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The Entropy Condition and the Admissibility of Shocks

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The author proposed (*Trans. Amer. Math. Soc.* **199** (1974), 89–112) the extended entropy condition (E) and solved the Riemann problem for general 2×2 conservation laws. The Riemann problem for 3×3 gas dynamics equations was treated by the author (*J. Differential Equations* 18 (1975), 218–231). In this paper we justify condition (E) by the viscosity method in the spirit of Gelfand [*Uspehi Mat. Nauk* **14** (1959), 87–158]. We show that a shock satisfies condition (E) if and only if the shock is admissible, that is, it is the limit of progressive wave solutions of the associated viscosity equations. For the "genuinely nonlinear" 2×2 conservation laws, Conley and Smoller [*Comm. Pure Appl. Math.* **23** (1970), 867–884] proved that a shock satisfies Lax's shock inequalities [cf. *Comm. Pure Appl. Math.* **14** (1957), 537–566] if and only if it is admissible. In this paper, we consider systems that are not necessarily genuinely nonlinear.

1. GENERAL 2×2 CONSERVATION LAWS

Consider general 2×2 conservation laws

$$\begin{aligned} &u_t + f(u, v)_x = 0, \\ &v_t + g(u, v)_x = 0, \qquad -\infty < x < \infty, \qquad t \geqslant 0, \end{aligned} \tag{1.1}$$

where $(u, v) = (u, v)(x, t)$ and $f, g \in C^2(U)$ for some open set U in $\mathbb{R}^2$. Assume that (1.1) is strictly hyperbolic, that is, $d(f, g)$ had real and distinct eigenvalues $\lambda_1 < \lambda_2$ at each point in U.

It is known that a weak solution (u, v) to (1.1) satisfies the *Hugonoit condition* (H) across any discontinuity at (x, t):

$$\text{(H)} \quad \frac{f(u_+, v_+) - f(u_-, v_-)}{u_+ - u_-} = \frac{g(u_+, v_+) - g(u_-, v_-)}{v_+ - v_-} = s,$$

where $(u_+, v_+) = (u, v)(x + 0, t)$, $(u_-, v_-) = (u, v)(x - 0, t)$, and S is is the speed of discontinuity. For any (u_0, v_0) in U, let the *shock set* through (u_0, v_0) be the set of points (u, v) satisfying Hugonoit condition

$$\frac{f(u, v) - f(u_0, v_0)}{u - u_0} = \frac{g(u, v) - g(u_0, v_0)}{v - v_0} \equiv \sigma(u_0, v_0; u, v),$$

where $\sigma = \sigma(u_0, v_0; u, v)$ is the *shock speed.*

In [5] it is proved that there exists a unique solution to the Riemann problem for (1.1) if across every discontinuity (u_-, v_-) and (u_+, v_+), the following *extended entropy condition* (E) is satisfied.

$$\text{(E)} \quad \sigma(u_-, v_-; u_+, v_+) \leqslant \sigma(u_-, v_-; u, v)$$

for all (u, v) on the shock curve through (u_-, v_-) between (u_-, v_-) and (u_+, v_+).

Suppose that (u_+, v_+) belongs to the shock set through (u_-, v_-) and $s = \sigma(u_-, v_-; u_+, v_+)$. Denote by $\{u_-, v_-; u_+, v_+; s\}$ the shock wave solution

$$(u, v)(x, t) = \begin{cases} (u_-, v_-) & \text{for} \quad x - st < 0, \\ (u_+, v_+) & \text{for} \quad x - st > 0. \end{cases}$$

DEFINITION. A shock $\{u_-, v_-; u_+, v_+; s\}$ is *admissible* if for each $\epsilon > 0$, there are sequences $\{f^\epsilon, g^\epsilon\}$, $(u_-^\epsilon, v_-^\epsilon)$, and $(u_+^\epsilon, v_-^\epsilon)$,

$$\begin{aligned} s^\epsilon &= \sigma(u_-^\epsilon, v_-^\epsilon; u_+^\epsilon, v_+^\epsilon) \\ &= \frac{f^\epsilon(u_+^\epsilon, v_+^\epsilon) - f^\epsilon(u_-^\epsilon, v_-^\epsilon)}{u_+^\epsilon - u_-^\epsilon} = \frac{g^\epsilon(u_+^\epsilon, v_+^\epsilon) - g^\epsilon(u_-^\epsilon, v_-^\epsilon)}{v_+^\epsilon - v_-^\epsilon}, \end{aligned}$$

and progressive wave solution $(u^\epsilon, v^\epsilon)(\xi)$, $\xi = (x - s^\epsilon t)/\epsilon$, connecting $(u_-^\epsilon, v_-^\epsilon)$ on the left and $(u_+^\epsilon, v_+^\epsilon)$ on the right of viscosity equations

$$\begin{aligned} u_t + f^\epsilon(u, v)_x &= \epsilon u_{xx}, \\ u_t + g^\epsilon(u, v)_x &= \epsilon v_{xx} \end{aligned} \tag{1.2}$$

such that $(f^\epsilon, g^\epsilon) \to (f, g)$ in the C_0^2 topology, $s^\epsilon \to s$, $(u_-^\epsilon, v_-^\epsilon) \to (u_-, v_-)$, and $(u_+^\epsilon, v_+^\epsilon) \to (u_+, v_+)$ as $\epsilon \to 0_+$.

Remarks. An admissible shock $\{u_-, v_-; u_+, v_+; s\}$ is the uniform limit of progressive waves $(u^\epsilon, v^\epsilon)\,(y(\epsilon)\,(x - s^\epsilon t))$ in the region $\{(x, t) \mid (x - st) > \delta\}$ for every $\delta > 0$. Here $y(\epsilon)$ is chosen so that $y(\epsilon) \to 0_+$ rapidly as $\epsilon \to 0_+$. To find a progressive wave solution to (1.2) connecting $(u_-^\epsilon, v_-^\epsilon)$ and $(u_+^\epsilon, v_+^\epsilon)$ we have to find a connecting orbit of the vector field (cf. [1])

$$\begin{aligned} u_\xi^\epsilon &= -s^\epsilon(u^\epsilon - u_-^\epsilon) + f^\epsilon(u^\epsilon, v^\epsilon) - f^\epsilon(u_-^\epsilon, v_-^\epsilon), \\ v_\xi^\epsilon &= -s^\epsilon(v^\epsilon - v_-^\epsilon) + g^\epsilon(u^\epsilon, v^\epsilon) - g^\epsilon(u_-^\epsilon, v_-^\epsilon) \end{aligned} \tag{1.3}$$

such that

$$\lim_{\xi\to-\infty}(u^\epsilon(\xi), v^\epsilon(\xi)) = (u_-^\epsilon, v_-^\epsilon) \qquad \text{and} \qquad \lim_{\xi\to+\infty}(u^\epsilon(\xi), v^\epsilon(\xi)) = (u_+^\epsilon, v_+^\epsilon).$$

We first find conditions that guarantee the hyperbolicity of (1.1).

PROPOSITION 1.1. *Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be any nonsingular constant real matrix. Suppose that for any (u, v) in U,*

$$\{a(af_v + bg_v) - b(af_u + bg_u)\}\{d(cf_u + dg_u) - c(cf_v + dg_v)\} > 0. \tag{1.4}$$

Then (1.1) is strictly hyperbolic.

Proof. By direct calculation we know that if $\beta\gamma > 0$, then the real matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ has real and distinct eigenvalues. Set

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} a & d \\ c & d \end{pmatrix}\begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}.$$

Then

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix} \equiv d(f, g)$$

have the same eigenvalues. But $\beta\gamma > 0$ if and only if (1.4) holds. This proves the proposition. Q.E.D.

In [4], Lax proved that for (u, v) near (u_0, v_0), the shock set through (u_0, v_0) contains two curves through (u_0, v_0) and each curve has one of the right eigenvectors of $d(f, g)$ as its tangent. Hereafter, we assume that for any (u_0, v_0) in U,

> The shock set through (u_0, v_0) consists of two curves $s_1(u_0, v_0)$ and $s_2(u_0, v_0)$ such that for any (u, v) on $\sigma(u_0, v_0; u, v) < \lambda_2(u, v)$ and for any (u, v) on $s_2(u_0, v_0)$, $\sigma(u_0, v_0; u, v) > \lambda_1(u, v)$. (1.5)

Write $m = au + bv$, $n = cu + db$, $k = af + bg$, $l = cf + dg$. Consider

$$\begin{aligned} m_t + k(m, n)_x &= 0, \\ n_t + l(m, n)_x &= 0, \end{aligned} \tag{1.1$'$}$$

where m and n are basic dependent variables. It is shown that condition (E) for (1.1)$'$ is equivalent to condition (E) for (1.1). In other words, condition

(E) is invariant under *linear* manipulations. It is noted, however, that nonlinear manipulations could change the shock set and condition (E).

In mn-coordinates, (1.2) takes the form $(\partial k/\partial n)(\partial l/\partial m) > 0$. For definiteness, set $(\partial k/\partial n) < 0$ and $(\partial l/\partial m) < 0$. Let $\mathrm{I}(m_0, n_0)$, $\mathrm{II}(m_0, n_0)$, $\mathrm{III}(m_0, n_0)$, and $\mathrm{IV}(m_0, n_0)$ be the quadrants defined by (m_0, n_0), e.g.,

$$\mathrm{I}(m_0, n_0) = \{(m, n) \mid m > m_0, n > n_0\}.$$

In mn-coordinates, the right eigenvectors of $d(f, g)$ corresponding to λ_i, $i = 1, 2$, can be taken to be $\gamma_i = (1, a_i)^t$, $a_1 > 0 > a_2$. Let $h_i = h_i(u_0, v_0; u, v)$ be a nonzero smooth tangent to $s_i(u_0, v_0)$ at (u, v), and

$$h_i(u_0, v_0; u, v) = \gamma_i(u_0, v_0).$$

Condition (1.5) implies that there is no point $(u, v) \neq (u_0, v_0)$ with $m = m_0$, or $n = n_0$ where $m_0 = au_0 + nv_0$ and $n_0 = cu_0 + dv_0$. Therefore

$$s_i(u_0, v_0) - \{(u_0, v_0)\} = s_i^+(u_0, v_0) \cup s_i^-(u_0, v_0), \qquad i = 1, 2,$$

such that $s_1^+(u_0, v_0) \subset \mathrm{I}(m_0, n_0)$, $s_1^-(u_0, v_0) \subset \mathrm{III}(m_0, n_0)$, $s_2^+(u_0, v_0) \subset \mathrm{IV}(m_0, n_0)$, and $s_2^-(u_0, v_0) \subset \mathrm{II}(m_0, n_0)$. Let $(d/d\mu_i)$ be the directional derivative along curve s_i in the direction h_i, $i = 1, 2$.

LEMMA 1.1. *Assume that* (1.4) *and* (1.5) *hold. Write* $h_i = \sum_{j=1}^{2} a_{ij}\gamma_j$, $i = 1, 2$. *Then* $a_{11} > 0$ *and* $a_{22} > 0$ *and* $(u, v) \in s(u_0, v_0)$, *and for* $(u, v) \in s_i^+(u_0, v_0)$ *we have*

(1) $(d\sigma/d\mu_i) > 0$ *if and only if* $\sigma < \lambda_i$ *and*

(2) $(d\sigma/d\mu_i) < 0$ *if and only if* $\sigma > \lambda_i$,

and for $(u, v) \in s_i^-(u_0, v_0)$, *we have*

(3) $(d\sigma/d\mu_i) > 0$ *if and only if* $\sigma > \lambda_i$ *and*

(4) $(d\sigma/d\mu_i) < 0$ *if and only if* $\sigma < \lambda_i$.

Proof. We only prove the lemma when $(u, v) \in s_2^+(u_0, v_0)$. The other cases are proved similarly. Since $h_i = \gamma_i$ at (u_0, v_0), to prove $a_{11} > 0$ and $a_{22} > 0$, we have only to show that $a_{11} \neq 0$ and $a_{22} \neq 0$. Differentiating

$$\sigma(u_0, v_0; a, v)\begin{pmatrix} u - u_0 \\ v - v_0 \end{pmatrix} = \begin{pmatrix} f(u, v) - f(u_0, v_0) \\ g(u, v) - g(u_0, v_0) \end{pmatrix}$$

along $s_i(u_0, v_0)$, we have

$$\frac{d\sigma}{d\mu_i}\begin{pmatrix} u - u_0 \\ v - v_0 \end{pmatrix} = (d(f, g) - \sigma)\, h_i$$
$$= \sum_j (\lambda_j - \sigma)\, a_{ij}\gamma_j. \tag{1.6}$$

Since $(u, v) \in \mathrm{IV}(m_0, n_0)$ and $\gamma_2{}^t \in \mathrm{IV}(m_0, n_0)$, it follows from (1.6) that if $a_{22} = 0$, then $(d\sigma/d\mu_2) = 0$ and $h_2 = 0$. But h_i is nonzero and so $a_{22} \neq 0$.
(1), (2), (3), and (4) are proved by using (1.6) and the fact that $a_{11} > 0$ and $a_{22} > 0$. We omit the details. Q.E.D.

LEMMA 1.2. *Assume that* (1.4) *and* (1.5) *hold. Let* (u_1, v_1) *be any point on* $s_2{}^+(u_0, v_0)$ *between* (u_0, v_0) *and* (u_2, v_2), $(u_2, v_2) \in s_2{}^+(u_0, v_0)$. *If*

$$\sigma(u_0, v_0; u_1, v_1) = \sigma(u_0, v_0; u_2, v_2),$$

then $(u_1, v_1) \in s_2{}^-(u_2, v_2)$.

Proof. For simplicity assume that there exist only finite many points (u^k, v^k), $k = 1, 2,..., q$ on $s_2{}^+(u_0, v_0)$ between (u_0, v_0) and (u_2, v_2) such that $\sigma(u_0, v_0; u^k, v^k) = \sigma(u_0, v_0; u_2, v_2)$. Without loss of generality assume that there is no point (u, v) on $s_2{}^+(u_0, v_0)$ between (u^k, v^k) and (u^{k+1}, v^{k+1}), $k = 1, 2,..., q-1$, such that $\sigma(u_0, v_0; u, v) = \sigma(u_0, v_0; u_2, v_2)$. Then $(d\sigma/d\mu_2) \leqslant 0$ at either (u^k, v^k) or (u^{k+1}, v^{k+1}) along $s_2{}^+(u_0, v_0)$ for each fixed k. By Lemma 1.1, $\sigma \geqslant \lambda_2$ at either (u^k, v^k) or (u^{k+1}, v^{k+1}). Since

$$\sigma(u_0, v_0; u^k, v^k) = \sigma(u_0, v_0; u^{k+1}, v^{k+1}),$$

it follows that (u^k, v^k) is in the shock through (u^{k+1}, v^{k+1}) and

$$\sigma(u^k, v^k; u^{k+1}, v^{k+1}) = \sigma(u_0, v_0; u^k, v^k) = \sigma(u_0, v_0; u^{k+1}, v^{k+1}) = \sigma.$$

But $\sigma \geqslant \lambda_2$ at either (u^k, v^k) or (u^{k+1}, v^{k+1}), (1.5) implies that $(u^k, v^k) \in s_2(u^{k+1}, v^{k+1})$. By Lemma 1.1, we conclude that

$$(u^k, v^k) \in s_2{}^-(u^{k+1}, v^{k+1}),$$

and so $(u^{k+1}, v^{k+1}) \in \mathrm{IV}(m^k, n^k)$, $m^k = au^k + bv^k$, $n^k = cu^k + dv^k$. By finite induction, we have $(u^i, v^i) \in \mathrm{IV}(m^j, n^j)$ for $q \geqslant i > j \geqslant 1$. In particular, $(u_2, v_2) \in \mathrm{IV}(u_1, v_1)$ and the lemma is proved.

We prove the lemma by contradiction when there exist infinite many points (u, v) on $s_2{}^+(u_0, v_0)$ between (u_0, v_0) and (u_2, v_2) with

$$\sigma(u_0, v_0; u, v) = \sigma(u_0, v_0; u_2, v_2).$$

We omit the details. Q.E.D.

Similarly, we have analogous results as Lemma 1.2 for curves $s_2{}^-$, $s_1{}^+$, and $s_1{}^-$.

THEOREM 1.1. *Assume that* (1.4) *and* (1.5) *hold. Then a shock* $\{u_-, v_-; u_+, v_+; s\}$ *is admissible if and only if the shock satisfies condition* (E).

Proof. We only proof the theorem when $m = u$, $n = v$, and (1.4) reduces to $f_v g_u > 0$, and when $(u_+, v_+) \in s_2{}^+(u_-, v_-)$. The general cases are proved analogously. For definiteness, assume that $f_v < 0$ and $g_u < 0$. This implies that $u_+ > u_-$ and $v_+ < v_-$.

Assume first that $s = \sigma(u_-, v_-; u_+, v_+) < \sigma(u_-, v_-; u, v)$ for every $(u, v) \neq (u_+, v_+)$ on $s_2{}^+(u_-, v_-)$ between (u_-, v_-) and (u_+, v_+). Let (1.3) take the form

$$\begin{aligned} u_\xi &= -s(u - u_-) + f(u, v) - f(u_-, v_-) \equiv \phi(u, v), \\ v_\xi &= -s(v - v_-) + g(u, v) - g(u_-, v_-) = \psi(u, v). \end{aligned} \tag{1.3$'$}$$

Since $\phi_v = f_v \neq 0$ and $\psi_u = g_u \neq 0$, it follows that $\phi = 0$ ($\psi = 0$) is a curve defined for $u(v)$. By Lemma 1.2 there is no point (u, v) on $s_2{}^+(u_0, v_0)$, $u_- < u < u_+$, or $v_+ < v < v_-$ with $\sigma(u_-, v_-; u, v) = s$. The inequality $s < \sigma$ implies that $\phi > 0$ on the curve $\psi = 0$ joining (u_-, v_-) and (u_+, v_+). Similarly $\psi < 0$ on the curve $\phi = 0$ joining (u_-, v_-) and (u_+, v_+). Thus $\phi = 0$ and $\psi = 0$ do not intercept between (u_-, v_-) and (u_+, v_+) and there is no critical point for (1.3)$'$ in the region bounded by $\phi = 0$ and $\psi = 0$. This is indicated in Fig. 1.

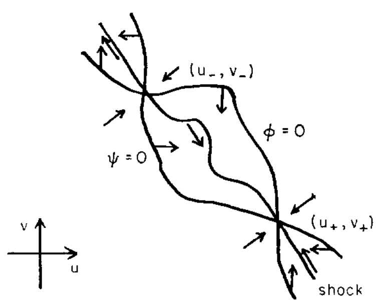

FIGURE 1

The point (u_-, v_-) is a critical point for (1.3)$'$ with index -1, and (u_-, v_+) is a critical point with index $+1$. Since the vector fields on the boundary $\phi = 0$ and $\psi = 0$ point toward the region and there is no critical point in this region bounded by $\phi = 0$ and $\psi = 0$, there is a connecting orbit from (u_-, v_-) to (u_+, v_+). Therefore $\{u_-, v_-; u_+, v_+; s\}$ is admissible.

We may have, however, $s = \sigma(u_-; v_-; u, v)$ for some (u, v) on $s_2{}^+(u_0, v_0)$ between (u_0, v_0) and (u_+, v_+). To deal with this case, we first note that by condition (E), $\sigma = (u_-, v_-; u, v)$ is nonincreasing at $(u, v) = (u_+, v_+)$. If

$(d\sigma/d\mu_2) < 0$ at (u_+, v_+), we then take $(f^\epsilon, g^\epsilon) = (f, g)$ and $s^\epsilon < s$, $| s^\epsilon - s |$ small. Since $(d\sigma/d\mu_2) < 0$ at (u_+, v_+), we can find $(u_+^\epsilon, v_+^\epsilon)$ on $s_2^+(u_0, v_0)$, $| u_+^\epsilon - u_+ |$ small, such that $\sigma(u_-, v_-; u_+^\epsilon, v_+^\epsilon) = s^\epsilon$. Consider the shock wave $\{u_-, v_-; u_+^\epsilon, v_+^\epsilon; s^\epsilon\}$ that satisfies condition (E) and $s^\epsilon < \sigma(u_-, v_-; u, v)$ for every (u, v) on $s_2^+(u, v)$ between (u_-, v_-) and $(u_+^\epsilon, v_+^\epsilon)$. The first part of this proof guarantees that $\{u_-, v_-; u_+^\epsilon, v_+^\epsilon; s^\epsilon\}$ is admissible and so is $\{u_-, v_-; u_+, v_+; s\}$ by definition.

If σ is stationary at (u_+, v_+) along $s_2^+(u_-, v_-)$, then we employ the standard perturbation theory to find (f^ϵ, g^ϵ) (cf. [7]). We omit the details.

Conversely, suppose that condition (E) fails for $\{u_-, v_-; u_+, v_+; s\}$; we want to show that the shock is not admissible. If $s > \lambda_2(u_-, v_-)$, then (u_-, v_-) is an unstable node and it is obvious that there is no connecting orbit from (u_-, v_-) to (u_+, v_+) (cf. Fig. 2). If $s < \lambda_2(u_-, v_-)$, then (u_-, v_-) is a saddle

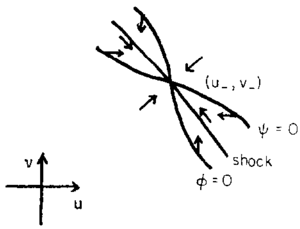

FIGURE 2

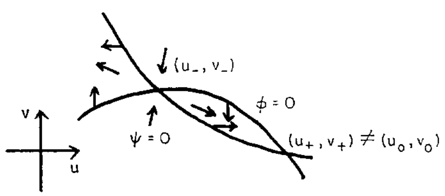

FIGURE 3

point for (1.3)′. Let (u_0, v_0) be the first point on $s_2^+(u_-, v_-)$ such that $\sigma(u_-, v_-; u_0, v_0) = s$. Since $\{u_-, v_-; u_+, v_+; s\}$ does not satisfy condition (E), we know that $(u_0, v_0) \neq (u_+, v_+)$, and by Lemma 1.2, there is no point (u, v) on $s_2^+(u_-, v_-)$, $u_- < u < u_0$, or $v_0 < v < v_-$ with $\sigma(u_-, v_-; u, v) = s$. Therefore, by the first part of this proof, there is a connecting orbit from (u_-, v_-) to (u_0, v_0); but there is no orbit connecting (u_-, v_-) and (u_+, v_+). If $s = \lambda_2(u_-, v_-)$, (u_-, v_-) is a singular critical point, and it can be proved analogously that there is no connecting orbit for (1.3)′ either. Moreover, for ϵ small, the shock $\{u_-^\epsilon, v_-^\epsilon; u_+^\epsilon, v_+^\epsilon; s^\epsilon\}$ does not satisfy condition (E), and there is no connecting orbit for (1.3). This proves that $\{u_-, v_-; u_+, v_+; s\}$ is not admissible. This completes the proof of Theorem 1.1. Q.E.D.

Remark. After obtaining the results of this paper I learned of the results of Professor Dafermos [2], in which he characterizes the solution of the Riemann problem he obtained earlier as the limit of centered wave solutions $(u^\epsilon, v^\epsilon)(x/t)$ of

$$u_t + f(u, v)_x = \epsilon t u_{xx},$$
$$v_t + g(u, v)_x = \epsilon t v_{xx},$$

where $\epsilon > 0$ is small. He showed that the solution $(\tilde{u}, \tilde{v})$ of the Riemann problem $\{(u_l, v_l), (u_r, v_r)\}$ thus obtained can be described by $\tilde{u} = \tilde{u}(v)$, v between v_r and v_l, or $\tilde{v} = \tilde{v}(u)$, u between u_r and u_l. In case $\tilde{v} = \tilde{v}(u)$, the following proposition holds [2, Proposition 3.4].

PROPOSITION. *Suppose that the solution* $(\tilde{u}, \tilde{v})$ *has a discontinuity* $(u_-, v_-) = (\tilde{u}, \tilde{v})(s - 0)$ *and* $(u_+, v_+) = (\tilde{u}, \tilde{v})(s + 0)$ *at* $x/t = s$. *Then for any* u *between* u_- *and* u_+, *the solution of the initial value problem* (1.3)′, $u(0) = u$ *and* $v(0) = \tilde{v}(u)$ *has the property that* $v(\xi) = \tilde{v}(u(\xi))$ *for* $-\infty < \xi < \infty$.

In other words, the vector field (1.3)′ restricted to the curve $\tilde{v} = \tilde{v}(u)$ is a tangent vector field pointing toward (u_+, v_+). Therefore it follows from Theorem 1.1 and its proof that the shock $\{u_-, v_-; u_+, v_+; s\}$ satisfies condition (E). By the uniqueness theorem proved in [5], the solution of the Riemann problem obtained in [2] is actually the same as the one in [5].

2. Gas Dynamics Equations

Consider gas dynamics equations in Lagrangian coordinates

$$\begin{aligned} u_t + p_x &= 0, \\ v_t - u_x &= 0, \\ E_t + (pu)_x &= 0, \qquad -\infty < x < \infty, \quad t \geqslant 0, \end{aligned} \tag{2.1}$$

where u is the velocity, v is the specific volume, $p = p(v, e) \geqslant 0$ is the pressure, e is the internal energy, and $E = \frac{1}{2}u^2 + e$ is the total energy. For the physical reasons, we consider the viscosity equations of the form

$$\begin{aligned} u_t + p_x &= (\epsilon u_x)_x, \\ v_t - u_x &= 0, \\ E_t + (pu)_x &= (\epsilon u u_x)_x, \end{aligned} \tag{2.2}$$

where $\epsilon > 0$ is the viscosity coefficient. The notion of admissibility and condition (E) are defined the same way as in Section 1, except we only consider the perturbation p^ϵ of p.

We assume that $p_v(v, S) < 0$ so that (2.1) is hyperbolic. Here S is the entropy. We also assume that the shock set through any point (u_0, v_0, E_0) consists of three curves $C_1(u_0, v_0, E_0)$, $C_2(u_0, v_0, E_0)$, and $C_3(u_0, v_0, E_0)$ such that C_1 and C_3 are defined for u and on C_2, u and p are constant. It is shown that these assumptions imply that the shock speed σ is positive on C_3, negative on C_1, and zero on C_2. Certain sufficient conditions are given in [5] to guarantee that C_1 and C_3 are defined for u.

Since the shock speed is constant along C_2 whether or not the convexity condition $p_{vv}(v, s) > 0$ is satisfied, we only justify condition (E) for shock curves C_1 and C_3.

THEOREM 2.1. *A shock $\{u_-, v_-, E_-; u_+, v_+, E_+; s\}$ is admissible if and only if the shock satisfies condition* (E).

Proof. We only prove the theorem when $(u_+, v_+, E_+) \in C_3(u_-, v_-, E_-)$ and $u_+ > u_-$ so that $s = \sigma(u_-, v_-, E_-; u_+, v_+, E_+) > 0$. The other cases are treated similarly.

Analogous to the proof of Theorem 1.1, we only consider the case

$$\sigma(u_-, v_-, E_-; u_+, v_+, E_+) < \sigma(u_-, v_-, E_-; u, v, E)$$

for every $(u, v, E) \in C_3(u_-, v_-, E_-)$ between (u_-, v_-, E_-) and (u_+, v_+, E_+). We have to find an orbit connecting (u_-, v_-, E_-) and (u_+, v_+, E_+) for the system of ordinary differential equations

$$\begin{aligned} u_\xi &= -s(u - u_-) + p - p_-, \\ 0 &= -s(v - v_-) - (u - u_-), \\ uu_\xi &= -s(E - E_-) + (pu - p_-u_-), \qquad \xi = (x - st)/\epsilon. \end{aligned} \tag{2.3}$$

From (2.3) we can eliminate v and E to write $u_\xi = f(u)$ for some smooth function f. It is noted that $f(u_-) = f(u_+) = 0$. We assert that there is no zero of f between u_- and u_+. Indeed, if $u_\xi = f(u) = 0$ at u between u_- and u_+, then $-s(u - u_-) + p - p_- = 0$, $-s(v - v_-) - (u - u_-) = 0$, and $-s(E - E_-) - (pu - p_-u_-) = 0$ by (2.3). This implies that

$$(u, v, E) \in C_3(u_-, v_-, E_-) \qquad \text{and} \qquad \sigma(u_-, v_-, E_-; u, v, E) = s.$$

This is a contradiction, since we assume that $s < \sigma$. Hence $f(u)$ is nonzero for $u_- < u < u_+$. We next assert that $f(u)$ is positive for $u_- < u < u_+$. We have only to show that $f(u)$ is positive for $u_- < u$, $| u_- - u |$ small. Indeed, by

(2.3), $E - E_- = (u - u_-)(u + (p_-/s))$ and $v - v_- = (1/s)(u - u_-)$ and so

$$\lim_{u \to u_-} \frac{p - p_-}{u - u_-} = \lim_{u \to u_-} \frac{p_\xi}{u_\xi} = \lim_{u \to u_-} \frac{p_v(v, e)\, v_\xi + p_e(v, e)(E_\xi - uu_\xi)}{u_\xi}$$

$$= \lim_{u \to u_-} -\frac{1}{s}\,(p_v(v, e) + p_e(v, e)\,p)$$

$$= \lim_{u \to u_-} -\frac{1}{s}\, p_v(v, S) = -\frac{1}{s}\, p_v(v_-, S_-),$$

where S_- is the entropy at (u_-, v_-, E_-). Since

$$s < \sigma(u_-, v_-, E_-; u_-, v_-, E_-) \equiv (-p_v)^{1/2}(v_-, S_-)$$

by assumption, we know that

$$\lim_{u \to u_-} \frac{u_\xi}{u - u_-} = \frac{1}{s}(-s^2 - p_v) = \frac{-1}{s}(s + (-p_v)^{1/2})(s - (-p_v)^{1/2}) > 0.$$

Therefore $u_\xi > 0$ at $u > u_-$, $|u - u_-|$ small. This completes the proof of Theorem 2.1. Q.E.D.

COROLLARY. *If* $\{u_-, v_-, E_-; u_+, v_+, E_+; s\}$ *satisfies condition* (E), *then*

(i) $S(u_-, v_-, E_-) \geqslant S(u_+, v_+, E_+)$ *when* $s > 0$, *and*

(ii) $S(u_-, v_-, E_-) \leqslant S(u_+, v_+, E_+)$ *when* $s < 0$.

In other words, the entropy S increases in the time direction across any shock satisfying condition (E).

Proof. We prove only (i). Condition (ii) is proved similarly.

By thermodynamic equations, $de = T\,ds - p\,dv$. Therefore

$$\begin{aligned} TS_\xi &= e_\xi + pv_\xi = (E - \tfrac{1}{2}u^2)_\xi + pv_\xi \\ &= (2u - u_- + (p_-/s))\, u_\xi - uu_\xi - (p/s)\, u_\xi \\ &= (u - u_- + (p_- - p/s))\, u_\xi \\ &= -(1/s)\, u_\xi^2 \leqslant 0. \end{aligned}$$

This proves the corollary. Q.E.D.

Remark. The author proved this corollary in [6] by investigating the behavior of the entropy S along the shock and rarefaction curves.

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