J_2)\right\} \\
&\quad + \sum\left\{|a| \,|\, a \text{ is partitioned from a wave in } V_t(J_1, J_2)\right. \\
&\qquad \text{and } a \text{ corresponds to a quadratic} \\
&\qquad \left.\text{incoming waves of } \Delta\right\} \\
&\leqq KQ(\Lambda_{J_1, J_2'}).
\end{aligned}$$

To show (4.4), we see, using (ii) of Lemma 4.3 and Definition 4.2, that, for $J_1 \leqq J \leqq J_2$,

$$\sum_{a \in W_l(J_1;\, J_1, J_2') \cup V_l(J_1, J_2')} \left\{ |\tilde{a}(n, a) - a| \,\middle|\, \tilde{a}(n, a) \text{ is on } J \right\}$$

$$\leqq \sum_{a \in W_l(J_1;\, J_1, J_2) \cup V_l(J_1, J_2)} \left\{ |\tilde{a}(n, a) - a| \,\middle|\, \tilde{a}(n, a) \text{ is on } J \right\}$$

$$\leqq KQ(\Lambda_{J_1, J})$$

and

$$\sum_{a \in W_l(J_1;\, J_1, J_2') \cup V_l(J_1, J_2')} \left\{ |\tilde{a}(n, a) - a| \,\middle|\, \tilde{a}(n, a) \text{ is on } J_2' \right\}$$

$$\text{(4.7)} \quad \leqq \sum_{a \in W_l(J_1;\, J_1, J_2) \cup V_l(J_1, J_2)} \left\{ |\tilde{a}(n, a) - a| \,\middle|\, \tilde{a}(n, a) \text{ is on } J_2 \text{ and not in } \Delta \right\}$$

$$+ \sum \left\{ |\tilde{a}(n, a) - a| \,\middle|\, \tilde{a}(n, a) \text{ is a translated outgoing wave of } \Delta \right\}.$$

The last term is bounded by

$$\sum \left\{ |\tilde{a}(n, a) - \tilde{a}(n-1, a)| + |\tilde{a}(n-1, a) - a| \,\middle|\, \tilde{a}(n-1, a) \text{ is on } J_2 \text{ and in } \Delta \right\}.$$

Thus, (4.7) is bounded by

$$\sum_{a \in W_l(J_1;\, J_1, J_2) \cup V_l(J_1, J_2)} \left\{ |\tilde{a}(n, a) - a| \,\middle|\, \tilde{a}(n, a) \text{ is on } J_2 \right\}$$

$$+ \sum \left\{ |\tilde{a}(n, a) - \tilde{a}(n-1, a)| \,\middle|\, \tilde{a}(n-1, a) \text{ is a translated incoming wave of } \Delta \right\}$$

$$\leqq KQ(\Lambda_{J_2, J_2'}).$$

The proof of (4.5) is the same as that of (4.4). This completes the proof of case 1

*Case 2.* $S_k$ *is in* $\Delta$. The proof of this case is similar to that of case 1 with the incoming waves replaced by the incoming waves $\{a_i\}_{i<k}, \{b_i\}_{i \neq k}$ and the scattered waves $\{a_{l,i}\}_{l \geqq k,\, i \neq k}$ (see Figure 3.2). We shall not repeat this proof.

**4.2. Truncation error.** We require the equidistributed sequence $\{\theta^n\}_{n=0}^{\infty}$ to satisfy the following condition; see [7][8]: There is a constant $C$ such that, given any subinterval $I \subset (-\frac{1}{2}, \frac{1}{2})$, any $0 \leqq n_1 \leqq n_2$,

$$(4.8)\quad |\#\{n_1 \leqq n \leqq n_2, \theta^n \in I\} - |I|(n_2 - n_1)| \leqq C \log(2 + n_2 - n_1).$$

Here, $|I|$ is the length of $I$.

THEOREM 4.6 (Consistency Theorem). *Under the assumption of Theorem* 3.1 *and assuming the sequence* $\{\theta^n\}_{n=0}^{\infty}$ *satisfies* (4.8), *the truncation errors of the Glimm approximate solutions satisfy*

$$(4.9)\quad |\langle \nu_{\Delta x}, \phi\rangle| \leqq C(\mathrm{TV} + S_k^0)\left((\Delta t)^{1/2}|\log(\Delta t)|\,|\phi|_\infty + (\Delta t)^{1/2}|\nabla\phi|_\infty\right)$$

*for any test function* $\phi \in C^1(\mathbb{R} \times [0, \infty))$ *with compact support. Here, C is some positive constant depending on* $f$, $u_l^0$, $u_r^0$, *and* $\eta$.

Proof: We shall basically follow the proof of Liu [11], [7]. The difference consists in the fact that we need to trace the translated scattered waves and the strong shock, and to trace backward the so called cancelled waves in [11] to simplify the proof.

Suppose the support of $\phi$ is contained in $\mathbb{R} \times [0, T]$ for some $T > 0$. Then,

$$\begin{aligned}|\langle \nu_{\Delta x}, \phi\rangle| &= \left|\sum_{n=1}^{\infty} E^n + \int [u_{\Delta x}(x, 0+) - u_{\Delta x}(x, 0)]\phi(x, 0)\, dx\right| \\ &\leqq \left|\sum_{p=0}^{L-1}\sum_{n=1}^{M} E^{pM+n}\right| + O(\Delta x)|\phi_x|_\infty(\mathrm{TV} + S_k^0).\end{aligned}$$

Here, $M$ is an arbitrary positive integer to be determined later, $L$ is an integer such that $LM\Delta t \geqq T$, and

$$\begin{aligned}E^n &= \int [u_{\Delta x}(x, n\Delta t+) - u_{\Delta x}(x, n\Delta t-)]\phi(x, n\Delta t)\, dx \\ &= \sum_{j=-\infty}^{\infty}\int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} [u_{\Delta x}(x, n\Delta t+) - u_{\Delta x}(x, n\Delta t-)] \qquad (4.10)\\ &\qquad\cdot\phi(j\Delta x, n\Delta t)\, dx + O_1,\end{aligned}$$

$$|O_1| \leqq O((\Delta x)^2)|\phi_x|_\infty(\mathrm{TV} + S_k^0).$$

The estimate of $O_1$ follows from (3.2)–(3.3). Let $J^n$ be the entire S-curve between $t=(n-1)\,\Delta t$ and $t=n\,\Delta t$, and $\Lambda^p$ the region between $J^{pM}$ and $J^{(p+1)M}$. We shall apply the wave tracing technique in $\Lambda^p$ to evaluate $\Sigma_{n=1}^{M} E^{pM+n}$. To demonstrate such a calculation, we may, without loss of generality, assume $p=0$. In $\Lambda^0$, we require that all rarefaction waves are partitioned so finely that the state difference of each of them has norm less than a given small number $\varepsilon>0$. In (4.10), we have

$$\begin{aligned}\int_{(j-1/2)\,\Delta x}^{(j+1/2)\,\Delta x}&[u_{\Delta x}(x, n\,\Delta t+) - u_{\Delta x}(x, n\,\Delta t-)]\,\phi(j\Delta x, n\,\Delta t)\,dx\\ &= \sum a[\lambda(a) + e^n(\lambda(a))]\,\phi(j\Delta x, n\,\Delta t)\\ &\quad + O(\varepsilon)\,\Delta t\,\phi(j(a)\,\Delta x, n\,\Delta t) \operatorname*{TV}_{x\in((j-1/2)\,\Delta x,(j+1/2)\,\Delta x)} u_{\Delta x}(x, n\,\Delta t-).\end{aligned}$$

The summation is over all partitioned waves $a$ issued from $(j\Delta x,(n-1)\,\Delta t)$, $\lambda(a)$ denotes the speed of wave $a$, $j(a)\equiv j$ is the $x$-mesh index where the wave $a$ is situated, and

$$e^n(\lambda) \equiv -\frac{\Delta x}{2\,\Delta t}\left(\operatorname{sign}(\theta^n) + \operatorname{sign}\left(\lambda\frac{\Delta t}{\Delta x} - \theta^n\right)\right). \tag{4.11}$$

Hence, in (4.10), instead of summing over $j$, we can sum over all partitioned waves $a$ passing through $J^n$. According to Theorem 4.5, the partitioned waves on $J^n$ are classified into the translated waves, the quadratic waves and the strong shock. Thus,

$$\sum_{n=1}^{M} E^n = \sum_{n=1}^{M}\left(E_t^n + E_q^n + E_{S_k}^n\right) + O_1 + O_2,$$

$$E_*^n \equiv \sum_{a\in W_*(J_n)} a[\lambda(a) + e^n(\lambda(a))]\,\Delta t\,\phi(j(a)\,\Delta x, n\,\Delta t), \quad * = t, q,$$

$$E_{S_k}^n \equiv S_k^{n-1}\left[\sigma_k^{n-1} + e^n\left(\sigma_k^{n-1}\right)\right]\phi\left(j\left(S_k^{n-1}\right)\Delta x, n\,\Delta t\right),$$

$$|O_2| \leqq O(\varepsilon)M\,\Delta t\,\mathrm{TV}|\phi|_\infty.$$

Here, $S_k^{n-1}$ is the state difference of the strong shock $S_k(J^n)$, $\sigma_k^{n-1}$ is its wave speed. From (4.1),

$$|O_3| \equiv \left|\sum_{n=1}^{M} E_q^n\right| \leqq \sum_{n=1}^{M} |E_q^n| \leqq M\Delta t\, Q(\Lambda^0) O(|\phi|_\infty).$$

To evaluate $\Sigma_{n=1}^{M} E_t^n$, we can trace, according to Theorem 4.5, each wave in $W_t(J^n)$ back to a wave on $W_t(J^1)$ or on $V_t(J^1, J^M)$. Let $\Sigma_a$ denote the summation over $a \in W_t(J^1) \cup V_t(J^1, J^M)$, then

$$\begin{aligned}
\sum_{n=1}^{M} E_t^n &= \sum_a \sum_{n=n_1(a)}^{n_2(a)} \tilde{a}(n,a)\big[\tilde{\lambda}(n,a) + e^n(\tilde{\lambda}(n,a))\big]\, \Delta t\, \phi(\tilde{j}(n,a)\,\Delta x, n\,\Delta t) \\
&= \sum_a \sum_{n=n_1(a)}^{n_2(a)} a\big[\lambda(a) + e^n(\lambda(a))\big]\, \Delta t\, \phi(j(a)\,\Delta x, n\,\Delta t) \qquad (4.12)\\
&\quad + O_4 + O_5 + O_6 + O_7,
\end{aligned}$$

with

$$\begin{aligned}
|O_4| &\equiv \Bigg|\sum_a \sum_{n=n_1(a)}^{n_2(a)} \tilde{a}(n,a)\big[\tilde{\lambda} + e^n(\tilde{\lambda})\big]\, \Delta t\big(\phi(\tilde{j}(n,a)\,\Delta x, n\,\Delta t) \\
&\qquad\qquad - \phi(j(a)\,\Delta x, n_1(a)\,\Delta t)\big)\Bigg| \\
&\leqq (M\Delta t)^2\, \mathrm{TV}\, O(|\nabla\phi|_\infty),\\
|O_5| &\equiv \Bigg|\sum_a \sum_{n=n_1(a)}^{n_2(a)} (\tilde{a}(n,a) - a)\big[\tilde{\lambda} + e^n(\tilde{\lambda})\big]\, \Delta t\, \phi(j(a)\,\Delta x, n_1(a)\,\Delta t)\Bigg| \\
&\leqq M\Delta t\, Q(\Lambda^0) O(|\phi|_\infty),\\
|O_6| &\equiv \Bigg|\sum_a \sum_{n=n_1(a)}^{n_2(a)} a\big[\tilde{\lambda}(n,a) - \lambda(a)\big]\, \Delta t\, \phi(j(a)\,\Delta x, n_1(a)\,\Delta t)\Bigg| \\
&\leqq M\Delta t\, Q(\Lambda^0) O(|\phi|_\infty),\\
|O_7| &\equiv \Bigg|\sum_a \sum_{n=n_1(a)}^{n_2(a)} a\big[e^n(\tilde{\lambda}(n,a)) - e^n(\lambda(a))\big]\, \Delta t\, \phi(j(a)\,\Delta x, n_1(a)\,\Delta t)\Bigg|.
\end{aligned}$$

In some of the above formulae, we have abbreviated $\tilde{a}(n, a)$ by $\tilde{a}$, and $\tilde{\lambda}(n, a)$

by $\tilde{\lambda}$. In $O_7$,

$$\left|\sum_{n_1(a)}^{n_2(a)} e^n(\tilde{\lambda}(n,a)) - e^n(\lambda)\right|$$

$$= \left|\sum_{n_1(a)}^{n_2(a)} \frac{\Delta t}{2\,\Delta x}\left(\operatorname{sign}\left(\lambda\frac{\Delta t}{\Delta x} - \theta^n\right) - \left(\tilde{\lambda}(n,a)\frac{\Delta t}{\Delta x} - \theta^n\right)\right)\right|$$

$$\leqq \frac{\Delta x}{2\,\Delta t}\#\left\{n_1(a) \leqq n \leqq n_2(a),\ \theta^n \in \left(\min_n \tilde{\lambda}(n,a)\frac{\Delta t}{\Delta x},\ \max_n \tilde{\lambda}(n,a)\frac{\Delta t}{\Delta x}\right)\right\}$$

$$\leqq \frac{\Delta x}{\Delta t}\Big((n_2(a) - n_1(a))\max_n|\tilde{\lambda}(n,a) - \lambda(a)|$$

$$+ C\log(2 + n_2(a) - n_1(a))\Big),$$

in view of (4.11), (4.8). From (4.5), it follows that

$$|O_7| \leqq (M\,\Delta t\,Q(\Lambda^0) + \Delta t\log(2 + M)\,\mathrm{TV})O(|\phi|_\infty).$$

In the first term of (4.12),

$$\sum_{n_1}^{n_2} e^n(\lambda) = \sum_{n_1}^{n_2} -\frac{\Delta x}{2\,\Delta t}\left\{\operatorname{sign}(\theta^n) + \operatorname{sign}\left(\lambda\frac{\Delta t}{\Delta x} - \theta^n\right)\right\}$$

$$= -\frac{\Delta x}{2\,\Delta t}\Big\{\#\{n_1 \leqq n \leqq n_2,\ \theta^n \in (0,\tfrac{1}{2})\}$$

$$- \#\{n_1 \leqq n \leqq n_2,\ \theta^n \in (-\tfrac{1}{2},0)\}$$

$$+ \#\{n_1 \leqq n \leqq n_2,\ \theta^n \in (-\tfrac{1}{2},\lambda\,\Delta t/\Delta x)\}$$

$$- \#\{n_1 \leqq n \leqq n_2,\ \theta^n \in (\lambda\,\Delta t/\Delta x,\tfrac{1}{2})\}\Big\}.$$

From (4.8),

$$\left|\sum_{n_1}^{n_2} e^n(\lambda) + \lambda(n_2 - n_1)\right| \leqq 2C\frac{\Delta x}{\Delta t}\log(2 + n_2 - n_1) \leqq 2C\frac{\Delta x}{\Delta t}\log(2 + M).$$

Hence, the first term of (4.12) satisfies

$$|O_8| \equiv \left|\sum_a \sum_{n=n_1(a)}^{n_2(a)} a[\lambda(a) + e^n(\lambda(a))]\phi(j(a)\,\Delta x, n\,\Delta t)\right|$$

$$\leqq \Delta t\log(2 + M)\,\mathrm{TV}\,O(|\phi|_\infty).$$

We now turn to the evaluation of $\Sigma_n E^n_{S_k}$. We shall trace the strong shock and use the following estimates for $S_k$:

$$|S_k^{n-1} - S_k^0| \leqq K[A(J^1, J^M) + Q(\Lambda^0)],$$

(see (3.12)) and

$$|\sigma_k^{n-1} - \sigma_k^0| \leqq K\left[A(J^1, J^M) + Q(\Lambda^0)\right]. \tag{4.13}$$

Inequality (4.13) follows from (3.12) and the implicit function theorem. With (3.3), (3.12), and (4.13), we have

$$\begin{aligned}\sum_{n=1}^{M} E_{S_k}^n = &\sum_{n=1}^{M} S_k^0\left[\sigma_k^0 + e^n(\sigma_k^0)\right] \Delta t\, \phi\left(j(S_k^0)\, \Delta x, \Delta t\right)\\ &+ O_9 + O_{10} + O_{11} + O_{12},\end{aligned} \tag{4.14}$$

with

$$\begin{aligned}|O_9| \equiv &\left| \sum_{n=1}^{M} S_k^{n-1}\left[\sigma_k^{n-1} + e^n(\sigma_k^{n-1})\right] \right.\\ &\left. \cdot \Delta t\left[\phi\left(j(S_k^{n-1})\, \Delta x, n\, \Delta t\right) - \phi\left(j(S_k^0)\, \Delta x, \Delta t\right)\right] \right|\\ \leqq\ &(M \Delta t)^2\left(|S_k^0| + \mathrm{TV}\right) O\left(|\nabla \phi|_\infty\right),\\ |O_{10}| \equiv &\left| \sum_{n=1}^{M} \left(S_k^{n-1} - S_k^0\right)\left[\sigma_k^{n-1} + e^n(\sigma_k^{n-1})\right] \Delta t\, \phi\left(j(S_k^0)\, \Delta x, \Delta t\right) \right|\\ \leqq\ &M \Delta t\left[A(J^1, J^M) + Q(\Lambda^0)\right] O\left(|\phi|_\infty\right),\\ |O_{11}| \equiv &\left| \sum_{n=1}^{M} S_k^0\left(\sigma_k^{n-1} - \sigma_k^0\right) \Delta t\, \phi\left(j(S_k^0)\, \Delta x, \Delta t\right) \right|\\ \leqq\ &M \Delta t\left(S_k^0 + \mathrm{TV}\right)\left[A(J^1, J^M) + Q(\Lambda^0)\right] O\left(|\phi|_\infty\right),\\ |O_{12}| \equiv &\left| \sum_{n=1}^{M} S_k^0\left[e^n(\sigma_k^{n-1}) - e^n(\sigma_k^0)\right] \Delta t\, \phi\left(j(S_k^0)\, \Delta x, \Delta t\right) \right|\\ \leqq\ &\Delta t\left(S_k^0 + \mathrm{TV}\right) O\left(|\phi|_\infty\right) \#\left\{1 \leqq n \leqq M,\right.\\ &\qquad\qquad \left. \theta^n \in \left(\min_n \sigma_k^{n-1} \frac{\Delta t}{\Delta x}, \max_n \sigma_k^{n-1} \frac{\Delta t}{\Delta x}\right)\right\}\\ \leqq\ &\Delta t\left(S_k^0 + \mathrm{TV}\right) O\left(|\phi|_\infty\right)\left(M\left[A(J^1, J^M) + Q(\Lambda^0)\right] + C \log(2 + M)\right).\end{aligned}$$

Let us denote the first term of (4.14) by $O_{13}$. From (4.8),

$$|O_{13}| \leqq \Delta t\big(S_k^0 + \mathrm{TV}\big)\log(2 + M)O\big(|\phi|_\infty\big).$$

Summarizing the above calculation, we have

$$\left|\sum_{n=1}^{M} E^n\right| \leqq \sum_1^{13} |O_i|.$$

This estimate is also valid for $|\sum_{n=1}^{M} E^{pM+n}|$ in a general strip $\Lambda^p$ with $Q(\Lambda^0)$ replaced by $Q(\Lambda^p)$ and $A(J^1, J^M)$ replaced by $A(J^{pM+1}, J^{(p+1)M})$.

Now, we are in a position to sum $\sum_{n=1}^{M} E^{pM+n}$ over $p$. From the definitions of $A$ and $Q$, and from (3.16)–(3.17),

$$\sum_{p=0}^{L-1} Q(\Lambda^p) = Q(\Lambda_{J^1, J^{LM}}) = O(\mathrm{TV}),$$

$$\sum_{p=0}^{L-1} A(J^{pM+1}, J^{(p+1)M}) = A(J^1, J^{LM}) = O(\mathrm{TV}).$$

Therefore,

$$\begin{aligned}
\left|\sum_1^{LM} E^n\right| &\leqq \sum_{p=0}^{L-1}\left|\sum_{n=1}^{M} E^{pM+n}\right| \\
&\leqq O\big(|\nabla\phi|_\infty\big)\big(\mathrm{TV} + S_k^0\big)\big(L(\Delta x)^2 + L(M\Delta t)^2\big) \\
&\quad + O\big(|\phi|_\infty\big)\big\{\varepsilon\,\mathrm{TV} + L\,\Delta t\log(2 + M)\big(\mathrm{TV} + S_k^0\big) \\
&\qquad + M\Delta t\big[A(J^1, J^{LM}) + Q(\Lambda_{J^1, J^{LM}})\big]\big(\mathrm{TV} + S_k^0\big)\big\}.
\end{aligned}$$

If we choose $\varepsilon = \Delta t$, $M = O(\Delta t)^{-1/2}$; therefore $L = O(\Delta t)^{-1/2}$ and we obtain

$$\left|\sum_1^{LM} E^n\right| = \big(\mathrm{TV} + S_k^0\big)\big((\Delta t)^{1/2} O\big(|\nabla\phi|_\infty\big) + (\Delta t)^{1/2}|\log(\Delta t)|O\big(|\phi|_\infty\big)\big).$$

This completes the proof.

## 5. The Front Tracking Scheme

From Theorem 4.6, we observe that the major truncation error of the Glimm scheme for the problem (1.1), (1.2) comes from $|S_k^0|(\Delta t)^{1/2}|\log \Delta t|$—an error due to the diffusion in the location of the strong shock. In this section, we shall introduce a simple front tracking scheme to reduce this error. We shall show that such a scheme is also stable and consistent. The error released from the strong shock is replaced by $O(\Sigma|a|\,\Delta t)$—an error caused by the scattered waves near the front. Here, $\Sigma|a|$ is a summation over all incident small waves.

**5.1. The front tracking scheme.** The scheme is a hybridization of front propagation to trace the location of the strong discontinuity and the random choice method for the rest of the flow. Let $\Delta x$, $\Delta t$ and the equidistributed sequence $\{\theta^n\}_{n=0}^{\infty}$ be given as in the Glimm scheme. We shall introduce a function $x_f(t)$ to represent the locus of the front. Let us define $j_f(n) \equiv [x_f(n\,\Delta t)/\Delta x]$ and call the interval $((j_f(n)-1)\Delta x, (j_f(n)+2)\,\Delta x)$ the front region at time step $n$. The scheme is defined inductively in time as follows (see Figure 5.1). Suppose the approximate solution $u_{\Delta x}$ has been defined for $t < n\,\Delta t$. At $t = n\,\Delta t+$, outside the front region, $u_{\Delta x}$ are to be the piecewise constant sample states defined by (2.1). Inside the front region, we first solve a Riemann problem with two side states being the sample states defined by (2.1) in the regions $((j_f(n)-1)\,\Delta x, j_f(n)\,\Delta x)$ and $((j_f(n)+1)\,\Delta x, (j_f(n)+2)\,\Delta x)$. The solution to this Riemann problem contains a strong $k$-shock. Let us call it $S_k^n$ and its two side states $u_l^n$ and $u_r^n$. Then define, in the front region,

$$u_{\Delta x}(x, n\,\Delta t+) = \begin{cases} u_l^n, & x \in \big((j_f(n)-1)\,\Delta x, x_f(n\,\Delta t)\big), \\ u_r^n, & x \in \big(x_f(n\,\Delta t), (j_f(n)+2)\,\Delta x\big), \end{cases}$$

For $n\,\Delta t < (n+1)\,\Delta t$, $u_{\Delta x}$ are defined to be the solutions to the sequence of Riemann problems emitted from $t = n\,\Delta t+$.

**5.2. Stability and consistency.** The core of a front tracking method lies in how the off-front flow communicates with the front. This information is hidden in the interaction diagram near the front. Lemma 5.1 below is devoted to the estimate of such interaction. We need some notation. Let us denote the waves issued from $(j\Delta x, n\,\Delta t)$ by $\{a_i(j,n)\}_{i=1}^{N}$. The sample point $((j+\theta^{n+1})\,\Delta x, (n+1)\,\Delta t)$ divides these waves into a left family $\{a_i^L(j,n)\}_{i=1}^{N}$ and a right family $\{a_i^R(j,n)\}_{i=1}^{N}$, which are, respectively, the solutions to the Riemann problems

$$\big(u_{\Delta x}(j\Delta x-, n\,\Delta t+), u_{\Delta x}((j+\theta^{n+1})\,\Delta x, (n+1)\,\Delta t-)\big)$$

and

$$\big(u_{\Delta x}((j+\theta^{n+1})\,\Delta x, (n+1)\,\Delta t-), u_{\Delta x}(j\Delta x+, n\,\Delta t+)\big).$$

Notice that $a_i(j,n) = a_i^L(j,n) + a_i^R(j,n)$. Let us consider Figure 5.1 for the interaction in the front region at $t = (n+1)\,\Delta t$. We notice that if $\theta^{n+1} \geq 0$, then only the waves $\{a_l^R(j_f(n)-1, n)\}_{l \geqq k}$ will interact with $S_k^n$, and if $\theta^{n+1} < 0$, then only the waves $\{a_l^L(j_f(n)+2, n)\}_{l \leqq k}$ will interact with $S_k^n$. Without loss of

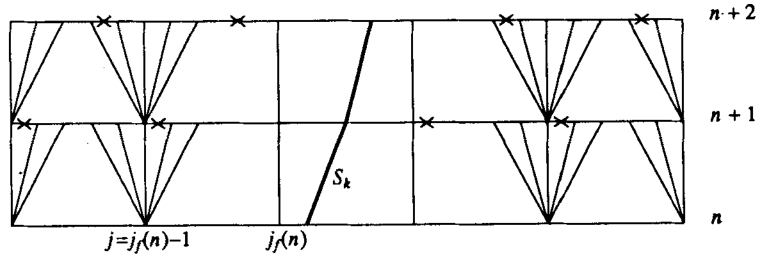


Figure 5.1

generality, we may assume $\theta^{n+1} \geqq 0$. Further, in Lemma 5.1 below, we shall only demonstrate the estimates for waves on the left-hand side of $S_k^n$. The estimates for the waves on the right-hand side can be treated similarly.

LEMMA 5.1. *Consider Figure 5.1 with $\theta^{n+1} \geqq 0$. Let us abbreviate $j_f(n) - 1$ by $j$. Under the assumptions of Lemma 3.3, for each wave $a_l^R$, $l \geqq k$, there exists a set of scattered waves $\{a_{l,i}^R\}_{i \neq k}$ such that*

$$(5.1)\qquad \begin{aligned} |a_{l,i}^R| &= O(1)|a_l^R| + O(1)Q_1, \qquad l \geqq k,\ i \neq k,\\ Q_1 &= \sum_{\substack{l \geqq k\\ l \neq m}} |a_l^R(j,n)||a_m^R(j,n)|. \end{aligned}$$

*The outgoing waves in the front region satisfy the following estimates:*

$$(5.2)\qquad |S_k^{n+1} - S_k^n| = O(1)\sum_{l \geqq k} |a_l^R(j,n)| + O(1)Q_1.$$

*The waves on the left-hand side of $S_k$ satisfy*

$$(5.3)\qquad \begin{aligned} a_i(j, n+1) &= a_i^R(j-1, n) + a_i(j,n)\\ &\quad + \sum_{l \geqq k} a_{l,i}^R(j,n) + O(1)(Q_1 + Q_2), \qquad i < k, \end{aligned}$$

$$(5.4)\qquad a_i(j, n+1) = a_i^R(j-1, n) + a_i^L(j,n) + O(1)(Q_1 + Q_2), \qquad i \geqq k,$$

with

$$\begin{aligned} Q_2 = &\sum_{\substack{l \geqq k\\ i}} |a_l^R(j,n)|\big(|a_i^L(j,n)| + |a_i^R(j-1,n)|\big)\\ &+ \sum{}' |a_i^R(j-1,n)||a_m^L(j,n)|. \end{aligned}$$

*Here, all $O(1)$ functions are bounded by the constant $K_2$ of Lemma 3.3.*

Proof: By Lemma 3.3, the interaction

$$\{a_l^R(j,n)\}_{l\geqq k} + S_k^n \to c_1,\cdots, S_k^{n+1},\cdots, c_N$$

yields the estimates (5.1)–(5.2) and

$$c_i = a_i^R(j,n) + a_{l,i}^R(j,n) + O(1)Q_1, \qquad i<k.$$

The waves $\{a_i(j,n+1)\}$ are the resulting waves of the interaction

$$\{a_i^R(j-1,n)\} + \{a_i^L(j,n)\} + \{c_i\}_{i<k}.$$

By applying Lemma 3.2, we can obtain (3.3)–(3.4).

THEOREM 5.2. *Under the assumptions of Theorem* 3.1, *the front tracking scheme defined in this section is stable and consistent for problem* (1.1), (1.2). *Moreover, its truncation error satisfies*

$$|\langle \nu_{\Delta x}, \phi\rangle| \leqq C\,\mathrm{TV}\big((\Delta t)^{1/2}|\log \Delta t\,\|\phi|_\infty + (\Delta t)^{1/2}|\nabla\phi|_\infty\big)$$

*for all test function* $\phi \in C^1(\mathbb{R}\times[0,\infty))$ *with compact support.*

Proof: The proof of the stability theorem of this front tracking scheme is almost identical to the proof of Theorem 3.1. The only modification is the following: In the induction process of the proof, as we meet the front region, instead of advancing one diamond, as in the proof of Theorem 3.1, we need to advance four diamonds in this region simultaneously and apply Lemma 5.1 there. We shall not repeat the proof. In the proof of the consistency theorem, the wave partition and classification are the same as before. The estimate of the truncation error has some modification. The error term $E_{S_k}^n$ is replaced by another error caused by the scattered waves near the strong shock. To be precise, from (4.10), the error $E^n$ can be written as

$$E^n = \int (u_{\Delta x}(x, n\,\Delta t+) - u_{\Delta x}(x, n\,\Delta t-))\phi(x, n\,\Delta t)\,dx$$

$$= \tilde{E}^n + \tilde{\tilde{E}}^n + O_1,$$

$$\tilde{E}^n = \Bigg(\sum_{j<j_f(n)} + \sum_{j>j_f(n)+1}\Bigg)\int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x}(u_{\Delta x}(x, n\,\Delta t+) - u_{\Delta x}(x, n\,\Delta t-))$$
$$\cdot\phi(j\Delta x, n\,\Delta t)\,dx,$$

$$\tilde{\tilde{E}}^n = \Bigg(\int_{(j_f(n)-1/2)\Delta x}^{x_f(n\,\Delta t)} + \int_{x_f(n\,\Delta t)}^{(j_f(n)+3/2)\Delta x}\Bigg)(u_{\Delta x}(x, n\,\Delta t+) - u_{\Delta x}(x, n\,\Delta t-))$$
$$\cdot\phi(j\Delta x, n\,\Delta t)\,dx.$$

$\tilde{E}^n$ can be broken up into $E_t^n$ and $E_q^n$ and admits the same estimates as before; and

$$\begin{aligned}|\tilde{\tilde{E}}^n| &\leqq \left|\left(u_l^n - u_l^{n-1}\right)\left(x_f(n\,\Delta t) - \left(j_f(n) - \tfrac{1}{2}\right)\Delta x\right)\right||\phi|_\infty \\ &\quad + \left|\left(u_r^n - u_r^{n-1}\right)\left(\left(j_f(n) + \frac{3}{2}\right)\Delta x\right) - x_f(n\,\Delta t)\right||\phi|_\infty \\ &\leqq \Delta x\left(A(J^{n-1}, J^n) + Q(\Lambda_{J^{n-1}, J^n})\right)O(|\phi|_\infty).\end{aligned}$$

Hence,

$$\begin{aligned}\left|\sum_{n=1}^{LM} \tilde{\tilde{E}}^n\right| &\leqq \Delta x\left(A(J^1, J^{LM}) + Q(\Lambda_{J^1, J^{LM}})\right)O(|\phi|_\infty) \\ &\leqq \Delta x\,\mathrm{TV}\,O(|\phi|_\infty).\end{aligned}$$

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## Bibliography

[1] Chern, I.-L., Glimm, J., McBryan, O., Plohr, B., and Yaniv, S., *Front tracking for gas dynamics*, J. Comp. Phys. 62, 1986, pp. 83–110.

[2] Chern, I.-L., and Colella, P., *A conservative front tracking method for hyperbolic conservation laws*, J. Comp. Phys., to appear.

[3] Courant, R., and Friedrichs, K. O., *Supersonic Flow and Shock Waves*, Springer-Verlag, 1984.

[4] Giusti, E., *Minimal Surfaces and Functions of Bounded Variation*, Birkhäuser Boston, 1984.

[5] Glimm, J., *Solutions in the large for nonlinear hyperbolic systems of equations*, Comm. Pure Appl. Math. 18, 1965, pp. 695–715.

[6] Glimm, J., Isaacson, E., Marchesin, D., and McBryan, O., *Front tracking for hyperbolic systems*, Adv. Appl. Math. 2, 1981, pp. 91–119.

[7] Hoff, D., and Smoller, J., *Error bounds for Glimm difference approximations for scalar conservation laws*, Trans. Amer. Math. Soc. 289, 1985, pp. 611–642.

[8] Kuipers, L., and Niederreiter, H., *Uniform Distribution of Sequences*, John Wiley & Sons, 1974.

[9] Kuznetsov, N. N., *On stable methods for solving first-order partial differential equations in the class of discontinuous functions*, in *Topics in Numerical Analysis II*, Proc. Roy. Irish Acad. Conf. on Numerical Analysis, J. J. H. Miller, ed., Academic Press, New York, 1976, pp. 183–197.

[10] Lax, P. D., *Hyperbolic Systems of Conservation Laws and the Mathematical Theory of Shock Waves*, Conf. Board Math. Sci. 11, SIAM, 1973.

[11] Liu, T.-P., *The deterministic version of the Glimm scheme*, Comm. Math. Phys. 57, 1977, pp. 135–148.

[12] Liu, T.-P., private communication.

[13] Smoller, J., *Shock Waves and Reaction-Diffusion Equations*, Springer-Verlag, New York, 1983.

[14] Volpert, A., *The space BV and quasilinear equations*, Mat. Sb. 73, 1967, pp. 255–302; English transl. in Math. USSR Sb. 2, 1967, pp. 225–267.

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