COMMUNICATIONS ON PURE AND APPLIED MATHEMATICS, VOL. XXX, 611–638 (1977)

# The Artificial Compression Method for Computation of Shocks and Contact Discontinuities. I. Single Conservation Laws

AMIRAM HARTEN*

## 1. Introduction

The pioneering work of Von Neumann and Richtmyer [16] gave birth to the shock capturing concept in the computation of discontinuous solutions of hyperbolic problems. In this approach a discontinuity is computed as part of the solution rather than, as in shock fitting, being considered as an internal boundary. The idea of Von Neumann and Richtmyer was to solve a system of equations modified by the addition of artificial viscosity terms. This modified system has a continuous transition wherever the original system has a jump discontinuity and therefore can be conceptually approached by finite differences.

Lax and Wendroff [12] showed that the limit solution of any finite difference scheme in a conservation form which is consistent with the conservation laws, satisfies the jump conditions across a discontinuity *automatically*. This was a conceptual breakthrough which enabled the direct discretization of the conservation laws by introducing the notion of *numerical* viscosity.

The shock capturing approach is not without faults: Standard finite difference schemes approximate a shock by a continuous transition which occupies 3–5 cells. This is a severe obstacle for the development of multi-dimensional computer codes. The computation of contact discontinuities by standard schemes is even worse. The width of the computed contact discontinuity behaves asymptotically like $n^{1/(R+1)}$, where $n$ is the number of time steps taken and $R$ is the order of accuracy of the scheme. This implies that steady state solutions of standard finite difference schemes cannot contain contact discontinuities.

Another unpleasant feature of the shock capturing is that finite difference schemes with second or higher-order accuracy produce overshoots or undershoots upon crossing the discontinuity. These oscillations not only damage the

* This paper was prepared as a result of work performed at the Courant Institute of Mathematical Sciences under U. S. Air Force Contract No. F44620-74-C-0062 and at the Institute for Computer Applications in Science and Engineering (ICASE) NASA Langley Research Center, Hampton, Virginia, under NASA Contract No. NASI-14101. Reproduction in whole or in part is permitted for any purpose of the United States Government.

accuracy but can also falsely trigger chemical reactions, cause nonlinear instabilities (cf. [2]), and force the scheme to pick a nonphysical solution (cf. [6]). Several numerical viscosity techniques have been designed in order to damp these oscillations (e.g. [12], [2]; [7], [8]); unfortunately these techniques further smear the discontinuity.

At present the name of the game is resolution without oscillations. Among the special schemes designed to achieve this goal are the anti-diffusion method of Boris and Book [1], the use of noncentered differencing by Van Leer [15] and Warming and Beam [17], the implementation of Glimm's scheme by Chorin [3], and the method of artificial compression (ACM) developed by the author. The ACM is a technique to modify standard finite difference schemes which prevents the smearing of contact discontinuities and improves the resolution of shocks. It can be easily implemented in existing computer programs.

This paper is the first in a series of several articles on the ACM and its applications; part of the material presented here is based on an earlier report [5]. In this first article the discussion is restricted to first-order accurate schemes for a single conservation law. Subsequent papers will describe the ACM extensions to systems of conservation laws in several space dimensions, to higher-order accurate schemes and to multi-fluid problems.

Since the ACM is closely related to the entropy condition for shocks, we start in Section 2 with a brief review of the relevant theory. In Section 3 we analyze the factors that control the smearing of shocks and linear discontinuities; this analysis is used in Section 4 to outline the principles of the ACM. In Sections 5 and 6 we describe the numerical implementation of the ACM; we conclude with a discussion of its properties in Section 7.

## 2. Discontinuous Solutions of a Conservation Law

We consider the initial value problem

$$u_t + f(u)_x = 0\,, \qquad u(x,0) = \phi(x)\,, \qquad -\infty < x < \infty\,. \tag{2.1}$$

The solution $u(x, t)$ of (2.1) is constant along the characteristic curve $x = x(t)$, where

$$\frac{dx}{dt} = \frac{df}{du} \equiv a(u)\,. \tag{2.2}$$

The constancy of $u$ along the characteristics implies that the characteristics are straight lines. Their slope, however, depends upon the initial data and therefore they may intersect, and where they do, no smooth solution will exist. To get existence in the large, we admit "weak solutions" which satisfy an integral version of (2.1):

$$\int_0^\infty \int_{-\infty}^\infty [w_t u + w_x f(u)]\, dx\, dt + \int_{-\infty}^\infty w(x, 0)\phi(x)\, dx = 0\,, \tag{2.3}$$

where $w(x, t)$ is any smooth test function of compact support. Equation (2.3) admits piecewise continuous solutions of (2.1) such that across the discontinuity curve the Rankine-Hugoniot relation is satisfied

$$f(u_R) - f(u_L) = S(u_R - u_L)\,. \tag{2.4}$$

Here $S$ is the speed of propagation of the discontinuity, and $u_L$ and $u_R$ are the states on the left and on the right of the discontinuity, respectively.

The class of all weak solutions is too wide in the sense that there is no uniqueness for the initial value problem; e.g. (2.1) with $f = \frac{1}{2}u^2$ and $\phi(x) = 0$ for $x < 0$, $\phi(x) = 1$ for $x \geqq 0$ has the discontinuous solution $u(x, t) = 0$ for $x < \frac{1}{2}t$, $u(x, t) = 1$ for $x \geqq \frac{1}{2}t$ as well as the continuous solution $u(x, t) = 0$ for $x \leqq 0$, $u(x, t) = x/t$ for $0 \leqq x \leqq t$, $u(x, t) = 1$ for $x \geqq t$. One approach for determining the physically relevant solutions is to admit only those which are obtainable as limits of solutions $u^\varepsilon(x, t)$, $\varepsilon \downarrow 0$, of the viscous problem

$$u_t^\varepsilon + f(u^\varepsilon)_x = \varepsilon[\beta(u^\varepsilon)u_x^\varepsilon]_x\,, \qquad \varepsilon > 0, \quad \beta > 0\,. \tag{2.5}$$

Equation (2.5) is a parabolic equation and it follows from the maximum principle that it has at most one solution. It is also true that a solution $u^\varepsilon$ exists for all $t$ and that, as $\varepsilon \to 0$, these solutions converge in the $L_1$ sense to a limit $u$.

Oleinik [13] gave the following characterization of an admissible discontinuity $(u_L, u_R, S)$, i.e., a discontinuity in the limit solution $u$ (see also [10]):

$$s(u, u_L) \equiv \frac{f(u) - f(u_L)}{u - u_L} \geqq S \geqq \frac{f(u) - f(u_R)}{u - u_R} \equiv s(u, u_R) \tag{2.6}$$

for all $u$ between $u_L$ and $u_R$. Inequality (2.6) is known also as the entropy condition, or condition E. It is useful to restate (2.6) in terms of the function $g_0(u) = f(u) - Su$, which is the flux in the coordinate system which moves with the discontinuity, and the constant $C \equiv g_0(u_L) = g_0(u_R)$ (equality follows from (2.4))

$$[g_0(u) - C]\operatorname{sgn}(u_R - u_L) \geqq 0 \tag{2.7}$$

for all $u$ between $u_L$ and $u_R$.

We say that a discontinuity $(u_L, u_R, S)$ possesses a viscous profile if there exists a solution to (2.5) of the form $u^\varepsilon(x, t) = V^\varepsilon(y)$, $y = (x - St)/\varepsilon$ such that $\lim_{y \to +\infty} V^\varepsilon = u_R$ and $\lim_{y \to -\infty} V^\varepsilon = u_L$. It was shown in [5] that a discontinuity $(u_L, u_R, S)$ possesses a viscous profile if and only if a strict inequality in (2.7) holds for all $u$ between $u_L$ and $u_R$. This profile is given implicitly by the expression

$$y(V) = \int_{v_0}^{V} \frac{\beta(w)}{g_0(w) - C}\, dw + y(v_0)\,. \tag{2.8a}$$

Let $y_\pm$, $y_- < y_+$ be such that $V^\varepsilon(y_\pm) = v_\pm$. Hence $W(v_-, v_+) = y_+ - y_-$ can be expressed as

$$W(v_-, v_+) = \int_{v_-}^{v_+} \frac{\beta(w)}{g_0(w) - C}\, dw\,. \tag{2.8b}$$

Equation (2.8b) depicts an important property of the viscous profile: the width of the transition from $v_-$ to $v_+$ is of order $\varepsilon$.

In the context of this paper we say that a discontinuity is a shock if a strict inequality holds in (2.7) for all $u$ between $u_L$ and $u_R$. A discontinuity is called a contact discontinuity or a linear discontinuity if an equality holds identically in (2.7), i.e., $f(u) = Su + \text{const.}$ for all $u$ between $u_L$ and $u_R$. Thus a shock possesses a viscous profile but a contact discontinuity does not. In the next section we shall show that a similar statement is true for solutions of finite difference equations.

## 3. Computation of Discontinuous Solutions by Finite Difference Schemes

Let $v(x, t)$ be a finite difference approximation to (2.1):

$$v_j^{n+1} = (L \cdot v^n)_j\,, \qquad v_j^n = v(j\,\Delta x, n\,\Delta t)\,; \tag{3.1}$$

$\Delta t$ and $\Delta x$ are the time and space increments. The finite difference scheme (3.1) is said to be in conservation form if it can be written in the following way:

$$v_j^{n+1} = v_j^n - \lambda(h_{j+1/2} - h_{j-1/2})\,, \qquad \lambda = \Delta t/\Delta x\,, \tag{3.2a}$$

where

$$h_{j+1/2} = h(v_{j-k+1}, v_{j-k+2}, \cdots, v_{j+k})\,, \qquad h_{j-1/2} = h(v_{j-k}, v_{j-k+1}, \cdots, v_{j+k-1}).$$

In order for (3.2a) to be consistent with (2.1), $h$ must be related to $f$ as follows:

(3.2b) $$h(w, w, \cdots, w) = f(w).$$

Lax and Wendroff [12] proved the following theorem: Let $v(x, t)$ be a solution of a finite difference scheme in conservation form. If $v(x, t)$ converges boundedly almost everywhere to some function $u(x, t)$ as $\Delta x$ and $\Delta t$ tend to zero, $\lambda = \Delta t/\Delta x = \text{fixed}$, then $u(x, t)$ is a weak solution of (2.1).

This theorem does not answer the question whether this limit solution $u(x, t)$ is the unique physically relevant solution, i.e., whether all its discontinuities satisfy the entropy condition (2.6). In the following we shall discuss the so-called monotone schemes, the limit solutions of which do satisfy the entropy condition (2.6).

A finite difference scheme

(3.3a) $$v_j^{n+1} = H(v_{j-k}^n, v_{j-k+1}^n, \cdots, v_{j+k}^n)$$

is said to be *monotone* if $H$ is a monotone increasing function of each of its arguments, i.e.,

(3.3b) $$\frac{\partial H}{\partial v_{j-m}^n} \geqq 0 \quad \text{for} \quad -k \leqq m \leqq k.$$

A truncation error analysis of monotone schemes in conservation form shows that for all smooth solutions of (2.1)

(3.4a) $$\begin{aligned} u(x, t+\Delta t) - H(u(x-k\,\Delta x, t), \cdots, u(x+k\,\Delta x, t)) \\ = -(\Delta t)^2[\beta(u, \lambda)u_x]_x + O((\Delta t)^3), \end{aligned}$$

where

(3.4b) $$\beta(u, \lambda) = \frac{1}{2\lambda^2} \sum_{j=-k}^{k} j^2 H_j(u, \cdots, u) - \tfrac{1}{2}a^2(u).$$

Here $H_j$ is the partial derivative of $H$ with respect to its $(k+j+1)$-th argument, and $a = df/du$; e.g., for the scheme of Lax and Friedrichs

(3.5a) $$v_j^{n+1} = \tfrac{1}{2}(v_{j-1}^n + v_{j+1}^n) - \tfrac{1}{2}\lambda[f(v_{j+1}^n) - f(v_{j-1}^n)]$$

which is monotone under the CFL condition

(3.5b) $$\lambda \max |a(u)| \leqq 1,$$

we have

$$\beta(u, \lambda) = \tfrac{1}{2}[1/\lambda^2 - a^2(u)]. \tag{3.5c}$$

It was shown in [6] that $\beta(u, \lambda) \geqq 0$ and $\beta(u, \lambda) \not\equiv 0$, except in the trivial case where $H_j(u, \cdots, u) \equiv 0$ for all $j \neq j_0$, i.e., the finite difference operator is a pure translation. Thus, monotone schemes are inherently first-order accurate. This result was well known for the linear case, see [11] and [4].

A finite difference operator $L$ is called *monotonicity preserving* if, for any monotone mesh function $v$, $w = L \cdot v$ is also monotone.

THEOREM 3.1. *A monotone scheme is monotonicity preserving.*

Proof: Assume that $v$ is any mesh function and denote $w_j = H(v_{j-k}, v_{j-k+1}, \cdots, v_{j+k})$. By the mean value theorem there exists a $\theta$, $0 \leqq \theta \leqq 1$, such that

$$\begin{aligned} w_{j+1} - w_j &= H(v_{j-k+1}, \cdots, v_{j+k+1}) - H(v_{j-k}, \cdots, v_{j+k}) \\ &= \sum_{m=-k}^{k} H_m(\theta v_{j-k+1} + (1-\theta)v_{j-k}, \cdots, \theta v_{j+k} + (1-\theta)v_{j+k+1}) \\ &\qquad \cdot (v_{j+m+1} - v_{j+m}), \end{aligned}$$

whereas before, $H_i$ is the partial derivative of $H$ with respect to its $(i+k+1)$-th argument; by (3.3b), $H_i \geqq 0$ for $-k \leqq i \leqq k$. Since $v$ is a monotone increasing (decreasing) function of $j$, $v_{j+1} - v_j \geqq 0$ ($\leqq 0$) for all $j$; it follows from the non-negativity of the coefficients $H_m$ that also $w_{j+1} - w_j \geqq 0$ ($\leqq 0$); thus $w$ is a monotone mesh function of the same kind.

Although the two notions of monotonicity are equivalent in the linear constant coefficients case (see [4]), the converse to Theorem 3.1 is not true in the nonlinear case. The class of monotonicity preserving schemes is larger than that of monotone schemes and it includes also second-order accurate schemes (Van Leer [14]). We remark that the notion of monotonicity preserving schemes, unlike the notion of monotone schemes, cannot be extended to nonlinear systems.

Solutions of monotone schemes behave very much like solutions of the corresponding modified parabolic equation

$$w_t + f(w)x = \Delta t[\beta(w, \lambda) w_x]_x, \tag{3.6}$$

where $\beta(w, \lambda) \geqq 0$ is given by (3.4a). Both approximations share many properties like maximum principle, $L_1$-contractiveness and convergence to solutions which satisfy the entropy condition (see [6]). Furthermore, Jennings

[9] was able to show that monotone schemes in conservation form possess steady progressing profiles (discrete shocks) which are discrete versions of the viscous profiles discussed in Section 2, and exist exactly under the same condition of a strict inequality in (2.7).

We conjecture that the width of the transition of these discrete shocks can be likewise approximated by (2.8b):

$$W(v_-, v_+) \approx \lambda \int_{v_-}^{v_+} \frac{\beta(w, \lambda)}{g_0(w) - C} \, dw . \tag{3.7}$$

Here $\beta(w, \lambda)$ is given by (3.4a) and $W(v_-, v_+)$, a dimensionless quantity, measures the number of cells occupied by values between $v_-$ and $v_+$. A numerical test reported in [5] showed excellent agreement between the approximation (3.7) and the actual width of the transition.

It is made clear in (3.7) that the width $W$ of the transition is inversely proportional to the size of $g_0(w) - C$, which measures the nonlinearity of the flux function for values between $u_L$ and $u_R$. If $[g_0(w) - C] \operatorname{sgn}(u_R - u_L)$ is positive but small, i.e., if the flux function is almost linear, then the spread $W$ will be unacceptably large. In fact, if the flux function is linear, then linear finite difference schemes cannot possess a steady progressing solution, that is $W \to \infty$ as $n \to \infty$.

For example let us consider the solution of the initial value problem

$$u_t + S u_x = 0 , \qquad S = \text{const.}, \tag{3.8a}$$

$$u(x, 0) = \begin{cases} u_L & \text{for } x \leqq 0 , \\ u_R & \text{for } x > 0 , \end{cases} \tag{3.8b}$$

by the Lax-Friedrichs scheme (3.5a). Because of the linearity of the problem and the scheme, the solution can be expressed explicitly by

$$\begin{aligned} v_j^{n+1} &= [(1-\theta) T_\Delta + \theta T_\Delta^{-1}]^n v_j^0 = \sum_{k=0}^{n} \binom{n}{k} \theta^k (1-\theta)^{n-k} v_{j+n-2k} \\ &= u_R + (u_L - u_R) \sum_{k \geqq (j+n)/2}^{n} \binom{n}{k} \theta^k (1-\theta)^{n-k} , \end{aligned} \tag{3.9}$$

where $T_\Delta v_j = v_{j+1}$, $T_\Delta^{-1} v_j = v_{j-1}$, and $0 \leqq \theta = \frac{1}{2}(1 + \lambda S) \leqq 1$. By the DeMoivre-Laplace limit theorem, for $n$ sufficiently large, the binominal distribution (3.9) can be approximated by

$$v_j^n \approx \tfrac{1}{2}(u_R + u_L) + \tfrac{1}{2}(u_R - u_L) \operatorname{Erf}\left(y/\sqrt{2n(1 - \lambda^2 S^2)}\right) , \tag{3.10}$$

where $y=(x-St)/\Delta x$. Thus $v_j^n$ is asymptotically constant along the curves $y/\sqrt{2n(1-\lambda^2S^2)}=\text{const}$. Therefore the width of the transition at time $T=n\cdot\Delta t$ is given by

(3.11)
$$W=\text{const.}\sqrt{n(1-\lambda^2S^2)}\,.$$

Observe that the right-hand side of (3.10) is the solution to the parabolic modified equation (3.6) with the coefficient $\beta(\lambda)$ of the Lax-Friedrichs scheme (3.5c). One can show in general that the leading asymptotic term in the transition width in the solution of an $R$-th order accurate linear finite difference scheme is given by the width of the transition in the solution of the modified equation (see [5]):

(3.12a)
$$u_t+Su_x=(\Delta x)^R\lambda\beta_R(\lambda)\frac{\partial^{R+1}u}{\partial x^{R+1}}+\cdots.$$

The latter, by a similarity argument, has the solution

(3.12b)
$$u(x,t)=F(Z)\,,\qquad Z=\frac{(x-St)/\Delta x}{[\lambda^2\beta_R(\lambda)t/\Delta t]^{1/(R+1)}}\,.$$

Thus the transition width in the solution of an $R$-th order accurate linear finite difference scheme is given by

(3.12c)
$$W=\text{const.}\,[\lambda^2\beta_R(\lambda)n]^{1/(R+1)}\,.$$

We conclude this section with a short summary: Solutions of monotone schemes behave similarly to solutions of their parabolic modified equation; likewise they possess steady progressing profiles under a strict inequality in the entropy condition (2.7) ($[g_0(u)-C]\,\text{sign}\,(u_R-u_L)>0$). The spread of the discrete shocks is inversely proportional to $|g_0(u)-C|$. In the case of linear discontinuity, ($g_0(u)-C\equiv 0$) no linear monotone scheme, nor any other linear finite difference scheme, can possess a steady progressing profile; the linear discontinuity is smeared as $n^{1/(R+1)}$, where $n$ is the number of time steps taken and $R$ is the order of accuracy.

## 4. Principles of Artificial Compression

In this section we shall discuss modifications of standard finite difference scheme by an *artificial compression method* (ACM). These modifications prevent the smearing of contact discontinuities and improve the resolution of shocks.

Let $u(x, t)$ be a solution of the single conservation law

$$u_t + f(u)_x = 0 \tag{4.1}$$

with either a shock or a contact discontinuity $(u_L(t), u_R(t), S(t))$ propagating with speed $S(t)$, across which the value of $u$ jumps from $u_L(t)$ to $u_R(t)$. To simplify the description we assume that at time $t$, the solution $u(x, t)$ does not take on values between $u_L(t)$ and $u_R(t)$.

The essence of the ACM is to solve a modified equation

$$u_t + [f(u) + g(u, t)]_x = 0 \tag{4.2}$$

rather than the original equation (4.1). Here, $g(u, t)$ is any function which has the following properties:

$$g(u, t) \equiv 0 \quad \text{for} \quad u \notin (u_L(t), u_R(t)), \tag{4.3a}$$

$$g(u, t) \cdot \operatorname{sgn}[u_R(t) - u_L(t)] > 0 \quad \text{for} \quad u \in (u_L(t), u_R(t)). \tag{4.3b}$$

A function $g(u, t)$ which satisfies (4.2) is called an *artificial compression flux* (ACF).

THEOREM 4.1. *$u(x, t)$, the solution of the original conservation law* (4.1), *is also a solution of the modified equation* (4.2).

Proof: It follows, from property (4.3a) and the assumption that $u(x, t)$ at time $t$ does not take on values between $u_L(t)$ and $u_R(t)$, that $\hat{f} = f + g$ is identical to $f$ except for $u$ in $(u_L(t), u_R(t))$. To complete the proof we show that $(u_L(t), u_R(t), S(t))$ is an admissible discontinuity for the modified equation (4.2). Property (4.3a) implies that $g(u_L, t) = g(u_R, t) \equiv 0$; thus the Rankine-Hugoniat relation (2.4), $\hat{f}(u_R, t) - \hat{f}(u_L, t) \equiv f(u_R) - f(u_L) = S(u_R - u_L)$, is satisfied by the same speed of propagation $S(t)$. Moreover, $(u_L, u_R, S)$ is an admissible discontinuity of (4.1), i.e., the entropy condition (2.7)

$$[g_0(u) - C] \operatorname{sgn}(u_R - u_L) \geqq 0$$

is satisfied for all $u \in (u_L, u_R)$; here $g_0 = f - Su$, $C = g_0(u_L) = g_0(u_R)$. Denote by $\hat{g}_0(u, t) = \hat{f}(u, t) - Su$ the corresponding function for the modified equation (4.2): $\hat{g}_0 = g_0 + g(u, t)$, $\hat{C} = \hat{g}_0(u_L, t) = \hat{g}_0(u_R, t) = C$. It follows from property (4.3b) that the entropy condition in the modified equation is also satisfied:

$$(\hat{g}_0 - \hat{C}) \operatorname{sgn}(u_R - u_L) = (g_0 - C) \operatorname{sgn}(u_R - u_L) + g \cdot \operatorname{sgn}(u_R - u_L) > 0 .$$

The original conservation equation (4.1) and the modified equation (4.2) have the same solution; however, when we apply the same finite difference scheme to both equations, the numerical solution is not the same. The solution of the modified equation (4.2) has better resolution of shocks and contact discontinuities. When $(u_L, u_R, S)$ is a contact discontinuity for the original equation, it is a shock for the modified equation. From the analysis presented in Section 3 we conclude that any monotone finite difference scheme applied to (4.1) will smear the discontinuity as $\sqrt{n}$; but by Jennings' theorem, the scheme when applied to the modified equation (4.2) will produce steady progressing profiles.

If $(u_L, u_R, S)$ is a shock for (4.1), then it is also a shock for the modified equation (4.2). The modification amounts to increasing the denominator $|g_0(u) - C|$ in (3.7). Since the spread of the transition is inversely proportional to this quantity, the modification has the effect of decreasing the spread.

For example, let us consider a computation of an admissible discontinuity $(u_L, u_R, S)$, by applying the Lax-Friedrichs scheme (3.5a) to the modified equation (4.2):

$$\text{(4.4a)}\quad v_j^{n+1} = \tfrac{1}{2}(v_{j+1}^n + v_{j-1}^n) - \tfrac{1}{2}\lambda(\alpha)[f(v_{j+1}^n) - f(v_{j-1}^n) + \alpha(g(v_{j+1}^n) - g(v_{j-1}^n))],$$

$$\text{(4.4b)}\qquad g(v) = \begin{cases} (v - u_L)(u_R - v)/(u_R - u_L) & \text{for } \; v \in [u_L, u_R], \\ 0 & \text{otherwise}, \end{cases}$$

$$\text{(4.4c)}\qquad v_j^0 = \begin{cases} u_L & \text{for } \; j \leqq 0, \\ u_R & \text{for } \; j > 0, \end{cases}$$

where $\alpha$ is a positive parameter.

Consider first a contact discontinuity, $f(u) = Su$. According to the CFL condition (3.5b), we must choose

$$\text{(4.5a)}\qquad \lambda = \lambda(\alpha) = \delta/(|S| + \alpha), \qquad 0 < \delta \leqq 1.$$

For $\alpha = 0$, $\delta \neq 1$, the initial discontinuity is smeared as $\sqrt{n}$. For all $\alpha > 0$, the numerical solution (4.4a) approaches a steady progressing profile. The spread of this profile is given approximately by formula (3.7). Using (3.5c) we can evaluate this integral:

$$\text{(4.5b)}\quad W(v_-, v_+) = \frac{2\delta\alpha}{|S| + \alpha}(1 - 2\eta) + \frac{(|S| + \alpha)^2 - \delta^2(S^2 + \alpha^2)}{\delta\alpha(|S| + \alpha)} \log \frac{1 - \eta}{\eta},$$

where

$$\eta = \frac{v_- - u_L}{u_R - u_L} = \frac{u_R - v_+}{u_R - u_L};$$

note that $1 < \eta < 0.5$. It is easy to see that $W$ as given by formula (4.5b) is a decreasing function of $\alpha$. The table below, calculated for $\eta = 0.1$, $\delta = 0.9$, $S = 1$, gives an evaluation of the size of $W$ for various choices of $\alpha$:

| $\alpha$ | 0.0 | 0.25 | 0.5 | 1.0 | $\infty$ |
|---|---|---|---|---|---|
| $W/\Delta x$ | $\infty$ | 5.77 | 4.50 | 3.62 | 1.63 |

When the initial discontinuity is a shock, e.g. $f = u^2$, $u_L = 1$, $u_R = 0$, the finite difference scheme (4.4) possesses a steady progressing profile for all $\alpha \geqq 0$, provided that $\lambda$ satisfies the CFL condition (3.5b):

$$\lambda = \lambda(\alpha) = \delta/(2+\alpha), \qquad 0 < \delta \leqq 1. \tag{4.6a}$$

The spread of the profile is given approximately by

$$W(\eta, 1-\eta) = \frac{(2+\alpha)^2 - \delta^2[2(1+\alpha)+\alpha^2]}{\delta(1+\alpha)(2+\alpha)} \log \frac{1-\eta}{\eta} + 2\delta \frac{1+\alpha}{2+\alpha}(1-2\eta); \tag{4.6b}$$

(4.6b) is a decreasing function of $\alpha$. The table below, calculated for $\eta = 0.1$, $\delta = 0.9$, gives an evaluation of the size of $W$ for various choices of $\alpha$:

| $\alpha$ | 0.0 | 2.0 | 4.0 | $\infty$ |
|---|---|---|---|---|
| $W/\Delta x$ | 3.62 | 2.69 | 2.42 | 1.9 |

The addition of the artificial compression flux increases the sound speed to the left of the shock and decreases it to the right of the shock. This changes the characteristic field so as to make it more convergent towards the line $dx/dt = S$. The maximum of the sound speed is thereby increased; according to the CFL condition this causes a reduction of the permissible time step, as can be seen from (3.5a) and (3.6a).

It is convenient to separate the artificial compression from the main calculation. This can be accomplished by splitting the solution of equation (4.3) in the following way: Let $L$ be any finite difference scheme in conservation form (3.2) for solving the original conservation law (4.1):

$$v_j^{n+1} = (L \cdot v^n)_j = v_j^n - \lambda(h_{j+1/2} - h_{j-1/2}),$$

$$h_{j+1/2} = h(v_{j-l+1}, \cdots, v_{j+l}), \qquad h(v, \cdots, v) = f(v),$$

and let $C$ be some finite difference scheme, in conservation form, for solving the equation

$$u_t + g(u)_x = 0, \tag{4.7a}$$

where $g(u)$ is an ACF (4.3). Thus

$$\text{(4.7b)} \qquad \begin{gathered} v_j^{n+1} = (C \cdot v^n)_j = v_j^n - \hat{\lambda}(d_{j+1/2} - d_{j-1/2}), \\ d_{j+1/2} = d(v_{j-m+1}, \cdots, v_{j+m}), \qquad d(v, \cdots, v) = g(v). \end{gathered}$$

We combine these schemes, i.e., we set

$$\text{(4.8a)} \qquad v^{n+1} = \hat{L}v^n = CLv^n.$$

$\hat{L}$ can be written explicitly as

$$\text{(4.8b)} \qquad v_j^{n+1} = (\hat{L}v^n)_j = v_j^n - \lambda(k_{j+1/2} - k_{j-1/2}),$$

where

$$\text{(4.8c)} \qquad \begin{aligned} k_{j+1/2} &= k(v_{j-m-l+1}^n, \cdots, v_{j+m+l}^n) \\ &= h(v_{j-l+1}^n, \cdots, v_{j+l}^n) + d((Lv^n)_{j-m+1}, \cdots, (Lv^n)_{j+m}), \\ k(v, \cdots, v) &= f(v) + \frac{\hat{\lambda}}{\lambda} g(v). \end{aligned}$$

Thus the composite scheme $\hat{L} = CL$ is in a conservation form consistent with a modified equation (4.2). Furthermore, if for some choice of $\lambda$ and $\hat{\lambda}$, both $L$ and $C$ are monotone (or monotonicity preserving) so is $\hat{L}$; when $\hat{L}$ is monotone, Jennings' theorem guarantees the existence of steady progressing profiles.

When $C$ is applied $P$ times, compression increases; thus the higher $P$, the more compressive $\hat{L} = C^P L$ is. Note that $\hat{L} = C^P L$ is consistent with the equation

$$\text{(4.9)} \qquad u_t + \left[ f(u) + P \cdot \frac{\hat{\lambda}}{\lambda} g(u, t) \right]_x = 0.$$

When $C$ is the one-sided scheme to be described in the following section, then $P = 1$ is adequate.

Another practical advantage of split-flux ACM is that *it can be easily added to an existing computer program* in order to improve resolution of shocks and to prevent the smearing of contact discontinuities. A subroutine for the artificial compressor $C$ and a statement calling it in the main program is all that is needed.

## 5. The Artificial Compressor $C$

The artificial compressor $C$ is some finite difference approximation to the solution operator of the equation $u_t + g_x = 0$, where $g$ is an ACF (4.3). Since the ACF $g$ vanishes at $u_L$ and $u_R$, the jump discontinuity $(u_L, u_R)$ is a *stationary shock* for

$$u_t + g_x = 0 . \tag{5.1a}$$

The solution to the initial data

$$u(x,0) = \begin{cases} u_L & \text{for} \quad x \leqq x_L , \\ U_0(x) & \text{for} \quad x_L \leqq x \leqq x_R , \\ u_R & \text{for} \quad x \geqq x_R , \end{cases} \tag{5.1b}$$

where $U_0(x)$ is monotone, $U_0(x_L) = u_L$, $U_0(x_R) = u_R$, tends to the stationary shock solution

$$U_\infty(x) = \begin{cases} u_L & \text{for} \quad x \leqq x_\infty , \\ u_R & \text{for} \quad x > x_\infty ; \end{cases} \tag{5.2a}$$

$x_\infty$ is determined from the conservation relation

$$\int_{x_L}^{x_R} U_0(x)\, dx = \int_{x_L}^{x_R} U_\infty(x)\, dx = (x_\infty - x_L)u_L + (x_R - x_\infty)u_R . \tag{5.2b}$$

Thus the split flux ACM, $\hat{L} = CL$, is a corrective type scheme: $L$ smears the discontinuity while propagating it; $C$ compresses the smeared transition towards a sharp discontinuity. Since $C$ does not involve physical motion and its application does not change the physical time in the solution of the original problem (4.1), the time-step $\hat{\Delta} t = \hat{\lambda}\, \Delta x$ associated with the artificial compressor (4.7) should be regarded as a dummy time-step.

The split flux approach allows the freedom to choose $C$ independently of the main calculation $L$. A good choice for $C$ is a finite difference scheme that has maximal resolution of stationary shocks. It will be shown at the end of this section that the following one-sided scheme has this property:

$$\begin{aligned} v_j^{n+1} &= v_j^n - \tfrac{1}{2}\hat{\lambda}(g_{j+1}^n - g_{j-1}^n) \\ &\quad + \tfrac{1}{2}\hat{\lambda}[|g_{j+1}^n - g_j^n| \operatorname{sgn}(v_{j+1}^n - v_j^n) - |g_j^n - g_{j-1}^n| \operatorname{sgn}(v_j^n - v_{j-1}^n)] . \end{aligned} \tag{5.3}$$

This scheme is in conservation form, i.e., it can be written as

$$v_j^{n+1} = v_j^n - \hat{\lambda}(G_{j+1/2}^n - G_{j-1/2}^n), \tag{5.4a}$$

$$G_{j+1/2}^n = \begin{cases} g_j & \text{if} \quad v_{j+1} > v_j, \\ g_{j+1} & \text{if} \quad v_{j+1} < v_j. \end{cases} \tag{5.4b}$$

Scheme (5.3) is indeed one-sided; this can be seen immediately by introducing the variable $\gamma$,

$$\gamma_{i+1/2} = \begin{cases} [g(v_{j+1}) - g(v_j)]/(v_{j+1} - v_j) & \text{if} \quad v_{j+1} \neq v_j, \\ \dfrac{dg}{du}(v_j) & \text{if} \quad v_j = v_{j+1}, \end{cases} \tag{5.5}$$

and rewriting (5.3) as

$$v_j^{n+1} = v_j^n - \hat{\lambda}\gamma_{j+1/2}^-(v_{j+1}^n - v_j^n) - \hat{\lambda}\gamma_{j-1/2}^+(v_j^n - v_{j-1}^n). \tag{5.6a}$$

The notations $\gamma^+$ and $\gamma^-$ are the standard ones,

$$\gamma^+ = \max(0, \gamma) = \tfrac{1}{2}(\gamma + |\gamma|), \tag{5.6b}$$

$$\gamma^- = \min(0, \gamma) = \tfrac{1}{2}(\gamma - |\gamma|). \tag{5.6c}$$

THEOREM 5.1. *Under the* CFL *condition*

$$\hat{\lambda} \max |\gamma_{j+1/2}| \leqq 1, \tag{5.7}$$

*the one-sided scheme* (5.4), (5.6) *is monotonicity preserving.*

Proof: Denote $\Delta_{i+1/2} W = W_{i+1} - W_i$. By subtracting (5.6) at $j = i$ from (5.6) at $j = i+1$, we get

$$\begin{aligned} \Delta_{i+1/2} v^{n+1} &= \Delta_{i+1/2} v^n - \hat{\lambda}\gamma_{i+3/2}^- \Delta_{i+3/2} v^n - \hat{\lambda}\gamma_{i+1/2}^+ \Delta_{i+1/2} v^n \\ &\quad + \hat{\lambda}\gamma_{i+1/2}^- \Delta_{i+1/2} v^n + \hat{\lambda}\gamma_{i-1/2}^+ \Delta_{i-1/2} v^n \\ &= -\hat{\lambda}\gamma_{i+3/2}^- \Delta_{i+3/2} v^n + (1 - \hat{\lambda}\,|\gamma_{i+1/2}|)\, \Delta_{i+1/2} v^n + \hat{\lambda}\gamma_{i-1/2}^+ \Delta_{i-1/2} v^n. \end{aligned} \tag{5.8}$$

The coefficients $\gamma_{i-1/2}^+$ and $-\gamma_{i+3/2}^- = (-\gamma_{i+3/2})^+$ are non-negative by definition; the coefficient $1 - \hat{\lambda}\,|\gamma_{i+1/2}|$ is non-negative by the CFL condition (5.7). Let $u^n$ be a monotone mesh function and let $s = 1$ if it is increasing and $s = -1$ if it is decreasing; thus $s \cdot \Delta_{i+1/2} v^n = |\Delta_{i+1/2} v^n| \geqq 0$ for all $i$. Multiply (5.8) by $s$; it follows from the non-negativity of the coefficients that likewise $s \cdot \Delta_{i+1/2} v^{n+1} \geqq 0$. Thus $v^{n+1}$ is also a monotone mesh function, and of the same kind.

We remark that if $\gamma_{j+1/2}$ in (5.5) does not change sign for all $v_j$ and $v_{j+1}$ in the domain of interest, then the one-sided scheme is not only monotonicity preserving but also monotone. However, if $\gamma$ changes sign, the one-sided scheme is not necessarily monotone. Consider for example the case $g(u) = u(1-u)$, $0 \leqq u \leqq 1$. The partial derivative of the right-hand side of (5.4b) with respect to $v_{j+1}$ is $\frac{1}{2}\hat{\lambda}(1-2v_{n+1})[1-\operatorname{sgn}(1-v_{j+1}-v_j)]$, which is negative for $v_{j+1} < \frac{1}{2}$, $v_j > 1 - v_{j+1}$. In the case of a stationary shock, $\gamma_{j+1/2}$ does change sign, and consequently one cannot use Jennings' theorem to prove existence of stationary profiles. Nevertheless the one-sided scheme (5.4) does possess stationary shock-like profiles as is evident from the following theorem.

THEOREM 5.2. *Let $g(u)$ be an* ACF (4.3) *for the discontinuity* $(u_L, u_R)$, *i.e.,* $g(u_L) = g(u_R) = 0$ *and* $g(u) \operatorname{sgn}(u_R - u_L) > 0$ *for all* $u \in (u_L, u_R)$. *The finite difference solution of the one-sided scheme* (5.4)–(5.7) *to the initial data*

$$v_j^0 = \begin{cases} u_L & \text{for } j \leqq J_L, \\ V_j & \text{for } J_L < j < J_R, \\ u_R & \text{for } J_R \leqq j, \end{cases} \tag{5.9a}$$

*where $V_j$ is a monotone function of $j$, converges pointwise to the stationary shock-like solution*

$$v_j^\infty = \begin{cases} u_L & \text{for } j < J_\infty, \\ u_L + \alpha(u_R - u_L) & \text{for } j = J_\infty, \\ u_R & \text{for } j > J_\infty, \end{cases} \tag{5.9b}$$

*where the integer $J_\infty$ and $\alpha$, $0 \leqq \alpha \leqq 1$, $J_L \leqq J_\infty \leqq J_R$, are uniquely determined by the conservation relation*

$$\sum_{j=J_L}^{J_R} v_j^0 = \sum_{j=J_L}^{J_R} v_j^\infty = (J_\infty - \alpha - J_L + 1)u_L + (J_R - J_\infty + \alpha)u_R. \tag{5.9c}$$

Proof: The initial data (5.9a) are monotone and the one-sided scheme is monotonicity preserving, thereby $v^n$ is a monotone mesh function for all $n$; thus $s \cdot \Delta_{j+1/2} v^n = |\Delta_{j+1/2} v^n|$ for all $j$ and $n$, $s = \operatorname{sgn}(u_R - u_L)$.

Since $g(u)$ is an ACF for $(u_L, u_R)$ we have $s \cdot g_j = |g_j|$ for all $j$; using this and the identity $\min(a, b) = \frac{1}{2}[a + b - |a-b|]$, we can write the conservation form (5.4) of the one-sided scheme as

$$v_j^{n+1} = v_j^n - \hat{\lambda}(G_{j+1/2} - G_{j-1/2}), \tag{5.10a}$$

$$G_{j+1/2} = s \cdot \min(|g_j|, |g_{j+1}|), \tag{5.10b}$$

$g(u_L) = g(u_R) = 0$; therefore $G_{j+1/2} = 0$ for $j \leqq J_L$ and $j \geqq J_R - 1$. Consequently, for all $n \geqq 0$, $v_j^n = u_L$ for $j \leqq J_L$, $v_j^n = u_R$ for $j \geqq J_R$, and

$$\sum_{j=J_L}^{J_R} v_j^n = \sum_{j=J_L}^{J_R} v_j^0 . \tag{5.10c}$$

Let $J_\infty$ be the largest integer such that, for all $j < J_\infty$, $\lim_{n\to\infty} v_j^n = u_L$. Hence $\lim_{n\to\infty} g_j^n = 0$, $\lim_{n\to\infty} G_{j+1/2}^n = 0$ for all $j < J_\infty$. Subtracting (5.10a) for $j = J_\infty - 1$ from (5.10a) for $j = J_\infty$ and then multiplying by $s$, we get

$$|\Delta_{J_\infty-1/2} v^{n+1}| = |\Delta_{J_\infty-1/2} v^n| - \hat{\lambda} \min(|g_{J_\infty}^n|, |g_{J_\infty+1}^n|) + \hat{\lambda}\varepsilon^n, \tag{5.11a}$$

$$\varepsilon^n = 2G_{J_\infty-1/2}^n - G_{J_\infty-3/2}^n, \qquad \lim_{n\to\infty} \varepsilon^n = 0. \tag{5.11b}$$

We complete the proof of the theorem by showing that

$$\lim_{n\to\infty} \min(|g_{J_\infty}^n|, |g_{J_\infty+1}^n|) = 0, \tag{5.12a}$$

which by the definition of $J_\infty$ and the strict positivity of $|g|$ in $(u_L, u_R)$ implies

$$\lim_{n\to\infty} v_{J_\infty+1}^n = u_R, \tag{5.12b}$$

i.e., the limit solution has the shock-like structure (5.9b). It is easy to see that $G_{j+1/2} \equiv 0$ for (5.9b), and therefore it is a stationary solution of (5.10a); letting $n \to \infty$ in (5.10c), we get (5.9c).

We prove (5.12a) by negation: Suppose the sequence in (5.12a) is bounded away from zero, i.e., there exists some $\eta > 0$ such that $\min(|g_{J_\infty}^n|, |g_{J_\infty+1}^n|) \geqq \eta$ for all $n \geqq 0$. Equation (5.11b) implies that there exists an $N \geqq 0$ such that, for all $n \geqq N$, $|\varepsilon^n| \leqq \frac{1}{2}\eta$. It follows from (5.11a) that

$$|\Delta_{J_\infty-1/2} v^{n+1}| \leqq |\Delta_{J_\infty-1/2} v^n| - \tfrac{1}{2}\hat{\lambda}\eta \quad \text{for all} \quad n \geqq N;$$

consequently,

$$|\Delta_{J_\infty-1/2} v^{N+k}| \leqq |\Delta_{J_\infty-1/2} v^N| - \tfrac{1}{2}\hat{\lambda}\eta \cdot k \quad \text{for all} \quad k.$$

Thus for $k$ sufficiently large, $|\Delta_{J_\infty-1/2} v^{N+k}|$ can be made negative; this is a contradiction.

Equation (5.9b) is the maximal resolution possible for schemes in conservation form. We remark that this maximal resolution of stationary shocks does not necessarily extend to progressing shocks.

## 6. Implementation of Artificial Compression

We turn now to the more general case where we do not know $u_L$ and $u_R$ ahead of time, nor the location of the discontinuity. This information needed for the construction of the ACF (4.3) has to be extracted from the numerical solution $v$ itself. We fit the ideas of the ACM into the numerical framework constructing a $g$ at the beginning of each time-step, such that

$$g(v)\operatorname{sgn}(u_R-u_L)\gtreqq 0 \quad \text{for all} \quad v\in(u_L,u_R)\,, \tag{6.1a}$$

$$g(v)=O(\Delta x)\,, \qquad g(v(x+\Delta x))-g(v(x))=O((\Delta x)^2)\,, \tag{6.1b}$$

for $v$ outside the transition region.

We observe that $\Delta_{j+1/2}v=v_{j+1}-v_j$ is a good candidate for the numerical ACF (6.1): It is $O(\Delta x)$ wherever $v$ is smooth and it has the same sign as $u_R-u_L$ in the transition region. Furthermore, in a region where $v_j$ is a strictly monotone function of $j$, the correspondence between $v$ and $j$ is one-to-one; therefore, $\Delta_{j+1/2}v$ can be regarded as a function of $v$. This is useful for purposes of analysis.

Any mesh function is piecewise monotone; denote by $J_m$ the endpoints of the intervals of monotonicity. To each monotone interval $J_L\leqq j\leqq J_R$, $s=\operatorname{sgn}(v_{J_R}-v_{J_L})$, we assign a function $g$ as follows:

$$g_j=\begin{cases}0 & \text{for } j\leqq J_L\,,\\ s\cdot\min(|\Delta_{j+1/2}v|,|\Delta_{j-1/2}v|) & \text{for } J_L<j<J_R\,,\\ 0 & \text{for } J_R<j\,.\end{cases} \tag{6.2}$$

THEOREM 6.1. (a) *For all $J_L\leqq j<J_R$ such that $v_j\neq v_{j+1}$,*

$$|\gamma_{j+1/2}|=|g_{j+1}-g_j|/|v_{j+1}-v_j|\leqq 1\,. \tag{6.3a}$$

(b) *If $v_{xx}$ is bounded, then*

$$|g_{j+1}-g_j|=O((\Delta x)^2)\,. \tag{6.3b}$$

Proof: It follows from the definition (6.2) that

$$|g_{j+1}-g_j|\leqq|\min(|\Delta_{j+3/2}v|,\ |\Delta_{j+1/2}v|)-\min(|\Delta_{j+1/2}v|,\ |\Delta_{j-1/2}v|)|\,,\quad J_L\leqq j<J_R\,,$$

i.e., an equality in (6.4a) for $J_L+1 \leqq j \leqq J_R-2$, and a strict inequality at $j=J_L$ and $j=J_R-1$. The right-hand side of (6.4a) is smaller than

$$\text{(6.4a)} \qquad \max\left[\min\left(|\Delta_{j+3/2}v|, |\Delta_{j+1/2}v|\right), \min\left(|\Delta_{j+1/2}v|, |\Delta_{j-1/2}v|\right)\right],$$

which is smaller than $|\Delta_{j+1/2}v|$; this proves (6.3a). The right-hand side of (6.4a) is also smaller than $\max\left[|\Delta_{j+3/2}v-\Delta_{j+1/2}v|, |\Delta_{j+1/2}v-\Delta_{j-1/2}v|\right]$; thus

$$\text{(6.4b)} \qquad |g_{j+1}-g_j| \leqq \max\left(|v_{j+2}-2v_{j+1}+v_j|, |v_{j+1}-2v_j+v_{j-1}|\right).$$

If $v$ has a bounded second derivative, then $|v_{l+1}-2v_l+v_{l-1}|=O((\Delta x)^2)$ and (6.3b) follows.

To each of the monotone intervals $[J_m, J_{m+1}]$ we assign an artificial compressor $C_\Delta^{(m)}$, which is the one-sided scheme (5.3) with the numerical ACF (6.2):

$$\text{(6.5a)} \quad w_j^{(m)}=(C_\Delta^{(m)}v)_j=\begin{cases} v_j-\frac{1}{2}\hat{\lambda}[g_{j+1}-g_{j-1}-(|g_{j+1}-g_j|-|g_j-g_{j-1}|)\cdot s_m] \\ \quad \text{for} \quad J_m+1 \leqq j \leqq J_{m+1}-1, \\ v_j \quad \text{otherwise}, \end{cases}$$

where $s_m=\operatorname{sgn}(v_{J_{m+1}}-v_{J_m})$.

THEOREM 6.2. *If* $\hat{\lambda} \leqq 1$, *then* $w^{(m)}=C_\Delta^{(m)}v$ *in* (6.5a) *is monotone in* $[J_m, J_{m+1}]$.

Proof: Due to inequality (6.3a), the CFL condition (5.7) is satisfied by $\hat{\lambda} \leqq 1$. Denote by $\tilde{v}$ the mesh function

$$\text{(6.5b)} \qquad \tilde{v}_j=\begin{cases} v_{J_m} & \text{for} \quad j \leqq J_m, \\ v_j & \text{for} \quad J_m \leqq j \leqq J_{m+1}, \\ v_{J_{m+1}} & \text{for} \quad J_{m+1} \leqq j. \end{cases}$$

The function $\tilde{v}$ is monotone; the one-sided scheme is monotonicity preserving under the CFL condition, therefore if $\hat{\lambda} \leqq 1$, then $C_\Delta^{(m)}\tilde{v}$ is also monotone. From the definitions (6.2) and (6.5a) it follows immediately that $(C_\Delta^{(m)}v)_j = (C_\Delta^{(m)}\tilde{v})_j$ for $J_m \leqq j \leqq J_{m+1}$.

$C_\Delta^{(m)}$ needs to be applied only for such $m$ that $[J_m, J_{m+1}]$ contains an admissible discontinuity. In most practical problems these monotone intervals are easily identified. However, if we cannot locate the relevant monotone regions, then we apply (6.5a) to all of them:

$$w_j = (C_\Delta v)_j = \begin{cases} w_j^{(m)} & \text{for} \quad J_m + 1 \leqq j \leqq J_{m+1} - 1\,, \\ v_j & \text{for} \quad j = J_m, J_{m+1}\,. \end{cases} \tag{6.6}$$

Applying $C_\Delta$ in regions where $v$ is smooth introduces an error of second order, and no harm is done to the first-order accuracy there. Note that $w = C_\Delta v$ is also monotone in $[J_m, J_{m+1}]$, and has exactly the same extrema as $v$; thus

$$\|C_\Delta\| = 1\,. \tag{6.7}$$

In practice one need not locate the partition $\{J_m\}$ of monotone intervals explicitly; (6.6) can be accomplished implicitly by replacing the definition of $g_j$ in (6.2) by

$$s_{j+1/2} = \operatorname{sgn}(\Delta_{j+1/2} v)\,, \tag{6.8a}$$

$$g_j = s_{j+1/2} \cdot \max[0, \min(|\Delta_{j+1/2} v|, s_{j+1/2}\, \Delta_{j-1/2} v)]\,, \tag{6.8b}$$

and applying the one-sided scheme for all $j$:

$$w_j = (C_\Delta v)_j = v_j - \tfrac{1}{2}\hat{\lambda}(g_{j+1} - g_{j-1}) + \tfrac{1}{2}\hat{\lambda}[|g_{j+1} - g_j| \cdot s_{j+1/2} - |g_j - g_{j-1}| \cdot s_{j-1/2}]\,. \tag{6.8c}$$

The conservation form (5.4) of the one-sided scheme

$$w_j (C_\Delta v)_j = v_j - \hat{\lambda}(G_{j+1/2} - G_{j-1/2})\,, \tag{6.9a}$$

$$G_{j+1/2} = \tfrac{1}{2}[(g_j + g_{j+1}) - |g_{j+1} - g_j| \cdot s_{j+1/2}] \tag{6.9b}$$

can be computed directly in a very convenient way. Observing that $\min(a, b) = \frac{1}{2}[a + b - |a - b|]$, one can rewrite (6.9b) as

$$\begin{aligned} G_{j+1/2} &= s_{j+1/2} \cdot \min(s_{j+1/2} \cdot g_j, s_{j+1/2} \cdot g_{j+1}) \\ &= s_{j+1/2} \max[0, \min(s_{j+1/2}\, \Delta_{j-1/2} v, |\Delta_{j+1/2} v|, s_{j+1/2}\, \Delta_{j+3/2} v)]\,. \end{aligned} \tag{6.9c}$$

Next we prove that $C_\Delta$ is indeed an artificial compressor, i.e., Theorem 5.2 is true also for the numerical ACF (6.2). The proof will follow the same general line of reasoning as that of Theorem 5.2, except that changes will be made to account for the fact that the numerical ACF (6.2) is a functional rather than a function.

LEMMA 6.3. *Let $v_j$ be a monotone function of $j$ and $w = C_\Delta v$; then*

$$|\Delta_{j+1/2} w| \geqq \min(|\Delta_{j-1/2} v|, |\Delta_{j+1/2} v|, |\Delta_{j+3/2} v|)\,. \tag{6.10}$$

Proof: Since $v$ is monotone, $\Delta v$ and $G$ have the same sign $s$. Since $C_\Delta$ is monotonicity preserving, $\Delta w$ has the same sign as $\Delta v$. Subtracting equation (6.9) at $j$ from (6.9) at $j+1$ and then multiplying by $s$, we obtain

$$|\Delta_{j+1/2} w| = |\Delta_{j+1/2} v| - \hat{\lambda}\,|G_{j+3/2}| + 2\hat{\lambda}\,|G_{j+1/2}| - \hat{\lambda}\,|G_{j-1/2}|\,. \tag{6.11a}$$

From the definitions (6.8b) and (6.9c) it follows immediately that

(i) $|g_j| + |g_{j+1}| \leqq |G_{j+1/2}| + |\Delta_{j+1/2} v|$,

(ii) $|\Delta_{j+1/2} v| \geqq |g_j| \geqq |G_{j+1/2}|$.

Writing $|\Delta_{j+1/2} v| = \hat{\lambda}\,|\Delta_{j+1/2} v| + (1-\hat{\lambda})\,|\Delta_{j+1/2} v|$ and using (i) for the first term and (ii) for the second one, we get

$$|\Delta_{j+1/2} v| > \hat{\lambda}(|g_j| + |g_{j+1}| - |G_{j+1/2}|) + (1-\hat{\lambda})\,|G_{j+1/2}|\,. \tag{6.11b}$$

Using (6.11b) on the right-hand side of (6.11a), we obtain

$$\begin{aligned} |\Delta_{j+1/2} w| &\geqq \hat{\lambda}(|g_j| + |g_{j+1}| - |G_{j+1/2}|) + (1-\hat{\lambda})\,|G_{j+1/2}| - \hat{\lambda}\,|G_{j+3/2}| \\ &\quad + 2\hat{\lambda}\,|G_{j+1/2}| - \hat{\lambda}\,|G_{j-1/2}| \\ &= |G_{j+1/2}| + \hat{\lambda}(|g_j| - |G_{j-1/2}|) + \hat{\lambda}(|g_{j+1}| - |G_{j+3/2}|)\,. \end{aligned}$$

By (ii), the last two terms in the right-hand side of the above equality are positive; thus $|\Delta_{j+1/2} w| \geqq |G_{j+1/2}|$. By definition (6.9c), (6.10) follows.

Let $v^0$ be initial data of the form

$$v_j^0 = \begin{cases} u_L & \text{for} \quad j \leqq J_L\,, \\ U_j & \text{for} \quad J_L \leqq j \leqq J_R\,, \\ u_R & \text{for} \quad J_R \leqq j\,, \end{cases} \tag{6.12}$$

where $U_j$ is a *strictly* monotone function of $j$, $U_{J_L} = u_L$ and $U_{J_R} = u_R$. Let $v^{n+1} = C_\Delta v^n$, $n \geqq 0$. Denote by $J_{L,n}$ the largest integer $J$ such that $v_j^n = u_L$ for $j \leqq J$ and denote by $J_{R,n}$ the smallest integer $J$ such that $v_j^n = u_R$ for $j \geqq J$; $J_{L,0} = J_L$, $J_{R,0} = J_R$. From (6.9) we see that $G^n_{J_{L,n} \pm 1/2} = G^n_{J_{R,n} \pm 1/2} = 0$, and therefore $v^{n+1}_{J_{L,n}} = u_L$ and $v^{n+1}_{J_{R,n}} = u_R$. Thus $J_{L,n}$ is nondecreasing, $J_{L,n+1} \geqq J_{L,n}$, and $J_{R,n}$ is nonincreasing, $J_{R,n+1} \leqq J_{R,n}$. Lemma 6.3 immediately implies

COROLLARY 6.4. *$v_j^{n+1}$ is strictly monotone in $J_{L,n} \leqq j \leqq J_{R,n}$.*

THEOREM 6.5. *Let $v^{n+1} = C_\Delta v^n$, $n \geqq 0$, where $C_\Delta$ is the artificial compressor* (6.8)–(6.9) *with $\hat{\lambda} \leqq 1$, and $v^0$ is the initial data* (6.12). *As $n \to \infty$, $v^n$ converges pointwise to the stationary shock-like solution* (5.9b)–(5.9c).

Proof: Let $J_{L,\infty}$ be the largest integer such that $\lim_{n\to\infty} v_j^n = u_L$ for all $j \leqq J_{L,\infty}$. Thus $\lim_{n\to\infty} \Delta_{j+1/2} v^n = 0$ for $j \leqq J_{L,\infty} - 1$, and consequently, by (6.9c), $\lim_{n\to\infty} G_{j+1/2}^n = 0$ for $j \leqq J_{L,\infty}$. Subtracting (6.9a) at $j = J_{L,\infty}$ from (6.9a) at $j = J_{L,\infty} + 1$ and then multiplying by $s = \operatorname{sgn}(u_R - u_L)$, we get

$$|\Delta_{J_{L,\infty}+1/2} v^{n+1}| = |\Delta_{J_{L,\infty}+1/2} v^n| - \hat{\lambda}\, |G_{J_{L,\infty}+3/2}^n| + \hat{\lambda}\varepsilon_n\,, \tag{6.13a}$$

where

$$\varepsilon_n = 2G_{J_{L,\infty}+1/2}^n - G_{J_{L,\infty}-1/2}^n \to 0 \quad \text{as} \quad n \to \infty\,. \tag{6.13b}$$

We claim that

$$\lim_{n\to\infty} G_{J_{L,\infty}+3/2}^n = 0\,. \tag{6.14a}$$

We prove (6.13c) by negation, as in Theorem 5.2: If there is a subsequence of $G_{J_{L,\infty}+3/2}^n$ which is bounded away from zero, we can make the left-hand side of (6.13a) negative, which is a contradiction. It follows now from (6.13c) and (6.13a) that $\lim_{n\to\infty} \Delta_{J_{L,\infty}+1/2} v^n \neq 0$ exists. From Corollary 6.4 it follows that $|\Delta_{J_{L,\infty}+1/2} v^n|$ is bounded away from zero, i.e., there exists a $\eta_L > 0$ such that

$$|\Delta_{J_{L,\infty}+1/2} v^n| \geqq \eta_L \quad \text{for all} \quad n \geqq 0\,. \tag{6.14b}$$

Let $J_{R,\infty}$ be the smallest integer such that $\lim_{n\to\infty} v_j^n = u_R$ for all $j \geqq J_{R,\infty}$. Using the same argument we can prove

$$\lim_{n\to\infty} G_{J_{R,\infty}-3/2}^n = 0\,. \tag{6.15a}$$

There exists a $\eta_R > 0$ such that

$$|\Delta_{J_{R,\infty}-1/2} v^n| \geqq \eta_R \quad \text{for all} \quad n \geqq 0\,. \tag{6.15b}$$

We complete the proof of Theorem 6.5 by showing that $J_{R,\infty} - J_{L,\infty} \leqq 2$. Suppose $J_{R,\infty} - J_{L,\infty} \geqq 3$, and denote

$$\eta_I = \min_{J_{L,\infty}+1 \leqq j \leqq J_{R,\infty}-2} (|\Delta_{j+1/2} v^0|)\,.$$

Since $J_{L,\infty} \geqq J_{L,0}$ and $J_{R,\infty} \leqq J_{R,0}$, $\eta_I > 0$. Denote $\eta = \min(\eta_L, \eta_I, \eta_R) > 0$. From Lemma 6.3 we get $|\Delta_{j+1/2} v^1| \geqq \eta$ for $J_{L,\infty}+1 \leqq j \leqq J_{R,\infty}-2$; from (6.14b) and (6.15b) it follows that this is also true for $j = J_{L,\infty}$ and $j = J_{R,\infty}-1$. By induction,

$$(6.16) \qquad |\Delta_{j+1/2} v^n| \geqq \eta \quad \text{for} \quad J_{L,\infty} \leqq j \leqq J_{R,\infty}-1 \quad \text{and} \quad \text{all} \quad n \geqq 0\,.$$

From (6.16) and the definition (6.9c) we see that, for all $n \geqq 0$, $|G^n_{J_{L,\infty}+3/2}| \geqq \eta$ and $|G^n_{J_{R,\infty}-3/2}| \geqq \eta$. This is a contradiction to (6.14a) and (6.15a).

*Remarks.* 1. For $\hat{\lambda} = 1$, the convergence to the stationary shock-like solution (5.9b) occurs in a finite number of time-steps (see [5]).

2. The assumption of a strictly monotone transition in the initial data (6.12) is necessary. If $v^0$ is not strictly monotone, then wherever $v_j^0 = v_{j+1}^0 = u_I$, the common value $u_I$ is treated by $C_\Delta$ as if it were an intermediate constant state. Provided that the points in which $\Delta_{j+1/2} v^0 = 0$ are distinct, the formation of these intermediate constant states can be prevented by replacing $\Delta_{j+1/2} v$ in all the $C_\Delta$ formulae by

$$(6.17) \quad \bar{\Delta}_{j+1/2} v = \begin{cases} s_{j+1/2} \min(|\Delta_{j-1/2} v|, |\Delta_{j+3/2} v|) & \text{if} \quad |\Delta_{j+1/2}| \\ \qquad = \min(\Delta_{j-1/2} v \cdot s_{j+1/2}, |\Delta_{j+1/2} v|, \Delta_{j+3/2} v \cdot s_{j+1/2})\,, \\ \Delta_{j+1/2} v & \text{otherwise}\,. \end{cases}$$

$\bar{\Delta} v$ is a nonlinear operation which differs from $\Delta v$ only at a minimum of the absolute value of the latter; there the value is increased to the minimum of the two immediate neighbors. Modifying $C_\Delta$ by replacing $\Delta_{j+1/2} v$ by $\bar{\Delta}_{j+1/2} v$ has the effect of increasing $|g_j|$ and by this improving the compressibility of $C_\Delta$. In case the strict monotonicity of the transition in the initial data (6.12) is violated at distinct points, the modification forces $|g_j|$ to be positive there. It is interesting to note that as in Jennings' theorem, as well as in the viscous equation (2.5), the strict positivity of $g$ is needed to prove the existence of a profile.

It follows immediately from the geometric interpretation of the operation $\bar{\Delta} v$ that, for all $j$,

$$(6.18a) \qquad |\bar{\Delta}_{j+1/2} v| \leqq |\Delta_{j+1/2} v|\,,$$

$$(6.18b) \qquad |\bar{\Delta}_{j+1/2} v - \bar{\Delta}_{j-1/2} v| \leqq |\Delta_{j+1/2} v - \Delta_{j-1/2} v|\,,$$

and therefore Theorem 6.1 remains valid after the proposed modification.

## 7. Properties of the Δ-Artificial Compression

In this section we discuss solutions of the corrective type ACM

$$L_\Delta = (C_\Delta)^k L, \qquad k \geqq 1, \tag{7.1}$$

where $C_\Delta$ is the Δ-artificial compressor of the previous section and $L$ is a monotone first-order scheme in conservation form. The application of $L$ to an admissible discontinuity has the effect of increasing spread. Successive application of $C$ to a smeared discontinuity transforms it back to a shock-like solution (Theorem 6.5). We find that if $L$ is an explicit scheme of $2J+1$ points, then $k = J$ is adequate (see [5]).

In the following we discuss solutions of (7.1) where $L$ is the Lax-Friedrichs scheme (3.5), the most diffusive among standard first-order schemes. The Lax-Friedrichs scheme has a support of 3 points $j-1$, $j$, $j+1$ but it does not use the central point $j$ explicitly; consequently, it is not a *strictly* monotone scheme. For example, its operation on the mesh step-function $u_L, \cdots, u_L, u_R, \cdots, u_R$ results in $u_L, \cdots, u_L, u_*, u_*, u_R, \cdots, u_R$, where $u_* = \frac{1}{2}(u_L + u_R) - \frac{1}{2}\lambda[f(u_R) - f(u_L)]$, which is not strictly monotone. Hence, we find it necessary to include in $C_\Delta$ the smoothing operation (6.17). Numerical experiments presented in the following indicate that $L_\Delta$, defined by (7.1) with $k = 1$, produces a steady progressing profile with almost maximal resolution.

We say that a finite difference scheme possesses a steady progressing profile traveling at speed $S$ if for any rational number $\delta = \lambda S$, where $\lambda = \Delta t/\Delta x$ is in the range of the stability condition, there exists an integer $P_\delta$ such that $\delta P_\delta$ is an integer and, for all $j$,

$$v_j^{n+P_\delta} = v_{j-\delta P_\delta}. \tag{7.2}$$

We refer to the smallest $P_\delta$ which satisfies (7.2) as the period.

We consider first $f(u) = u$, the linear case. We take discontinuous initial data

$$u(x, 0) = \begin{cases} 1 & \text{for} \quad x < 0, \\ 0 & \text{for} \quad x \geqq 0. \end{cases} \tag{7.3}$$

Let $w^n(\lambda, \hat{\lambda})$ be the solution obtained by $L_\Delta$ defined by (7.1) with $k = 1$, $0 < \lambda = \Delta t/\Delta x \leqq 1$, and $0 \leqq \hat{\lambda} \leqq 1$. Our numerical tests with rational numbers $\lambda$ show that $w^n(\lambda, 1)$ becomes, after a relatively small number of time steps, a steady progressive solution. Table I lists the values of $w^n$ for four consecutive values of $n$ calculated with $\lambda = \frac{1}{2}$. The results clearly show that the period $P_{1/2}$ is 4. In the last column we list for comparison $w^{100}$ computed without any artificial compression ($\hat{\lambda} = 0$).

Table I

**A Steady Progressing Profile for $\lambda = \frac{1}{2}$, $\hat{\lambda} = 1$**[1]

| $\lambda = \frac{1}{2}, \hat{\lambda} = 1$ <br> $n = P_{1/2}$ | $\lambda = \frac{1}{2}, \hat{\lambda} = 1$ <br> $n = P_{1/2} + 1$ | $\lambda = \frac{1}{2}, \hat{\lambda} = 1$ <br> $n = P_{1/2} + 2$ | $\lambda = \frac{1}{2}, \hat{\lambda} = 1$ <br> $n = P_{1/2} + 3$ | $\lambda = \frac{1}{2}, \hat{\lambda} = 1$ <br> $n = 2P_{1/2}$ | $\lambda = \frac{1}{2}, \hat{\lambda} = 0$ <br> $n = 100$ |
|---|---|---|---|---|---|
| 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 0.896 |
| · | · | · | · | · | |
| · | · | · | · | · | 0.850 |
| · | · | · | · | · | |
| | | | | | 0.850 |
| 1.000 | | | | | 0.792 |
| 0.999 | 1.000 | 1.000 | | | 0.792 |
| 0.983 | 0.992 | 0.997 | 1.000 | 1.000 | 0.722 |
| 0.972 | 0.988 | 0.995 | 0.998 | 0.999 | 0.722 |
| → 0.759 | 0.861 | 0.922 | 0.959 | 0.983 | 0.642 |
| 0.288 → | 0.660 → | 0.852 | 0.938 | 0.972 | 0.642 |
| 0.0 | 0.0 | 0.235 → | 0.607 → | 0.759 | 0.553 |
| | | 0.0 | 0.0 | 0.288 → | 0.553 |
| | | | | 0.0 | 0.462 |
| | | | | | 0.462 |
| · | · | · | · | · | |
| · | · | · | · | · | 0.462 |
| · | · | · | · | · | |
| | | | | | 0.371 |
| | | | | | 0.371 |
| | | | | | 0.286 |
| | | | | | 0.286 |
| | | | | | 0.211 |
| 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.211 |

Observe that when ACM is used, two cells around the exact location of the discontinuity contain on the average 85% of the jump; the minimum is 76% and the maximum is 94%. This is in sharp contrast to the Lax-Friedrichs solution without ACM, where after only 100 time steps 68% of the jump occupies 20 cells. Additional computations not listed here show that after 400 time steps 68% of the jump occupies 40 cells; this verifies the $\sqrt{n}$ behavior of the smearing.

In Table II we give the steady progressing solutions of $w^n(\lambda, 1)$ for $\lambda = 0$, $\frac{1}{4}$, $\frac{1}{2}$, 1, the limiting case $\lambda = 0$ corresponds to pure diffusion without convection. As $\lambda$ increases, the dissipation term (3.5c) decreases and the resolution is slightly improved.

Next we consider the shock problem $f = \frac{1}{2}u^2$, $u(x, 0)$ given by (7.3). In this case the Lax-Friedrichs scheme itself possesses steady progressing solutions. In Table III we give the steady progressing solutions $w(\lambda, \hat{\lambda})$ with $k = 1$ for $\hat{\lambda} = 1$ and $\lambda = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$; for comparison's sake $w(\frac{1}{2}, \frac{1}{2})$ and $w(\frac{1}{2}, 0)$ are also

[1] Arrow indicates the exact location of the discontinuities.

Table II
**Profiles of the ACM for a Contact Discontinuity**[1]

| $\lambda=0,\ \hat{\lambda}=1$ $P_0=1$ | $\lambda=\frac{1}{4},\ \hat{\lambda}=1$ $P_{1/4}=8$ | $\lambda=\frac{1}{2},\ \hat{\lambda}=1$ $P_{1/2}=4$ | $\lambda=\frac{3}{4},\ \hat{\lambda}=1$ $P_{3/4}=8$ | $\lambda=1,\ \hat{\lambda}=1$ $P_1=1$ |
|---|---|---|---|---|
| 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| | 1.000 | 0.999 | 1.000 | |
| | 0.989 | 0.983 | 0.992 | |
| | 0.982 | 0.972 | 0.975 | |
| 1.000 | 0.737 | 0.759 | 0.816 | 1.000 |
| → | → | → | → | → |
| 0.0 | 0.293 | 0.288 | 0.217 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |

Table III
**Profiles of the ACM for a Shock**[1]

| $\lambda=0,\ \hat{\lambda}=1$ $P_0=1$ | $\lambda=\frac{1}{4},\ \hat{\lambda}=1$ $P_{1/4}=8$ | $\lambda=\frac{1}{2},\ \hat{\lambda}=1$ $P_{1/2}=4$ | $\lambda=\frac{3}{4},\ \hat{\lambda}=1$ $P_{3/4}=1$ | $\lambda=1,\ \hat{\lambda}=1$ $P_1=1$ | $\lambda=\frac{1}{2},\ \hat{\lambda}=\frac{1}{2}$ $P_{1/2}=4$ | $\lambda=\frac{1}{2},\ \hat{\lambda}=0$ $P_{1/2}=4$ |
|---|---|---|---|---|---|---|
| 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 0.996 | 0.896 |
| · | | | | | 0.989 | 0.896 |
| · | | | | | | |
| · | 1.000 | 1.000 | 1.000 | | 0.969 | 0.740 |
| | 0.999 | 0.997 | 0.999 | | 0.916 | 0.740 |
| 1.000 | 0.934 | 0.959 | 0.973 | 1.000 | 0.785 | 0.494 |
| → | → | → | → | → | → | → |
| 0.0 | 0.567 | 0.543 | 0.528 | 0.0 | 0.525 | 0.494 |
| | 0.0 | 0.0 | 0.0 | | 0.214 | 0.254 |
| · | | | | | 0.073 | 0.254 |
| · | | | | | | |
| · | | | | | 0.024 | 0.107 |
| | | | | | 0.008 | 0.107 |
| 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.002 | 0.040 |

given. Observe that the resolution of the shock in $w(\lambda, 1)$ is better than that of the contact discontinuity (Table II). We can see from Table III that the resolution of the shock depends strongly on the amount of artificial compression applied: 96% of the jump in $w(\frac{1}{2}, \frac{1}{2})$ occupy 7 cells compared to 2 cells in $w(\frac{1}{2}, 1)$.

The numerical results presented in Tables I–III show that the ACM is capable of maximal resolution of shocks and contact discontinuities. The ACM is essentially a modification of the characteristic field of the given initial value problem by the addition of the convergent characteristic field $dx/dt = \gamma$

((5.5)) of a stationary shock. This process results in an improvement of the numerical solution only in regions containing an admissible discontinuity. When the ACM is applied to a smooth solution (bounded $u_{xx}$), the characteristic speeds are modified by a term which is $O(\Delta x)$. Thus a compression wave will develop a shock somewhat earlier than it should, and an expansion wave will expand with an $O(\Delta x)$ delay. In the linear case with a constant characteristic speed, the ACM modification results in having a slightly convergent characteristic field in each monotone strip of the solution. Hence, as $n \to \infty$, the ACM solution approaches a square-wave structure. (We remark that without artificial compression, the Lax-Friedrichs solution tends pointwise to a constant. Both schemes give unsatisfactory results for long term computation in this case.)

The ACM in its present form should not be applied to inadmissible discontinuities, e.g. a rarefraction wave developing from initial discontinuity, because the addition of the $\Delta$-ACF might make it an admissible discontinuity for the modified equation (4.2). In practice such regions are easily identified. There is also the possibility of designing an automatic way of eliminating the application of the ACM to rarefaction waves by using $\alpha_{j+1/2}\,\Delta_{j+1/2}v$ instead of $\Delta_{j+1/2}v$ in the $\Delta$-ACM formulae (6.8) and (6.9); here $\alpha_{j+1/2}$ is a switch: $0 \leqq \alpha_{j+1/2} \leqq 1$, $\alpha_{j+1/2} \approx 1$ in shocks and contact discontinuities, but $\alpha_{j+1/2} \approx 0$ in a rarefaction wave. In regions of smoothness, $\alpha_{j+1} - \alpha_j = O(\Delta x)$. In the case of one-dimensional Eulerian gas equations, one can use the fact that entropy jumps across a shock and a contact discontinuity, but is constant across a rarefaction wave to construct such a switch; this will be described in detail in a subsequent article.

Finally we would like to point out a certain similarity between the ACM and the anti-diffusion method of Boris and Book [1]. The latter is composed of two steps: convection and anti-diffusion. The convection $L_1$ is performed by a monotone scheme, e.g.

$$w_j = (L_1 \cdot v)_j = (L_2 \cdot v)_j + \tfrac{1}{8}(v_{j+1} - 2v_j + v_{j-1}), \tag{7.4a}$$

where $L_2$ is the second-order accurate Lax-Wendroff scheme. Equation (7.4a) is monotone under the condition $|\lambda\, df/du| \leqq \sqrt{3}/2$. In the anti-diffusion step A that follows, the diffusion term $\frac{1}{8}(v_{j+1} - 2v_j + v_{j-1})$ is removed wherever it is possible to do so without violating the monotonicity of $w_j$. This is done by

$$(Aw)_j = w_j - (f^c_{j+1/2} - f^c_{j-1/2}), \tag{7.4b}$$

where $f^c_{j+1/2}$ is not $\frac{1}{8}(w_{j+1} - w_j)$ but rather

$$f^c_{j+1/2} = s_{j+1/2} \max\left[0, \min\left(\Delta_{j-1/2}w \cdot s_{j+1/2}, \tfrac{1}{8}\,|\Delta_{j+1/2}w|, \Delta_{j+3/2}w \cdot s_{j+1/2}\right)\right]. \tag{7.4c}$$

Here $\Delta_{j+1/2} w = w_{j+1} - w_j$ and $s_{j+1/2} = \operatorname{sgn}(\Delta_{j+1/2} w)$. The anti-diffusion step (7.4b)–(7.4c) is a monotonicity preserving operator. We observe that the so called flux-limiter (7.4c) has formal resemblance to the ACM in conservation form (6.9c); it differs only by having $\frac{1}{8}|\Delta_{j+1/2} w|$ rather than $|\Delta_{j+1/2} w|$ as the second argument in the min function. In general, the artificial compression (6.9) is different from the anti-diffusion (7.4b)–(7.4c). However, when applied to a sharp transition between two constant states, the anti-diffusion is about the same as the artificial compression (6.9) with $\lambda = \frac{1}{8}$. This suggests that the flux limitation, which is due to monotonicity requirement, introduced some element of artificial compression into the scheme. We remark that the anti-diffusion method improves the resolution of shocks and contact discontinuities but it does not allow a complete control over their spread. Numerical solutions of the Eulerian Equations of gas presented in [1] show that the anti-diffusion does not prevent the smearing of a contact discontinuity.

*Acknowledgment.* I thank Peter Lax for many sessions of constructive criticism. Many helpful discussions with Sam Burstein and Gary Sod are gratefully acknowledged.

The computations were made on the CDC-6600 at the ERDA Mathematics and Computing Laboratory, Courant Institute, under Contract No. EY-76-C-02-3077*000.

## Bibliography

[1] Boris, J. P., and Book, D. L., *Flux corrected transport* I. SHASTA, *A fluid transport algorithm that works*, Jour. Comp. Phys., Vol. 11, 1973, pp. 38–69.

[2] Burstein, S. Z., *Finite difference calculations for hydrodynamic flows containing discontinuities*, Jour. Comp. Phys., Vol. 1, 1966, pp. 198–222.

[3] Chorin, A. J., *Random choice solution of hyperbolic systems*, Jour. Comp. Phys., Vol. 22, 1976, pp. 517–533.

[4] Godunov, S. K., *Finite difference methods for numerical computations of discontinuous solutions of equations of fluid dynamics*, Mat. Sb., Vol. 47, 1959, pp. 271–295. (In Russian.)

[5] Harten, A., *The Method of Artificial Compression: I. Shocks and Contact Discontinuities*, AEC Research and Development Report C00-3077-50, Courant Institute, New York University, June 1974.

[6] Harten, A., Hyman, J. M., and Lax, P. D., *On finite-difference approximations and entropy conditions for shocks*, (with Appendix by B. Keyfitz), Comm. Pure Appl. Math., Vol. 29, 1976, pp. 297–322.

[7] Harten, A., and Zwas, G., *Switched numerical Shuman filters for shock calculations*, Jour. Eng. Math. Vol. 6, 1972, pp. 207–216.

[8] Harten, A., and Zwas, G., *Self-adjusting hybrid schemes for shock computations*, Jour. Comp. Phys., Vol. 6, 1972, pp. 568–583.

[9] Jennings, G., *Discrete shocks*, Comm. Pure Appl. Math. Vol. 27, 1974, pp. 25–37.

[10] Keyfitz, B., *Solutions with shocks, and example of a $L_1$-contractive semigroup*, Comm. Pure Appl. Math., Vol. 24, 1971, pp. 125–132.

[11] Lax, P. D., *On the stability of difference approximations to solutions of hyperbolic equations with variable coefficients*, Comm. Pure Appl. Math., Vol. 14, 1961, pp. 497–520.

[12] Lax, P. D., and Wendroff, B., *Systems of conservation laws*, Comm. Pure Appl. Math., Vo 13, 1960, pp. 217–237.

[13] Oleinik, O. A., *Discontinuous solutions of nonlinear differential equations*, Uspekhi Ma Nauk, Vol. 12, 1957, pp. 3–73 (Amer. Math. Soc. Transl. Ser., 2, 26, pp. 95–172).

[14] Van Leer, B., *Towards the ultimate conservative difference scheme* II. *Monotonicity an conservation combined in a second order scheme*, Jour. Comp. Phys., Vol. 14, 1974, p| 361–370.

[15] Van Leer, B., *Towards the ultimate conservative difference scheme* III. *Upsteam-centere finite-difference schemes for ideal compressible flows*, to appear in Jour. Comp. Phys.

[16] Von Neumann, J., and Richtmyer, R. D., *A method for the numerical calculation hydrodynamic shocks*, Jour. Appl. Phys., Vol. 21, 1950, pp. 232–237.

[17] Warming, R. F., and Beam, R. M., *Upwind second-order difference schemes and applicatio in unsteady aerodynamic flows*, Proceedings of the AIAA 2nd Computational Flui Dynamics Conference, June 1975.

Received February, 1977.