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On the existence, uniqueness and stability of entropy solutions to scalar conservation laws

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ABSTRACT

We consider one-dimensional scalar conservation laws with and without viscosity where the flux function $F(x,t,u)$ is only assumed to be absolutely continuous in x, locally integrable in t and continuous in u. The existence and uniqueness of entropy solutions for the associated initial-value problem are obtained through the vanishing viscosity method and the doubling variables technique. We also prove the stability of entropy solutions in $C([0,T];L^1_{loc}(\mathbb{R}))$ and in $C([0,T];L^1(\mathbb{R}))$ with respect to both initial data and flux functions.

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1. Introduction

In this paper we study the existence, uniqueness and stability of entropy solutions to the quasilinear parabolic–hyperbolic equation

$$\begin{cases} \partial_t u + \partial_x\big[F(x,t,u)\big] = \lambda u_{xx} & \text{in } Q_T := \mathbb{R}\times(0,T), \\ u(\cdot,0) = u^0 & \text{in } \mathbb{R}, \end{cases} \tag{1}$$

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where $\lambda \geqslant 0$ and the flux function F depends not only on u but also on x and t. When F is assumed to be Lipschitz continuous in u and is independent of x and t, Eq. (1) becomes a classical problem in scalar conservation laws (see for example [7,12,16,30]). More recently (1) has been studied in full generality by many authors (see [2,3,11,17–19,29]) as it appears in several applications including porous media flow [10,14], sedimentation-consolidation processes [8], traffic flow and blood flow. It is also interesting to note that the scalar equation (1) can be used to analyze the following system of equations:

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = \lambda \partial^2_{xx}\rho & \text{in } \mathbb{R} \times (0,T), \\ \partial_t(\rho v) + \partial_x\big(\rho v^2\big) = \rho(\alpha \partial_x \Phi + \beta) + \lambda \partial_x(v \partial_x \rho) & \text{in } \mathbb{R} \times (0,T), \\ \partial^2_{xx}\Phi = \rho & \text{in } \mathbb{R}, \\ \rho(\cdot,0) = \rho^0 \in \mathcal{P}_2(\mathbb{R}), \quad v(\cdot,0) = v^0 \in L^2(\rho_0) \end{cases} \tag{2}$$

where $\alpha, \beta \in \mathbb{R}$, $\lambda \geqslant 0$ are given numbers and $\mathcal{P}_2(\mathbb{R})$ is the set of all Borel probability measures on $\mathbb{R}$ with finite quadratic moments. A special case of (2) is the pressureless Euler system ($\alpha = \beta = \lambda = 0$) which was first introduced by Zeldovich [31] to model the evolution of a sticky particle system. A system of this type consists of a finite collection of particles that move freely along a given line in the absence of forces. Moreover the particles stick to each other upon collision to form compound particles with masses equal to the total mass of particles involved in the collision and velocities are determined by the conservation of momentum. The system (2) also includes the pressureless, attractive/repulsive Euler–Poisson system with zero background charge ($\alpha = \pm 1$ and $\beta = \lambda = 0$) used to model gravitationally interacting particles that stick upon collision. Based on previous works [6,13,25], we show in [24] that a solution (ρ, v) of (2) can be obtained from a solution u of the scalar conservation law (1) with $u^0(x) := \rho_0(-\infty, x]$ and the flux function $F : [0,\infty) \times [0,1] \longrightarrow \mathbb{R}$ is given by

$$F(t,u) := \int_0^u v^0\big(N_0(\omega)\big)\,d\omega + t \int_0^u (\alpha\omega + \beta)\,d\omega. \tag{3}$$

Here N_0 is the generalized inverse of u^0 and coincides with the right-continuous optimal map pushing the probability measure $\chi_{(0,1)}\,dx$ forward to ρ_0. Notice that $u \mapsto F(t,u)$ is a function in $W^{1,2}(0,1)$ that is neither Lipschitz continuous nor non-degenerate.

One of the main purposes of this paper is to extend the uniqueness results in [17,19] to more general flux functions. There are many works devoted to studying the uniqueness of entropy solutions to Eq. (1) by using the celebrated doubling variables technique of Kruzhkov [20] and its extension by Carrillo [9]. On one hand, Karlsen and Ohlberger in [17] and Karlsen and Risebro in [19] considered the general flux $F(x,t,u)$ and were able to prove the L^1-contraction property for entropy solutions under some conditions on the flux function which imply, in particular, that $F(x,t,u)$ has to be Lipschitz continuous in both x and u and bounded in t. On the other hand, it was shown by Panov [27] and Maliki and Toure [23] that if $F(x,t,u) = \varphi(u)$ and φ is continuous then (1) has a unique entropy solution (see also [1,4,5,21]). These authors actually considered the problem in $\mathbb{R}^n$, however it is known from the counterexample of Kruzhkov and Panov [21] that, when $n > 1$ and $F(x,t,u) = \varphi(u)$, mere continuity of the flux in u is *not sufficient* to obtain L^1-contractivity. For this reason and as we are interested in results which include the specific flux (3), we restrict our study to the real line $\mathbb{R}$ and consider flux functions of the form

$$F(x,t,u) = K_1(x,t)f_1(u) + K_2(x,t)f_2(u) + \cdots + K_N(x,t)f_N(u). \tag{4}$$

We then prove in Theorem 2.2 that if $\lambda \geqslant 0$, $K_i \in L^1(0,T; W^{1,1}_{loc}(\mathbb{R}))$ and $f_i \in C(\mathbb{R})$, then entropy solutions of (1) have the L^1-contraction property which yields uniqueness as a consequence. This is one of our main results and the conditions on K_i and f_i are very minimal in view of the well-definedness of the definition of entropy solutions (see Definition 2.1). The result is obtained by carefully adapting the doubling of variables technique of Kruzhkov and through an approximation step. We also use some ideas previously employed by the authors of [1,9,17,19,27,23]. We remark that it is possible to include a source term into Eq. (1) and allow for a more general diffusion function (i.e. $A(u)$ instead of λu) as in [17,19,23] but, for simplicity, we choose not to consider these generalities here.

We also prove the existence of entropy solutions of (1) for flux functions of the form (4) and for bounded initial data. There are scattering results about existence of solutions under different assumptions and using different methods, beginning with the pioneering works of Oleinik [26] and Kruzhkov [20]. The case $\lambda = 0$ and $F(x,t,u) = \varphi(u)$ with continuous φ was considered by Andreianov, Benilan and Kruzhkov [1], and Panov [27]. Maliki and Toure established in [23] the existence result for a general diffusion function and $F(x,t,u) = \varphi(u)$. In [18], Karlsen and Risebro studied the situation where $F(x,t,u) = f(k(x),u)$ and, by proving convergence of finite difference schemes, they obtained the existence of an entropy solution when f satisfies strong smoothness assumptions (see Section 2.1 in [18]) and u^0, k, k' are in $BV \cap L^1 \cap L^\infty$. We note that the dependence on x makes the existence problem more difficult as the equation is no longer translation invariant. Recently, Panov investigated in [29] the hyperbolic case (i.e. $\lambda = 0$) for a very general flux function $F(x,t,u)$ and he was able to prove the existence of an entropy solution by taking appropriate smooth approximations $F_m(x,t,u)$ of $F(x,t,u)$ and studying the limit via localization principle for H-measures of entropy solutions to the equation $\partial_t u^m + \partial_x F_m(x,t,u^m) = 0$. Our existence result in the particular case $\lambda = 0$ can follow from [29], however our proof is via the vanishing viscosity method which is of independent interest because it shows in addition that solutions of the viscous problems converge to that of the inviscid problem (Theorem 3.6). As a consequence, we are able to avoid imposing the non-degenerate condition on F as in [29] when $F(x,t,u)$ is independent of x (see Theorem 3.7); thus allows for flux functions such as (3).

Besides the results about existence and uniqueness, in this work we also investigate the continuous dependence of the unique entropy solution with respect to the flux function $F(t,u) = \sum_{i=1}^N A_i(t) f_i(u)$ and the bounded initial data u^0. Our main stability result (Theorem 4.1) generalizes the result by Maliki in [22] where $F = \varphi(u)$ was considered. We show that if $F_n(\cdot,u) \to F(\cdot,u)$ in $L^1(0,T)$, $F_n(t,\cdot) \to F(t,\cdot)$ in $C(\mathbb{R})$ and $u_n^0 \to u^0$ in $L^1_{loc}(\mathbb{R})$, then the corresponding entropy solutions u_n converge in $C([0,T]; L^1_{loc}(\mathbb{R}))$ to the entropy solution u of the initial-value problem $\partial_t u + \partial_x F(t,u) = \lambda u_{xx}$, $u(\cdot,0) = u^0$. Moreover, the convergence is in $C([0,T]; L^1(\mathbb{R}))$ if u_n^0 converges to u^0 in $L^1(\mathbb{R})$.

We end the introduction by pointing out that there is some recent progress in studying the well-posedness for inviscid scalar conservation laws when the flux depends discontinuously on the space variable x. Chen, Even and Klingenberg considered in [11] flux functions of the form $F(x,u)$ under very special structural conditions, while Andreianov, Karlsen and Risebro studied in [3] the flux function defined by $F(x,u) := \chi_{(-\infty,0)}(x) f^l(u) + \chi_{(0,\infty)}(x) f^r(u)$. It would be interesting to know what happen in the case $F(x,t,u)$ is given by (6) with $K_i \in L^1(0,T; BV_{loc}(\mathbb{R}))$ and $f_i \in C(\mathbb{R})$. One of the main obstacles is to find a correct notion of solutions noticing that the second integral in Definition 2.1 does not make sense anymore due to the presence of the term $\int_{Q_T} \operatorname{sign}(u-k)\phi \, dF_x^s(x,t,k)$, where F_x^s is the singular part of F_x. Panov introduced in [29] a notion of entropy solutions for very general flux functions by replacing the undefined term by its well-defined upper bound $\int_{Q_T} \phi \, d|F_x^s|(x,t,k)$. This makes the existence of solutions plausible but one should not expect uniqueness when F_x has a singular part.

The paper is organized as follows. In Section 2 we prove the Kato-type inequality and then use it to establish the L^1-contraction principle for entropy solutions. We study the existence of an entropy solution for bounded initial data in Section 3: first for the viscous scalar conservation law in Section 3.1 and then for the inviscid scalar conservation law in Section 3.2. Finally, Section 4 contains results about continuous dependence of the entropy solution with respect to the flux function and the initial data.

2. Uniqueness of entropy solutions

We consider the quasilinear parabolic–hyperbolic equation

$$\begin{cases} \partial_t u + \partial_x\big[F(x,t,u)\big] = \lambda u_{xx} & \text{in } Q_T := \mathbb{R}\times(0,T), \\ u(\cdot,0) = u^0 & \text{in } \mathbb{R}, \end{cases} \tag{5}$$

where $\lambda \geqslant 0$ and the flux function has the form

$$F(x,t,z) = K_1(x,t)f_1(z) + K_2(x,t)f_2(z) + \cdots + K_N(x,t)f_N(z) \tag{6}$$

with $f_i : \mathbb{R}\to\mathbb{R}$ being continuous and $K_i \in L^1(0,T; W^{1,1}_{loc}(\mathbb{R}))$. Let us recall the definition of entropy solutions.

Definition 2.1. Let $u^0 \in L^\infty(\mathbb{R})$. A function $u \in L^\infty(Q_T)$ is an entropy solution of (5) if $\lambda u \in L^2(0,T; H^1_{loc}(\mathbb{R}))$ and

$$\int_{Q_T} \big\{|u-k|\phi_t + \operatorname{sign}(u-k)\big[F(x,t,u) - F(x,t,k)\big]\phi_x\big\}\,dt\,dx - \int_{Q_T} \operatorname{sign}(u-k)F_x(x,t,k)\phi\,dt\,dx$$
$$+ \int_{\mathbb{R}} \big|u^0(x) - k\big|\phi(x,0)\,dx \geqslant \lambda \int_{Q_T} \operatorname{sign}(u-k)u_x\phi_x\,dt\,dx$$

for all $k\in\mathbb{R}$ and all nonnegative test functions $\phi \in C_0^\infty(\mathbb{R}\times[0,T))$.

In this section we establish the following L^1-contraction principle which yields in particular the uniqueness of entropy solutions to Eq. (5).

Theorem 2.2. *Assume $\lambda \geqslant 0$, $f_i \in C(\mathbb{R})$ and $K_i \in L^1(0,T; L^\infty(\mathbb{R})) \cap L^2(0,T; L^2_{loc}(\mathbb{R})) \cap L^1(0,T; W^{1,1}_{loc}(\mathbb{R}))$. Suppose u, v are entropy solutions of (5) with initial data $u^0, v^0 \in L^\infty(\mathbb{R})$ respectively. Then*

$$\int_{\mathbb{R}} \big(u(x,t) - v(x,t)\big)^+\,dx \leqslant \int_{\mathbb{R}} \big(u^0(x) - v^0(x)\big)^+\,dx \quad \textit{for a.e. } t\in(0,T). \tag{7}$$

The proof of Theorem 2.2 is based on the Kato-type inequality discussed in the next subsection.

2.1. Kato-type inequality

For each $\eta > 0$, we define the continuous approximations to the sign^+ and sign^- functions:

$$\operatorname{sign}^+_\eta(z) := \begin{cases} 1 & \text{for } z \geqslant \eta, \\ \frac{z}{\eta} & \text{for } 0 \leqslant z \leqslant \eta, \\ 0 & \text{for } z \leqslant 0 \end{cases} \quad \text{and} \quad \operatorname{sign}^-_\eta(z) := \begin{cases} 0 & \text{for } z \geqslant 0, \\ \frac{z}{\eta} & \text{for } -\eta \leqslant z \leqslant 0, \\ -1 & \text{for } z \leqslant -\eta. \end{cases}$$

We will need the following result about the entropy dissipation term which can be found in [15,19,23] and originates from an important observation by Carrillo [9, Lemma 5].

Lemma 2.3. *Assume $\lambda > 0$, $f_i \in C(\mathbb{R})$ and $K_i \in L^2(0,T;L^2_{loc}(\mathbb{R})) \cap L^1(0,T;W^{1,1}_{loc}(\mathbb{R}))$. Suppose $u^0 \in L^\infty(\mathbb{R})$ and $u \in L^\infty(Q_T) \cap L^2(0,T;H^1_{loc}(\mathbb{R}))$ satisfies*

$$\begin{cases} \int\limits_{Q_T} \big(u\varphi_t + \big[F(x,t,u) - \lambda u_x\big]\varphi_x\big)\,dt\,dx = 0 & \forall \varphi \in C_0^\infty(Q_T), \\ \operatorname{esslim}_{t\to 0^+} u(\cdot,t) = u^0 & \text{in } L^1_{loc}(\mathbb{R}). \end{cases} \tag{8}$$

Then for any $k \in \mathbb{R}$ and any nonnegative function $\phi \in C_0^\infty(\mathbb{R}\times[0,T))$, we have

$$\int\limits_{Q_T} \big\{(u-k)^+\phi_t + \operatorname{sign}^+(u-k)\big[F(x,t,u) - F(x,t,k) - \lambda u_x\big]\phi_x - \operatorname{sign}^+(u-k)F_x(x,t,k)\phi\big\}\,dt\,dx$$
$$+ \int\limits_{\mathbb{R}} \big(u^0(x)-k\big)^+\phi(x,0)\,dx = \lim_{\eta\downarrow 0}\int\limits_{Q_T} |\lambda u_x|^2\big(\operatorname{sign}^+_\eta\big)'\big(\lambda(u-k)\big)\phi\,dt\,dx$$

and

$$\int\limits_{Q_T} \big\{(u-k)^-\phi_t + \operatorname{sign}^-(u-k)\big[F(x,t,u) - F(x,t,k) - \lambda u_x\big]\phi_x - \operatorname{sign}^-(u-k)F_x(x,t,k)\phi\big\}\,dt\,dx$$
$$+ \int\limits_{\mathbb{R}} \big(u^0(x)-k\big)^-\phi(x,0)\,dx = \lim_{\eta\downarrow 0}\int\limits_{Q_T} |\lambda u_x|^2\big(\operatorname{sign}^-_\eta\big)'\big(\lambda(u-k)\big)\phi\,dt\,dx.$$

It is well known that (see for example [27]) any entropy solution of (5) satisfies (8). Therefore by combining with Lemma 2.3 we deduce that:

Remark 2.4. Assume $\lambda > 0$ and $u \in L^\infty(Q_T) \cap L^2(0,T;H^1_{loc}(\mathbb{R}))$. Then u is an entropy solution of (5) if and only if (8) holds.

Lemma 2.3 together with the doubling of variables technique of Kruzhkov gives the following Kato-type inequality.

Lemma 2.5. *Assume $f_i : \mathbb{R} \to \mathbb{R}$ are continuous and $K_i \in L^2(0,T;L^2_{loc}(\mathbb{R})) \cap L^1(0,T;W^{1,1}_{loc}(\mathbb{R}))$. Suppose u, v are entropy solutions of (5) with initial data $u^0, v^0 \in L^\infty(\mathbb{R})$ respectively. Then for any nonnegative test function $\psi \in C_0^\infty(\mathbb{R}\times[0,T))$, we have*

$$\int\limits_{Q_T} \big\{(u-v)^+\psi_t + \operatorname{sign}^+(u-v)\big[F(x,t,u) - F(x,t,v)\big]\psi_x + (\lambda u - \lambda v)^+\psi_{xx}\big\}\,dt\,dx$$
$$+ \int\limits_{\mathbb{R}} \big(u^0 - v^0\big)^+\psi(x,0)\,dx \geqslant 0. \tag{9}$$

Proof. Let $\phi \in C_0^\infty(\hat{Q}_T \times \hat{Q}_T)$, $\phi \geqslant 0$, $\phi = \phi(x,t,y,s)$, where $\hat{Q}_T := \mathbb{R}\times[0,T)$. We will write $u = u(x,t)$ and $v = v(y,s)$. Assume for the moment that $\lambda > 0$. Then it follows from Remark 2.4 and Lemma 2.3 that

$$\int_{Q_T}\int_{Q_T}\big\{(u-v)^+\phi_t+\operatorname{sign}^+(u-v)\big[F(x,t,u)-F(x,t,v)-\lambda u_x\big]\phi_x\big\}\,dt\,dx\,ds\,dy$$
$$-\int_{Q_T}\int_{Q_T}\operatorname{sign}^+(u-v)F_x(x,t,v)\phi\,dt\,dx\,ds\,dy+\int_{Q_T}\int_{\mathbb{R}}\big(u^0(x)-v(y,s)\big)^+\phi(x,0,y,s)\,dx\,ds\,dy$$
$$=\lim_{\eta\downarrow 0}\int_{Q_T}\int_{Q_T}|\lambda u_x|^2\big(\operatorname{sign}_\eta^+\big)'\big(\lambda(u-v)\big)\phi\,dt\,dx\,ds\,dy.$$

Moreover, integrating by parts in the y variable gives

$$-\int_{Q_T}\int_{Q_T}\operatorname{sign}^+(u-v)(\lambda u_x)\phi_y\,dt\,dx\,ds\,dy=-\lim_{\eta\downarrow 0}\int_{Q_T}\int_{Q_T}\operatorname{sign}_\eta^+\big(\lambda(u-v)\big)(\lambda u_x)\phi_y\,dt\,dx\,ds\,dy$$
$$=-\lim_{\eta\downarrow 0}\int_{Q_T}\int_{Q_T}(\lambda u_x)(\lambda v_y)\big(\operatorname{sign}_\eta^+\big)'\big(\lambda(u-v)\big)\phi\,dt\,dx\,ds\,dy.$$

Therefore by adding together, we obtain

$$\int_{Q_T}\int_{Q_T}\big\{(u-v)^+\phi_t+\operatorname{sign}^+(u-v)\big[\big(F(x,t,u)-F(x,t,v)\big)\phi_x-\lambda u_x(\phi_x+\phi_y)\big]\big\}\,dt\,dx\,ds\,dy$$
$$-\int_{Q_T}\int_{Q_T}\operatorname{sign}^+(u-v)F_x(x,t,v)\phi\,dt\,dx\,ds\,dy+\int_{Q_T}\int_{\mathbb{R}}\big(u^0(x)-v(y,s)\big)^+\phi(x,0,y,s)\,dx\,ds\,dy$$
$$=\lim_{\eta\downarrow 0}\int_{Q_T}\int_{Q_T}\big(|\lambda u_x|^2-(\lambda u_x)(\lambda v_y)\big)\big(\operatorname{sign}_\eta^+\big)'\big(\lambda(u-v)\big)\phi\,dt\,dx\,ds\,dy. \tag{10}$$

Using the second identity in Lemma 2.3 and arguing similarly as above, we also have

$$\int_{Q_T}\int_{Q_T}\big\{(v-u)^-\phi_s+\operatorname{sign}^-(v-u)\big[\big(F(y,s,v)-F(y,s,u)\big)\phi_y-\lambda v_y(\phi_x+\phi_y)\big]\big\}\,dt\,dx\,ds\,dy$$
$$-\int_{Q_T}\int_{Q_T}\operatorname{sign}^-(v-u)F_y(y,s,u)\phi\,dt\,dx\,ds\,dy+\int_{Q_T}\int_{\mathbb{R}}\big(v^0(y)-u(x,t)\big)^-\phi(x,t,y,0)\,dy\,dt\,dx$$
$$=\lim_{\eta\downarrow 0}\int_{Q_T}\int_{Q_T}\big(|\lambda v_y|^2-(\lambda u_x)(\lambda v_y)\big)\big(\operatorname{sign}_\eta^-\big)'\big(\lambda(v-u)\big)\phi\,dt\,dx\,ds\,dy. \tag{11}$$

As $z^-=(-z)^+$, $\operatorname{sign}^-(z)=-\operatorname{sign}^+(-z)$ and $(\operatorname{sign}_\eta^-)'(z)=(\operatorname{sign}_\eta^+)'(-z)$, we obtain by adding (10) and (11) that

$$\int_{Q_T}\int_{Q_T}(u-v)^+(\phi_t+\phi_s)\,dt\,dx\,ds\,dy$$
$$+\int_{Q_T}\int_{Q_T}\operatorname{sign}^+(u-v)\big[F(x,t,u)-F(y,s,v)\big](\phi_x+\phi_y)\,dt\,dx\,ds\,dy$$

$$
\begin{aligned}
&+\int\limits_{Q_T}\int\limits_{Q_T} \operatorname{sign}^+(u-v)\big\{\big[F(y,s,v)-F(x,t,v)\big]\phi_x-\big[F(x,t,u)-F(y,s,u)\big]\phi_y\big\}\,dt\,dx\,ds\,dy\\
&+\int\limits_{Q_T}\int\limits_{Q_T} \big\{\operatorname{sign}^+(u-v)(\lambda v_y-\lambda u_x)(\phi_x+\phi_y)\big\}\,dt\,dx\,ds\,dy\\
&+\int\limits_{Q_T}\int\limits_{Q_T} \operatorname{sign}^+(u-v)\big[F_y(y,s,u)-F_x(x,t,v)\big]\phi\,dt\,dx\,ds\,dy\\
&+\int\limits_{Q_T}\int\limits_{\mathbb{R}} \big(u^0(x)-v(y,s)\big)^+\phi(x,0,y,s)\,dx\,ds\,dy+\int\limits_{Q_T}\int\limits_{\mathbb{R}} \big(u(x,t)-v^0(y)\big)^+\phi(x,t,y,0)\,dy\,dt\,dx\\
&=\lim_{\eta\downarrow 0}\int\limits_{Q_T}\int\limits_{Q_T} |\lambda u_x-\lambda v_y|^2\big(\operatorname{sign}^+_\eta\big)'\big(\lambda(u-v)\big)\phi\,dt\,dx\,ds\,dy\geqslant 0.
\end{aligned}
$$

Also it follows from

$$
\begin{aligned}
\operatorname{sign}^+(u-v)(\lambda v_y-\lambda u_x)&=\operatorname{sign}^+\big(\lambda(u-v)\big)\lambda v_y-\operatorname{sign}^+\big(\lambda(u-v)\big)\lambda u_x\\
&=-(\lambda u-\lambda v)^+_y-(\lambda u-\lambda v)^+_x
\end{aligned}
$$

and integration by parts that

$$
\begin{aligned}
&\int\limits_{Q_T}\int\limits_{Q_T} \big\{\operatorname{sign}^+(u-v)(\lambda v_y-\lambda u_x)(\phi_x+\phi_y)\big\}\,dt\,dx\,ds\,dy\\
&\quad=-\int\limits_{Q_T}\int\limits_{Q_T}(\lambda u-\lambda v)^+_y(\phi_x+\phi_y)\,dt\,dx\,ds\,dy-\int\limits_{Q_T}\int\limits_{Q_T}(\lambda u-\lambda v)^+_x(\phi_x+\phi_y)\,dt\,dx\,ds\,dy\\
&\quad=\int\limits_{Q_T}\int\limits_{Q_T}(\lambda u-\lambda v)^+(\phi_{xx}+2\phi_{xy}+\phi_{yy})\,dt\,dx\,ds\,dy.
\end{aligned}
$$

Thus the above relation can be rewritten as

$$
\begin{aligned}
&\int\limits_{Q_T}\int\limits_{Q_T}(u-v)^+(\phi_t+\phi_s)\,dt\,dx\,ds\,dy\\
&\quad+\int\limits_{Q_T}\int\limits_{Q_T} \operatorname{sign}^+(u-v)\big[F(x,t,u)-F(y,s,v)\big](\phi_x+\phi_y)\,dt\,dx\,ds\,dy\\
&\quad+\int\limits_{Q_T}\int\limits_{Q_T}(\lambda u-\lambda v)^+(\phi_{xx}+2\phi_{xy}+\phi_{yy})\,dt\,dx\,ds\,dy\\
&\quad+\int\limits_{Q_T}\int\limits_{Q_T} \operatorname{sign}^+(u-v)\big[F_y(y,s,u)-F_x(x,t,v)\big]\phi\,dt\,dx\,ds\,dy\\
&\quad+\int\limits_{Q_T}\int\limits_{Q_T} \operatorname{sign}^+(u-v)\big\{\big[F(y,s,v)-F(x,t,v)\big]\phi_x-\big[F(x,t,u)-F(y,s,u)\big]\phi_y\big\}\,dt\,dx\,ds\,dy
\end{aligned}
$$

$$+\int_{Q_T}\int_{\mathbb{R}}\big(u^0(x)-v(y,s)\big)^+\phi(x,0,y,s)\,dx\,ds\,dy+\int_{Q_T}\int_{\mathbb{R}}\big(u(x,t)-v^0(y)\big)^+\phi(x,t,y,0)\,dy\,dt\,dx$$
$$\geqslant 0. \tag{12}$$

We derived (12) for $\lambda > 0$. In the case $\lambda = 0$, Definition 2.1 implies that

$$\partial_t(u-k)^+ + \partial_x\big[\operatorname{sign}^+(u-k)\big(F(x,t,u)-F(x,t,k)\big)\big] + \operatorname{sign}^+(u-k)F_x(x,t,k) \leqslant 0, \quad \text{and}$$
$$\partial_t(u-k)^- + \partial_x\big[\operatorname{sign}^-(u-k)\big(F(x,t,u)-F(x,t,k)\big)\big] + \operatorname{sign}^-(u-k)F_x(x,t,k) \leqslant 0$$

in $\mathcal{D}'(Q_T)$. Hence by inspecting the arguments leading to (12), we see that inequality (12) also holds for $\lambda = 0$ as well.

Next for $\rho > 0$, set $\delta_\rho(z) := \frac{1}{\rho}\delta(\frac{z}{\rho})$ where $\delta \in C_0^\infty(\mathbb{R})$ is given by

$$\delta(z) := \begin{cases} Ce^{\frac{-1}{1-z^2}} & \text{if } |z| < 1, \\ 0 & \text{if } |z| \geqslant 1 \end{cases} \tag{13}$$

with $C > 0$ being chosen such that $\int_{\mathbb{R}} \delta(\sigma)\,d\sigma = 1$. Consider any nonnegative test function $\psi \in C_0^\infty(\hat{Q}_T)$, and define $\phi \in C_0^\infty(\hat{Q}_T \times \hat{Q}_T)$ by $\phi(x,t,y,s) := \psi(\frac{x+y}{2}, \frac{t+s}{2})\delta_h(\frac{x-y}{2})\delta_\rho(\frac{t-s}{2})$. If we let ψ_X and ψ_T denote the derivatives of ψ in its first and second variables respectively, then a direct computation yields

$$(\phi_t+\phi_s)(x,t,y,s) = \psi_T\left(\frac{x+y}{2},\frac{t+s}{2}\right)\delta_h\left(\frac{x-y}{2}\right)\delta_\rho\left(\frac{t-s}{2}\right),$$
$$\phi_x(x,t,y,s) = \frac{1}{2}\left[\psi_X\left(\frac{x+y}{2},\frac{t+s}{2}\right)\delta_h\left(\frac{x-y}{2}\right)+\psi\left(\frac{x+y}{2},\frac{t+s}{2}\right)\delta_h'\left(\frac{x-y}{2}\right)\right]\delta_\rho\left(\frac{t-s}{2}\right),$$
$$\phi_y(x,t,y,s) = \frac{1}{2}\left[\psi_X\left(\frac{x+y}{2},\frac{t+s}{2}\right)\delta_h\left(\frac{x-y}{2}\right)-\psi\left(\frac{x+y}{2},\frac{t+s}{2}\right)\delta_h'\left(\frac{x-y}{2}\right)\right]\delta_\rho\left(\frac{t-s}{2}\right),$$
$$(\phi_x+\phi_y)(x,t,y,s) = \psi_X\left(\frac{x+y}{2},\frac{t+s}{2}\right)\delta_h\left(\frac{x-y}{2}\right)\delta_\rho\left(\frac{t-s}{2}\right),$$
$$(\phi_{xx}+2\phi_{xy}+\phi_{yy})(x,t,y,s) = \psi_{XX}\left(\frac{x+y}{2},\frac{t+s}{2}\right)\delta_h\left(\frac{x-y}{2}\right)\delta_\rho\left(\frac{t-s}{2}\right).$$

Using this test function, it follows from (12) that

$$\int_{Q_T}\int_{Q_T}(I_0+I_1+I_2+I_3+I_4)\delta_h\left(\frac{x-y}{2}\right)\delta_\rho\left(\frac{t-s}{2}\right)dt\,dx\,ds\,dy$$
$$+\frac{1}{2}\int_{Q_T}\int_{Q_T} I_5\psi\left(\frac{x+y}{2},\frac{t+s}{2}\right)\delta_h'\left(\frac{x-y}{2}\right)\delta_\rho\left(\frac{t-s}{2}\right)dt\,dx\,ds\,dy$$
$$+\int_0^T\int_{\mathbb{R}^2}\left\{\big(u^0(x)-v(y,t)\big)^+ + \big(u(x,t)-v^0(y)\big)^+\right\}\psi\left(\frac{x+y}{2},\frac{t}{2}\right)\delta_h\left(\frac{x-y}{2}\right)\delta_\rho\left(\frac{t}{2}\right)dx\,dy\,dt$$
$$\geqslant 0, \tag{14}$$

where

$$I_0 := (u-v)^+\psi_T\left(\frac{x+y}{2}, \frac{t+s}{2}\right),$$

$$I_1 := \operatorname{sign}^+(u-v)\big[F(x,t,u)-F(y,s,v)\big]\psi_X\left(\frac{x+y}{2}, \frac{t+s}{2}\right),$$

$$I_2 := (\lambda u-\lambda v)^+\psi_{XX}\left(\frac{x+y}{2}, \frac{t+s}{2}\right),$$

$$I_3 := \operatorname{sign}^+(u-v)\big[F_y(y,s,u)-F_x(x,t,v)\big]\psi\left(\frac{x+y}{2}, \frac{t+s}{2}\right),$$

$$I_4 := \operatorname{sign}^+(u-v)\big\{\big[F(y,s,v)-F(x,t,v)\big]-\big[F(x,t,u)-F(y,s,u)\big]\big\}\frac{1}{2}\psi_X\left(\frac{x+y}{2}, \frac{t+s}{2}\right),$$

$$I_5 := \operatorname{sign}^+(u-v)\big\{\big[F(y,s,v)-F(x,t,v)\big]+\big[F(x,t,u)-F(y,s,u)\big]\big\}.$$

Using the change of variables $\tilde{x}=\frac{x+y}{2}$, $\tilde{y}=\frac{x-y}{2}$, we have

$$\begin{aligned}
&\lim_{h\downarrow 0}\lim_{\rho\downarrow 0}\int_0^T\int_{\mathbb{R}^2}\big\{\big(u^0(x)-v(y,t)\big)^+ + \big(u(x,t)-v^0(y)\big)^+\big\}\psi\left(\frac{x+y}{2},\frac{t}{2}\right)\delta_h\left(\frac{x-y}{2}\right)\delta_\rho\left(\frac{t}{2}\right)dx\,dy\,dt \\
&= 2\lim_{h\downarrow 0}\lim_{\rho\downarrow 0}\int_0^T\int_{\mathbb{R}^2}\big\{\big(u^0(\tilde{x}+\tilde{y})-v(\tilde{x}-\tilde{y},t)\big)^+ + \big(u(\tilde{x}+\tilde{y},t)-v^0(\tilde{x}-\tilde{y})\big)^+\big\} \\
&\quad\times\psi\left(\tilde{x},\frac{t}{2}\right)\delta_h(\tilde{y})\delta_\rho\left(\frac{t}{2}\right)d\tilde{x}\,d\tilde{y}\,dt \\
&= 2\lim_{h\downarrow 0}\int_{\mathbb{R}^2}\big\{\big(u^0(\tilde{x}+\tilde{y})-v^0(\tilde{x}-\tilde{y})\big)^+ + \big(u^0(\tilde{x}+\tilde{y})-v^0(\tilde{x}-\tilde{y})\big)^+\big\}\psi(\tilde{x},0)\delta_h(\tilde{y})\,d\tilde{x}\,d\tilde{y} \\
&= 4\int_{\mathbb{R}}\big(u^0(x)-v^0(x)\big)^+\psi(x,0)\,dx. \qquad (15)
\end{aligned}$$

Similarly, we now use the change of variables $\tilde{x}=\frac{x+y}{2}$, $\tilde{y}=\frac{x-y}{2}$, $\tilde{t}=\frac{t+s}{2}$, $\tilde{s}=\frac{t-s}{2}$, which maps $Q_T\times Q_T$ into $\mathbb{R}\times\mathbb{R}\times\{(\tilde{t},\tilde{s})\colon\ 0\leqslant\tilde{t}+\tilde{s}\leqslant T,\ 0\leqslant\tilde{t}-\tilde{s}\leqslant T\}$. We employ this to obtain the following limits:

$$\begin{aligned}
&\lim_{h\downarrow 0}\lim_{\rho\downarrow 0}\int_{Q_T}\int_{Q_T} I_0\delta_h\left(\frac{x-y}{2}\right)\delta_\rho\left(\frac{t-s}{2}\right)dt\,dx\,ds\,dy \\
&\quad= \lim_{h\downarrow 0}\lim_{\rho\downarrow 0}\int_{Q_T}\int_{Q_T}(u-v)^+\psi_T\left(\frac{x+y}{2},\frac{t+s}{2}\right)\delta_h\left(\frac{x-y}{2}\right)\delta_\rho\left(\frac{t-s}{2}\right)dt\,dx\,ds\,dy \\
&\quad= 4\int_{Q_T}\big(u(x,t)-v(x,t)\big)^+\psi_t(x,t)\,dt\,dx, \qquad (16)
\end{aligned}$$

$$
\begin{aligned}
&\lim_{h\downarrow 0}\lim_{\rho\downarrow 0}\int_{Q_T}\int_{Q_T} I_1\delta_h\left(\frac{x-y}{2}\right)\delta_\rho\left(\frac{t-s}{2}\right)dt\,dx\,ds\,dy\\
&\quad=4\int_{Q_T}\operatorname{sign}^+\big(u(x,t)-v(x,t)\big)\big[F\big(x,t,u(x,t)\big)-F\big(x,t,v(x,t)\big)\big]\psi_x(x,t)\,dt\,dx,
\end{aligned}
\tag{17}
$$

$$
\lim_{h\downarrow 0}\lim_{\rho\downarrow 0}\int_{Q_T}\int_{Q_T} I_2\delta_h\left(\frac{x-y}{2}\right)\delta_\rho\left(\frac{t-s}{2}\right)dt\,dx\,ds\,dy=4\int_{Q_T}\big(\lambda u(x,t)-\lambda v(x,t)\big)^+\psi_{xx}(x,t)\,dt\,dx,
\tag{18}
$$

$$
\begin{aligned}
&\lim_{h\downarrow 0}\lim_{\rho\downarrow 0}\int_{Q_T}\int_{Q_T} I_3\delta_h\left(\frac{x-y}{2}\right)\delta_\rho\left(\frac{t-s}{2}\right)dt\,dx\,ds\,dy\\
&\quad=4\int_{Q_T}\operatorname{sign}^+\big(u(x,t)-v(x,t)\big)\big[F_x\big(x,t,u(x,t)\big)-F_x\big(x,t,v(x,t)\big)\big]\psi(x,t)\,dt\,dx,
\end{aligned}
\tag{19}
$$

$$
\lim_{h\downarrow 0}\lim_{\rho\downarrow 0}\int_{Q_T}\int_{Q_T} I_4\delta_h\left(\frac{x-y}{2}\right)\delta_\rho\left(\frac{t-s}{2}\right)dt\,dx\,ds\,dy=0
\tag{20}
$$

and

$$
\begin{aligned}
&\lim_{h\downarrow 0}\lim_{\rho\downarrow 0}\int_{Q_T}\int_{Q_T} I_5\psi\left(\frac{x+y}{2},\frac{t+s}{2}\right)\delta_h'\left(\frac{x-y}{2}\right)\delta_\rho\left(\frac{t-s}{2}\right)dt\,dx\,ds\,dy\\
&=\lim_{h\downarrow 0}\lim_{\rho\downarrow 0}\int_{Q_T}\int_{Q_T}\operatorname{sign}^+(u-v)\big\{\big[F(y,s,v)-F(x,t,v)\big]+\big[F(x,t,u)-F(y,s,u)\big]\big\}\\
&\quad\times\psi\left(\frac{x+y}{2},\frac{t+s}{2}\right)\delta_h'\left(\frac{x-y}{2}\right)\delta_\rho\left(\frac{t-s}{2}\right)dt\,dx\,ds\,dy\\
&=2\lim_{h\downarrow 0}\int_{\mathbb{R}}\int_{Q_T}\operatorname{sign}^+\big(u(x,t)-v(y,t)\big)\big\{\big[F\big(y,t,v(y,t)\big)-F\big(x,t,v(y,t)\big)\big]\\
&\quad+\big[F\big(x,t,u(x,t)\big)-F\big(y,t,u(x,t)\big)\big]\big\}\psi\left(\frac{x+y}{2},t\right)\delta_h'\left(\frac{x-y}{2}\right)dt\,dx\,dy\\
&=2\sum_{i=1}^{N}\lim_{h\downarrow 0}\int_{\mathbb{R}}\int_{Q_T}\operatorname{sign}^+\big(u(x,t)-v(y,t)\big)\big[f_i\big(u(x,t)\big)-f_i\big(v(y,t)\big)\big]\big[K_i(x,t)-K_i(y,t)\big]\\
&\quad\times\psi\left(\frac{x+y}{2},t\right)\delta_h'\left(\frac{x-y}{2}\right)dt\,dx\,dy\\
&=8\int_{Q_T}\operatorname{sign}^+\big(u(x,t)-v(x,t)\big)\big[F_x\big(x,t,v(x,t)\big)-F_x\big(x,t,u(x,t)\big)\big]\psi(x,t)\,dt\,dx,
\end{aligned}
\tag{21}
$$

where we have employed Lemma 2.6 below to obtain the last identity.

By first letting $\rho\downarrow 0$ and then $h\downarrow 0$ in (14), we conclude from (15)–(21) that the inequality (9) holds. $\square$

In the above proof, the following lemma was used to obtain (21). We prove it by approximating the flux function and the entropy solution so that we can perform integration by parts to handle the term δ_h'.

Lemma 2.6. *Assume $f \in C(\mathbb{R})$ and $K \in L^1(0,T;L^\infty(\mathbb{R})) \cap L^1(0,T;W^{1,1}_{loc}(\mathbb{R}))$. Let*

$$A_h := \int_{\mathbb{R}} \int_{Q_T} H\big(u(x,t), v(y,t)\big)\big[K(x,t) - K(y,t)\big]\psi\left(\frac{x+y}{2}, t\right)\delta_h'\left(\frac{x-y}{2}\right) dt\, dx\, dy$$

where $H(u,v) := \operatorname{sign}^+(u-v)[f(u) - f(v)]$. Then we have

$$\lim_{h\downarrow 0} A_h = -4\int_{Q_T} H\big(u(x,t), v(x,t)\big)\psi(x,t)K_x(x,t)\, dt\, dx. \tag{22}$$

Proof. Since $f \in C(\mathbb{R})$, there exists a sequence $\{f_\epsilon\}$ of Lipschitz functions on $\mathbb{R}$ such that $f_\epsilon \to f$ uniformly on compact subsets of $\mathbb{R}$. Let $H^\epsilon(u,v) := \operatorname{sign}^+(u-v)[f_\epsilon(u) - f_\epsilon(v)]$, and

$$A_h^\epsilon := \int_{\mathbb{R}} \int_{Q_T} H^\epsilon\big(u(x,t), v(y,t)\big)\big[K(x,t) - K(y,t)\big]\psi\left(\frac{x+y}{2}, t\right)\delta_h'\left(\frac{x-y}{2}\right) dt\, dx\, dy.$$

We claim that

$$\lim_{h\downarrow 0} A_h^\epsilon = -4\int_{Q_T} H^\epsilon\big(u(x,t), v(x,t)\big)\psi(x,t)K_x(x,t)\, dt\, dx. \tag{23}$$

To see this, let $\{u^n\}$ be a sequence of smooth functions on Q_T such that $u^n \to u$ a.e. on Q_T and $\|u^n\|_{L^\infty(Q_T)} \leqslant \|u\|_{L^\infty(Q_T)}$ for all n. Then we have

$$\begin{aligned}
&A_h^\epsilon + 4\int_{Q_T} H^\epsilon\big(u(x,t), v(x,t)\big)\psi(x,t)K_x(x,t)\, dt\, dx \\
&\quad = \int_{\mathbb{R}} \int_{Q_T} H^\epsilon\big(u^n(x,t), v(y,t)\big)\big[K(x,t) - K(y,t)\big]\psi\left(\frac{x+y}{2}, t\right)\delta_h'\left(\frac{x-y}{2}\right) dt\, dx\, dy \\
&\qquad + 4\int_{Q_T} H^\epsilon\big(u^n(x,t), v(x,t)\big)\psi(x,t)K_x(x,t)\, dt\, dx \\
&\qquad + 4\int_{Q_T} \big[H^\epsilon\big(u(x,t), v(x,t)\big) - H^\epsilon\big(u^n(x,t), v(x,t)\big)\big]\psi(x,t)K_x(x,t)\, dt\, dx \\
&\qquad + \int_{\mathbb{R}} \int_{Q_T} \big[H^\epsilon\big(u(x,t), v(y,t)\big) - H^\epsilon\big(u^n(x,t), v(y,t)\big)\big]\big[K(x,t) - K(y,t)\big] \\
&\qquad \times \psi\left(\frac{x+y}{2}, t\right)\delta_h'\left(\frac{x-y}{2}\right) dt\, dx\, dy.
\end{aligned} \tag{24}$$

Observe that $|H^\epsilon(u_1,v) - H^\epsilon(u_2,v)| \leqslant \|f_\epsilon\|_{Lip}|u_1-u_2|$. That is for each fixed v, the function $u \mapsto H^\epsilon(u,v)$ is Lipschitz continuous. Therefore by using integration by parts and the chain rule $\frac{\partial}{\partial x}H^\epsilon(u^n(x,t),v(y,t)) = H^\epsilon_u(u^n(x,t),v(y,t))u^n_x(x,t)$, we obtain

$$\begin{aligned}
&\lim_{h\downarrow 0}\int_{\mathbb{R}}\int_{Q_T} H^\epsilon\big(u^n(x,t),v(y,t)\big)\big[K(x,t)-K(y,t)\big]\psi\left(\frac{x+y}{2},t\right)\delta_h'\left(\frac{x-y}{2}\right)dt\,dx\,dy\\
&\quad= -2\lim_{h\downarrow 0}\int_{\mathbb{R}}\int_{Q_T} H^\epsilon_u\big(u^n(x,t),v(y,t)\big)u^n_x(x,t)\big[K(x,t)-K(y,t)\big]\psi\left(\frac{x+y}{2},t\right)\delta_h\left(\frac{x-y}{2}\right)dt\,dx\,dy\\
&\qquad- 2\lim_{h\downarrow 0}\int_{\mathbb{R}}\int_{Q_T} H^\epsilon\big(u^n(x,t),v(y,t)\big)K_x(x,t)\psi\left(\frac{x+y}{2},t\right)\delta_h\left(\frac{x-y}{2}\right)dt\,dx\,dy\\
&\qquad- \lim_{h\downarrow 0}\int_{\mathbb{R}}\int_{Q_T} H^\epsilon\big(u^n(x,t),v(y,t)\big)\big[K(x,t)-K(y,t)\big]\psi_x\left(\frac{x+y}{2},t\right)\delta_h\left(\frac{x-y}{2}\right)dt\,dx\,dy\\
&\quad= -4\int_{Q_T} H^\epsilon\big(u^n(x,t),v(x,t)\big)K_x(x,t)\psi(x,t)\,dt\,dx.
\end{aligned}$$

Thus by taking the limit $h \to 0$ in (24) we arrive at

$$\begin{aligned}
&\limsup_{h\downarrow 0}\left|A_h^\epsilon + 4\int_{Q_T} H^\epsilon\big(u(x,t),v(x,t)\big)\psi(x,t)K_x(x,t)\,dt\,dx\right|\\
&\quad\leqslant \|f_\epsilon\|_{Lip}\|\psi\|_\infty\left\{4\int_0^T\int_{-R_1}^{R_1}\big|u^n(x,t)-u(x,t)\big|\big|K_x(x,t)\big|\,dx\,dt\right.\\
&\qquad\left.+\limsup_{h\downarrow 0}\int_0^T\int_{-R_1}^{R_1}\int_{-R_1}^{R_1}\big|u^n(x,t)-u(x,t)\big|\big|K(x,t)-K(y,t)\big|\left|\delta_h'\left(\frac{x-y}{2}\right)\right|dy\,dx\,dt\right\}
\end{aligned}\tag{25}$$

for some constant $R_1 > 0$ depending only on the support of ψ. We have

$$\begin{aligned}
&\int_0^T\int_{-R_1}^{R_1}\int_{-R_1}^{R_1}\big|u^n(x,t)-u(x,t)\big|\big|K(x,t)-K(y,t)\big|\left|\delta_h'\left(\frac{x-y}{2}\right)\right|dy\,dx\,dt\\
&\quad=\int_0^T\int_{-R_1}^{R_1}\int_{x-2h}^{x+2h}\big|u^n(x,t)-u(x,t)\big|\big|K(x,t)-K(y,t)\big|\frac{1}{h^2}\left|\delta'\left(\frac{x-y}{2h}\right)\right|dy\,dx\,dt\\
&\quad=4\int_{-1}^{1}\left\{\int_0^T\int_{-R_1}^{R_1}\big|u^n(x,t)-u(x,t)\big|\big|g_{hz}(x,t)\big|\,dx\,dt\right\}|z|\big|\delta'(z)\big|\,dz
\end{aligned}$$

where $g_{hz}(x,t) := \frac{K(x,t)-K(x-2hz,t)}{2hz}$. Moreover, $g_{hz} \longrightarrow K_x$ in $L^1((-R_1,R_1)\times(0,T))$ as h tends to zero because $g_{hz}(x,t) \longrightarrow K_x(x,t)$ for a.e. (x,t) in $\Omega := (-R_1,R_1)\times(0,T)$ and

$$\|g_{hz}\|_{L^1(\Omega)} = \int_0^T \left[\frac{1}{2h|z|} \int_{-R_1}^{R_1} \left|K(x,t) - K(x-2hz,t)\right| dx \right] dt$$
$$\longrightarrow \int_0^T \mathbf{TV}_{R_1}\big(K(\cdot,t)\big)\, dt = \int_0^T \left\|K_x(\cdot,t)\right\|_{L^1(-R_1,R_1)} dt = \|K_x\|_{L^1(\Omega)}.$$

Here $\mathbf{TV}_R(\xi)$ denotes the total variation of the function ξ on the interval $(-R,R)$. Therefore, since $u^n - u \in L^\infty(Q_T)$ we infer that

$$\lim_{h\downarrow 0} \int_0^T \int_{-R_1}^{R_1} \int_{-R_1}^{R_1} \left|u^n(x,t) - u(x,t)\right| \left|K(x,t) - K(y,t)\right| \left|\delta_h'\left(\frac{x-y}{2}\right)\right| dy\, dx\, dt$$
$$= 4 \int_{-1}^{1} \left\{ \int_0^T \int_{-R_1}^{R_1} \left|u^n(x,t) - u(x,t)\right| \left|K_x(x,t)\right| dx\, dt \right\} |z| \left|\delta'(z)\right| dz.$$

This together with (25) gives

$$\limsup_{h\downarrow 0} \left| A_h^\epsilon + 4 \int_{Q_T} H^\epsilon\big(u(x,t), v(x,t)\big) \psi(x,t) K_x(x,t)\, dt\, dx \right|$$
$$\leqslant C \|f_\epsilon\|_{Lip} \|\psi\|_\infty \int_0^T \int_{-R_1}^{R_1} \left|u^n(x,t) - u(x,t)\right| \left|K_x(x,t)\right| dx\, dt.$$

Since $u^n \to u$ almost everywhere on $(-R_1,R_1) \times (0,T)$ and $\|u^n\|_{L^\infty(Q_T)} \leqslant \|u\|_{L^\infty(Q_T)}$, by letting $n \to \infty$ and using the dominated convergence theorem we then obtain (23) as desired.

Next we have

$$\left| A_h + 4 \int_{Q_T} H\big(u(x,t), v(x,t)\big) \psi(x,t) K_x(x,t)\, dt\, dx \right|$$
$$\leqslant \left|A_h - A_h^\epsilon\right| + \left| A_h^\epsilon + 4 \int_{Q_T} H^\epsilon\big(u(x,t), v(x,t)\big) \psi(x,t) K_x(x,t)\, dt\, dx \right|$$
$$+ 4 \int_{Q_T} \left| H\big(u(x,t), v(x,t)\big) - H^\epsilon\big(u(x,t), v(x,t)\big) \right| \left|\psi(x,t)\right| \left|K_x(x,t)\right| dt\, dx. \tag{26}$$

To estimate $|A_h^\epsilon - A_h|$, note that as $\delta(z)$ is given by (13) we get $\delta_h'(z) = \frac{\frac{-2z}{h^2}}{[1-(\frac{z}{h})^2]^2} \delta_h(z)$. Hence

$$\left|A_h^\epsilon - A_h\right| \leqslant \int_0^T \int_{-R_1}^{R_1} \int_{x-2h}^{x+2h} \left| H^\epsilon\big(u(x,t), v(y,t)\big) - H\big(u(x,t), v(y,t)\big) \right| \left|K(x,t) - K(y,t)\right| \psi\left(\frac{x+y}{2}, t\right)$$
$$\times \frac{\left|\frac{x-y}{h^2}\right|}{[1 - (\frac{x-y}{2h})^2]^2} \delta_h\left(\frac{x-y}{2}\right) dy\, dx\, dt.$$

It is easy to see that $|H^\epsilon(u(x,t),v(y,t)) - H(u(x,t),v(y,t))| \leqslant 2\|f_\epsilon - f\|_{L^\infty(-R_0,R_0)}$, where $R_0>0$ is a constant satisfying $\|u\|_\infty, \|v\|_\infty \leqslant R_0$. It follows that

$$\begin{aligned}\left|A_h^\epsilon - A_h\right| &\leqslant C\|f_\epsilon - f\|_{L^\infty(-R_0,R_0)}\|\psi\|_\infty \int_0^T \int_{-R_1}^{R_1} \int_{-1}^{1} \frac{|K(x,t)-K(x-2hz,t)|}{2h|z|} \frac{z^2 e^{\frac{-1}{1-z^2}}}{(1-z^2)^2}\,dz\,dx\,dt \\ &\leqslant C\|f_\epsilon - f\|_{L^\infty(-R_0,R_0)}\|\psi\|_\infty \int_0^T \int_{-1}^{1} \mathbf{TV}_{R_1}\big(K(\cdot,t)\big) \frac{z^2 e^{\frac{-1}{1-z^2}}}{(1-z^2)^2}\,dz\,dt \\ &\leqslant C\|f_\epsilon - f\|_{L^\infty(-R_0,R_0)}\|\psi\|_\infty \int_0^T \int_{-R_1}^{R_1} \big|K_x(x,t)\big|\,dx\,dt,\end{aligned}$$

where we have used the fact that $e^{\frac{1}{2(1-z^2)}} \geqslant \frac{1}{8(1-z^2)^2}$. This together with (26) and (23) gives

$$\begin{aligned}&\limsup_{h\downarrow 0}\left|A_h + 4\int_{Q_T} H\big(u(x,t),v(x,t)\big)\psi(x,t)K_x(x,t)\,dt\,dx\right| \\ &\quad \leqslant C\|f_\epsilon - f\|_{L^\infty(-R_0,R_0)}\|\psi\|_\infty \int_0^T \int_{-R_1}^{R_1} \big|K_x(x,t)\big|\,dx\,dt \quad \forall \epsilon>0.\end{aligned}$$

Letting $\epsilon \to 0$, we conclude that

$$\limsup_{h\downarrow 0}\left|A_h + 4\int_{Q_T} H\big(u(x,t),v(x,t)\big)\psi(x,t)K_x(x,t)\,dt\,dx\right| \leqslant 0$$

yielding (22). □

2.2. Proof of the L^1-contraction principle

We are now ready to prove the L^1-contraction principle. The following well-known version of Gronwall's inequality will be needed.

Lemma 2.7 *(Gronwall's inequality). Let $t_0 < T \leqslant +\infty$. Assume $x, h \in L^\infty(t_0,T)$ and $k \in L^1(t_0,T)$ are non-negative functions satisfying*

$$x(t) \leqslant h(t) + \int_{t_0}^{t} k(s)x(s)\,ds \quad \text{for a.e. } t \in (t_0,T).$$

Then

$$x(t) \leqslant h(t) + \int_{t_0}^{t} h(s)k(s)\exp\left[\int_s^t k(u)\,du\right]ds \quad \text{for a.e. } t \in (t_0,T).$$

In addition, if $h(t)$ is nonincreasing on $[t_0, T)$, then

$$x(t) \leqslant h(t_0) \exp\left[\int_{t_0}^{t} k(s)\,ds\right] \quad \textit{for a.e. } t \in (t_0, T).$$

Proof of Theorem 2.2. Let $M > 0$ be such that $\|u\|_{L^\infty(Q_T)} \leqslant M$ and $\|v\|_{L^\infty(Q_T)} \leqslant M$. For each i, as f_i is uniformly continuous on $[-M, M]$ there exists a nondecreasing subadditive function $\omega_i : [0,\infty) \to [0,\infty)$ satisfying

$$\left|f_i(z_1) - f_i(z_2)\right| \leqslant \omega_i\big(|z_1 - z_2|\big) \quad \forall z_1, z_2 \in [-M, M].$$

Also by replacing $\omega_i(\epsilon)$ by $\omega_i(\epsilon) + \sqrt{\epsilon}$ if necessary, we can assume further that $\lim_{\epsilon\to 0^+} \frac{\omega_i(\epsilon)}{\epsilon} = +\infty$ for all $i = 1, 2, \dots, N$. Then

$$\begin{aligned}
\left|\operatorname{sign}^+(u-v)\big[F(x,t,u) - F(x,t,v)\big]\right| &\leqslant \sum_{i=1}^{N} \left|K_i(x,t)\right| \left|\operatorname{sign}^+(u-v)\big[f_i(u) - f_i(v)\big]\right| \\
&\leqslant \sum_{i=1}^{N} \left|K_i(x,t)\right| \omega_i\big((u-v)^+\big).
\end{aligned}$$

Let $\epsilon > 0$. Since ω_i is nondecreasing and subadditive, we have

$$\omega_i(r) \leqslant (r+\epsilon)\frac{\omega_i(\epsilon)}{\epsilon} \quad \text{for any } r \geqslant 0.$$

Hence by setting $w := (u-v)^+$ and $w^0 := (u^0 - v^0)^+$, we obtain from Lemma 2.5 that

$$\int_{Q_T} \left\{ w\psi_t + (w+\epsilon)\left(\sum_{i=1}^{N} \left|K_i(x,t)\right| \frac{\omega_i(\epsilon)}{\epsilon}\right)|\psi_x| + \lambda w|\psi_{xx}| \right\} dt\,dx \geqslant 0 \tag{27}$$

for all $\epsilon > 0$ and all nonnegative test functions $\psi \in C_0^\infty(Q_T)$. We claim that (27) yields

$$\int_{\mathbb{R}} w(x,\tau)\,dx \leqslant \int_{\mathbb{R}} w^0(x)\,dx \quad \text{for a.e. } \tau \in (0,T). \tag{28}$$

To see this, let us consider a function $\beta \in C_0^\infty(\mathbb{R})$, $\operatorname{spt}(\beta) \subset [0,1]$, $\beta(s) \geqslant 0$ and $\int \beta(s)\,ds = 1$. For each $\nu \in \mathbb{N}$, define $\delta_\nu(s) := \nu\beta(\nu s)$ and $\theta_\nu(t) := \int_{-\infty}^{t} \delta_\nu(s)\,ds$. It is clear that the sequence $\delta_\nu(s)$ converges to the Dirac δ-measure in $\mathcal{D}'(\mathbb{R})$ as $\nu \to \infty$. Since $0 \leqslant \theta_\nu(t) \leqslant 1$, $\theta_\nu(t) = 0$ if $t \leqslant 0$ and $\theta_\nu(t) = 1$ if $t \geqslant 1/\nu$, the sequence $\theta_\nu(t)$ converges pointwise to the Heaviside function $\operatorname{sign}^+(t)$ as $\nu \to \infty$. Now for any nonnegative function $\xi \in C_0^\infty(\mathbb{R})$ and any $\sigma, \tau \in (0,T)$ satisfying $\sigma < \tau$, by applying (27) for the test function $\psi(x,t) := \xi(x)\theta_\nu(\tau - t)\theta_\nu(t - \sigma)$ we get

$$\begin{aligned}
&\int_{Q_T} w(x,t)\xi(x)\delta_\nu(\tau - t)\theta_\nu(t-\sigma)\,dt\,dx - \int_{Q_T} w(x,t)\xi(x)\theta_\nu(\tau - t)\delta_\nu(t-\sigma)\,dt\,dx \\
&\quad \leqslant \int_{Q_T} \left\{ (w+\epsilon)\left(\sum_{i=1}^{N} \left|K_i(x,t)\right| \frac{\omega_i(\epsilon)}{\epsilon}\right)|\xi_x| + \lambda w|\xi_{xx}| \right\} \theta_\nu(\tau - t)\theta_\nu(t-\sigma)\,dt\,dx.
\end{aligned}$$

It follows by sending $\nu \to \infty$ that

$$\int_{\mathbb{R}} w(x,\tau)\xi(x)\,dx \leqslant \int_{\mathbb{R}} w(x,\sigma)\xi(x)\,dx + \int_{\sigma}^{\tau}\int_{\mathbb{R}} (w+\epsilon)\left(\sum_{i=1}^{N}\left|K_i(x,t)\right|\frac{\omega_i(\epsilon)}{\epsilon}\right)|\xi_x|\,dx\,dt + \int_{\sigma}^{\tau}\int_{\mathbb{R}} \lambda w|\xi_{xx}|\,dx\,dt$$

for a.e. $\sigma,\tau \in (0,T)$ with $\sigma \leqslant \tau$. Hence by letting $\sigma \to 0^+$ and using $\operatorname{esslim}_{t\to 0^+}[w(t)-w^0]=0$ in $L^1_{loc}(\mathbb{R})$, we conclude that

$$\int_{\mathbb{R}} w(x,\tau)\xi(x)\,dx \leqslant \int_{\mathbb{R}} w^0(x)\xi(x)\,dx + \sum_{i=1}^{N}\left[\frac{\omega_i(\epsilon)}{\epsilon}\int_0^{\tau}\int_{\mathbb{R}} w\left|K_i(x,t)\right||\xi_x|\,dx\,dt\right] + \sum_{i=1}^{N}\left[\omega_i(\epsilon)\int_0^{\tau}\int_{\mathbb{R}}\left|K_i(x,t)\right||\xi_x|\,dx\,dt\right] + \int_0^{\tau}\int_{\mathbb{R}} \lambda w|\xi_{xx}|\,dx\,dt \tag{29}$$

for any nonnegative $\xi \in C_0^\infty(\mathbb{R})$ and a.e. $\tau \in (0,T)$. But as $K_i \in L^1(0,T;L^\infty(\mathbb{R}))$, we infer from standard approximations that (29) holds for all nonnegative $\xi \in W_0^{2,1}(\mathbb{R}) \equiv W^{2,1}(\mathbb{R})$.

Following [23], we take $\psi \in C^2(\mathbb{R}) \cap L^1(\mathbb{R})$ be such that $0 \leqslant \psi \leqslant 1$, $\psi \equiv 1$ in $[-1,1]$, $|\psi'| \leqslant \psi$ and $|\psi''| \leqslant \psi$. For each $\epsilon > 0$, let $\xi(x) := \psi(\frac{|x|}{R})$ where $R := R(\epsilon) > 0$ will be chosen later. Then

$$\xi \equiv 1 \quad \text{in } K := [-R,R], \qquad \left|\xi_x(x)\right| \leqslant \frac{\xi(x)}{R} \quad \text{and} \quad \left|\xi_{xx}(x)\right| \leqslant \frac{\xi(x)}{R^2}.$$

Set $\tilde{K} := \mathbb{R} \setminus K$, then by (29) and since $\int_{\tilde{K}} \xi\,dx = 2R\int_1^\infty \psi(x)\,dx =: C_0 R$, we have

$$\begin{aligned}\int_{\mathbb{R}} w(x,\tau)\xi(x)\,dx &\leqslant \int_{\mathbb{R}} w^0(x)\xi(x)\,dx + \sum_{i=1}^{N}\left[\frac{\omega_i(\epsilon)}{\epsilon R}\int_0^{\tau}\int_{\tilde{K}}\left|K_i(x,t)\right|w\xi\,dx\,dt\right] \\ &\quad + \sum_{i=1}^{N}\left[\frac{\omega_i(\epsilon)}{R}\int_0^{\tau}\int_{\tilde{K}}\left|K_i(x,t)\right|\xi\,dx\,dt\right] + \frac{\lambda}{R^2}\int_0^{\tau}\int_{\tilde{K}} w\xi\,dx\,dt \\ &\leqslant \int_{\mathbb{R}} w^0(x)\xi(x)\,dx + \frac{1}{\epsilon R}\left(\frac{\lambda\epsilon}{R} + \sum_{i=1}^{N}\omega_i(\epsilon)\right)\int_0^{\tau}\int_{\tilde{K}}\left\|M(\cdot,t)\right\|_{L^\infty(\tilde{K})} w\xi\,dx\,dt \\ &\quad + C_0\sum_{i=1}^{N}\omega_i(\epsilon)\int_0^{\tau}\left\|M(\cdot,t)\right\|_{L^\infty(\tilde{K})}\,dt,\end{aligned}$$

where $M(x,t) := \max\{|K_1(x,t)|, \dots, |K_N(x,t)|, 1\}$. We now choose $R = R(\epsilon) > 0$ such that $\frac{1}{\epsilon R}(\frac{\lambda\epsilon}{R} + \sum_{i=1}^N \omega_i(\epsilon)) = 1$. It follows that

$$R = \frac{\sum_{i=1}^N \omega_i(\epsilon) + \sqrt{[\sum_{i=1}^N \omega_i(\epsilon)]^2 + 4\lambda\epsilon^2}}{2\epsilon}. \tag{30}$$

In particular, we get $R \to \infty$ as $\epsilon \to 0^+$ since $\lim_{\epsilon\to 0^+} \frac{\omega_i(\epsilon)}{\epsilon} = +\infty$. Using this choice of R we infer from the above inequality that

$$\begin{aligned}\int_{\mathbb{R}} w(x,\tau)\xi(x)\,dx \leqslant \int_{\mathbb{R}} w^0(x)\xi(x)\,dx + C_0 \sum_{i=1}^N \omega_i(\epsilon) \int_0^T \|M(\cdot,t)\|_{L^\infty(\mathbb{R})}\,dt \\ + \int_0^\tau \|M(\cdot,t)\|_{L^\infty(\mathbb{R})} \biggl(\int_{\tilde{K}} w\xi\,dx\biggr)\,dt\end{aligned} \tag{31}$$

for any $\epsilon > 0$ and a.e. $\tau \in (0,T)$. Hence Gronwall's inequality (see Lemma 2.7) gives

$$\int_{\mathbb{R}} w(x,\tau)\chi_{[-R,R]}(x)\,dx \leqslant \Biggl\{\int_{\mathbb{R}} w^0(x)\,dx + C_0 \sum_{i=1}^N \omega_i(\epsilon) \int_0^T \|M(\cdot,t)\|_{L^\infty(\mathbb{R})}\,dt\Biggr\} e^{\int_0^T \|M(\cdot,t)\|_\infty\,dt}.$$

It follows by letting $\epsilon \to 0^+$ and using Fatou's lemma that

$$\int_{\mathbb{R}} w(x,\tau)\,dx \leqslant \|w^0\|_{L^1(\mathbb{R})} e^{\int_0^T \|M(\cdot,t)\|_{L^\infty(\mathbb{R})}\,dt} < \infty \quad \text{for a.e. } \tau \in (0,T).$$

We infer from this estimate that the functions $h_\epsilon(t) := \int_{\tilde{K}} w(x,t)\,dx$ satisfy $\|h_\epsilon\|_{L^\infty(0,T)} \leqslant C$ uniformly in ϵ and $\lim_{\epsilon\to 0^+} h_\epsilon(t) = 0$ for a.e. $t \in (0,T)$. Therefore by letting $\epsilon \to 0^+$ in (31) and using the dominated convergence theorem, we obtain (28) as claimed. □

The following results are immediate consequences of Theorem 2.2.

Corollary 2.8. *Assume $\lambda \geqslant 0$, $f_i \in C(\mathbb{R})$ and $K_i \in L^1(0,T;L^\infty(\mathbb{R})) \cap L^2(0,T;L^2_{loc}(\mathbb{R})) \cap L^1(0,T;W^{1,1}_{loc}(\mathbb{R}))$. Let $u^0 \in L^\infty(\mathbb{R})$ and suppose that u is an entropy solution of* (5)*. Then*

$$\|u(\cdot,t)\|_{L^1(\mathbb{R})} \leqslant \|u^0\|_{L^1(\mathbb{R})} \quad \textit{for a.e. } t \in (0,T). \tag{32}$$

In case $K_i(x,t)$ are independent of x, we also have

$$\mathbf{TV}\bigl(u(\cdot,t)\bigr) \leqslant \mathbf{TV}\bigl(u^0\bigr) \quad \textit{for a.e. } t \in (0,T). \tag{33}$$

Proof. Inequality (32) is obtained from Theorem 2.2 by taking $v \equiv 0$. On the other hand, (33) follows from Theorem 2.2 by taking $v = u(x+h,t)$, where $h > 0$ is arbitrary. □

Corollary 2.9. *Assume $\lambda \geqslant 0$, $f_i \in C(\mathbb{R})$ and $K_i \in L^1(0,T;L^\infty(\mathbb{R})) \cap L^2(0,T;L^2_{loc}(\mathbb{R})) \cap L^1(0,T;W^{1,1}_{loc}(\mathbb{R}))$. Let $u^0, v^0 \in L^\infty(\mathbb{R})$ satisfy $u^0 - v^0 \in L^1(\mathbb{R})$. Suppose that u and v are entropy solutions of (5) with initial data u^0 and v^0 respectively. Then for almost every $t \in (0,T)$, we have $u(t) - v(t) \in L^1(\mathbb{R})$ and*

$$\int_{\mathbb{R}} \big[u(x,t) - v(x,t)\big]\,dx = \int_{\mathbb{R}} \big[u^0(x) - v^0(x)\big]\,dx. \tag{34}$$

Proof. Theorem 2.2 gives $u(t) - v(t) \in L^1(\mathbb{R})$ and $\|u(t) - v(t)\|_{L^1(\mathbb{R})} \leqslant \|u^0 - v^0\|_{L^1(\mathbb{R})}$ for a.e. $t \in (0,T)$. Next we express that u and v are weak solutions of (5). Choosing a test function of the form $\phi_\epsilon(x,t) = \chi(t)\theta(\epsilon x)$ where $\chi \in C_0^\infty(-\infty,T)$ and $\theta \in C_0^\infty(\mathbb{R})$ with $\theta \equiv 1$ in a neighbourhood of the origin, we have

$$\begin{aligned}
&\int_{Q_T} (u-v)\partial_t\phi_\epsilon\,dx\,dt + \int_{Q_T} \big[F(x,t,u) - F(x,t,v)\big]\partial_x\phi_\epsilon\,dx\,dt \\
&\quad + \int_{\mathbb{R}} \big(u^0(x) - v^0(x)\big)\phi_\epsilon(x,0)\,dx = -\lambda \int_{Q_T} (u-v)\partial^2_{xx}\phi_\epsilon\,dx\,dt,
\end{aligned}$$

that is to say,

$$\begin{aligned}
&\int_{Q_T} (u-v)\chi'(t)\theta(\epsilon x)\,dx\,dt + \epsilon \int_{Q_T} \big[F(x,t,u) - F(x,t,v)\big]\chi(t)\theta'(\epsilon x)\,dx\,dt \\
&\quad + \chi(0)\int_{\mathbb{R}} \big(u^0(x) - v^0(x)\big)\theta(\epsilon x)\,dx = -\lambda\epsilon^2 \int_{Q_T} (u-v)\chi(t)\theta''(\epsilon x)\,dx\,dt.
\end{aligned}$$

Since $\|u(t) - v(t)\|_{L^1(\mathbb{R})} \leqslant \|u^0 - v^0\|_{L^1(\mathbb{R})}$ and $u^0 - v^0 \in L^1(\mathbb{R})$, the theorem of dominated convergence allows us to pass to the limit in each of the four integrals when $\epsilon \to 0^+$ to obtain

$$\int_{Q_T} (u-v)\chi'(t)\,dx\,dt + \chi(0)\int_{\mathbb{R}} \big(u^0(x) - v^0(x)\big)\,dx = 0. \tag{35}$$

Indeed, to see the second term tends to zero observe that

$$\begin{aligned}
\big|F(x,t,u) - F(x,t,v)\big| &\leqslant \sum_{i=1}^N \big|K_i(x,t)\big|\big|f_i(u) - f_i(v)\big| \leqslant \sum_{i=1}^N \big|K_i(x,t)\big|\omega_i\big(|u-v|\big) \\
&\leqslant \sum_{i=1}^N \big|K_i(x,t)\big|\big(|u-v| + \epsilon\big)\frac{\omega_i(\epsilon)}{\epsilon} \quad \text{for any } \epsilon > 0,
\end{aligned}$$

where ω_i are chosen as in the proof of Theorem 2.2. Therefore,

$$\limsup_{\epsilon\to 0^+}\epsilon\left|\int_{Q_T}\big[F(x,t,u)-F(x,t,v)\big]\chi(t)\theta'(\epsilon x)\,dx\,dt\right|$$

$$\leqslant\sum_{i=1}^N\limsup_{\epsilon\to 0^+}\omega_i(\epsilon)\left\{\left\|\theta'\right\|_\infty\int_0^T\left\|K_i(t)\right\|_{L^\infty(\mathbb{R})}\left\|u(t)-v(t)\right\|_{L^1(\mathbb{R})}\left|\chi(t)\right|dt\right.$$

$$\left.+\,\epsilon\int_0^T\left\|K_i(t)\right\|_{L^\infty(\mathbb{R})}\left|\chi(t)\right|\left(\int_{\mathbb{R}}\left|\theta'(\epsilon x)\right|dx\right)dt\right\}$$

$$\leqslant\sum_{i=1}^N\limsup_{\epsilon\to 0^+}\omega_i(\epsilon)\left\{\left\|\theta'\right\|_\infty\left\|u^0-v^0\right\|_{L^1(\mathbb{R})}+\left\|\theta'\right\|_{L^1(\mathbb{R})}\right\}\int_0^T\left\|K_i(t)\right\|_{L^\infty(\mathbb{R})}\left|\chi(t)\right|dt=0$$

as desired.

Let $h(t):=\int_{\mathbb{R}}[u(x,t)-v(x,t)]\,dx$ for $t\in[0,T]$, which is a function in $L^\infty(0,T)$. Then (35) can be rewritten as $\int_0^T h(t)\chi'(t)\,dt+\chi(0)\int_{\mathbb{R}}(u^0-v^0)\,dx=0$ for all $\chi\in C_0^\infty(-\infty,T)$. We deduce that $h(t)=\int_{\mathbb{R}}(u^0-v^0)\,dx$ for a.e. $t\in(0,T)$ and hence (34) is proved. $\square$

3. Existence of entropy solutions

In this section we show that Eq. (5) with $\lambda\geqslant 0$ admits an entropy solution for any given bounded initial data. The flux function $F(x,t,u)$ is assumed to be of the form (6).

3.1. Viscous scalar conservation law

We first consider the case $\lambda>0$ and for simplicity let us take $\lambda=1$, that is,

$$\begin{cases}\partial_t u+\partial_x\big[F(x,t,u)\big]=u_{xx} & \text{in } Q_T:=\mathbb{R}\times(0,T),\\ u(\cdot,0)=u^0 & \text{in } \mathbb{R}.\end{cases}\tag{36}$$

Let $W(0,T):=\{u\in L^2(0,T;H^1(\mathbb{R})):\ u_t\in L^2(0,T;H^{-1}(\mathbb{R}))\}$. Note that since $W(0,T)\subset C(0,T;L^2(\mathbb{R}))$, we have $\lim_{t\to 0^+}\|u(t)-u(0)\|_{L^2(\mathbb{R})}=0$ for all $u\in W(0,T)$.

Proposition 3.1. *Assume $f_i:\mathbb{R}\to\mathbb{R}$ are Lipschitz continuous and $K_i\in L^\infty(\mathbb{R}\times(0,T))$. Then for each $u^0\in L^2(\mathbb{R})$, the initial-value problem*

$$\begin{cases}\partial_t u+\partial_x\big[F(x,t,u)-F(x,t,0)\big]=u_{xx} & \text{in } Q_T,\\ u(\cdot,0)=u^0 & \text{in } \mathbb{R}\end{cases}\tag{37}$$

has a unique solution $u\in W(0,T)$ with $u(0)=u^0$ in the L^2 sense.

Proof. For $v\in L^2(0,T;L^2(\mathbb{R}))$, let $h(x,t):=-e^{-\Theta t}[F(x,t,e^{\Theta t}v(x,t))-F(x,t,0)]$ where $\Theta>0$ is a constant determined later. Then $h\in L^2(0,T;L^2(\mathbb{R}))$ due to f_i are Lipschitz continuous and K_i are bounded. It follows that there is exactly one solution $\tilde{w}\in W(0,T)$ of the equation

$$\begin{cases}\tilde{w}_t-\tilde{w}_{xx}+\Theta\tilde{w}=h_x & \text{in } Q_T,\\ \tilde{w}(\cdot,0)=u^0 & \text{in } \mathbb{R}.\end{cases}$$

Moreover for a.e. $t \in (0,T)$, we have

$$\langle \tilde{w}_t(t), z\rangle + \int_{\mathbb{R}} \big(\tilde{w}_x(x,t) z_x + \Theta \tilde{w}(x,t) z\big)\,dx = -\int_{\mathbb{R}} h(x,t) z_x\,dx \quad \forall z \in H^1(\mathbb{R}). \tag{38}$$

Here and what follows, $\langle\cdot,\cdot\rangle$ denotes the standard pairing between $H^{-1}(\mathbb{R})$ and $H^1(\mathbb{R})$.

Define $T_\Theta(v) := \tilde{w}$. Then $T_\Theta : L^2(0,T;L^2(\mathbb{R})) \longrightarrow L^2(0,T;L^2(\mathbb{R}))$. We next show that T_Θ is a contraction for Θ large enough. Indeed, let $w_1 = T_\Theta(v_1)$ and $w_2 = T_\Theta(v_2)$. If we set $w := w_1 - w_2$, then it follows from (38) that

$$\begin{aligned}
&\langle w_t(t), z\rangle + \int_{\mathbb{R}} \big[w_x(x,t) z_x + \Theta w(x,t) z\big]\,dx \\
&\quad = e^{-\Theta t} \int_{\mathbb{R}} \big[F\big(x,t,e^{\Theta t} v_1(x,t)\big) - F\big(x,t,e^{\Theta t} v_2(x,t)\big)\big] z_x\,dx \quad \text{for all } z \in H^1(\mathbb{R}).
\end{aligned}$$

By taking $z(x) := w(x,t)$, this gives for a.e. t

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\langle w(t), w(t)\rangle_{L^2} + \|w_x(t)\|^2_{L^2(\mathbb{R})} + \Theta\|w(t)\|^2_{L^2(\mathbb{R})} \\
&\quad = e^{-\Theta t}\int_{\mathbb{R}} \sum_{i=1}^N K_i(x,t)\big[f_i\big(e^{\Theta t} v_1(x,t)\big) - f_i\big(e^{\Theta t} v_2(x,t)\big)\big] w_x(x,t)\,dx \\
&\quad \leqslant \int_{\mathbb{R}} C\big|v_1(x,t) - v_2(x,t)\big|\big|w_x(x,t)\big|\,dx,
\end{aligned}$$

where $C := \sum_{i=1}^N \|K_i\|_{L^\infty(\mathbb{R}\times(0,T))}\|f_i\|_{Lip(\mathbb{R})}$. Integrating with respect to t and noting that $w(0) = 0$, we obtain

$$\begin{aligned}
&\frac{1}{2}\|w(T)\|^2_{L^2(\mathbb{R})} + \int_0^T \big(\|w_x(t)\|^2_{L^2(\mathbb{R})} + \Theta\|w(t)\|^2_{L^2(\mathbb{R})}\big)\,dt \\
&\quad \leqslant \frac{C^2}{4}\int_0^T\int_{\mathbb{R}} \big|v_1(x,t) - v_2(x,t)\big|^2\,dx\,dt + \int_0^T\int_{\mathbb{R}} \big|w_x(x,t)\big|^2\,dx\,dt
\end{aligned}$$

yielding

$$\int_0^T \|w(t)\|^2_{L^2(\mathbb{R})}\,dt \leqslant \frac{C^2}{4\Theta}\int_0^T\int_{\mathbb{R}} \big|v_1(x,t) - v_2(x,t)\big|^2\,dx\,dt.$$

Thus for $\Theta > C^2/4$, the map $T_\Theta : L^2(0,T;L^2(\mathbb{R})) \longrightarrow L^2(0,T;L^2(\mathbb{R}))$ is a contraction. So there exists a unique $v \in L^2(0,T;L^2(\mathbb{R}))$ such that $T_\Theta(v) = v$. This implies that $v \in W(0,T)$ and $u(x,t) := e^{\Theta t} v(x,t)$ is the desired unique solution to Eq. (37). $\square$

The next result gives more information about the solution when the initial data is bounded.

Proposition 3.2. *Assume that the functions $f_i : \mathbb{R} \to \mathbb{R}$ are locally Lipschitz continuous and $K_i \in L^\infty(\mathbb{R} \times (0,T))$. Suppose there exist some $a, b \in \mathbb{R}$ with $a \leqslant 0 \leqslant b$ such that*

$$\partial_x\big[F(x,t,a) - F(x,t,0)\big] = \partial_x\big[F(x,t,b) - F(x,t,0)\big] = 0 \quad \text{for a.e. } (x,t) \in \mathbb{R} \times (0,T).$$

Then for any $u^0 \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ satisfying $a \leqslant u^0(x) \leqslant b$, Eq. (37) has a unique solution $u \in W(0,T) \cap L^\infty(\mathbb{R} \times [0,T])$ with $u(0) = u^0$ in the L^2 sense. Moreover for a.e. $t \in (0,T)$, we have $a \leqslant u(\cdot,t) \leqslant b$.

Proof. Let $\Psi(z) = \Psi(|z|) \in C^\infty(\mathbb{R})$ be such that $0 \leqslant \Psi \leqslant 1$, $\Psi(z) \equiv 1$ on $[a,b]$, and $\Psi(z) \equiv 0$ on $(-\infty, a-1] \cup [b+1, \infty)$. Define

$$\tilde f_i(z) := \Psi(z) f_i(z) \quad \text{and} \quad \tilde F(x,t,z) := K_1(x,t)\tilde f_1(z) + \cdots + K_N(x,t)\tilde f_N(z).$$

Then $\tilde f_i : \mathbb{R} \to \mathbb{R}$ are Lipschitz continuous. Therefore by Proposition 3.1, the equation

$$\begin{cases} \partial_t u + \partial_x\big[\tilde F(x,t,u) - \tilde F(x,t,0)\big] = u_{xx} & \text{in } Q_T, \\ u(\cdot,0) = u^0 & \text{in } \mathbb{R} \end{cases}$$

has a unique solution $u \in W(0,T)$. We claim that $a \leqslant u(\cdot,t) \leqslant b$ for a.e. $t \geqslant 0$. This together with the definitions of $\tilde f_i$ yields that u is also a solution to (37). In order to show the first inequality in the claim, define $v(x,t) := -u(x,t) + a$. Let $C := \sum_{i=1}^N \|K_i\|_{L^\infty(\mathbb{R}\times(0,T))} \|\tilde f_i\|_{Lip(\mathbb{R})}$. Then for a.e. t, we have

$$\begin{aligned} -\big\langle u_t(t), z\big\rangle - \int_{\mathbb{R}} u_x(x,t) z_x \, dx &= \int_{\mathbb{R}} \big[\tilde F(x,t,0) - \tilde F\big(x,t,u(x,t)\big)\big] z_x \, dx \\ &= \int_{\mathbb{R}} \big[\tilde F(x,t,a) - \tilde F\big(x,t,-v(x,t)+a\big)\big] z_x \, dx \leqslant C \int_{\mathbb{R}} \big|v(x,t)\big| |z_x| \, dx \end{aligned}$$

for all $z \in H^1(\mathbb{R})$ with compact support. Notice that since $a \leqslant 0$, $v^+(t) \in H^1(\mathbb{R})$ for a.e. $t \in (0,T)$ where $v^+ := \max\{v, 0\}$. Therefore, if we choose $\zeta \in C_0^\infty(\mathbb{R})$ be such that $0 \leqslant \zeta \leqslant 1$, $\zeta(x) = 1$ if $|x| < 1$ and $\zeta(x) = 0$ if $|x| \geqslant 2$, then by taking $z(x) := v^+(x,t)\zeta(x/n)$ in the above inequality and letting $n \to \infty$ we obtain

$$\big\langle v_t(t), v^+(t)\big\rangle + \int_{\mathbb{R}} v_x(x,t) v_x^+(x,t) \, dx \leqslant C \int_{\mathbb{R}} \big|v(x,t)\big| \big|v_x^+(x,t)\big| \, dx \quad \text{for a.e. } t \in (0,T).$$

Hence it follows by integrating from 0 to t and using the facts $2\int_0^t \langle v_t, v^+\rangle \, ds = \|v^+(\cdot,t)\|_{L^2}^2 - \|v^+(\cdot,0)\|_{L^2}^2$, $v^+(\cdot,0) \equiv 0$ and $\int_{\mathbb{R}} v_x v_x^+ \, dx = \int_{\mathbb{R}} (v_x^+)^2 \, dx$ that

$$\begin{aligned} \frac{1}{2}\big\|v^+(\cdot,t)\big\|_{L^2(\mathbb{R})}^2 + \int_0^t \big\|v_x^+(\cdot,s)\big\|_{L^2(\mathbb{R})}^2 \, ds &\leqslant C \int_0^t \int_{\mathbb{R}} \big|v(x,s)\big| \big|v_x^+(x,s)\big| \, dx \, ds \\ &= C \int_0^t \int_{\mathbb{R}} \big|v^+(x,s)\big| \big|v_x^+(x,s)\big| \, dx \, ds \\ &\leqslant \frac{C^2}{4} \int_0^t \big\|v^+(\cdot,s)\big\|_{L^2(\mathbb{R})}^2 \, ds + \int_0^t \big\|v_x^+(\cdot,s)\big\|_{L^2(\mathbb{R})}^2 \, ds. \end{aligned}$$

Thus $\|v^+(\cdot,t)\|^2_{L^2(\mathbb{R})} \leqslant \frac{C^2}{2}\int_0^t \|v^+(\cdot,s)\|^2_{L^2(\mathbb{R})}\,ds$ for a.e. $t \in (0,T)$ and so we conclude from Gronwall's inequality (Lemma 2.7) that $v^+(\cdot,t) \equiv 0$ for a.e. $t \in (0,T)$. That is, $a \leqslant u(\cdot,t)$ for a.e. $t \in (0,T)$. By considering the function $\tilde{v}(x,t) := u(x,t) - b$ and arguing as above, we also obtain $u(\cdot,t) \leqslant b$ for a.e. $t \in (0,T)$. Therefore, the claim is proved.

To prove uniqueness, suppose that u_1 and u_2 are two solutions in $W(0,T) \cap L^\infty(\mathbb{R}\times[0,T])$ of (37). Then by truncating f_i outside the region $|z| \leqslant \max\{\|u_1\|_{L^\infty(\mathbb{R}\times[0,T])}, \|u_2\|_{L^\infty(\mathbb{R}\times[0,T])}\}$ and using Proposition 3.1, we see that $u_1 \equiv u_2$ as desired. □

The following lemma shows that the weak solution obtained in Proposition 3.2 is indeed an entropy solution.

Lemma 3.3. *Assume $f_i : \mathbb{R} \to \mathbb{R}$ are locally Lipschitz continuous and $K_i \in L^\infty(\mathbb{R}\times(0,T)) \cap L^1(0,T; W^{1,1}_{loc}(\mathbb{R}))$. Suppose there exist some $a, b \in \mathbb{R}$ with $a \leqslant 0 \leqslant b$ such that*

$$F_x(x,t,a) = F_x(x,t,b) = F_x(x,t,0) = 0 \quad \textit{for a.e. } (x,t) \in \mathbb{R}\times(0,T). \tag{39}$$

Then for any $u^0 \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ satisfying $a \leqslant u^0(x) \leqslant b$, the unique solution $u \in W(0,T) \cap L^\infty(\mathbb{R}\times[0,T])$ of (37) *given by Proposition* 3.2 *is also an entropy solution of* (36)*.*

Proof. It follows from Proposition 3.2 and the assumptions that $u \in L^\infty(Q_T) \cap L^2(0,T; H^1(\mathbb{R}))$ satisfies Eq. (36) in the sense of distributions. Therefore, u is an entropy solution of (36) by Remark 2.4. □

Using the above preliminary results, we obtain the following theorem which is the main result of this subsection.

Theorem 3.4. *Suppose $\lambda > 0$, $f_i \in C(\mathbb{R})$, $K_i \in L^2_{loc}(\mathbb{R}\times(0,T)) \cap L^1(0,T; W^{1,1}_{loc}(\mathbb{R}))$ and condition* (39) *holds. Then for each $u^0 \in L^\infty(\mathbb{R})$ with $a \leqslant u^0(x) \leqslant b$, the initial-value problem* (5) *admits an entropy solution u satisfying $a \leqslant u(\cdot,t) \leqslant b$ for a.e. $t \in (0,T)$.*

Proof. For simplicity we can assume that $\lambda = 1$. Let us fix $\tilde{T} > T$ and consider K_i as functions in $L^1(\mathbb{R}; W^{1,1}_{loc}(\mathbb{R}))$ by taking $K_i(t) \equiv 0$ for $t \notin (0,T)$. For each i, choose a sequence $\{K_i^\epsilon\}$ as follows:

$$K_i^\epsilon(x,t) := \int_{\mathbb{R}} \hat{K}_i^\epsilon(x,t-s)\eta_\epsilon(s)\,ds,$$

where $\{\eta_\epsilon\}$ is the standard sequence of mollifiers on $\mathbb{R}$ and $x \mapsto \hat{K}_i^\epsilon(x,t)$ is the absolutely continuous function given by

$$\hat{K}_i^\epsilon(x,t) := \begin{cases} K_i(\frac{1}{\epsilon},t) & \text{if } x \geqslant \frac{1}{\epsilon}, \\ K_i(x,t) & \text{if } |x| < \frac{1}{\epsilon}, \\ K_i(-\frac{1}{\epsilon},t) & \text{if } x \leqslant -\frac{1}{\epsilon}. \end{cases}$$

Clearly, $K_i^\epsilon \in C(\mathbb{R}\times\mathbb{R})$ and $\|K_i^\epsilon\|_{L^\infty(\mathbb{R}\times(0,\tilde{T}))} = \|K_i^\epsilon\|_{L^\infty([-\frac{1}{\epsilon},\frac{1}{\epsilon}]\times(0,\tilde{T}))}$. In fact, we have $\{K_i^\epsilon\} \subset L^\infty(\mathbb{R}\times(0,\tilde{T})) \cap L^1(0,\tilde{T}; W^{1,1}_{loc}(\mathbb{R}))$ and $K_i^\epsilon \longrightarrow K_i$ in $L^1(0,\tilde{T}; W^{1,1}_{loc}(\mathbb{R}))$ as $\epsilon \to 0^+$. Next observe that without loss of generality we can assume $f_i(x) = f_i(a)$ for $x \leqslant a$ and $f_i(x) = f_i(b)$ for $x \geqslant b$. Then select $f_i^\epsilon : \mathbb{R} \to \mathbb{R}$ be a sequence of locally Lipschitz continuous functions such that $f_i^\epsilon(a-\epsilon) = f_i(a)$, $f_i^\epsilon(0) = f_i(0)$, $f_i^\epsilon(b+\epsilon) = f_i(b)$ and $f_i^\epsilon \longrightarrow f_i$ uniformly on compact subsets of $\mathbb{R}$. This can be achieved by taking $f_i^r(x) := f_i(x)$ if $x \leqslant 0$, $f_i^r(x) := f_i(0)$ if $x \geqslant 0$ and $f_i^l(x) := f_i(x)$ if $x \geqslant 0$, $f_i^l(x) := f_i(0)$ if $x \leqslant 0$, and defining

$$f_i^\epsilon(x) := \begin{cases} (\xi_\epsilon^l * f_i^r)(x) & \text{if } x \leqslant 0, \\ (\xi_\epsilon^r * f_i^l)(x) & \text{if } x \geqslant 0. \end{cases}$$

Here $\xi_\epsilon^l(x) := \epsilon^{-1}\xi^l(x/\epsilon)$, $\xi_\epsilon^r(x) := \epsilon^{-1}\xi^r(x/\epsilon)$ where ξ^l and ξ^r are nonnegative functions in $C_0^\infty(\mathbb{R})$ satisfying $\int \xi^l(x)\,dx = \int \xi^r(x)\,dx = 1$, $\operatorname{spt}(\xi^l) = [-1, 0]$ and $\operatorname{spt}(\xi^r) = [0, 1]$.

Define $F^\epsilon(x,t,z) := \sum_{i=1}^N K_i^\epsilon(x,t) f_i^\epsilon(z)$. Then it follows from the above construction and the assumption (39) that

$$F_x^\epsilon(x,t,a-\epsilon) = F_x^\epsilon(x,t,b+\epsilon) = F_x^\epsilon(x,t,0) = 0 \quad \text{for a.e. } (x,t) \in \mathbb{R} \times (0,T).$$

For $\epsilon > 0$, let $u_\epsilon^0 := u^0 \chi_{(-\frac{1}{\epsilon}, \frac{1}{\epsilon})}$ and $u^\epsilon \in W(0,\tilde{T}) \cap L^\infty(\mathbb{R} \times [0,\tilde{T}])$ be the unique solution given by Proposition 3.2 of the equation

$$\begin{cases} \partial_t u^\epsilon + \partial_x\big[F^\epsilon\big(x,t,u^\epsilon\big) - F^\epsilon(x,t,0)\big] = u_{xx}^\epsilon & \text{in } Q_{\tilde{T}} := \mathbb{R} \times (0,\tilde{T}), \\ u^\epsilon(\cdot,0) = u_\epsilon^0 & \text{in } \mathbb{R}. \end{cases} \tag{40}$$

Notice that $a - \epsilon \leqslant u^\epsilon(\cdot,t) \leqslant b + \epsilon$ for a.e. $t \in (0,\tilde{T})$. Let $K \subset \mathbb{R}$ be an arbitrary compact set and choose $\phi \in C_0^\infty(\mathbb{R} \times [0,\tilde{T}))$ such that $0 \leqslant \phi \leqslant 1$ and $\phi = 1$ on $K \times [0,T]$. Multiplying (40) by $u^\epsilon\phi$ and integrating over $\mathbb{R} \times (0,\tilde{T})$, we obtain

$$\frac{1}{2}\int\limits_{Q_{\tilde{T}}} \big(u^\epsilon\big)^2 \phi_t\,dt\,dx + \frac{1}{2}\int\limits_{\mathbb{R}} \big(u_\epsilon^0\big)^2 \phi(x,0)\,dx + \int\limits_{Q_{\tilde{T}}} F^\epsilon\big(x,t,u^\epsilon\big) u^\epsilon \phi_x\,dt\,dx$$
$$- \sum_{i=1}^N \int\limits_{Q_{\tilde{T}}} \left(\int\limits_0^{u^\epsilon} f_i^\epsilon(z)\,dz\right) \big(K_i^\epsilon(x,t)\phi\big)_x\,dt\,dx = \int\limits_{Q_{\tilde{T}}} u_x^\epsilon\big(u_x^\epsilon\phi + u^\epsilon\phi_x\big)\,dt\,dx.$$

Consequently,

$$\int\limits_{Q_{\tilde{T}}} \big(u_x^\epsilon\big)^2 \phi\,dt\,dx = \frac{1}{2}\int\limits_{Q_{\tilde{T}}} \big(u^\epsilon\big)^2 (\phi_t + \phi_{xx})\,dt\,dx + \frac{1}{2}\int\limits_{\mathbb{R}} \big(u_\epsilon^0\big)^2 \phi(x,0)\,dx$$
$$+ \sum_{i=1}^N \int\limits_{Q_{\tilde{T}}} \left\{ \left(u^\epsilon f_i^\epsilon\big(u^\epsilon\big) - \int\limits_0^{u^\epsilon} f_i^\epsilon(z)\,dz \right) K_i^\epsilon \phi_x - \left(\int\limits_0^{u^\epsilon} f_i^\epsilon(z)\,dz \right) \big(K_i^\epsilon\big)_x \phi \right\} dt\,dx.$$

As $\phi \equiv 1$ on $K \times [0,T]$, $\|u_\epsilon^0\|_\infty \leqslant \|u^0\|_\infty$, $\|u^\epsilon(\cdot,t)\|_\infty \leqslant \max\{-a,b\} + \epsilon$ for a.e. $t \in (0,\tilde{T})$ and $\{K_i^\epsilon\}$ converges to K_i in $L^1(0,\tilde{T}; W_{loc}^{1,1}(\mathbb{R}))$, we infer that

$$\int\limits_K \int\limits_0^T \big(u_x^\epsilon\big)^2\,dt\,dx \leqslant C(\phi, K_i, f_i, a, b) \quad \text{for all } \epsilon > 0. \tag{41}$$

This together with the boundedness of $\{u^\epsilon\}$ implies that there exists a subsequence $\{u^\epsilon\}$ converging weakly to some function u in $L^2(0,T;H^1_{loc}(\mathbb{R}))$. We next show that up to a further subsequence $u^\epsilon(x,t) \to u(x,t)$ for a.e. $(x,t) \in \mathbb{R} \times (0,T)$. For any $k \in \mathbb{R}$, define

$$\begin{aligned} L_\epsilon &:= \partial_t\big(u^\epsilon - k\big)^+ + \partial_x\big[\operatorname{sign}^+\big(u^\epsilon - k\big)\big(F\big(x,t,u^\epsilon\big) - F(x,t,k)\big)\big] - \partial_x\big[\operatorname{sign}^+\big(u^\epsilon - k\big)u^\epsilon_x\big] \\ &= A_\epsilon + B_\epsilon, \end{aligned}$$

where

$$\begin{aligned} A_\epsilon &:= \partial_t\big(u^\epsilon - k\big)^+ + \partial_x\big[\operatorname{sign}^+\big(u^\epsilon - k\big)\big(F^\epsilon\big(x,t,u^\epsilon\big) - F^\epsilon(x,t,k)\big)\big] - \partial_x\big[\operatorname{sign}^+\big(u^\epsilon - k\big)u^\epsilon_x\big], \\ B_\epsilon &:= \partial_x\big\{\operatorname{sign}^+\big(u^\epsilon - k\big)\big[F\big(x,t,u^\epsilon\big) - F^\epsilon\big(x,t,u^\epsilon\big) - F(x,t,k) + F^\epsilon(x,t,k)\big]\big\}. \end{aligned}$$

Since $K_i \in L^2_{loc}(Q_T)$, we get $K_i^\epsilon \to K_i$ in $L^2_{loc}(Q_T)$ as $\epsilon \to 0^+$. This implies that $\operatorname{sign}^+(u^\epsilon - k)[F(x,t,u^\epsilon) - F^\epsilon(x,t,u^\epsilon) - F(x,t,k) + F^\epsilon(x,t,k)]$ converges to zero in $L^2_{loc}(Q_T)$ because

$$\begin{aligned} &\big|F\big(x,t,u^\epsilon\big) - F^\epsilon\big(x,t,u^\epsilon\big) - F(x,t,k) + F^\epsilon(x,t,k)\big| \\ &\leqslant \sum_{i=1}^N \big|\big[K_i(x,t) - K_i^\epsilon(x,t)\big]\big[f_i\big(u^\epsilon\big) - f_i(k)\big] + K_i^\epsilon(x,t)\big[f_i\big(u^\epsilon\big) - f_i^\epsilon\big(u^\epsilon\big) - f_i(k) + f_i^\epsilon(k)\big]\big| \\ &\leqslant \sum_{i=1}^N \big\{C\big|K_i(x,t) - K_i^\epsilon(x,t)\big| + \big(\big\|f_i^\epsilon - f_i\big\|_{L^\infty(a-1,b+1)} + \big|f_i^\epsilon(k) - f_i(k)\big|\big)\big|K_i^\epsilon(x,t)\big|\big\} \quad \forall\epsilon > 0. \end{aligned}$$

Consequently,

$$B_\epsilon \longrightarrow 0 \quad \text{in } W^{-1,2}_{loc}(Q_T). \tag{42}$$

We know from Lemma 3.3 that u^ϵ is the entropy solution of $\partial_t u^\epsilon + \partial_x[F^\epsilon(x,t,u^\epsilon)] = u^\epsilon_{xx}$. Thus by following the proof of Lemma 3.5 below and using (41), the fact that $\{F^\epsilon(x,t,u^\epsilon) - F^\epsilon(x,t,k)\}$ and $\{F^\epsilon_x(x,t,k)\}$ are bounded in $L^1_{loc}(Q_T)$, we see that $\{A_\epsilon\}$ lies in a compact set of $W^{-1,p}_{loc}(Q_T)$ for any $1 < p < 2$. This and (42) imply that the sequence $\{L_\epsilon\}$ lies in a compact set of $W^{-1,p}_{loc}(Q_T)$ for any $1 < p < 2$. Hence we can apply [28, Corollary 27] with $n = 2$, $\varphi(x,t,u) = (F(x,t,u),u)$, $A(x,t) \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $g(u) = u$ to obtain a further subsequence $\{u^\epsilon\}$ satisfying $u^\epsilon(x,t) \longrightarrow u(x,t)$ for a.e. (x,t) in Q_T. In particular, $u \in L^\infty(Q_T)$ and $a \leqslant u(\cdot,t) \leqslant b$ for a.e. $t \geqslant 0$. Note that in our case the non-degenerate condition required in [28, Corollary 27] is always satisfied since for almost all $(x,t) \in Q_T$ and for all nonzero vectors $(\xi_1,\xi_2) \in \mathbb{R}^2$, the functions $\lambda \mapsto F(x,t,\lambda)\xi_1 + \lambda\xi_2$ and $\lambda \mapsto \lambda\xi_1^2$ are not constant simultaneously on non-degenerate intervals.

Since u^ϵ is the entropy solution, we have for any $k \in \mathbb{R}$ and any nonnegative test function $\varphi \in C^\infty_0(\mathbb{R} \times [0,T))$ that

$$\begin{aligned} &\int_{Q_T} \big\{\big|u^\epsilon - k\big|\varphi_t + \operatorname{sign}\big(u^\epsilon - k\big)\big[F^\epsilon\big(x,t,u^\epsilon\big) - F^\epsilon(x,t,k)\big]\varphi_x\big\}\,dt\,dx \\ &\quad - \int_{Q_T} \operatorname{sign}\big(u^\epsilon - k\big)F^\epsilon_x(x,t,k)\phi\,dt\,dx + \int_{\mathbb{R}} \big|u^0_\epsilon(x) - k\big|\varphi(x,0)\,dx \geqslant \int_{Q_T} \operatorname{sign}\big(u^\epsilon - k\big)u^\epsilon_x\varphi_x\,dt\,dx. \end{aligned}$$

In addition, for almost all $k \in \mathbb{R}$ the Lebesgue measure of the level set $\{(x,t) \in Q_T \colon\ u(x,t) = k\}$ is zero, whence

$$\operatorname{sign}\bigl(u^\epsilon(x,t) - k\bigr) \longrightarrow \operatorname{sign}\bigl(u(x,t) - k\bigr) \quad \text{a.e. on } Q_T.$$

Therefore, by letting $\epsilon \to 0^+$ and using the pointwise convergence of u^ϵ together with the fact that $u_x^\epsilon \to u_x$ weakly in $L^2(0,T; L^2_{loc}(\mathbb{R}))$, we conclude that u is an entropy solution of (36). Notice that it is sufficient to verify the inequality in the definition of entropy solution (Definition 2.1) for all k in a dense subset of $\mathbb{R}$. $\square$

3.2. Inviscid scalar conservation law via vanishing viscosity limit

We next show that Eq. (5) with $\lambda = 0$ admits an entropy solution which is the limit in $L^1_{loc}(Q_T)$ of entropy solutions u^ϵ to the viscous conservation laws

$$\begin{cases} \partial_t u^\epsilon + \partial_x\bigl[F(x,t,u^\epsilon)\bigr] = \epsilon u^\epsilon_{xx} & \text{in } Q_T, \\ u^\epsilon(\cdot,0) = u^0 & \text{in } \mathbb{R}. \end{cases} \tag{43}$$

To achieve this we need the following compactness lemma whose proof is an adaptation of the arguments by Panov [28,29].

Lemma 3.5. *Suppose $f_i \in C(\mathbb{R})$, $K_i \in L^2(0,T; L^2_{loc}(\mathbb{R})) \cap L^1(0,T; W^{1,1}_{loc}(\mathbb{R}))$ and condition (39) holds. Let $u^0 \in L^\infty(\mathbb{R})$ be such that $a \leqslant u^0(x) \leqslant b$ and for each $\epsilon > 0$ let u^ϵ be an entropy solution of (43). Then for any $k \in \mathbb{R}$, the sequence*

$$I_\epsilon := \partial_t\bigl(u^\epsilon - k\bigr)^+ + \partial_x\bigl[\operatorname{sign}^+\bigl(u^\epsilon - k\bigr)\bigl(F\bigl(x,t,u^\epsilon\bigr) - F(x,t,k)\bigr)\bigr]$$

lies in a compact set of $W^{-1,p}_{loc}(Q_T)$ for any $1 < p < 2$.

Proof. As a consequence of Lemma 2.3, we have

$$\begin{aligned} &\partial_t\bigl(u^\epsilon - k\bigr)^+ + \partial_x\bigl[\operatorname{sign}^+\bigl(u^\epsilon - k\bigr)\bigl(F\bigl(x,t,u^\epsilon\bigr) - F(x,t,k)\bigr)\bigr] \\ &\quad + \operatorname{sign}^+\bigl(u^\epsilon - k\bigr)F_x(x,t,k) - \epsilon\,\partial_x\bigl[\operatorname{sign}^+\bigl(u^\epsilon - k\bigr)u^\epsilon_x\bigr] \leqslant 0 \quad \text{in } \mathcal{D}'(Q_T). \end{aligned}$$

Therefore by the Riesz representation theorem, there exist locally finite and positive Borel measures μ^ϵ on Q_T such that

$$I_\epsilon = -\mu^\epsilon - \operatorname{sign}^+\bigl(u^\epsilon - k\bigr)F_x(x,t,k) + \epsilon\,\partial_x\bigl[\operatorname{sign}^+\bigl(u^\epsilon - k\bigr)u^\epsilon_x\bigr] \quad \text{in } \mathcal{D}'(Q_T). \tag{44}$$

Let $H \subset Q_T$ be a compact set. Then (44) gives

$$\begin{aligned} \mu^\epsilon(H) \leqslant \int_{Q_T} \phi\, d\mu^\epsilon(x,t) = &\int_{Q_T} \operatorname{sign}^+\bigl(u^\epsilon - k\bigr)\bigl(F\bigl(x,t,u^\epsilon\bigr) - F(x,t,k)\bigr)\phi_x\, dt\, dx \\ &+ \int_{Q_T} \bigl(u^\epsilon - k\bigr)^+\phi_t\, dt\, dx - \int_{Q_T} \operatorname{sign}^+\bigl(u^\epsilon - k\bigr)F_x(x,t,k)\phi\, dt\, dx \\ &- \epsilon \int_{Q_T} \operatorname{sign}^+\bigl(u^\epsilon - k\bigr)u^\epsilon_x\phi_x\, dt\, dx \end{aligned}$$

for any nonnegative $\phi \in C_0^\infty(Q_T)$ satisfying $\phi \equiv 1$ on H. Notice that by the same arguments leading to (41), we have

$$\epsilon \int\limits_H \left(u_x^\epsilon\right)^2 dt\, dx \leqslant C_H \quad \text{uniformly in } \epsilon > 0. \tag{45}$$

Hence it follows from the above inequality and the fact $a \leqslant u^\epsilon(x,t) \leqslant b$ by Theorem 3.4 that $\mu^\epsilon(H) \leqslant C_H$ uniformly in ϵ. This together with $K_i \in L^1(0,T; W^{1,1}_{loc}(\mathbb{R}))$ and $f_i \in C(\mathbb{R})$ yields

$$-\mu^\epsilon - \operatorname{sign}^+\left(u^\epsilon - k\right) F_x(x,t,k) \quad \text{is bounded in } \mathcal{M}_{loc}(Q_T), \tag{46}$$

where $\mathcal{M}_{loc}(Q_T)$ is the space of locally finite Borel measures on Q_T. Also (45) implies that

$$\epsilon\, \partial_x\left[\operatorname{sign}^+\left(u^\epsilon - k\right) u_x^\epsilon\right] \longrightarrow 0 \quad \text{in } W^{-1,2}_{loc}(Q_T). \tag{47}$$

By using (44), (46), (47) and the fact that $\mathcal{M}_{loc}(Q_T)$ is compactly embedded in $W^{-1,p}_{loc}(Q_T)$ for any $1 < p < 2$, we obtain the conclusion of the lemma. $\square$

Theorem 3.6. *Suppose $f_i \in C(\mathbb{R})$, $K_i \in L^2(0,T; L^2_{loc}(\mathbb{R})) \cap L^1(0,T; W^{1,1}_{loc}(\mathbb{R}))$ and condition (39) holds. Assume in addition that $F(x,t,u)$ is non-degenerate in the sense that for almost every $(x,t) \in Q_T$ the function $u \mapsto F(x,t,u)$ is not affine on non-degenerate intervals. Let $u^0 \in L^\infty(\mathbb{R})$ be such that $a \leqslant u^0(x) \leqslant b$ and u^ϵ be an entropy solution of (43). Then there exists a subsequence $\{u^\epsilon\}$ converging in $L^1_{loc}(Q_T)$ to some function u satisfying $a \leqslant u(\cdot,t) \leqslant b$ for a.e. $t \in (0,T)$. Moreover, u is an entropy solution for the initial-value problem (5) with $\lambda = 0$. If we also have $K_i \in L^1(0,T; L^\infty(\mathbb{R}))$, then u is the unique entropy solution and the whole sequence $\{u^\epsilon\}$ converges to u in $L^1_{loc}(Q_T)$.*

Proof. Since $a \leqslant u^\epsilon(x,t) \leqslant b$ and $F(x,t,u)$ is non-degenerate, the existence of a convergent subsequence $\{u^\epsilon\}$ in $L^1_{loc}(Q_T)$ follows from Lemma 3.5 and [28, Corollary 27] (see also [29, Corollary 2] for a similar result). By standard arguments, we then conclude that the limit function u is an entropy solution of (5) with $\lambda = 0$. If we assume in addition that $K_i \in L^1(0,T; L^\infty(\mathbb{R}))$, then u is unique by Theorem 2.2 and hence we infer that the whole sequence $\{u^\epsilon\}$ converges to u in $L^1_{loc}(Q_T)$. $\square$

The requirement in Theorem 3.6 that $F(x,t,u)$ is non-degenerate excludes some interesting flux functions such as the one given by (4) corresponding to the pressureless Euler–Poisson system. This restriction is common in using the compensated compactness method. However when F is independent of x as (4), by employing a different method we are able to remove the non-degenerate condition in the next theorem.

Theorem 3.7. *Assume $\lambda \geqslant 0$ and $F(t,z) = \sum_{i=1}^N A_i(t) f_i(z)$ with $A_i \in L^2(0,T)$, $f_i \in C(\mathbb{R})$. Then for any $u^0 \in L^\infty(\mathbb{R})$, the problem (5) has a unique entropy solution $u \in C([0,T]; L^1_{loc}(\mathbb{R}))$ satisfying $\|u\|_{L^\infty(Q_T)} \leqslant \|u^0\|_\infty$. If in addition $u^0 \in L^1(\mathbb{R})$, then $u \in C([0,T]; L^1(\mathbb{R}))$.*

Proof. The uniqueness is guaranteed by Theorem 2.2, so we only need to show the existence of an entropy solution. Also it remains to consider the case $\lambda = 0$ since Theorem 3.4 and similar arguments as below yield the desired results for any $\lambda > 0$.

We first assume $u^0 \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ and notice that condition (39) is satisfied for any $a \leqslant 0 \leqslant b$. For each $\epsilon > 0$, let u^ϵ be the unique entropy solution given by Theorem 3.4 of

$$\begin{cases} \partial_t u^\epsilon + \partial_x\left[F\left(t, u^\epsilon\right)\right] = \epsilon u^\epsilon_{xx} & \text{in } Q_T, \\ u^\epsilon(\cdot, 0) = u^0 & \text{in } \mathbb{R}. \end{cases} \tag{48}$$

By Theorem 3.4, Corollary 2.8 and Theorem 2.2, we have for all $\epsilon > 0$ and for a.e. $t \in (0, T)$

$$\left\|u^\epsilon(\cdot,t)\right\|_\infty \leqslant \left\|u^0\right\|_\infty \quad \text{and} \quad \left\|u^\epsilon(\cdot,t)\right\|_{L^1(\mathbb{R})} \leqslant \left\|u^0\right\|_{L^1(\mathbb{R})}; \tag{49}$$

$$\int_{\mathbb{R}} \left|u^\epsilon(x+h,t) - u^\epsilon(x,t)\right| dx \leqslant \int_{\mathbb{R}} \left|u^0(x+h) - u^0(x)\right| dx \quad \forall h \in \mathbb{R}. \tag{50}$$

For each $\phi \in C_0^\infty(B_r)$ and any $t_1, t_2 \in [0, T]$ with $t_1 < t_2$, we have

$$\begin{aligned}\left|\int_{B_r} \left[u^\epsilon(x,t_1) - u^\epsilon(x,t_2)\right]\phi(x)\,dx\right| &= \left|\int_{t_1}^{t_2} \left\langle u_t^\epsilon(t), \phi\right\rangle dt\right| \\ &= \left|\int_{t_1}^{t_2}\int_{\mathbb{R}} F\left(t,u^\epsilon\right)\phi_x\,dx\,dt + \epsilon\int_{t_1}^{t_2}\int_{\mathbb{R}} u^\epsilon\phi_{xx}\,dx\,dt\right| \\ &\leqslant M|B_r|\left(\|\phi_x\|_\infty + \|\phi_{xx}\|_\infty\right)\omega(t_1,t_2) \end{aligned} \tag{51}$$

where $M := \|u^0\|_\infty + \max_i \|f_i\|_{L^\infty(-\|u^0\|_\infty, \|u^0\|_\infty)}$ and $\omega(t_1,t_2) := \int_{t_1}^{t_2} \sum_{i=1}^N |A_i(t)|\,dt + |t_2 - t_1|$. Hence by a simple variation of Kruzhkov's interpolation lemma (see [16, Lemma 4.10]) we get for all $r > 0$ and all $t_1, t_2 \in [0, T]$ with $t_1 < t_2$ that

$$\int_{B_r} \left|u^\epsilon(x,t_1) - u^\epsilon(x,t_2)\right| dx \leqslant C_r\left[\omega(t_1,t_2)^{\frac{1}{3}} + \nu\left(\omega(t_1,t_2)^{\frac{1}{3}}\right)\right] \quad \text{uniformly in } \epsilon. \tag{52}$$

Here $\nu(\cdot)$ is a spatial modulus continuity given by (50). It follows from (49), (50), (52) and the L^1_{loc} compactness lemma (see [18,16]) that there exist a function $u \in L^1(Q_T) \cap L^\infty(Q_T) \cap C([0,T]; L^1_{loc}(\mathbb{R}))$ and a subsequence still denoted by $\{u^\epsilon\}$ such that u^ϵ converges to u in $C([0,T]; L^1_{loc}(\mathbb{R}))$ and also a.e. on Q_T. Now let $\phi \in C_0^\infty(\mathbb{R} \times [0,T))$ be a nonnegative test function. Since u^ϵ is an entropy solution of (48), we have

$$\begin{aligned}&\int_{\mathbb{R}}\int_0^T \left\{\left|u^\epsilon - k\right|\phi_t + \operatorname{sign}\left(u^\epsilon - k\right)\left[F\left(t,u^\epsilon\right) - F(t,k)\right]\phi_x\right\} dt\,dx + \int_{\mathbb{R}} \left|u^0(x) - k\right|\phi(x,0)\,dx \\ &\quad \geqslant \epsilon\int_{\mathbb{R}}\int_0^T \left|u^\epsilon - k\right|_x \phi_x\,dt\,dx = -\epsilon\int_{\mathbb{R}}\int_0^T \left|u^\epsilon - k\right|\phi_{xx}\,dt\,dx.\end{aligned}$$

Hence by letting $\epsilon \to 0^+$ and using the fact $\|u^\epsilon(\cdot,t)\|_\infty \leqslant \|u^0\|_\infty$, we obtain as in the proof of Theorem 3.4 that

$$\int_{\mathbb{R}}\int_0^T \left\{|u - k|\phi_t + \operatorname{sign}(u-k)\left[F(t,u) - F(t,k)\right]\phi_x\right\} dt\,dx + \int_{\mathbb{R}} \left|u^0(x) - k\right|\phi(x,0)\,dx \geqslant 0$$

for almost all $k \in \mathbb{R}$ implying that u is an entropy solution of (5). Moreover, the uniqueness of entropy solutions for (5) yields that the whole sequence u^ϵ converges to u in $C([0,T]; L^1_{loc}(\mathbb{R}))$.

Now suppose that $u^0 \in L^\infty(\mathbb{R})$. For each $(n,m) \in \mathbb{N}^2$, set

$$u^0_{n,m} := (u^0)^+ \chi_{\{|x| \leqslant n\}} - (u^0)^- \chi_{\{|x| \leqslant m\}} \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}).$$

Let $u_{n,m} \in L^1(Q_T) \cap L^\infty(Q_T) \cap C([0,T]; L^1_{loc}(\mathbb{R}))$ be the unique entropy solution of (5) with $\lambda = 0$ corresponding to initial data $u^0_{n,m}$. Since $-(u^0)^- \leqslant u^0_{n,m+1} \leqslant u^0_{n,m} \leqslant u^0_{n+1,m} \leqslant (u^0)^+$, Theorem 2.2 implies that $-\|u^0\|_\infty \leqslant u_{n,m+1} \leqslant u_{n,m} \leqslant u_{n+1,m} \leqslant \|u^0\|_\infty$ a.e. on Q_T. Hence for each positive integer m, there exists $\tilde{u}_m \in L^\infty(Q_T)$ such that $u_{n,m} \uparrow \tilde{u}_m$ a.e. on Q_T as $n \to \infty$. Of course, we have $-\|u^0\|_\infty \leqslant \tilde{u}_{m+1} \leqslant \tilde{u}_m \leqslant \|u^0\|_\infty$ a.e. on Q_T. Thus there exists $u \in L^\infty(Q_T)$ with $-\|u^0\|_\infty \leqslant u \leqslant \|u^0\|_\infty$ a.e. on Q_T such that $\tilde{u}_m \downarrow u$ as $m \to \infty$. We claim that u is an entropy solution of (5) with $\lambda = 0$. Indeed, by the above almost everywhere convergence and since

$$\int_{Q_T} \big\{ |u_{n,m} - k|\phi_t + \operatorname{sign}(u_{n,m} - k)\big[F(t,u_{n,m}) - F(t,k)\big]\phi_x \big\}\, dt\, dx + \int_{\mathbb{R}} \big|u^0_{n,m}(x) - k\big|\phi(x,0)\, dx \geqslant 0$$

for all $k \in \mathbb{R}$ and all nonnegative test functions $\phi \in C_0^\infty(\mathbb{R} \times [0,T))$, we conclude by first letting $n \to \infty$ and then letting $m \to \infty$ that u is an entropy solution. By arguing similarly and using the uniqueness of entropy solutions of (5), we also have $u_{n,m} \downarrow \hat{u}_n$ a.e. on Q_T as $m \to \infty$ and $\hat{u}_n \uparrow u$ a.e. on Q_T as $n \to \infty$. Now let $u_j(x,t) := u_{j,j}(x,t)$ and $u^0_j(x) := u^0_{j,j}(x)$. Then for each $n \geqslant 1$, by the monotonicity we have

$$\begin{aligned} \big|u_j(x,t) - u(x,t)\big| &\leqslant \big|u_{j,j}(x,t) - \tilde{u}_j(x,t)\big| + \big|\tilde{u}_j(x,t) - u(x,t)\big| \\ &\leqslant \big[\tilde{u}_j(x,t) - u_{n,j}(x,t)\big] + \big|\tilde{u}_j(x,t) - u(x,t)\big| \quad \forall j \geqslant n \end{aligned}$$

giving $\limsup_{j\to\infty} |u_j(x,t) - u(x,t)| \leqslant u(x,t) - \hat{u}_n(x,t)$ for a.e. $(x,t) \in Q_T$. Therefore we obtain that $u_j \to u$ almost everywhere on Q_T as $j \to \infty$.

Next we show that $u \in C([0,T]; L^1_{loc}(\mathbb{R}))$. Lemma 2.5 together with the translation invariance of the equation gives

$$\begin{aligned} &-\int_0^T \int_{\mathbb{R}} \big|u_j(x+h,t) - u_j(x,t)\big|\psi_t(x,t)\, dx\, dt \\ &\quad \leqslant \int_{\mathbb{R}} \big|u^0_j(x+h) - u^0_j(x)\big|\psi(x,0)\, dx + \int_0^T \int_{\mathbb{R}} \big|F\big(t,u_j(x+h,t)\big) - F\big(t,u_j(x,t)\big)\big|\big|\psi_x(x,t)\big|\, dx\, dt \end{aligned} \tag{53}$$

for any $\psi(x,t) \in C_0^\infty(\mathbb{R} \times [0,T))$, $\psi \geqslant 0$. Consider functions $\delta_\nu(t)$ and $\theta_\nu(t) := \int_{-\infty}^t \delta_\nu(s)\, ds$ as in the proof of Theorem 2.2. Let $\xi \in C_0^\infty(\mathbb{R})$ be an arbitrary nonnegative function. Using inequality (53) with $\psi(x,t) := \xi(x)\theta_\nu(t_0 - t) \in C_0^\infty(\mathbb{R} \times [0,T))$, $t_0 \in (0,T)$, we obtain

$$\begin{aligned} &\int_0^T \delta_\nu(t_0 - t) \int_{\mathbb{R}} \big|u_j(x+h,t) - u_j(x,t)\big|\xi(x)\, dx\, dt \\ &\quad \leqslant \int_{\mathbb{R}} \big|u^0_j(x+h) - u^0_j(x)\big|\xi(x)\, dx + \int_0^{t_0} \int_{\mathbb{R}} \big|F\big(t,u_j(x+h,t)\big) - F\big(t,u_j(x,t)\big)\big|\big|\xi_x(x)\big|\, dx\, dt. \end{aligned}$$

It then follows by letting $\nu \to \infty$ that

$$\sup_{t\in[0,T]} \int_{\mathbb{R}} \left|u_j(x+h,t) - u_j(x,t)\right|\xi(x)\,dx \leqslant \int_{\mathbb{R}} \left|u_j^0(x+h) - u_j^0(x)\right|\xi(x)\,dx + \int_0^T \int_{\mathbb{R}} \left|F\big(s,u_j(x+h,s)\big) - F\big(s,u_j(x,s)\big)\right|\left|\xi_x(x)\right|dx\,ds$$

for all $\xi \in C_0^\infty(\mathbb{R})$, $\xi \geqslant 0$. Observe that the last term tends to $\int_0^T \int_{\mathbb{R}} |F(s,u(x+h,s)) - F(s,u(x,s))| \times |\xi_x(x)|\,dx\,ds$ and $\int_{\mathbb{R}} |u_j^0(x+h) - u_j^0(x)|\xi(x)\,dx$ tends to $\int_{\mathbb{R}} |u^0(x+h) - u^0(x)|\xi(x)\,dx$ uniformly in $|h| < 1$. Therefore we infer that for any $r > 0$,

$$\int_{B_r} \left|u_j(x+h,t) - u_j(x,t)\right|dx \leqslant \nu_{F,u^0,r}(h)$$

uniformly in $j \in \mathbb{N}$ and $t \in [0,T]$, where $\nu_{F,u^0,r}$ is a modulus of continuity. Moreover, as u_j are limits of solutions u^ϵ of (48) with initial data u_j^0 it follows from (51) that

$$\left|\int_{B_r} \left[u_j(x,t_1) - u_j(x,t_2)\right]\phi(x)\,dx\right| = \left|\int_{t_1}^{t_2} \int_{\mathbb{R}} F(t,u_j)\phi_x\,dx\,dt\right| \leqslant M|B_r|\|\phi_x\|_{L^\infty(B_r)}\omega(t_1,t_2)$$

for any $\phi \in C_0^\infty(B_r)$ and any $t_1 < t_2$. Thus we obtain from Kruzhkov's interpolation lemma that $\int_{B_r} |u_j(x,t_1) - u_j(x,t_2)|\,dx \leqslant C_r[\omega(t_1,t_2)^{\frac{1}{2}} + \nu_{F,u^0,r}(\omega(t_1,t_2)^{\frac{1}{2}})]$ for all j. This gives $u \in C([0,T]; L^1_{loc}(\mathbb{R}))$ as desired.

It remains to prove that $u \in C([0,T]; L^1(\mathbb{R}))$ if $u^0 \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$. We first claim that if $v^0, w^0 \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ satisfy $v^0 \geqslant w^0$ a.e. in $\mathbb{R}$ and v, w are entropy solutions of (5) with initial data v^0 and w^0 respectively, then

$$v - w \in C\big([0,T]; L^1(\mathbb{R})\big). \tag{54}$$

Observe that by the L^1 contraction we have $h(t) := v(t) - w(t) \geqslant 0$ for all $t \in [0,T]$. Also,

$$\int_{\mathbb{R}} \left|h(x,t)\right|dx = \int_{\mathbb{R}} h(x,t)\,dx = \int_{\mathbb{R}} \left[v^0(x) - w^0(x)\right]dx \quad \forall t \in [0,T]. \tag{55}$$

To see (55), let us fix $t \in [0,T)$. By Theorem 2.2 and Corollary 2.9, there exists a set $E \subset (t,T)$ of full measure such that $\int_{\mathbb{R}} h(x,s)\,dx = \int_{\mathbb{R}} [v^0(x) - w^0(x)]\,dx$ and $\int_{\mathbb{R}} |h(x,s)|\,dx \leqslant \int_{\mathbb{R}} |h(x,t)|\,dx$ for all $s \in E$. Moreover as $h \in C([0,T]; L^1_{loc}(\mathbb{R}))$, we can choose a sequence $\{t_n\}$ in E satisfying $t_n \to t$ and $h(x,t_n) \to h(x,t)$ for a.e. $x \in \mathbb{R}$. Then

$$\begin{aligned}\int_{\mathbb{R}} \left[v^0(x) - w^0(x)\right]dx &= \int_{\mathbb{R}} \left|h(x,t_n)\right|dx \leqslant \int_{\mathbb{R}} \left|h(x,t)\right|dx \\ &\leqslant \liminf_{n\to\infty} \int_{\mathbb{R}} \left|h(x,t_n)\right|dx = \int_{\mathbb{R}} \left[v^0(x) - w^0(x)\right]dx,\end{aligned}$$

where the last inequality is due to Fatou's lemma. Thus $\int_{\mathbb{R}} |h(x,t)|\,dx = \int_{\mathbb{R}} [v^0(x) - w^0(x)]\,dx$ and (55) holds for all $t \in [0,T)$. But then by extending A_i to be functions in $L^2(0,T')$ for some $T' > T$ and using the uniqueness of entropy solutions together with the above arguments, we see that h has an extension in $C([0,T']; L^1_{loc}(\mathbb{R}))$ satisfying (55) for any $t < T'$. In particular, we deduce that (55) also holds at $t = T$. Next let $t \in [0,T]$ and $\{t_n\} \in [0,T]$ be such that $t_n \to t$. Since $h(\cdot,t_n) \to h(\cdot,t)$ in $L^1_{loc}(\mathbb{R})$, there exists a subsequence $\{t_{n_k}\}$ such that $h(x,t_{n_k}) \to h(x,t)$ for a.e. $x \in \mathbb{R}$. This and the fact $\|h(\cdot,t_{n_k})\|_{L^1(\mathbb{R})} \to \|h(\cdot,t)\|_{L^1(\mathbb{R})}$ from (55) give $\int_{\mathbb{R}} |h(x,t_{n_k}) - h(x,t)|\,dx \longrightarrow 0$ as $k \to \infty$. We then infer that $h(\cdot,t_n) \to h(\cdot,t)$ in $L^1(\mathbb{R})$ as $n \to \infty$, and so $h \in C([0,T]; L^1(\mathbb{R}))$ yielding claim (54). Now for $u^0 \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$, we take $v^0(x) := (u^0)^+(x)$ and let v be the entropy solution of (5) with initial data v^0. Then since $u = -(v-u) + v$, $v^0 \geqslant u^0$ and $v^0 \geqslant 0$, it follows from claim (54) that $u \in C([0,T]; L^1(\mathbb{R}))$. $\square$

4. Stability of entropy solutions

Estimate (7) in Theorem 2.2 gives an explicit dependence estimate for the entropy solution u upon the initial data u^0. In this section, we consider also the dependence of u with respect to the flux function $F(t,z)$ and obtain results which generalize and strengthen those considered by Maliki in [22]. Fix $N \in \mathbb{N}$, $T > 0$ and define

$$\mathcal{X} := \big\{ (\mathbf{A}, \mathbf{f}, u^0):\ A_i \in L^2(0,T),\ \ f_i \in C(\mathbb{R}) \text{ and } u^0 \in L^\infty(\mathbb{R}) \big\}$$

where $\mathbf{A} := (A_1, \dots, A_N)$ and $\mathbf{f} := (f_1, \dots, f_N)$.

Theorem 4.1. *Assume $\lambda \geqslant 0$ and let $(\mathbf{A}, \mathbf{f}, u^0) \in \mathcal{X}$. Suppose $\{(\mathbf{A}^n, \mathbf{f}^n, u^0_n)\}_n$ is a sequence in $\mathcal{X}$ such that:*

$$\begin{cases} A_i^n \to A_i \quad \text{in } L^1(0,T), \qquad f_i^n \to f_i \quad \text{in } C(\mathbb{R}) \quad \text{and} \quad u_n^0 \to u^0 \quad \text{in } L^1_{loc}(\mathbb{R}), \\ \{u_n^0\} \quad \text{is bounded in } L^\infty(\mathbb{R}). \end{cases} \tag{56}$$

Let $F_n(t,z) := \sum_{i=1}^N A_i^n(t) f_i^n(z)$ and u_n be the unique entropy solution of

$$\begin{cases} \partial_t u_n + \partial_x\big[F_n(t,u_n)\big] = \lambda (u_n)_{xx} & \text{in } Q_T, \\ u_n(\cdot,0) = u_n^0 & \text{in } \mathbb{R}. \end{cases} \tag{57}$$

Then u_n converges to u in $C([0,T]; L^1_{loc}(\mathbb{R}))$, where u is the unique entropy solution of (5) with flux function $F(t,z) := \sum_{i=1}^N A_i(t) f_i(z)$.

Proof. Let $M > 0$ be such that $\|u_n^0\|_{L^\infty(\mathbb{R})} \leqslant M$ for all n. For each $\epsilon \geqslant 0$, define

$$\omega_i^n(\epsilon) := \sqrt{\epsilon} + \sup_{x,y \in [-M,M],\ |x-y| \leqslant \epsilon} \big| f_i^n(x) - f_i^n(y) \big|,$$

$$\omega_i(\epsilon) := \sqrt{\epsilon} + \sup_{x,y \in [-M,M],\ |x-y| \leqslant \epsilon} \big| f_i(x) - f_i(y) \big|.$$

Then $\omega_i^n, \omega_i : [0,\infty) \to [0,\infty)$ are nondecreasing subadditive functions satisfying $\lim_{\epsilon \to 0^+} \omega_i^n(\epsilon) = \lim_{\epsilon \to 0^+} \omega_i(\epsilon) = 0$ and $\lim_{\epsilon \to 0^+} \frac{\omega_i^n(\epsilon)}{\epsilon} = +\infty$ uniformly in n. Moreover, $|\omega_i^n(\epsilon) - \omega_i(\epsilon)| \leqslant 2\|f_i^n - f_i\|_{L^\infty(-M,M)}$ for all $\epsilon \geqslant 0$ giving $\omega_i^n(\epsilon) \to \omega_i(\epsilon)$ uniformly on $[0,\infty)$ as $n \to \infty$.

Let $K \subset \mathbb{R}$ be a compact set and we need to show that

$$\lim_{n \to \infty} \sup_{t \in [0,T]} \int_K \big| u_n(x,t) - u(x,t) \big|\,dx = 0.$$

Take $\hat{u}^0$ be an arbitrary function in $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Let $\hat{u}$ denote the entropy solution of (5) with initial data $\hat{u}^0$. Also for each n, let $\hat{u}_n$ be the entropy solution to (57) corresponding to initial data $\hat{u}^0$. Then $\hat{u}, \hat{u}_n \in L^\infty(Q_T) \cap C([0,T]; L^1(\mathbb{R}))$ by Theorem 3.7. We have

$$\int_K \left|u_n(x,t) - u(x,t)\right| dx$$
$$\leqslant \int_K \left|u_n(x,t) - \hat{u}_n(x,t)\right| dx + \int_K \left|\hat{u}_n(x,t) - \hat{u}(x,t)\right| dx + \int_K \left|\hat{u}(x,t) - u(x,t)\right| dx.$$

As $f_i^n \to f_i$ on compact subsets of $\mathbb{R}$ and there exist a subsequence $\{n_k\}$ and functions $B_i \in L^1(0,T)$ such that $A_i^{n_k} \to A_i$ a.e. in $(0,T)$ and $|A_i^{n_k}(t)| \leqslant B_i(t)$ for a.e. $t \in (0,T)$, it follows from the proof of Theorem 3.7 that $\hat{u}_n$ converges to $\hat{u}$ in $C([0,T]; L^1_{loc}(\mathbb{R}))$. Therefore, the above inequality yields

$$\limsup_{n\to\infty} \sup_{t\in[0,T]} \int_K \left|u_n(x,t) - u(x,t)\right| dx$$
$$\leqslant \limsup_{n\to\infty} \sup_{t\in[0,T]} \int_K \left|u_n(x,t) - \hat{u}_n(x,t)\right| dx + \sup_{t\in[0,T]} \int_K \left|\hat{u}(x,t) - u(x,t)\right| dx. \tag{58}$$

From Lemma 2.5, we can deduce that

$$-\int_0^T \int_{\mathbb{R}} \left|u_n(x,t) - \hat{u}_n(x,t)\right| \psi_t(x,t)\, dx\, dt \leqslant \int_{\mathbb{R}} \left|u_n^0(x) - \hat{u}^0(x)\right| \psi(x,0)\, dx$$
$$+ \int_0^T \int_{\mathbb{R}} \operatorname{sign}(u_n - \hat{u}_n)\big[F_n(t,u_n) - F_n(t,\hat{u}_n)\big] \psi_x\, dx\, dt$$
$$+ \int_0^T \int_{\mathbb{R}} \lambda |u_n - \hat{u}_n| \psi_{xx}\, dx\, dt$$

for any nonnegative test function $\psi(x,t) \in C_0^\infty(\mathbb{R} \times [0,T))$. Hence by arguing as in the proof of Theorem 3.7 we obtain for each $t \in [0,T]$

$$\int_{\mathbb{R}} \left|u_n(x,t) - \hat{u}_n(x,t)\right| \xi(x)\, dx \leqslant \int_{\mathbb{R}} \left|u_n^0(x) - \hat{u}^0(x)\right| \xi(x)\, dx$$
$$+ \int_0^t \int_{\mathbb{R}} \left|F_n(s,u_n) - F_n(s,\hat{u}_n)\right| |\xi_x|\, dx\, ds$$
$$+ \int_0^t \int_{\mathbb{R}} \lambda |u_n - \hat{u}_n| |\xi_{xx}|\, dx\, ds \quad \forall \xi \in C_0^\infty(\mathbb{R}),\ \xi \geqslant 0.$$

Define $w_n(x,t) := |u_n(x,t) - \hat{u}_n(x,t)|$ and $w_n^0(x) := |u_n^0(x) - \hat{u}^0(x)|$. Then from the above inequality and by the subadditive property of ω_i^n, we get for any $t \in [0,T]$ that

$$\int_{\mathbb{R}} w_n(x,t)\xi(x)\,dx \leqslant \int_{\mathbb{R}} w_n^0\xi\,dx + \sum_{i=1}^{N}\left[\frac{\omega_i^n(\epsilon)}{\epsilon}\int_0^t\int_{\mathbb{R}} \left|A_i^n(s)\right| w_n|\xi_x|\,dx\,ds\right]$$
$$+\sum_{i=1}^{N}\left[\omega_i^n(\epsilon)\int_0^t\int_{\mathbb{R}}\left|A_i^n(s)\right||\xi_x|\,dx\,ds\right] + \int_0^t\int_{\mathbb{R}}\lambda w_n|\xi_{xx}|\,dx\,ds \tag{59}$$

for all $\epsilon > 0$ and all $\xi \in C_0^\infty(\mathbb{R})$, $\xi \geqslant 0$.

For each $\epsilon > 0$, let $R = R(\epsilon)$ be given by (30) and $\xi(x) := \psi(\frac{|x|}{R})$ as in the proof of Theorem 2.2. By using ω_i^n instead of ω_i, we similarly define $R_n = R_n(\epsilon)$ and $\xi_n(x) := \psi(\frac{|x|}{R_n})$. Set $\tilde{K}_n := \mathbb{R} \setminus [-R_n, R_n]$ and $E_n(t) := 1 + \sum_{i=1}^N |A_i^n(t)|$, then it follows from (59) and the calculations after (29) that

$$\int_{\mathbb{R}} w_n(x,t)\xi_n(x)\,dx \leqslant \int_{\mathbb{R}} w_n^0\xi_n\,dx + C_0\sum_{i=1}^N \omega_i^n(\epsilon)\int_0^t E_n(s)\,ds$$
$$+\frac{1}{\epsilon R_n}\left(\frac{\lambda\epsilon}{R_n} + \sum_{i=1}^N\omega_i^n(\epsilon)\right)\int_0^t\int_{\tilde{K}_n} E_n(s)w_n\xi_n\,dx\,ds$$
$$\leqslant \int_{\mathbb{R}} w_n^0\xi_n\,dx + C_0\sum_{i=1}^N\omega_i^n(\epsilon)\int_0^T E_n(s)\,ds + \int_0^t E_n(s)\left(\int_{\tilde{K}_n} w_n\xi_n\,dx\right)ds$$

for all $\epsilon > 0$ and all $t \in [0, T]$. Using Gronwall's inequality, this implies that

$$\int_{\mathbb{R}} w_n(x,t)\xi_n(x)\,dx \leqslant e^{\int_0^T E_n(s)\,ds}\left\{\int_{\mathbb{R}} w_n^0\xi_n\,dx + C_0\sum_{i=1}^N\omega_i^n(\epsilon)\int_0^T E_n(s)\,ds\right\}.$$

But as $\chi_{[-R_n,R_n]}(x) \leqslant \xi_n(x)$ and $\lim_{\epsilon\to 0^+} R_n = +\infty$ uniformly in n, we obtain for all $0 < \epsilon < \epsilon_0$ ($\epsilon_0 > 0$ depends only on the compact set K) that

$$\sup_{t\in[0,T]}\int_K w_n(x,t)\,dx \leqslant e^{\int_0^T E_n(s)\,ds}\left\{\int_{\mathbb{R}} w_n^0\xi_n\,dx + C_0\sum_{i=1}^N\omega_i^n(\epsilon)\int_0^T E_n(s)\,ds\right\} \quad \forall n \in \mathbb{N}.$$

Since $\{w_n^0\}$ is bounded in $L^\infty(\mathbb{R})$, $w_n^0 \to |u^0 - \hat{u}^0|$ in $L^1_{loc}(\mathbb{R})$, $R_n \to R$ and $\psi \in L^1(\mathbb{R})$, it is easy to see that

$$\lim_{n\to\infty}\int_{\mathbb{R}} w_n^0\xi_n\,dx = \lim_{n\to\infty}\int_{\mathbb{R}} w_n^0\psi\left(\frac{|x|}{R_n}\right)dx = \int_{\mathbb{R}}\left|u^0 - \hat{u}^0\right|\psi\left(\frac{|x|}{R}\right)dx = \int_{\mathbb{R}}\left|u^0 - \hat{u}^0\right|\xi\,dx.$$

This together with $E_n \to E := 1 + \sum_{i=1}^N |A_i|$ in $L^1(0, T)$ allows us to pass to the limit as $n \to \infty$ in the above inequality to conclude that

$$\limsup_{n\to\infty}\sup_{t\in[0,T]}\int_K\left|u_n(x,t) - \hat{u}_n(x,t)\right|dx \leqslant e^{\int_0^T E(s)\,ds}\left\{\int_{\mathbb{R}}\left|u^0 - \hat{u}^0\right|\xi\,dx + C_0\sum_{i=1}^N\omega_i(\epsilon)\int_0^T E(s)\,ds\right\}.$$

Similarly, we also get

$$\sup_{t\in[0,T]}\int_K \left|u(x,t)-\hat u(x,t)\right|dx \leqslant e^{\int_0^T E(s)\,ds}\left\{\int_{\mathbb{R}} \left|u^0-\hat u^0\right|\xi\,dx + C_0\sum_{i=1}^N \omega_i(\epsilon)\int_0^T E(s)\,ds\right\}.$$

Thus by combining these with (58), we deduce for all $0<\epsilon<\epsilon_0$ that

$$\limsup_{n\to\infty}\sup_{t\in[0,T]}\int_K \left|u_n(x,t)-u(x,t)\right|dx \leqslant 2e^{\int_0^T E(s)\,ds}\left\{\int_{\mathbb{R}} \left|u^0-\hat u^0\right|\xi\,dx + C_0\sum_{i=1}^N \omega_i(\epsilon)\int_0^T E(s)\,ds\right\}.$$

Because $\hat u^0$ is an arbitrary function in $L^1(\mathbb{R})\cap L^\infty(\mathbb{R})$, by an approximation this yields

$$\limsup_{n\to\infty}\sup_{t\in[0,T]}\int_K \left|u_n(x,t)-u(x,t)\right|dx \leqslant 2C_0 e^{\int_0^T E(s)\,ds}\left(\sum_{i=1}^N \omega_i(\epsilon)\int_0^T E(s)\,ds\right) \quad \forall 0<\epsilon<\epsilon_0.$$

The result then follows by letting ϵ tend to 0. □

If the sequence of initial data $\{u_n^0\}$ converges to u^0 in $L^1(\mathbb{R})$ instead of $L^1_{loc}(\mathbb{R})$, then we not only can simplify the conditions in Theorem 4.1 but also obtain a stronger conclusion as stated in the next proposition.

Proposition 4.2. *Let $\lambda\geqslant 0$ and let $(\mathbf{A},\mathbf{f},u^0)\in\mathcal{X}$. Assume in addition that $u^0\in L^1(\mathbb{R})$. Suppose $\{(\mathbf{A}^n,\mathbf{f}^n,u_n^0)\}_n$ is a sequence in $\mathcal{X}$ such that*:

$$A_i^n\to A_i \quad \text{in } L^1(0,T), \qquad f_i^n\to f_i \quad \text{in } C(\mathbb{R}), \quad \text{and} \quad u_n^0\to u^0 \quad \text{in } L^1(\mathbb{R}).$$

Let u_n be the entropy solution of (57) *and u be the entropy solution of* (5)*, where $F_n(t,z):=\sum_{i=1}^N A_i^n(t)f_i^n(z)$ and $F(t,z):=\sum_{i=1}^N A_i(t)f_i(z)$. Then u_n converges to u in $C([0,T];L^1(\mathbb{R}))$.*

Proof. Note that $u_n,u\in C([0,T];L^1(\mathbb{R}))$ by Theorem 3.7. For each n, let $\hat u_n$ be the entropy solution to (57) corresponding to initial data u^0. Then by using Theorem 2.2, we get

$$\begin{aligned}\int_{\mathbb{R}}\left|u_n(x,t)-u(x,t)\right|dx &\leqslant \int_{\mathbb{R}}\left|u_n(x,t)-\hat u_n(x,t)\right|dx+\int_{\mathbb{R}}\left|\hat u_n(x,t)-u(x,t)\right|dx\\ &\leqslant \int_{\mathbb{R}}\left|u_n^0(x)-u^0(x)\right|dx+\int_{\mathbb{R}}\left|\hat u_n(x,t)-u(x,t)\right|dx.\end{aligned}$$

Therefore, the proposition follows if we can show that $\hat u_n$ converges to u in $C([0,T];L^1(\mathbb{R}))$. In order to prove this, we first claim that if $v^0,w^0\in L^\infty(\mathbb{R})\cap L^1(\mathbb{R})$ satisfy $v^0\geqslant w^0$ a.e. in $\mathbb{R}$ and v_n,w_n are entropy solutions of (57) with initial data v^0 and w^0 respectively, then

$$v_n-w_n\longrightarrow v-w \quad \text{in } C\big([0,T];L^1(\mathbb{R})\big) \tag{60}$$

where v and w are entropy solutions of (5) with initial data v^0 and w^0. Indeed, let $h_n(t) := v_n(t) - w_n(t)$ and $h(t) := v(t) - w(t)$. Then $h_n, h \in L^\infty(Q_T) \cap C([0,T]; L^1(\mathbb{R}))$ by Theorem 3.7, and they are all nonnegative by Theorem 2.2. Hence by combining with Corollary 2.9, we obtain

$$\big\|h_n(\cdot,t)\big\|_{L^1(\mathbb{R})} = \big\|h(\cdot,t)\big\|_{L^1(\mathbb{R})} = \int_{\mathbb{R}} \big[v^0(x) - w^0(x)\big]\,dx \quad \text{for all } t \in [0,T]. \tag{61}$$

Since $A_i^n \to A_i$ in $L^1(0,T)$ and $f_i^n \to f_i$ in $C(\mathbb{R})$, we have as in the proof of Theorem 4.1 that v_n and w_n converge in $C([0,T]; L^1_{loc}(\mathbb{R}))$ to v and w respectively. Thus

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \int_K \big|h_n(x,t) - h(x,t)\big|\,dx = 0 \quad \text{for all compact sets } K \subset \mathbb{R}. \tag{62}$$

Define

$$a_n := \sup_{t\in[0,T]} \int_{\mathbb{R}} \big|h_n(x,t) - h(x,t)\big|\,dx = \int_{\mathbb{R}} \big|h_n(x,t_n) - h(x,t_n)\big|\,dx, \quad t_n \in [0,T].$$

Select a subsequence $\{t_{n_k}\}$ of $\{t_n\}$ such that $t_{n_k} \to t_0 \in [0,T]$ as $k \to \infty$. As

$$\big\|h_{n_k}(\cdot,t_{n_k}) - h(\cdot,t_0)\big\|_{L^1(K)} \leqslant \big\|h_{n_k}(\cdot,t_{n_k}) - h(\cdot,t_{n_k})\big\|_{L^1(K)} + \big\|h(\cdot,t_{n_k}) - h(\cdot,t_0)\big\|_{L^1(K)},$$

we conclude from (62) and the fact $h \in C([0,T]; L^1(\mathbb{R}))$ that $h_{n_k}(\cdot,t_{n_k}) - h(\cdot,t_0) \to 0$ in $L^1_{loc}(\mathbb{R})$. Thus by taking a further subsequence still labelled as $\{n_k\}$ we can assume that $h_{n_k}(x,t_{n_k}) \to h(x,t_0)$ for almost every x in $\mathbb{R}$. This together with (61) yields

$$\int_{\mathbb{R}} \big|h_{n_k}(x,t_{n_k}) - h(x,t_0)\big|\,dx \longrightarrow 0 \quad \text{as } k \to \infty$$

implying $\lim_{k\to\infty} a_{n_k} = 0$. Consequently, we infer that $a_n \to 0$ as $n \to \infty$ and hence claim (60) is proved.

We are ready to prove that $\hat{u}_n$ converges to u in $C([0,T]; L^1(\mathbb{R}))$. For this, let $\tilde{u}_n$ be the entropy solution of (57) with initial data $(u^0)^+$ and $\tilde{u}$ be the entropy solutions of (5) with initial data $(u^0)^+$. Then by claim (60), we have $\tilde{u}_n - \hat{u}_n \longrightarrow \tilde{u} - u$ in $C([0,T]; L^1(\mathbb{R}))$ and $\tilde{u}_n \longrightarrow \tilde{u}$ in $C([0,T]; L^1(\mathbb{R}))$. Therefore, $\hat{u}_n$ converges to u in $C([0,T]; L^1(\mathbb{R}))$ as desired. $\square$

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