Convergence of Approximate Solutions to Conservation Laws

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1. Introduction

We are concerned with the problem of convergence of approximate solutions to hyperbolic systems of conservation laws. The general setting is a system of n conservation laws in one space dimension

$$u_t + f(u)_x = 0, \tag{1.1}$$

where $u = u(x, t) \in R^n$ and f is a smooth nonlinear mapping from R^n to R^n. We shall assume that f is strictly hyperbolic in the sense that its Jacobian $\nabla f(u)$ has n real and distinct eigenvalues,

$$\lambda_1(u) < \lambda_2(u) < \ldots < \lambda_n(u),$$

at each state $u \in R^n$.

We are interested here in families of approximate solutions generated by parabolic systems of the form

$$u_t + f(u)_x = \varepsilon\, D(u)_{xx}, \qquad u = u_\varepsilon(x, t)$$

and by finite difference schemes

$$D_t u + D_x f(u) = 0, \qquad u = u_{\Delta x}(x, t)$$

which are conservative in the sense of Lax & Wendroff [11]. One strategy for a convergence proof is to try to establish uniform estimates on both the amplitude and derivatives of the approximate solutions in relevant metrics and then appeal to an appropriate compactness argument to extract a subsequence that converges in the strong topology, regarding convergence of the entire family as a question of uniqueness of the limit.

In the context of conservation laws we recall that the L^∞ norm and the total variation norm provide a natural pair of metrics in which to investigate stability (in the sense of uniform boundedness) of families of exact and approximate solutions. The L^∞ norm serves as an appropriate measure of the solution amplitude while the total variation norm serves as an appropriate measure of the solution gradient. The role of these norms for conservation laws is indicated by GLIMM'S theorem [6] concerning the stability and convergence of the approximate solutions generated by his random choice method. If the total variation norm of the initial data u_0 is sufficiently small, then the family of random choice approximations $\{u_{\Delta x}\}$ is stable in the sense that

$$|u_{\Delta x}(\cdot, t)|_\infty \leqq \text{const.}\ |u_0|_\infty, \tag{1.2}$$

$$TVu_{\Delta x}(\cdot, t) \leqq \text{const.}\ TVu_0, \tag{1.3}$$

where the constants are independent of the mesh length Δx and depend only on the flux function f. Furthermore, there exists a subsequence which converges pointwise a.e. to a globally defined distributional solution u.

The proof of this theorem is based on a general study of wave interactions in the exact solution and in the associated random choice approximations. It remains an open problem to establish the corresponding estimates for parabolic systems and finite difference schemes. The difficulty stems in part from the fact that the exact and approximate solution operators do not obey a maximum principle in the norms that are naturally associated with the system.

In this paper we shall establish compactness theorems for families of approximate solutions generated by parabolic systems and conservative finite difference schemes with the aid of the theory of compensated compactness. The main step in the program consists of the proof of a conjecture of TARTAR [19] which we shall discuss below. The analysis here employs the weak topology and averaged quantities rather than the strong topology and the fine scale features. In the setting of systems of two conservation laws we prove that, for a general class of approximation methods which respect the Lax entropy condition, L^∞ stability alone guarantees compactness: if $\{u_n\}$ is a family of approximate solutions which satisfy the uniform bound

$$|u_n|_\infty \leqq \text{const.},$$

then there exists a subsequence which converges pointwise a.e. to an exact solution of (1.1). Thus the set of associated approximate solution operators forms a family of mappings which is uniformly compact from L^∞ to L^1_{loc}.

In connection with earlier work we recall that TARTAR [19] has established new compactness theorems for approximations to scalar conservation laws $(n = 1)$ as a corollary of his work on the theory of compensated compactness. In particular, the paper [19] contains a theorem which characterizes composite weak limits in terms of expected values of probability measures: Suppose $v_\varepsilon(y)$: $R^m \to R^n$ is a family of functions uniformly bounded in L^∞. There exists a subsequence v_{ε_k} and an associated family

$$\{\nu_y : y \in R^m\}$$

of probability measures $\nu_y = \nu_y(\lambda)$ on the range space R^n such that for all continuous maps $g: R^n \to R$ the limit of the composition $g(v_{\varepsilon_k})$ exists in the weak-star topology of L^∞ and equals a.e. in y the expected values of g with respect to

ν_y, *i.e.*

$$\lim g(v_k^\varepsilon(y)) = \int g(\lambda)\, d\nu_y(\lambda),$$

where λ denotes the generic point of R^n. It follows that strong convergence corresponds to the statement that ν_y equals a point mass for a.e. y. The deviation between weak and strong convergence is measured by the spreading of the support of ν_y.

In setting of hyperbolic conservation laws TARTAR [19] has conjectured ν_y will reduce to a point mass under appropriate assumptions of nonlinearity or concentrate itself on a set whose geometry allows the continuity with respect to weak limits of the particular nonlinear functions appearing in the system. In [19] it is shown that for a scalar conservation law $\nu_{(x,t)}$ is supported on an interval where f is affine. For systems of two conservation laws we prove that ν reduces to a point mass if the system is genuinely nonlinear in the sense of LAX [9]. Systems of this type include the isentropic equations of gas dynamics and the equations of shallow water waves. We also show that ν reduces to a point mass for the equations of elasticity which provide the main example of a non-genuinely nonlinear system. Statements of the compactness theorems for exact solutions and parabolic regularizations are given in Section 2. In Section 7 finite difference schemes are treated.

Our analysis is based on a study of the progressing entropy waves of state space introduced by LAX [10], and in particular, on relationships between their structure and the structure of the support of $\nu_{(x,t)}$. We recall that the structure of the individual measures $\nu_{(x,t)}$ is restricted by a commutativity relation which follows as a corollary of the entropy condition and a continuity theorem of TARTAR [19] and MURAT [14] for bilinear maps in the weak topology. Specifically, almost all of the measures $\nu_{(x,t)}$ commute with a special antisymmetric bilinear form B acting on entropy-entropy flux pairs (η, q), *i.e.* solutions of the linear hyperbolic system of partial differential equations which serves as the compatibility relation between entropy and its flux,

$$\nabla\eta\, \nabla f = \nabla q.$$

TARTAR [19] conjectured that any probability measure ν which commutes with B,

$$\nu \circ B = B \circ \nu,$$

must reduce to a point mass if the system under consideration has appropriate nonlinear structure; for example, if the system is genuinely nonlinear in the sense of LAX or if the eigenvalues are not badly degenerate. Our proof that $\nu_{(x,t)}$ reduces to a point mass is based on explicit connections which can be established between the coefficient structure of the progressing entropy waves and the action of the bilinear form B; *cf.* Sections 4 and 5.

With regard to the application of the reduction theorem for ν to the compactness of approximate solutions to parabolic systems and finite difference schemes, it is necessary from the point of view of the current theory of compensated compactness to show that the approximate solutions u_n satisfy the entropy inequality within an error E_n which lies in a compact subset of the negative Sobolev space H^{-1}. In the case of parabolic approximations to a scalar conservation law, Tartar [19] verified the H^{-1} condition with the aid of a lemma of Murat [19, 16] concerned with the positive cone of H^{-1}. For parabolic systems with positive definite diffusion, Murat's lemma can again be applied in a straightforward fashion; *cf.* Section 2. For general parabolic systems and finite difference schemes additional estimates are required to establish the H^{-1} condition; *cf.* Sections 6 and 7. In particular, we recall that for the parabolic systems arising in continuum mechanics such as the compressible Navier-Stokes equations diffusion is absent in the mass equation and the problem arises of showing that the diffusion present in the momentum equations induces, roughly speaking, an equal amount of diffusion of mass to guarantee that both components of the error of the approximate solution lie in a compact subset of H^{-1}. A result on the convergence of solutions of the compressible Navier-Stokes equations to compressible Euler equations is stated in Section 2 and proof of the H^{-1} condition is given in Section 3. The H^{-1} condition for a class of first order accurate conservative finite difference schemes including the Lax-Friedrichs scheme and Godunov's scheme is established in Section 7.

The problem of establishing a uniform L^∞ estimate for approximate solutions remains open under general circumstances.

In the special case of the equations of elasticity with a stress-strain relation admitting one inflection point, there exists a family of arbitrarily large bounded invariant regions for the exact solution operator. This family is preserved by approximation methods which employ equal rates of diffusion in each of the primitive variables, *e.g.* the Lax-Friedrichs scheme, Godunov's scheme and the standard method of artificial viscosity. For methods of this type the presence of a family of invariant regions leads to a uniform L^∞ estimate and a convergence theorem for initial data in L^∞. As a corollary, one obtains a large data existence theorem for the equations of elasticity.

It remains an open problem to establish global existence of solutions to general systems with arbitrarily large initial data. In this connection we refer the reader to Nishida's paper [17] which establishes a global existence theorem using the random choice method on the isothermal equations of gas dynamics ($n = 2$) with initial data having arbitrarily large but locally finite total variation. Existence results for the equations of gas dynamics with a polytropic gas and initial data having "moderately" large total variation are established in [4, 12, 17, 18]. For large data results for systems of mainly mathematical interest we refer the reader to [2, 3].

We recall that in the case of isentropic gas dynamics ($n = 2$) there also exists a family of bounded invariant regions which yields an *a priori* L^∞ estimate of the form

$$0 \leqq \varrho(x, t) \leqq M, \qquad |v(x, t)| \leqq M$$

for the density ϱ and velocity v. Unfortunately, the vacuum state $\varrho = 0$ represents a singularity from several points of view; for example, the eigenvalues λ_1 and λ_2 coalesce at $\varrho = 0$ and strict hyperbolicity is lost. In addition the state $\varrho = 0$ forms a singular line for the entropy equation in the sense that half of all entropy waves fail to lie in a compact subset of H^{-1}. In this paper our attention is restricted to exact and approximate solutions (ϱ_n, v_n) for which the density is bounded uniformly away from zero. A forthcoming paper will deal with the special singularity at the vacuum state.

In connection with the general role of the weak topology in the setting of conservation laws we recall that LAX [8] showed that the solution operator S_ε of the scalar diffusion equation

$$u_t + f(u)_x = \varepsilon u_{xx}$$

is continuous with respect to the initial data in the weak topology uniformly with respect to ε, and as a consequence, completely continuous in the strong L^1 topology in the genuinely nonlinear case $f'' \neq 0$. This result was established by appealing to a maximum principle for the associated potential function which satisfies the Hamilton-Jacobi equation. Thus, in this situation a form of compactness is established without explicit estimates on the derivatives. It would be interesting to study the action of the solution operator to hyperbolic systems with respect to the weak topology.

In the setting of the scalar conservation it remains an open problem to establish convergence of conservative finite difference schemes which are accurate to second order. Stability and convergence results have been obtained so far only for methods which are precisely accurate to first order. It would be interesting to consider the work of MAJDA & OSHER [1] on the L^2-stability of second order methods applied to scalar equations, together with the family measures $\{\nu_{(x,t)}\}$, with an eye to a convergence theorem through compensated compactness.

In connection with the weak topology and the calculus of variations we refer the reader to the work of BALL [1, 24], BALL, CURRIE & OLVER [25], DACOROGNA [27, 28, 29, 30] and to references cited therein. With regard to the general theory of compensated compactness we refer the reader to TARTAR [19, 26] and MURAT [14, 15, 16] and to recent lecture notes of DACOROGNA [27].

2. Exact Solutions and Parabolic Approximations

In this section we shall consider the compactness of exact solutions with small oscillation. We shall begin in the context which exhibits maximum regularization: strictly hyperbolic genuinely nonlinear systems. We recall that an eigenvalue λ_j is genuinely nonlinear in LAX's sense [9] if its derivative in the corresponding eigendirection never vanishes, *i.e.*

$$r_j \nabla \lambda_j \neq 0,$$

where

$$\nabla f r_j = \lambda_j r_j.$$

In this situation, the wave speed associated with waves in the j^{th} characteristic field changes monotonically with wave amplitude. The system is called genuinely nonlinear if all of its eigenvalues λ_j are genuinely nonlinear. Examples of genuinely nonlinear systems are provided by the equations of isentropic gas dynamics,

$$\begin{aligned} \varrho_t + (\varrho u)_x &= 0, \\ (\varrho u)_t + (\varrho u^2 + p(\varrho))_x &= 0, \end{aligned} \tag{2.1}$$

for a polytropic gas $p = \text{const.}\,\varrho^\gamma$, $\gamma > 1$ and the equations of elasticity in Lagrange coordinates

$$(2.2)\qquad \begin{aligned} u_t - \sigma(v)_x &= 0, \\ v_t - u_x &= 0, \end{aligned}$$

provided that the stress σ is either a convex or concave function of the strain v. We remark that the system (2.1) remains genuinely nonlinear for all choices of $\gamma > 0$. The special case $\gamma = 2$ produces the equations for shallow water waves; the range $1 < \gamma \leqq 5/3$ corresponds to an isentropic gas.

First we shall recall the definition of generalized entropy as formulated by Lax [10] and discuss the role it plays in the regularization of solutions. Consider a system (1) of n conservation laws. A pair of mappings

$$\eta : R^n \to R, \qquad q : R^n \to R$$

is called an entropy-entropy flux pair if all smooth solutions satisfy an additional conservation law of the form

$$\eta(u)_t + q(u)_x = 0.$$

For the purposes of this paper we shall restrict our attention to the class of systems endowed with an entropy pair in which η is strictly convex. As observed by Lax & Friedrichs [23] this class includes the basic systems of continuum mechanics. Furthermore Lax showed that all strictly hyperbolic systems of two equations possess locally defined strictly convex entropies and that a broad class of systems of two equations possesses globally defined strictly convex entropies [10]. The structure of the entropy η is discussed in greater detail in Section 4. Within this class it is standard to impose the Lax entropy inequality

$$(2.3)\qquad \eta(u)_t + q(u)_x \leqq 0$$

on weak solutions $u = n(x, t)$. Here we shall restrict our attention to solutions in $L^\infty \cap \mathrm{BV}_{\mathrm{loc}}$. Here BV denotes the space of functions of several variables which have bounded variation in the sense of Cesari [5, 20], *i.e.* locally L^1 functions whose distributional gradient is represented by locally bounded measures. Experience with conservation laws has indicated that $L^\infty \cap \mathrm{BV}_{\mathrm{loc}}$ is a natural function space for the solution operator. In this regard we recall that solutions produced by the random choice method lie in $L^\infty \cap \mathrm{BV}_{\mathrm{loc}}$ by virtue of the estimates (1.2) and (1.3). Within this category, one can prove that the measure

$$(2.4)\qquad \theta_u \equiv \eta(u)_t + q(u)_x$$

is concentrated on the shock set $\Gamma(u)$ of u, *i.e.* the set of points of approximate jump discontinuity [5, 20], and consequently that the entropy inequality (2.3) holds if and only if all shock waves dissipate generalized entropy *i.e.*

$$\theta_u(E) \leqq 0$$

for all Borel subsets $E \subset \Gamma(u)$. We remark that, for the purpose of formulating an admissibility criterion on the shock waves of a solution, it is standard in the general theory of conservation laws to employ a (strictly) convex entropy; in mechanics it is traditional to work with a concave entropy. We shall see in subsequent sections that non-convex entropies play a useful role in studying the compactness of solutions. Second, we remark that the admissibility criterion (2.3) can be posed with respect to an arbitrary strictly convex entropy. It is easy to show that if a given solution u in $L^\infty \cap \mathrm{BV}_{\mathrm{loc}}$ satisfies (2.3) for one strictly convex entropy η then it satisfies (2.3) for all convex entropies η.

Theorem. *Consider a strictly hyperbolic, genuinely nonlinear system of two equations. There exists a (small) constant M with the following property. If u^ε is a sequence of admissible solutions in* $\mathrm{BV}_{\mathrm{loc}}$ *such that*

$$|u^\varepsilon|_\infty \leqq M,$$

then there exists a subsequence u^{ε_k} converging pointwise a.e. *to an admissible solution u.*

Under an additional technical assumption on the system we shall in the next section establish the corresponding result in the case that M is an arbitrary finite number. Hence, the exact solution operator restricted to admissible solutions is a compact mapping from L^∞ to L^1_{loc}.

With an eye to approximation methods it is natural to consider the extent to which the entropy condition can be relaxed while still maintaining compactness. In this connection we shall first recall a well-known example of a solution sequence to a system of conservation laws which contains no strongly convergent subsequence in any L^p space. Consider two states a and b in R^n which satisfy the Rankine-Hugoniot conditions

$$\sigma(a-b) = f(a) - f(b).$$

Fix $0 < \gamma < 1$ and define a solution $u(x, t)$ in the strip

$$\{(x, t): \sigma t - \gamma < x < \sigma t + (1-\gamma)\}$$

by setting

$$u \equiv a \text{ in } \{(x, t): \sigma t < \gamma - x < \sigma t\},$$

$$u \equiv b \text{ in } \{(x, t): \sigma t < x < \sigma t + (1-\gamma)\}$$

and continue u to a globally defined solution through a periodic extension

$$u(x+1, t) = u(x, t).$$

Thus u consists of a family of parallel shock waves propagating with speed σ and spends a distance γ in the state a and a distance $1-\gamma$ in the state b. Since the equations are invariant under uniform dilations

$$(x, t) \to (\varepsilon x, \varepsilon t),$$

the functions $u^\varepsilon = u(x/\varepsilon, t/\varepsilon)$ form a family of periodic solutions which converges weakly to the integral average

$$\gamma a + (1-\gamma) b$$

but contains no strongly convergent subsequence. In the case of a genuinely nonlinear system, either the shocks connecting a to b from left to right are admissible or the shocks connecting b to a from left to right are admissible, provided that $|a-b|$ is sufficiently small. Thus one can not allow an arbitrary mixing between the two types of waves and still guarantee compactness. On the other hand it turns out that a substantial mixing is permissible. One can allow, for example, an infinite number of non-admissible shock waves into the solution sequence, indeed, an infinite number whose total variation becomes unbounded and still guarantee the existence of a pointwise convergent subsequence. The only condition which is needed is that the total amount of entropy dissipated by the admissible waves plus the total amount of entropy produced by the non-admissible waves remains uniformly bounded with respect to ε, *i.e.*

$$\text{total mass } \theta_{u^\varepsilon} \leqq M \tag{2.5}$$

for some constant M. We note that condition (2.5) does not imply a uniform bound on the total variation. For example, the weight which θ_u assigns to a strip

$$S_T = \{(x, t): 0 < t < T\}$$

is given by the formula

$$\theta_u(S_T) = \int_0^T \Sigma\{\sigma[\eta] - [q]\}\, dt,$$

where the summation is taken over all shock waves in the solution u at time. Here the square bracket denotes the jump in the enclosed quantity from left to right across the shock and σ the corresponding speed of propagation. The formula is a simple consequence of Green's theorem for BV functions; *cf.* [5, 20]. As shown by Lax [10] the local rate of entropy dissipation (or production), *i.e.*,

$$\sigma[\eta] - [q]$$

associated with a shock wave connecting u_l to u_r and propagating with speed σ is cubic in the strength $|u_l - u_r|$ of the shock:

$$\sigma[\eta] - [q] = \sigma\{\eta(u_l) - \eta(u_r)\} - q(u_l) + q(u_r) = O\,|u_l - u_r|^3.$$

In general the p^{th}-order variation will only dominate the total (first order) variation of those shocks with magnitude greater than a fixed constant.

Second, with regard to the reversibility of the problem, we recall that system (1.1) is invariant under reflections through an arbitrary point of the x–t plane. In the case of the origin, we note that if $u(x, t)$ is a solution then so is $u(-x, -t)$. The hypotheses of the following compactness theorem are completely reversible.

Theorem. *Consider a strictly hyperbolic genuinely nonlinear system of two conservation laws. There exists a small constant M with the following property. If η is a strictly convex entropy defined in a neighborhood of the origin and if u^ε is a sequence of solutions in* $\mathrm{BV}_{\mathrm{loc}}$ *such that*

$$|u^\varepsilon|_\infty \leqq M$$

and such that $\theta(u^\varepsilon)$ has a total mass uniformly bounded with respect to ε, then there exists a subsequence converging pointwise a.e. *to a solution u.*

Under an additional assumption on the system we shall establish the corresponding result with arbitrary finite M. The proofs of the theorem above will be given in the subsequent sections.

Next, we shall discuss the compactness of solutions to associated parabolic systems of the form

$$u_t + f(u)_x = \varepsilon\, D u_{xx}, \tag{2.6}$$

where for simplicity we take D to be a constant $n \times n$ matrix. Here, in order to ensure correct entropy production in the limit as ε approaches zero, it is necessary in general that the diffusion matrix D be non-negative with respect to the second derivative of a strictly convex entropy η, *i.e.*

$$\nabla^2 \eta\, D \geqq 0,$$

where $\nabla^2 \eta$ denotes the Hessian matrix of η. This can be easily seen by multiplying system (2.6) by the gradient of η. The identity

$$\nabla \eta \{u_t + \nabla f(u)\, u_x\} = \varepsilon\, \nabla \eta\, D u_{xx}$$

can be rewritten as

$$\eta(u)_t + q(u)_x = \varepsilon(\nabla \eta\, D u_x)_x - \varepsilon u_x^t\, \nabla^2 \eta\, D u_x. \tag{2.7}$$

Suppose that η is a non-negative entropy and that

$$\nabla^2 \eta\, D \geqq \delta > 0.$$

Then integration of (2.7) over a strip S_T yields

$$\int \eta(u(x, T))\, dx - \int \eta(u(x, 0))\, dx = -\varepsilon \int_0^T \int_{-\infty}^{\infty} u_x^t\, \nabla^2 \eta\, D u_x\, dx\, dt,$$

provided $u(x, t)$ vanishes sufficiently fast at infinity. Hence we obtain an estimate

$$\varepsilon \int_0^T \int_{-\infty}^{\infty} u_x^2\, dx\, dt \leqq \text{const.}$$

where the constant depends only on δ and the total amount of entropy in the initial data. Consequently, the term

$$\varepsilon(\nabla \eta\, D u_x)_x$$

approaches zero in the sense of distributions as ε approaches zero, if we restrict our attention to solution sequences which are uniformly bounded in L^∞. We conclude that for all non-negative test functions ϕ with compact support

$$\iint \phi_t \eta(u) + \phi_x q(u)\, dx\, dt \geqq 0,$$

if u is the limit pointwise a.e. of solutions u^ε to system (2.6) and if $\nabla^2 \eta\, D$ is positive definite. Thus, while the diffusion problem for (2.6) is properly posed under the condition that D be non-negative with respect to the Euclidean inner product

$$(D\xi, \xi) \geqq 0,$$

correct entropy production requires that D be non-negative with respect to the inner product induced by the Hessian of a strictly convex entropy η, *i.e.*

$$[D\xi, \xi] \geqq 0,$$

where

$$[\xi, \tau] = (\nabla^2\eta\xi, \tau). \tag{2.8}$$

We remark that (2.8) is the symmetrizing inner product for the hyperbolic system (1.1): it has been observed by LAX & FRIEDRICHS [23] that multiplication of the quasilinear form of (1.1) by the Hessian of η transforms the system into symmetric hyperbolic form:

$$\nabla^2\eta u_t + \nabla^2\eta\, \nabla f u_x = 0.$$

Here $\nabla^2\eta$ is positive definite while $\nabla^2\eta\,\nabla f$ is symmetric, *i.e.* ∇f is a symmetric matrix with respect to the inner product (2.8). It would be interesting to investigate further the admissibility criterion in connection with the various symmetry relations which arise in the equations. As one final remark we observe that in the case of weak shock waves it is necessary for the purpose of obtaining the shock as the limit of traveling waves

$$u^\varepsilon = \psi\left(\frac{x - \sigma t}{\varepsilon}\right)$$

that the diffusion matrix D be non-negative in the eigendirections r_j of ∇f, *i.e.*

$$[Dr_j, r_j] \geqq 0. \tag{2.9}$$

It would be interesting to determine whether or not (2.9) is sufficient for all limits (boundedly a.e.) of solutions of the parabolic system (2.6) to satisfy the entropy condition (2.3).

Theorem. *Consider a genuinely nonlinear strictly hyperbolic system of two equations and the associated parabolic system with D positive definite with respect to the inner product induced by some strictly convex entropy η. There exists a constant M with the following property. If u^ε is a sequence of smooth solutions of* (2.6) *vanishing sufficiently fast at infinity and satisfying*

$$|u^\varepsilon|_\infty \leqq M,$$

then there exists a subsequence which converges pointwise a.e. *to an admissible solution of the corresponding hyperbolic system* (1.1).

Thus the solution operators of the diffusion equation form a family of operators which is compact from L^∞ to L^1_{loc} uniformly with respect to the parameter ε. In the case of systems which admit a coordinate system of quasi-convex Riemann invariants we shall establish the same compactness result in the case where M is an arbitrary finite number.

In connection with the general character of the diffusion operator we note that theorems above generalize immediately to the situation where the second order operator is based on a smooth mapping $D = D(u)$ from R^n to R^n. The term

$$\varepsilon\, D(u)_{xx}$$

is handled in exactly the same way. We need only require that for all states u under consideration the matrix

$$\nabla^2 \eta(u)\, \nabla D(u)$$

be positive definite. We recall that system (2.1) is genuinely nonlinear and admits a coordinate system of quasi-convex Riemann invariants:

$$u - \int \frac{c(\varrho)}{\varrho}\, d\varrho \quad u + \int \frac{c(\varrho)}{\varrho}\, d\varrho,$$

where $c(\varrho)$ denotes the speed of sound. For this system strict hyperbolicity is maintained provided that the density ϱ is bounded uniformly away from zero.

Corollary. *If* $(\varrho^\varepsilon, u^\varepsilon)$ *is a sequence of admissible solutions to system* (2.1) *in* $\mathrm{BV}_{\mathrm{loc}}$ *satisfying*

$$0 < m \leqq \varrho^\varepsilon \leqq M \quad \text{and} \quad |u^\varepsilon| \leqq M,$$

then there exists a subsequence converging pointwise a.e. *to an admissible solution* (ϱ, u).

With regard to the structure of the associated parabolic systems of physical interest, we recall that in mechanics the diffusion matrix is merely semi-definite since diffusion is absent in the equation for the conservation of mass. However, with some additional work we establish the following result in Section 3.

Theorem. *Let* $(\varrho^\varepsilon, u^\varepsilon)$ *be a sequence of solutions of the compressible Navier-Stokes equations*

$$\varrho_t + (\varrho u)_x = 0,$$

$$(\varrho u)_t + (\varrho u^2 + p)_x = \varepsilon u_{xx},$$

with $p = \text{const.}\,\varrho^\gamma$, $\gamma > 1$. *If* $(\varrho^\varepsilon, u^\varepsilon)$ *vanishes sufficiently fast at infinity and if*

$$0 < m \leqq \varrho^\varepsilon \leqq M, \quad |u^\varepsilon|_\infty \leqq M,$$

then there exists a subsequence converging pointwise a.e. *to an admissible solution of the compressible Euler equations* (2.1).

One is again lead to ask how much mixing between admissible and non-admissible waves can be allowed without destroying compactness. It turns out that a sufficient condition can be given which is the analogue of the hyperbolic condition of uniformly bounded entropy production and which is phrased in terms of the structure of the viscous layers which develop into shock waves: we need only require that the spatial gradient of u^ε grow in L^2_{loc} no faster than the square of the (Reynolds) number ε. For example in the case of large oscillation we obtain the following.

Theorem. *Consider a genuinely nonlinear strictly hyperbolic system of two equations with a coordinate system of quasi-convex Riemann invariants together with a parabolic system of the form* (2.6) *where D is an arbitrary 2×2 matrix. If u^ε is a sequence of smooth solutions of* (2.6) *satisfying*

$$|u^\varepsilon|_\infty + \sqrt{\varepsilon}\ |u^\varepsilon_x|_{L^2(K)} \leqq M,$$

where the constant M depends only on the compact subset $K \subset R^2$, then there exists a subsequence which converges pointwise a.e. *to a solution u of the hyperbolic system* (1.1).

As before, if we restrict attention to solution sequences with uniformly small oscillation, we can drop the requirement of quasi-convex Riemann invariants. We remark that the local condition

$$\sqrt{\varepsilon}\ |u^\varepsilon_x|_{L^2(K)} \leqq \text{const.}$$

follows immediately if the diffusion matrix D is either positive definite (or negative definite) with respect to a given entropy η, provided that the solutions u^ε under consideration vanish sufficiently fast at infinity so that the formal integration of the identity (2.7) can be accomplished in a rigorous fashion.

In the case of the equations of mechanics, in which the diffusion matrix D or operator $D(u)$ is merely semi-definite, there is a separate problem of showing that the diffusion of certain distinguished components induces an "equal" amount of diffusion of all components, at least to the extent that all components of the spatial derivative grow in L^2_{loc} no faster than $\sqrt{\varepsilon}$. In Section 3 we establish this fact for the compressible Navier-Stokes equations using certain estimates of Greenberg, MacCamy & Mizel [31].

We conclude this section with a discussion of the technical conditions we need for the analysis of the compactness of exact solutions in BV_{loc} with large oscillation. We shall first of all require that the system admit a coordinate system of quasi-convex Riemann invariants, *i.e.* a pair of smooth functions $z = z(u_1, u_2)$, $w = w(u_1, u_2)$ such that

$$\begin{aligned} &r_1 \cdot \nabla w = 0, \quad r_2 \cdot \nabla z = 0, \\ &r_2 \cdot \nabla w \neq 0, \quad r_1 \cdot \nabla z \neq 0, \end{aligned} \tag{2.10}$$

i.e. w is a 1-Riemann invariant, z is a 2-Riemann invariant with w monotone along the level curves of z and z monotone along the level curves of w and such that sets of the form

$$\{(u_1, u_2) : w < w_0\} \quad \text{and} \quad \{(u_1, u_2) : z < z_0\}$$

are convex. Analytically, this is equivalent to the condition that w and z be convex in the directions tangential to their level curves:

$$\nabla^2 w(r_1, r_1) \geqq 0, \quad \nabla^2 z(r_2, r_2) \geqq 0.$$

Under the assumption of quasi-convex Riemann invariants, Lax showed that one can construct a strictly convex entropy η on each compact subset of the state space R^2. The assumption of a global coordinate system of quasi-convex Riemann invariants is the simplest condition we know that guarantees the existence of a strictly convex entropy in the large. It turns out that we can weaken somewhat this condition by asking for a pair of functions (z, w) which satisfy (2.10) and for an entropy pair (η_K, q_K), with η_K strictly convex, for each compact subset $K \subset R^2$ and still prove the same compactness theorems. However, for the purpose of simplicity in the statement of the results we choose to phrase the hypotheses in terms of a condition such as quasi-convexity which can be readily checked in the examples of interest. For example, if we consider the parabolic system (2.6) and a solution sequence u^ε whose range lies in a ball of B, we need only postulate the existence of one strictly convex entropy defined on B together with a coordinate system (z, w) of Riemann invariants.

For the purpose of extending the compactness result for small solutions of hyperbolic systems to solutions with large oscillation, we impose an additional condition beyond that the existence of entropy pairs (η_K, q_K) with η_K strictly convex and a coordinate system of Riemann invariants. This technical condition requires that there exist one entropy pair (η, q) with the following property. If (σ, u^-, u^+) is any solution of the Rankine-Hugoniot relations

$$\sigma(u^+ - u^-) = f(u^+) - f(u^-) \tag{2.11}$$

with $u^\pm$ in a compact set $K \subset R^2$, then the rate of entropy production

$$\sigma[\eta] - [q] \equiv \sigma\{\eta(u^+) - \eta(u^-)\} - q(u^+) - q(u^-)$$

associated with the shock (σ, u^-, u^+) is definite in the sense that

$$|\sigma[\eta] - [q]| \geqq \text{const.}\, |u^+ - u^-|^3. \tag{2.12}$$

We recall that in the situation where the oscillation of the solutions under consideration is small, an arbitrary strictly convex entropy η has the property that

$$|\sigma[\eta] - [q]| \geqq \text{const.}\, |u^+ - u^-|^3;$$

cf. LAX [10]. Furthermore, in the case of small oscillation, if, say, we fix the state u^- and consider the collection of all states u^+ which can be connected to u^- through a shock propagating with speed σ, *i.e.* (σ, u^+, u^-) satisfies (2.11), then we find a set of n curves passing through u^- each formed as the union of two branches, one along which

$$\sigma[\eta] - [q] \leqq -\text{const.}\, |u^+ - u^-|^3,$$

called the admissible branch, and one along which

$$\sigma[\eta] - [q] \geqq -\text{const.}\, |u^+ - u^-|^3,$$

called the inadmissible branch. Thus, the lower bound (2.12) holds if, for example, the solution set of the Rankine-Hugoniot relation consists of curves along which the rate of entropy production

$$\sigma[\eta] - [q] \tag{2.13}$$

never vanishes. We note that this is typically the situation in mechanics. The isentropic equations of gas dynamics (2.1), under the condition of strict hyperbolicity $p' > 0$, admits a global convex entropy η given by the mechanical energy

$$\eta = \frac{1}{2}\varrho u^2 + \varrho\varepsilon(\varrho)$$

with associated flux

$$q = u\eta + up(\varrho),$$

where, from the thermodynamic relation

$$T\,dS = de + p\,d\tau,$$

we obtain, at constant classical entropy S, the relation

$$e'(\varrho) = \frac{p}{\varrho^2}$$

for the specific internal energy ε in terms of the density ϱ. Here η is convex function of ϱ and the momentum $m = \varrho u$. In Lagrange coordinates

$$u_t + p(v)_x = 0,$$

$$v_t - u_x = 0$$

with $p'(v) < 0$; it is seen more easily that the mechanical energy

$$\eta = \frac{1}{2}u^2 + \int^v p(v)\,dv$$

is a strictly convex function of the primitive variables u and v. At least in the case of a polytropic gas $p = \text{const.}\,\varrho^\gamma$ it is known classically that the solution of the Rankine-Hugoniot relations consisting of (two) smooth curves for each fixed state u^- in R^2 for which the quantity (2.13) does not vanish unless $u^+ = u^-$.

If one is given an entropy η^* which satisfies (2.12), then the associated rate of entropy production dominates the rate of entropy production for an arbitrary entropy η in the sense that

$$|\sigma[\eta^*] - [q^*]| \geqq \text{const.}\, |\sigma[\eta] - [q]|$$

for an appropriate constant depending on the compact set in which the analysis takes place. In particular if the total mass of the dissipation measure

$$\theta_u^* = \eta^*(u)_t + q^*(u)_x$$

is bounded uniformly on a solution sequence for the particular entropy η^*, then so is the dissipation measure

$$\theta_u = \eta(u)_t + q(u)_x$$

corresponding to an arbitrary entropy. Thus, control on the total entropy production is, roughly speaking, independent of the choice of entropy. We shall discuss this point further and the role which it plays in the compactness of solutions in the following section.

3. Generalized Measures and Compensated Compactness

In this section we shall describe the background involving TARTAR's work on weak convergence and compensated compactness and his general conjecture concerning the entropy structure and compactness of exact and approximate solutions [19]. Consider an arbitrary sequence of functions

$$u^\varepsilon(y): R^m \to R^n$$

which lie in a bounded set of L^∞. There exists a subsequence which converges in the weak-star topology of L^∞ to a function u, *i.e.* the average value with respect to each bounded set $B \subset R^\infty$ converges to the corresponding average value of u:

$$\lim \int_B u^\varepsilon(y)\, dy = \int_B u(y)\, dy.$$

In general u^ε need not contain any subsequence converging pointwise a.e. Suppose that g is a continuous real-valued map in the range space R^n. In [19] it is shown that composite weak limits can be expressed as expected values of associated probability measures: after passing to a subsequence, one can produce a family of probability measures

$$\{\nu_y : y \in R^m\}$$

such that for all continuous g,

$$\lim g(v^\varepsilon(y)) = \int g(\lambda)\, d\nu_y(\lambda),$$

where the limit exists in the weak-star topology of L^∞ and the equality holds for almost all y in R^m. Here λ denotes a generic point of R^n. For simplicity in printing we shall write

$$\langle \nu, g\rangle = \int g(\lambda)\, d\nu(\lambda)$$

as an abbreviation for the expected value. It follows that strong convergence corresponds to the situation where ν_y is a point mass. The deviation between weak and strong convergence is measured by the spreading of the support of ν_y; for example, if g is Lipschitz continuous, then

$$|g(\lim v^\varepsilon) - \lim g(v^\varepsilon)|_\infty \leqq \text{const.} \max_y \operatorname{diam} \operatorname{spt} \nu_y.$$

In the setting of conservation laws the goal is to show that either ν_y reduces to a point mass or is concentrated on a set whose structure permits one to deduce the continuity with respect to weak limits of the particular nonlinear functions appearing in the equations.

One of the main restrictions placed on the family of measures ν_y is a commutation relation due to the boundedness of the entropy production. Almost all of the measures ν_y commute with an antisymmetric bilinear form acting on entropy pairs. We recall that from the theory of compensated compactness that the inner product is a (bilinear) weakly continuous mapping when restricted to pairs ϕ_ε, ψ_ε of vectors which lie in a bounded set of L^2 and which have a controlled expansion and rotation in the sense that both

$$\operatorname{div} \phi_\varepsilon \quad \text{and} \quad \operatorname{curl} \psi_\varepsilon$$

lie in a compact subset of the negative Sobolev space H^{-1}_{loc}; more precisely if ϕ_ε and ψ_ε converge weakly in L^2 to functions ϕ and ψ then

$$\lim \langle \phi_\varepsilon, \psi_\varepsilon \rangle = \langle \phi, \psi \rangle$$

provided that the respective divergence and curl lie in a compact subset of H^{-1}_{loc} (*cf.* [14, 19]). This fact can be applied to conservation laws in the following way. Suppose u^ε is a family of functions which has uniformly bounded entropy production with respect to entropy pairs $(\eta_j, q_j), j = 1, 2$, in the sense that the sequence

$$\eta_j(u^\varepsilon)_t + q_j(u^\varepsilon)_x$$

lies in a compact subset of H^{-1}_{loc}. Thus

$$\operatorname{div}(\eta_1, q_1) \quad \text{and} \quad \operatorname{curl}(q_2, \eta_2)$$

lie in a compact subset of H^{-1}_{loc} and we deduce that for almost all y

$$\langle \nu_y, \eta_1 q_2 - \eta_2 q_1 \rangle = \langle \nu_y, \eta_1 \rangle \langle \nu_y, q_2 \rangle - \langle \nu_y, \eta_2 \rangle \langle \nu_y, q_1 \rangle .$$

The conjecture of Tartar is the following. Suppose that $u^\varepsilon = u^\varepsilon(x, t)$ is a sequence of functions which lie in a bounded set of L^∞ and satisfy the condition that

$$\eta(u^\varepsilon)_t + q(u^\varepsilon)_x$$

lies in a compact subset of H^{-1}_{loc} for all entropy pairs (η, q), *i.e.* all solutions of the compatibility relation

$$\nabla \eta \, \nabla f = \nabla g;$$

then u^ε contains a subsequence converging pointwise a.e. In the case of a scalar conservation law Tartar [19] showed that ν_y reduces to point mass in the genuinely nonlinear case $f'' \neq 0$ and that, in general, ν_y is supported on an interval where f is affine. In the latter situation one deduces that f is weakly continuous

$$f(u^\varepsilon) \rightharpoonup f(u)$$

and that the speed of sound f' is strongly continuous

$$f'(u^\varepsilon) \to f'(u).$$

We refer the reader to [19] for further discussion. In the setting of systems of two equations we prove that ν_y reduces to a point mass for genuinely nonlinear systems and for the quasilinear wave equation (3.2) in the case (corresponding to elasticity) where σ'' vanishes at only one point. Our analysis is based on a study of progressing entropy waves; *cf.* Sections 4 and 5.

We remark that some aspect of nonlinearity must be present if ν_y reduces to point mass. For general systems with eigenvalues which degenerate on "large" sets one may expect to establish the continuity of the flux map f if the support of ν_y has the correct structure.

We conclude ths section by discussing the circumstances under which the dissipation measure

$$\eta(u^\varepsilon)_t + q(u^\varepsilon)_x \tag{3.0}$$

associated with an entropy pair (η, q) and a uniformly bounded sequence of exact and/or approximate solutions u^ε lies in a compact subset of H^{-1}_{loc}. Let us first consider the case of parabolic systems

$$u_t + f(u)_x = \varepsilon \, D u_{xx}$$

for which the diffusion matrix D is positive definite with respect to one fixed entropy η:

$$\nabla^2 \eta \, D \geqq \delta > 0.$$

In this situation we have seen in Section 2 that

$$\varepsilon \int\limits_0^T \int\limits_{-\infty}^{\infty} u_x^2 \, dx \, dt \leqq \text{const.}$$

It then follows easily that (3.1) lies in a compact subset of H^{-1}_{loc}. The right hand side of the identity

$$\eta(u^\varepsilon)_t + q(u^\varepsilon)_x = \varepsilon(\nabla\eta \, Du^\varepsilon_x)_x - \varepsilon u^\varepsilon_x \, \nabla^2\eta \, Du_x$$

is expressed as the sum of two terms: The first lies in a compact subset of H^{-1} since it is the derivative of the function

$$\varepsilon \, \nabla\eta \, Du^\varepsilon_x$$

whose L^2 norm goes to zero as ε approaches zero $\left(\text{like } \sqrt{\varepsilon}\right)$ while the second term forms a family of functions uniformly bounded in L^1

$$\varepsilon \iint |u^\varepsilon_x \, \nabla^2\eta \, Du_x| \, dx \, dt \leqq \text{const.}$$

One then concludes from the following lemma of MURAT [16, 19] that (3.0) lies in a compact subset of H^{-1}_{loc}: If v^ε is a family of distributions which lie in a bounded set of $w^{-1,\infty}$ and which admit a decomposition

$$v^\varepsilon = v^\varepsilon_1 + v^\varepsilon_2,$$

where v^ε_1 lies in a compact subset of H^{-1}_{loc} and v^ε_2 lies in a bounded subset of the space $BM = C^*$ of measures with finite total variation then v^ε lies in a compact subset of H^{-1}_{loc}. There are, of course, a number of variations of this lemma. We remark that the applicability of MURAT'S lemma to scalar parabolic problems was pointed out by TARTAR [19]. Our objective here is merely to point out some straightforward generalizations.

The treatment of exact solutions u^ε in $L^\infty \cap \mathrm{BV}_{\mathrm{loc}}$ is analogous in spirit but requires several remarks concerning the structure of dissipation measures

$$\theta_u \equiv \eta(u)_t + q(u)_x.$$

In the case of sequences of solutions of (1.1) with uniformly small oscillation we observe that postulating the entropy inequality

$$\eta(u^\varepsilon)_t + q(u^\varepsilon)_x \leqq 0,$$

for just one strictly convex entropy η, implies that for all entropy pairs (η, q) the dissipation measure θ_{u^ε} lies in a compact subset of H^{-1}_{loc}. This can be shown as follows. From the Green's theorem for measures [5, 20] it follows that the weight which θ_{u^ε} gives to any rectangle is uniformly bounded since u^ε lies in a bounded set of L^∞:

$$\{\eta(u^\varepsilon)_t + q(u^\varepsilon)_x\}(R) = \int_{\partial R} n_t \eta(u^\varepsilon) + n_x q(u^\varepsilon)\, ds.$$

If θ_{u^ε} is either a positive or negative measure, then its total variation (over any compact set) is bounded uniformly in ε. If θ_{u^ε} is a signed measure then we appeal to the fact that the

$$\text{(3.1)} \qquad \text{total mass } \theta_{u^\varepsilon} \leqq \text{const. total mass of } \theta^*$$

if θ^* corresponds to a strictly convex entropy. Indeed we have

$$\theta_u\{S_T\} = \int_0^T \Sigma\sigma[\eta] - [q]\, dt,$$

where the summation is taken over all shock waves, *i.e.* points of approximate jump discontinuity in u at time t. Here S_T denotes the strip

$$S_T = \{(x, t) 0 : \leqq t \leqq T\}.$$

If the oscillation is small we have

$$\sigma[\eta] - [q] = O(u^+ - u^-)^3,$$

for arbitrary entropy pairs (η, q) applied to shocks connecting say u^- to u^+ through a wave with propagation speed σ, while

$$\sigma[\eta^*] - [q^*] \leqq -\text{const.}\, |u^+ - u^-|^3$$

if η^* is strictly convex.

In the case of solutions with large oscillation, we observe that under fairly general circumstances, the total amount of entropy production associated with a given strictly convex entropy η^* and measured by the total mass of θ_u^* dominates the total amount of entropy production associated with an arbitrary (non-convex) entropy in the sense of the inequality (3.1). Of course, for a given η^* the constant appearing in (3.1) depends upon the choice of η and on a compact set containing the range of u^ε. As we remarked in Section 2 it is sufficient for this purpose to postulate the existence of one entropy η^* such that

$$|\sigma[\eta^*] - [q^*]| \geqq \text{const.}\, |u^+ - u^-|^3$$

for all solutions (σ, u^-, u^+) to the Rankine-Hugoniot relations. In this case a uniform bound on the total mass of $\theta^*_{u^\varepsilon}$ implies a uniform bound on the total mass of θ_{u^ε} for arbitrary pairs (η, q).

Next, we turn to the problem of showing that θ_{u^ε} lies in a compact subset of H^{-1}_{loc} in the case of parabolic systems where the diffusion matrix is merely semi-definite with respect to entropy. For concreteness, we shall consider the quasilinear wave equation

$$\text{(3.2)} \qquad \begin{aligned} u_t - \sigma(v)_x &= \varepsilon(d(v)\, u_x)_x, \\ v_t - u_x &= 0, \end{aligned}$$

where σ and d are smooth functions such that $\sigma' > 0$ and $d > 0$. The choice $d \equiv 1$ corresponds to the standard equations of viscoelasticity while the choice $d = 1/v$ corresponds to the compressible Navier-Stokes equations written in Lagrange coordinates. One may work in Eulerian coordinates just as well. We shall assume that we are given a family of smooth solutions $(u^\varepsilon, v^\varepsilon)$ which are uniformly bounded in L^∞ and which approach a constant value (u_0, v_0) sufficiently fast as x approaches $\pm\infty$ so that the following formal manipulations are justified and show that

$$\varepsilon \int\int u_x^2 + v_x^2 \, dx \, dt \leqq \text{const.},$$

where the constant depends only on the initial data. By appealing to the mechanical energy

$$\eta = \frac{1}{2} u^2 + \int_{v_0}^{v} \sigma(v) \, dv$$

and its flux

$$q = u\eta - u\sigma$$

we shall first show that

$$\varepsilon \int\int u_x^2 \, dx \, dt \leqq \text{const.} \tag{3.3}$$

We recall that if (η, q) is an entropy pair with η convex, then

$$\eta^*(u) = \eta(u) - \eta(w) - \nabla\eta(w)(u - w),$$
$$q^*(u) = q(u) - q(w) - \nabla\eta(w)\{f(u) - f(w)\}$$

is also an entropy pair with η^* non-negative. Here w denotes a fixed vector in R^n and u the basic state variable. Applying this observation to the mechanical energy, we obtain

$$\eta_t^* + q_x^* = \varepsilon(u - u_0)(d(v)\, u_x)_x \tag{3.4}$$

by multiplying system (3.2) by the gradient (with respect to the vector (u, v)) of

$$\eta^* = \frac{1}{2}(u - u^-)^2 + \int_{v_0}^{v} \sigma(v) \, dv - \sigma(v_0)(v - v_0).$$

Integration of (3.6) over the strip S_T yields the bound (3.3). In order to show that

$$\varepsilon \int\int v_x^2 \, dx \, dt \leqq \text{const.}$$

we follow Greenberg, MacCamy & Mizel [31] and write the diffusion term as an exact space-time derivative:

$$\begin{aligned} u_t - \sigma(v)_x &= \varepsilon D_{xt}, \\ D'(v) &= d(v), \end{aligned} \tag{3.5}$$

We remark that without loss of generality we may take $u_0 = 0$. Now, multiplication of (3.5) by D_x yields

$$D_x u_t - D_x \sigma_x = \frac{\varepsilon}{2}(D_x^2)_t.$$

Integration over S_T implies

$$\iint D_x\sigma_x\,dx\,dt + \frac{\varepsilon}{2}\int D_x^2(x,T) - D_x^2(x,0)\,dx = \iint D_x u_t\,dx\,dt$$

$$= \int D_x u(x,T) - D_x u(x,0)\,dx - \iint D_{xt}u\,dx\,dt$$

$$\geqq \text{const.} + \frac{\varepsilon}{2}\int D_x^2(x,T)\,dx + \frac{2}{\varepsilon}\int u^2(x,T)\,dx - \iint (d(v)\,u_x)_x\,u\,dx\,dt.$$

One integration by parts yields

$$\iint D_x\sigma_x\,dx\,dt \leqq \text{const.} + \frac{2}{\varepsilon}\int u^2(x,T)\,dx + \iint d(v)\,u_x^2\,dx\,dt.$$

But

$$\iint D_x\sigma_x\,dx\,dt = \iint d(v)\,\sigma'(v)\,v_x\,dx\,dt \geqq \text{const.}\iint v_x^2\,dx\,dt.$$

Thus

$$\varepsilon\iint v_x^2\,dx\,dt \leqq \text{const.} + \text{const.}\int u^2(x,T)\,dx + \text{const.}\iint u^2\,dx\,dt$$

and the desired result follows since we have *a priori* control of u in L^2.

4. The Structure of Entropy Waves

Consider a hyperbolic system of n equations

$$u_t + f(u)_x = 0. \tag{4.1}$$

A mapping $\eta : R^n \to R$ is called an entropy for (1) with entropy flux $q : R^n \to R$ if all smooth solutions $u = u(x,t)$ satisfy an additional conservation law of the form

$$\eta(u)_t + q(u)_x = 0. \tag{4.2}$$

Expressing (1) and (2) in quasilinear form,

$$u_t + \nabla f(u)\,u_x = 0, \qquad \nabla\eta(u)\,u_t + \nabla q(u)\,u_x = 0,$$

one obtains a compatibility condition between η and q

$$\nabla\eta\,\nabla f = \nabla q \tag{4.3}$$

which is necessary and sufficient for (4.2). We recall that (4.3) represents a linear hyperbolic system of n equations in two unknowns η and q. Taking the inner product of (4.3) with the right eigenvectors r_j of ∇f produces, from the relation

$$\nabla f r_j = \lambda_j r_j,$$

the characteristic form

$$(\lambda_j\,\nabla\eta - \nabla q)\cdot r_j = 0,$$

which is equivalent to (4.3). In the following discussion we shall be primarily interested in the determinate case $n = 2$.

We recall from LAX [10] that the compatibility equation (4.3) admits two families of progressing wave solutions (η, q). Let r denote a right eigenvector of ∇f and $\phi = \phi(u)$ a corresponding Riemann invariant:

$$\nabla f r = \varrho r, \qquad r \cdot \nabla \phi = 0.$$

Let s and λ denote the eigenvector and eigenvalue of the opposite field

$$\nabla f s = \lambda s, \qquad \lambda \neq \varrho.$$

Equation (4.3) possesses formal asymptotic solutions of the form

$$\eta_k = e^{k\phi} \sum_{n=0}^{\infty} V_n/k^n, \qquad q_k = e^{k\phi} \sum_{n=0}^{\infty} H_n/k^n.$$

For concreteness, we take $k > 0$. Substitution of the formal series into (4.3) yields the defining relations for the coefficients V_n and H_n:

$$\begin{aligned} &\lambda V_0 - H_0 = 0, \\ &(\lambda V_n - H_n) \nabla \phi = \nabla H_{n-1} - \nabla V_{n-1} \nabla f, \quad n \geqq 1. \end{aligned} \tag{4.4}$$

A convenient form for (4.4) is obtained by dotting with r and s:

$$0 = (\nabla H_{n-1} - \varrho \nabla V_{n-1}) \cdot r, \tag{4.5}$$

$$\lambda V_n - H_n = (\nabla H_{n-1} - \lambda \nabla V_{n-1}) \cdot s/(\nabla \phi \cdot s). \tag{4.6}$$

In the case $n = 1$, equation (4.5) is satisfied by substituting $H_0 = \lambda V_0$ and solving for V_0, *i.e.* by taking V_0 to be a solution of

$$(\lambda - \varrho)\, r \cdot \nabla V_0 + (r \cdot \nabla \lambda)\, V_0 = 0.$$

The pair (4.5)–(4.6) is then solved recursively for $n \geqq 1$.

With regard to general properties, we recall that the phase function of a progressing wave is constant along the corresponding characteristic curves and hence, in the context of conservation laws, coincides with a Riemann invariant. Second, the leading coefficients V_0 and H_0 depend only on the eigenvectors and eigenvalues of ∇f. In particular, V_0 and H_0 are independent of the particular choice of phase function ϕ within the class of Riemann invariants of a fixed type; we recall that if ϕ is a j-Riemann invariant,

$$r_j \cdot \nabla \phi = 0,$$

then so is any smooth function of ϕ. Third, we recall an observation of LAX [10] that the leading coefficient V_0 may be taken to be a positive solution of (4.6) by providing positive data along a non-characteristic curve. In this situation it follows that the ratio of entropy flux to entropy asymptotically equals the corresponding eigenvalue:

$$q_k/\eta_k = \lambda + O(1/k).$$

For the purposes of this paper we shall assume that the flux function f admits a coordinate system of Riemann invariants (w, z):

$$r_1 \cdot \nabla w = 0, \qquad r_2 \cdot \nabla z = 0,$$

$$r_2 \cdot \nabla w \neq 0, \qquad r_1 \cdot \nabla z \neq 0.$$

For orientation in terms of waves in physical space, we recall that if $u = u(x, t)$ is a smooth solution of (4.1) then the composition of u with the 1-Riemann invariant z is constant along 1-characteristics

$$\frac{dx(t)}{dt} = \lambda_1(u(x(t)), t)),$$

while the composition of u with the 2-Riemann invariant w is constant along 2-characteristics

$$\frac{dx(t)}{dt} = \lambda_2(u(x(t)), t)),$$

i.e.,

$$w_t + \lambda_1 w_x = 0,$$

$$z_t + \lambda_2 z_x = 0.$$

In the special case of isentropic gas dynamics we have

$$\lambda_1 = u - c, \qquad \lambda_2 = u + c,$$

$$z = u - \int \frac{c}{\varrho}\, d\varrho, \qquad w = u + \int \frac{c}{\varrho}\, d\varrho.$$

In the presence of a coordinate system of Riemann invariants, Lax [10] has constructed exact progressing wave solutions. For each N there exists in a neighborhood of an arbitrary compact set K in R^2, solutions (η_k, q_k) which satisfy

$$\eta_k = e^{k\phi}\left[\sum_{n=0}^{N} V_n/k^n + O(1/k^{N+1})\right],$$

$$q_k = e^{k\phi}\left[\sum_{n=0}^{N} H_n/k^n + O(1/k^{N+1})\right]$$

for large k. The initial data for (η_k, q_k) is taken along a non-characteristic curve which does not intersect K and which is appropriately oriented relative to K; *cf.* [10].

With regard to the nonlinear structure of the equations we observe that the presence or absence of genuine nonlinearity is reflected in the coefficients V_1 and H_1 of the first order terms. It follows from relation (3.6) that

$$\lambda H_1 - V_1 = (r \cdot \nabla\lambda)\, V_0/(r \cdot \nabla\phi).$$

We note that the coefficient of $r \cdot \nabla\lambda$ maintains one sign. Hence, for a given phase ϕ, the deviation between the ratio of entropy flux to entropy and the corresponding eigenspeed λ is asymptotically determined by the derivative of λ in the corresponding eigendirection:

$$q_k/\eta_k - \lambda = -(r \cdot \nabla\lambda)/(r \cdot \nabla\phi)\, k + O(1/k^2).$$

The latter property is basic to our analysis.

5. The Reduction of ν

Consider a compactly supported probability measure ν on R^2 such that

$$\langle \nu, \eta_1 \rangle \langle \nu, q_2 \rangle - \langle \nu, \eta_2 \rangle \langle \nu, q_1 \rangle = \langle \nu, \eta_1 q_2 - \eta_2 q_1 \rangle \tag{5.1}$$

for all C^2 entropy pairs (η_j, q_j). We shall prove that ν reduces to a point mass if the underlying system (1.1) is genuinely nonlinear or if it is given by the quasilinear wave equation with one inflection point. In order to determine the restrictions on the spreading of the support of ν, we shall apply the commutativity relation (5.1) to families of entropy waves with minimal dispersion, namely the progressing entropy waves at large wave number. These waves provide a smooth and technically convenient approximation to the discontinuous planar entropy pairs (η_p, q_p) with a sharp wave front which are natural test waves for (5.1). Unfortunately, without C^2 smoothness the family of dissipation measures

$$\eta_p(u^\varepsilon)_t + q_p(u^\varepsilon)_x$$

associated with a sequence of exact or approximate solutions u^ε may not lie in a compact subset of H^{-1}.

Additional motivation for the use of the progressing waves is the following. From the point of view of regularization it is natural to consider the structure of the characteriestic form of system (1.1). It is not difficult to show that a continuous function u in $\mathrm{BV}_{\mathrm{loc}}$ is a solution of (1.3) if and only if it satisfies the characteristic system

$$l_j(u) \cdot \{u_t + \lambda_j(u)\, u_x\} = 0, \qquad 1 \leqq j \leqq n,$$

where l_j denote the left eigenvectors of ∇f. More generally, a function u in $L^\infty \cap \mathrm{BV}_{\mathrm{loc}}$ is a weak solution if and only if

$$l_j(u) \cdot \{u_t + \lambda_j(u)\, u_x\} = \mu_j,$$

where μ_j are appropriate measures concentrated on the shock set Γ of u, *i.e.* the set of points of approximate jump discontinuity [5, 20]. In the case $n = 2$, the structure is somewhat simplified by the fact that the eigenvectors l_j can be represented as gradients of Riemann invariants w_j, leading to the characteristic form

$$w_j(u)_t + \lambda_j(u)\, w_j(u)_x = \mu_j. \tag{5.2}$$

Furthermore, if u^ε is a solution sequence in $L^\infty \cap \mathrm{BV}_{\mathrm{loc}}$ satisfying the entropy condition (or at least having uniformly bounded entropy production; *cf.* Section 2), then it is not difficult to show that the associated measures μ_j^ε have uniformly bounded total mass and as consequence of Sobolev imbedding lie in a compact subset of $W_{\text{loc}}^{-1,p}$ if $p < 2$. Now, if the coefficients $\lambda_j(u^\varepsilon)$ were somewhat more regular, uniformly in ε, one could deduce from the characteristic form (5.2) and compensated compactness that the product of the Riemann invariants is weakly continuous, *i.e.* if

$$w_j(u^\varepsilon) \rightharpoonup w_j^*,$$

then

$$w_1(u^\varepsilon)\, w_2(u^\varepsilon) \rightharpoonup w_1^* w_2^*,$$

indicating the presence of regularization and restrictions on the structure of ν. It would be interesting, for general systems, to consider the structure of the characteristic equations in light of the theory of compensated compactness. Here we observe that the progressing entropy wave serves as a divergence from approximation to the non-divergence form characteristic systems which permits the application of the theory of compensated compactness as developed in [19].

We shall begin by locating approximately the boundary of the support of ν. We note that, in general, the asymptotic behavior of the Laplace transform of a measure with phase ϕ is determined by the weight which the measure gives to sets near the bounding level curves of ϕ. Let R denote the smallest characteristic rectangle

$$R = \{(u, v) : w_- \leqq w(u, v) \leqq w_+, \quad z_- \leqq z(u, v) \leqq z_+\}$$

which contains the support of ν. As usual (w, z) denotes the coordinate system of Riemann invariants. Let $(\eta_{\pm k}, q_{\pm k})$ denote progressing waves, say defined on a sphere containing $\operatorname{spt} \nu$, with the phase function $\phi = \pm w(u, v)$:

$$\eta_{\pm k} = e^{\pm kw}\{V_0 + V_1/k + O(1/k^2)\},$$

$$q_{\pm k} = e^{\pm kw}\{\lambda_2 V_0 + H_1/k + O(1/k^2)\}.$$

Using these waves, we shall show that $w_+ = w_-$. To this end we shall appeal only to the structure of V_j, H_j for $j = 0$ and $j = 1$. In order to distinguish between distinct level lines of the Riemann invariants, we shall introduce boundary probability measures by taking the "trace" of ν near the boundary of R. For this purpose we introduce probability measures $\mu_k^\pm$ on R defined by

$$\langle \mu_k^\pm, h\rangle = \langle \nu, h\eta_{\pm k}\rangle / \langle \nu, \eta_{\pm k}\rangle,$$

where $h = h(u, v)$ denotes an arbitrary continuous function. As a consequence of weak-star compactness, there exist probability measures $\mu^\pm$ on R such that

$$\langle \mu^\pm, h\rangle = \lim_{k\to\infty} \langle \mu_k^+, h\rangle,$$

after the selection of an appropriate subsequence. We observe the measure μ^+ and μ^- are respectively concentrated on the boundary sections of R associated with w, *i.e.*

$$R \cap \{(u, v) : w = w^+\} \quad \text{and} \quad R \cap \{(u, v) : w = w^-\}.$$

Next, we shall show that $\mu^\pm$ represent the action of ν on the fundamental form $q - \lambda_j \eta$ as follows:

$$\langle \mu^+, q - \lambda_2 \eta\rangle = \langle \nu, q - \lambda_2^+ \eta\rangle,$$

$$\langle \mu^-, q - \lambda_2 \eta\rangle = \langle \nu, q - \lambda_2^- \eta\rangle,$$

where the constants $\lambda_2^\pm$ denote the average value of λ_2 with respect to $\mu^\pm$:

$$\lambda_2^\pm = \langle \mu^\pm, \lambda_2 \rangle .$$

We remark that $q - \lambda_j \eta$ serve as the defining functions for the classical Riemann function for the second order hyperbolic equation

$$\operatorname{curl}(\nabla\eta\, \nabla f) = 0,$$

which governs the entropy η. Indeed, the quantities $q - \lambda_j \eta$ must vanish along the edge of planar entropy pair with a front moving into the zero state ($\eta = 0$, $q = 0$). For this reason one anticipates that taking a trace by letting k approach infinity will produce the average values of $q - \lambda_j \eta$.

In order to establish (5.2) and (5.3) we apply the commutatively relation (5.1) to (η_k, q_k) and (η, q) where (η, q) represents an arbitrary entropy pair and pass to the limit. We obtain

$$\langle \nu, q \rangle - \langle \nu, \eta \rangle \frac{\langle \nu, q_k \rangle}{\langle \nu, \eta_k \rangle} = \frac{\langle \nu, q\eta_k - q_k \eta \rangle}{\langle \nu, \eta_k \rangle} . \tag{5.4}$$

Since

$$q_k = (\lambda_2 + O(1/k))\, \eta_k ,$$

we obtain from (5.4) in the limit

$$\langle \nu, q \rangle - \langle \nu, \eta \rangle \langle \mu^+, \lambda_2 \rangle = \langle \mu^+, q - \lambda_2 \eta \rangle .$$

The identity (5.3) is derived in a similar way using (η_{-k}, q_{-k}). Next, we assert that

$$\lambda_2^+ = \lambda_2^- , \tag{5.5}$$

and as a consequence

$$\langle \mu^+, q - \lambda_2 \eta \rangle = \langle \mu^-, q - \lambda_2 \eta \rangle . \tag{5.6}$$

The equality of the average values can be established by applying (5.1) to pairs with opposite phase. We have

$$\frac{\langle \nu, q_k \rangle}{\langle \nu, \eta_k \rangle} - \frac{\langle \nu, q_{-k} \rangle}{\langle \nu, \eta_{-k} \rangle} = \frac{\langle \nu, q_k \eta_{-k} - q_{-k} \eta_k \rangle}{\langle \nu, \eta_k \rangle \langle \nu, \eta_{-k} \rangle} . \tag{5.7}$$

If $w^+ = w^-$, equality (5.5) is immediate. If not, then for sufficiently small ε we have

$$\langle \nu, \eta_k \rangle \geqq \text{const. } e^{+k(w^+ - \varepsilon)} \quad \text{and} \quad \langle \nu, \eta_{-k} \rangle \geqq \text{const. } e^{-k(w^- + \varepsilon)},$$

for large k and the denominator on the right hand side of (5.7) approaches infinity. The numerator is in general $O(1/k)$. Thus the right hand side approaches zero while the left approaches $\lambda_2^+ - \lambda_2^-$.

We remark that the equality $\lambda_2^+ = \lambda_2^-$ of the average values is leaning in the direction of the conclusion $w_+ = w_-$ in the case where λ_2 is genuinely nonlinear. Indeed, with respect to this coordinate system (w, z), the condition of genuine nonlinearity of $\lambda_2 = \lambda_2(w, z)$ is expressed by monotonicity with respect to w:

$$\frac{\partial \lambda_2}{\partial w} \neq 0.$$

Similarly, the condition of genuine nonlinearity of λ_1 is expressed by

$$\frac{\partial \lambda_1}{\partial z} \neq 0.$$

Thus, for genuinely nonlinear systems, the eigenvalues change monotonically with the corresponding phase functions.

First, we shall show that ν is a point mass if the system is genuinely nonlinear. We claim that $w^+ = w^-$. Substitute (η_k, q_k) into (5.6) and use the fact that

$$q_k - \lambda_2 \eta_k = e^{kw}\{(H_1 - \lambda_2 V_1)/k + O(1/k^2)\}.$$

Since the coefficient $H_1 - \lambda_2 V_1$ does not vanish under the assumption of genuine nonlinearity (*cf.* Section 4) and since $\mu^{\pm}$ are probability measures we obtain the estimates

$$\langle \mu^-, q_k - \lambda_2 \eta_k \rangle \leqq \text{const. } e^{kw^-}/k,$$

$$\langle \mu^+, q_k - \lambda_2 \eta_k \rangle \geqq \text{const. } e^{kw^+}/k,$$

with non-zero constants of the same sign and deduce that $w_+ = w_-$. In the same fashion, if λ_1 is genuinely nonlinear we conclude that $z_+ = z_-$. This completes the proof for genuinely nonlinear systems.

Remark. For the special case of isentropic gas dynamics for a polytropic gas with $p = \text{const.}\, \varrho^\gamma$, $1 < \gamma \leqq 3$, the fact that ν is a point mass can be deduced using only the leading terms V_0, H_0 of the progressing waves. For this system equation (5.7) and its analogue with phase function $\phi = z$ implies immediately that

$$\max \{\lambda_2 \mid R \cap (w = w^-)\} \geqq \min \{\lambda_2 \mid R \cap (w = w^+)\},$$

$$\max \{\lambda_1 \mid R \cap (z = z^-)\} \leqq \min \{\lambda_1 \mid R \cap (z = z^+)\}.$$

These constraints together with the fact that the eigenvalues are linear combinations of the Riemann invariants implies that the support of ν is a point.

Next, we shall consider the quasilinear wave equation

$$u_t - \sigma(v)_x = 0,$$

$$v_t - u_x = 0,$$

where $\operatorname{sgn} v\sigma''$ is of one sign and σ'' vanishes at just one point, say $v = 0$. In terms of the classical Riemann invariants

$$w = u + \int_0^v \sqrt{\sigma'}\, dv, \qquad z = u - \int_0^v \sqrt{\sigma'}\, dv,$$

we note that each of the eigenvalues

$$\lambda_1 = -\lambda_2 = -\sqrt{\sigma'}$$

is genuinely nonlinear when restricted to either of the sets

$$\{(u, v) : w > z\} \quad \text{or} \quad \{(u, v) : w < z\},$$

i.e. the derivatives $r_j \cdot \nabla\lambda_j$ vanish only on the curve

$$\Gamma = \{(u, v) : v = 0\} = \{(u, v) : w = z\}.$$

The key to the situation lies in the fact that the curve Γ of degeneracy of the eigenvalues is transverse to the level curves of the Riemann invariants and consequently that the effect of projecting onto the boundary "minimizes" the set of degenerate values. Indeed, this is the principal motivation for our introduction of the boundary measures. Let R denote the smallest characteristic rectangle containing the support of ν and define

$$I_w^{\pm} = R \cap \{w = w^{\pm}\} \quad \text{and} \quad I_z^{\pm} = R \cap \{z = z^{\pm}\}. \tag{5.7}$$

The only configuration which presents a new situation is the one where the points

$$P^+ = I_w^+ \cap I_z^+ \quad \text{and} \quad P^- = I_w^- \cap I_z^-$$

both lie on Γ. If not then at least one of the boundary arcs (5.7) is contained in a component of Γ^c, *i.e.*, in an open set where both of the eigenvalues λ_j are genuinely nonlinear. In this situation we can apply the argument above for genuinely nonlinear systems without modification. For example, if P^+ lies in Γ^c, then either

$$I_w^+ \subset \Gamma^c \quad \text{or} \quad I_z^+ \subset \Gamma^c.$$

If $I_w^+ \subset \Gamma^c$, we appeal to the fact that the restriction of the quantity

$$H_1 - \lambda_2 V_1,$$

associated with the phase w, to the set I_z^+ does not vanish and conclude that the support of ν is contained in I_z^+. In both cases we arrive at a situation where the support of ν is contained in a region where both eigenvalues are genuinely nonlinear and thus where the argument above for genuinely nonlinear systems holds.

Let us therefore consider the case where P^+ and P^- lie on Γ. We observe that the restriction of $r_2 \cdot \nabla\lambda_2$ to I_w^+ vanishes at only one point, namely P^+ while the restriction of $r_2 \cdot \nabla\lambda_2$ to I_w^- vanishes at only one point, namely P^-. Thus, the supports of the boundary measures μ^+ and μ^- are contained within arcs I_w^+ and I_w^- along which the derivative $\sigma_2 \cdot \nabla\lambda_2$ maintains one sign and vanishes at only one point. Similarly, the supports of the boundary measures associated with the phase function z, call them θ^+ and θ^-, are contained in arcs I_z^+ and I_z^- along which $r_1 \cdot \nabla\lambda_1$ maintains one sign and vanishes at only one point. Thus, we may again apply the argument for genuinely nonlinear systems with the exception of the situation where

$$\operatorname{spt} \mu^+ = \operatorname{spt} \theta^+ = P^+, \qquad \operatorname{spt} \mu^- = \operatorname{spt} \theta^- = P^-.$$

Therefore, the only new situation arises when all of the boundary measures $\mu^{\pm}$, $\theta^{\pm}$ reduce to a point mass. In the presence of such a reduction we obtain from the general relations

$$\langle \mu^+, q - \lambda_2 \eta \rangle = \langle \mu^-, q - \lambda_2 \eta \rangle,$$
$$\langle \theta^+, a - \lambda_1 \eta \rangle = \langle \theta^-, q - \lambda_1 \eta \rangle$$

the condition that

$$q(P^+) - \lambda_j(P^+)\,\eta(P^+) = q(P^-) - \lambda_j(P^-)\,\eta(P^-) \tag{5.8}$$

for $j = 1$ and $j = 2$ and for all pairs (η, q). From (5.8) we conclude that

$$P^+ = P^- = \operatorname{spt} \nu .$$

We conclude this section by sketching a somewhat different proof that ν reduces to a Dirac mass in the case of genuinely nonlinear systems of two equations. The proof deals with the asymptotic rates of convergence as $k \to \infty$ of the ratios

$$\pi_k = \frac{\langle \nu, q_k \rangle}{\langle \nu, \eta_k \rangle} \quad \text{and} \quad \pi_{-k} = \frac{\langle \nu, q_{-k} \rangle}{\langle \nu, \eta_{-k} \rangle}.$$

Let us restrict our attention to the entropy pairs $(\eta_{\pm k}, q_{\pm k})$ corresponding to a phase function w in the form of a 1-Riemann invariant. The commutation relation applied to pairs

$$(\eta_k, q_k) \quad \text{and} \quad (\eta_j, q_j)$$

and to pairs

$$(\eta_{-k}, q_{-k}) \quad \text{and} \quad (\eta_{-j}, q_{-j})$$

shows that for an appropriately large p the subsequences

$$\Pi_{np} \quad \text{and} \quad \Pi_{-np}$$

with indices differing by p depend monotonically on n with opposite parity for large n, *i.e.* if Π_{np} increases then Π_{-np} decreases while if Π_{np} decreases then Π_{-np} increases. This fact follows from the relation

$$\frac{\langle \nu, q_k \rangle}{\langle \nu, \eta_k \rangle} - \frac{\langle \nu, q_j \rangle}{\langle \nu, \eta_j \rangle} = \frac{\langle \nu, q_k \eta_j - q_j \eta_k \rangle}{\langle \nu, \eta_k \rangle \langle \nu, \eta_j \rangle}$$

and the observation that the numerator on the right hand side takes the form

$$q_k \eta_j - q_j \eta_k = \tau e^{(k+j)w} \left[\frac{1}{k} - \frac{1}{j} + O\left(\frac{1}{k^2} + \frac{1}{j^2} \right) \right],$$

where τ is either a strictly positive or strictly negative function. On the other hand the relation (5.7) between ratios with opposite index shows that for all $\varepsilon > 0$ there is a constant c_ε such that

$$|\Pi_k - \Pi_{-k}| \leqq c_\varepsilon e^{-(\alpha - \varepsilon)k}$$

for large k where $\alpha = \omega^+ - \omega^-$ denotes the width of the support of ν with respect to ω. Now since the sequences $\Pi_{\pm np}$ are monotone, it follows that

$$|\Pi_{\pm np} - \Pi| \leqq |\Pi_{np} - \Pi_{-np}|,$$

where Π denotes their common limit

$$\Pi = \lim_{n\to\infty} \Pi_{\pm np}.$$

We deduce that k and j are large and $|k-j| \geqq p$; then

$$|\langle \nu, q_k\eta_j - q_j\eta_k\rangle / \langle \nu, \eta_k\rangle \langle \nu, q_j\rangle| \leqq c_\varepsilon e^{-(\alpha-\varepsilon)k}.$$

However, the left hand side is a quantity on the order of

$$\langle \nu, e^{(k+j)w}\rangle / \langle \nu, e^{kw}\rangle \langle \nu, e^{jw}\rangle (k+j)$$

which is greater than

$$C_\delta\, e^{-\delta k}$$

for all δ. This contradicts the fact that $\alpha \neq 0$. We conclude that the support of ν must be contained in a line $w = \text{const.}$ In similar fashion we deduce that the support of ν must be contained in a line $z = \text{const.}$

6. The Equations of Elasticity

In this section we discuss the compactness of exact and approximate solutions to the equations of elasticity

$$\begin{aligned} u_t - \sigma(v)_x &= 0, \\ v_t - u_x &= 0, \end{aligned} \tag{6.1}$$

and establish a gobal existence theorem with large initial data. We recall that system (6.1) is strictly hyperbolic provided that $\sigma' > 0$. In elasticity, genuine nonlinearity is typically precluded by the fact that the medium in question can sustain discontinuities in both the compressive and expansive phases of the motion. In the simplest model for common rubber, one postulates that the stress σ as a function of the strain v switches from concave in the compressive mode $v < 0$ to convex in the expansive mode $v > 0$, *i.e.*

$$\operatorname{sgn}(v\sigma'') > 0 \tag{6.2}$$

if $v \neq 0$. In contrast, a gas admits only compressive shocks, an empirical fact reflected in the hypotheses $\sigma'' \neq 0$ which is placed on (6.1) for the purpose of studying the motion of a gas.

We observe that system (6.1) admits a coordinate system of (standard) Riemann invariants

$$z = u - \int \sqrt{\sigma'}\, dv, \quad w = u + \int \sqrt{\sigma'}\, dv \tag{6.3}$$

and admits a strictly convex entropy

$$\eta = \frac{1}{2}u^2 + \int \sigma(v)\, dv$$

with flux $q = u\eta - u\sigma$.

Theorem. *Suppose that* $(u^\varepsilon, v^\varepsilon)$ *is a sequence of solutions in* $\mathrm{BV}_{\mathrm{loc}}$ *to system* (6.1) *where* σ'' *vanishes at just one point. If*

$$|u^\varepsilon|_\infty + |v^\varepsilon|_\infty + \text{total mass } \theta^\varepsilon \leqq M$$

for some constant M *independent of* ε, *then there exists a subsequence which converges pointwise* a.e. *to a solution* (6.1).

Here we take the standard measure of dissipation of mechanical energy

$$\theta = \eta(u^\varepsilon, v^\varepsilon)_t + q(u^\varepsilon, v^\varepsilon)_x.$$

As we remarked earlier, one of the simplest ways to ensure that entropy production is uniformly bounded is to postulate the entropy inequality

$$\eta_t + q_x \leqq 0.$$

We note that in the case of genuinely nonlinear systems, the entropy inequality with respect to one strictly convex entropy η is sufficient to rule out all of the non-physical or unstable waves, at least in the case of solutions with small oscillation. For general systems, additional constraints must be placed on the shock waves. In this connection we refer the reader to T.-P. Liu [21, 22]. We emphasize the point here that compactness of solution to genuinely nonlinear or non-genuinely nonlinear systems does not require correct entropy production but only bounded entropy production.

We remark that the analogous result holds for parabolic systems associated with (6.1). In particular, we can establish compactness of solutions to the system of physical interest

$$\begin{aligned} u_t - \sigma(v)_x &= \varepsilon(d(v)\, u_x)_x, \\ v_t - u_x &= 0. \end{aligned} \tag{6.4}$$

Theorem. *Suppose that* $\sigma' > 0$, $d > 0$ *and* σ'' *vanishes at just one point. If* $(u^\varepsilon, v^\varepsilon)$ *is a sequence of smooth solutions of* (6.4) *which are uniformly bounded in* L^∞ *and which approach a constant* (u_0, v_0) *sufficiently fast as* x *approaches* $\pm\infty$, *then there exists a subsequence which converges pointwise* a.e. *to a solution of the hyperbolic system with* $\varepsilon = 0$.

We remark that there is one special case of interest for which a uniform L^∞ estimate can be established. If σ is convex for positive v and concave for negative v, *i.e.* if

$$\operatorname{sgn} v\sigma'' > 0 \tag{6.5}$$

then one can appeal to the invariant regions defined by the level surfaces of the Riemann invariants (6.3) in order to establish an estimate of the form

$$|u(\cdot, t)| + |v(\cdot, t)|_\infty \leqq \text{const.}$$

where the constant is independent of t, provided that the initial data lie in L^∞. Indeed, it follows from theorem of Chueh, Conley & Smoller [32], which characterizes the invariant regions for general parabolic systems, that the regions of the form

$$\{(u, v) : |z| \leqq N, |w| \leqq N\}$$

are invariant for positive N with respect to the solution operator of the system

$$\begin{aligned} u_t - \sigma(v)_x &= \varepsilon u_{xx}, \\ v_t - u_x &= \varepsilon v_{xx}, \end{aligned} \tag{6.6}$$

which employs equal dissipation of each of the solution components. We also note that the fact that the hyperbolic system ($\varepsilon = 0$) possesses bounded invariant regions under the assumption (6.5) was pointed out earlier by T. NISHIDA. Thus we obtain the following:

Corollary. *Suppose that* $\sigma' > 0$, *and* σ'' *vanishes at just one point. If* $(u^\varepsilon, v^\varepsilon)$ *is sequence of smooth solutions to system* (6.6) *which have initial data uniformly bounded in* L^∞ *and which approach a constant* (u_0, v_0) *sufficiently fast as x approaches* $\pm\infty$, *then there exists a subsequence which converges pointwise* a.e. *to a solution of the hyperbolic system* ($\varepsilon = 0$).

In particular one can establish a large data existence theorem for the equations of elasticity by taking, say, initial data which equals a constant outside of a bounded interval, solving the parabolic system (6.6) and passing to the limit. In the case of elasticity one usually normalizes the variables so that $v > -1$. It is sufficient in this situation to consider initial data such that u equals a constant outside u_0 a bounded interval while v equals a constant $-1 + \delta$ with $\delta > 0$ outside of a bounded interval.

To my knowledge the only other large data existence theorem for equations of physical interest is the one established by T. NISHIDA [17] for the isothermal equations of gas dynamics, which form a genuinely nonlinear system of two equations of the form (6.1) in Lagrange coordinates with $\sigma = -1/v$. In connection with the existence problem with "moderately" large data the reader is referred to [4, 12, 17, 18] concerning equations of physical interest. With regard to existence theorems with arbitrarily large data for systems of mainly mathematical interest the reader is referred to [2, 3].

7. Finite Difference Schemes

In this section we shall consider finite difference schemes which are conservative in the sense of LAX & WENDROFF [11]. For simplicity we consider one-step schemes with a three-point domain of dependence:

$$\begin{aligned} u(x, t + \Delta t) - u(x, t) &+ \frac{1}{2}\lambda G\{u(x, t), u(x + \Delta x, t)\} \\ &- \frac{1}{2}\lambda G\{u(x - \Delta x, t), u(x, t)\} = 0, \end{aligned} \tag{7.1}$$

with the standard consistency condition that the numerical flux

$$G: R^n \times R^n \to R^n$$

reduces on the diagonal to the exact flux f:

$$G(a, a) = f(a). \tag{7.2}$$

For simplicity in printing we shall suppress the dependence, if any, of G on the ratio of mesh lengths, $\lambda = \Delta t/\Delta x$. Alternatively, we shall write (7.1) in the form

$$u_j^{n+1} - u_j^n + \frac{\lambda}{2} G(u_j^n, u_{j+1}^n) - \frac{\lambda}{2} G(u_{j-1}^n, u_j^n) = 0. \tag{7.3}$$

For background we recall that the Lax-Friedrichs scheme

$$u_j^{n+1} - \frac{1}{2}(u_{j+1}^n + u_{j-1}^n) + \frac{\lambda}{2}(f(u_{j+1}^n) - f(u_{j-1}^n)) = 0$$

is based on a numerical flux

$$G(a, b) = \frac{1}{2}(f(a) + f(b)) + \frac{1}{\lambda}(a - b)$$

and may be regarded as a formal approximation to the parabolic system

$$u_t + f(u)_x = \Delta t\, u_{xx},$$

in which the time derivative is replaced by a forward difference quotient while the spatial derivatives are replaced by central differences:

$$\frac{1}{\Delta t}(u_j^{n+1} - u_j^n) + \frac{1}{2\Delta x}(f(u_{j+1}^n) - f(u_{j-1}^n)) = \frac{\Delta t}{2\Delta x^2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n).$$

In a similar fashion, one may consider the class of numerical fluxes which are linear in $a, b, f(a)$ and $f(b)$:

$$G = \theta f(a) + (1 - \theta) f(b) + \frac{D}{\lambda}(a - b), \tag{7.4}$$

where θ and D denote $n \times n$ matrices. Here the difference scheme corresponds to a formal approximation of

$$u_t + f(u)_x = \Delta t\, D u_{xx}$$

with linear forward time differences and linear central space differences. For the purposes of this section we shall restrict our attention to the class of numerical fluxes G which are linear in $a, f, b(a)$ and $f(b)$ modulo a term which is quadratic in $a - b$:

$$G = \theta f(a) + (1 - \theta) f(b) + \frac{D}{\lambda}(a - b) + O(a - b)^2. \tag{7.5}$$

We remark that a representation of the form (7.2) holds if and only if the partial derivative G_a restricted to the diagonal is affine in the Jacobian of f, *i.e.* if and only if

$$G_a(a, a) = \theta\, \nabla f(a) + D/\lambda \tag{7.6}$$

for some matrices θ and D. Indeed if (7.6) holds then differentiation of the consistency condition (7.2) implies

$$G_b(a, a) = (1 - \theta)\, \nabla f(a) - D/\lambda$$

and we deduce (7.5). In contrast the most general smooth function G satisfying (7.2) has the property that for each matrix θ there exists a matrix $D_\theta(a)$ such that

$$G(a + b) = \theta f(a) + (1 - \theta) f(b) + D_\theta(a)\,(a - b) + O(a - b)^2 .$$

Thus we are restricting our attention to schemes for which the diffusion coefficient matrix $D_\theta(a)$ is constant for some θ. We note that such schemes are precisely accurate to first order.

Next we turn to the entropy condition. Let (η, q) denote an entropy pair for system (1.1), *i.e.*

$$\nabla\eta\, \nabla f = \nabla q .$$

Motivated by LAX [10], we shall refer to a mapping

$$Q : R^n \times R^n \to R^n$$

as a numerical entropy flux for scheme (7.1) if the local discrete entropy production associated with the mesh point $(j\Delta x, n\Delta t)$,

$$E_{jn} \equiv \eta(u_j^{n+1}) - \eta(u_j^n) + \frac{\lambda}{2}\, Q(u_j^n, u_{j+1}^n) - \frac{\lambda}{2}\, Q(u_{j-1}^n, u_{j-1}^n, u_j^n),$$

is quadratic with respect to the values at level n, *i.e.*,

$$E_{jn} = O(|u_{j-1}^n - u_j^n|^2 + |u_j^n - u_{j+1}^n|^2).$$

Here u_j^{n+1} is defined by (7.3). It is easy to show that the following conditions along the diagonal are necessary and sufficient for E_{jn} to be quadratic:

$$Q(a, a) = q(a),$$

$$\nabla\eta(a)\, G_a(a, a) = Q_a(a, a), \qquad \nabla\eta(a)\, G_b(a, a) = Q_b(a, a).$$

With regard to the limiting entropy production, we recall from [10] that if a sequence of difference approximations generated by the Lax-Friedrichs scheme converges boundedly almost everywhere to a function $u(x, t)$, then $u(x, t)$ is an exact solution satisfying the entropy inequality

$$\eta(u)_t + q(u)_x \leqq 0, \tag{7.6}$$

where η is a strictly convex entropy and q the associated entropy flux, provided that the C-F-L number is appropriately restricted. The entropy inequality follows in the limit from a local inequality

$$E_{jn} \leqq 0, \tag{7.7}$$

where the numerical flux is taken as

$$Q(a, b) = \frac{1}{2}(q(a) + q(b)) + \frac{1}{\lambda}(\eta(a) - \eta(b)).$$

In a similar fashion one can establish the entropy inequality (7.6) for schemes with numerical flux of the form (7.4) in the case where θ is a scalar and $D = I$, *i.e.* if all components undergo equal rates of numerical diffusion. Here one takes

$$Q = \theta q(a) + (1 - \theta)\, q(b) + \frac{1}{\lambda}(\eta(a) - \eta(b)).$$

It would be interesting to classify the family of matrices D for which a local entropy inequality of the form (7.7) holds for some numerical flux Q. Here one might begin by considering

$$Q = \theta q(a) + (1 - \theta) + \nabla\eta\left(\frac{a+b}{2}\right) D(a - b)/\lambda.$$

In treating second-order accurate methods one anticipates the need for considering non-local discrete entropy inequalities.

We shall begin our discussion of compactness by considering difference schemes which admit an interpretation as integral-average layering methods. The treatment here is somewhat less technical than the general case. Let $w = w(x, t, a, b)$ denote the classical solution of the Riemann problem with data (a, b), *i.e.*

$$w(x, 0) = a \text{ if } x < 0, \qquad w(x, 0) = b \text{ if } x > 0.$$

For simplicity we shall restrict our attention to genuinely nonlinear systems with the property that the Riemann problem has a similarity solution $w = w(x/t)$ consisting of constant states, centered rarefaction waves and shock waves satisfying the entropy condition. For systems of two equations, such solutions to the Riemann problem are available under rather general circumstances [24].

We recall that the value u_j^{n+1} determined by the Lax-Friedrichs scheme can be interpreted as the average value of the solution to the Riemann problem with data (u_{j-1}^n, u_{j+1}^n):

$$u_j^{n+1} = \frac{1}{2\Delta x} \int_{-\Delta x}^{\Delta x} w(x, \Delta t, u_{j-1}^n, u_{j+1}^n)\, dx.$$

In working with difference schemes with a two-point domain of dependence, as in the Lax-Friedrichs scheme, we shall restrict our attention to the sublattice $\{(j\Delta x, r\Delta t) : j + n = \text{even}\}$. Thus, given a lattice function $\{u_j^r\}$ generated by the Lax-Friedrichs scheme one may associate an everywhere defined approximate solution $u = u(x, t; \Delta x)$ with the following properties. The function u is an exact solution in each strip of the form

$$S_n = \{(x, t) : r\Delta t \leqq t < (n + 1)\Delta t\},$$

and coincides with the Riemann solution $w(x, t, u_{j-1}^n, u_{j+1}^n)$ in the rectangle

$$S_n \cap \{(x, t) : (j - 1)\Delta x < x < (j + 1)\Delta x\}.$$

Thus, the approximate solution u has the same local structure as the random choice approximations of GLIMM [6]. Likewise, u satisfies the exact equation within an error which depends upon the difference between the limiting values of u at the interfaces $t = m\,\Delta t$. If ϕ is a smooth function with compact support in the strip $0 \leqq t \leqq T = r\,\Delta t$, then

(7.8)

$$\int\int \phi_t u + \phi_x f(u)\,dx\,dt = \int \phi u(x, T)\,dx - \int \phi u(x, 0)\,dx + \sum_{m=1}^{n-1} \int \phi(x, m\,\Delta t)\,[u^m]\,dx,$$

where

$$[u^m] = u(x, m\,\Delta t - 0) - u(x, m\,\Delta t + 0).$$

A similar identity holds for an arbitrary entropy pair (η, q):

(7.9)
$$\int\int \phi_t \eta(u) + \phi_x q(u)\,dx\,dt = \int \phi\eta(x, T)\,dx - \int \phi\eta(x, 0)\,dx + \sum_{m=1}^{n-1} \int \phi(x, m\,\Delta t)\,[\eta^m]\,dx + \int_0^T \Sigma(\phi)\,dt.$$

Here

$$[\eta^m] = \eta\{u(x, m\,\Delta t - 0)\} - \eta\{u(x, m\,\Delta t + 0)\}$$

and $\Sigma(\phi)$ denotes the total rate of entropy production weighted by ϕ, a quantity which is defined as follows. Let $\gamma = (x(t), t)$ denote a shock wave in u and let $[\eta]$ and $[q]$ denote the jump of η and q across γ from left to right, *e.g.*

$$[\eta] = \eta\{u(x(t) - 0, t)\} - \eta\{u(x(t) + 0, t)\}.$$

Then

$$\Sigma(\phi) = \Sigma\{\sigma[\eta] - [q]\}\,\phi(x(t), t),$$

where the summation is taken over all shock waves γ in u at a fixed time t.

We observe in passing that if η is convex then $[\eta^m] \geqq 0$ by Jensen's inequality and hence the Lax entropy inequality holds, *i.e.*

$$\int\int \phi_t \eta + \phi_x q\,dx\,dt \geqq 0,$$

for all smooth non-negative ϕ with compact support in the upper half-plane. The same inequality holds, of course, for pointwise a.e. limits of the approximate solutions u.

Second, we recall that GODUNOV's method [25] can be expressed as either a conservative difference scheme based on a numerical flux $G(a, b)$ given by the value at the wake point $(0, 1)$ of the solution of the Riemann problem with data (a, b):

$$G(a, b) = w(0, 1; a, b),$$

or as a layering method based on integral averages at the interfaces $t = m\,\Delta t$. The layered solution u associated with the lattice function $\{u_j^n\}$ of Godunov's method coincides in the rectangle

$$S_n \cap \{(x, t) : j\,\Delta x < x < (j + 1)\,\Delta x\}$$

with the Riemann solution $w(x, t, u_j^n, u_{j+1}^n)$. Here u assumes the constant value

$$u_j^{n+1} = \frac{1}{\Delta x} \int_{\left(j-\frac{1}{2}\right)\Delta x}^{\left(j+\frac{1}{2}\right)\Delta x} u(x, (n+1)\, t - 0)\, dx$$

on the set

$$\left\{(x, t) : \left(j - \frac{1}{2}\right)\Delta x < x < \left(j + \frac{1}{2}\right)\Delta x,\ t = (n+1)\,\Delta t\right\}.$$

We observe that the same identities (7.8) and (7.9) hold for Godunov's method.

The main results concerning the Lax-Friedrichs scheme and Godunov's scheme are the following. Consider a strictly hyperbolic genuinely nonlinear system of two equations.

Theorem. *Let u^ε be a sequence of approximate solutions generated by either the Lax-Friedrichs scheme or Godunov's scheme and suppose that u^ε has uniformly small oscillation, i.e.*

$$|u^\varepsilon|_\infty \leqq M,$$

where M is a sufficiently small constant. Then there exists a subsequence converging pointwise a.e. *to an exact solution u (satisfying the entropy condition).*

The same compactness result can be established for sequences of approximate solutions which are uniformly bounded in L^∞ provided one restricts attention to systems with quasi-convex Riemann invariants for which the Riemann problem is solvable in the standard fashion in terms of constant states, centered rarefaction waves and admissible shocks.

Proof. We shall show that for all entropy pairs (η, q) the measure

$$\eta_t(u^\varepsilon) + q(u^\varepsilon)_x \tag{7.10}$$

lies in a compact subset of H_{loc}^{-1}. For concreteness, assume that u^ε is associated with the Lax-Friedrichs scheme and that the initial data have compact support. Let us write the entropy identity in the form

$$\iint \phi_t \eta + \phi_x q\, dx\, dt = M(\phi) + L(\phi) + \Sigma(\phi),$$

where the interface error L is decomposed as follows:

$$L = \Sigma L_{jn}, \qquad L_{jn}(\phi) = \int_{(j-1)\Delta x}^{(j+1)\Delta x} \phi(x, nt)\, [\eta^r]\, dx$$

with

$$[\eta^n] = \eta(u_-^n) - \eta(u_j^n), \qquad u_-^n = u(x, n\,\Delta t - 0).$$

For simplicity in printing we suppress the dependence of the operators M, L an Σ on the particular choice of approximate solution u^ε. Let $Q\eta(a, b)$ denote the quadratic part of η at the point b:

$$Q\eta(a, b) = \eta(a) - \eta(b) - \nabla\eta(b)\,(a - b).$$

In general $Q = O(a-b)^2$ while

$$Q\eta(a,b) \geqq \text{const.}\,|a-b|^2,$$

if η is strictly convex. We remark that substitution of the function $\phi \equiv 1$ leads to the identity

$$L_{jn}(1) = \int Q\eta(u_-^n, u_j^n)\,dx$$

and to the lower bound

$$L_{jn}(1) \geqq \text{const.} \int |u_-^n - u_j^n|^2\,dx,$$

if η is strictly convex. Thus, by taking η to be a non-negative strictly convex entropy and setting $\phi \equiv 1$ we obtain an L^2 bound on the interface error of the form

$$\sum_{j,n} \int_{(j-1)\Delta x}^{(j+1)\Delta x} |u_-^n - u_j^n|^2\,dx \leqq \text{const.}, \tag{7.11}$$

by appealing to the fact that Σ is non-positive. In this regard we recall that if η is an arbitrary strictly convex entropy with entropy flux q, then its quadratic part

$$\eta(u) - \eta(v) - \nabla\eta(v)\,(u-v)$$

for each fixed state v is a non-negative strictly convex entropy with entropy flux

$$q(u) - q(v) - \nabla\eta(v)\,(f(u) - f(v)).$$

The estimate (7.11) states, roughly speaking, that the total amount of numerical entropy dissipated across the interfaces is bounded in terms of the data.

We shall show that the measure (7.10) lies in a compact subset of H_{loc}^{-1} by showing that M, L and Σ lie in a compact subset of $W_{\text{loc}}^{-1,1}$. Hence (7.10) lies in a compact subset of $W_{\text{loc}}^{-1,1}$ and in a bounded subset of $W^{-1,\infty}$. First we remark that

$$|M(\phi)| \leqq \text{const.}\,|\phi|_\infty \quad \text{and} \quad |L(\phi)| \leqq \text{const.}\,|\phi|_\infty.$$

Hence M and L lie in a bounded set of C^* and in a compact subset of $W_{\text{loc}}^{-1,1}$. We complete the proof by showing that L can be written in the form

$$L = L_1 + L_2,$$

where

$$|L_1(\phi)| \leqq \text{const.}\,|\phi|_\infty \quad \text{and} \quad |L_2(\phi)| \leqq \text{const.}\,(\Delta x)^\beta\,|\phi|_{C^\alpha},$$

for appropriate α and β. By the Sobolev imbedding theorem

$$|L_2(\phi)| \leqq \text{const.}\,(\Delta x)^\beta\,|\phi|_{W^{1,p}}$$

for an appropriate p and appropriate constant depending on the support of ϕ. Hence L_2 lies in a compact subset of $W_{\text{loc}}^{-1,q}$. In order to establish these estimates we split the local interface error as follows:

$$L_{jn} = L_{1jn} + L_{2jn} = \phi_j^n \int [\eta^r]\,dx + \int (\phi(x, n\Delta t) - \phi_j^n)\,[\eta^r]\,dx,$$

where $\phi_j^n = \phi(j\Delta x, n\Delta t)$. Since

$$\Sigma\left|\int_{(j-1)\Delta x}^{(j+1)\Delta x}[\eta^r]\,dx\right| = \Sigma\,|\int Q\eta(u_-^n, u_j^n)\,dx| \leqq \text{const.},$$

we obtain

$$|L_1(\phi)| \leqq \Sigma\,|L_{1jn}(\phi)| \leqq \text{const.}\ |\phi|_\infty.$$

On the other hand

$$|L_{2jn}(\phi)| \leqq |\phi|_{C^\alpha}\int \Delta x^2\,|[\eta^r]|\,dx.$$

Applying the simple inequality

$$ab \leqq \frac{a^2}{\delta} + \delta b^2$$

with $\delta = \Delta x^\theta$ we obtain

$$|L_{2jn}(\phi)| \leqq |\phi|_{C^\alpha}\int \frac{\Delta x^{2\alpha}}{\Delta x^\theta} + \Delta x^\theta[\eta^r]^2\,dx$$

and

$$|L_2(\phi)| \leqq |\phi|_{C^\alpha}\left(\frac{\Delta x^{2\alpha+1}}{\Delta x^{2+\theta}} + \text{const.}\ \Delta x^\theta\right)$$

since there are only on the order of $1/\Delta x^2$ terms in the summation ΣL_{2jn} if ϕ has compact support. The desired bound is obtained by choosing $\alpha > \frac{1}{2}$ and θ sufficiently small. This completes the proof.

For the equations of elasticity (6.1) one can establish convergence for the Lax-Friedrichs scheme and the Godunov scheme with initial data having arbitrarily large oscillation by appealing to the *a priori* L^∞ estimate afforded by the presence of a family of bounded invariant regions. We recall that approximation methods which employ equal rates of dissipation for each of the basic variables preserve the invariant regions which exist at the hyperbolic level.

Theorem. *Suppose that σ'' vanishes only at $v = 0$ and satisfies*

$$\operatorname{sgn}(v\sigma'') > 0.$$

Consider initial data in L^∞ which (for simplicity) equal a constant value outside a bounded interval. Then any sequence of approximate solutions generated by either the Lax-Friedrichs scheme or Godunov's scheme applied to system (6.1) *contains a subsequence converging pointwise* a.e.

Next, we shall turn to general conservative schemes. For the purpose of discussing pointwise convergence we shall associate with each difference approximation $\{u_j^n\}$ on the lattice $(j\Delta x, n\Delta t)$ a piecewise constant function $u(x, t)$ by setting

$u = u_j^n$ in $R_{jn} =$

$$\left\{(x,t): \left(j-\frac{1}{2}\right)\Delta x < x < \left(j+\frac{1}{2}\right)\Delta x, \left(j-\frac{1}{2}\right)\Delta t < t < \left(j+\frac{1}{2}\right)\Delta t\right\}.$$

We shall study the measures

$$\eta(u)_t + Q_x^+ ,$$

where

$$Q^+ = Q\{u(x, t), u(x + \Delta x, t)\}$$

and show, under certain assumptions to be stated below, that they lie in a compact subset of H_{loc}^{-1} and that the functions Q^+ and

$$q\{u(x, t; \Delta x)\}$$

have the same weak limits as Δx approaches zero. These two facts permit a treatment of differences which is analogous to that given for parabolic systems. In particular we shall show that a uniform bound on the L^∞ norm of a family of difference approximations together with a uniform bound on the total amount of discrete entropy production implies compactness. The following remarks provide the motivation for our main hypothesis concerning the total entropy production. We recall that the difference approximations $\{u_j^n\}$ generated by the Lax-Friedrichs scheme satisfy an estimate of the form

$$\Delta x \sum_{j,n} |u_{j-1}^n - u_{j+1}^n|^2 \leqq \text{const.} \tag{7.12}$$

if, for example, the initial data have compact support. Here the constant depends only on the maximum norm of the difference approximation. It is shown in [10] that the local rate of entropy production is negative definite in the sense that

$$E_{jn} \leqq \text{const.}\, |u_{j-1}^n - u_{j+1}^n|^2$$

and (7.12) follows by summation. In an analogous fashion one can prove that difference approximations $\{u_j^n\}$ generated by Godunov's method satisfy an estimate of the form

$$\Delta x \sum_{j,n} |u_{j-1}^n - u_j^n|^2 + |u_j^n - u_{j+1}^n|^2 \leqq \text{const.} \tag{7.13}$$

where the constant depends only on the maximum norm of $\{u_j^n\}$. We shall postpone the proof of this bound until the end of this section.

Our main results concerning compactness of difference approximations are the following. Consider a genuinely nonlinear strictly hyperbolic system of two equations together with a conservative difference scheme employing a numerical flux of the form (7.5).

Theorem. *Let u^ε be a sequence of difference approximations with uniformly small maximum norm and uniformly bounded entropy production in the sense of* (7.13). *Then there exists a subsequence which converges piontwise* a.e *to an exact solution of* (1.1).

As before we identify a lattice function $\{u_j^n\}$ with the corresponding piecewise constant function that assumes the value u_j^n on the rectangle centered at $(j\Delta x, r\,\Delta t)$. In the case of large oscillation we have the following.

Theorem. *Assume, in addition, that the system* (1.1) *admits a coordinate system of quasi-convex Riemann invariants. If the difference approximations* $u^\varepsilon = \{u_j^n(\varepsilon)\}$ *satisfy estimate* (7.13) *and*

$$\max_{j,n} |u_j^n(\varepsilon)| \leqq M$$

for some constant M which is independent of ε, then there exists a subsequence converging pointwise a.e. *to an exact solution.*

Proof. We shall first make an observation concerning weak limits of the numerical flux. Suppose that

$$Q: R^n \times R^n \to R^n$$

is an arbitrary smooth function and let

$$q(a) = Q(a, a).$$

Suppose that v^ε is an arbitrary sequence of functions uniformly bounded in L^∞ such that

$$\iint |v^\varepsilon(x+\varepsilon, t) - v^\varepsilon(x, t)|^p \, dx \, dt \tag{7.14}$$

approaches zero for some $p \geqq 1$ as ε approaches zero. If

$$\lim q(v^\varepsilon) = q^*,$$

$$\lim Q^\varepsilon = Q^*,$$

where $Q^\varepsilon = Q(v^\varepsilon(x+\varepsilon, t),\ v^\varepsilon(x, t))$ and where the limits are taken in the weak star topology of L^∞, then $q^* = Q^*$. We can apply this remark to the difference approximations under consideration since the entropy bound (7.13) implies that (7.14) holds with $\varepsilon = \Delta x$ and $p = 2$ and then conclude that the numerical flux

$$Q^+ = Q\{u(x + \Delta x, t), u(x, t)\}$$

and the exact flux $q\{u(x, t)\}$ associated with a family of difference approximations $u = u(x, t; \Delta x)$ have the same closure in L^∞ weak star.

In order to show that the measures $\eta_t + Q_x^+$ lie in a compact subset of H_{loc}^{-1} we make the following remarks concerning their local structure. The weight which $\eta_t + Q_x^+$ gives to the rectangle

$$R_{jn} = \left\{(x, t): \left(j - \frac{1}{2}\right)\Delta x \leqq x \leqq \left(j + \frac{1}{2}\right)\Delta x,\ \left(n - \frac{1}{2}\right)\Delta t < t \leqq \left(n + \frac{1}{2}\right)\Delta t\right\}$$

is quadratic in the quantity

$$\delta_{jn} = |u_{j-1}^n - u_j^n| + |u_j^n - u_{j+1}^n|,$$

i.e.

$$\{\eta_t + Q_x^+\}(R_{jn}) = O(\Delta x)\, \delta_{jn}^2,$$

if η is an arbitrary entropy and Q an associated numerical entropy flux. In the notation the difference approximations are defined by the equation

$$\{u_t + G_x^+\}(R_{jn}) = 0.$$

Thus, the action of the measure $\eta_t + Q_x^+$ on a test function ϕ satisfies

$$\left| \int_{R_{jn}} \int \phi(\eta_t + Q_x^+) \right| \leqq O(\Delta x) \, |\phi - \phi_{jn}| \, \delta_{jn} + O(\Delta x) \, |\phi|_\infty \, \delta_{jn}^2,$$

where $\phi_{jn} = \phi(j\Delta x, n\Delta t)$. Hence for appropriate α and β we obtain an estimate of the form

$$\left| \int_{R^2} \int \phi(\eta_t + Q_x^+) \right| \leqq \text{const.} \, |\phi|_\infty + \text{const.} \, \Delta x^\beta \, |\phi|_{C^\alpha},$$

if ϕ is a test function with compact support in a fixed strip $0 \leqq t \leqq T$, as we did in the case of layering methods. This completes the proof.

We conclude this section with the remarks which are required to show that Godunov's scheme produces difference approximations which satisfy the bound (7.13). We recall that the choice $\phi \equiv 1$ in the entropy identity of Godunov's method viewed as a layering method lead to control on the interface error in L^2:

$$\Delta x \sum_{j,n} \int_{\left(j-\frac{1}{2}\right)\Delta x}^{\left(j+\frac{1}{2}\right)\Delta x} |u(x, nt - 0) - u_j^n|^2 \, dx \leqq \text{const.}$$

This estimate can be converted into (7.13) as follows. We observe that if $g(x)$ is a piecewise constant function defined on some interval $[a, b]$ having at most N jumps, then the deviation between g and its average value

$$\bar{g} = \frac{1}{b-a} \int_a^b g(x) \, dx$$

in L^p, *i.e.*

$$\int_a^b |g - \bar{g}|^p \, dx$$

is equivalent to the p^{th} order variation:

$$\text{const.} \, \Sigma \, |\varepsilon|^p \leqq \int_a^b |g - \bar{g}|^p \, dx \leqq \text{const.} \, \Sigma \, |\varepsilon|^p,$$

where ε denotes a jumping at a typical discontinuity. Indeed, the same estimate holds for any value $\bar{g}$ in the convex hull of the range of g. An analogous remark applies to solutions of the Riemann problem where ε is interpreted as the standard wave magnitude. Hence the sum of the squares of the wave magnitudes arising in the Riemann solution at lattice point say $((j + \frac{1}{2}) \Delta x, n \Delta t)$ is bounded by the quantity

$$\int_{\left(j-\frac{1}{2}\right)\Delta x}^{\left(j+\frac{1}{2}\right)\Delta x} |u(x, (n+1)\Delta t - 0) - u_j^{n+1}|^2 \, dx$$
$$+ \int_{\left(j+\frac{1}{2}\right)\Delta x}^{\left(j+\frac{3}{2}\right)\Delta x} |u(x, (n+1)\Delta t - 0) - u_{j+1}^{n+1}|^2 \, dx.$$

The estimate (7.13) follows.

8. Inhomogeneous Systems

The results obtained in the previous sections extend virtually without modification to inhomogeneous systems of the form

$$u_t + f(u)_x = g(x, t, u),$$

where g is a smooth function of its arguments. The same entropy condition is imposed and one postulates a uniform bound on the L^∞ norm of the exact or approximate solutions under consideration. Among the noteworthy examples is the quasilinear wave equation with friction

$$w_{tt} = \sigma(w_x)_x - \alpha w_t$$

which corresponds to an inhomogeneous system of two equations of the form

$$\begin{aligned} u_t - \sigma(v)_x &= -\alpha u, \\ v_t - u_x &= 0 \end{aligned} \tag{8.1}$$

by setting $u = w_t$, $v = w_x$. In the case of physical interest ($\alpha > 0$), the hyperbolic system (8.1) admits a family of bounded invariant regions in the setting of elasticity where $\operatorname{sgn}(v\sigma'') > 0$. These regions assume the form

$$R = \{(u, v) : w_1 \leqq w \leqq w_2,\ z_1 \leqq z \leqq z_2\} \tag{8.2}$$

with the requirement that all of the vertices of the region R lie on one of the coordinate axes. We recall that in the case $\alpha = 0$, all regions of the form (8.2) are invariant provided that just two vertices lie on the line $v = 0$. Thus the addition of a friction term destroys some but not all of the bounded invariant regions.

By appealing to an *a priori* L^∞ estimate based on the presence of invariant regions, one can establish convergence of approximate solutions to methods such as Lax-Friedrichs, Godunov and artificial viscosity

$$\begin{aligned} u_t - \sigma(v)_x &= \varepsilon u_{xx} - \alpha u, \\ v_t - u_x &= \varepsilon v_{xx}, \end{aligned}$$

applied to initial data with large oscillation, and obtain as a corollary a global existence theorem for system (8.1). As before, we need only remark that approximation methods which diffuse all of the primitive variables at an equal rate respect the invariant regions present at the hyperbolic level. We remark that previous results of the dissipative wave equation were limited to data with small oscillation.

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