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A POSTERIORI ERROR ESTIMATES FOR GENERAL NUMERICAL METHODS FOR SCALAR CONSERVATION LAWS.

BERNARDO COCKBURN* AND HUIING GAU†

Abstract. In this paper, we obtain a simple *a posteriori* error estimate for numerical methods for nonlinear scalar conservation laws. The estimate is totally independent (i) of the dimension of the space, (ii) of the type of nonlinearity f, and (iii) of the numerical method. It can thus be used, regardless of the numerical schemes used to compute the approximate solution, to define mathematically sound adaptivity algorithms for conservation laws. Preliminary numerical results displaying the sharpness of the estimate are presented.

Key words. A posteriori error estimates, scalar conservation laws

AMS(MOS) subject classifications. 65M60, 65N30, 35L65

1. Introduction. In this paper, we show how a straightforward use of Kuznetsov approximation theory [12] leads to a simple *a posteriori* error estimate for numerical methods for the scalar conservation law [11]

$$v_t + \nabla \cdot f(v) = 0 \qquad \text{in } (0,T) \times \mathbb{R}^d, \tag{1.1a}$$

$$v(0) = v_0 \qquad \text{on } \mathbb{R}^d. \tag{1.1b}$$

The main advantages of the *a posteriori* estimate we obtain are that its form is totally independent (i) of the dimension of the space, (ii) of the type of nonlinearity f, and (iii) of the numerical method used to produce approximations to the entropy solution v of (1.1); this is the single *a posteriori* error estimate with the three above properties in the available literature. Hence, it can be used, regardless of the numerical scheme, to define mathematically sound adaptivity algorithms for conservation laws.

To understand why such a straightforward application of the work of Kuznetsov [12] took almost twenty years to come to light, we briefly display the history of the ideas leading to the estimate presented in this paper; for a more elaborated exposition, see the recent work by Cockburn and Gremaud [9].

Kuznetsov [12] was the first to obtain error estimates for numerical schemes for scalar conservation. He proved that monotone schemes defined in uniform Cartesian grids converge to the entropy solution in $\mathrm{L}^\infty(0,T;\mathrm{L}^1(\mathbb{R}^d))$ at the optimal rate of $(\Delta x)^{1/2}$. To do so, he first obtained an upper bound for the error that depends on the approximate solution (that is, he obtained an *a posteriori* error estimate) and then used the regularity properties of the approximate solution to obtain a rate of convergence. Since almost every author interested in obtaining error estimates for scalar conservation laws (Sanders [18], Lucier [14,15,16], Cockburn [3,4,5], Cockburn, LeFloch and Coquel [7], Cockburn and Gremaud [8], and Vila [20]) followed Kuznetsov's original approach, they all obtained an *a posteriori* error estimate and then, in order to get a rate of convergence, they all were bound to obtain regularity properties of the approximate solution – a very hard task indeed for even the simplest schemes in general triangulations.

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Only recently, Cockburn and Gremaud [9] found a way to suitably modify Kuznetsov's original approach to obtain *a priori* error estimates for conservation laws, that is, error estimates which do not depend on the regularity properties of the approximate solution – something that was thought to be not possible until now. As a by product, it became more natural to view Kuznetsov's approach as an *a posteriori* error estimates approach that should be used for defining adaptivity strategies. (Let us point out, however, that Lucier [13,16] did use Kuznetsov approximation theory for defining adaptivity methods for conservatin laws, and that Cockburn and Chen [1] obtained a result similar to ours (Theorem 2.1) to prove optimal error estimates for the one-dimensional drift-diffusion model for semiconductor device simulation.) In this paper, we fully exploit this point of view and use Kuznetsov's approach to obtain a simple and powerful *a posteriori* error estimate for arbitrary numerical schemes for the general scalar conservation law.

Other *a posteriori* error estimates for the nonlinear case have been obtained by Tadmor [19] and Nessyahu and Tadmor [17] for the one-dimensional case and a strictly convex nonlinearity f. These authors use a duality technique (which leads also to local error estimates) which allows them to obtain an *a posteriori* estimate similar to ours, [17, (2.10a)], the main difference being that we can use the L^1-norm where they use the Lip'-norm. We want also so to mention that recently Johnson and Szepessy [10] obtained *a posteriori* for the much more difficult system of gas dynamics for the Shock-Capturing Streamline Diffusion method.

The paper is organized as follows. In §2, we display the *a posteriori* error estimate, Theorem 2.1, which is a straightforward application of Kuznetsov's approximation theory [12] (for the sake of completeness, we display a proof of the result in the Appendix). In §3, we use Theorem 2.1 to show that the problem of obtaining an *a posteriori* can be rewritten as a minimization problem (involving interpolation operators only), which can be easily used to define adaptivity strategies. This line of work will be further developed elsewhere. In this paper, we restrict ourselves to displaying preliminary numerical results that show that the obtained upper bound for the error is sharp. We end in §4 with some concluding remarks.

2. The *a posteriori* estimate. In what follows, we denote by u any continuous function from $[0,T]$ to $\mathrm{L}^1(\mathbb{R}^d)$. Moreover, we assume that u is a smooth function, except on a finite number of $(d-1)$-dimensional surfaces $\mathcal{C}_i, i = 1,\dots,I$. We denote by $n_{\mathcal{C}_i}$ the unit normal to $\mathcal{C}_i$ such that its component on the t-axis is negative. The jump of $G(u)$ at the point P on surface $\mathcal{C}_i$, $[G(u)](P)$, is defined as follows:

$$[G(u)](P) = \lim_{\rho \downarrow 0} \left(G(u(P + \rho\, n_{\mathcal{C}_i})) - G(u(P - \rho\, n_{\mathcal{C}_i})) \right).$$

We set $\mathcal{C}_i(T) = (0,T) \times \mathbb{R}^d \cap \mathcal{C}_i$ and $\mathcal{C}(T) = \cup_{i=1}^{I} \mathcal{C}_i(T)$. Finally, we set $\Omega(T) = (0,T) \times \mathbb{R}^d \setminus \mathcal{C}(T)$.

With this notation, we have the following approximation result.

THEOREM 2.1 (THE *a posteriori* ESTIMATE). *Let u be as above and let v be the entropy solution of the conservation law* (1.1). *Then,*

$$\| u(T) - v(T) \|_{L^1(\mathbb{R}^d)} \le \Phi(v_0, u; T),$$

where

$$\Phi(v_0, u; T) = \| u(0) - v_0 \|_{L^1(\mathbb{R}^d)} + \| \mathit{Residual}(u) \|_{L^1(\Omega(T))} + \sum_{i=1}^{I} \| \mathit{Residual}(u) \|_{L^1(\mathcal{C}_i(T))},$$

where

$$Residual(u) = \begin{cases} u_t + \nabla \cdot f(u), & \text{on } \Omega(T), \\ \max\left\{0, \sup_{c\in[a,b]}\left\{([F(u,c)], [U(u-c)]) \cdot n_{\mathcal{C}_i}\right\}\right\} & \text{on } \mathcal{C}_i(T), \end{cases}$$

and $U(u-c) = |u-c|$, $F(u,c) = sign(u-c)\big(f(u)-f(c)\big)$. *The interval* $[a,b]$ *is the range of the initial data* v_0.

The term $\| Residual(u) \|_{L^1(\Omega(T))}$ is the truncation error on the smooth regions of the approximation defined by u. The term $\| Residual(u) \|_{L^1(\mathcal{C}_i(T))}$ measures how close to satisfying the entropy condition are the discontinuities of u on the curve $\mathcal{C}_i(T)$. To see this, let us consider the one-dimensional case, $d = 1$, for the sake of simplicity. If we write $C_i(T) = \{(x,t), x = x_i(t), 0 \le t_{i1} \le t \le t_{i2} \le T\}$, we have the familiar expression

$$\| Residual(u) \|_{L^1(\mathcal{C}_i(T))} = \int_{t_{i1}}^{t_{i2}} \max\left\{0, \sup_{c\in[a,b]}\left\{([F(u,c)] - \frac{dx_i}{dt}[U(u-c)])(s, x_i(s))\right\}\right\} ds.$$

It is well-known that $\| Residual(u) \|_{L^1(\mathcal{C}_i)} = 0$ if and only if the discontinuities of u on the curve $C_i(T)$ are discontinuities of an entropy solution of the conservation law (1.1a). Moreover, this only happens if the so-called Ranquine-Hugoniot condition, $\frac{dx_i}{dt} = \frac{[f(u)]}{[u]}$, is verified on the curve $C_i(T)$.

In this way, if u is an entropy solution of the conservation law (1.1a), then $Residual(u) \equiv 0$ and the estimate of Theorem 2.1 becomes

$$\| u(T) - v(T) \|_{L^1(\mathbb{R}^d)} \le \| u(0) - v_0 \|_{L^1(\mathbb{R}^d)}.$$

as expected.

Now, suppose that we have a numerical scheme that produces an approximate solution u_h at times t^n, $n = 1, \ldots, N$. Let us denote by $\mathcal{U}$ the set of all the *interpolates* u such that $u(t^n) = u_h(t^n)$ for $n = 1, \ldots, N$. Then, a simple application of Theorem 2.1, gives the following *a posteriori* error estimate:

$$\| u_h(t^n) - v(t^n) \|_{L^1(\mathbb{R}^d)} \le \inf_{u\in\mathcal{U}} \Phi(v_0, u; t^n), \quad n = 1, \ldots, N. \qquad (\star)$$

This estimate is the main contribution of this paper. As we claimed in the introduction, this estimate is totally independent (i) of the dimension of the space, (ii) of the type of nonlinearity f, and (iii) of the numerical method used to produce approximations to the entropy solution v of (1.1).

3. On the sharpness of the *a posteriori* estimate. The main application of the estimate $(\star)$ is the devising of adaptivity strategies. The goal of these strategies is to ensure that the error is smaller than a given tolerance, say ϵ, with the smallest possible computational effort. In order to achieve this goal, it is natural to try to enforce the following inequality:

$$\Phi(v_0, u^\star; t^N) \le \epsilon,$$

where $u^\star$ is the member of $\mathcal{U}$ we chose to work with. It is clear that the first task we have to face, before attempting to define any adaptivity strategies, is to study, at least numerically, the behavior of the ratio

$$r(u^\star, t^N) = \frac{\Phi(v_0, u^\star; t^N)}{\| u_h(t^N) - v(t^N) \|_{L^1(\mathbb{R}^d)}},$$

since we do not want this ratio to be too big. The ideal situation occurs, of course, when we can find a simple interpolation $u^\star$ such that $r(u^\star, t^N) \equiv 1$. In what follows, we present some preliminary numerical results that show that this ideal situation can be easily achieved.

We consider the approximate solution given by the well-known Engquist-Osher scheme in uniform grids $\{t^n = n\,\Delta t\}_{n=0}^{N_t} \times \{x_i = i\,\Delta x\}_{i=0}^{N_x}$ defined on the space domain is the interval $[0,1)$ with periodic boundary conditions. We use two ways of interpolating the values u_i^n generated by the Engquist-Osher scheme. We denote by $u_1^\star$ the standard piecewise-bilinear function such that $u_1^\star(t^n, x_i) = u_i^n$. We also define a more sophisticated interpolation function $u_2^\star$ which takes into account the nature of the conservation law. We define $u_2^\star$ as follows. First, we take $u^\star(n\,\Delta t)$ to be the piecewise-linear function such that $u_2^\star(t^n, x_i) = u_i^n$. We extend $u_2^\star$ to the strips $(n\,\Delta t, (n+1)\,\Delta t) \times [0,1)$ as follows. We consider all the trapezoids formed by the lines $t = t^n$, $t = t^{n+1}$ and the approximate characteristics $x = x_i + f'(u(t^n, x_i))\,(t - t^n)$ and $x = x_i + f'(u(t^{n+1}, x_i))\,(t - t^{n+1})$. We then defined u inside of each of these trapezoids as the standard bilinear isoparametric interpolate of the values of u at the vertices; see Ciarlet [2, pages 229– 230]. This construction is well-defined for the examples considered below.

All the integrals are computed by using the mid-point rule.

Example 1. We take $f(v) = v$, $v_0(x) = 1. + .5\sin(\pi(2x-1))$, and $T = .1$. The behavior of the ratio $r(u^\star, T)$ is displayed in Tables 1 and 2. We can see that with the simple interpolate $u_1^\star$, the ratio $r(u_1^\star, T)$ is not bigger that 1.3 and that with the more sophisticated interpolate $u_2^\star$, the ratio $r(u_2^\star, T)$ differs very little from 1. These results confirm that the error estimate is sharp and that even with a very simple interpolation procedure very good results can be obtained.

Example 2. We take $f(v) = v^2/2$, $v_0(x) = 1. + .5\sin(\pi(2x-1))$, and $T = .1$. The results obtained for the previous example also hold in this case, as we can see in Tables 3 and 4. (Note that ratios less than 1 might be obtained because we are evaluating the integrals in an approximate way.)

Example 3. We take $f(v) = v$, $T = 1/\sqrt{2}$, and $v_0(x) = \begin{cases} 1, & \text{for } x \in (.4, .6), \\ 0, & \text{elsewhere.} \end{cases}$

The results are displayed in Tables 5 and 6. We see, once more, that the ratio $r(u^\star, T)$ is close to 1 and that, unlike the previous cases, the results with $u_2^\star$ are only slightly better than the results obtained with $u_1^\star$. This might be explained by the fact that (i) the 'smooth region' of the exact solution is flat and so the contribution of the *Residual* in that region is negligible, and by the fact that (ii) the approximate solution is very smooth around the location of the exact discontinuity.

4. Concluding remarks. The preliminary results we have presented in §3 show that the error estimate $(\star)$ in §2 is sharp indeed. In a forthcoming paper, we study the behavior of the ratio $r(u, T)$ for several numerical schemes in the case of general nonlinearities. It must be pointed out that, in order to treat the general case, we must use interpolates u that have discontinuities. If only continuous functions u are used, there is no way to distinguish entropy-violating discontinuities from entropy-satisfying discontinuities.

Table 1

Example 1: Behavior of $r(u_1^\star, T)$ with respect to Δx and $\lambda = \Delta t/\Delta x$.

$1/\Delta x$	$\lambda = .25$	$\lambda = .50$	$\lambda = .75$
10	1.149	1.157	1.146
20	1.184	1.180	1.155
40	1.206	1.202	1.183
80	1.220	1.216	1.204
160	1.227	1.225	1.218
320	1.230	1.229	1.226
640	1.232	1.231	1.230

Table 2

Example 1: Behavior of $r(u_2^\star, T)$ with respect to Δx and $\lambda = \Delta t/\Delta x$.

$1/\Delta x$	$\lambda = .25$	$\lambda = .50$	$\lambda = .75$
10	1.02357	1.03430	1.04607
20	1.00642	1.00997	1.00633
40	1.00169	1.00275	1.00225
80	1.00044	1.00073	1.00043
160	1.00011	1.00019	1.00013
320	1.00003	1.00005	1.00004
640	1.00002	1.00002	1.00003

Table 3

Example 2: Behavior of $r(u_1^\star, T)$ with respect to Δx and $\lambda = \Delta t/\Delta x$.

$1/\Delta x$	$\lambda = .25$	$\lambda = .50$	$\lambda = .75$
10	1.177	1.161	1.165
20	1.197	1.187	1.175
40	1.232	1.218	1.197
80	1.249	1.238	1.215
160	1.257	1.249	1.225
320	1.262	1.253	1.231
640	1.264	1.257	1.234

Table 4
Example 2: Behavior of $r(u_2^\star, T)$ with respect to Δx and $\lambda = \Delta t/\Delta x$.

$1/\Delta x$	$\lambda = .25$	$\lambda = .50$	$\lambda = .75$
10	1.04012	1.03034	1.0504
20	1.00009	1.00025	1.01097
40	1.00071	0.99700	1.00065
80	1.00055	0.99875	0.99878
160	0.99947	0.99931	0.99895
320	1.00014	0.99968	0.99935
640	1.00019	0.99990	0.99964

Table 5
Example 3: Behavior of $r(u_1^\star, T)$ with respect to Δx and $\lambda = \Delta t/\Delta x$.

$1/\Delta x$	$\lambda = .25$	$\lambda = .50$	$\lambda = .75$
10	1.385	1.347	1.361
20	1.233	1.212	1.248
40	1.155	1.149	1.186
80	1.116	1.112	1.135
160	1.087	1.081	1.096
320	1.064	1.059	1.069
640	1.045	1.042	1.049

Table 6
Example 3: Behavior of $r(u_2^\star, T)$ with respect to Δx and $\lambda = \Delta t/\Delta x$.

$1/\Delta x$	$\lambda = .25$	$\lambda = .50$	$\lambda = .75$
10	1.301	1.265	1.287
20	1.159	1.141	1.184
40	1.088	1.085	1.132
80	1.057	1.059	1.094
160	1.038	1.040	1.066
320	1.028	1.029	1.048
640	1.020	1.021	1.034

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Appendix. Proof of Theorem 2.1. By Kuznetsov [12, Lemma 2], we have that, for $0 < \epsilon_t < T$ and ϵ_x,

$$\begin{aligned}\| u(T) - v(T) \|_{L^1(\mathbb{R}^d)} \le{}& \| u(0) - v_0 \|_{L^1(\mathbb{R}^d)} + E(u, v; T) \\ &+ 2\,\epsilon_x \left\{ | v_0 |_{TV(\mathbb{R}^d)} + | \nabla u |_{L^\infty(0,T;L^1(\mathbb{R}^d))} \right\} \\ &+ 2\,\epsilon_t \left\{ \| f' \| \, | v_0 |_{TV(\mathbb{R}^d)} + | u_t |_{L^\infty(0,T;L^1(\mathbb{R}^d))} \right\},\end{aligned}$$

where $\| f' \| = \sup_{|n|=1} \sup_{v \in [a,b]} \left| f'(v) \cdot n \right|$ and

$$E(u, v; \tau) = \int_0^\tau \int_{\mathbb{R}^d} \Theta_\tau(u, v(t,x); t, x)\, dx\, dt,$$

where

$$\begin{aligned}\Theta_\tau(u, c; t, x) = {}& - \int_0^\tau \int_{\mathbb{R}^d} U(u(t', x') - c)\, \partial_{t'} \varphi(t, x, t', x')\, dx'\, dt' \\ &+ \int_{\mathbb{R}^d} U(u(\tau, x') - c)\, \varphi(t, x, \tau, x')\, dx' - \int_{\mathbb{R}^d} U(u(0, x') - c)\, \varphi(t, x, 0, x')\, dx' \\ &- \int_0^\tau \int_{\mathbb{R}^d} F(u(t', x'), c) \cdot \nabla_{x'} \varphi(t, x, t', x')\, dx' dt'.\end{aligned}$$

The function $\varphi = \varphi(t, x, t', x')$ is taken as follows:

$$\varphi = w_{\epsilon_t}(t - t') \prod_{i=1}^{d} w_{\epsilon_x}(x_i - x_i'), \qquad (x, t), (x', t') \in \mathbb{R}^d \times \mathbb{R}^+,$$

where ϵ_t and ϵ_x are two arbitrary positive numbers and $w_\lambda(s) = \frac{1}{\lambda}\, w(\frac{s}{\lambda})$, for any $s \in \mathbb{R}$, $\lambda = \epsilon_t, \epsilon_x$. The function w is a nonnegative, even, smooth, compact supported function of unit mass.

It is enough to prove Theorem 2.1 in the case in which the function u has a single discontinuity surface, $\mathcal{C}(T) = \mathcal{C}_1(T)$, which divides $(0, T) \times \mathbb{R}^d$ in two disjoint subsets whose union is $\Omega(T)$. In this case, after performing integration by parts, we obtain

$$\begin{aligned}\Theta(u, v; \tau) = {}& \int_{\Omega(T)} Residual(u(t', x'))\, sign(u(t', x') - v(t, x))\, \varphi(t, x, t', x')\, dx'\, dt' \\ &+ \int_{\mathcal{C}(T)} ([F(u, c)], [U(u - c)])(t', x') \cdot n_{\mathcal{C}}(t', x')\ \varphi(t, x, t', x')\, d\Gamma'.\end{aligned}$$

Since, by the definition of φ, $\int_0^\tau \int_{\mathbb{R}^d} \varphi(t, x, t', x')\, dx\, dt \le 1$, we obtain

$$\begin{aligned}E(u, v; \tau) \le{}& \int_{\Omega(T)} | Residual(u(t', x')) |\, dx'\, dt' \\ &+ \int_{\mathcal{C}(T)} \max \left\{ 0, \sup_{c \in \mathbb{R}} \left\{ ([F(u, c)], [U(u - c)])(t', x') \cdot n_{\mathcal{C}}(t', x') \right\} \right\} d\Gamma' \\ ={}& \| Residual(u) \|_{L^1(\Omega(T))} + \| Residual(u) \|_{L^1(\mathcal{C}(T))}.\end{aligned}$$

We can now let the parameters ϵ_t and ϵ_x go to zero to obtain Theorem 2.1.

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