STATE OF THE ART: LIMIT EQUILIBRIUM AND FINITE-ELEMENT ANALYSIS OF SLOPES[a]

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ABSTRACT: In the past 25 years great strides have been made in the area of static stability and deformation analysis. The widespread availability of microcomputers has brought about considerable change in the computational aspects of slope stability analysis. Analyses can be done much more thoroughly, and, from the point of view of mechanics, more accurately than was possible without computers. Still, engineers performing slope stability analyses must have more than a computer program. They must have a thorough mastery of soil mechanics and soil strength, a solid understanding of the computer programs they use, and the ability and patience to test and judge the results of their analyses to avoid mistakes and misuse. Realistic analyses of deformations of slopes and embankments were not possible until about 25 years ago. They are possible now mainly because the finite-element method has been developed and adapted to these applications. The principal requirement for achieving reasonably accurate and useful results from these analyses is suitable representation of the stress-strain behavior of the soils involved. In the past 25 years the finite-element method has been used to analyze a large number of dams, as well as other embankments and slopes. The experience gained over this period of time provides a number of valuable lessons concerning the advantages and limitations of the finite-element method for use in practical engineering problems.

INTRODUCTION

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Realistic analyses of deformations of slopes and embankments were not possible until about 25 years ago. They are possible now mainly because the finite-element method has been developed and adapted to these applications. The principal requirement for achieving reasonably accurate and useful results from these analyses in suitable representation of the stress-strain behavior of the soils involved. In the past 25 years the finite-element method has been used to analyze a large number of dams, as well as other embankments and slopes. The experience gained over this period of time provides a number of valuable lessons concerning the advantages and limitations of the finite-element method for use in practical engineering problems.

STABILITY ANALYSIS CONDITIONS AND SHEAR STRENGTHS

The first prerequisite for performing effective slope stability analyses is to formulate the right problem, and to formulate it correctly. Selecting appropriate conditions for analysis of slopes requires consideration of the shear strengths of soils under drained and undrained conditions, and consideration of the conditions that will control drainage in the field. The principals involved in selecting analysis conditions and shear strengths are summarized in Table 1.

Free draining soils are those that are able to drain completely within the construction or loading period. Impermeable soils are those for which essentially no drainage can take place during construction or loading.

The most logical basis for estimating the degree of drainage during construction or loading is the value of the dimensionless time factor, T, which is expressed as:

$$T = C_v t/D^2$$

in which C_v = coefficient of consolidation (sq ft/yr or m^2/yr); t = construction or loading time (years); and D = length of drainage path (feet or meters).

If the value of T exceeds 3.0, it is reasonable to treat the material as drained. If the value of T is smaller than 0.01, it is reasonable to treat the material as undrained. If the value of T is between these limits, both possibilities should be considered. If the data required to calculate T is not available, it is usually assumed, for problems that involve normal rates of loading, that soils with permeabilities greater than 10^{-4} cm/s will be drained, and soils with permeabilities (hydraulic conductivities) less than 10^{-7} cm/s will be undrained.

Drained conditions are analyzed in terms of effective stresses, using values of c' and ϕ' determined from drained tests, or from undrained tests with pore pressure measurement. Performing drained triaxial tests on clays is frequently impractical because the required testing time is so long. Direct shear tests or CU tests with pore pressure measurement are often used because the testing time is shorter. Values of c' and ϕ' determined from CU tests with pore pressure measurement have been found to be essentially the same as values determined from drained triaxial or direct shear tests. Values of ϕ' for natural deposits of cohesionless soils are usually estimated using correlations with standard penetration or cone penetration tests.

Undrained conditions are analyzed in terms of total stress in order to avoid having to rely on estimated values of pore pressure for undrained loading conditions, which cannot be predicted accurately. Undrained shear strengths for total stress analyses can be evaluated using in situ tests, unconsolidated-

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TABLE 1. Shear Strengths, Water Pressures, and Unit Weights for Slope Stability Analyses

Soil type (1)	Parameter (2)	Condition: End of construction (3)	Condition: Multistage loading[a] (4)	Condition: Long-term (5)
All soils	External water pressures	Include	Include	Include
All soils	Unit weights	Total	Total	Total
Free-draining	Shear strength parameters	Effective stress envelope, c' and ϕ'	Effective stress envelope, c' and ϕ'	Effective stress envelope, c' and ϕ'
Free-draining	Internal pore pressures	u from steady state seepage analyses	u from steady state seepage analyses	u from steady state seepage analyses
"Impermeable"	Shear strength parameters	Total stress envelope, c and ϕ from in situ tests, UU lab tests, or CU lab tests	Total stress envelope, using $\phi_u = 0$ and s_u from CU lab tests and estimated consolidation pressures	Effective stress envelope, c' and ϕ'
"Impermeable"	Internal pore pressures	No internal pore pressures, set u equal to zero in computer input	No internal pore pressures, set u equal to zero in computer input	u from steady state seepage analyses

[a]Multistage loading includes rapid drawdown, stage construction, and any other condition where a period of consolidation under one set of loads is followed by a change in load under undrained conditions.

undrained (UU) tests, or consolidated-undrained (CU) tests in conjunction with a strength normalizing procedure such as SHANSEP (Ladd and Foott 1974). For multistage loading conditions, such as staged construction or rapid drawdown, the undrained strength is estimated using CU tests results together with values of consolidation pressure estimated by means of consolidation analysis (Ladd 1991).

Stability analysis involves solution of a problem involving force and/or moment equilibrium. The equilibrium problem can be formulated in terms of (1) total unit weights and boundary water pressures; or (2) buoyant unit weights and seepage forces. The first of these alternatives (total unit weights and boundary water pressures) is the better choice, because it is more straightforward. This procedure is outlined in Table 1. Although it is possible, in principle, to use buoyant unit weights and seepage forces, that procedure is fraught with conceptual difficulties, and offers no particular advantage when analyses are performed using a computer program.

The forces involved in the equilibrium problem include those due to the strength of the soil. The strength can be expressed in terms of either the total stress or the effective stress on the failure plane. When the strength of a soil is expressed in terms of total stresses, the pore pressures used as input for that soil should be specified as zero. This results in correct evaluation of the total stress when total unit weights and external water pressures are used. Internal pore pressures for effective stress analyses are determined by seepage analyses for long-term steady-state conditions, or by hydrostatic pressure distributions if there is no flow. External water pressures are included in both total stress and effective stress analyses, because forces due to external water pressures are components that must be included in the overall force and moment equilibrium of the slope. External water pressures can be included in analyses by representing the water outside the slope as a "soil" with $c = 0$, $\phi = 0$, and unit weight $= \rho_w g = 9.81$ kN/m^3 (62.4 lb/cu ft). Alternatively, many computer programs provide a means of representing external water pressures as distributed loads on the external boundaries of the slope.

For embankments and multistage loading conditions where the loading results in increased stresses in the soil, the short-term condition is critical. This is because these types of loads result in positive changes in pore pressures, and, as these positive excess pore pressures dissipate over time, the effective stresses and the strength of the soil increase.

The reverse is true of excavations. An excavation results in negative changes in pore pressures. When these dissipate, the effective stresses and the strength of the soil decrease, and the slope becomes less stable. In cases where it is not clear whether the short-term or the long-term condition will be more critical, both should be analyzed, to ensure that the slope will have adequate stability under the most critical loading condition it will experience.

For natural slopes, the most severe conditions are often associated with high pore pressures, and water pressures in cracks, during wet periods. These are drained conditions, and are analyzed using effective stresses, with water pressures determined from seepage analyses.

Criteria for acceptable values of safety factor should be established with two important considerations in mind. These are: (1) what is the degree of uncertainty involved in evaluating the conditions and shear strengths for analysis; and (2) what are the possible consequences of failure? When the uncertainty and the consequences of failure are both small, it is acceptable to use small factors of safety, on the order of 1.3 or even smaller in some circumstances. When the uncertainties or the consequences of failure increase, larger factors of safety are necessary. Large uncertainties coupled with large consequences of failure represent an unacceptable condition, no matter what the calculated value of the factor of safety. Typical minimum acceptable values of factor of safety are about 1.3 for end-of-construction and multistage loading, 1.5 for normal long-term loading conditions, and 1.0 to 1.2 for rapid drawdown, in cases where rapid drawdown represents an improbable or infrequent loading condition.

STABILITY ANALYSIS PROCEDURES

Impact of Computers

Whitman and Bailey (1967) prefaced a section of their paper for the first Conference on Stability and Performance of Slopes and Embankments with these words:

> "Let us begin by imagining how we might wish to perform slope stability analyses using a computer."

They described how an engineer, using a keyboard and monitor connected to a large central computer, could enter data efficiently and view results nearly instantaneously. The computer would search automatically for the slip surface with the lowest factor of safety, and the engineer would be able to modify the analysis conditions easily to study the effects of changes in various parameters, or to improve the design. The calculation process would be very quickly and efficiently accomplished by the computer, freeing the experienced engineer to concentrate on the validity of the input and the reasonableness of the output.

When Whitman and Bailey wrote their 1967 paper, the scenario they described was, in large part, science fiction. At that time most soil and foundation engineers were using slide rules for their engineering calculations. Few could imagine actually having access to powerful and convenient computer equipment. Where computers were available the common means for input and output were card punch machines, card readers, line printers, and teletype machines.

Now, of course, engineering computational methods have been changed dramatically by the proliferation of computers. Inexpensive microcomputers afford virtually every geotechnical engineer and every geotechnical engineering student computing power and convenience approaching what Whitman and Bailey wished for in the late 1960s.

The easy availability of this computing power has revolutionized slope stability analyses in two ways:

1. Analyses can be done using "advanced" methods that satisfy all conditions of equilibrium. A corollary benefit has been that computers have made it possible to study with great thoroughness some fundamental aspects of the accuracy of slope stability computations.
2. A large number of slip surfaces can be analyzed, making it possible to locate critical slip surfaces with a high degree of reliability. It is possible to find critical noncircular surfaces as well as circular surfaces.

TABLE 2. Characteristics of Equilibrium Methods of Slope Stability Analysis (Duncan and Wright, 1980)

Method (1)	Characteristics (2)
Slope Stability Charts (Janbu 1968; Duncan et al. 1987)	Accurate enough for many purposes Faster than detailed computer analyses
Ordinary Method of Slices (Fellenius 1927)	Only for circular slip surfaces Satisfies moment equilibrium Does not satisfy horizontal or vertical force equilibrium
Bishop's Modified Method (Bishop 1955)	Only for circular slip surfaces Satisfies moment equilibrium Satisfies vertical force equilibrium Does not satisfy horizontal force equilibrium
Force Equilibrium Methods (e.g. Lowe and Karafiath 1960; U.S. Army Corps of Engineers 1970)	Any shape of slip surfaces Do not satisfy moment equilibrium Satisfies both vertical and horizontal force equisibrium
Janbu's Generalized Procedure of Slices (Janbu 1968)	Any shape of slip surfaces Satisfies all conditions of equilibrium Permits side force locations to be varied More frequent numerical problems than some other methods
Morgenstern and Price's Method (Morgenstern and Price 1965)	Any shape of slip surfaces Satisfies all conditions of equilibrium Permits side force orientations to be varied
Spencer's Method (Spencer 1967)	Any shape of slip surfaces Satisfies all conditions of equilibrium Side force are assumed to be parallel

Practical Methods of Analysis

Many methods for analyzing slope stability have been developed. The limit equilibrium methods most often used for practical problems are listed in Table 2. A geotechnical engineer faced with so many methods of analyzing stability problems naturally wants to know:

1. Which of these methods are accurate, which are inaccurate, and for what conditions?
2. Which of the accurate methods can be applied most easily?

The issues related to these important questions are discussed in the following sections.

Fundamental Features and Limitations

The methods of slope stability analysis listed in Table 2 share some common features and limitations that should be understood by engineers who apply these methods to engineering problems.

All of the methods employ the same definition of the factor of safety, F

$$F = \frac{\text{Shear strength of soil}}{\text{Shear stress required for equilibrium}} \tag{1}$$

Another way to state this definition is that F is "the factor by which the shear strength of the soil would have to be divided to bring the slope into a state of barely stable equilibrium." As Lowe (1976) pointed out, defining the factor of safety as a factor on shear strength is logical, because shear strength is usually the quantity that involves the greatest degree of uncertainty.

An implicit assumption in equilibrium analyses of slope stability is that the stress-strain behavior of the soil is ductile, i.e., that the soil does not have a brittle stress-strain curve (where the shearing resistance drops off after reaching a peak). This limitation results from the fact that the methods provide no information regarding the magnitudes of the strains within the slope, nor any indication about how they may vary along the slip surface. As a consequence, unless the strengths used in the analysis can be mobilized over a wide range of strains (i.e., unless the stress-strain behavior is ductile) there is no guarantee that peak strength can be mobilized simultaneously along the full length of the slip surface. If the shearing resistance drops off after reaching the peak, progressive failure can occur, and the shearing resistance that can be mobilized at some points may be smaller than the peak strength. The only fully reliable approach in this case is to use the residual strength rather than the peak strength in the analysis.

The number of equations of equilibrium available is smaller than the number of unknowns in slope stability problems. As a result, all equilibrium methods of slope stability analysis employ assumptions to render the problem determinate. In the case of methods that satisfy all conditions of equilibrium, it has been found that these assumptions do not have a significant effect on the value of the factor of safety. In the case of force equilibrium methods (methods that satisfy only force equilibrium and not moment equilibrium), the value of the factor of safety is affected significantly by the assumed inclinations of the side forces between slices. As a result, force equilibrium methods do not afford as high a degree of accuracy as do methods that satisfy all conditions of equilibrium.

Wright et al. (1973), Tavenas et al. (1980), and others have noted that the factor of safety actually varies from place to place along the slip surface, whereas, in most equilibrium analyses the factor of safety is assumed to be constant. It is worth noting that the average value of F is the same for all practical purposes, even if the factor of safety is assumed to vary from place to place along the slip surface (Chugh 1986). The average value of F defined by (1) is thus insensitive to the assumption that F is the same for every slice.

The value of F as defined by (1) affords a useful index of the margin of stability for a slope. Values of F computed using equilibrium analyses are as reliable as the data that define the conditions analyzed. Minimum values of factor of safety used for various conditions need to be based on experience, considering the likely uncertainties involved in defining the conditions analyzed, and the possible consequences of failure.

Computational Accuracy

Within the last 25 years many studies have been done to evaluate the computational accuracy of the methods listed in Table 2, and other methods of slope stability analysis. Such studies have been done by Spencer (1967), Chen and Giger (1971), Wright et al. (1973), Chen and Snitbhan (1975), Huang and Avery (1976), Fredlund and Krahn (1977), Garber and

Baker (1979), Sarma (1979), Duncan and Wright (1980), Fredlund (1980), Fredlund et al. (1981), Baker and Frydman (1983), Chen and Morgenstern (1983), Ching and Fredlund (1983), Leshchinsky (1990), and Leshchinsky and Huang (1991). These studies have been of great value in clarifying the issue of accuracy in slope stability analyses.

The studies mentioned dealt only with what is called here the "computational" accuracy of the methods, i.e., how accurate are the methods with respect to the way they treat the mechanics of the problem? This is evaluated by comparing factors of safety for the methods to what are believed to be the correct answers for a range of conditions where the slope geometry, water pressures, unit weights, and shear strengths are precisely defined. Inaccuracies that inevitably result from difficulties in defining these quantities for a real slope are not included in studies of computational accuracy.

It is important to note that, for such an evaluation to be valid, the minimum factors of safety for the different methods should be compared, not factors of safety calculated for arbitrarily chosen slip surfaces. This is because different methods may have different critical slip surfaces. If factors of safety are compared for arbitrarily chosen slip surfaces, the results of the comparison will depend on which slip surface is chosen, and may be misleading (Duncan and Wright 1980). Some of the comparative studies that have been published are invalid for this reason.

The crux of the issue with regard to evaluating the computational accuracy of the methods is: What is the correct answer to which other answers should be compared? Although it has proven difficult to find absolutely correct answers, it is possible to decide what is the correct answer with sufficient accuracy for all practical purposes. This conclusion is based on the finding that accurate methods of slices give essentially the same values of F as do friction circle analyses, log spiral analyses, and finite-element analyses, all of which involve different approaches to solution of the equilibrium problem. The maximum difference between factors of safety calculated by methods that satisfy all conditions of equilibrium is about 12%, usually less. Thus, with an accuracy of about $\pm 6\%$, factors of safety calculated using methods that satisfy all conditions of equilibrium can be considered to be the correct answer. This is certainly close enough for practical purposes, because slope geometry, water pressures, unit weights and shear strengths can seldom, if ever, be defined with an accuracy as good as $\pm 6\%$.

Thus, if an engineer performs slope stability analyses using methods that satisfy all conditions of equilibrium, it is justified in virtually every case to conclude that the accuracy of the analyses is as good as, or better than, the accuracy with which the analysis conditions are defined. The engineer can then devote his or her attention to the most important and most difficult issues involved in analyses of slope stability—those of defining geometry, shear strengths, unit weights, and water pressures, and of determining the possible uncertainties in these quantities.

The findings concerning the accuracy of the methods listed in Table 2 can be summarized as follows:

1. The basic accuracy achievable with slope stability charts is as good as the accuracy with which slope geometry, unit weights, shear strengths, and pore pressures can be defined in many cases. The principal limitation of slope stability charts is that they are developed for simple conditions, and approximations are necessary to apply them to real conditions. Nevertheless, if the necessary approximations are made judiciously, accurate results can be achieved more quickly with slope stability charts than by using a computer program. A very effective procedure is to perform preliminary analyses using charts, and final analyses using a computer program.

2. The ordinary method of slices (OMS) is highly inaccurate for effective stress analyses of flat slopes with high pore pressures—the computed factor of safety is too low. The method is perfectly accurate for $\phi = 0$ analyses, and quite accurate for any type of total stress analysis using circular slip surfaces. The method does not have numerical problems.

3. Bishop's modified method is accurate for all conditions (except when numerical problems are encountered). Its limitations are that it is applicable only to circular slip surfaces, and that it has numerical problems under some conditions. If a factor of safety is calculated using Bishop's modified method that is smaller than the factor of safety for the same circle calculated using the ordinary method of slices, it can be concluded that there are numerical problems with the Bishop's modified method analysis. The ordinary method of slices factor of safety is a better answer in these cases. For this reason it is a good idea to calculate the OMS value of F for each circle when the Bishop's modified method is used, so that the values can be compared.

4. Factors of safety calculated using force equilibrium methods are sensitive to the assumed inclinations of side forces between slices. A poor assumption regarding side force inclination can result in a computed factor of safety that is seriously in error. Like all methods that include consideration of side forces on slices, these methods have numerical problems in some cases.

5. Methods that satisfy all conditions of equilibrium (e.g., Janbu's, Morgenstern and Price's, and Spencer's) are accurate for any conditions (except when numerical problems are encountered). The factor of safety computed using any of these methods differ by no more than about 12% from the factor of safety calculated by any other method that satisfies all conditions of equilibrium, and no more than about 6% from what can fairly be considered to be the correct answer. All of these methods have numerical problems under some conditions.

Avoiding Mistakes and Misuse

Another important aspect of accuracy in stability analyses is avoiding mistakes and misuse. When a computer program is used, mistakes in analyses result mainly from these sources:

1. Failure of the person performing the analysis to understand soil mechanics well enough to know how to define the water pressures, unit weights and shear strengths appropriately for the analysis.
2. Failure of the person performing the analysis to understand the computer program well enough to define these quantities correctly in the input.
3. Failure of the person performing the analysis, and the person reviewing the results to check the results and properly evaluate their reasonableness.

The results of slope stability analyses can be checked by experience; by performing extra analyses to compare with a known result; by performing extra analyses to be sure that changes in input cause changes in results that makes sense; and by comparing key results with computations performed using another computer program, slope stability charts, or detailed hand computations. Unless they have been checked carefully, computer results should not be relied upon. The Corps of Engineers requires detailed hand calculations for each critical circle to verify computer results.

Ease of Application

Another matter of considerable interest to an engineer selecting a method for analysis of slope stability is ease of application. Factors that are related to ease of application include: (1) the amount of engineer time required to arrive at an answer; (2) the frequency of numerical problems that require special attention; and (3) the number of steps necessary to develop results in final form for reports or other documents.

Morgenstern and Price (1965) suggested that a number of analyses be performed, examining the internal stress distribution for each one, and varying the parameter controlling the side force inclination, $f(x)$, until a solution was found with an internal stress distribution that was reasonable in all respects. While this procedure was certainly logical and prudent in 1965, it subsequently has been found that factors of safety for solutions with reasonable and unreasonable internal stress distributions are not significantly different (Duncan and Wright 1980; Chen and Morgenstern 1983). Because slope stability analyses are performed to calculate factors of safety, and not internal stresses, it does not matter in the end whether the internal stress distribution is reasonable or not, provided the analysis is done using a method that satisfies all conditions of equilibrium. If the internal stress distribution implicit in the analysis is unreasonable, the engineer can rely on the fact that there is another solution, with a reasonable internal stress distribution, that would give essentially the same factor of safety.

Because the amount of engineering time involved in examining internal stress distributions and developing alternative solutions is so large, and because the effect on the computed factor safety is so small, the writer and others have come to the conclusion that the most effective procedure is not even to examine the internal stress distribution. This makes the analysis process much more efficient. Chen and Morgenstern (1983) developed a computer program which automated the process of examining internal stresses, and used this program for a detailed examination of the issue. They concluded that their research "confirms the view that variations in the factor of safety between several methods in common use are of little practical significance."

As noted previously, all of the methods of analysis that consider side forces between slices are subject to numerical problems under some conditions. When numerical problems arise, the solution may fail to converge, or the calculated value of the factor of safety may be unreasonable. These numerical problems are often associated with slip surfaces that have unreasonable shapes, as noted by Ching and Fredlund (1983). Problems are most frequently encountered where there is a layer of soil with large cohesion at the top of the slope, and tension tends to develop in the upper part of the slip surface, or where there is a layer of soil with a high friction angle at the base of the slope, and the slip surface emerges through this layer at an angle that is too near vertical. Adding a tension crack at the top of the slope can eliminate the tension, and flattening the exit angle at the lower end of the slip surface often eliminates problems that arise there. The computer program developed by Chen and Morgenstern (1983) incorporated sophisticated analytical methods that promoted stable convergence as well as automating the process of finding solutions with physically reasonable internal stress distributions.

To be a good choice for practical engineering applications, the method of analysis selected should be incorporated in an easy-to-use computer program that measures up to the ideal described by Whitman and Bailey (1967). In 1996, we are on the verge of realizing this goal. Computer programs are being developed that will allow an engineer to combine graphical and numerical input in an extremely effective way, to follow the analyses by watching a changing graphical screen, and to get graphical or numerical output in camera-ready form from the computer program. This generation of computer programs, now on the horizon, will permit geotechnical engineers to concentrate on engineering, and to take the computations for granted. As mentioned previously, the engineer will still have to understand soil mechanics and the computer program thoroughly to avoid misuse.

Techniques for Searching for Critical Slip Surface

Locating the slip surface that has the lowest factor of safety is an important part of analyzing slope stability, and a large number of computer techniques have been developed to automate as much of this process as possible. A variety of different procedures have been used for locating the critical circle, or the critical noncircular slip surface.

The problem of locating the critical circular slip surface is the less difficult problem. Most computer programs use systematic changes in the position of the center of the circle and the length of the radius to find the critical circle. For conditions where the geometry is complex, as in most real problems, local minima may exist, and it is necessary to perform multiple searches using different starting points and different searching strategies, to be sure that the overall minimum value of F has been found.

The problem of locating the critical noncircular surface is more complex, and a variety of different approaches have been developed and used. Most are applicable to any method of analysis that can be used to calculate the factor of safety for noncircular slip surfaces:

- Boutrop and Lovell (1980), and Siegel et al. (1981) used random slip surface generators to generate kinematically admissible slip surfaces. The surface with the smallest factor of safety was selected from those generated.
- Baker (1980) coupled dynamic programming minimization techniques with Spencer's method to find critical noncircular slip surfaces.
- Celestino and Duncan (1981) developed a technique of moving one point at a time on a slip surface in one specified direction to find the most critical noncircular surface. Li and White (1987) proposed techniques for improving the efficiency and the robustness of the method. Arai and Tagyo (1985) used similar procedures.
- Optimization techniques were used by Nguyen (1985), and Chen and Shao (1988). Nguyen used the simplex reflection technique, and Chen and Shao used simplex, steepest descent and Davidson-Fletcher-Powell methods, all to good advantage.

All of these methods appear to work fairly efficiently and effectively for problems that do not involve extremely complicated slope geometries. Regardless of what procedure is used within a computer program, the engineer performing the analyses should apply tests of reasonableness to the results, and perform multiple searches, to be sure that the critical slip surface has indeed been located.

A useful by-product of these studies to locate critical slip surfaces is the finding that, unless there are geological controls that constrain the slip surface to a noncircular shape, it can be assumed with little inaccuracy that the critical slip surface is circular. Spencer (1969) found that circular slip surfaces were as critical as logarithmic spiral slip surfaces for all practical purposes. Celestino and Duncan (1981), and Spencer (1981) found that, in analyses where the slip surface was allowed to take any shape, the critical slip surface found by the search was essentially circular. Chen (1970), and Baker and Garber (1977) maintained that the critical slip surface is actually a log spiral. The difference between the minimum factor of safety for the critical circle and the minimum factor of safety for the critical log spiral is in any case too small to be of practical consequence.

A second interesting sidelight that has emerged from these studies is the issue of the validity of slope stability analyses based on variational calculus. Variational calculus has been used for analysis of slope stability problems by Baker and Garber (1977, 1978), Castillo and Revilla (1977), Revilla and Castillo (1977), and Ramamurthy et al. (1977). The technique, which is mathematically complex, purports to provide a means of minimizing the factor of safety for a slope, treating the slip surface, the normal stress distribution, and the interslice force distribution as variables. Some promising-looking results have been achieved, but also some puzzling results. For example, Revilla and Castillo (1977) found that the method resulted in factors of safety for simple slopes in $\phi = 0$ material that were about 30% lower than the value found by Taylor (1948) and subsequently confirmed by many investigators.

De Josselin De Jong (1980) concluded that these applications of variational calculus, which assume the existence of slip lines, are incorrect. These analyses require defining a class of slip lines, and assuming that the class includes the real slip line. Luceno and Castillo (1980), and Castillo and Luceno (1982) concluded that the method used by Baker and Garber (1977, 1978) was incorrectly formulated, and that the method used by Castillo and Revilla (1977) can be shown to be correct only for $\phi = 0$ conditions.

Thus, in view of the fact that both the practical results and the theoretical basis are questionable, it appears that these types of applications of the calculus of variations have not resulted in significant advancement of the practical state of the art of slope stability analysis. The calculus of variations appears to be of greater value when combined with conventional limit equilibrium procedures, for the purpose of locating the most critical slip surface.

Three-Dimensional Analyses of Slope Stability

A large number of studies of three-dimensional slope stability problems have been done since the late 1960s. The types of problems addressed in these studies fall into three categories: (1) slopes that are curved in plan, or that contain corners; (2) slopes that are subjected to loads of limited extent at the top; and (3) slopes in which the potential failure surface is constrained by physical boundaries, such as a dam in a narrow rock-walled valley, or a waste repository with a bowl-shaped liner.

Studies of three-dimensional (3D) slope stability are summarized in Table 3. Based on these studies, three important conclusions can be drawn:

First, it seems clear that the factor of safety for three-dimensional analysis is greater than the factor of safety for two-dimensional analyses, i.e. $F_3 > F_2$, provided that F_2 is calculated for the most critical two-dimensional (2D) section. The only studies that indicate otherwise are those by Hovland (1977), Chen and Chameau (1983), and Seed et al. (1990). Hovland's analyses were based on an extension of the Ordinary Method of Slices, which is inaccurate because it assumes zero normal stress on vertical surfaces. Azzouz and Baligh (1978) showed that results calculated using this method are illogical for some conditions, and that extension of the Ordinary Method

TABLE 3. Methods of Analyzing 3D Slope Stability

Authors (1)	Method (2)	Strength (3)	Geometry of slope/slip surface (4)	3D effects found (5)
Anagnosti (1969)	Extended Morgenstern and Price	c, ϕ	Unrestricted/unrestricted	$F_3 = 1.5\ F_2$ in one case
Baligh and Azzouz (1975)	Extended circular arc	$\phi = 0$	Simple slopes/surfaces of revolution	$F_3 > F_2$
Giger and Krizek (1975)	Upper bound theory of perfect plasticity	c, ϕ	Slopes with corners/log spiral	$F_3 > F_2$
Giger and Krizek (1976)	Upper bound theory of perfect plasticity	c, ϕ	Slopes with corners/log spiral (with loads on top of slope)	$F_3 > F_2$
Baligh et al. (1977)	Extended circular arc	$\phi = 0$	Simple loaded slopes/surfaces of revolution	$F_3 > F_2$
Hovland (1977)	Extended ordinary method of slices	c, ϕ	Unrestricted/unrestricted	$F_3 < F_2$ for some cases
Azzouz et al. (1981)	Extended Swedish circle	$\phi = 0$	Four real embankments/surfaces of revolution	$F_3 = 1.07\ F_2$ to $1.3\ F_2$
Chen and Chameau (1982)	Extended Spencer, and finite element	c, ϕ	Unrestricted/unrestricted	Spencer results are similar to FEM
Chen and Chameau (1983)	Extended Spencer	c, ϕ	Unrestricted/unrestricted	$F_3 < F_2$ for some cases
Azzouz and Baligh (1983)	Extended Swedish circle	$\phi = 0$	Same as Baligh and Azzouz with loads on top	$F_3 > F_2$
Dennhardt and Forster (1985)	Assumed s on slip surface	c, ϕ	Slopes with loads/unrestricted	$F_3 > F_2$
Leshchinsky et al. (1985)	Limit equilibrium and variational analysis	c, ϕ	Unrestricted	$F_3 > F_2$
Ugai (1985)	Limit equilibrium and variational analysis	$\phi = 0$	Vertical slopes/cylindrical	$F_3 > F_2$
Leshchinsky and Baker (1986)	Limit equilibrium and variational analysis	c, ϕ	Slopes constrained in 3rd dimension/unrestricted	$F_3 > F_2$ for $c > 0$, $F_3 = F_2$ for $c = 0$
Baker and Leshchinsky (1987)	Limit equilibrium and variational analysis	c, ϕ	Conical heaps/unrestricted	$F_3 > F_2$
Cavounidis (1987)	Limit equilibrium	c, ϕ	Unrestricted/unrestricted	F_3 must be $> F_2$
Hungr (1987)	Extended Bishop's modified	c, ϕ	Unrestricted/surfaces of revolution	$F_3 > F_2$
Gens et al. (1988)	Extended Swedish circle	$\phi = 0$	Simple slopes/surfaces of revolution	$F_3 > F_2$
Leshchinsky and Mullet (1988)	Limit equilibrium and variational analysis	c, ϕ	Vertical slopes with corners/unrestricted	$F_3 > F_2$
Ugai (1988)	Extended ordinary method of slices, Bishop's modified, Janbu, and Spencer	c, ϕ	Unrestricted/unrestricted	$F_3 > F_2$, except for OMS
Xing (1988)	Limit equilibrium	c, ϕ	Unrestricted/ellipsoidal	$F_3 > F_2$
Michalowski (1989)	Kinematical theorem of limit plasticity	c, ϕ	Unrestricted/unrestricted	$F_3 > F_2$
Seed et al. (1990)	Ad hoc 2D and 3D	c, ϕ	One particular case, the Kettleman Hills failure	$F_3 < F_2$
Leshchinsky and Huang (1991)	Limit equilibrium and variational analysis	$c. \phi$	Unrestricted/unrestricted	$F_3 > F_2$

of Slices is not an adequate approach to 3D analysis. Hutchinson and Sarma (1985) questioned some of the assumptions used by Chen and Chameau, and Ugai (1988), who also extended Spencer's method to 3D, found that F_3 was greater than F_2, rather than smaller, as Chen and Chameau had found. Seed et al. compared results of 2D and 3D analyses that did not satisfy all conditions of equilibrium. The horizontal force imbalance in their approximate 3D analysis was 3.7% of the weight of the sliding mass. Because the friction angles along the slip surface they studied were so small (8° to 9°), this force imbalance could result in as much as a 25% difference in the calculated factor of safety. Thus, all of the cases where F_3 was found to be smaller than F_2 appear to involve significant potential inaccuracies. It appears that Cavounidis (1987) was correct in concluding that "Methods that give F_3/F_2 ratios that are smaller than unity either compare inappropriate factors or, more probably, contain simplifying assumptions that neglect important aspects of the problem."

Second, Hutchinson and Sharma (1985) and Leshchinsky and Baker (1986) pointed out that 2D and 3D analyses should give the same factor of safety for slopes in homogeneous cohesionless soils, because the critical slip surface is a shallow plane parallel to the surface of the slope.

Third, Azzouz et al. (1981), and Leshchinsky and Huang (1991) noted that, if 3D effects are neglected in analyses to back calculate shear strengths, the back calculated strengths will be too high.

Analyses of Stability of Reinforced Slopes

Research and field studies during the past 25 years have developed a solid basis for analysis and design of reinforced slopes and embankments. Based on the research findings, and experience with field installations, it is possible now to design with confidence slopes and embankments that are reinforced with geotextiles, geogrids, and steel mesh (Christie and El Hadi 1977; Rowe and Soderman 1985; Mitchell and Villet 1987; Schaefer and Duncan 1988; Christopher 1987; Franks et al. 1988).

The principal function of reinforcement in slopes and embankments is to provide stabilizing forces. Through friction between the reinforcement and the soil, acting over a sufficient development length, it is possible to develop enough resistance to pullout so that the full tensile strength of the reinforcement can be mobilized to help stabilize the slope. Within lengths of reinforcement where sufficient pullout resistance has developed to prevent slip, the reinforcement and the soil undergo essentially the same strains.

The properties important to the effective function of reinforcement materials in soil are their tensile strength, tensile stiffness, time-dependent deformation characteristics, durability, interface friction, and resistance to damage during construction (Christopher and Holtz 1984). For purposes of selecting allowable loads for reinforcement in soil, it is important to consider the amount of strain that can be tolerated in the reinforcement without excessive deformations in the reinforced slope or embankment. These vary from 2% for reinforcement embedded in brittle soils susceptible to strain softening, up to 10% for slopes in ductile, nonsensitive soils (Jewell 1985; Haliburton 1981; Fowler 1982; Christopher and Holtz 1984; Rowe and Soderman 1985; Bonaparte et al. 1987).

Modes of failure of reinforced slopes and embankments include tensile failure of the reinforcement, pullout of the reinforcement from the soil, excessive deformation of the reinforcement (Haliburton et al. 1978), and also raveling of the soil from between layers of reinforcement at the face of steep slopes (Ingold 1982). If the reinforcement fails in tension or deforms excessively, the same type of shear failure can take place as would occur without reinforcement. In addition to modes of failure that involve rupture or excessive elongation of the reinforcement, embankments on weak foundations must be designed to be safe against bearing capacity failure, and against sliding of the embankment across the top of the reinforcement (Haliburton et al. 1978).

Slope stability analyses can be used to calculate the factor of safety of a slope or embankment including the stabilizing effect of the reinforcement force. Through repeated trials, these analyses can be used to determine the reinforcement force needed to achieve a given factor of safety with respect to soil failure. Stability analyses of reinforced slopes and embankments can be performed using the same methods that are used for slopes without reinforcement. The effect of the reinforcement is included in these analyses through tensile forces, of specified magnitudes, acting at the location where the reinforcement will be installed in the slope.

Low (1989) developed slope stability charts for analysis of reinforced embankments on weak clay foundations.

Studies by Wright and Duncan (1991) have shown that any of the methods that satisfy all conditions of equilibrium result in essentially the same factor of safety, provided that the reinforcement forces are included in the equations of horizontal, vertical, and moment equilibrium. Bishop's modified method also results in essentially the same factor of safety provided that the reinforcement forces are included in the equations of vertical and moment equilibrium. (Bishop's modified method does not use the equations of horizontal force equilibrium.) These studies also showed that the manner in which the reinforcement force is distributed along the length of the reinforcement has no significant effect on the calculated factor of safety. Whether the force is applied as a concentrated force acting at the base of the slice through which it crosses the slip surface, or is applied as a concentrated force at the surface of the slope where it terminates, the calculated factor of safety is for all practical purposes the same. These two points of application represent the extremes possible, and it appears that any other reasonable method of representing the reinforcing force will also lead to the same value of factor of safety.

Factors of safety are incorporated in design calculations for reinforced slopes and embankments in two ways. One is by applying a factor of safety to the limiting load in the reinforcement. The other is by applying a factor of safety to the shear strength of the soil. Selection of acceptable minimum values of factor of safety is guided by understanding of the behavior of the reinforcement and the soil, and by consideration of the consequences of failure. The values of factor of safety used in the reinforcement strength and the soil strength need not be the same (Ingold 1991).

Conclusions Regarding Slope Stability Analyses

Nearly universal availability of computers and much improved understanding of the mechanics of slope stability analyses have brought about considerable change in the computational aspects of slope stability analysis. Analyses can be done much more thoroughly, and, from the point of view of mechanics, more accurately than was possible previously. However, the easy availability of computing power has not reduced at all the need for engineers performing slope stability analyses to have (1) a thorough command of soil mechanics and soil strength; (2) a solid understanding of the computer programs they use; and (3) the ability to evaluate the results of their analyses to avoid mistakes and misuse.

It is important to understand the limitations of slope stability analyses, and also to see these limitations in perspective. The fact that equilibrium analyses of slope stability involve assumptions and limitations does not mean that they are valueless. It means that they cannot be used without understanding and judgment. The factor of safety defined by (1) affords a useful index of the stability of a slope, provided that its value is considered with due regard for the uncertainties involved in evaluating the shear strengths and other quantities involved in the analysis.

It was not possible to perform three-dimensional stability analyses of slopes in soil before computers were available, and the first of these analyses was done only in the 1960s. Studies of 3D slope stability have progressed far enough to conclude that the factor of safety calculated using 3D analyses will always be greater than, or equal to, the factor of safety calculated using 2D analyses.

DEFORMATION ANALYSES

Introduction

This review of deformation analyses of slopes and embankments is focused primarily on the finite-element method. Although the finite-element method is the most generally applicable and the most widely used method of analyzing deformations, it is not the only method that can be used. For some conditions, simpler methods have been developed. These are discussed subsequently.

The finite-element method was introduced to the geotechnical engineering profession by Clough and Woodward (1967), in a paper written for the first Berkeley conference on the stability and performance of slopes and embankments. The most significant aspect of their paper was the use of nonlinear stress-strain relationships in their analyses of an embankment dam. Geotechnical engineers had long been aware of the limited usefulness of linear elastic analyses of stresses and movements in earth masses, and it was immediately apparent that the ability to consider nonlinear stress-strain behavior gave the finite-element method great potential for use in geotechnical engineering problems.

Advantages and Limitations of Finite-Element Method

The finite-element method is a general-purpose method that has many desirable characteristics for application to analysis of stresses and movements in earth masses.

- It has been used to calculate stresses, movements, and pore pressures in embankments and slopes.
- It has been used for analyses of conditions during construction, and also following construction, as consolidation or swelling occur and excess pore pressures dissipate.
- It has been used to investigate the likelihood of cracking, hydraulic fracturing, local failure, and overall stability of slopes.

The method is so general that it is possible to model many complex conditions with a high degree of realism, including in the analyses such things as nonlinear stress-strain behavior, nonhomogeneous conditions, and changes in geometry during construction of an embankment or an excavation.

This generality and power does not come without its price. The effort and the cost of finite-element analyses are high. Each analysis takes a considerable amount of engineering time to develop property values, to perform the computer analyses, and to evaluate the results. The amount of engineering time required has been reduced through development of graphical preprocessors and postprocessors, but it is still very significant. In addition, a considerable amount of time is required to learn to use the method effectively.

The cost of computer time for finite-element analyses has been reduced greatly by the fact that they can be performed on the powerful new microcomputers that are now available. This has not had a very significant effect on the overall cost of performing analyses, however, because even when computer time was an order of magnitude more expensive than it is now, its cost was typically no more than about 10% of the total cost of an analysis.

Although some types of finite-element analyses can now be performed on microcomputers, mainframe computers are required for problems that involve very large numbers of elements, three-dimensional analyses, and iterative techniques that involve extremely large numbers of calculations for accurate simulation of nonlinear behavior.

Incremental Analyses

The key to realistic simulation of static geotechnical problems in finite-element analyses has been the use of incremental analysis techniques. These involve simulating the overall problem as a series of events, and analyzing each event as a simple linear problem. For example, construction of an embankment can be modeled in increments, each involving addition of a layer of elements to the mesh. Digging an excavation can be modeled in increments, each involving removal of a layer of elements from the mesh. Applying a load to an anchor within a slope can be modeled in increments, each involving application of an increment of load to the anchor. Six to ten increments are usually enough to achieve reasonable accuracy.

Incremental analyses afford a convenient means of modeling two very important aspects of geotechnical engineering problems, namely changes in geometry and nonlinear stress-strain behavior. Changes in geometry, such as the height of an embankment or the depth of an excavation, are modeled by adding elements to the mesh or removing them from the mesh. Nonlinear and stress-dependent stress-strain behavior is modeled by changing the stiffness values assigned to each element during each increment of the analysis.

Information Required for Analyses

Finite-element analyses require definition of initial conditions, stress-strain, properties, and the construction or loading sequence.

For problems that involve natural soil deposits, or already existing fills, it is necessary to specify the state of stress in the soil mass prior to the beginning of construction or loading. The initial stresses are needed for three reasons. First, in incremental analyses, the changes in stress calculated during each increment are added to the stresses at the beginning of the increment to evaluate the stresses at the end. To begin this process, it is necessary to know the initial stresses. Second, the stiffness of the soil depends on the stresses in the soil. Third, in analyses of excavation, the forces that are applied to simulate excavation of the soil are calculated using the before-excavation stresses on the boundary of the excavation. To calculate these forces, it is necessary to know the initial stresses.

The initial stresses can be measured, but are usually estimated. For level ground, where at-rest pressure conditions would be expected to prevail, the vertical stresses are usually taken to be equal to the overburden pressure, and the horizontal stresses are taken as k_0 times the overburden pressure. The value of k_0 is frequently estimated using the empirical relationships of Jaky (1944), Brooker and Ireland (1965), or Mayne and Kulhawy (1982). For initially nonlevel ground the difficulties in estimating the initial stresses are greater. The main problem is in establishing a reasonable initial state of stress that satisfies equilibrium. One simple procedure that has been used is to perform a gravity turn-on analysis (apply vertical forces representing the weight of the material to an initially unstressed mesh), and then change the horizontal stresses to be equal to k_0 times the calculated vertical stresses, using values of k_0 based on the best information available.

The reference state for stresses and the reference state for displacements are fundamentally different. The reference state for stresses must satisfy equilibrium. The reference state for displacements, on the other hand, is arbitrary. It is permissible to define any condition as zero displacement (i.e. as the reference state for displacements) provided it is used consistently. When the initial conditions are established for a natural soil deposit or a preexisting fill, the stresses should satisfy equilibrium. The displacements, however, can be defined as zero. If a gravity turn-on analysis is used to establish initial stress conditions, the displacements calculated in the analysis can be ignored (set equal to zero).

Strains, like displacements, have arbitrary reference states. Therefore it is preferable to relate the values of stress-strain parameters used in nonlinear stress-strain relationships to stresses rather than strains.

The stress-strain properties of the soils play a critical role in finite-element analyses. In some simple cases where soils are not stressed close to failure, and where strains are small, it is possible to represent soils as linear elastic materials. More often, it is necessary to use stress-strain relationships that account for nonlinear behavior, and that account for the fact that soil modulus values vary with confining pressure.

In some cases it is necessary to use stress-strain relationships based on plasticity theory in order to model important aspects of soil behavior. These cases include problems where the undrained behavior is being analyzed in terms of effective stresses, and it is necessary to calculate accurate values of changes in pore pressure resulting from undrained loading, and problems where local failure occurs, and the behavior is controlled to a significant degree by the properties assigned to material that has already failed. Experience has shown that elastic stress-strain relationships have significant shortcomings in these cases, even if they model nonlinear and stress-dependent behavior.

The analysis should simulate as closely as possible the actual construction or loading sequence of the problem being analyzed. Adding elements to simulate fill placement, removing elements to simulate excavation, and applying loads in increments have been mentioned previously. Other processes that can be modeled in finite-element analyses include raising or lowering water levels within a soil mass, changing the temperature of structures, and consolidation. Of some interest and importance is the fact that there is no fully rational way of analyzing a natural slope unless the processes involved in its formation are known and can be modeled. As discussed previously, the stress conditions in a natural slope can be approximated using gravity turn-on analyses, with empirical adjustments to the horizontal stress.

In designing a finite-element mesh it is important to consider the number of steps to be used in the analysis as well as the significant geometric features that need to be included. In problems that model nonlinear behavior or changes in geometry, the accuracy of the analyses is affected by the number of steps used. Studies have shown that about eight layers are enough to model construction of an embankment, or an excavation. Acceptable accuracy can sometimes be achieved with fewer layers. The number of steps needed for accuracy is independent of the embankment height or the excavation depth.

Stress-Stain Relationships Used in Practice

Selecting an appropriate soil stress-strain relationship is primarily involved with balancing simplicity and accuracy. Which type of relationship is most suitable for a given case depends on the conditions being analyzed and the purpose of the analysis. While it seems reasonable that more complex stress-strain relationships should be able to model the behavior of soils more accurately, there is no benefit in using a very complex relationship to analyze a problem where the simplest representation of the stress-strain behavior of the soil would result in acceptable accuracy. In order of increasing complexity, the choices of stress-strain relationships include linear elastic, multilinear elastic, hyperbolic (elastic), elastoplastic, and elastoviscoplastic.

Linear Elastic

The principal advantage of linear elastic analyses is simplicity. Only two elastic parameters (e.g. Young's modulus, E, and Poisson's ratio, ν) are needed to characterize the stress-strain behavior of isotropic elastic materials. The shortcoming of linear elasticity is that it is not a good model for the actual stress-strain behavior of soils, except at low stress levels and small strains. Because the most appropriate values of E and ν depend on confining pressure and deviator stress, there is no fully rational way of selecting single values of E and ν for a linear analysis.

In spite of these shortcomings, linear elastic analyses have been used in a number of cases for analysis of embankments and excavated slopes. Several of these analyses are summarized in Table 4. It can be noted that in many of the cases where calculated movements were compared to measured values, the agreement was found to be good. The writer believes that this is affected by the fact that the analyses were done after the fact (after the field measurements were made). According to the system suggested by Lambe (1973), these are "Class C1" predictions. If an analysis is done after the fact and the agreement is not good initially, there is a natural tendency to find out what is "wrong," and to make changes that improve the agreement.

It can be concluded, however, that linear elastic finite-element analyses result in reasonable values of stress and patterns of displacement in many cases. If only stresses are of interest, and if there are not zones of strongly differing stiffness, like a stiff shell and a soft core in a zoned dam, linear elastic analyses may be about as good as nonlinear analyses. Displacements from linear elastic analyses may also be reasonably accurate, provided that the right values of modulus and Poisson's ratio are used in the analyses. The references in Table 4 provide guidance on selecting modulus values for use in finite element analyses based on various types of laboratory tests.

Multilinear Elastic

Modeling stress-strain curves for soil using two or more straight lines improves the accuracy with which laboratory stress-strain curves can be represented in analyses. Reduction in modulus with increasing strain can be modeled, and local failure within an element can be represented by assigning a very small modulus value to the element once the calculated shear stress becomes equal to the strength of the soil. A number of analyses performed using such procedures are summarized in Table 5. In many of the cases summarized in Table 5, there was reasonable agreement between the results of the analyses and field measurements, although there seems to be a tendency for calculated movements to be larger than measured movements. The reasons suggested for this include:

1. Where stress-strain parameter values are determined from tests on remolded specimens, these specimens, freshly compacted in the lab, will be less stiff than the material in the field, because the material in the field will have aged somewhat and grown stiffer by the time it is subjected to the full load it will bear.
2. If laboratory test specimens are compacted to the specified minimum density that is allowable in the field, they

TABLE 4. Finite Element Analyses of Slopes and Embankments Performed Using Linear Elastic Stress-Strain Relationships

Authors (1)	Type of slope (2)	Purpose of analysis (3)
Clough and Woodward (1967)	Embankments	Examined effects of incremental construction and foundation deformations on stresses and deformations
Duncan and Dunlop (1969)	Excavations	Examined effects of initial horizontal stresses on stresses and deformations after excavation
Penman et al. (1971)	Scammonden Dam	Compared movements calculated after the fact with measured movements, found reasonable agreement
Poulos et al. (1972)	Embankments	Developed a simplified means of estimating deformations without performing finite-element analyses
Lefebvre et al. (1973)	Dams in V-shaped valleys	Performed 3D analyses to evaluate the accuracy of 2D plane strain analyses for various valley wall slopes
Penman and Charles (1973)	Llyn Brianne Dam	Compared movements calculated after the fact with measured movements, found reasonable agreement
Eisenstein and Simmons (1975)	Mica Dam	Compared movements calculated after the fact using 2D and 3D analyses with measured movements, found reasonable agreement
Naylor (1975)	Embankment	Compared results of linear and nonlinear analyses
Thoms et al. (1976)	Highway embankment on saturated clay foundation	Compared settlements calculated in FEM consolidation analyses, after the fact, with measured settlements, found reasonable agreement.
Cavounidis and Hoeg (1977)	Consolidation of core in zoned dam	Compared results of linear and nonlinear analyses
Klym et al. (1977)	Powerhouse excavation	Deduced elastic modulus value for stiff glacial deposits from measured heave during excavation.
Cathie and Dungar (1978)	Llyn Brianne Dam	Compared linear and nonlinear analyses; Compared movements calculated after the fact using 2D and 3D analyses with measured movements, found reasonable agreement
Martin (1978)	Storvass Dam	Presented 3D analyses of concrete-faced rockfill dam; no comparisons with measured behavior
Eisenstein and Law (1979)	Embankments	Studied effects of anisotropic stress-strain behavior
Adikari and Parkin (1981)	Talbingo Dam	Compared linear and nonlinear analyses; Compared movements calculated after the fact with measured movements, found reasonable agreement
Justo and Saura (1983)	El Infernillo Dam	Compared movements calculated after the fact using 2D and 3D analyses with measured movements, found reasonable agreement
Penman and Charles (1985)	Winscar Dam	Presented analyses of asphalt-faced rockfill dam; compared movements calculated after the fact with measured movements, found reasonable agreement for movements during construction, poorer agreement for movements during reservoir filling
Koga et al. (1988)	Reinforced embankment	Compared calculated behavior of reinforced and unreinforced embankments
Mino et al. (1988)	Reinforced cut slope	Compared movements calculated after the fact with measured movements, found reasonable agreement
Naylor and Mattar (1988)	Fills	Investigated effect of number of layers of elements used in finite element mesh

will be less dense than the average material in the field, because the average field density is always higher than the specified minimum. Being less dense they will tend to be less stiff.

3. Where stress-strain parameter values are determined from tests on samples retrieved from the field, the lab specimens will tend to be less stiff than the material in the field because disturbance will tend to flatten the stress-strain curves and reduce the stiffness.
4. In most cases triaxial tests are used to develop stress-strain properties for analyses whereas many field conditions are closer to plane strain. Because triaxial stress-strain curves are not as steep as plane strain stress-strain curves, the tendency is to underestimate the stiffness of the soil by using triaxial test data.
5. In cases where dams are constructed in V-shaped valleys with steep valley walls, 2D analyses can significantly overestimate movements because they ignore the effects of cross-valley arching. In the case of a dam in a valley with 1:1 valley wall slopes, the difference is on the order of 40% (Lefebvre et al. 1973).

Because they model nonlinear stress-strain behavior, the analyses summarized in Table 5 offer some potential for studying the development of zones of local failure within and around slopes and embankments, and for examining behavior after local failure has begun. The ability of these analyses to model postfailure behavior is limited, however, because the properties assigned to elements that have failed are more representative of the properties of air or water than of soil after failure. Once a stage is reached where an important aspect of the behavior is controlled by the properties assigned to elements that have already failed, the results usually become erratic, unrealistic, and unreliable. Numerical problems may also occur.

It is possible to infer likely crack locations based on zones of tension computed in the analyses. A successful example is described in the paper by Eisenstein and Simmons (1975). To study the possibility of transverse cracking in embankment dams, it is necessary to perform 3D analyses, or plane strain analyses of the longitudinal section (Lefebvre et al. 1973).

Hyperbolic Elastic

Hyperbolic stress-strain relationships have a great deal in common with the methods discussed in the previous section. They use the generalized Hooke's Law, and thus relate strain increments to stress increments. The main difference from the methods covered in the previous section is that the hyperbolic model affords a systematic method for relating modulus values to stresses, whereas the methods in the previous section are more ad hoc, and usually less convenient. The parameters in the hyperbolic model that relate modulus values to stresses have physical meaning, and their values can be determined by means of conventional laboratory tests. Primarily because of their convenience and practicality, they have been used quite widely. Examples of their use are summarized in Table 6. All of the applications listed in Table 6 use the hyperbolic modulus relationship developed by Duncan and Chang (1970), but they use four different methods of characterizing Poisson's ratio or bulk modulus.

TABLE 5. Finite Element Analyses of Slopes and Embankments Performed Using Multilinear Elastic Stress-Strain Relationships

Authors (1)	Type of slope (2)	Purpose of analysis (3)
Clough and Woodward (1967)	Otter Brook Dam	Compared movements calculated after the fact with measured movements, found reasonable agreement
Duncan and Dunlop (1970)	Excavated slopes	Studied effect of K_0 and strength profile on development of failure zones
Boughton (1970)	Wilmot Dam	Compared settlements of concrete-faced rockfill dam calculated after the fact with measured settlements, found reasonable agreement
Alberro (1972)	El Infiernillo Dam	Compared movements calculated after the fact with measured movements, discussed causes of differences
Eisenstein et al. (1972)	Duncan Dam	Compared zones of tension calculated after the fact with observed cracks, found good agreement
Raymond (1972)	Embankment loads on saturated clay foundations	Compared pore pressures and movements calculated after the fact with measured values, discussed limitations of analyses
Skermer (1973)	El Infiernillo Dam	Compared movements calculated after the fact with measured movements, found good agreement in the core, not so good in the shell
Eisenstein and Simmons (1975), also Eisenstein and Law (1977)	Mica Dam	Compared movements calculated after the fact using 2D and 3D analyses with measured movements, found reasonable agreement, found refinements in stress-strain were more important than 3D effects
Shibata et al. (1976)	Test embankment on loose sand and soft silt	Compared settlements calculated after the fact with measured settlements, found reasonable agreement
Cathie and Dungar (1978)	Llyn Brianne Dam	Compared movements calculated after the fact using 2D and 3D analyses with measured movements, found nonlinear analysis overestimated movements
Adikari and Parkin (1981)	Talbingo Dam	Compared linear and nonlinear analyses, compared movements calculated after the fact with measured movements, found reasonable agreement
Dolezalova and Leitner (1981)	Dalesice Dam	Compared movements calculated with softening due to wetting, after the fact, with measured movements, found reasonable agreement
Rossi and Medeiros (1985)	Foz do Areia Dam	Compared settlements of concrete-faced rockfill dam, calculated after the fact with measured settlements, found reasonable agreement
Viega Pinto and Marahna Das Neves (1985)	Beliche Dam	Compared movements calculated after the fact with softening due to wetting with measured movements, found reasonable agreement
Naylor et al. (1986)	Beliche Dam	Compared movements calculated by two methods, comparison with measured movements will be made in a later paper
Alonso et al. (1988)	Earth dam	Discussed two-stage analyses of flow and deformation in partly saturated dam materials
Miki et al. (1988)	Reinforced test embankments	Compared strains in reinforcement calculated after the fact with measured values, found reasonable agreement
Taki et al. (1988)	Reinforced and unreinforced test embankments	Compared movements calculated after the fact with measured movements, found reasonable agreement
Naylor et al. (1989)	Earth dam	Discussed analyses of movements due to wetting, method is applicable to other stress-strain relationships as well

1. The simplest procedure is to use a constant value of Poisson's ratio, ν. One of the problems with using a constant value of ν is that it is difficult to select a value of ν in a logical way, because logic and test results indicate that the value of ν depends on the stress conditions to which the soil is subjected. Another problem with using a constant value of ν is that, when a soil element fails, and is assigned a small modulus value to represent the fact that it has reached the top of the stress-strain curve, its properties are much like the properties of air—it can undergo large strains of any type without significant change in stress. This is not a good representation of the properties of real soils after failure. Even after failure real soils remain resistant to all types of changes in stress except the type by which they reached failure.
2. To model the measured volumetric strains in triaxial tests more accurately, Kulhawy and Duncan (1972) developed a procedure for relating the value of ν to the confining pressure and the deviator stress. Using this procedure affords a logical means of deriving the parameters used in the analyses from test results. However, it has been found in some cases that this formulation leads to erratic or unstable results (e.g. Cathie and Dungar 1978).
3. A procedure that involves relating the value of bulk modulus to the confining pressure was developed by Duncan at al. (1980). However this formulation subsequently was found to have problems under some conditions also (e.g. Boscardin et al. 1990).
4. An alternative procedure which involves relating the value of bulk modulus to the mean normal stress, was developed by Boscardin et al. (1990). They found that this formulation avoids many of the problems involved in other formulations.

Most of the cases summarized in Table 6 involve analyses that were done after the fact—they were type C1 predictions according to Lambe's classification. In the case of the LG4 Main Dam, analyzed by Pare et al. (1984), the analyses were performed before the dam was constructed (a type A prediction, according to Lambe's classification), and the comparison with measured behavior was done and published later by Verma et al. (1985). This particular case thus deserves special attention. It is interesting to note that the measured movements were somewhat smaller than those that had been predicted. The difference between the calculated and measured movements was attributed to Verma et al. (1985) to the fact that the compaction achieved when the dam was constructed was better than was anticipated when the analyses were done. Another type A prediction using hyperbolic stress-strain relationships—the analysis of New Melones Dam—is discussed in a subsequent section of this paper.

Elastoplastic and Elastoviscoplastic

Cases analyzed using elastoplastic and elastoviscoplastic stress-strain relationships are summarized in Table 7. These types of stress-strain relationships are more complex than those discussed previously, and would be expected to be capable of modeling the behavior of real soils more closely. This seems to be most important where the conditions analyzed are

TABLE 6. Finite Element Analyses of Slopes and Embankments Performed Using Hyperbolic Elastic Stress-Strain Relationships

Authors (1)	Type of slope (2)	Purpose of analysis (3)
Chang and Duncan (1970)	Buena Vista Pumping Plant excavation	Compared movements calculated after the fact with measured movements, found good agreement
Kulhawy and Duncan (1972)	Oroville Dam	Compared movements and earth pressures on core block calculated after the fact with field measurements, found good agreement
Nobari and Duncan (1972)	Oroville Dam	Compared movements during reservoir filling with softening due to wetting calculated after the fact with field measurements, agreement good for horizontal, fair for vertical, difference attributed to creep
Palmerton (1972)	Earth dam	Compared 2D and 3D analyses
Sharma et al. (1975, 1976, 1977, 1979)	Tehri Dam	Compared calculated behavior for sections with vertical and inclined cores, found interface elements improved behavior, no field measurements
Quigley et al. (1976)	New Melones Dam	Calculated movements before the fact using 2D and 3D analyses with hyperbolic model, Comparisons with measured movements are shown in the present paper, see Chang and Duncan (1977) in Table 7.
Stille et al. (1976)	Embankment loads on soft clay	Compared movements calculated after the fact with measured movements, found good agreement when Poisson's ratio was equal to 0.4
Tanaka and Nakano (1976)	Miyama Dam	Compared movements calculated after the fact with measured movements, found good agreement
Cavounidis and Hoeg (1977)	Consolidation of core in zoned dam	Compared results of linear and hyperbolic analyses, no field measurements
Cathie and Dungar (1978)	Llyn Brianne Dam	Compared movements calculated after the fact using 2D and 3D analyses with measured movements, found hyperbolic analysis overestimated movements, constant Poisson's ratio better than variable
Adikari and Parkin (1981) and (1982)	Talbingo Dam	Compared movements calculated after the fact with measured movements, found reasonable agreement
Cole and Cummins (1981)	Dartmouth Dam	Compared stresses and movements calculated after the fact with measured values, found good agreement for stresses, poor for movements
Adikari et al. (1982)	Dartmouth Dam	Compared movements calculated after the fact with measured movements, found reasonable agreement
Yuan and Mao (1982)	Dam in China	Compared analyses performed using secant and tangent modulus versions of the hyperbolic model
Li and Desai (1983)	Oroville Dam	Compared movements calculated after the fact, with measured movements, found reasonable agreement
Pare et al. (1984)	LG4 Main Dam	Examined effects of abutment steepness and axis curvature on dam behavior, in analyses done before the fact, see Verma et al. (1985) for comparison with measured values
Seco E Pinto and Marahna Das Neves (1985a)	Alvito Dam	Compared movements calculated after the fact with measured movements, found reasonable agreement
Verma et al. (1985)	LG4 Main Dam	Found measured settlements smaller than calculated, attributed to better than expected compaction
Lianshi and Jinsheng (1988)	Concrete core rockfill dam	Soil parameters were adjusted to achieve agreement between calculated and measured movements
Shen and Zhang (1988)	Lubuge Dam	Compared hyperbolic, modified Cam clay, other elastoplastic stress-strain relationship, no field measurements
Khalid et al. (1990)	Cethana Dam	Compared movements and stresses in concrete-faced rockfill dam calculated after the fact with measured values, found reasonable agreement for face movements, calculated stresses higher than measured, attributed to creep

close to failure. The greatest differences between elastic and plastic behavior occur at high stress levels. At high stress levels, strains that result from an increment of stress are strongly affected by the stresses existing in the soil before the new stress increment is applied. Plasticity relationships, which model this aspect of behavior, afford more realistic approximations of the behavior of real soils than do elastic stress-strain relationships. Note that Kohgo and Yamashita (1988), for example, found that the results of their elastoplastic finite-element analyses agreed well with the results of conventional equilibrium analyses of slope stability.

Where the conditions are not so close to failure, however, there may be little advantage in using elastoplastic analyses rather than the multilinear or hyperbolic elastic analyses. This is because, at low stress levels, the strains that result from an increment of stress are represented fairly accurately by elastic stress-strain relationships. The differences between calculated and measured movements for stable slopes and embankments computed using elastoplastic stress-strain relationships (Table 7) do not appear to be significantly better than found in analyses that were performed using nonlinear elastic stress-strain relationships (Tables 5 and 6).

Elastoplastic stress-strain relationships are useful in cases where undrained conditions are analyzed in terms of effective stresses, and the accuracy of the analyses depends on reasonable predictions of the changes in pore pressures caused by changes in total stress. Elastic stress-strain relationships are not very useful for such analyses because the change in pore pressure in a saturated elastic material is always equal to the change in mean normal total stress (Chang and Duncan 1977).

Example—Analysis of New Melones Dam

The finite-element analyses described here were performed before New Melones dam was built (Quigley et al. 1976). The purpose of the analyses was to predict the stresses, strains, and displacements in the dam so that these results could be used in laying out an instrumentation program, and to establish estimates of performance that could be compared with field measurements during and after construction, as a means of determining if the behavior of the dam was within what would be considered the normal range. The analyses were completed in December 1976 (Quigley et al. 1976), and construction of the embankment was completed in November 1978. The analysis was therefore a class A prediction according to Lambe's classification.

TABLE 7. Finite Element Analyses of Slopes and Embankments Performed Using Elasto-Plastic and Elasto-Visco-Plastic Stress-Strain Relationships

Authors (1)	Type of slope (2)	Purpose of analysis (3)
Wroth and Simpson (1972)	Embankment on soft clay	Compared movements and pore pressures calculated after the fact with measured values, found better agreement for movements than for pore pressures
Smith and Hobbs (1974)	Hypothetical slopes and embankments	Simulated centrifuge slopes behaved differently from simulated field slopes due to drainage and tensile failure
Smith and Hobbs (1976)	Highway embankments	Compared results of elastoviscoplastic consolidation analyses to field measurements, found good agreements for settlement, mixed results for pore pressures
Snitbhan and Chen (1976)	Vertical slope	Compared results with limit analyses, found good agreement
Tanaka and Nakano (1976)	Miyama Dam	Compared movements calculated after the fact with measured values, found good agreement
Chang and Duncan (1977)	New Melones Dam	Developed modified Cam clay model, applicable to compacted clay and rockfill, calculated movements and pore pressures before the fact
Cathie and Dungar (1978)	Llyn Brianne Dam	Compared movements calculated after the fact using 2D and 3D analyses with measured movements, found elastoplastic analysis not better than linear
Sekiguchi and Shibata (1979)	Embankment loads on soft clay	Used elastoviscoplastic consolidation analyses to study various possible indicators of impending failure
Thamm (1979)	Embankment loads on soft clay	Compared results of elastoviscoplastic consolidation analyses to field measurements, found reasonable agreement
Zangl (1979)	Hypothetical asphalt core dam	Found results reasonable as compared with qualitative observations of behavior
Cavounidis and Vaziri (1982)	Zoned dam	Studied load transfer effects between core and shell, found different results using elastoplastic and elastic
Rowe (1984)	Reinforced embankments	Compared settlements calculated after the fact with measured movements, found calculated settlements a little larger than measured
Seco E Pinto and Marahna Das Neves (1985)	Alvito Dam	Compared movements calculated after the fact using modified Cam clay model, with measured movements, found reasonable agreement for settlements, horizontal movements not as good as hyperbolic
Duncan et al. (1987)	Mohicanville Dike No. 2	Compared movements and stresses in steel reinforcement calculated after the fact using modified Cam clay model with measured values, found good agreement
Kohgo and Yamashita (1988)	Homogeneous embankment	Compared elastoplastic analyses to limit equilibrium stability analyses, found good agreement
Mylleville and Rowe (1988), and Rowe and Myllevile (1988)	Reinforced embankments on soft clay	Investigated modes of failure and discussed use of partial safety factors
Borja et al. (1990)	I-95 embankment	Compared movements calculated after the fact, with measured values, found good agreement. Model includes viscous behavior of the soil

FIG. 1. New Melones Dam during Construction

The analyses discussed in the following paragraphs were performed using hyperbolic stress-strain relationships. The results of the analyses, and comparisons of the actual conditions with those anticipated before construction of the dam, illustrate a number of important issues involved in the use of finite-element analyses for practical engineering purposes.

New Melones Dam is located on the Stanislaus River, about 64 km (40 mi) east of Stockton, Calif. With a height of 190.5 m (625 ft), it is the highest rockfill dam constructed by the U.S. Army Corps of Engineers. It has compacted rockfill shells and an impervious clay core. It has a crest length of about 500 m (1,600 ft) and the total volume of the embankment is about 12,250,000 m^3 (16,000,000 cu yd). A photo of the dam is shown in Fig. 1. A plan view of the dam, and a longitudinal section are shown in Fig. 2. The axis of the dam is curved on

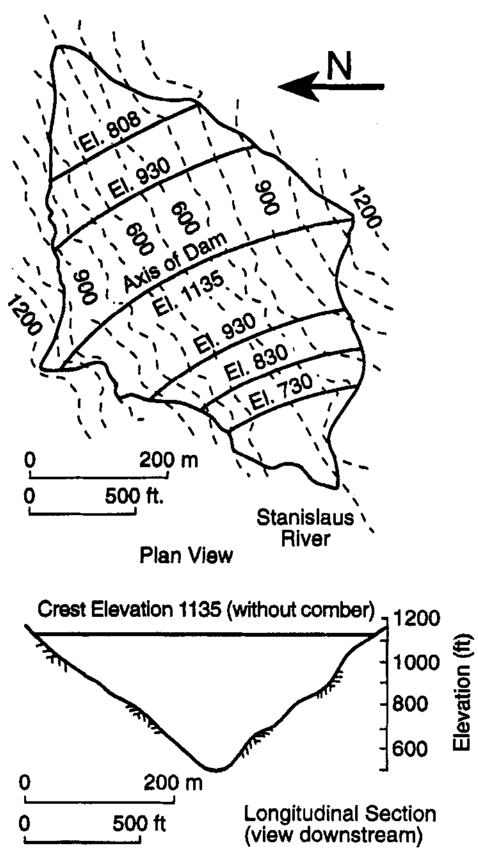

FIG. 2. Plan View and Longitudinal Section of New Melones Dam and Abutments (Elevations in Feet; 1 ft = 0.3048 m)

a 610 m (2,000 ft) radius. The steep valley walls have average slopes of about 1.2 horizontal on 1.0 vertical. The abutments and the stream bottom were stripped of soil overburden, and the dam was founded on hard metavolcanic rocks.

Both 2D and 3D analyses were performed. The purpose of the 3D analyses was to determine the likely effects of abutment geometric irregularities on the potential for transverse cracking in the dam. The mesh shown in Fig. 3 is the one used in the 2D analyses, and it is also the centerline section through the 3D mesh, which represented half of the nearly symmetrical dam. Fig. 4 shows a horizontal section through the 3D mesh. Laying out and numbering the 3D mesh was a large undertaking, involving several weeks of effort. When the analyses were completed, it was concluded that the results of the 2D analyses could have been adjusted to provide reasonable estimates of the 3D behavior, even though the valley wall slopes were very steep, and somewhat irregular. Previous studies by Lefebvre and Duncan (1971) and by Simmons (1974) could have been used as the basis for the adjustments. The results reported here are from the 3D analysis, which provided the most realistic representation of the actual geometry of the dam.

It can be noted in Fig. 3 that the dam has two cores, one within the upstream shell, and one at the center of the dam. This is because the dam-within-the-dam that is visible in the upstream part of the mesh was constructed first, and served as a cofferdam during construction of the main embankment. This sequence of construction was modeled in the finite-element analyses. The actual construction sequence was close to the one used in the finite-element analyses, except that the contractor was able to place core material faster than expected, and constructed a portion of the main dam core at the same time as the cofferdam. No adjustments to the finite-element analysis results have been made as a result of this difference.

A range of material properties was used in the analyses, to bracket the expected possible range of behavior. The stiffest properties corresponded to what was called the "dry shell–stiff core" condition, and the least stiff properties corresponded to what was called the "wet shell–soft core" condition. The dry shell–stiff core condition was considered to represent the likely lower bound for movements, and the wet shell–soft core condition was considered to represent the probable upper bound, consistent with the specifications for fill placement in the dam.

An indication of the range of results from these analyses is shown by the values of maximum calculated settlement in the core:

Dry shell–stiff core, maximum settlement = 2.2 m (7.2 ft)
Wet shell–soft core, maximum settlement = 3.4 m (11.2 ft)

The wet shell–soft core value is about 55% larger than the dry shell–stiff core value. It was expected that the maximum settlement in the dam would fall somewhere in the range defined by these values. It is interesting to note that for the steep valley wall slopes at New Melones Dam, 2D analyses of the maximum section resulted in settlements that exceeded those described previously by about 40%.

The measured movements in the dam proved to be close to the values calculated in the dry shell–stiff core analyses. End-of-construction settlements measured at six inclinometers on the axis of the dam are compared to the results of the dry shell–stiff core analyses in Fig. 5. The variations with depth of both the calculated and measured settlements are typical for an embankment of an unyielding rock foundation: Settlements are zero at the bottom of the embankment because there is no compressible material below. Settlements are zero at the top as well, because there is no overlying material that would increase stresses and cause compression in the underlying fill. The largest settlements occur near midheight, where the thickness of compressible fill beneath, and the magnitude of the load from fill above, combine to produce the maximum settlement. The settlement at midheight [about 2.1 m (7 ft)] is approximately one percent of the height 190 m (625 ft).

It can be seen that the calculated and measured values are in close agreement. At the slope indicator closest to the centerline, the measured (and also the calculated) settlements correspond to the time when the top of the fill had reached elevation 305 m (1,000 ft), rather than the end of construction. After this stage, it was no longer possible to measure settle-

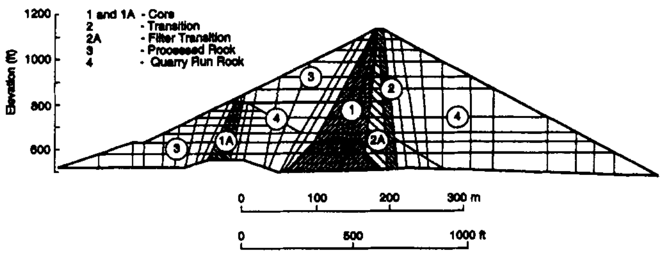

FIG. 3. Finite-Element Mesh for New Melones Dam (Also Centerline Section through Three-Dimensional Mesh) (1 ft = 0.3048 m)

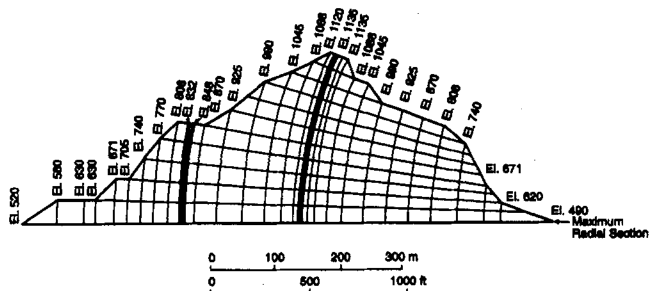

FIG. 4. Plan View of Three-Dimensional Mesh for New Melones Dam (Elevations in Feet; 1 ft = 0.3048 m)

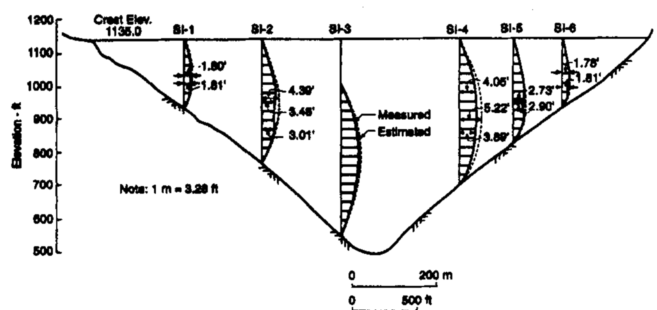

FIG. 5. Comparison of Predicted and Measured Settlements during Construction of New Melones Dam (1 ft = 0.3048 m)

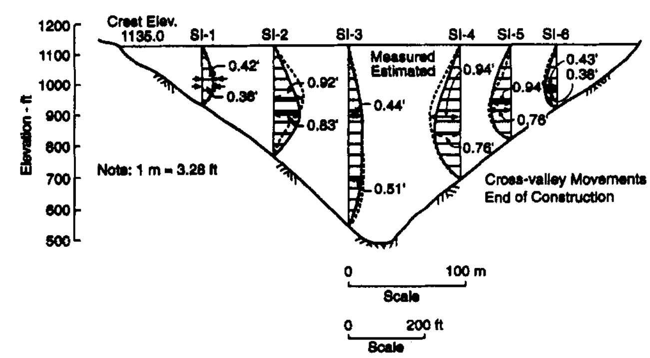

FIG. 6. Comparison of Predicted and Measured Cross-Valley Movements during Construction for New Melones Dam (1 ft = 0.3048 m)

ments at this slope indicator. By the time this stage was reached, the strains in the bottom of the core were so large that the joints in the slope indicator casing had closed, and it was no longer possible to locate them with the fishhook tool used for that purpose.

The cross-valley movements measured at the same slope indicators at the end of construction are shown in Fig. 6, together with the values calculated in the dry shell–stiff core analyses. It can be seen that there is good agreement between the calculated and the measured values at most locations.

Upstream-downstream movements at the slope indicator locations are compared to the measured values at the end of construction in Fig. 7. It may be noted that there is a bow-shaped deviation between the calculated and the measured movements about 30 m (100 ft) below the crest of the dam. This difference is believed to be due to a construction detail. For a time when the top of the embankment was in this range, the fill was consistently higher on the upstream side of the slope indicator casing. This bias in the elevation of the fill had the effect of crowding the slope indicator toward the downstream direction during this period.

Measured and calculated movements at the locations of Radiosonde tubes in the upstream and the downstream shell are shown in Fig. 8. While the predicted movements are in reasonable agreement with those measured, it can be seen that the calculated settlements are significantly larger than those

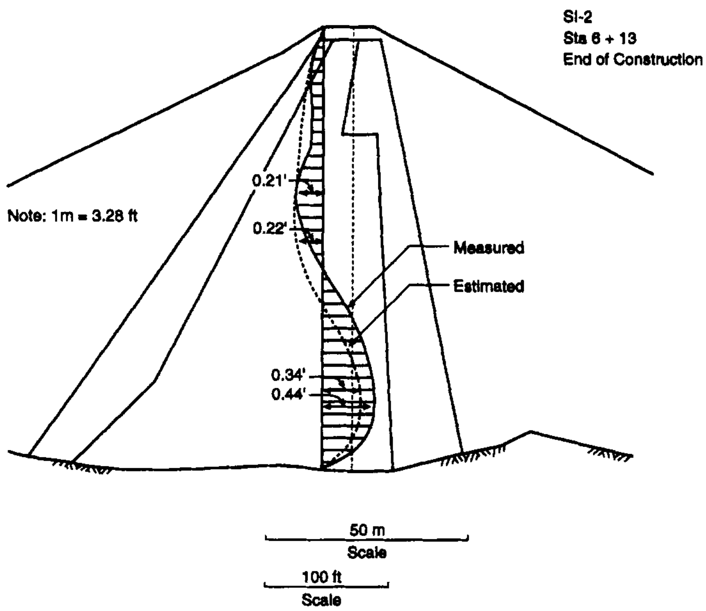

FIG. 7. Comparison of Predicted and Measured Upstream-Downstream Movements during Construction for New Melones Dam (1 ft = 0.3048 m)

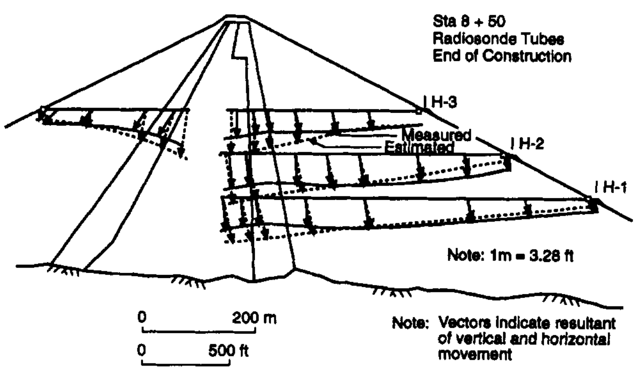

FIG. 8. Comparison of Predicted and Measured Movements at Radiosonde Tube Locations during Construction for New Melones Dam (1 ft = 0.3048 m)

measured in zone 2A, downstream from the core. No tests were done on this material. It was considered to have the same stiffness and strength as the core, based primarily on the similarities in its appearance and the specifications for its grain size, plasticity, and compaction specifications. In retrospect, however, it appears that zone 2A was significantly stiffer than the core.

Overall, it was concluded that the finite-element analyses predicted the possible range of behavior of the dam with a useful degree of accuracy. The results of the analyses were used to good advantage in designing the instrumentation system for the dam.

A notable example of the usefulness of the finite-element analysis is the Radiosonde tube that was placed in the upstream shell. Originally it was not planned to install this instrument, because an instrument located there would be submerged and lost when the reservoir rose. However, the finite-element analyses indicated that the greatest strains at any location in the dam would be in the upstream shell. It was therefore decided to install a Radiosonde tube in this area, even though it would have a short useful life.

Another example worth noting is the decision not to install earth pressure cells in the dam. Originally it was planned to install earth pressure cells in the transition zone downstream from zone 2A to study the degree of stress transfer from the core to the stiffer transition and shell. The earth pressure cells had already been purchased. After the analyses had been performed, when the details of the instrumentation program were being planned, the writer recommended that the Sacramento District omit the earth pressure cells from the embankment. One reason for this recommendation was the experience gained from a similar installation in Oroville Dam, located northeast of Sacramento, Calif. The earth pressure cells installed in Oroville Dam had produced no useful data—in fact the readings were not believable. Another reason for the recommendation to omit earth pressure cells from New Melones Dam was the fact that the finite-element analyses predicted both movements and stresses. It was reasoned that, if the movements were found to be within the range predicted (as they eventually were), then the stresses would have been calculated more accurately than they could be measured.

The analyses were very useful for judging the significance of the field measurements. The process of comparing the calculated and measured values was not very elaborate. Once it became clear that the measured movements were in substantial agreement with the results of the dry shell–stiff core analyses, it was felt that the behavior for the dam was good, and evaluations of the measurements from that time on consisted primarily of comparing the calculated and measured movements and noting that they continued to track each other in most areas.

Sources of Uncertainty in Finite-Element Analyses

A discussion of the techniques and uses of finite-element analyses should consider carefully the sources of uncertainty, or possible inaccuracy, in the results of these analyses. Some of these have already been discussed in previous sections of this paper. It is useful, however, to note specifically the factors that contributed to uncertainties in the particular case of New Melones Dam, because this example offers a number of lessons. Based on this experience there are a number of factors that can contribute to differences between the calculated results and the results of field measurements of behavior. These are:

1. Degree of Compaction. End-result specifications require that the dry density of the in-place material be at least equal to a specified minimum. The average dry density is always higher than this minimum. In the case of New Melones Dam

the specified minimum dry density of the core was 95% of standard Proctor. For the stiff core case it was estimated that the average degree of compaction would be 96% and for the soft core case it was estimated that the average would be 95%. The contractor found the core very easy to compact with the equipment used, and the average dry density of the core was 100% of standard Proctor.

2. Compaction Water Content. The specifications for New Melones Dam required that the water content of the core as compacted in the field be no drier than 2% dry of optimum and no wetter than 3% wet of optimum. This range in water content corresponds to a very large difference in the undrained strength and stress-strain characteristics of the core material. The stability analyses were based on the conservative approach of using strength values corresponding to an average field water content 3% wet of optimum. If the same "let's be conservative" approach had been followed in selecting deformation properties for finite-element analyses, the calculated movements would have been several times as large as the real movements. It was considered highly improbable that the average water content in the core would be near the extremes permitted by the specifications. For the stiff core case it was estimated that the average water content would be one percent dry of optimum, and for the soft core case it was estimated that the average would be equal to the optimum water content. The actual average was about one percent dry.

3. No Test Data. Even in the best-engineered earth structures, there are zones of materials on which no tests are performed. The properties of these materials must be estimated using data for scalped or similar materials, making allowances for the differences in gradation. In the case of New Melones Dam, all of the zones contained particles larger than the maximum that could be included in the triaxial test specimens. A comparison of the maximum grain sizes permitted by the specifications, and the grain sizes of the samples that were tested in the laboratory, are listed in Table 8.

Only two series of triaxial tests were performed, one on material that represented zones 1 and 2A (the cohesive, impermeable materials), and the other on material that represented zones 2, 3, 4, and 5 (the cohesionless, permeable materials). In evaluating the results of these tests to develop hyperbolic parameters for the analyses, adjustments were made for the larger amount of rock fragments in the field than in the lab samples, for the higher average degree of compaction estimated for the stiff core condition, and for the lower water content estimated for the stiff core condition. In consideration of all of these differences, the modulus number used for the stiff core condition was twice as high as the value determined from laboratory tests.

4. Construction Sequence. It was planned that the upstream part of New Melones Dam would be constructed first, so the main dam area would not be flooded during construction. However, the contractor found it possible to place fill faster than anticipated, and so constructed a portion of the main dam at the same time as the integral cofferdam, rather than later as planned. This change did not have a major effect on the nature of the deformations in the dam.

5. Simplified Approximations of Stress-Strain Behavior. As discussed earlier in this paper, the various stress-strain relationships used in finite-element analyses are simplified approximations of the real behavior of the materials they represent. The hyperbolic stress-strain relationship used in the New Melones Dam analyses approximated the behavior of the dam fill as nonlinear and stress-dependent, but as intrinsically elastic in the sense described earlier. As a result, these analyses are not capable of representing volume changes caused by changes in shear stress, and they do not reflect the influences of stress path in the same way that an elastoplastic stress-strain relationship would. In the case of New Melones Dam, a stable embankment not close to failure, these simplifications in the hyperbolic stress-strain relationships were probably of less importance than the uncertainties associated with estimates of density, water content, behavior of untested materials, and construction sequence.

6. Reliability of Field Measurements. When results of analyses are compared to field measurements, it is usually assumed that the field measurements represent the "truth" by which the analytical results can be judged. It should be kept in mind, however, that there are also uncertainties associated with field measurements. In the case of New Melones Dam there was a period during construction when the pore pressures measured by a group of pneumatic piezometers all registered readings of about 862 kPa (125 psi). After a great deal of study over a period of weeks, and consideration and rejection of a number of highly inventive theories about the behavior of the core material, it was found that the measurements were being made using an erroneous procedure. The input air pressure was being regulated to 862 kPa (125 psi) when the measurements were made. When the inlet pressures were increased, the measured pore pressures increased to normal values.

TABLE 8. Maximum Particle Sizes Specified for New Melones Dam, and Maximum Particle Sizes Used in Laboratory Triaxial Test Specimens

Material (1)	Maximum Particle Size mm (2)	Maximum Particle Size in. (3)	Maximum Particle Size for Laboratory Triaxial Tests mm (4)	Maximum Particle Size for Laboratory Triaxial Tests in. (5)
Zone 1 (impermeable core)	76	3	25	1
Zone 2A (filter transition downstream from core)	152	6	25	1
Zone 2 (core-shell transition)	152	6	64	2.5
Zone 3 (processed rock, upstream, top half of shell)	457	18	64	2.5
Zone 4 (quarry run rock, main part of shell)	457	18	64	2.5
Zone 5 (oversize rock, optional, outer parts of shell)	457	48	64	2.5

Simplified Methods of Estimating Deformations

Due to the fact that finite-element analyses require considerable time and effort, it is useful to have available some simpler procedures for estimating embankment deformations without performing computer analyses.

Poulos et al. (1972) used linear elastic analyses to develop a simple procedure for estimating embankment deformations by means of nondimensional factors. They found that deformations estimated using the simplified procedure were in reasonable agreement with measured deformations of a rockfill embankment and with the results of detailed finite-element analyses.

Resendiz and Romo (1972) used hyperbolic analyses to incorporate nonlinear effects in simplified estimates of deformations of homogeneous embankments. The nonlinear effects are keyed to the value of the factor of safety calculated using conventional equilibrium methods of slope stability analysis. They illustrated their method through an application to Otter Brook Dam, in New Hampshire, and found good agreement.

Clements (1984) proposed a simple procedure for estimating the postconstruction settlements of rockfill dams based entirely on field measurements for 68 dams. He found that the magnitudes of the settlements varied widely from one dam to another, and that none of the semiempirical equations he investigated for estimating deformations was accurate for all of the dams. He suggested that the data he compiled could be used directly to estimate postconstruction settlements. He recommended using measurements on dams with similar characteristics as the basis for estimating postconstruction settlements in new dams.

Walker and Duncan (1984) studied lateral bulging in three dams with fills compacted wet of optimum. They used the data from these dams, results of finite-element analyses of embankments, and simple approximations for the shapes of stress-strain curves to develop a simple procedure for predicting the amount of bulging deformation in an embankment during construction. The method uses stress-strain curves for unconsolidated-undrained tests on the embankment fill, the factor of safety calculated using conventional equilibrium methods of slope stability analyses, and a simple semiempirical equation. Bulging deformations calculated using this procedure were in good agreement with the measured deformations in the three dams studied.

Conclusions Regarding Deformation Analyses

The availability of computers, and of finite-element analysis computer programs, has made it possible to perform rational analyses of stress and deformations in slopes and embankments. These analyses are capable of modeling several important aspects of the actual conditions, including nonlinear stress-strain behavior, stress-dependent stress-strain behavior, sequential changes in geometry during construction, and dissipation of excess pore pressures following construction. While finite-element analyses have great potential for modeling field conditions realistically, they require large amounts of effort and involve high costs.

Four different types of stress-strain relationships have been used in practice to analyze a large number of dams, embankments, and slopes. These analyses, and the principal conclusions drawn from them, are summarized in Tables 4, 5, 6, and 7. The stress-strain relationships used in these analyses are linear elastic, multilinear elastic, hyperbolic, and elastoplastic. Each of these relationships has its own advantages and limitations. Linear elastic stress-strain relationships have the advantage of simplicity, and the limitation that they only model the behavior of real soils well at low stress levels and small strains. Multilinear elastic stress-strain relationships have the advantage that they can be used to model any shape of stress-strain curve for ductile materials, and the limitation that they must be developed on a case-by-case basis to approximate the particular stress-strain characteristics of the soils under consideration. Hyperbolic stress-strain relationships have the advantages that they model nonlinear behavior, and that the parameters involved have physical significance and can be evaluated using the results of conventional triaxial tests. They have the limitation that they are inherently elastic, and do not model plastic deformations in a fully logical way. Elastoplastic, and elastoviscoplastic stress-strain relationships have the advantage that they model more realistically the behavior of soils close to failure, at failure, and after failure. They have the limitation that they are more complex.

Comparisons of the results of finite-element analyses with field measurements have shown that there is a tendency for calculated deformations to be larger than measured deformations. The reasons for this difference include: (1) soils in the field tend to be stiffer than soils at the same density and water content in the lab because of aging effects; (2) average field densities are higher than the specified minimum dry density, which is often used for preparing lab triaxial test specimens; (3) samples of inplace materials suffer disturbance during sampling, and are less stiff as a result; (4) many field conditions approximate plane strain, whereas triaxial tests are almost always used to evaluate stress-strain behavior and strength; and (5) two-dimensional finite-element analyses overestimate deformations of dams constructed in V-shaped valleys with steep valley walls.

By the time of this writing there has been about 30 years of experience with using the finite-element method to estimate stresses and deformations in slopes and embankments. This experience has shown the considerable potential of the method for use in engineering practice, and it has pointed up clearly the sources of uncertainty in the results of these analyses. These are related primarily to difficulties involved in being able to predict the actual densities and water contents of soils in the field, and with being able to anticipate the sequence of operations that will be followed during construction. With due allowance for the uncertainties, finite-element analyses afford a powerful method of analysis that is applicable to a wide variety of engineering problems.

ACKNOWLEDGMENTS

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