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ERROR ESTIMATES FOR FINITE ELEMENT METHODS FOR SECOND ORDER HYPERBOLIC EQUATIONS*

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Abstract. The standard Galerkin method for a mixed initial-boundary value problem for a linear second order hyperbolic equation is analysed.

Optimal estimates for the error in $L^\infty(L^2)$ are derived using L^2-projections of the initial data as starting values, and minimal smoothness requirements on the solution.

1. Introduction. Let Ω be a bounded domain in $\mathbb{R}^n$, a generic point of which will be denoted by $x = (x_1, x_2, \cdots, x_n)$ and let $\partial\Omega$ denote the boundary of Ω which will be assumed to be an $(n-1)$-dimensional manifold of class C^∞.

For fixed $0 < T < \infty$, we shall be interested in approximating the solution of the following mixed initial-boundary value problem. A function $u(x, t)$ defined on $\bar{\Omega} \times [0, T]$ is sought which satisfies

$$\text{(1.1)} \quad \begin{cases} \dfrac{\partial^2 u}{\partial t^2}(x, t) - \displaystyle\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}(x, t)\right) = f(x, t), & (x, t) \in \Omega \times (0, T], \\ u(x, t) = 0, \quad (x, t) \in \partial\Omega \times [0, T], \end{cases}$$

$$\text{(1.2)} \quad u(x, 0) = u_0(x), \qquad x \in \bar{\Omega},$$

$$\text{(1.3)} \quad \frac{\partial u}{\partial t}(x, 0) = q_0(x), \qquad x \in \bar{\Omega}.$$

f, u_0 and q_0 are given functions,

$$\text{(1.4)} \quad a_{ij} = a_{ji} \in C^\infty(\bar{\Omega}), \qquad i, j = 1, 2, \cdots, n,$$

and there exists a constant $\alpha > 0$ such that

$$\text{(1.5)} \quad \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geqq \alpha \sum_{i=1}^{n} \xi_i^2,$$

for all $x \in \bar{\Omega}$ and all $(\xi_1, \xi_2, \cdots, \xi_n) \in \mathbb{R}^n$.

Dupont [2] has analyzed both the continuous-time and a fully discrete three-level Galerkin method for the problem (1.1)–(1.3). For the continuous-time method, Dupont obtains optimal $L^\infty(L^2)$ estimates for the error, $O(h^r)$ using subspaces of piecewise polynomial functions of degree $\leqq r - 1$, for $r \geqq 2$, assuming that the starting values are $O(h^r)$ close to the H^1-projections of the initial data u_0 and q_0 [2, Th. 1].

In this work it is shown that the optimal $L^\infty(L^2)$ estimates for the error are obtainable using L^2-projections of the initial data as starting values, and with less assumptions on the smoothness of the solution. This is the content of Theorem 3.1.

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Here, estimates $O(h^r + \tau^2)$ are derived for the error in L^2 for a fully discrete method, using the above subspaces, where τ denotes the discrete time step. The L^2-projections of the initial data are used as starting values, which eliminates the relative computational difficulties of choosing starting values in [2]. Also, the proof of Theorem 4.2, where these estimates are derived, reveals the correct smoothness assumptions for the solution.

Throughout the paper, C will denote a general constant, not necessarily the same in any two places.

2. Notation. For $s \geqq 0$, $H^s(\Omega)$ will denote the Sobolev space $W_2^s(\Omega)$ of real-valued functions on Ω; the norm on $H^s(\Omega)$ will be denoted by $\|\cdot\|_s$. For definitions and the relevant properties of these spaces, we refer to [3].

In particular, $H^0(\Omega) = L^2(\Omega)$, the inner product and norm on which will be denoted by

$$(u, v) = \int_\Omega uv\,dx, \qquad u, v \in L^2(\Omega),$$

and

$$\|u\| = \{(u, u)\}^{1/2}, \qquad u \in L^2(\Omega).$$

$C_0^\infty(\Omega)$ will denote the space of infinitely differentiable functions on Ω which have support compactly contained in Ω and $\mathring{H}^1(\Omega)$ will denote the subspace of $H^1(\Omega)$ obtained by completing $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_1$.

Also following [3], $H^{-1}(\Omega)$ is defined as the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|v\|_{-1} = \sup_{\substack{\psi \in C_0^\infty(\Omega) \\ \psi \neq 0}} \frac{|(v, \psi)|}{\|\psi\|_1}, \qquad v \in C_0^\infty(\Omega).$$

Again, following [3], for H a Banach space with norm $\|\cdot\|_H$, and $v : [0, T] \to H$ Lebesgue measurable, the following norms are defined:

$$\|v\|_{L^2(0,T;H)} = \left(\int_0^T \|v(\cdot, t)\|_H^2\,dt\right)^{1/2},$$

and

$$\|v\|_{L^\infty(0,T;H)} = \sup_{0 \leqq t \leqq T} \|v(\cdot, t)\|_H.$$

We adopt the notation

$$L^p(0, T; H) = \{v : [0, T] \to H : \|v\|_{L_p(0,T;H)} < \infty\}, \qquad p = 2, \infty.$$

Associated with (1.1) is the bilinear form

$$a(u, v) = \int_\Omega \left\{\sum_{i,j=1}^n a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j}\right\} dx, \qquad u, v \in H^1(\Omega).$$

From (1.4) and (1.5) it follows that there exist constants $C_1 < \infty$ and $C_2 > 0$ such that

$$|a(u, v)| \leqq C_1 \|u\|_1 \|v\|_1 \quad \text{for all } u, v \in H^1(\Omega), \tag{2.1}$$

and

$$a(u, u) \geqq C_2 \|u\|_1^2 \quad \text{for all } u \in \mathring{H}^1(\Omega). \tag{2.2}$$

The boundary value problem (1.1)–(1.3) has the following weak formulation: a mapping $u \in L^2(0, T; \mathring{H}^1(\Omega))$ is sought with

$$\frac{\partial^2 u}{\partial t^2} \in L^2(0, T; H^{-1}(\Omega)),$$

such that

$$\left(\frac{\partial^2 u}{\partial t^2}(\cdot, t), v\right) + a(u(\cdot, t), v) = (f(\cdot, t), v) \quad \text{for all } v \in \mathring{H}^1(\Omega), \quad t > 0, \tag{2.3}$$

and

$$(u(\cdot, 0), v) = (u_0, v) \quad \text{for all } v \in \mathring{H}^1(\Omega), \tag{2.4}$$

$$\left(\frac{\partial u}{\partial t}(\cdot, 0), v\right) = (q_0, v) \quad \text{for all } v \in \mathring{H}^1(\Omega). \tag{2.5}$$

Existence and uniqueness of a solution u of (2.3)–(2.5) for $f \in L^2(0, T; H^{-1}(\Omega))$ and $u_0, q_0 \in \mathring{H}^1(\Omega)$ is proved, for example, in [3].

Henceforth, it will be assumed that the problem (2.3)–(2.5) has a unique solution u, and in the appropriate places to follow, additional conditions on the regularity of u which guarantee the convergence results, will be imposed.

Let $r \geqq 2$ be a fixed integer. In the notation of [1], we assume the existence of families $\{S_h^r(\Omega)\}_{0<h\leqq 1}$ of finite-dimensional subspaces of $\mathring{H}^1(\Omega)$ which possess the following approximation properties.

There exists a constant C such that if $v \in H^s(\Omega) \cap \mathring{H}^1(\Omega)$, $1 \leqq s \leqq r$, then

$$\inf_{\chi \in S_h^r(\Omega)} \{\|v - \chi\| + h\|v - \chi\|_1\} \leqq Ch^s \|v\|_s. \tag{2.6}$$

The following result is a consequence of the above properties of $\{S_h^r(\Omega)\}_{0<h\leqq 1}$, and the error estimation techniques initiated in [4].

LEMMA 2.1. *Let u be the solution of* (2.3)–(2.5). *Then there exists a unique mapping $\omega_h \in L^2(0, T; S_h^r(\Omega))$ which satisfies*

$$a(\omega_h(\cdot, t), v) = a(u(\cdot, t), v) \quad \textit{for all } v \in S_h^r(\Omega), \quad t \geqq 0. \tag{2.7}$$

Furthermore, if for some integer $k \geqq 0$

$$\frac{\partial^k u}{\partial t^k} \in L^p(0, T; H^s(\Omega)),$$

then

$$\frac{\partial^k \omega_h}{\partial t^k} \in L^p(0, T; S_h^r(\Omega))$$

and

$$\left\| \left(\frac{\partial}{\partial t}\right)^k [u - \omega_h] \right\|_{L_p(0,T;L^2(\Omega))} \leqq C_3 h^s \left\| \left(\frac{\partial}{\partial t}\right)^k u \right\|_{L_p(0,T;H^s(\Omega))},$$

for some constant C_3 independent of h and u, and $1 \leqq s \leqq r$.

3. The continuous-time Galerkin approximation. The following theorem defines the continuous time Galerkin approximation and derives the optimal $L^\infty(L^2)$ error estimates. Together with Theorem 4.1, this is the essential result of the paper. Again, the technique of error estimation here consists of a special manipulation of an argument initiated by Wheeler [5] for parabolic equations, of comparing the Galerkin approximation with a so-called elliptic projection, already defined by (2.7).

THEOREM 3.1. *Let u be the solution of* (2.3)–(2.5); *then for each $h \in (0, 1]$, there exists a unique mapping*

$$U_h \in L^2(0, T; S_h^r(\Omega))$$

which satisfies

$$\left(\frac{\partial^2 U_h}{\partial t^2}(\cdot, t), v\right) + a(U_h(\cdot, t), v) = (f(\cdot, t), v) \tag{3.1}$$

for all $v \in S_h^r(\Omega)$, $t > 0$,

$$(U_h(\cdot, 0), v) = (u_0, v) \qquad \textit{for all } v \in S_h^r(\Omega), \tag{3.2}$$

$$\left(\frac{\partial U_h}{\partial t}(\cdot, 0), v\right) = (q_0, v) \quad \textit{for all } v \in S_h^r(\Omega). \tag{3.3}$$

Furthermore, if $u \in L^\infty(0, T; H^r(\Omega))$ and $\partial u/\partial t \in L^2(0, T; H^r(\Omega))$, then there exists a constant $C = C(T)$ such that

$$\|u - U\|_{L^\infty(0,T;L^2(\Omega))} \leqq Ch^r \left\{ \|u\|_{L^\infty(0,T;H^r(\Omega))} + \left\| \frac{\partial u}{\partial t} \right\|_{L^2(0,T;H^r(\Omega))} \right\}.$$

Proof. The existence and uniqueness of the mapping U_h follows from the fact that the equations (3.1)–(3.3) are equivalent to an initial value problem for a system of linear ordinary differential equations of the second order, the unknown functions being the coefficients of U_h relative to the chosen basis for $S_h^r(\Omega)$. It is easily shown that the system possesses a unique solution.

Now let ω_h be defined by (2.7), and set

$$\eta = u - \omega_h, \quad \psi = U_h - \omega_h \quad \text{and} \quad \ell = u - U_h.$$

From (3.1), (2.7) and (2.3), for any $v \in S_h^r(\Omega)$, and $0 < t \leqq T$,

$$
\begin{aligned}
\left(\frac{\partial^2 \psi}{\partial t^2}(\cdot, t), v\right) + a(\psi(\cdot, t), v) &= (f(\cdot, t), v) - \left(\frac{\partial^2 \omega_h}{\partial t^2}(\cdot, t), v\right) - a(\omega_h(\cdot, t), v) \\
&= (f(\cdot, t), v) - a(u(\cdot, t), v) - \left(\frac{\partial^2 \omega_h}{\partial t^2}(\cdot, t), v\right) \\
&= \left(\frac{\partial^2 \eta}{\partial t^2}(\cdot, t), v\right).
\end{aligned} \tag{3.4}
$$

In (3.4), the possible dependence of v on t has been suppressed for brevity. Now (3.4) may be rewritten

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{\partial \psi}{\partial t}(\cdot, t), v(\cdot, t)\right) - \left(\frac{\partial \psi}{\partial t}(\cdot . t), \frac{\partial v}{\partial t}(\cdot, t)\right) + a(\psi(\cdot, t), v(\cdot, t)) \\
&\qquad = \frac{d}{dt}\left(\frac{\partial \eta}{\partial t}(\cdot, t), v(\cdot, t)\right) - \left(\frac{\partial \eta}{\partial t}(\cdot, t), \frac{\partial v}{\partial t}(\cdot, t)\right)
\end{aligned} \tag{3.5}
$$

for all $v \in S_h^r(\Omega)$.

Noting that $\ell = \eta - \psi$, we see that (3.5) becomes

$$
\begin{aligned}
&-\left(\frac{\partial \psi}{\partial t}(\cdot, t), \frac{\partial v}{\partial t}(\cdot, t)\right) + a(\psi(\cdot, t), v(\cdot, t)) \\
&\qquad = \frac{d}{dt}\left(\frac{\partial \ell}{\partial t}(\cdot, t), v(\cdot, t)\right) - \left(\frac{\partial \eta}{\partial t}(\cdot, t), \frac{\partial v}{\partial t}(\cdot, t)\right)
\end{aligned} \tag{3.6}
$$

for all $v \in S_h^r(\Omega)$, $t > 0$.

Now let $0 < \xi \leqq T$. We now make the particular choice

$$
\hat{v}(\cdot, t) = \int_t^\xi \psi(\cdot, \tau)\, d\tau, \qquad 0 \leqq t \leqq T. \tag{3.7}
$$

Then clearly $\hat{v}(\cdot, \xi) = 0$, and

$$
\frac{\partial \hat{v}}{\partial t}(\cdot, t) = -\psi(\cdot, t), \qquad 0 \leqq t \leqq T.
$$

Hence, using (3.7) in (3.6), we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\{\|\psi(\cdot, t)\|^2\} - \frac{1}{2}\frac{d}{dt}a(\hat{v}(\cdot, t), \hat{v}(\cdot, t)) \\
&\qquad = \frac{d}{dt}\left(\frac{\partial \ell}{\partial t}(\cdot, t), \hat{v}(\cdot, t)\right) + \left(\frac{\partial \eta}{\partial t}(\cdot, t), \psi(\cdot, t)\right).
\end{aligned} \tag{3.8}
$$

Now integrating (3.8) from $t = 0$ to $t = \xi$, we have

$$
\begin{aligned}
&\|\psi(\cdot, \xi)\|^2 - \|\psi(\cdot, 0)\|^2 + a(\hat{v}(\cdot, 0), \hat{v}(\cdot, 0)) \\
&\qquad = -2\left(\frac{\partial \ell}{\partial t}(\cdot, 0), \hat{v}(\cdot, 0)\right) + 2\int_0^\xi \left(\frac{\partial \eta}{\partial t}(\cdot, t), \psi(\cdot, t)\right) dt.
\end{aligned} \tag{3.9}
$$

Now from (3.3) it follows that

$$\left(\frac{\partial \ell}{\partial t}(\cdot, 0), v\right) = 0 \quad \text{for all } v \in S_h^r(\Omega). \tag{3.10}$$

Hence, using (3.10) and (2.2), we reduce (3.9) to

$$\begin{aligned}
\|\psi(\cdot, \xi)\|^2 &\leqq \|\psi(\cdot, 0)\|^2 + 2\int_0^\xi \left(\frac{\partial \eta}{\partial t}(\cdot, t), \psi(\cdot, t)\right) dt \\
&\leqq \|\psi(\cdot, 0)\|_0^2 + 2\sqrt{T}\|\psi\|_{L^\infty(0,T;L^2(\Omega))}\left\|\frac{\partial \eta}{\partial t}\right\|_{L^2(0,T;L^2(\Omega))} \qquad (3.11)\\
&\leqq \|\psi(\cdot, 0)\|_0^2 + \frac{1}{2}\|\psi\|^2_{L^\infty(0,T;L^2(\Omega))} + 2T\left\|\frac{\partial \eta}{\partial t}\right\|^2_{L^2(0,T;L^2(\Omega))}
\end{aligned}$$

Now taking the supremum in (3.11) over the variable $0 \leqq \xi \leqq T$, we obtain

$$\frac{1}{2}\|\psi\|^2_{L^2(0,T;L^2(\Omega))} \leqq \|\psi(\cdot, 0)\|^2 + 2T\left\|\frac{\partial \eta}{\partial t}\right\|^2_{L^2(0,T;L^2(\Omega))}, \tag{3.12}$$

or

$$\|\psi\|_{L^\infty(0,T;L^2(\Omega))} \leqq \sqrt{2}\|\psi(\cdot, 0)\| + 2\sqrt{T}\left\|\frac{\partial \eta}{\partial t}\right\|_{L^2(0;T;L^2\ (\Omega))} \tag{3.13}$$

From (3.13),

$$\begin{aligned}
\|\ell\|_{L^\infty(0,T;L^2(\Omega))} &\leqq \|\eta\|_{L^\infty(0,T;L^2(\Omega))} + \|\psi\|_{L^\infty(0,T;L^2(\Omega))} \\
&\leqq \|\eta\|_{L^\infty(0,T;L^2(\Omega))} + 2\sqrt{T}\left\|\frac{\partial \eta}{\partial t}\right\|_{L^2(0,T;L^2(\Omega))} + \sqrt{2}\|\psi(\cdot, 0)\| \qquad (3.14)\\
&\leqq \|\eta\|_{L^\infty(0,T;L^2(\Omega))} + 2\sqrt{T}\left\|\frac{\partial \eta}{\partial t}\right\|_{L^2(0,T;L^2(\Omega))} + \sqrt{2}\|\eta(\cdot, 0)\| \\
&\qquad + \sqrt{2}\|\ell(\cdot, 0)\|.
\end{aligned}$$

Now from (3.2) and (2.6), we have

$$\|l(\cdot, 0)\| \leqq Ch^r\|u_0\|_r \leqq Ch^r\|u\|_{L^\infty(0,T;H^r(\Omega))}. \tag{3.15}$$

Hence, finally, using (3.15) and Lemma 2.1 in (3.14), we arrive at

$$\|\ell\|_{L^\infty(0,T;L^2(\Omega))} \leqq C(T)h^r\left\{\|u\|_{L^\infty(0,T;H^r(\Omega))} + \left\|\frac{\partial u}{\partial t}\right\|_{L^2(0,T;H^r(\Omega))}\right\}.$$

The result of Theorem 3.1 now follows. □

4. A fully discrete Galerkin scheme. Let $T = J\tau$ for some integer $J \geqq 1$; for a sequence $\{V^n\}_{n=0}^J \subset L^2(\Omega)$, we define

$$\partial_\tau V^n = \tau^{-1}[V^{n+1} - V^n] \quad \text{and} \quad V^{n+1/2} = \tfrac{1}{2}[V^{n+1} + V^n], \quad n = 0, 1, \cdots, J-1.$$

Also for a continuous mapping $V: [0, T] \to H^1(\Omega)$, we define $V^n = V(\cdot, n\tau)$, $0 \leqq n \leqq J$.

The discrete Galerkin approximation is defined as follows. We seek a sequence $\{U^n\}_{n=0}^J \subset S_h^r(\Omega)$ such that U^n approximates u^n optimally in $L^2(\Omega)$.

The following lemma defines the Galerkin approximations $\{U^n\}_{n=0}^J$, in terms of an auxiliary sequence $\{Q^n\}_{n=0}^J \subset S_h^r(\Omega)$ and in fact gives a computational algorithm for finding $\{U^n\}_{n=0}^J$.

LEMMA 4.1. *There exists a unique sequence* $\{U^n\}_{n=0}^J \subset S_h^r(\Omega)$ *and a corresponding unique sequence* $\{Q^n\}_{n=0}^J \subset S_h^r(\Omega)$ *which simultaneously satisfy the equations*

$$(U^0, \chi) = (u_0, \chi) \quad \textit{for all } \chi \in S_h^r(\Omega), \tag{4.1}$$

$$(Q^0, \chi) = (q_0, \chi) \quad \textit{for all } \chi \in S_h^r(\Omega), \tag{4.2}$$

and

$$(\partial_\tau Q^n, \chi) + a(U^{n+1/2}, \chi) = (f^{n+1/2}, \chi) \tag{4.3}$$

for all $\chi \in S_h^r(\Omega)$,

$$\partial_\tau U^n = Q^{n+1/2}, \qquad 0 \leqq n \leqq J-1. \tag{4.4}$$

Proof. Clearly U^0 and Q^0 exist uniquely.
From (4.3) and (4.4), for $n \geqq 0$, Q^{n+1} satisfies

$$A_\tau(Q^{n+1}, \chi) = F^n\chi \quad \text{for all } \chi \in S_h^r(\Omega),$$

where $A_\tau(\cdot\,, \cdot)$ is the bilinear form given by

$$A_\tau(U, V) = \frac{\tau^2}{2}a(U, V) + (U, V), \quad U, V \in \mathring{H}^1(\Omega),$$

and F^n is the linear functional given by

$$F^nV = \tau[(f^{n+1/2}, V) - a(U^n, V)] + (Q^n, V) - \frac{\tau^2}{4}a(Q^n, V), \qquad V \in H^1(\Omega).$$

From (2.2), $A_\tau(\cdot\,, \cdot)$ is positive definite, and so Q^{n+1} exists uniquely, and hence from (4.4), U^{n+1} exists uniquely, $n = 0, 1, \cdots, J-1$. □

Towards estimating the errors $\|u^n - U^n\|$, we define the auxiliary functions

$$\xi^n = U^n - \omega_h^n, \tag{4.5}$$

$$P^n = Q^n - \left(\frac{\partial \omega_h}{\partial t}\right)^n, \qquad 0 \leqq n \leqq J, \tag{4.6}$$

and again $\eta = u - \omega_h$, where ω_h is defined by (2.7). We now present in Lemma 4.2 and Theorem 4.1 combined, a discrete analogue of the argument of Theorem 3.1.

LEMMA 4.2. *Let u be the solution of* (2.3)–(2.5), *and suppose that*

$$\frac{\partial u}{\partial t} \in L^2(0, T; H^r(\Omega)) \quad \textit{and} \quad \left(\frac{\partial}{\partial t}\right)^k u \in L^2(0, T; L^2(\Omega))$$

for $k = 3, 4$; *then for some constant* $C_4 = C_4(T)$, *independent of* h *and* τ,

$$\max_{0 \leqq n \leqq J} \|\xi^n\| \leqq \sqrt{2}\|\xi^0\| + C_4\left\{h^r\left\|\frac{\partial u}{\partial t}\right\|_{L^2(0,T;H^r(\Omega))} + \tau^2\left[\left\|\frac{\partial^3 u}{\partial t^3}\right\|_{L^2(0,T;L^2(\Omega))} + \left\|\frac{\partial^4 u}{\partial t^4}\right\|_{L^2(0,T;L^2(\Omega))}\right]\right\}.$$

Proof. From (2.3) it follows that

$$\left(\partial_\tau\left(\frac{\partial u}{\partial t}\right)^n, \chi\right) + a(u^{n+1/2}, \chi) = (f^{n+1/2} + \rho^n, \chi) \tag{4.7}$$

for all $\chi \in \mathring{H}^1(\Omega)$, where

$$\rho^n = \partial_\tau\left(\frac{\partial u}{\partial t}\right)^n - \left(\frac{\partial^2 u}{\partial t^2}\right)^{n+1/2}. \tag{4.8}$$

Now from (4.3), (4.5), (4.6), (2.7) and (4.7), for any $\chi \in S_h^r(\Omega)$,

$$\begin{aligned}(\partial_\tau P^n, \chi) + a(\xi^{n+1/2}, \chi) &= (\partial_\tau Q^n, \chi) + a(U^{n+1/2}, \chi) - \left(\partial_\tau\left(\frac{\partial \omega_h}{\partial t}\right)^n, \chi\right) - a(\omega_h^{n+1/2}, \chi)\\ &= (f^{n+1/2}, \chi) - a(u^{n+1/2}, \chi) - \left(\partial_\tau\left(\frac{\partial \omega_h}{\partial t}\right)^n, \chi\right)\\ &= \left(\partial_\tau\left(\frac{\partial \eta}{\partial t}\right)^n - \rho^n, \chi\right), \qquad 0 \leqq n \leqq J-1.\end{aligned} \tag{4.9}$$

Also, from (4.4) and (4.6),

$$\begin{aligned}\partial_\tau \xi^n = Q^{n+1/2} - \partial_\tau \omega_h^n &= P^{n+1/2} - \left[\partial_\tau \omega_h^n - \left(\frac{\partial \omega_n}{\partial t}\right)^{n+1/2}\right]\\ &= P^{n+1/2} + \partial_\tau \eta^n - \left(\frac{\partial \eta}{\partial t}\right)^{n+1/2} - \sigma^n,\end{aligned} \tag{4,10}$$

where

$$\sigma^n = \partial_\tau u^n - \left(\frac{\partial u}{\partial t}\right)^{n+1/2}, \qquad 0 \leqq n \leqq J-1. \tag{4.11}$$

Hence, from (4.10),

$$\partial_\tau \xi^0 = P^0 + \frac{\tau}{2}\partial_\tau P^0 + \partial_\tau \eta^0 - \left(\frac{\partial \eta}{\partial t}\right)^{1/2} - \sigma^0, \tag{4.12}$$

and

$$\partial_\tau \xi^n = P^0 + \frac{\tau}{2}\sum_{k=0}^{n} \partial_\tau P^k + \frac{\tau}{2}\sum_{k=0}^{n-1} \partial_\tau P^k + \partial_\tau \eta^n - \left(\frac{\partial \eta}{\partial t}\right)^{n+1/2} - \sigma^n, \quad 1 \leqq n \leqq J-1. \tag{4.13}$$

Now, define a sequence $\{\varphi^n\}_{n=0}^{J}$ via

$$\varphi^0 = 0; \quad \varphi^n = \tau \sum_{k=0}^{n-1} \xi^{k+1/2}, \qquad 1 \leqq n \leqq J. \tag{4.14}$$

Then

$$\varphi^{1/2} = \frac{\tau}{2}\xi^{1/2} \tag{4.15}$$

and

$$\varphi^{n+1/2} = \frac{\tau}{2}\left[\sum_{k=0}^{n} \xi^{k+1/2} + \sum_{k=0}^{n-1} \xi^{k+1/2}\right], \qquad 1 \leqq n \leqq J-1. \tag{4.16}$$

Hence, from (4.12), (4.15) and (4.9), for any $\chi \in S_h^r(\Omega)$,

$$\begin{aligned}(\partial_\tau \xi^0, \chi) + a(\varphi^{1/2}, \chi) &= \left(P_0 + \partial_\tau \eta^0 - \left(\frac{\partial \eta}{\partial t}\right)^{1/2} - \sigma^0, \chi\right) + \frac{\tau}{2}\left(\partial_\tau\left(\frac{\partial \eta}{\partial t}\right) - \rho^0, \chi\right) \\ &= \left(P^0 - \left(\frac{\partial \eta}{\partial t}\right)^0, \chi\right) + \left(\partial_\tau \eta^0 - \sigma^0 - \frac{\tau}{2}\rho^0, \chi\right) \\ &= \left(\partial_\tau \eta^0 - \sigma^0 - \frac{\tau}{2}\rho^0, \chi\right),\end{aligned} \tag{4.17}$$

where we have used the fact that from (4.6) and (4.2),

$$\left(P^0 - \left(\frac{\partial \eta}{\partial t}\right)^0, \chi\right) = \left(Q^0 - \left(\frac{\partial u}{\partial t}\right)^0, \chi\right) = 0 \quad \text{for all } \chi \in S_h^r(\Omega).$$

Similarly, from (4.13), (4.16) and (4.9) and the last equation, for any $\chi \in S_h^r(\Omega)$, and $1 \leqq n \leqq J-1$,

$$
\begin{aligned}
&(\partial_\tau \xi^n, \chi) + a(\varphi^{n+1/2}, \chi) \\
&= \left(P^0 + \partial_\tau \eta^n - \left(\frac{\partial\eta}{\partial t}\right)^{n+1/2} - \sigma^n, \chi\right) \\
&\quad + \left(\frac{\tau}{2}\left[\sum_{k=0}^{n} \partial_\tau\left(\frac{\partial\eta}{\partial t}\right)^k - \rho^k\right] + \frac{\tau}{2}\left[\sum_{k=0}^{n-1} \partial_\tau\left(\frac{\partial\eta}{\partial t}\right)^k - \rho^k\right], \chi\right) \\
&= \left(P^0 + \partial_\tau \eta^n - \left(\frac{\partial\eta}{\partial t}\right)^{n+1/2} - \sigma^n, \chi\right) \\
&\quad + \left(\left(\frac{\partial\eta}{\partial t}\right)^{n+1/2} - \left(\frac{\partial\eta}{\partial t}\right)^0 - \frac{\tau}{2}\sum_{k=0}^{n} \rho^k - \frac{\tau}{2}\sum_{k=0}^{n-1} \rho^k, \chi\right) \\
&= \left(P^0 - \left(\frac{\partial\eta}{\partial t}\right)^0, \chi\right) + \left(\partial_\tau \eta^n - \sigma^n - \frac{\tau}{2}\left[\sum_{k=0}^{n} \rho^k + \sum_{k=0}^{n-1} \rho^k\right], \chi\right) \\
&= \left(\partial_\tau \eta^n - \sigma^n - \frac{\tau}{2}\left[\sum_{k=0}^{n} \rho^k + \sum_{k=0}^{n-1} \rho^k\right], \chi\right).
\end{aligned}
\tag{4.18}
$$

Hence if we define

$$
\varepsilon^0 = \partial_\tau \eta^0 - \frac{\tau}{2}\rho^0 - \sigma^0 \quad \text{and} \quad \varepsilon^n = \partial_\tau \eta^n - \frac{\tau}{2}\rho^0 - \tau \sum_{k=0}^{n-1} \rho^{k+1/2} - \sigma^n, \qquad 1 \leqq n \leqq J-1,
\tag{4.19}
$$

then (4.17) and (4.18) reduce to

$$
(\partial_\tau \xi^n, \chi) + a(\varphi^{n+1/2}, \chi) = (\varepsilon^n, \chi), \qquad 0 \leqq n \leqq J-1,
\tag{4.20}
$$

for all $\chi \in S_h^r(\Omega)$.

In (4.20), we now make the choice

$$
\hat{\chi} = \partial_\tau \varphi^n = \xi^{n+1/2}, \qquad 0 \leqq n \leqq J-1;
$$

then we obtain

$$
\tfrac{1}{2}\|\xi^{n+1}\|^2 - \tfrac{1}{2}\|\xi^n\|^2 + \tfrac{1}{2}a(\varphi^{n+1}, \varphi^{n+1}) - \tfrac{1}{2}a(\varphi^n, \varphi^n) = \tau(\varepsilon^n, \xi^{n+1/2}), \qquad 0 \leqq n \leqq J-1.
\tag{4.21}
$$

Summing in (4.21) from $n = 0$ to $n = l-1$, for any $1 \leqq l \leqq J$, and using (4.14) and (2.2), we obtain

$$
\begin{aligned}
\|\xi^l\|^2 &\leqq \|\xi^0\|^2 + 2\tau \sum_{n=0}^{l-1} (\varepsilon^n, \xi^{n+1/2}) \\
&\leqq \|\xi^0\|^2 + 4T\tau \sum_{n=0}^{l-1} \|\varepsilon^n\|^2 + \frac{\tau}{4T}\sum_{n=0}^{l-1} \|\xi^{n+1/2}\|^2 \\
&\leqq \|\xi^0\|^2 + 4T\tau \sum_{n=0}^{l-1} \|\varepsilon^n\|^2 + \tfrac{1}{2}\max_{0 \leqq n \leqq J} \|\xi^n\|^2
\end{aligned}
\tag{4.22}
$$

Hence (4.22) gives

$$\max_{0 \leqq n \leqq J} \|\xi^n\|^2 \leqq 2\|\xi^0\|^2 + 8T\tau \sum_{n=0}^{J-1} \|\varepsilon^n\|^2. \tag{4.23}$$

Now, starting from (4.8), a computation which simply involves integrating by parts twice shows that

$$\rho^k = \frac{1}{2\tau} \int_{k\tau}^{(k+1)\tau} [(k+1)\tau - s][k\tau - s] \frac{\partial^4 u}{\partial t^4}(\cdot, s)\, ds,$$

and hence, by Schwarz' inequality,

$$\|\rho^k\|^1 \leqq \frac{1}{5!}\tau^3 \int_{k\tau}^{(k+1)\tau} \left\|\frac{\partial^4 u}{\partial t^4}(\cdot, s)\right\|^2 ds.$$

Hence

$$\tau \sum_{k=0}^{J \leqq 1} \|\rho^k\|^2 \leqq \frac{1}{5!}\tau^4 \left\|\frac{\partial^4 u}{\partial t^4}\right\|^1_{L^2(0,T;L^2(\Omega))} \tag{4.24}$$

Similarly, from (4.11),

$$\sigma^k = \frac{1}{2\tau} \int_{k\tau}^{(k+1)\tau} [(k+1)\tau - s][k\tau - s] \frac{\partial^3 u}{\partial t^3}(\cdot, s)\, ds$$

and so

$$\tau \sum_{k=0}^{J-1} \|\sigma^k\|^2 \leqq \frac{1}{5!}\tau^4 \left\|\frac{\partial^3 u}{\partial t^3}\right\|^2_{L^2(0,T;L^2(\Omega))} \tag{4.25}$$

Also from Lemma 2.1 and the fact that

$$\partial_\tau \eta^k = \frac{1}{\tau} \int_{k\tau}^{(k+1)\tau} \frac{\partial \eta}{\partial t}(\cdot, s)\, ds,$$

$$\|\partial_\tau \eta^k\|^2 \leqq \frac{1}{\tau} \int_{k\tau}^{(k+1)\tau} \left\|\frac{\partial \eta}{\partial t}(\cdot, s)\right\|^2 ds,$$

and so

$$\tau \sum_{k=0}^{J-1} \|\partial_\tau \eta^k\|^2 \leqq \left\|\frac{\partial \eta}{\partial t}\right\|^2_{L^2(0,T;L^2(\Omega))} \leqq Ch^{2r} \left\|\frac{\partial u}{\partial t}\right\|^2_{L^2(0,T;H^r(\Omega))} \tag{4.26}$$

Now, from (4.19), with empty sums set to zero, for $0 \leqq n \leqq J - 1$,

$$\begin{aligned} \|\varepsilon^n\|^2 &\leqq 4\left\{\|\partial_\tau \eta^n\|^2 + \frac{\tau^2}{4}\left\|\sum_{k=0}^{n} \rho^k\right\|^2 + \frac{\tau^2}{4}\left\|\sum_{k=0}^{n-1} \rho^k\right\|^2 + \|\sigma^n\|^2\right\} \\ &\leqq 4\left\{\|\partial_\tau \eta^n\|^2 + \frac{\tau^2}{4} J \sum_{k=0}^{J-1} \|\rho^k\|^2 + \|\sigma^n\|^2\right\}, \\ &= 4\left\{\|\partial_\tau \eta^n\|^2 + \frac{T}{2}\left(\tau \sum_{k=0}^{J-1} \|\rho^k\|^2\right) + \|\sigma^n\|^2\right\}, \end{aligned} \tag{4.27}$$

Hence, from (4.24)–(4.27),

$$\tau \sum_{k=0}^{J-1} \|\varepsilon^n\|^2 \tag{4.28}$$
$$\leqq 4Ch^{2r}\left\|\frac{\partial u}{\partial t}\right\|^2_{L^2(0,t;H^r(\Omega))} + \frac{2T^2}{5!}\tau^4\left\|\frac{\partial^4 u}{\partial t^4}\right\|^2_{L^2(0,T;L^2(\Omega))} + \frac{4}{5!}\tau^4\left\|\frac{\partial^3 u}{\partial t^3}\right\|^2_{L^2(0,T;L^2(\Omega))}$$

Finally, combining (4.28) and (4.23), we obtain

$$\max_{0\leqq n\leqq J} \|\xi^n\|$$
$$\leqq \sqrt{2}\|\xi^0\| + C(T)\left\{h^r\left\|\frac{\partial u}{\partial t}\right\|_{L^2(0,T;H^r(\Omega))} + \tau^2\left[\left\|\frac{\partial^3 u}{\partial t_3}\right\|_{L^2(0,T;L^2(\Omega))} + \left\|\frac{\partial^4 u}{\partial t^4}\right\|_{L^2(0,T;L^2(\Omega))}\right]\right\}.$$

The result of Lemma 4.2 now follows. □

THEOREM 4.1. *Let u be the solution of* (2.3)–(2.5), *and let* $\{U^n\}_{n=0}^J \subset S_h^r(\Omega)$ *be the sequence defined by* (4.1)–(4.4).

Suppose that $u \in L^\infty(0, T; H^r(\Omega))$,

$$\frac{\partial u}{\partial t} \in L^2(0, T; H^r(\Omega)) \quad \textit{and} \quad \left(\frac{\partial}{\partial t}\right)^k u \in L^2(0, T; L^2(\Omega))$$

for $k = 3, 4$. *Then there exists a constant* $C_5 = C_5(T)$ *independent of h and* τ *such that*

$$\max_{0\leqq n\leqq J} \|u(\cdot, n\tau) - U^n\| \leqq C_5\{h^r + \tau^2\}.$$

Proof. From (4.1) and (2.6), we have

$$\|U_0 - u(\cdot, 0)\| \leqq Ch^r\|u_0\|_r \leqq Ch^r\|u\|_{L^\infty(0,T;H^r(\Omega))},$$

and so from Lemma 2.1,

$$\|\xi^0\| \leqq \|\eta^0\| + \|U^0 - u(\cdot, 0)\| \leqq Ch^r\|u\|_{L^\infty(0,T;H^r(\Omega))}. \tag{4.29}$$

From Lemma 2.1 and Lemma 4.2 with (4.29),

$$\|u(\cdot, n\tau) - U^n\|$$
$$\leqq \|\eta^n\| + \sqrt{2}\|\xi^0\| + C\left\{h^r\left\|\frac{\partial u}{\partial t}\right\|_{L^2(0,T;H^r(\Omega))} + \tau^2\left[\left\|\frac{\partial^3 u}{\partial t^3}\right\|_{L^2(0,T;L^2(\Omega))} + \left\|\frac{\partial^4 u}{\partial t^4}\right\|_{L^2(0,T;L^2(\Omega))}\right]\right\}$$
$$\leqq C(T)\left\{h^r\left[\|u\|_{L^\infty(0,T;H^r(\Omega))} + \left\|\frac{\partial u}{\partial t}\right\|_{L^2(0,T;H^r(\Omega))}\right] + \tau^2\left[\left\|\frac{\partial^3 u}{\partial t^3}\right\|_{L^2(0,T;L^2(\Omega))} + \left\|\frac{\partial^4 u}{\partial t^4}\right\|_{L^2(0,T;L^2(\Omega))}\right]\right\}$$

The result of Theorem 4.1 now follows. □

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