Hierarchical Model Adaptivity

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Hierarchical Model adaptivity

Definition

- Hierarchy of models
- Adaptivity of models

Hierarchical Stucture

- Physical phenomena can be described by various systems of PDEs.
- These models differ in complexity (perhaps **incrementally**)

Hierarchy of models

What is a hierarchy of models?

Get a complicated model and simplify it using physical reasoning, if possible. This results in a model hierarchy.

Example of hierarchical model structure

$$
\begin{aligned}
\textbf{Navier-Stokes Equations:} \quad & \frac{D\mathbf{u}}{Dt} - \Delta\mathbf{u} + \nabla p = \mathbf{f} \\
\textbf{Stokes Flow:} \quad & -\Delta\mathbf{u} + \nabla p = \mathbf{f} \\
\textbf{Euler Equations:} \quad & \frac{D\mathbf{u}}{Dt} + \nabla p = \mathbf{f}
\end{aligned}
$$

Adaptivity of models

What do we mean by adaptivity of models?

The capability to choose the PDE system that is most appropriate **locally**, based on certain criteria as the computation goes on.

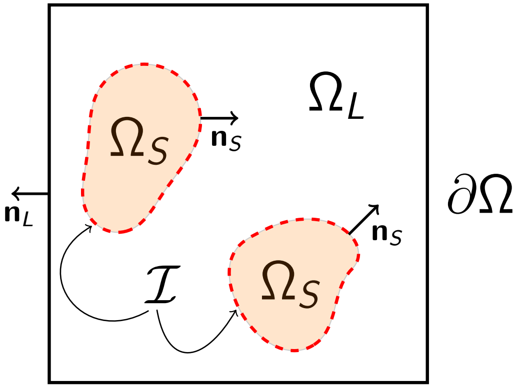

Figure 1: A combined problem with different PDE systems in Ω_L and in Ω_S.

Hierarchical Model Adaptivity

Definition of *Hierarchical* Model Adaptivity

A hierarchy of PDE systems modelling a physical phenomenon is created by succesively simplifying the complicated system. Models are adaptively selected from a hierarchy of models.

Further considerations

- Complicated models are more descriptive but also slower.
- Simpler models are faster but contain less detail.

Research questions

- How do we couple models?
- How do we switch between models (in real-time)?

Combined Stokes-Laplace problem

Model Problem: Individual Equations

Stokes Equations

Find $(\mathbf{u}, p)$ in $\Omega \subset \mathbb{R}^2$ such that:

$$\begin{aligned} -\Delta \mathbf{u} + \nabla p &= \mathbf{f}, \quad \text{in } \Omega \\ \nabla \cdot \mathbf{u} &= 0, \quad \text{in } \Omega \\ \mathbf{u}|_{\partial\Omega} &= 0. \end{aligned} \tag{1}$$

Poisson Equation

Find $\mathbf{u}$ in $\Omega \subset \mathbb{R}^2$ such that:

$$\begin{aligned} -\Delta \mathbf{u} &= \mathbf{f}, \quad \text{in } \Omega \\ \mathbf{u}|_{\partial\Omega} &= 0. \end{aligned} \tag{2}$$

The variables $\mathbf{u}$, $\mathbf{f}$ and p represent the velocity, the forcing and the pressure respectively.

Combined Stokes-Laplace problem

A coupling of the Stokes and Laplace problems

The Laplace problem is a good approximation to the Stokes problem if the divergence of the velocity is zero or small.

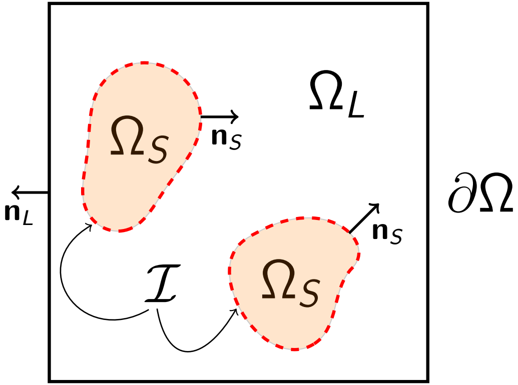

Figure 2: The combined Stokes-Laplace problem with the Stokes equation in Ω_S and Laplace's Equation in Ω_L

Finite Element Method

What does the FEM do?

- Approximates the PDE
- Approximates the solution's Function Space
- Translate our problem into a finite-dimensional linear algebra problem
- Can approximate very complicated domains

Weak Formulation

Multiply our problem by a test function and integrate by parts. This is the weak form of the problem.

Model Problem: Individual Weak Formulations

Stokes Equations Weak form

$$\left.\begin{aligned} \int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v} - p\,(\nabla \cdot \mathbf{v}) = \int_\Omega \mathbf{f} \cdot \mathbf{v} \quad & \forall \mathbf{v} \in \mathbb{V} \\ \int_\Omega q\,(\nabla \cdot \mathbf{u}) = 0 \qquad\qquad & \forall q \in \mathbb{P}, \end{aligned}\right\} \text{ in } \Omega$$

Laplace's Equation Weak form

$$\int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v} = \int_\Omega \mathbf{f} \cdot \mathbf{v} \quad \forall \mathbf{v} \in \mathbb{V} \quad \text{ in } \Omega.$$

Model Problem: Combined Weak Formulation

Stokes Equation Weak form in Ω_S

$$\int_{\Omega_S} \nabla \mathbf{u}_S : \nabla \mathbf{v}_S - p\,(\nabla \cdot \mathbf{v}_S) + \int_{\mathcal{I}} (pI\mathbf{n}_S - \nabla \mathbf{u}_S \mathbf{n}_S) \cdot \mathbf{v}_S = \int_{\Omega_S} \mathbf{f} \cdot \mathbf{v}_S \quad \forall \mathbf{v} \in \mathbb{V}_S$$

$$\int_{\Omega_S} q\,(\nabla \cdot \mathbf{u}_S) = 0 \qquad \forall q \in \mathbb{P}.$$

Laplace's Equation Weak form in Ω_L

$$\int_{\Omega_L} \nabla \mathbf{u}_L : \nabla \mathbf{v}_L - \int_{\mathcal{I}} \mathbf{v}_L \cdot (\nabla \mathbf{u}_L \mathbf{n}_L) = \int_{\Omega_L} \mathbf{f} \cdot \mathbf{v}_L \quad \forall \mathbf{v}_L \in \mathbb{V}_L.$$

Combined Weak form

Weak form for combined problem

$$\begin{aligned} \int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v} - \int_{\Omega_S} p\,(\nabla \cdot \mathbf{v}) + \int_{\mathcal{I}} (pI\mathbf{n}_S - [\![\nabla \mathbf{u}]\!]) \cdot \mathbf{v} = \int_\Omega \mathbf{f} \cdot \mathbf{v} \quad & \forall \mathbf{v} \in \mathbb{V} \\ \int_{\Omega_S} q\,(\nabla \cdot \mathbf{u}) = 0 \qquad\qquad & \forall q \in \mathbb{P}, \end{aligned} \tag{3}$$

where the jump accross $\mathcal{I}$ is given by:

$$[\![\nabla \mathbf{u}]\!] \coloneqq \mathbf{n}_S \cdot \nabla \mathbf{u}_S + \mathbf{n}_L \cdot \nabla \mathbf{u}_L. \tag{4}$$

Does my problem have a solution?

inf-sup conditions

A problem is well-posed if it accepts a unique solution which is also stable - e.g. that can be controlled by the problem data (see [2]). Inf-sup conditions are necesary and sufficient conditions for our problem to be well-posed (see [1]). This condition, for an abstract variational problem is

$$\inf_{\mathbf{u}\in\mathbb{V}} \sup_{\mathbf{v}\in\mathbb{V}} \frac{a\left(\mathbf{u},\mathbf{v}\right)}{\left|\mathbf{u}\right|_1 \left|\mathbf{v}\right|_1} \quad \geq \quad \alpha > 0 \tag{5}$$

Solvability Condition

For the *inf-sup* conditions to hold in this case, we require that on $\mathcal{I}$

$$\int_{\mathcal{I}} \left(pI\mathbf{n}_S - [\![\nabla \mathbf{u}]\!]\right) \cdot \mathbf{v} = 0. \tag{6}$$

Implementation of Adaptivity

Model and Mesh Adaptivity

Model and Mesh Adaptivity are driven by an a-posteriori error indicator. This is a quantity that can be computed from available information

The a-posteriori error indicator

$$\eta_{R,K} = \begin{cases} \left(h_K^2 \,||\mathbf{f}_T + \Delta \mathbf{u}_T - \nabla p_T||_K^2 + ||\mathrm{div}\mathbf{u}_T||_K^2 + \frac{1}{2} \sum_{E \in \mathcal{E}_{K,\Omega} \setminus \mathcal{I}} h_E \,||[\![\nabla \mathbf{u}_T - p_T]\!]| \right. \\ \left(h_K^2 \,||\mathbf{f}_T + \Delta \mathbf{u}_T||_K^2 + \frac{1}{2} \sum_{E \in \mathcal{E}_{K,\Omega} \setminus \mathcal{I}} h_E \,||[\![\nabla \mathbf{u}_T]\!]||_E^2 \right)^{1/2} \text{ if } K \in \Omega_L \end{cases}$$

We implement model and mesh adaptivity by breaking the indicator into a **modelling error** and a **discretisation error**.

Numerical simulations

Simulations: Analytical Solution

How do we check that our model works?

We manufacture a solution and use it on a simple problem.

Analytical Solution

$$\mathbf{u}(\mathbf{x}) = \begin{bmatrix} 200x^2(1-x)^2 y(1-y)(1-2y) \\ -200y^2(1-y)^2 x(1-x)(1-2x) \end{bmatrix} \text{ and } \tag{7}$$

$$p = x(x-0.5)(x-1)\left(y^3 - 1.5y^2 + 0.625y - 0.0625\right) \tag{8}$$

Simulations: Stokes Solution

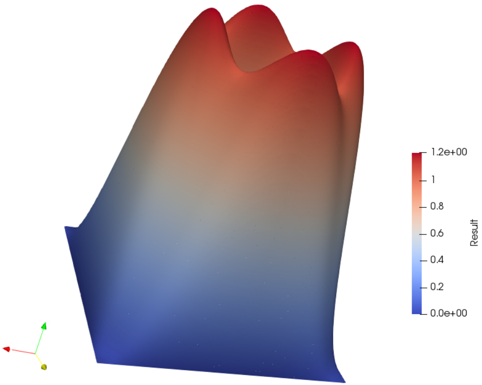

Figure 3: Stokes velocity magnitude

Simulations: Stokes with Laplace

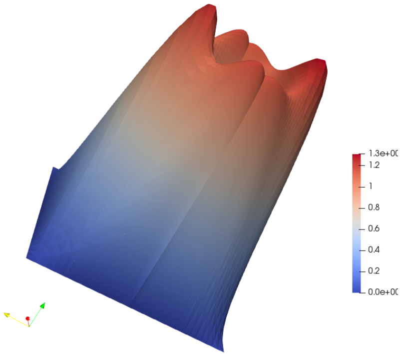

Figure 4: Velocity Mangitude

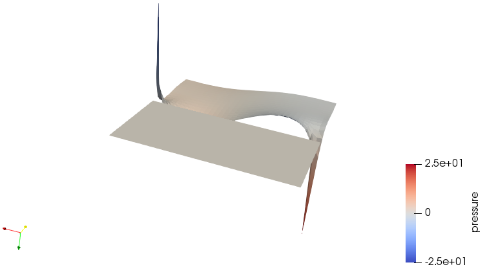

Figure 5: Pressure

Future-steps/Questions?

Future steps

- Identify the reason for the non-zero velocity gradient jump
- Model Mesh adaptivity with non-fixed domains
- Try a different model hierarchy

Thank you! Questions?

References I

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