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GLIMM'S SCHEME FOR SYSTEMS WITH ALMOST-PLANAR INTERACTIONS

Steven Schochet

School of Mathematical Sciences
Raymond and Beverly Sackler Faculty of Exact Sciences
Tel Aviv University
Ramat Aviv 69978 Israel

I. Introduction

Consider a system of nonlinear conservation laws

$$u_t + f(u)_x = 0 \tag{1.1}$$

such that, in a neighborhood of some $u^{\sim}$,

(1.2) f is sufficiently smooth, the system is strictly-hyperbolic, and each field is either genuinely nonlinear or linearly degenerate .

(For definitions of these terms see e.g. [3].) Let $\sigma_j(u)$ be the jth eigenvalue of $f_u(u)$, let $r_j(u)$ denote the corresponding right eigenvector, and define

$$R_j = r_j \cdot \nabla_u \ . \tag{1.3}$$

Glimm's [2] existence theorem says that if the commutators

$$[R_i(u), R_j(u)] = 0 \quad \text{for all } i \text{ and } j \text{ and all } u \text{ in a neighborhood of } u^\sim \tag{1.4}$$

then system (1.1) with initial data

$$u(0,x) = u_0(x) \tag{1.5}$$

has a weak solution for all time provided that

$$\|u_0(x) - u^\sim\|_{L^\infty}\{1 + TV(u_0)\} \le \varepsilon_1 \ , \tag{1.6}$$

while if condition (1.4) does not hold then such a weak solution exists under the stronger condition

$$\|u_0(x) - u^\sim\|_{L^\infty} + TV(u_0) \le \varepsilon_2 \ , \tag{1.7}$$

where the ε_i depend only on f and $u^\sim$.

From the proof of this theorem in [2] it is not too hard to see that it suffices for (1.4) to hold just for $u = u^\sim$; i.e. (1.7) may be replaced by (1.6) whenever

$$[R_i(u), R_j(u)]\Big|_{u=u^\sim} = 0 \quad \text{for all } i \text{ and } j \ . \tag{1.8}$$

It will be shown here that (1.8) can be further relaxed to the requirement that

$$C_{ij}^k(u^\sim) = 0 \quad \text{for } i, j, \text{ and } k \text{ all distinct} \ , \tag{1.9}$$

where

$$[R_i(u), R_j(u)] = \sum_k C_{ij}^k(u) R_k(u) \ . \tag{1.10}$$

Assuming that $f_u(u^\sim)$ is diagonal, which can always be arranged by a linear change of the dependent variables, condition (1.9) (unlike (1.8)) can be expressed simply in terms of f as

$$\frac{\partial^2 f^i}{\partial u_j \partial u_k}\bigg|_{u=u^\sim} = 0 \text{ for } i, j, k \text{ distinct} \ . \tag{1.11}$$

It was shown in [2] that the wave-strength vector n resulting from the interaction of two wave-sets α and β is given by

$$\varepsilon_k = \alpha_k + \beta_k + \sum_{i>j} C_{ij}^k(u_M)\alpha_i\beta_j + 0(|\alpha| + |\beta|)D(\alpha,\beta) \ , \tag{1.12}$$

where the coefficients C_{ij}^k are defined in (1.10), u_M is the state separating the wave-sets α and β, and

$$D(\alpha,\beta) = \sum_{\alpha_j \to \beta_k} |\alpha_j| \, |\beta_k| \ , \tag{1.13}$$

where $\alpha_j \to \beta_k$ indicates that the waves α_j and β_k approach each other, i.e. would eventually collide if only those two waves were present. Thus, condition (1.9) amounts to the assumption that the new waves produced by the interaction of two given waves have strengths that are third order in the strengths of the original waves. However, in contrast to (1.4) or (1.8), condition (1.9) allows the interaction to produce changes in the strengths of the interacting waves themselves that are of second order. Since the largest effect of the interaction of two individual waves then occurs in the plane of their original wave-strengths, the interaction may be called almost-planar. Although (1.9) limits the interaction coefficients C_{ij}^k only at the point $u^\sim$, at nearby points u the nonplanar interactions are still third order in the quantities α_i, β_j, and $|u_M - u^\sim|$.

The sufficiency of condition (1.9) will be proven by reduction to the case when (1.4) or (1.8) holds; specifically, we will define modified wave-strengths $\varepsilon^\wedge$ in such a way that (1.12) becomes

$$\varepsilon_k^\wedge = \alpha_k^\wedge + \beta_k^\wedge + \sum_{\substack{i>j \\ i,j,k \\ \text{distinct}}} C_{ij}^k(u^\sim)\alpha_i^\wedge\beta_j^\wedge + 0(\|u - u^\sim\|)D(\alpha^\wedge,\beta^\wedge) \ , \tag{1.14}$$

where here and later $\| \ \|$ denotes the L^∞ norm. Hence when (1.9) holds then $\varepsilon^\wedge = \alpha^\wedge + \beta^\wedge$ up to terms of third order, which is the essential ingredient of Glimm's proof of BV and L^∞ bounds under assumption (1.4). By the further results of [2] these bounds suffice to show global existence.

Condition (1.4) holds for all 2×2 systems if the normalizations of the eigenvectors r_j are chosen properly, because of the existence of a coordinate system of

Riemann invariants. Hence the existence result obtained here is nontrivial only for systems of at least 3 equations. It would be nice if one could apply this existence result to the best-known system of three equations, the nonisentropic Euler equations of fluid dynamics. For this system fields 1 and 3 are genuinely nonlinear, while field 2 is linearly degenerate. Now $C^2_{31} \equiv 0$ but unfortunately C^1_{32} and C^3_{21} never vanish. However, using the fact that C^k_{ij} is nonzero only when one of the interacting waves is from the 2-field, together with the fact that the entropy S is a k-Riemann invariant for both of the other fields, we will show global existence for the Euler equations assuming that

$$TV(S(0,x)) \leq \varepsilon_3 \tag{1.15}$$

$$\|u_0(x) - u^{\sim}\|\{1 + TV(u_0)^2\} \leq \varepsilon_4 \ . \tag{1.16}$$

This result is thus a hybrid of the two types mentioned above, since the 2-field is required to satisfy an estimate like (1.7), but the other fields need only satisfy a slightly strengthened version of (1.6). Condition (1.15) was also required in previous global existence results for the Euler equations [4,6], which require in addition that the equation of state be sufficiently close to that of isothermal flow, in place of our requirement in (1.16) that the oscillation of the initial data be small.

The reader will be presumed to be familiar with the basic properties of conservation laws [3], Glimm's scheme [2], and, for the last section, certain facts about the Euler equations [1]; most of the necessary background on these topics may also be found in [5].

II. The Existence Theorem

Theorem 2.1. *Assume that system (1.1) satisfies (1.2) and (1.9). Let the constant* ε_i *be sufficiently small. If the initial data (1.5) satisfies (1.6) then there exists for all time a weak solution of (1.1,1.5).*

Proof: By the results of [2], it suffices to show that the Glimm approximants satisfy a uniform TV bound and a sufficiently small uniform sup-norm bound. Define the difference scheme, mesh curves, and diamonds as in [2]. In [2] the wave-strengths are defined as follows: For each field j, let $w_j(u)$ be a function satisfying the differential equation

$$R_j w_j \equiv 1 \tag{2.1}$$

together with the partially specified initial conditions

$$w_j(u^\sim) = 0 \tag{2.2}$$

$$R_k w_j \Big|_{u=u^\sim} = 0 \quad \text{for } k \neq j\ ; \tag{2.3}$$

the strength δ_j of a j-wave separating u_L on the left from u_R on the right is then given by

$$\delta_j = w_j(u_R) - w_j(u_L)\ . \tag{2.4}$$

In this article we will let w_j denote the particular function determined by (2.1) together with the initial condition

$$w_j(u) = 0 \quad \text{for } u \in \{u | u = T_\delta(u^\sim) \quad \text{with } \delta_j = 0\}\ , \tag{2.5}$$

where

$$T_\delta(u) = T^n_{\delta_n}(\ldots(T^1_{\delta_1}(u))\ldots) \tag{2.6}$$

and $T^k_{\delta_k}(u)$ is the right state of a k-wave of strength δ_k whose left state is u. Note that the set in (2.5) is independent of the way the wave-strengths are measured. Furthermore, the initial surface defined by (2.5) is noncharacteristic for (2.1) in a neighborhood of $u^\sim$, and (2.5) implies (2.3), since [3]

$$\frac{d}{d\delta_k} T^k_{\delta_k}(u)\Big|_{\delta_k=0} = r_k(u)\ . \tag{2.7}$$

By the proof of theorem 4.1 of [2], the desired BV and L^∞ bounds would hold if the functions w_j satisfied

$$w_j(T^k_{\varepsilon_k}(u)) - w_j(u) = \varepsilon_k\{\delta_{jk} + O(\|u - u^\sim\|^2)\} \tag{2.8}$$

and the wave-vector ε produced by the interaction of waves α and β satisfied

$$\varepsilon = \alpha + \beta + O(\|u - u^{\sim}\|)D(\alpha,\beta) , \tag{2.9}$$

where the sup in the L^{∞} norm is taken over all states present before the interaction. Equations (2.8-9) are essentially the same as [2; 2.16, 2.9]; the fact that in [2] the remainder terms are $0(|\text{wave-strengths}|^{\text{power}})$ instead of the $0(\|u - u^{\sim}\|^{\text{power}})$ occurring here is not important since in the proof of theorem 4.1 of [2] the former is simply estimated by the latter. In order to prove our theorem it therefore suffices to define modified functions and wave-strengths that satisfy (2.8-9), while maintaining the connection (2.4) between the two. Besides properties (2.4,2.8,2.9) the proof of theorem 4.1 in [2] also uses the fact that the wave- strengths δ_k satisfy

$$\delta_k \begin{matrix} < 0 \\ \\ > 0 \end{matrix} \quad \text{for} \quad \begin{matrix} k\text{-shocks} \\ \\ k\text{-rarefactions} \end{matrix} \tag{2.10}$$

However, (2.8) implies that $\delta_k = 0$ for the the trivial k-wave with right state equal to left state, and (2.9) ensures that δ_k is monotonic along the k-shock curve for small values of δ_k, so that, after multiplying w_k by -1 if necessary, (2.10) indeed holds. (Condition (2.10) also follows immediately from estimate (2.24) below.)

By a theorem of Lax [3], for any state u near $u^{\sim}$ there is a unique small wave-vector $\mu(u)$ such that $T_{\mu(u)}(u^{\sim}) = u$. Define the function w_j^L by

$$w_j^L(u) = \mu_j(u) . \tag{2.11}$$

In similar fashion, there is a unique wave-vector $\nu(u)$ such that $T_{\nu(u)}(u) = u^{\sim}$. Define $w^R(u)$ by

$$w_j^R(u) = -\nu_j(u) . \tag{2.12}$$

Finally, define

$$w_j^{\wedge}(u) = w^L(u) + w^R(u) - w(u) \tag{2.13}$$

and, as in (2.4), define the strength of a j-wave joining u on the left to v on the right by

$$\alpha_j^{\wedge}(u) = w_j^{\wedge}(v) - w_j^{\wedge}(u) . \tag{2.14}$$

The point of (2.11-14) is that in a diamond in which α and β interact to form ε the left state of the wave-vector β is shifted by an amount $O(|\alpha|)$ from the common left state of α and ε, whereas the right state of α is shifted by an amount $O(|\beta|)$ from the common right state of β and ε. Hence (2.11-2.14) create additional second-order terms, which it turns out exactly cancel those terms in (1.12) due to $C^p_{rs}(u^\sim)$ with r or s equal to p.

As noted above, the following proposition completes the proof of Theorem 2.1.

Proposition 2.2. *If the assumptions of Theorem 2.1 hold, except possibly (1.9), then $\varepsilon^\wedge$ satisfies (1.14) and $w^\wedge$ satisfies*

$$w^\wedge_j(T^k_{\varepsilon_k}(u)) - w^\wedge_j(u) =$$
$$\varepsilon^\wedge_k(\delta_{jk} + \sum_{\substack{p<k \\ p,k,j \\ \text{distinct}}} C^j_{pk}\mu(u)_p + O(\|u-u^\sim\|^2)\ .$$

Hence if (1.9) holds then $w^\wedge$ and $\varepsilon^\wedge$ satisfy (2.8-9), i.e.

$$w^\wedge_j(T^k_{\varepsilon k}(u)) - w^\wedge_j(u) = \varepsilon^\wedge_k\{\delta_{jk} + O(\|u-u^2\|^2)\} \tag{2.16}$$

$$\varepsilon^\wedge = \alpha^\wedge + \beta^\wedge + O(\|u-u^\sim\|)D(\alpha^\wedge,\beta^\wedge)\ . \tag{2.17}$$

Remark. One could obtain a modified version of (2.15), with the sum over $p<k$ replaced by the sum over $p>k$, if one replaced w_j, defined via (2.1,2.5), by the function $w^\#_j$ defined via (2.1) with the initial condition of being zero for left states reachable from the right state $u^\sim$ by a wave-vector with no j-wave. Similarly, one could obtain a symmetric form of (2.15), with the sum over $p<k$ replaced by half the sum over $p\neq k$, by replacing w_j by $\frac{1}{2}w_j + \frac{1}{2}w^\#_j$. The proof of these modified versions of (2.15) is completely analogous to that of the original (2.15) given below. In contrast, the unsymmetric form of (1.14) is essential and is due to the fact that the waves α_i and β_j interact only if $i>j$.

Proof: Define, for a j-wave joining u on the left to v on the right,

$$\alpha^L_j = w^L_j(v) - w^L_j(u) \tag{2.18}$$

$$\alpha_j^R = w_j^R(v) - w_j^R(u) \ , \tag{2.19}$$

so that

$$\alpha^\wedge = \alpha^L + \alpha^R - \alpha \ . \tag{2.20}$$

Now by (1.12),

$$\varepsilon_p = \alpha_p + \beta_p + \sum_{r>s} C_{rs}^p \alpha_r \beta_s + O(\|u - u^\sim\|) D(\alpha, \beta) \ , \tag{2.21}$$

where here and later $C_{rs}^p \equiv C_{rs}^p(u^\sim)$. In order to obtain (1.14) it therefore suffices to show that

$$\varepsilon_p^L = \alpha_p^L + \beta_p^L + \sum_{\substack{r>s \\ s \neq p}} C_{rs}^p \alpha_r \beta_s + O(\|u - u^\sim\|) D(\alpha, \beta) \tag{2.22}$$

$$\varepsilon_p^R = \alpha_p^R + \beta_p^R + \sum_{\substack{r>s \\ r \neq p}} C_{rs}^p \alpha_r \beta_s + O(\|u - u^\sim\|) D(\alpha, \beta) \tag{2.23}$$

$$\delta_p^\wedge = \delta_p(1 + O(\|u - u^\sim\|) \qquad \text{for any wave} \quad \delta_p \ , \tag{2.24}$$

since combining (2.22)+(2.23)-(2.21) and using (2.24) to replace $D(\alpha, \beta)$ by $D(\alpha^\wedge, \beta^\wedge)$ yields (1.14), which in turn reduces to (2.17) under assumption (1.9).

We now turn to proving (2.22). Suppose that the wave-vector α separates states u_L and u_M, and the wave-vector β separates states u_M and u_R. Let ε be the wave-vector separating states u_L and u_R, i.e ε is the wave-vector resulting from the interaction of α and β. Define $u_0 = u_L$, u_j =the state on the right side of α_j, $v_0 = u_M = u_n$, v_j =the state on the right of β_j, $z_0 = u_L$, z_j =the state on the right of ε_j. Then by (2.18),

$$\alpha_p^L = \mu_p(u_p) - \mu_p(u_{p-1}) \tag{2.25}$$

$$\beta_p^L = \mu_p(v_p) - \mu_p(v_{p-1}) \tag{2.26}$$

$$\varepsilon_p^L = \mu_p(z_p) - \mu_p(z_{p-1}) \ . \tag{2.27}$$

Since $\mu(u_j)$ is the wave-vector resulting from the interaction of $\mu(u_L)$ and the portion of α up to and including α_j, we find from (2.21) that

$$\mu_p(u_p) = \mu_p(u_L) + \alpha_p + \sum_{\substack{r>s \\ s\leq p}} C^p_{rs}\mu_r(u_L)\alpha_s + O(|\alpha|\;\|u - u^\sim\|^2) \tag{2.28}$$

$$\mu_p(u_{p-1}) = \mu_p(u_L) + \sum_{\substack{r>s \\ s\leq p-1}} C^p_{rs}\mu_r(u_L)\alpha_s + O(|\alpha|\;\|u - u^\sim\|^2)\ , \tag{2.29}$$

and hence by (2.25) that

$$\alpha^L_p = \alpha_p + \sum_{r>p} C^p_{rp}\mu_r(u_L)\alpha_p + O(|\alpha|\;\|u - u^\sim\|^2)\ . \tag{2.30}$$

In particular, if we consider the case when there is only one wave in α then we obtain (2.24), since a formula analogous to (2.30) holds for α^R. Similarly,

$$\varepsilon^L_p = \varepsilon_p + \sum_{r>p} C^p_{rp}\mu_r(u_L)\varepsilon_p + O(|\varepsilon|\|u - u^\sim\|^2)\ . \tag{2.31}$$

$$\beta^L_p = \beta_p + \sum_{r>p} C^p_{rp}\mu_r(u_M)\beta_p + O(|\beta|\;\|u - u^\sim\|^2)\ . \tag{2.32}$$

But $\mu(u_M)$ is the wave-vector resulting from the interaction of $\mu(U_L)$ and α, so

$$\mu_r(u_M) = \mu_r(u_L) + \alpha_r + O(|\alpha|\;\|u - u^\sim\|)\ . \tag{2.33}$$

Combining (2.31)–(2.30)–(2.32) and substituting (2.21) and (2.33) in the result yields (2.22) except that the error term is $O(\{|\alpha| + |\beta|\}\|u - u^\sim\|^2)$ instead of $O(\|u - u^\sim\|D(\alpha,\beta))$. However, since $\varepsilon^\wedge = \alpha^\wedge + \beta^\wedge$ when $D(\alpha,\beta) = 0$, the inductive proof from [2] shows that the remainder actually does have the desired form. Since the proof of (2.23) is completely analogous, this proves (2.17).

Finally, we will prove (2.15). Since (2.15) holds by definition (2.14) when $j = k$, it suffices to show it for $j \neq k$. Let α_k be a k-wave having left state u and right state v. Then (2.15) follows from the following estimates: For $j \neq k$,

$$\begin{aligned} w_j(v) = w_j(u) = &\sum_{\substack{p>k \\ p\neq j}} C^j_{pk}\mu(u)_p\alpha_k \\ &+ C^j_{jk}\mu(u)_j\alpha_k + O(|\alpha_k|\;\|u - u^\sim\|^2) \end{aligned} \tag{2.34}$$

$$w_j^L(v) - w_j^L(u) \equiv \mu_j(v) - \mu_j(u) \tag{2.35}$$
$$= \sum_{p>k} C^j_{pk}\mu(u)_p\alpha_k \; + O(|\alpha_k|\ \|u - u^{\sim}\|^2)$$

$$w_j^R(v) - w_j^R(u) \equiv \nu_j(u) - \nu_j(v) \tag{2.36}$$
$$= \sum_{p<k} C^j_{kp}\alpha_k\nu(v)_p + O(|\alpha_k|\ \|u - u^{\sim}\|^2)$$

$$\nu_j(v) = \nu_j(u) + O(\|u - u^{\sim}\|^2) = -\mu_j(u) + O(\|u - u^{\sim}\|^2) \tag{2.37}$$

$$C^j_{kp} = -C^j_{pk}\ . \tag{2.38}$$

I.e., substituting (2.37–2.38) into (2.35)+(3.36)–(2.34) yields (2.15). Estimates (2.35-2.36) are immediate consequences of interaction estimate (2.21), and (2.37) also follows from (2.21) since the interaction of α_k and $\nu(v)$ produces $\nu(u)$, while the interaction of $\mu(u)$ and $\nu(u)$ produces the zero wave-set. Equation (2.38) is simply a statement of the anti-symmetry of the commutators (1.10). To prove (2.34), we will expand $w_j(v) - w_j(u)$ in powers of α_k using (2.7) and the fact [3] that

$$\frac{d^2}{d\delta_k^2}T^k_{\delta_k}(u)\Big|_{\delta_k=0} = R_k(u)r_k(u)\ , \tag{2.39}$$

then expand u in powers of $\mu(u)$ and substitute (1.10,2.1-5) into the result. Thus,

$$\begin{aligned}
w_j(v) - w_j(u) &= w_j(T^k_{\alpha_k}(u)) - w_j(u) = \alpha_k\nabla w_j(u)\cdot r_k(u)\\
&\quad + \frac{1}{2}\alpha_k^2\{[D^2w_j(u)](r_k(u), r_k(u)) + \nabla w_j(u)\cdot(R_k(u)r_k(u))\} + O(|\alpha_k|^3)\\
&= \alpha_k\{R_k(u)w_j(u)\big|_{u=u^{\sim}} + \sum_p \mu_p(u)R_p(u)R_k(u)w_j(u)\big|_{u=u^{\sim}}\}\\
&\quad + \frac{1}{2}\alpha_k^2 R_k(u)R_k(u)w_j(u)\big|_{u=u^{\sim}} + O(|\alpha_k|\ \|u - u^{\sim}\|^2)\\
&= \alpha_k\{0 + \mu_j(u)R_j(u)R_k(u)w_j(u)\big|_{u=u^{\sim}}\\
&\quad + \sum_{\substack{p\neq j\\ p\geq k}} \mu_p(u) * R_p(u)R_k(u)w_j(u)\big|_{u=u^{\sim}}\\
&\quad + \sum_{\substack{p\neq j\\ p<k}} \mu_p(u) * R_k(u)R_p(u)w_j(u)\big|_{u=u^{\sim}}
\end{aligned}$$

$$\begin{aligned}
&+ \sum_{\substack{p\neq j\\ p>k}} \mu_p(u)[R_p(u), R_k(u)]w_j(u)\big|_{u=u^\sim}\}\\
&+ w_j(T^k_{\alpha_k}(u^\sim)) + O(|\alpha_k|\ \|u-u^\sim\|^2)
\end{aligned} \tag{2.40}$$

$$\begin{aligned}
&= \alpha_k\mu_j(u)\{R_k(u)R_j(u)w_j(u)\big|_{u=u^\sim} + [R_j(u), R_k(u)]w_j(u)\big|_{u=u^\sim}\}\\
&\quad + \sum_{\substack{p\neq j\\ p\geq k}} w_j(T^p_{\mu_p(u)}(T^k_{\alpha_k}(u^\sim)))\\
&\quad + \sum_{\substack{p\neq j\\ p<k}} w_j(T^k_{\alpha_k}(T^p_{\mu_p(u)}(u^\sim)))\\
&\quad + \alpha_k \sum_{\substack{p\neq j\\ p>k}} \mu_p(u)\sum_r C^r_{pk}R_r(u)w_j(u)\big|_{u=u^\sim} + 0 + O(|\alpha_k|\ \|u-u^\sim\|^2)\\
&= \alpha_k\{0 + \mu_j(u)\sum_r C^r_{jk}R_r(u)w_j(u)\big|_{u=u^\sim}\}\\
&\quad + 0 + 0 + \alpha_k \sum_{\substack{p\neq j\\ p>k}} \mu_p(u)C^j_{pk} + O(|\alpha_k|\ \|u-u^\sim\|^2)\\
&\quad = C^j_{jk}\mu_j(u)\alpha_k + \sum_{\substack{p>k\\ p\neq j}} C^j_{pk}\mu_p(u)\alpha_k + O(|\alpha_k|\ \|u-u^\sim\|^2)\ ,
\end{aligned}$$

which is (2.34).

III. The Euler Equations

The Euler equations of compressible $1-D$ fluid dynamics are [1]

$$\rho_t + (\rho v)_x = 0 \tag{3.1}$$

$$(\rho v)_t + (\rho v^2 + P)_x = 0 \tag{3.2}$$

$$\begin{aligned}
&(\frac{1}{2}\rho v^2 + \rho e(\rho, P))_t\\
&\qquad + ((\frac{1}{2}\rho v^3 + \rho v e(\rho, P) + vP)_x = 0\ ,
\end{aligned} \tag{3.3}$$

which have the form of (1.1) with $u = (\rho, \rho v, \frac{1}{2}\rho v^2 + e)$. Under standard assumptions [1] about the constitutive function $e(\rho, P)$, this system is strictly hyperbolic with fields 1 and 3 genuinely nonlinear and field 2 linearly degenerate, and there exists a function $S(\rho, P)$, called the entropy, such that any smooth solution of (3.1-3) satisfies the additional conservation law

$$(\rho S)_t + (\rho v S)_x = 0 \; . \tag{3.4}$$

Furthermore, S is a k-Riemann invariant of fields 1 and 3, and hence undergoes a change that is third-order in the shock strength across shocks of those fields [1]. On the other hand, a direct calculation (using the fact that the standard assumptions imply $\partial S/\partial \rho < 0$) shows that the change in S is first-order in the strength of the contact discontinuities of field 2.

Lemma 3.1. $C^2_{31} \equiv 0$.

Proof: Instead of calculating C^2_{31} directly using either (1.10) or (1.11), we note that when a 3-wave of strength α interacts with a 1-wave of strength β, the total change in the entropy S across both shocks is $O(|\alpha|^3 + |\beta|^3)$. By Glimm's interaction estimate (1.12), the total change in S across the 1-wave and 3-wave resulting from the interaction is of this order too. Hence the change in S across the 2-wave resulting from the interaction is of order $O(|\alpha|^3 + |\beta|^3)$ as well. But if $C^2_{31}(u)$ were ever nonzero then the strength of the resulting 2-wave, and hence also of the change in S across it, would be $\geq O(|\alpha|\ |\beta|)$. Keeping the middle state u fixed while $|\alpha| = |\beta| \to 0$ then yields a contradiction.

Theorem 3.2. Assume that the standard axioms about the constitutive function hold ([1; 2.02-2.07] plus the positivity of e, P and the temperature T). If ε_3 and ε_4 are sufficiently small and the initial data (1.5) satisfy (1.15-6), then (3.1-3) has a weak solution for all time with initial data (1.5).

Proof: *Just as for theorem 2.1, it suffices to show that the Glimm approximants satisfy a uniform TV bound and a sufficiently small uniform L^∞ bound. These bounds will be obtained via estimates similar to a combination of those in [2] for the cases when (1.6) holds and when (1.7) holds, but using the modified wave- strengths from section 2. Since (1.15-6) imply that the desired estimates hold at time zero, it suffices to show that if the estimates hold on a mesh curve I then they hold on any immediate successor mesh curve J. The following two lemmas show this fact for the TV and L^∞ bounds, respectively, and hence complete the proof of the theorem.*

Lemma 3.3. *Define, analogously to [2],*

$$L = \sum_{\alpha} \sum_{j} |\alpha_j^{\wedge}| \tag{3.5}$$

$$Q = \sum_{\alpha,\beta} D(\alpha^{\wedge}, \beta^{\wedge}) \ , \tag{3.6}$$

where the interaction function D is defined in (1.13). Here and later, sums over wave-sets are to be taken over some mesh curve; when necessary we will use the notation $L(I)$, etc. to indicate which mesh curve is meant. In addition, define

$$L_2 = \sum_{\alpha} |\alpha_2^{\wedge}| \tag{3.7}$$

$$D_2(\alpha, \beta) = \sum_{i \text{ or } j=2} D(\alpha_i^{\wedge}, \beta_j^{\wedge}) \tag{3.8}$$

$$Q_2 = \sum_{\alpha,\beta} D_2(\alpha, \beta) \ . \tag{3.9}$$

Let J be an immediate successor curve to I, separated by the diamond Ω, and define $D(\Omega) \equiv D(\alpha^{\wedge}, \beta^{\wedge})$, etc., where α and β are the wave-sets entering Ω. Then

$$L(J) \leq L(I) + c\{D_2(\Omega) + \|u(I) - u^{\sim}\|D(\Omega)\} \tag{3.10}$$

$$L_2(J) \leq L_2(I) + c\|u(I) - u^{\sim}\|D(\Omega)\} \tag{3.11}$$

$$Q(J) \leq Q(I) - D(\Omega) + c\{D_2(\Omega) + \|u(I) - u^{\sim}\|D(\Omega)\}L(\Omega) \tag{3.12}$$

$$Q_2(J) \leq Q_2(I) - D_2(\Omega) + c\{D_2(\Omega)L_2(\Omega) + \|u(I) - u^{\sim}\|D(\Omega)L(\Omega)\} \ . \tag{3.13}$$

Furthermore, if for mesh curves I' up to and including I

$$\|u(I') - u^{\sim}\| \leq c_1\|u_0 - u^{\sim}\| \tag{3.14}$$

$$L_2(I') \le c_2\{L_2(t=0) + \|u_0 - u^\sim\|(1 + L(t=0)^2)\} \le c_3\{\varepsilon_3 + \varepsilon_4\} \tag{3.15}$$

$$L(I') \le C_4 L(t=0) \tag{3.16}$$

then for ε_5 *sufficiently small,* k_1 *and* $k_2 = k_2(c_1)$ *sufficiently large, and* ε_3 *and* ε_4 *sufficiently small*

$$Q^\wedge \equiv \varepsilon_5 Q_2 + \|u_0 - u^\wedge\|[1 + L(t=0)]Q_1 \tag{3.17}$$

satisfies

$$Q^\wedge(J) \le Q^\wedge(I) - \frac{1}{2}\{\varepsilon_5 D_2(\Omega) + \|u_0 - u^\sim\|[1 + L(t=0)]D(\Omega)\} \tag{3.18}$$

$$L(J) + k_1 Q^\wedge(J) \le L(I) + k_1 Q^\wedge(I) \tag{3.19}$$

$$L_2(J) + k_2 Q^\wedge(J)/[1 + L(t=0)] \le L_2(I) + k_2 Q^\wedge(I)/[1 + L(t=0)]\,, \tag{3.20}$$

and, consequently, (3.15-6) hold for $I' = J$ *(provided the* c_i *satisfy the restrictions noted below).*

Proof: Estimates (3.10-11) follows immediately from (1.14) since the fact that $C_{31}^2 = 0$ implies that the second-order terms in (1.14) always involve an incoming 2-wave, and never involve an outgoing 2-wave. Estimate (3.12) is analogous to the estimate for Q in [2], while the variant in (3.13) follows from the fact that the second-order increases in wave-strengths in Ω occur only in 1- and 3-waves, and hence in Q_2 these increases are always multiplied by 2-waves only; i.e the factor of L multiplying D_2 in the estimate for Q can be replaced by L_2 in the estimate for Q_2.

Next, using (3.14-16) we find from (3.12-13) that $Q^\wedge$ satisfies

(3.21)
$$\begin{aligned}
&Q^\wedge(J) \le Q^\wedge(I) \\
&-D_2(\Omega)\{\varepsilon_5[1 - cL_2(I)] - \|u_0 - u^\sim\|L(t=0)L(I)\} \\
&-D(\Omega)\{\|u_0 - u^\sim\|[1 + L(t=0)]\{1 - \|u(I) - u^\sim\|L(I)\} - c\varepsilon_5\|u(I) - u^\sim\|L(I)\} \\
&\le Q^\wedge(I) - D_2(\Omega)\{\varepsilon_5[1 - cc_3\{\varepsilon_3 + \varepsilon_4\}] - c_4\varepsilon_4\} \\
&-D(\Omega)\|u_0 - u^\sim\|[1 + L(t=0)]\{1 - c_1c_4\varepsilon_4 - cc_1c_4\varepsilon_5\}\,.
\end{aligned}$$

Picking first ε_5 and then ε_3 and ε_4 sufficiently small, this yields (3.18). By combining (3.10-3.11) with (3.18) and choosing k_1 and k_2 sufficiently large we then obtain (3.19-20); since the increase in L_2 is bounded by $\|u(I) - u^{\sim}\|D(\Omega) \leq c_1\|u_0 - u^{\sim}\|D(\Omega)$, while the decrease in $Q^{\wedge}/[1 + L(t=0)]$ is at least $-\frac{1}{2}\|u_0 - u^{\sim}\|D(\Omega)$, the value of k_2 depends only on c_1.

Finally, we will show that (3.19-3.20) imply that (3.15-16) hold for $I' = J$. By the same method of proof, estimates analogous to (3.19-20) hold for previous diamonds. Hence we obtain from (3.19) that

$$
\begin{aligned}
L(J) &\leq L(t=0) + k_1 Q^{\wedge}(t=0) \\
&= L(t=0) + k_1\{\varepsilon_5 Q_2(t=0) + \|u_0 - u^{\sim}\|L(t=0)Q_1(t=0)\} \\
&\leq L(t=0) + k_1\{c\varepsilon_5 L(t=0)L_2(t=0) + \|u_0 - u^{\sim}\|L(t=0)^3\} \\
&\leq L(t=0)\{1 + c_5\varepsilon_5[\varepsilon_3 + \varepsilon_4]\} ,
\end{aligned}
\tag{3.22}
$$

where we have made use of the inequality $L_2 \leq cTV(S)$, which follows from the fact that the change in S is of first-order across 2- waves. Hence for a fixed constant c_4 larger than 1, (3.16) will also hold for $I' = J$ provided ε_3 and ε_4 are small enough. Similarly,

(3.23)

$$
\begin{aligned}
L_2(J) &\leq L_2(0) + k_2 Q(t=0)/[1 + L(t=0)] \\
&\leq L_2(t=0) + k_2\{c\varepsilon_5 L(t=0)L_2(t=0) + \|u_0 - u^{\sim}\|L(t=0)^3\}/[1 + L(t=0)] \\
&\leq L_2(t=0)\{1 + c_6\varepsilon_5\} + k_2\|u_0 - u^{\sim}\|(1 + L(t=0)^2) ,
\end{aligned}
$$

so that (3.15) will hold for $I' = J$ provided $c_3 > \max\{1, k_2(c_1)\}$ and ε_5 is small enough.

Lemma 3.4. *If (3.14) and the conclusions of lemma 3.3 hold for mesh curves I' up to and including I, c_1 is greater than one, and ε_3 and ε_4 are sufficiently small, then (3.14) holds for $I' = J$.*

Proof: From (2.15) and lemma 3.1 we find that

$$
w_2^{\wedge}(T_{\varepsilon_k}^k(u)) - w_2^{\wedge}(u) = \varepsilon_k^{\wedge}\{\delta_{jk} + O(\|u - u^{\sim}\|^2)\}
\tag{3.24}
$$

$$
\begin{aligned}
&w_3^{\wedge}(T_{\varepsilon_k}^k(u)) - w_3^{\wedge}(u) \\
&= \varepsilon_k^{\wedge}\{\delta_{jk} + \delta_{k2}C_{12}^3\mu(u)_1 + O(\|u - u^{\sim}\|^2)\} \\
&= \varepsilon_k^{\wedge}\{\delta_{jk} + \delta_{k2}O(\|u - u^{\sim}\|) + O(\|u - u^{\sim}\|^2)\} .
\end{aligned}
\tag{3.25}
$$

We would like to obtain the latter estimate for $w_1^\wedge$ also, but a straightforward application of (2.15) would yield an estimate having δ_{k2} replaced by δ_{k3}. However, by the remark after the statement of proposition 2.2, by replacing w_1 by $w_1^\#$ we replace the condition $p < k$ in (2.15) by $p > k$, and hence do obtain the desired estimate:

$$\begin{aligned} &w_1^\wedge(T_{\varepsilon_k}^k(u)) - w_1^\wedge(u) \\ &= \varepsilon_k^\wedge\{\delta_{jk} + \delta_{k2}C_{32}^1\mu(u)_3 + O(\|u - u^\sim\|^2)\} \\ &= \varepsilon_k^\wedge\{\delta_{jk} + \delta_{k2}O(\|u - u^\sim\|) + O(\|u - u^\sim\|^2)\}\ . \end{aligned} \tag{3.26}$$

Proceeding in similar fashion to [2], let $a_{m,n}$ be a mesh point on some curve I, and define

$$L_{i,m}(I) = \sum_\alpha \sum_{j \neq i} |\alpha_j| \tag{3.27}$$

$$L_{i,m}^{(2)}(I) = \sum_\alpha |\alpha_2|\ , \quad i = 1,3\ , \tag{3.28}$$

where the sum is taken over all waves crossing I that approach an imaginary i-wave located at the mesh point $a_{m,n}$. Define further

(3.29)
$$\begin{aligned} F_2(I) = \sup_{i,m} F_2(I,i,m) = \sup_{i,m}\{&|w_i^\wedge(u(a_{m,n})| \\ &+ k_3[\|u_0 - u^\sim\|L_{i,m}^{(2)}(I) \\ &+ \|u_0 - u^\sim\|^2 L_{i,m}(I) + k_4\|u_0 - u^\sim\|Q^\wedge(I)/\{1 + L(t=0)\}]\}\ . \end{aligned}$$

It now suffices to show that F_2 is nonincreasing on successive mesh curves, since the resultant inequality

$$\|u(J) - u^\sim\| \le F_2(J) \le F_2(t=0) \le \|u_0 - u^\sim\|\{1 + k_3[\varepsilon_3 + \varepsilon_4 + k_4 c(\varepsilon_3 + \varepsilon_4)]\}\ . \tag{3.30}$$

then yields estimate (3.14) for $I' = J$ provided that $c_1 > 1$ and ε_3 and ε_4 are small enough. The proof that F_2 is nonincreasing is similar to that in [2]. The decrease in $\|u_0 - u^\sim\|Q^\wedge/[1 + L(t=0)]$ is, by (3.18), at least $\frac{1}{2}\|u_0 - u^\sim\|^2 D(\Omega)$, whereas the increase in $\|u_0 - u^\wedge\|L_{i,m}^{(2)} + \|u_0 - u^\sim\|^2 L_{i,m}$ is no more that of $\|u_0 - u^\sim\|L_2 + \|u_0 - u^\sim\|^2 L$, which by (3.10-11) is $\le c\|u_0 - u^\sim\|^2 D(\Omega)$, so that the combination of these functionals appearing in the definition of F_2 decreases by at least $k_3\|u_0 - u^\sim\|^2 D(\Omega)/4$ if k_4 is large enough. Hence $F_2(J,i,m) \le F_2(I,i,m)$ at all common mesh points of I and J, since w remains the same there. Finally, at the mesh point $a_{m,n+2}$ of $J\backslash I$ the argument of [2] together with (3.24-26) shows that for the appropriate choice of $\sigma = \pm 1$,

$$
\begin{aligned}
|w_i(u(a_{m,n+2}))| \le {} & |w_i(u(a_{m+\sigma,n+1}))| \\
& + c\{\|u - u^\sim\| \, |\varepsilon_2| + \|u - u^\sim\|^2 \sum |\varepsilon_k|\} ,
\end{aligned}
\tag{3.31}
$$

where the term ε_2 and the terms ε_k are to be included only if this (outgoing) wave does not approach $a_{m,n+2}$ but the incoming waves α and β of the same index do approach $a_{m+\sigma,n+1}$. Thus, the terms involving these α and β waves appear in $F_2(I,i,m+\sigma)$ but not in $F_2(J,i,m)$. Consequently, from estimates (3.10–3.11), the above estimate for the rest of F_2, and (3.14) for $I' = I$ we obtain

(3.32)
$$
\begin{aligned}
F_2(J,i,m) \le {} & F_2(I,i,m+\sigma) \\
& + c\{\|u(I) - u^\sim\| \, |\varepsilon_2| + \|u(I) - u^\sim\|^2 \Sigma|\varepsilon_k|\} \\
& - k_3\{\|u_0 - u^\sim\|(|\alpha_2| + |\beta_2|) + \|u_0 - u^\sim\|^2 \Sigma(|\alpha_k| + |\beta_k|)\} \\
& - k_3\|u_0 - u^\sim\|^2 D(\Omega)/4 \\
\le {} & F_2(I,i,m+\sigma) + c\{c_1\|u_0 - u^\sim\|[|\alpha_2| + |\beta_2| + cc_1\|u_0 - u^\sim\|D(\Omega)] \\
& + c_1\|u - u^\sim\|^2 \Sigma[|\alpha_k| + |\beta_k| + cD(\Omega)]\} \\
& - k_3\{\|u_0 - u^\sim\|(|\alpha_2| + |\beta_2|) + \|u_0 - u^\sim\|^2 \Sigma(|\alpha_k| + |\beta_k|)\} \\
& - k_3\|u_0 - u^\sim\|^2 D(\Omega)/4 \\
\le {} & F_2(I,i,m+\sigma)
\end{aligned}
$$

if k_3 is sufficiently large. Hence F_2 is indeed nonincreasing, which as noted above completes the proof of the lemma.

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