INF-SUP CONDITION FOR OPERATOR EQUATIONS

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We study the well-posedness of the operator equation

$$Tu = f. \tag{1}$$

where T is a linear and bounded operator between two linear vector spaces. We give equivalent conditions on the existence and uniqueness of the solution and apply to variational problems to obtain the so-called inf-sup condition (also known as Babuska condition). When the linear system is in the saddle point form, we derive another set of inf-sup conditions (known as Brezzi conditions).

We shall skip the subscript of the norm for different spaces. It should be clear from the context.

1. Preliminary from Function Analysis

In this section we recall some basic facts in functional analysis, notablely three theorems: Hahn-Banach theorem, Closed Range Theorem, and Open Mapping Theorem. For detailed explanation and sketch of proofs, we refer to *Chapter: Minimal Functional Analysis for Computational Mathematicians*.

1.1. **Spaces.**

A complete normed space will be called a *Banach space*. A complete inner product space will be called a *Hilbert space*. Completeness means every Cauchy sequence will have a limit and the limit is in the space. Completeness is a nice property so that we can safely take the limit.

Functional analysis is studying $\mathscr{L}(U, V)$: the linear space consisting of all linear operators between two vector spaces U and V. When U and V are topological vector spaces (TVS), the subspace $\mathscr{B}(U, V) \subset \mathscr{L}(U, V)$ consists of all continuous linear operators. An operator T is bounded if T maps bounded sets into bounded sets. When U and V are normed spaces, for a bounded operator, there exists a constant M s.t. $\|Tu\| \leq M\|u\|, \forall u \in U$. The smallest constant M is defined as the norm of T. For $T \in \mathscr{B}(U, V)$, $\|T\| = \sup_{u\in U, \|u\|=1} \|Tu\|$. With such norm, the space $\mathscr{B}(U, V)$ becomes a normed vector space.

One can easily show that, for a linear operator, T is continuous iff T is bounded. Indeed by the translation invariance and the linearity of T, it suffices to prove such result at 0 for which the proof is straightforward by definition.

An important and special example is $V = R$. The space $\mathscr{L}(U, R)$ is called the (algebraic) dual space of U and denoted by U^*. For an operator $T \in \mathscr{L}(U, V)$, it induces an operator $T^* \in \mathscr{L}(V^*, U^*)$ by $\langle T^* f, u\rangle = \langle f, Tu\rangle$, where $\langle \cdot, \cdot\rangle$ is the duality pair, and T^* is called the transpose of T. The continuous linear functional of a normed space V will be denoted by V', i.e., $V' = \mathscr{B}(V, R)$ and the continuous transpose T' is defined as T^*. The algebraic dual space only uses the linear structure while the continuous dual space needs a topology (to define the continuity). Note that since R is a Banach space, V' is a Banach

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space no matter V is or not. In general, when the target space is complete, we can define the limit of a sequence of operators and show that the space of bounded operators is also complete. Namely if U is a normed linear space and V is a Banach space, then $\mathscr{B}(U,V)$ is Banach.

Question: Why are the dual space and the dual operator so important?

(1) For an inner product space, the inner product structure is quite useful. For normed linear spaces, the duality pair $\langle\cdot,\cdot\rangle : V' \times V \to R$ can play the (partial) role of the inner product.
(2) Since U' and V' are Banach spaces, T' is "nicer" than T. Many theorems are available for continuous linear operators between Banach spaces.

1.2. **Hahn-Banach Theorem.** A subspace S of a linear space V is a subset such that itself is a linear space with the addition and the scalar product defined for V. For a normed TVS, a closed subspace means the subspace is also closed under the norm topology, i.e., for every convergent sequence, the limit also lies in the subspace.

Theorem 1.1 (Hahn-Banach Extension)**.** *Let V be a normed linear space and $S \subset V$ a subspace. For any $f \in S' = \mathscr{B}(S,R)$ it can be extended to $f \in V' = \mathscr{B}(V,R)$ with preservation of norms.*

For a continuous linear functional defined on a subspace, the natural extension by density can extend the domain of the operator to the closure of S. So we can take the closure of S and consider closed subspaces only. The following corollary says that we can find a functional to separate a point with a closed subspace.

Corollary 1.2. *Let V be a normed linear space and $S \subset V$ a closed subspace. Let $v \in V$ but $v \notin S$. Then there exists a $f \in V'$ such that $f(S) = 0$ and $f(v) = 1$ and $\|f\| = \operatorname{dist}^{-1}(v,S)$.*

The corollary is obvious in an inner product space. We can use the vector $\tilde{f} = v - \operatorname{Proj}_S v$ which is orthogonal to S and scale $\tilde{f}$ with the distance such that $f(v) = 1$. The extension of f is through the inner product. Thanks to the Hahn-Banach theorem, we can prove it without the inner product structure.

Proof. Consider the subspace $S_v = \operatorname{span}(S,v)$. For any $u \in S_v$, $u = u_s + \lambda v$ with $u_s \in S, \lambda \in R$, we define $f(u) = \lambda$ and use Hahn-Banach theorem to extend the domain of f to V. Then $f(S) = 0$ and $f(v) = 1$ and it is not hard to prove the norm of $\|f\| = 1/d$. □

Another corollary resembles the Reisz representation theorem.

Corollary 1.3. *Let V be a normed linear space. For any $v \in V$, there exists a $f \in V'$ such that $f(v) = \|v\|^2$ and $\|f\| = \|v\|$.*

Proof. For a Hilbert space, we simply chose $f_v = v$ and for a norm space, we can apply Corollary 1.2 to $S = \{0\}$ and rescale the obtained functional. □

The norm structure in Hahn-Banach theorem is not necessary. It can be relaxed to a sub-linear functional and the preservation of norm can be relaxed to an inequality.

1.3. **Closed Range Theorem.** For an operator $T : U \to V$, denoted by $R(T)$ the range of T and $N(T)$ the null space of T. For a matrix $A_{m\times n}$ treating as a linear operator from R^n to R^m, there are four fundamental subspaces $R(A), N(A^T) \subset R^m, R(A^T), N(A) \subset R^n$ and the following relation (named the fundamental theorem of linear algebra by G. Strang [2]) holds

$$R(A) \oplus^{\perp} N(A^T) = R^m, \tag{2}$$

$$R(A^T) \oplus^{\perp} N(A) = R^n. \tag{3}$$

We shall try to generalize (2)-(3) to operators $T \in \mathscr{B}(U, V)$ between normed/inner product spaces.

For $T \in \mathscr{B}(U, V)$, the *null space* $N(T) := \{u \in U, Tu = 0\}$ is a closed subspace. For a subset S in a Hilbert space H, the *orthogonal complement* $S^{\perp} := \{u \in H, (u, v) = 0, \forall v \in S\}$ is a closed subspace. For Banach spaces, we do not have the inner product structure but can use the duality pair $\langle \cdot, \cdot \rangle : V' \times V \to R$ to define an "orthogonal complement" in the dual space which is called *annihilator*. More specifically, for a subset S in a normed space V, the annihilator $S^{\circ} = \{f \in V', \langle f, v\rangle = 0, \forall v \in S\}$. Similarly for a subset $F \subset V'$, we define ${}^{\circ}F = \{v \in V, \langle f, v\rangle = 0, \forall f \in F\}$. Similar to the orthogonal complement, annihilators are closed subspaces.

For a subset (not necessarily a subspace) $S \subset V$, $S \subseteq S^{\perp\perp}$ if V is an inner product space or $S \subseteq {}^{\circ}(S^{\circ})$ if V is a normed space. *The equality holds if and only if S is a closed subspace* (which can be proved using Hahn-Banach theorem). The space $S^{\perp\perp}$ or ${}^{\circ}(S^{\circ})$ is the smallest closed subspace containing S.

The range $R(T)$ is not necessarily closed even T is continuous. As two closed subspaces, the relation $N(T') = R(T)^{\circ}$ can be easily proved by definition. But the relation $R(T) = {}^{\circ}N(T')$ may not hold since ${}^{\circ}N(T')$ is closed but $R(T)$ may not. It is easy to show $R(T) \subseteq {}^{\circ}N(T')$. The equality holds if and only $R(T)$ is closed.

Theorem 1.4 (Closed Range Theorem)**.** *Let U and V be Banach spaces and let $T \in \mathscr{B}(U, V)$. Then the following conditions are equivalent*

(1) $R(T)$ is closed in V.
(2) $R(T')$ is closed in U'.
(3) $R(T) = {}^{\circ}N(T')$
(4) $R(T') = N(T)^{\circ}$.

Closeness is a nice property. An *operator is closed* if its graph is closed in the product space. More precisely, let $T : U \to V$ be a function and the graph of T is $G(T) = \{(u, Tu) : u \in U\} \in U \times V$. Then T is closed if its graph G is closed in $U \times V$ in the product topology. The definition of closed operators only uses the topology of the product space. The operator is not necessarily linear or continuous. One can easily show a linear and continuous operator is closed. When $T \in \mathscr{L}(U, V)$ and U, V are Banach spaces, these two properties are equivalent which is known as the closed graph theorem.

The range of a closed linear operator between Banach spaces (and thus continuous) is not necessarily closed. Just compare their definitions:

- Graph is closed: if $(u_n, Tu_n) \to (u, v)$, then $v = Tu$.
- Range is closed: if $Tu_n \to v$, then there exists a u such that $v = Tu$.

The difference is: in the second line, we do not know if u_n converges or not. But in the first line, we assume such limit exists.

1.4. **Open Mapping Theorem.** The stability of the equation can be ensured by the open mapping theorem.

Theorem 1.5 (Open Mapping Theorem)**.** *For $T \in \mathscr{B}(U, V)$ and both U and V are Banach spaces. If T is onto, then T is open.*

More explanation. Need Baire Category Theorem.

2. Inf-sup Condition: Babuška Theory

The well-posedness of the operator equation $Tu = f$ consists of three questions: existence, uniqueness, and stability.

2.1. **Operator Equations.** For the uniqueness, a useful criterion to check is wether T is bounded below.

Lemma 2.1. *Let U and V be Banach spaces. For $T \in \mathscr{B}(U, V)$, the range $R(T)$ is closed and T is injective if and only if T is bounded below, i.e., there exists a positive constant c such that*

$$\|Tu\| \geq c\|u\|, \quad \text{for all } u \in U. \tag{4}$$

Proof. **Sufficient**. If $Tu = 0$, inequality (4) implies $u = 0$, i.e., T is injective. Choosing a convergent sequence $\{Tu_k\}$, by (4), we know $\{u_k\}$ is also a Cauchy sequence and thus converges to some $u \in U$. The continuity of T shows that Tu_k converges to Tu and thus $R(T)$ is closed.

Necessary. When the range $R(T)$ is closed, as a closed subspace of a Banach space, it is also Banach. As T is injective, T^{-1} is well defined on $R(T)$. Apply Open Mapping Theorem to $T : U \to R(T)$, we conclude T^{-1} is continuous. Then

$$\|u\| = \|T^{-1}(Tu)\| \leq \|T^{-1}\|\|Tu\|$$

which implies (4) with constant $c = \|T^{-1}\|^{-1}$. □

A trivial answer to the existence of the solution to (1) is: if $f \in R(T)$, then it is solvable. When is it solvable for all $f \in V$? The answer is $V = R(T)$, i.e., T is surjective. A characterization can be obtained using the dual of T.

Lemma 2.2. *Let U and V be Banach spaces and let $T \in \mathscr{B}(U, V)$. Then T is surjective if and only if T' is an injection and $R(T')$ is closed.*

Proof. **Sufficient**. By closed range theorem, $R(T)$ is also closed. Suppose $R(T) \neq V$, i.e., there exists a $v \in V$ but $v \notin R(T)$. By Hahn-Banach theorem, there exists a $f \in V'$ such that $f(R(T)) = 0$ and $f(v) = 1$. Then $T'f \in U'$ satisfies

$$\langle T'f, u\rangle = \langle f, Tu\rangle = 0, \quad \forall u \in U. \tag{5}$$

So $T'f = 0$ which implies $f = 0$ contradicts with the fact $f(v) = 1$.

Necessary. When T is surjective, i.e., $R(T) = V$ is closed. By closed range theorem, so is $R(T')$. We then show if $T'f = 0$, then $f = 0$. Indeed by (5), $\langle f, Tu\rangle = 0$. As $R(T) = V$, this equivalent to $\langle f, v\rangle = 0$ for all $v \in V$, i.e., $f = 0$. □

Combination of Lemma 2.1 and 2.2, we obtain a useful criteria for the operator T to be surjective.

Corollary 2.3. *Let U and V be Banach spaces and let $T \in \mathscr{B}(U, V)$. Then T is surjective if and only if T' is bounded below, i.e. $\|T'f\| \geq c\|f\|$ for all $f \in V'$.*

2.2. **Abstract Variational Problems.** Let

$$a(\cdot,\cdot) : U \times V \mapsto \mathbb{R}$$

be a bilinear form on two Banach spaces U and V, i.e., it is linear to each variable. It will introduce two linear operators

$$A : U \mapsto V', \quad \text{and} \quad A' : V \mapsto U'$$
$$\text{by} \quad \langle Au, v\rangle = \langle u, A'v\rangle = a(u, v).$$

We consider the operator equation: Given a $f \in V'$, find $u \in U$ such that

$$Au = f \qquad \text{in } V', \tag{6}$$

or equivalently

$$a(u, v) = \langle f, v\rangle \quad \text{for all } v \in V.$$

To begin with, we have to assume both A and A' are continuous which can be derived from the continuity of the bilinear form.

(C) The bilinear form $a(\cdot,\cdot)$ is continuous in the sense that

$$a(u, v) \leq C\|u\|\|v\|, \quad \text{for all } u \in U, v \in V.$$

The minimal constant satisfies the above inequality will be denoted by $\|a\|$. With this condition, it is easy to check that A and A' are bounded operators and $\|A\| = \|A'\| = \|a\|$. The following conditions discuss the existence and the uniqueness.

(E)

$$\inf_{v\in V} \sup_{u\in U} \frac{a(u, v)}{\|u\|\|v\|} = \alpha_E > 0.$$

(U)

$$\inf_{u\in U} \sup_{v\in V} \frac{a(u, v)}{\|u\|\|v\|} = \alpha_U > 0.$$

Theorem 2.4. *Assume the bilinear form $a(\cdot,\cdot)$ is continuous, i.e., (C) holds, the problem (6) is well-posed if and only if (E) and (U) hold. Furthermore if (E) and (U) hold, then*

$$\|A^{-1}\| = \|(A')^{-1}\| = \alpha_U^{-1} = \alpha_E^{-1} = \alpha^{-1},$$

and thus for the solution to $Au = f$

$$\|u\| \leq \frac{1}{\alpha}\|f\|.$$

Proof. We can interpret (E) as $\|A'v\| \geq \alpha_E\|v\|$ for all $v \in V$ which is equivalent to A is surjective. Similarly (U) is $\|Au\| \geq \alpha_U\|u\|$ which is equivalent to A is injective. So $A : U \to V$ is isomorphism and by open mapping theorem, A^{-1} is bounded and it is not hard to prove the norm is α_U^{-1}. Prove for A' is similar. □

Let us take the inf-sup condition (E) as an example to show how to verify it. It is easy to show (E) is equivalent to

$$\text{for any } v \in V, \text{ there exists } u \in U, \ s.t.\ a(u, v) \geq \alpha\|u\|\|v\|. \tag{7}$$

We shall present a slightly different characterization of (E). With this characterization, it is transformed to a construction of a suitable function.

Theorem 2.5. *The inf-sup condition (E) is equivalent to that for any $v \in V$, there exists $u \in U$, such that*

$$a(u,v) \geq C_1\|v\|^2, \quad \text{and } \|u\| \leq C_2\|v\|. \tag{8}$$

Proof. Obviously (8) will imply (7) with $\alpha = C_1/C_2$. We now prove (E) implies (8). For any $v \in V$, by Corollary 1.3, there exists $f \in V'$ s.t. $f(v) = \|v\|^2$ and $\|f\| = \|v\|$. Since A is onto, we can find u s.t. $Au = f$ and by open mapping theorem, we can find a u with $\|u\| \leq \alpha_E^{-1}\|f\| = \alpha_E^{-1}\|v\|$ and $a(u,v) = \langle Au, v\rangle = f(v) = \|v\|^2$. □

For a given v, the desired u satisfying (8) could dependent on v in a subtle way. A special and simple case is $u = v$ when $U = V$ which is known the coercivity. The corresponding result is known as Lax-Milgram Theorem.

Corollary 2.6 (Lax-Milgram)**.** *For a bilinear form $a(\cdot,\cdot)$ on $V \times V$, if it satisfies*

(1) Continuity: $a(u,v) \leq \beta\|u\|\|v\|$;
(2) Coercivity : $a(u,u) \geq \alpha\|u\|^2$,

then for any $f \in V'$, there exists a unique $u \in V$ such that

$$a(u,v) = \langle f, v\rangle,$$

and

$$\|u\| \leq \beta/\alpha\|f\|.$$

The simplest case is the bilinear form $a(\cdot,\cdot)$ is symmetric and positive definite on V. Then $a(\cdot,\cdot)$ defines a new inner product. Lax-Milgram theorem is simply the Riesz representation theorem.

2.3. **Conforming Discretization of Variational Problems.** We consider conforming discretizations of the variational problem

$$a(u,v) = \langle f, v\rangle \tag{9}$$

in the finite dimensional subspaces $U_h \subset U$ and $V_h \subset V$. Find $u_h \in U_h$ such that

$$a(u_h, v_h) = \langle f, v_h\rangle, \quad \text{for all } v_h \in V_h. \tag{10}$$

The existence and uniqueness of (10) is equivalent to the following discrete inf-sup conditions:

$$(\mathbf{D}) \qquad \inf_{u \in U_h} \sup_{v \in V_h} \frac{a(u_h, v_h)}{\|u_h\|\|v_h\|} = \inf_{v \in V_h} \sup_{u \in U_h} \frac{a(u_h, v_h)}{\|u_h\|\|v_h\|} = \alpha_h > 0.$$

With appropriate choice of basis, (10) has a matrix form. To be well defined, first of all the matrix should be square. Second the matrix should be full rank (non singular). For a squared matrix, two inf-sup conditions are merged into one. To be uniformly stable, the constant α_h should be uniformly bounded below.

An abstract error analysis can be established using inf-sup conditions. The key property for the conforming discretization is the following Garlerkin orthogonality

$$a(u - u_h, v_h) = 0, \quad \text{for all } v_h \in V_h.$$

Theorem 2.7. *If the bilinear form $a(\cdot,\cdot)$ satisfies (C), (E), (U) and (D), then there exists a unique solution $u \in U$ to (9) and a unique solution $u_h \in U_h$ to (10). Furthermore*

$$\|u - u_h\| \leq \frac{\|a\|}{\alpha_h} \inf_{v_h \in U_h} \|u - v_h\|.$$

Proof. With those assumptions, we know for a given $f \in V'$, the corresponding solutions u and u_h are well defined. Let us define a projection operator $P_h : U \mapsto U_h$ by $P_h u = u_h$. Note that $P_h|_{U_h}$ is identity. In operator form $P_h = A_h^{-1} Q_h A$, where $Q_h : V' \to V_h'$ is the natural inclusion of dual spaces. We prove that P_h is a bounded linear operator and $\|P_h\| \leq \|a\|/\alpha_h$ as the following:

$$\begin{aligned} \|u_h\| &\leq \frac{1}{\alpha_h} \sup_{v_h \in V_h} \frac{a(u_h, v_h)}{\|v_h\|} \\ &= \frac{1}{\alpha_h} \sup_{v_h \in V_h} \frac{a(u, v_h)}{\|v_h\|} \\ &\leq \frac{1}{\alpha_h} \sup_{v \in V} \frac{a(u, v)}{\|v\|} \\ &\leq \frac{\|a\|}{\alpha_h} \|u\|. \end{aligned}$$

Then for any $w_h \in U_h$, note that $P_h w_h = w_h$,

$$\|u - u_h\| = \|(I - P)(u - w_h)\| \leq \|I - P_h\| \|u - w_h\|.$$

Since $P_h^2 = P_h$, we use the identity in [3]:

$$\|I - P_h\| = \|P_h\|,$$

to get the desired result. □

3. Inf-Sup Conditions for Saddle Point System: Brezzi Theory

3.1. **Variational problem in the mixed form.** We shall consider an abstract mixed variational problem first. Let V and P be two Banach spaces. For given $(f, g) \in V' \times P'$, find $(u, p) \in V \times P$ such that:

$$a(u, v) + b(v, p) = \langle f, v \rangle, \qquad \text{for all } v \in V, \tag{11}$$

$$b(u, q) = \langle g, q \rangle, \qquad \text{for all } q \in P. \tag{12}$$

Let us introduce linear operators

$$A : V \mapsto V', \text{ as } \langle Au, v \rangle = a(u, v)$$

and

$$B : V \mapsto P', B' : P \mapsto V', \text{ as } \langle Bv, q \rangle = \langle v, B'q \rangle = b(v, q).$$

Written in the operator form, the problem becomes

$$Au + B'p = f, \tag{13}$$

$$B\,u = g, \tag{14}$$

or in short

$$\begin{pmatrix} A & B' \\ B & 0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}. \tag{15}$$

3.2. **inf-sup conditions.** We shall study the well posedness of this abstract mixed problem.

First we assume all bilinear forms are continuous so that all operators A, B, B' are continuous.

(C) The bilinear form $a(\cdot,\cdot)$, and $b(\cdot,\cdot)$ are continuous

$$\begin{aligned} a(u,v) &\leq C\|u\|\|v\|, \quad \text{for all } u,v\in V,\\ b(v,q) &\leq C\|v\|\|q\|, \quad \text{for all } v\in V, q\in P. \end{aligned}$$

The solvable of the second equation (14) is equivalent to B is surjective or B' is injective and $R(B')$ closed which is equivalent to the following inf-sup condition **(B)**

$$\inf_{q\in P}\sup_{v\in V}\frac{b(v,q)}{\|v\|\|q\|} = \beta > 0$$

With condition (B), we have $B : V/N(B) \to P$ is an isomorphism. So given $g \in P'$, we can chose $u_1 \in V/N(B)$ such that $Bu_1 = g$ and $\|u_1\|_V \leq \beta^{-1}\|g\|_{P'}$.

After we get a unique u_1, we restrict the test function v in (11) to $N(B)$. Since $\langle v, B'q\rangle = \langle Bv, q\rangle = 0$ for $v \in N(B)$, we get the following variational form: find $u_0 \in N(B)$ such that

$$a(u_0, v) = \langle f, v\rangle - a(u_1, v), \quad \text{for all } v \in N(B). \tag{16}$$

The existence and uniqueness of u_0 is then equivalent to the two inf-sup conditions for $a(u, v)$ on space $Z = N(B)$.

(A)

$$\inf_{u\in Z}\sup_{v\in Z}\frac{a(u,v)}{\|u\|\|v\|} = \inf_{v\in Z}\sup_{u\in Z}\frac{a(u,v)}{\|u\|\|v\|} = \alpha > 0.$$

After we determine a unique $u = u_0 + u_1$ in this way, we solve

$$B'p = f - Au \tag{17}$$

to get p. Since u_0 is the solution to (16), the right hand side $f - Au \in N(B)^\circ$. Thus we require $B' : V \mapsto N(B)^\circ$ is an isomorphism which is also equivalent to the condition (B).

Theorem 3.1. *Assume the bilinear forms $a(\cdot,\cdot), b(\cdot,\cdot)$ are continuous, i.e., (C) holds. The mixed variational problem (15) is well-posed if and only if (A) and (B) hold. When (A) and (B) hold, we have the stability result*

$$\|u\|_V + \|p\|_P \lesssim \|f\|_{V'} + \|g\|_{P'}.$$

The following characterization of the inf-sup condition for the operator B is useful. The verification is again transferred to a construction of a suitable function. The proof is similar to that in Theorem 2.5 and thus skipped here.

Theorem 3.2. *The inf-sup condition (B) is equivalent to that: for any $q \in P$, there exists $v \in V$, such that*

$$b(v,q) \geq C_1\|q\|^2, \quad \textit{and} \quad \|v\| \leq C_2\|q\|. \tag{18}$$

Note that in general a construction of desirable $v = v(q)$, especially the control of norm $\|v\|$, may not be straightforward.

3.3. **Conforming Discretization.** We consider finite element approximation to the mixed problem: Find $u_h \in V_h$ and $p_h \in P_h$ such that

$$a(u_h, v_h) + b(v_h, p_h) = \langle f, v_h \rangle, \qquad \text{for all } v_h \in V_h, \tag{19}$$

$$b(u_h, q_h) = \langle g, q_h \rangle, \qquad \text{for all } q_h \in P_h. \tag{20}$$

We shall mainly consider the conforming case $V_h \subset V$ and $P_h \subset P$. We denote $B_h : V_h \to P_h'$ which can be written as $Q_h B I_h$ with natural embedding $I_h : V_h \hookrightarrow V$ and $Q_h : P' \hookrightarrow P_h'$, and denote $Z_h = N(B_h)$. Recall that $Z = N(B)$. In the application to Stokes equations $B = -\operatorname{div}$, so Z is called divergence free space and Z_h is discrete divergence free space.

Remark 3.3. In general $Z_h \not\subset Z$. Namely a discrete divergence free function may not be exactly divergence free. Just compare the meaning of $B_h u_h = 0$ in $(P_h)'$

$$\langle B_h u_h, q_h \rangle = 0, \quad \text{for all } q_h \in P_h,$$

with $Bu_h = 0$ in P'

$$\langle Bu_h, q \rangle = 0, \quad \text{for all } q \in P.$$

If we can identify $P = P'$ and $P_h = (P_h)'$ using Riesz representation theorem, then $N(B_h) \in (P_h)^\perp$ which may contains non-trivial elements in P. Namely it is possible that $Bu_h \in \ker(Q_h) \cap B(V_h)$. To enforce $Z_h \subset Z$, it suffices to have $B(V_h) \subset P_h$. Indeed when $B(V_h) \subset P_h$, $Q_h B u_h = B u_h$ and thus $B_h u_h = 0$ implies $Bu_h = 0$. $\Box$

The discrete inf-sup conditions for the finite element approximation will be
(D)

$$(\mathbf{A_h}) \qquad \inf_{u_h \in Z_h} \sup_{v_h \in Z_h} \frac{a(u_h, v_h)}{\|u_h\|_V \|v_h\|_V} = \alpha_h > 0,$$

$$(\mathbf{B_h}) \qquad \inf_{q_h \in P_h} \sup_{v_h \in V_h} \frac{b(v_h, q_h)}{\|v_h\|_V \|q_h\|_P} = \beta_h > 0.$$

Theorem 3.4. *If (A), (B), (C) and (D) hold, then the discrete problem is well-posed and*

$$\|u - u_h\|_V + \|p - p_h\|_P \le C \inf_{v_h \in V_h, q_h \in P_h} \|u - v_h\|_V + \|p - q_h\|_P.$$

Exercise 3.5. Let $U = V \times P$ and rewrite the mixed formulation using one bilinear form defined on U. Then use Babuska theory to prove the above theorem. Write explicitly how the constant C depends on the constants in all inf-sup conditions.

3.4. **Fortin operator.** Note that the inf-sup condition (B) in the continuous level implies: for any $q_h \in P_h$, there exists $v \in V$ such that $b(v, q_h) \ge \beta \|v\|_V \|q_h\|_P$ and $\|v\| \le C\|q_h\|$. For the discrete inf-sup condition, we need a $v_h \in V_h$ satisfying such property. One approach is to use the so-called Fortin operator [1] to get such a v_h from v.

Definition 3.6 (Fortin operator)**.** *A linear operator* $\Pi_h : V \to V_h$ *is called a* Fortin operator *if*

(1) $b(\Pi_h v, q_h) = b(v, q_h)$ *for all* $q_h \in P_h$
(2) $\|\Pi_h v\|_V \le C\|v\|_V$.

Theorem 3.7. *Assume the inf-sup condition (B) holds and there exists a Fortin operator* Π^h*, then the discrete inf-sup condition* (B_h) *holds.*

Proof. The inf-sup condition (B) in the continuous level implies: for any $q_h \in P_h$, there exists $v \in V$ such that $b(v, q_h) \geq \beta \|v\| \|q_h\|$ and $\|v\| \leq C\|q_h\|$. We choose $v_h = \Pi_h v$.

By the definition of Fortin operator

$$b(v_h, q_h) = b(v, q_h) \geq \beta \|v\|_V \|q_h\|_P \geq \beta C \|v_h\|_V \|q_h\|_P.$$

The discrete inf-sup condition then follows. □

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