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Solutions in the Large for Nonlinear Hyperbolic Systems of Equations*

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1. Introduction

We consider the quasilinear system of equations

$$v(x, t)_t + f(v(x, t))_x = 0 \,. \tag{1.1}$$

Here $-\infty < x < +\infty$, $t \geqq 0$, and v and f are vector-valued functions. The function

$$v(x, t) = \text{const.} \tag{1.2}$$

is always a solution of (1.1). We assume that (1.1) is hyperbolic and that f is a strictly nonlinear function. Our main result, Theorem 1.1, is an existence theorem for solutions of (1.1) with prescribed Cauchy data near the constant data $v(x, 0) =$ const. The solution is defined for all $t \geqq 0$ and all x and is near the corresponding constant solution (1.2). This answers in a fairly satisfactory manner the existence portion of a problem posed by Lax [4], p. 6.

The solutions we obtain are weak. They are not differentiable or continuous in general. Even for the simplest equations and initial data, the nonlinear nature of f forces the characteristic speeds to differ at differing x and t in such a way that a continuous solution is in general overdetermined by its initial data and does not exist for all $t \geqq 0$. However, a smooth solution will exist in some strip $0 \leqq t \leqq T$, even without restricting the (smooth) initial data to be near a constant, see [1]. For $t \geqq T$ there will in general be curves in the x, t-plane across which the solution has jump discontinuities (shocks). Presumably nothing worse can occur. We prove the weaker statement that there is a bound on the total variation in x (for fixed t) of the solution v, and that the bound is independent of t. In particular, v is a measurable function. Apparently the concept of solutions weaker than this (e.g., distributions) has no meaning because f is not linear.

Our solutions are limits of difference approximations; the difference approximation involves a random choice. This can be illustrated roughly as follows. Suppose we have determined the difference approximation u at the points

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$x_0 - h, t_0$ and $x_0 + h, t_0$. We take

$$u_1 = u(x_0 - h, t_0) \qquad \text{for} \qquad x < x_0 ,$$

$$u_2 = u(x_0 + h, t_0) \qquad \text{for} \qquad x > x_0 ,$$

as Cauchy data on the line $t = t_0$. The equation (1.1) has a (weak) solution v in some strip $t_0 \leqq t \leqq t_0 + \theta h$ with this Cauchy data. We define

$$u(x_0 , t + \theta h) = v(a, t + \theta h) ,$$

where a is chosen at random in the interval $[x_0 - h, x_0 + h]$. A similar difference scheme, but lacking this probabilistic feature, has been considered by Godunov [2].

Our proof requires two series of estimates, one for v and one for the first derivatives of v. For the estimates involving v we use the sup norm on v; for those involving the first derivatives we use the total variation norm on v. We define as the distance between the initial data $v(\cdot, 0)$ and a constant $\tilde{v}$ either

$$d_0 = \|v(\cdot, 0) - \tilde{v}\|_\infty \,(1 + \text{total variation } v(\cdot, 0)) ,$$

or

$$d_1 = \|v(\cdot, 0) - \tilde{v}\|_\infty + (\text{total variation } v(\cdot, 0)) .$$

We must assume that one of these distances d_0 or d_1 is small. Assuming d_0 is small is less restrictive and this will be permitted only for a restricted class of equations. The restricted class includes all 2×2 systems under consideration. In particular, it includes the equations of gas dynamics (for constant entropy and with one space variable).

We set $A(v) = \partial f(v)/\partial v$. Then (1.1) can be written as

$$v_t + A(v)v_x = 0 . \tag{1.3}$$

Let N be the order of the system. In saying that (1.1) or (1.3) is hyperbolic we mean that, for each value of v, the matrix $A(v)$ has N real and distinct eigenvalues

$$\lambda_1 = \lambda_1(v) < \lambda_2 < \cdots < \lambda_N .$$

Let $r_j = r_j(v)$ and $l_j = l_j(v)$ be the right and left eigenvectors of A for the eigenvalue λ_j. Following Lax [3], we assume that the system (1.1) or (1.3) is strictly nonlinear in the sense that

$$r_j \cdot \nabla \lambda_j \neq 0 .$$

We say that a local coordinate system $w_1, \cdots, w_N$ defined in an open set in R^N consists of Riemann invariants if

$$r_i \cdot \nabla w_j = 0 , \qquad i \neq j .$$

In general such a coordinate system will not exist; but, if $N = 2$, then it will necessarily exist. We say (1.1) is smooth if f is smooth.

THEOREM 1.1. *Let the equation* (1.1) *be hyperbolic, strictly nonlinear and smooth in a neighborhood of* $\tilde{v}$. *There is a* $K < \infty$ *and a* $\delta > 0$ *with the following property.*

If the initial data $v(x, 0)$ *are given so that* $d_1 \leqq \delta$, *then there is a weak solution* $v(x, t)$ *of* (1.1) *defined for all* x *and all* $t \geqq 0$ *with initial data* $v(x, 0)$ *such that*

$$\|v - \tilde{v}\|_\infty \leqq K \|v(\cdot, 0) - \tilde{v}\|_\infty , \tag{1.4}$$

$$\text{total variation } v(\cdot, t) \leqq K \, (\text{total variation } v(\cdot, 0)) , \qquad t \geqq 0 , \tag{1.5}$$

$$\int_{-\infty}^{\infty} |v(x, t_1) - v(x, t_2)| \, dx \leqq K \, |t_1 - t_2| \, (\text{total variation } v(\cdot, 0)) . \tag{1.6}$$

If there is a coordinate system $w_1, \cdots, w_N$ *defined in a neighborhood of* $\tilde{v}$ *which consists of Riemann invariants (for example, if* $N = 2$*), and if the initial data* $v(x, 0)$ *are given so that* $d_0 \leqq \delta$, *then there is a weak solution* $v(x, t)$ *of* (1.1) *defined for all* x *and all* $t \geqq 0$ *with initial data* $v(x, 0)$ *such that* (1.4) *and* (1.5) *hold.*

That v is a weak solution with initial data $v(x, 0)$ means that v is a bounded measurable function and that

$$\int_0^\infty \int_{-\infty}^\infty (\phi_t v + \phi_x f(v)) \, dx \, dt + \int_{-\infty}^\infty \phi(x, 0) v(x, 0) \, dx = 0$$

for all smooth functions ϕ which are identically zero outside some bounded set. The remaining sections of this paper are devoted to the proof of this theorem.

By simple examples it can be shown that the weak solutions of (1.1) are not unique in general. This is due to the existence of "extraneous" solutions of (1.1) which are not stable (with respect to small perturbations of initial data) and which contain "unnecessary" discontinuities not of shock type. We conjecture that our solution v of Theorem 1.1 is not of the "extraneous" type, [4], [5]. For general background and for the more fully developed theory of a single equation we refer the reader to [3], [4], [6] and to the references quoted there. The author is grateful to P. Lax for several stimulating discussions.

2. Bounds for Solutions of the Riemann Problem

The eigenvector r_j has a unique direction but its length is arbitrary. Near $\tilde{v}$ we choose its length in a smooth manner so that $r_j \cdot \nabla \lambda_j = 1$. Each r_k defines a vector field R_k,

$$R_k = r_k \cdot \nabla .$$

The integrals of R_k (i.e., w for which $r_k \cdot \nabla w = 0$) are called k-Riemann invariants of (1.1). There are $N - 1$ independent Riemann invariants defined in a neighborhood of $\tilde{v}$; the integral curves of R_k in this neighborhood are exactly the curves along which all k-Riemann invariants are constant.

If v_l and v_r are any two N-vectors near $\tilde{v}$, then (1.1) has a weak solution v with Cauchy data

$$v(x, 0) = \begin{cases} v_l , & x < 0 , \\ v_r , & x > 0 . \end{cases}$$

This problem is called a Riemann problem and its solution v exists by Theorem 9.1 of [3]. Furthermore, v takes on constant values $v_0 = v_l, \cdots, v_j, \cdots, v_N = v_r$ in sectors $\cdots, \Sigma_j = \{x, t: \xi_j t < x < \xi'_j t\}, \cdots$. The sectors Σ_{j-1}, Σ_j are consecutive. Also by [3] we have two possibilities for v in the sector between Σ_{j-1} and Σ_j :

(a_j) Σ_{j-1} and Σ_j have a common boundary $x = \xi_j t$, also $\lambda_j(v(t\xi_j - 0, t)) > \xi_j > \lambda_j(v(t\xi_j + 0, t))$,

(b_j) between Σ_{j-1} and Σ_j, $\lambda_j(v(t\xi, \xi)) = \xi$ and all j-Riemann invariants are constant.

In both cases v is a function of x/t. In case (b_j), v is continuous on and between Σ_{j-1} and Σ_j ; in case (a_j) there is a jump discontinuity. This jump discontinuity is called a j-shock wave. The configuration (b_j) is called a j-rarefaction wave, while a j-wave refers to either. The v_j are called intermediate states. They are near $\tilde{v}$ and are C^2 functions of v_l and v_r, see [3], p. 563. Third derivatives are piecewise continuous (with a jump at the transitions from (a_k) to (b_k)). We call v the resolution of the discontinuity v_l, v_r into j-waves, $1 \leqq j \leqq N$. We choose functions w_j defined near $\tilde{v}$ such that

$$R_j w_j = r_j \cdot \nabla w_j = 1 , \tag{2.1}$$

and we let the quantity

$$\varepsilon_j = w_j(v_j) - w_j(v_{j-1})$$

denote the magnitude of the j-wave in v. In the case (b_j) of rarefaction waves the range of v restricted to the sector between Σ_{j-1} and Σ_j lies on an integral curve of R_j, and ε_j represents the change of parameter along this curve. Along the curve, $\lambda_j(v)$ increases. Since sign $R_j\lambda_j =$ sign $R_j w_j$, $w_j(v)$ increases and $\varepsilon_j \geqq 0$. In case (a_j), let v_j^0 be the point on the integral curve of R_j through v_{j-1} with

$$w_j(v_j^0) = w_j(v_j) .$$

Then $v_j^0 = v_j + O(\varepsilon_j^3)$; see [3], p. 563. We suppose v_l and v_r are close to each other. Then ε_j is small, and

$$\text{sign}\,[\lambda_j(v_j) - \lambda_j(v_{j-1})] = \text{sign}\,[\lambda_j(v_j^0) - \lambda_j(v_{j-1})] = \text{sign}\,\varepsilon_j .$$

Since $\lambda_j(v_j) - \lambda_j(v_{j-1}) < 0$, we have $\varepsilon_j < 0$ also. Hold v_{j-1} fixed and consider v_j as a function of ε_j. The tangent $dv_j/d\varepsilon_j$ to this curve is

$$\begin{array}{lll} r_j(v_j) & \text{if} & \varepsilon_j \geqq 0 , \\ r_j(v_j) + O(\varepsilon_j^2) & \text{if} & \varepsilon_j < 0 , \end{array}$$

(see [3]), and the derivative of the tangent is

$$\begin{array}{lll} R_j(r_j(v_j)) = (r_j \cdot \nabla)r_j & \text{if} & \varepsilon_j \geqq 0 , \\ R_j(r_j(v_j)) + O(\varepsilon_j) & \text{if} & \varepsilon_j < 0 . \end{array}$$

By Taylor's expansion about v_{j-1}, we have

$$v_j = v_{j-1} + \varepsilon_j r_j(v_{j-1}) + 2^{-1}\varepsilon_j^2(r_j \cdot \nabla)r_j + O(\varepsilon_j^3) .$$

By induction on j we obtain[1]

$$(2.2_j) \qquad v_j = v_l + \sum_{1 \leq i \leq j} \varepsilon_i r_i(v_l) + \sum_{1 \leq h \leq i \leq j} \varepsilon_h \varepsilon_i (r_h \cdot \nabla) r_i (1 - 2^{-1}\delta_{hi}) + O(|\varepsilon|^3) ,$$

with all the r evaluated at v_l and with $\varepsilon = \varepsilon_1, \cdots, \varepsilon_N$.

Now suppose that v_m is some other state near $\tilde{v}$ and that v' (respectively v'') is the resolution of the discontinuity v_l, v_m (respectively v_m, v_r) into j-waves with intermediate states v'_j (respectively v''_j). Let the j-wave in v' (respectively v'') have magnitude γ_j (respectively δ_j). Let $\gamma = \gamma_1, \cdots, \gamma_N$ and $\delta = \delta_1, \cdots, \delta_N$. Applying (2.2_N) to the case v'', we have

$$(2.3) \qquad v_r = v''_N = v_m + \sum \delta_i r_i + \sum_{h \leq i} \delta_h \delta_i (r_h \cdot \nabla) r_i (1 - 2^{-1}\delta_{hi}) + O(|\delta|^3) ,$$

with all the r evaluated at v_m. To analyze v' we proceed in the order of decreasing j:

$$v'_{j-1} = v'_j - \gamma_j r_j(v_j) + 2^{-1}\gamma_j^2(r_j \cdot \nabla)r_j + O(\gamma_j^3) .$$

By an induction we obtain

$$(2.4) \qquad v_l = v'_0 = v_m - \sum_i \gamma_i r_i + \sum_{h \geq i} \gamma_h \gamma_i (r_h \cdot \nabla) r_i (1 - 2^{-1}\delta_{hi}) + O(|\delta|^3) ,$$

with all the r evaluated at v_m. We hold v_m fixed and regard ε as a function of γ and δ. Then ε has continuous second derivatives and piecewise continuous third derivatives. Since $\varepsilon = 0$ if $\gamma = 0 = \delta$, $\varepsilon = O(|\gamma| + |\delta|)$. Since

$$(2.5) \qquad r_i(v_l) = r_i(v_m) - \sum_h \gamma_h (r_h \cdot \nabla) r_i + O(|\gamma|^2) ,$$

we get from (2.2_N)

$$v_r - v_l = \sum_i \varepsilon_i r_i(v_m) + O([|\gamma| + |\delta|]^2) .$$

If we compare this to (2.3) and (2.4), we obtain

$$(2.6) \qquad \varepsilon_i = \gamma_i + \delta_i + O([|\gamma| + |\delta|]^2) .$$

We substitute (2.6) and (2.5) in (2.2_N) and get

$$v_r - v_l = \sum_i \varepsilon_i r_i + \sum_{h \leq i} \varepsilon_h \varepsilon_i (r_h \cdot \nabla) r_i (1 - 2^{-1}\delta_{hi})$$
$$- \sum_{h,i} \gamma_h \varepsilon_i (r_h \cdot \nabla) r_i + O([|\gamma| + |\delta|]^3) ,$$

[1] $\delta_{hi} = \begin{cases} 1 \text{ if } h = i, \\ 0 \text{ if } h \neq i. \end{cases}$

with all the r evaluated at v_m. Comparing this to (2.3) and (2.4), we get

$$
\begin{aligned}
\sum_i (\varepsilon_i - \gamma_i - \delta_i)r_i = \sum_i [&-2^{-1}(\gamma_i + \delta_i)^2 + (\gamma_i + \delta_i)\gamma_i \\
&+ 2^{-1}\delta_i^2 - 2^{-1}\gamma_i^2](r_i \cdot \nabla)r_i \\
&+ \sum_{h<i} [\delta_h\delta_i - (\gamma_h + \delta_h)(\gamma_i + \delta_i) + (\gamma_i + \delta_i)\gamma_h](r_h \cdot \nabla)r_i \\
&+ \sum_{h>i} [-\gamma_h\gamma_i + (\gamma_i + \delta_i)\gamma_h](r_h \cdot \nabla)r_i + O([|\gamma| + |\delta|]^3) \\
= &\sum_{h<i} \gamma_i\delta_h[(r_i \cdot \nabla)r_h - (r_h \cdot \nabla)r_i] + O([|\gamma| + |\delta|]^3) .
\end{aligned}
$$

$$\sum_i (\varepsilon_i - \gamma_i - \delta_i)R_i = \sum_{h<i} \gamma_i\delta_h[R_i , R_h] + O([|\gamma| + |\delta|]^3) .$$

In the case where there is a coordinate system w_j near $\tilde{v}$ with $R_j w_k = 0$ if $j \neq k$, we take (2.1) as a determination of the length of R_j (rather than a restriction on w_j). In this coordinate system $R_j = \partial/\partial w_j$ and $[R_i , R_h] = 0$; thus

$$\varepsilon_i = \gamma_i + \delta_i + O([|\gamma| + |\delta|]^3) . \tag{2.7}$$

We say that an i-wave from v' and a j-wave from v'' are approaching if

$$i > j$$

or if

$$i = j \qquad \text{and} \qquad \gamma_i < 0 \qquad \text{or} \qquad \delta_i < 0 .$$

Let

$$D(\gamma, \delta) = \sum |\gamma_i|\,|\delta_j| ,$$

where the sum is over all pairs i and j for which the i-wave from v' and the j-wave from v'' are approaching. The principal result of this section is the following refinement of (2.6) and (2.7).

THEOREM 2.1. *We have*

$$\varepsilon_i = \gamma_i + \delta_i + D(\gamma, \delta)O(1) \tag{2.8}$$

as $|\gamma| + |\delta| \to 0$ *and if* (2.7) *holds we have*

$$\varepsilon_i = \gamma_i + \delta_i + D(\gamma, \delta)O(|\gamma| + |\delta|) . \tag{2.9}$$

Proof: First we observe that the theorem is true in the case $D = 0$. In fact by the results of [3], the waves in v' and v'' can be fitted together exactly to produce a solution of the desired type, which by uniqueness must be v. Next we prove the theorem with D replaced by $|\gamma|\,|\delta|$. Fix i, γ, and δ and let

$$F(\sigma, \tau) = \varepsilon_i(\sigma\gamma, \tau\delta) - \sigma\gamma_i - \tau\delta_i .$$

Then $F(0, \tau) = 0 = F(\sigma, 0)$, F is C^2 and is C^3 where $\sigma \neq 0 \neq \tau$, and the bounds on F and its derivatives are uniform in γ and δ, $|\gamma| + |\delta| \to 0$. We have $F_\sigma(\sigma, 0) = 0$, thus $\tau^{-1}F_\sigma(\sigma, \tau)$ is bounded and continuous, uniformly in γ and δ. In the same way $\tau^{-1}F(\sigma, \tau)$ is bounded and continuous. Since $\tau^{-1}F(0, \tau) = 0$, it follows that $F = \sigma\tau O(1)$ uniformly in γ and δ. Therefore,

$$\varepsilon_i = \gamma_i + \delta_i + |\gamma|\,|\delta|\,O(1)\,. \tag{2.10}$$

Let $G(\gamma, \delta)$ be the $O(1)$ in (2.10). Then G is continuous and has piecewise continuous and bounded first derivatives. If (2.7) holds we must have $G(0, 0) = 0$. However this implies $G = O(|\gamma| + |\delta|)$ so that

$$\varepsilon_i = \gamma_i + \delta_i + |\gamma|\,|\delta|\,O(|\gamma| + |\delta|) \tag{2.11}$$

as asserted. The O in (2.10) and (2.11) is uniform in v_m for v_m near $\tilde{v}$.

We suppose $0 = \delta_{p+1} = \cdots = \delta_N$. To complete the proof we use induction on p. For $p = 0$ the theorem is true; we suppose it true for $p - 1$, $0 \leqq p - 1 \leqq N - 1$. Let

$$\Delta = \delta_i\,, \cdots, \delta_{p-1}\,, 0, \cdots, 0\,,$$

$$\mu_i = 0,\ \nu_i = \varepsilon_i(\gamma, \Delta) \text{ if the } i\text{-wave in } v' \text{ and the } p\text{-wave in } v'' \text{ approach,}$$

$$\mu_i = \varepsilon_i(\gamma, \Delta),\ \nu_i = 0 \text{ otherwise}\,.$$

We want to consider the interaction of the waves with magnitude ν_j, and the p-wave from v''. In order to do this we must take v''_{p-1} in the role of $\tilde{v}_m$. We indicate the dependence of ε on v_m by writing

$$\varepsilon_i = \varepsilon_i(\gamma, \delta, v_m)\,.$$

The result of this interaction is the solution of a Riemann problem: it consists of j-waves with magnitudes π_j, where

$$\pi_j = \pi_j(\nu, \cdots, 0, \delta_p\,, 0, \cdots, v''_{p-1})\,.$$

Next we consider the interaction of the waves with magnitudes μ and π. The result of this interaction is v. Hence if we let $\tilde{v}_m$ denote the intermediate state between the "μ-waves" and the "π-waves", we have

$$\varepsilon_i(\gamma, \delta, v_m) = \varepsilon_i(\mu, \pi, v_m)\,. \tag{2.12}$$

Now $D(\gamma, \Delta) \leqq D(\gamma, \delta)$, and by our induction hypothesis we have

$$\varepsilon_i(\gamma, \Delta, v_m) + \delta_{ip}\,\delta_p = \gamma_i + \delta_i + D(\gamma, \delta)O(1)\,.$$

By (2.10) we have

$$\pi_i = \nu_i + \delta_{ip}\,\delta_p + |\nu|\,|\delta_p|\,O(1)\,.$$

For $i < p$, $|\nu_i|\,|\delta_p| = 0$. For $i = p$, $\nu_p = 0$ or else

$$\nu_p = \gamma_p + D(\gamma, \delta)\,O(1)\,,$$

$$|\nu_p|\,|\delta_p| \leqq D(\gamma, \delta)O(1)\,.$$

For $i > p$, the i-wave from v' approaches the p-wave from v'' and

$$\begin{aligned}|\nu_i|\,|\delta_p| &\leqq (|\gamma_i| + D(\gamma, \delta)O(1))\,|\delta_p| \\ &\leqq D(\gamma, \delta)O(1)\,.\end{aligned}$$

Combining all the cases, we see that

$$\begin{aligned}|\nu|\,|\delta_p| &\leqq D(\gamma, \delta)O(1)\,, \\ \gamma_i + \delta_i &= \mu_i + \nu_i + \delta_{ip}\,\delta_p + D(\gamma, \delta)O(1) \\ &= \mu_i + \pi_i + D(\gamma, \delta)O(1)\,.\end{aligned}$$

We define

$$\tilde{\pi}_i = \begin{cases} 0 & \text{if} \quad i < p\,, \\ \nu_i + \delta_i & \text{if} \quad i \geqq p\,. \end{cases}$$

None of the "μ-waves" are approaching a "$\tilde{\pi}$-wave"; hence we have

$$\begin{aligned}\varepsilon_i(\mu, \tilde{\pi}, \tilde{v}_m) &= \mu_i + \tilde{\pi}_i \\ &= \gamma_i + \delta_i + D(\gamma, \delta)O(1)\,.\end{aligned} \tag{2.13}$$

If (2.7) holds, we obtain

$$\varepsilon_i(\mu, \tilde{\pi}, \tilde{v}_m) = \gamma_i + \delta_i + D(\gamma, \delta)O(|\gamma| + |\delta|)\,. \tag{2.14}$$

It follows from (2.10) that

$$\begin{aligned}|\varepsilon_i(\mu, \pi, \tilde{v}_m) - \varepsilon_i(\mu, \tilde{\pi}, \tilde{v}_m)| &\leqq |\pi - \tilde{\pi}|\, O(1) \\ &\leqq \begin{cases} DO(1)\,, & \\ DO(|\gamma| + |\delta|) & \text{if} \quad (2.7) \text{ holds}\,. \end{cases}\end{aligned}$$

This combined with (2.12)–(2.14) completes the proof.

We suppose without loss of generality that the coordinates w_j have the property

$$(R_k w_j)(\tilde{v}) = \delta_{jk}\,.$$

We take

$$\sup_j |w_j(v_A) - w_j(v_B)|$$

as a metric in R^N near $\tilde{v}$. We assume that $w_j(\tilde{v}) = 0$ and we define $|v| = \sup_i |w_i(v)|$. This is a norm with respect to the linear structure defined by the coordinates w_i but it is not a norm in the original coordinates in general. For intermediate states v_{k-1} and v_k of the resolution of v_l, v_r into j-waves we have

$$w_j(v_k) - w_j(v_{k-1}) = \varepsilon_k(\delta_{jk} + O(|v_l| + |v_r|))\,, \tag{2.15}$$

and

$$w_j(v_r) - w_j(v_l) = \varepsilon_j + |\varepsilon|\, O(|v_l| + |v_r|)\,.$$

If the w_j are Riemann invariants for R_k, $k \neq j$, so that $R_k w_j \equiv \delta_{jk}$, we have

$$w_j(v_k) - w_j(v_{k-1}) = \varepsilon_k \, \delta_{jk} + O(|\varepsilon_k|^3) \tag{2.16}$$

and

$$w_j(v_r) - w_j(v_l) = \varepsilon_j + O(|\varepsilon|^3) \,.$$

3. The Difference Scheme

Our difference scheme depends upon a random choice. Given a bounded neighborhood U_1 of the state $\tilde{v}$, we can find neighborhoods $U_3 \subset U_2 \subset U_1$ such that for any two states u_l and u_r in U_3 there is a unique resolution of u_l, u_r into j-waves with intermediate states $u_1, \cdots, u_{N-1}$ in U_2. We choose mesh lengths r and s such that

$$\sup \{|\lambda_j(u)| : u \in U_3 \,,\, 1 \leqq j \leqq N\} < r/s \,.$$

We hold the ratio r/s fixed, so s is a function of r. Let

$$Y = \{(m, n) : \ m, n \text{ integers}, \ m + n \text{ is even and } n \geqq 0\} \,,$$

$$A = \prod_{(m,n) \in Y} [(m-1)r, (m+1)r \times \{ns\}] \,; \tag{3.1}$$

each factor is a horizontal line segment in the plane. Give each factor the measure $1/2r$ times Lebesgue measure and give A the product measure, denoted by da. We choose a point $a = \{a_{m,n}\} \in A$. This is our random choice and the points $a_{m,n}$ are our mesh points.

Suppose that our difference approximation $u = u(x, t)$ has been defined for $x, t = a_{m-1,n-1}$ and $x, t = a_{m+1,n-1}$ and that these values of u lie in U_3. Let v be the solution of equation (1.1), with range $v \in U_3$ and with initial values

$$v(x, (n-1)s) = \begin{cases} u(a_{m-1,n-1}) \,, & (m-1)r \leqq x < mr \,, \\ u(a_{m+1,n-1}) \,, & mr < x \leqq (m+1)r \,, \end{cases}$$

which is a resolution of $u(a_{m-1,n-1})$, $u(a_{m+1,n-1})$ into j-waves. Let $u(a_{m,n}) = v(a_{m,n})$. This defines our difference scheme. It is convenient to set

$$u(x, t) = v(x, t) \,, \qquad (m-1)r \leqq x \leqq (m+1)r \,, \qquad (n-1)s \leqq t < ns \,.$$

Then u is a solution in this rectangle. For x near the boundary $(m-1)r$ (respectively $(m+1)r$) we have $u(x, t) = u(a_{m-1,n-1})$ (respectively $u(x, t) = u(a_{m+1,n-1})$), because of the restriction on r/s. Thus if u is defined in a strip $(n-1)s \leqq t < ns$, $-\infty < x < +\infty$, then u is constant across the lines $x = (m-1)r$, $m + n$ even, where its mode of definition changes and u is a solution in the strip.

We cannot show in a simple and direct fashion that the difference approximation u can be defined for all x and t. Instead, this will be proved simultaneously with the proof of the bounds on u, and requires that the initial data be a small perturbation from the constant state $\tilde{v}$.

In place of horizontal lines, it is convenient to consider curves consisting of line segments of the form $L_{m,n,m+1,n+1}$, $L_{m,n,m+1,n-1}$ joining $a_{m,n}$ to $a_{m+1,n+1}$ and joining $a_{m,n}$ to $a_{m+1,n-1}$, respectively. If the mesh index m increases monotonically on such a curve, we call it an I curve. We can partially order the I curves by saying that the larger curves lie toward larger time. Let O denote the unique I curve passing through the mesh points on $t = 0$ and $t = s$. We remark that these I curves could be replaced by a simpler class of curves. For example, the "polygons" joining six points of the form

$$(-\infty, ns + 0), \quad a_{m,n}, \quad a_{m+1,n+1},$$
$$a_{m+2k+1,n+1}, \quad a_{m+2k+2,n}, \quad (+\infty, ns + 0)$$

would suffice if the subsequent treatment were modified slightly.

4. Bounds for the Difference Equation

We obtain our bounds on the difference solution u by means of functionals. These functionals $F = F(u \mid J) = F(J)$ are functionals of the restriction of u to an I curve J, and they are monotone decreasing as functions of J. According to Section 3, $u \mid J$ consists of various shock and rarefaction waves, and the functionals F will be expressed as functions of these waves. Let α and β be two waves in u which cross J; suppose α is a j-wave and β is a k-wave. When $j \neq k$, we say α and β *approach* (each other) if the wave belonging to the faster family (larger index) lies to the left on J (toward $x = -\infty$). When $j = k$, we say α and β approach if $\alpha \neq \beta$ and if they are not both rarefaction waves. Let $|\alpha|$ be the absolute value of the magnitude of α as defined in Section 2. Let

$$Q(J) = \sum \{|\alpha|\,|\beta| \colon \ \alpha \text{ and } \beta \text{ cross } J \text{ and approach}\},$$
$$L(J) = \sum \{|\alpha| \colon \ \alpha \text{ crosses } J\}.$$

Let $a_{m,n}$ be a mesh point on J. We also define

$$L_{im}(J) = \sum |\alpha| \ ;$$

here the summation is over all j-waves α crossing $J, j \neq i$, which if $j > i$ (respectively $j < i$) lie to the left (respectively right) of $a_{m,n}$ on J. In other words, the sum is over j-waves, $j \neq i$, which approach a (nonexistent) i-wave at $a_{m,n}$. Let

$$F_1(J) = L(J) + K_0 Q(J),$$
$$F_2(J, i, m) = |w_i(u(a_{m,m}))| + \delta_0 L_{im}(J) + \delta_0 K_0 Q(J),$$
$$F_2(J) = \sup \{F(J, i, m) \colon \ 1 \leqq i \leqq N, m \text{ is an integer}\}.$$

Here δ_0 and K_0 are positive constants to be chosen later.

F_1 is equivalent to the total variation norm, and will be used to obtain the estimates (4.4). F_2 dominates the sup norm deviation of u from the constant $\tilde{v}$, and will be used in proving (4.3).

THEOREM 4.1. (A) *There are positive constants* δ_3 *and* K_3 *such that if*

$$\|u(\cdot, 0) - \tilde{v}\|_\infty \leqq \delta_3 , \tag{4.1}$$

$$\textit{total var. } u(\cdot, 0) \leqq \delta_3 , \tag{4.2}$$

then the difference solution u *can be defined for all* x *and all* $t \geqq 0$. *Furthermore,*

$$\|u(\cdot, \cdot) - \tilde{v}\|_\infty \leqq K_3 \|u(\cdot, 0) - \tilde{v}\|_\infty , \tag{4.3}$$

$$\textit{total var. } u(\cdot, t) \leqq K_3 \textit{ total var. } u(\cdot, 0) , \tag{4.4}$$

$$\int |u(x, t_1) - u(x, t_2)| \, dx \leqq K_3(|t_2 - t_1| + 4s) \textit{ total var. } u(\cdot, 0) . \tag{4.5}$$

(B) *Suppose the* w_j *are Riemann invariants. Then* (4.2) *may be replaced by*

$$\|u(\cdot, 0) - \tilde{v}\|_\infty \textit{ total var. } u(\cdot, 0) \leqq \delta_3 . \tag{4.6}$$

Proof: It is sufficient to prove the theorem using the linear structure of the coordinates w_i. In this case $\tilde{v}$ is the origin and

$$\|u(\cdot\,, \cdot) - \tilde{v}\|_\infty - \|u(\cdot\,, \cdot)\|_\infty = \sup_{ixt} |w_i(u(x, t))| .$$

We choose neighborhoods U_4 and U_5 of $\tilde{v}$, $U_3 \supset U_4 \supset U_5$, so that if the v' and v'' of Section 2 have their range in U_4, then the bounds of Section 2 hold with a constant K_1, $K_1 \geqq 1$, and also

$$|\varepsilon_i| \leqq 2 \, |v_r - v_l| , \tag{4.7}$$

$$\|v\|_\infty \leqq 2 \max (|v_l| , |v_r|) . \tag{4.8}$$

We further require that if v_l and v_r are in U_5, then all intermediate states lie in U_4. We suppose U_5 contains a sphere with center $\tilde{v}$. Let δ_1 be its radius. Let

$$\delta_2 = \min \{2^{-12}N^{-2}K_1^{-1} , 2^{-3}\delta_1\} .$$

We choose later our δ_3 in $(0, \delta_2)$. Assume (4.1) and (4.2) or (4.6) with δ_3 replaced by δ_2. Let

$$K_0 = 4NK_1 , \qquad \delta_0 = 2^6K_1 \|u(\cdot, 0)\|_\infty .$$

We can define u up to the I curve O and, for x, t between the curve O and the line $t = 0$, we have

$$|u(x, t)| \leqq 2 \, \|u(\cdot, 0)\|_\infty .$$

With the help of (4.7) and (4.8) we get

$$\begin{aligned}
L(O) &\leqq 2N \text{ total var. } u(\cdot, 0) , \\
F_1(O) &\leqq L(O) + K_0L(O)^2 \leqq 2L(O) \\
&\leqq 4N \text{ total var. } u(\cdot, 0) , \\
\delta_0 L_{im}(O) + \delta_0 K_0 Q(O) &\leqq \delta_0 \, 4N \text{ total var. } u(\cdot, 0) \\
&= 2^8 NK_1 \text{ total var. } u(\cdot, 0) \; \|u(\cdot, 0)\|_\infty \\
&\leqq \|u(\cdot, 0)\|_\infty , \\
F_2(O) &\leqq 3 \, \|u(\cdot, 0)\|_\infty .
\end{aligned}$$

Suppose that u has been defined up to some I curve J_1, that u is in U_5 below J_1 and that

$$F_1(J_1) \leqq 4N\delta_2 ,$$

$$F_2(J_1) \leqq 3\delta_2 .$$

This is true if we take J_1 to be the curve O. Let J_2 be an I curve which is an immediate successor to J_1. Thus J_2 and J_1 pass through the same mesh points except two. In fact for some $(m, n) \in Y$ we have

$$a_{m,n} \in J_1 \sim J_2 ,$$

$$a_{m,n+2} \in J_2 \sim J_1 .$$

Now u can be defined up to J_2 and is in U_4 below J_2. We shall show that $F_i(J_2) \leqq F_i(J_1)$. Then for x, t on J_2,

$$|u(x, t)| \leqq 2F_2(J_2) \leqq \delta_1 .$$

This implies that u is in U_5 below J_2. An induction on J would then prove that u can be defined for all x and $t \geqq 0$ and that

$$\|u(\cdot\,,\cdot)\|_\infty \leqq 2F_2(O) \leqq 6\,\|u(\cdot, 0)\|_\infty ,$$

$$\text{total var. } u(\cdot, t) \leqq F_1(O) \leqq 4N \text{ total var. } u(\cdot, 0) .$$

We show that $F_1(J_2) \leqq F_1(J_1)$. Let ε_j denote the j-wave in u crossing $J_2 \sim J_1$ and let γ_j (respectively δ_j) denote the j-wave in u, if any, crossing $J_1 \sim J_2$ left (respectively right) of $a_{m,n}$. Let

$$D = \sum |\gamma_j|\,|\delta_k| ,$$

where the sum is over the approaching γ's and δ's. This agrees with the terminology of Section 2, and by (2.8) we have

$$\begin{aligned} |\varepsilon_j| &\leqq |\gamma_j| + |\delta_j| + K_1 D , \\ L(J_2) &\leqq L(J_1) + NK_1 D . \end{aligned} \tag{4.9}$$

Let α be a k-wave which crosses $J_1 \cap J_2$ and approaches ε_j. If $k \neq j$ or if α is a shock wave, then α approaches γ_j and δ_j and we use the estimate (4.9). If α is a j-rarefaction wave, then ε_j is a shock wave and α approaches γ_j and δ_j if they are shocks. Let $|\gamma_j^-| = |\gamma_j|$ if γ_j is a shock wave, let it be zero otherwise; we define $|\delta_j^-|$ similarly. By (2.8) we have in this case

$$|\varepsilon_j| \leqq |\gamma_j^-| + |\delta_j^-| + K_1 D ,$$

and so we see that if $|\alpha|\,|\varepsilon_j|$ occurs in $Q(J_2)$, then it is dominated by (terms in $Q(J_1)$) $+ |\alpha|\,K_1 D$.

Since D occurs in $Q(J_1)$ but not in $Q(J_2)$, we have

$$\begin{aligned} Q(J_2) &\leqq Q(J_1) + NL(J_1)K_1D - D\,, \\ F_1(J_2) &\leqq F_1(J_1) + (NK_1D + 4N^2\,\delta_2K_0K_1D - K_0D) \\ &\leqq F_1(J_1) + K_0D(2^{-2} + 2^{-2} - 1) \\ &\leqq F_1(J_1) - 2^{-1}K_0D \leqq F_1(J_1)\,. \end{aligned}$$

We show that $F_2(J_2) \leq F_2(J_1)$. Consider an $m' \neq m$. Evidently ε_j lies on the same side of $a_{m',n'}$ (with respect to J_2) as both γ_j and δ_j do (with respect to J_1). Thus $|\varepsilon_j|$ occurs in $L_{im'}(J_2)$ if and only if $|\gamma_j| + |\delta_j|$ occurs in $L_{im'}(J_1)$. Using (4.9) as above, we see that

$$L_{im'}(J_2) + K_0Q(J_2) \leqq L_{im'}(J_1) + K_0Q(J_1)$$

and

$$F_2(J_2\,, i, m') \leqq F_2(J_1\,, i, m')\,. \tag{4.10}$$

Now we consider $F_2(J_2\,, i, m)$. First suppose that ε_i crosses J_2 on both sides of $a_{m,n+2}$ (and so ε_i is a rarefaction wave). Let b_1 (respectively b_2) be the left (respectively right) edge of ε_i at the point of crossing J_2. Since w_i changes monotonically along J_2 between b_1 and b_2 we have

$$|w_i(u(a_{m,n+2}))| \leqq |w_i(u(b_l))| \tag{4.11}$$

for either $l = 1$ or $l = 2$. Let $c = b_l$ for this choice of l. If ε_i crosses J_2 on one side of $a_{m,n+2}$ only, let $c = a_{m,n+2}$.

We compare $F_2(J_2\,, i, m)$ to $F_2(J_1\,, i, m + \sigma)$ where $\sigma = \pm 1$. We choose $\sigma = +1$ if ε_i crosses on both sides of $a_{m,n+2}$ and if $l = 2$, or if ε_i crosses to the left of $a_{m,n+2}$; we choose $\sigma = -1$ otherwise. Let u_{J_i} denote the restriction of u to J_i. Then

$$\begin{aligned} \|u_{J_2}(\cdot\,,\cdot)\|_\infty &\leqq 2\,\|u_{J_1}(\cdot,\cdot)\|_\infty \leqq 2^2F_2(O) \\ &\leqq 2^4\,\|u(\cdot\,,0)\|_\infty\,. \end{aligned}$$

Using this and (2.15) we see that

$$|w_i(u(c))| \leqq |w_i(u(a_{m+\sigma,n+1}))| + 2^5K_1\,\|u(\cdot,0)\|_\infty \sum|\varepsilon_k| \tag{4.12}$$

and

$$\sum|\varepsilon_k| \leqq \sum(|\gamma_k| + |\delta_k|) + NK_1D\,,$$

where the summation is over all ε_k crossing J_2 between c and $a_{m+\sigma,n+1}$. For such a k, ε_k is not approaching an i-wave at $a_{m,n+2}$ and so $|\varepsilon_k|$ does not occur in $L_{im}(J_2)$. However γ_k and δ_k are approaching a (nonexistent) i-wave at $a_{m+\sigma,n+1}$ and so the terms $|\gamma_k| + |\delta_k|$ occur in $L_{im+\sigma}(J_1)$. By (4.9) we have

$$L_{im}(J_2) \leqq L_{im+\sigma}(J_1) + NK_1D - \sum(|\gamma_k| + |\delta_k|)\,,$$

where the range of summation is as before. It follows that

$$L_{im}(J_2) + K_0Q(J_2) \leqq L_{im+\sigma}(J_1) + K_0Q(J_1) - \sum(|\gamma_k| + |\delta_k|) - 2^{-1}K_0D\,.$$

Combining this with (4.11) and (4.12), we have

$$
\begin{aligned}
F_2(J_2, i, m) &\leqq F_2(J_1, i, m+\sigma) + 2^5 K_1 \|u(\cdot, 0)\|_\infty \sum |\varepsilon_k| \\
&\qquad - \delta_0 \sum (|\gamma_k| + |\delta_k|) - \delta_0 2^{-1} K_0 D \\
&\leqq F_2(J_1, i, m+\sigma) + N K_1 D (2^5 K_1 \|u(\cdot, 0)\|_\infty - 2\delta_0) \\
&\qquad + \sum (|\gamma_k| + |\delta_k|)(2^5 K_1 \|u(\cdot, 0)\|_\infty - \delta_0) .
\end{aligned} \tag{4.13}
$$

From the definition of δ_0 it follows that

$$
F_2(J_2, i, m) \leqq F_2(J_1, i, m+\sigma) ,
$$
$$
F_2(J_2) \leqq F_2(J_1) .
$$

This shows that u can be defined by induction for all x and all $t \geqq 0$ and that (4.3) and (4.4) are satisfied.

If the w_j are Riemann invariants, we change our definition of K_0 and δ_0. Let

$$
K_0 = 2^5 N K_1 \|u(\cdot, 0)\|_\infty ,
$$
$$
\delta_0 = 2^{10} K_1 \|u(\cdot, 0)\|_\infty^2 .
$$

We use (4.6)–(4.8) and obtain as before

$$
\begin{aligned}
F_1(O) &\leqq 4N \text{ total var. } u(\cdot, 0) , \\
\delta_0(L_{im}(O) + K_0 Q(O)) &\leqq \delta_0 F_1(O) \\
&\leqq \|u(\cdot, 0)\|_\infty , \\
F_2(O) &\leqq 3 \|u(\cdot, 0)\|_\infty .
\end{aligned}
$$

We suppose inductively that u is defined and in U_5 below J_1 and that

$$
F_2(J_1) \leqq 4\delta_2 ,
$$
$$
F_1(J_1) F_2(J_1) \leqq 2^4 N \delta_2 .
$$

As before, we need only show that F_i is nonincreasing. We replace (4.9) by

$$
\begin{aligned}
|\varepsilon_j| &\leqq |\gamma_j| + |\delta_j| + 2 \|u_{J_1}\|_\infty K_1 D \\
&\leqq |\gamma_j| + |\delta_j| + 2^4 \|u(\cdot, 0)\|_\infty K_1 D
\end{aligned}
$$

and we obtain

$$
\begin{aligned}
L(J_2) &\leqq L(J_1) + 2^4 N \|u(\cdot, 0)\|_\infty K_1 D , \\
Q(J_2) &\leqq Q(J_1) + 2^4 N \|u(\cdot, 0)\|_\infty L(J_1) K_1 D - D , \\
F_1(J_2) &\leqq F_1(J_1) + 2^4 N \|u(\cdot, 0)\|_\infty K_1 D (1 + K_0 L(J_1)) - K_0 D \\
&\leqq F_1(J_1) + K_0 D (2^4 N \|u(\cdot, 0)\|_\infty K_1 L(J_1) - 2^{-1}) \\
&\leqq F_1(J_1) + K_0 D (2^6 N^2 K_1 \delta_2 - 2^{-1}) \\
&\leqq F_1(J_1) - 2^{-2} K_0 D .
\end{aligned}
$$

We consider F_2. Formula (4.10) follows as before. We replace (4.12) and (4.13) by

$$|w_i(c)| \leqq |w_i(u(a_{m+\sigma,n+1}))| + 2^{10}K_1 \|u(\cdot, 0)\|_\infty^2 \sum |\varepsilon_k|$$

and

$$\begin{aligned} F_2(J_2, i, m) \leqq & F_2(J_1, i, m+\sigma) \\ & + 2^{10}K_1 \|u(\cdot, 0)\|_\infty^2 \sum |\varepsilon_k| - \delta_0 \sum (|\gamma_k| + |\delta_k|) - 2^{-2}\delta_0 K_0 D \\ \leqq & F_2(J_1, i, m+\sigma) + NDK_1(2^{10}K_1 \|u(\cdot, 0)\|_\infty^2 - 2^3\delta_0) \\ & + \sum (|\gamma_k| + |\delta_k|)(2^{10}K_1 \|u(\cdot, 0)\|_\infty^2 - \delta_0) . \end{aligned}$$

It follows that, if (4.6) holds, μ can be defined for all x and $t \geqq 0$ when the w_j are Riemann invariants, and in this case (4.3) and (4.4) are satisfied.

We prove (4.5). Choose $\delta_3 < 2^{-4}N^{-1}\delta_2$ and so that the estimates (4.3) and (4.4) remain valid uniformly in $\tilde{v}$ if the $\tilde{v}$ of Theorem 4.1 is allowed to vary in a disk of radius $2^3N\delta_3$. This is possible because K_1 and U_5 can be chosen uniformly in $\tilde{v}$ as $\tilde{v}$ varies in a small neighborhood. We assume (4.1) and (4.2) or (4.6) without replacing δ_3 by δ_2 now. We suppose without loss of generality that $t_2 > t_1$. Let

$$t_0 = \sup\{t\colon\ t \leqq t_1, t = ns\},$$

$$S = [(t_2 - t_0)/s] + 1 .$$

Then $sS \leqq t_2 - t_1 + 2s$. For any x, $u(x, t_i)$ is determined entirely by the Cauchy data

$$u(y, t_0), \qquad y \in [x - Sr, x + Sr] .$$

This Cauchy data satisfy (4.1) and (4.2) or (4.6) with δ_3 replaced by δ_2 and $\tilde{v}$ replaced by $u(x, t_0)$. By the part of the theorem already proved, we get

$$|u(x, t_2) - u(x, t_1)| \leqq K_2 \sup\{|u(y, t_0) - u(x, t_0)|\colon\ x - Sr \leqq y \leqq x + Sr\}$$

for some constant K_2. Hence

$$|u(x, t_2) - u(x, t_1)| \leqq K_2 \text{ total var. } \big(u(\cdot, t_0) \,\big|\, [x - Sr, x + Sr]\big) ,$$

where $u \mid B$ is the restriction of u to the set B. Thus the integral of $|u(x, t_2) - u(x, t_1)|$ over x in one mesh interval $[mr, m + 2r]$ in the line $t = t_0$ is bounded by

$$2rK_2 \text{ total var. } \big(u(\cdot, t_0) \,\big|\, [(m - S)r, (m + 2 + S)r]\big) .$$

It follows that

$$\begin{aligned} \int_{-\infty}^{\infty} |u(x, t_2) - u(x, t_1)|\, dx & \leqq 2rK_2(2S + 2) \text{ total var. } u(\cdot, t_0) \\ & \leqq 8N(r/s)K_2(2S + 2)s \text{ total var. } u(\cdot, 0) \\ & \leqq K_3(|t_2 - t_1| + 4s) \text{ total var. } u(\cdot, 0) . \end{aligned}$$

This completes the proof of Theorem 4.1.

5. Convergence of the Approximate Solutions

Let $\psi(x)$ be the initial value of our desired solution v of (1.1). We assume ψ is bounded and has bounded total variation. For each mesh length r let ψ_r be defined by the equation

$$\psi_r(x) = \psi(mr)\,, \qquad (m-1)r \leqq x < (m+1)r\,, \quad m \text{ even}\,.$$

Then ψ_r is bounded and has bounded total variation with the same bounds as ψ and $\psi_r \to \psi$ at points of continuity of ψ. Hence $\psi_r \to \psi$ a.e. and $\psi_r \to \psi$ in L_1 on bounded intervals. Let $\tilde{v} = \psi(0)$; δ_3 is the δ of Theorem 1.1. We assume the hypothesis of Theorem 1.1 with this δ and we conclude from Theorem 4.1 that u is defined for all $t \geqq 0$. The difference scheme, and thus u, depend on the mesh length r and the choice of a in A. We express this by writing $u = u_{ra}$. Let ϕ be a C^2 function with compact support, and let

$$\mathfrak{d}(r, a, \phi) = \int_0^\infty \int_{-\infty}^\infty (\phi_t u_{ra} + \phi_x f(u_{ra}))\, dx\, dt + \int_{-\infty}^\infty \phi(x, 0)\psi_r(x)\, dx\,.$$

Since u is a (weak) solution of the equation in each horizontal strip $ns < t < (n+1)s$, we can compute

$$\mathfrak{d}(r, a, \phi) = \sum_{n=1}^\infty \int_{-\infty}^\infty \phi(x, ns)(u(x, ns) - u(x, ns-0))\, dx\,. \tag{5.1}$$

Our measure space A depends on r; however there is an obvious isomorphism

$$A \approx \prod [0, 1]$$

of A with an infinite product of copies of the interval $[0, 1]$ given by the isomorphism

$$a_{m,n} \to 2^{-1}((\text{first component } a_{m,n})r^{-1} - m + 1)$$

in each factor. By means of this isomorphism we regard the functions $\mathfrak{d}$ as defined on a fixed probability space. In the next lemma we choose a null set N in A, independently of r. This means that, under the above isomorphism, N is mapped onto a null set in $\Pi[0, 1]$ which is independent of r. This result can be regarded as the weak law of large numbers in the present context.

Lemma 5.1. *There is a null set $N \subset A$ and a sequence $r_i \to 0$ such that for any $a \in A \sim N$ and any test function ϕ, we have $\mathfrak{d}(r_i, a, \phi) \to 0$.*

Proof: Let $\mathfrak{d}(r, a, \varphi, n)$ denote a summand in (5.1). We regard this summand as a function of a (i.e., a random variable). We assert that, for any continuous φ,

$$\|\mathfrak{d}(r, \cdot, \varphi, n)\| \leqq Kr\, \|\varphi\|_\infty \text{ (total variation } \psi). \tag{5.2}$$

Now suppose φ has compact support and is piecewise linear and is linear in each of the triangles bounded by

$$((m-1)r, ns)\,, \qquad ((m+1)r, ns)\,, \qquad (mr, (n+1)s)\,, \qquad m+n \text{ even}\,.$$

We consider from now on only r of the form $r = 2^{-k}$. In this case our piecewise linear φ has the required form for all smaller r also, and we assert further that

$$\mathfrak{d}(r, \cdot, \varphi, n_1) \perp \mathfrak{d}(r, \cdot, \varphi, n_2)\,, \qquad n_1 \neq n_2\,, \tag{5.3}$$

$$\|\mathfrak{d}(r, \cdot, \varphi)\|_2^2 \to 0\,. \tag{5.4}$$

The inequality (5.2) follows from the inequality

$$\sum \int_{(m-1)r}^{(m+1)r} \varphi(x, ns)(u(x, ns) - u(x, ns - 0))\, dx$$
$$\leqq r\, \|\varphi\|_\infty \text{ (total variation } u(\cdot, ns - 0))$$

and from (4.4). The orthogonality in (5.3) is with respect to $L_2(A)$. To prove this suppose $n_1 < n_2$ and let $\hat{A}$, $d\hat{a}$ be the measure space product with a factor corresponding to m_2, n_2 omitted. Let

$$u(x, n) = u(x, ns) - u(x, ns - 0)\,.$$

The inner product of $\mathfrak{d}(r, \cdot, \varphi, n_1)$ and $\mathfrak{d}(r, \cdot, \varphi, n_2)$ is a sum of terms of the form

$$\int_{\hat{A}}\int \left| \int_{(m_2-1)r}^{(m_2+1)r} \varphi(x, n_2 s)\, \Delta u(x, n_2)\, dx \right| \left| \int_{-\infty}^{\infty} \varphi(x, n_1 s)\, \Delta u(x, n_1)\, dx \right| da_{m_2 n_2}\, d\hat{a}\,. \tag{5.5}$$

However φ is constant on $[(m_2 - 1)r, (m_2 + 1)r] \times \{n_2 s\}$ and

$$C = \int_{-\infty}^{\infty} \varphi(x, n_1 s)\, \Delta u(x, n_1)\, dx$$

is independent of $a_{m_2 n_2}$. Hence (5.5) is equal to

$$\int_{\hat{A}} C\varphi(m_2 r, n_2 s) \left[\iint_{(m_2-1)r}^{(m_2+1)r} \Delta u(x, n_2)\, dx\, da_{m_2 n_2} \right] d\hat{a}\,.$$

By our definitions,

$$\iint \Delta u(x, n_2)\, dx\, da_{m_2 n_2}$$
$$= \iint u(a_{m_2 n_2}, ns - 0) - u(x, ns - 0)\, dx\, da_{m_2 n_2}$$
$$= 0\,,$$

which proves (5.3). Also

$$\begin{aligned} \|\mathfrak{d}(r, \cdot, \varphi)\|_2^2 &= \sum_n \|\mathfrak{d}(r, \cdot, \varphi, n)\|_2^2 \\ &\leqq \sum_n \|\mathfrak{d}(r, \cdot, \varphi, n)\|_\infty^2\,. \end{aligned} \tag{5.6}$$

Since φ has compact support, there are $O(r^{-1})$ nonzero terms in this sum. Thus (5.6) follows from (5.2). In the same manner we obtain

$$\|\mathfrak{d}(r, \cdot, \varphi)\| \leqq \text{const. } \|\varphi\|_\infty . \tag{5.7}$$

For each piecewise linear ϕ there is a sequence $r_i \to 0$ such that $\mathfrak{d}(r_i, \cdot, \phi) \to 0$ a.e. By the diagonal process we can achieve this for a sequence $\phi_1, \phi_2, \cdots$ of the ϕ. We choose the sequence to be dense; let N be the null subset of A such that

$$\mathfrak{d}(r_i, \cdot, \phi_k) \to 0 \quad \text{on} \quad A \sim N \quad \text{as} \quad i \to \infty , \qquad k = 1, 2, \cdots .$$

We apply (5.2) with ϕ replaced by $\phi - \phi_k$ and conclude that $\mathfrak{d}(r_i, \cdot, \phi) \to 0$ on $A \sim N$ for any ϕ. This proves Lemma 5.1.

For any a in $A \sim N$ let $u^i = u_{r_i a}$. By Theorem 4.1, u^i is bounded and has bounded variation on horizontal lines, uniformly in i. By Helly's theorem a subsequence of the u^i converges in L_1 on bounded intervals of any given horizontal line. By the diagonal process we can achieve the same result for a countable number of horizontal lines, located at (say) rational times $t = p/q$. For an arbitrary time t, we have (by (4.5))

$$\begin{aligned}\int_{|x| \leqq M} |u^i(x, t) - u^j(x, t)|\, dx &\leqq \int_{|x| \leqq M} |u^i(x, t) - u^i(x, p/q)|\, dx \\ &+ \int_{|x| \leqq M} |u^i(x, p/q) - u^j(x, p/q)|\, dx + \int_{|x| \leqq M} |u^j(x, p/q) - u^j(x, t)|\, dx \\ &\leqq \text{const. } (|t - (p/q)| + r_i + r_j) + \int_{|x| \leqq M} |u^i(x, p/q) - u^j(x, p/q)|\, dx .\end{aligned}$$

Thus the subsequence $u^{i(j)}$ converges (uniformly for bounded t) on the intervals $|x| \leqq M$ of any horizontal line.[2] Therefore,

$$\lim_j f(u^{i(j)}) = f(v) \text{ (in } L_1 \text{ on bounded sets)}$$

and so

$$\int_0^\infty \int_{-\infty}^\infty \phi_t v + \phi_x f(v)\, dx\, dt + \int_{-\infty}^\infty \phi(x, 0) \psi(x)\, dx = \lim_{j \to 0} \mathfrak{d}'(r_{i(j)}, a, \phi) = 0 .$$

Hence v is a weak solution of equation (1.1). The bounds on v in Theorem 1.1 follow from those on u given by Theorem 4.1.

We remark that since f is nonlinear, it need not be continuous with respect to weak limits. Thus the technique from linear equations in which one first takes a weak L_2 limit of approximate solutions and then proves that the limit has the required derivatives does not seem to be applicable here. In fact, it appears that such a weak limit need not be a weak solution. Our approximate solutions converge in the strong L_2 topology on bounded sets.

[2] This compactness argument was used by Oleinik [6]. I am grateful to her for finding an error in my original proof at this point.

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