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TECHNICAL NOTE

On the ratio of factors of safety in slope stability analyses

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Some of the analyses of slope stability in three dimensions, for c, φ soil conditions, produce three-dimensional factors of safety that, in some cases, are smaller than the corresponding two-dimensional factors. If the comparison is to be meaningful it should be between minimum factors. In this case it is proven that the three-dimensional factor is always greater than or equal to the two-dimensional factor. Furthermore the same holds, *a fortiori*, for three-dimensional factors corresponding to a surface generated to include the two-dimensional critical line. Arguments about the influence of normal forces on the slip surface in two and three dimensions do not provide valid explanations about how the three-dimensional factor of safety can be smaller than the two-dimensional factor. Instead, the reason for this error should rather be attributed to some of the simplifying assumptions that some methods employ.

INTRODUCTION

Recently increased interest has been shown in treating the stability analysis of slopes as a three-dimensional problem, as it really is. This interest is justified because it may constitute a refinement that would increase the accuracy of the analysis and consequently lead to more economic and safer design. Also, the current computational capacities provide greatly increased possibilities for otherwise tedious and time-consuming solutions.

The undrained stability of slopes in three dimensions has been examined by Baligh & Azzouz (1975) for cylindrical slides with conical or ellipsoidal ends. Recently an analytical solution for cylindrical slip surfaces with plane ends has been proposed by Gens, Hutchinson & Cavounidis (1987), some results of which have also been presented at the 11th International Conference on Soil Mechanics and Foundation Engineering (Cavounidis, 1985). In these analyses the ratio of the factor of safety in three dimensions (F_3) to the factor of safety in two dimensions (F_2) has been found to be always greater than unity but decreases when the width of the slide increases, tending to unity for infinite width.

This is not always the case in the analyses presented for c, φ soil conditions. In one of the early studies of this kind Hovland (1977) extended the ordinary method of slices to three dimensions and suggested that in certain cases the ratio F_3/F_2 can be smaller than unity. This result has been questioned by Azzouz & Baligh (1978). Likewise Chen & Chameau (1983) in what resembles a three-dimensional version of Spencer's (1967) method showed results where the ratio F_3/F_2 is smaller than unity. In discussing their results they suggested that cohesionless material may, under certain circumstances, lead to such a result. Their conclusion was strongly questioned by Hutchinson & Sarma (1985). In reply to this criticism Chen & Chameau (1985) presented an explanation based on the inclination of the normal force on the base of a column for the three-dimensional case compared with the two-dimensional case. Reference to this argument will be made later.

The ratio F_3/F_2 will be discussed here and it will be argued that it must always be greater than or equal to unity.

THE PROBLEM

Certain qualifications and clarification need to be made from the outset to avoid confusion about the meaning of certain terms and the context in which they are used.

(*a*) The arguments proposed are solely in terms of limit equilibrium analyses.

(*b*) The two-dimensional factor of safety $F_{2\min}$ is computed in the 'worst' cross-section of a slope, i.e. it is the minimum factor that can be achieved, using a certain method, not on a particular cross-section but on any cross-section of a slope to give the $F_{2\min}$ value of that slope. Given the geometric characteristics and the material properties there is only one $F_{2\min}$.

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(*c*) Any three-dimensional factor can be expressed as

$$F_3 = \frac{\int_z R\,dz + P(E)}{\int_z D\,dz} \quad (1)$$

where R and D are respectively resisting and driving moments per unit length. ($R = R(x, y)$ and $D = D(x, y)$.) $P(E)$ is the additional resistance due to side forces and end resistance and axis z is perpendicular to the direction of movement (parallel to the axis of rotation). Equation (1) can be written in discrete form as

$$F_3 = \frac{\Sigma R\,\Delta z + P(E)}{\Sigma D\,\Delta z} \quad (2)$$

(*d*) The minimum three-dimensional factor of safety $F_{3\min}$ is not uniquely defined by the slope and the material properties. In addition the side boundaries (width) and the shape of the three-dimensional slip surface (e.g. a cylinder with ellipsoidal ends) have to be given.

(*e*) The three-dimensional method for infinite slide width is virtually the same as the corresponding two-dimensional method. Both Hovland (1977) and Chen & Chameau (1983) conformed to this. Any comparison has to take this into account or to use a two-dimensional method that gives even lower factors of safety. Otherwise, a very 'poor' two-dimensional method could always be used—or even devised—which gives high factors of safety and these could be compared with the three-dimensional factors computed using a 'good' three-dimensional method giving relatively low factors.

(*f*) The critical three-dimensional surface does not, in general, contain the two-dimensional critical slip line.

(*g*) Any meaningful comparison of factors of safety is between $F_{3\min}$ and $F_{2\min}$.

The proposition to prove is that the ratio of factors of safety is greater than or equal to unity, i.e.

$$F_{3\min}/F_{2\min} \geqslant 1 \quad (3)$$

PROOF

Assume that $F_{3\min}$ has been found and that it corresponds to surface S_3. Also assume that $F_{2\min}$ corresponds to line L_2 where in general L_2 is not a line on S_3.

Let S_2 be a line on a section of S_3 (a section taken parallel to the movement, i.e. normal to the axis of rotation) such that F_2 corresponding to S_2 is the smallest of all the two-dimensional factors of safety which correspond to lines S on other sections of S_3.

This means that

$$F_2 = R_0/D_0 \leqslant R_i/D_i \quad (4)$$

where R and D are respectively the resisting and the driving moments and the index 0 indicates the section giving the line S_2 while index i indicates any other section.

From equation (4)

$$R_0/D_0 \leqslant R_1/D_1 \quad (5)$$

and, since all R and D values are positive, by simple algebra

$$\frac{R_0}{D_0} \leqslant \frac{R_0 + R_1}{D_0 + D_1} \quad (6)$$

$$\frac{R_0}{D_0} \leqslant \frac{R_0 + \Sigma R_i}{D_0 + \Sigma D_i} \quad (7)$$

but according to equation (4) $R_0/D_0 = F_2$ and

$$\frac{R_0 + \Sigma R_i}{D_0 + \Sigma D_i} \leqslant F_{3\min} \quad (8)$$

Then

$$F_2 \leqslant F_{3\min} \quad (9)$$

but F_2 corresponding to S_2 is not the minimum for the slope. Instead, as already defined, $F_{2\min}$ corresponds to line L_2 which is not a line on S_3. Consequently

$$F_{2\min} \leqslant F_2 \quad (10)$$

and thus from inequalities (9) and (10)

$$F_{2\min} \leqslant F_{3\min} \quad (11)$$

USUAL PRACTICE

Very often, the three-dimensional surface considered is that which corresponds and contains the two-dimensional slip line (see for example Chen & Chameau (1983)), i.e. initially line L_2 corresponding to $F_{2\min}$ is found and then surface L_3 is generated as a surface with the given shape and boundaries (in accordance with the method and the geometry of the problem) and containing line L_2. Then L_3 corresponds to a factor F_3.

L_3 is in general not the same as S_3 which corresponds to $F_{3\min}$. Consequently

$$F_3 \geqslant F_{3\min} \quad (12)$$

and thus

$$F_3 \geqslant F_{2\min} \quad (13)$$

in accordance with inequality (11).

The same result can be obtained directly. If index 0 refers to the critical cross-section for two-

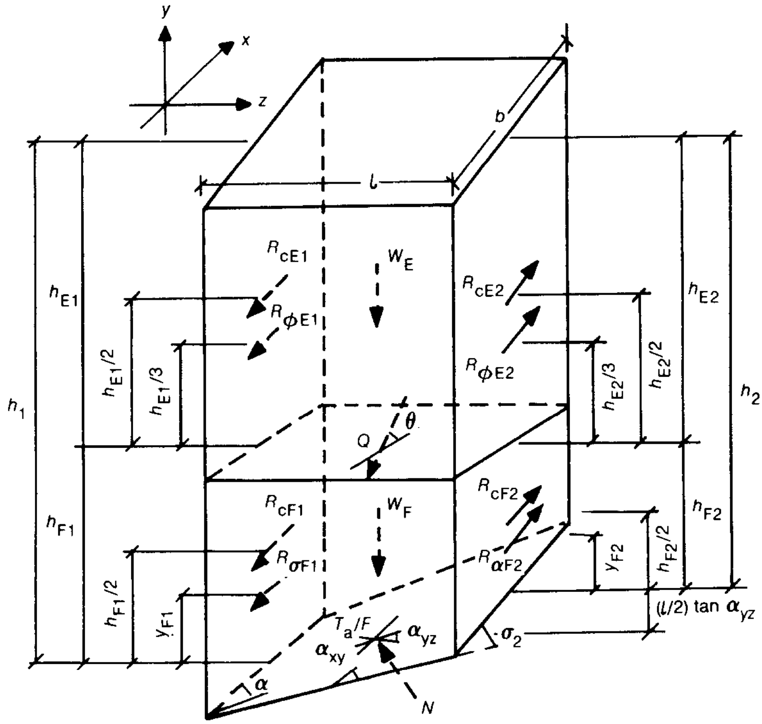

Fig. 1. Free-body diagram (with assumptions) (after Chen & Chameau (1983))

dimensional analysis and index i to any other cross-section then

$$F_{2\min} = R_0/D_0 \tag{14}$$

$$R_i/D_i \geqslant R_0/D_0 \tag{15}$$

Since all R and D values are positive, by simple algebra

$$\frac{R_1 + R_0}{D_1 + D_0} \geqslant \frac{R_0}{D_0} \tag{16}$$

$$\frac{R_0 + \Sigma R_i}{D_0 + \Sigma D_i} \geqslant \frac{R_0}{D_0} \tag{17}$$

but

$$F_3 \geqslant \frac{R_0 + \Sigma R_i}{D_0 + \Sigma D_i} \tag{18}$$

and consequently from inequalities (17) and (18) and equation (14)

$$F_3 \geqslant F_{2\min} \tag{19}$$

NORMAL FORCE

In their reply to Hutchinson & Sarma's (1985) discussion, Chen & Chameau (1985) state that the normal force N at the base of the column (Fig. 1) is

$$N_{3D} = \frac{W}{(1 + \tan^2 a_{xy} + \tan^2 a_{yz})^{1/2}} \tag{20}$$

for the three-dimensional case while for the two-dimensional case, for unit thickness

$$N_{2D} = \frac{W}{(1 + \tan^2 a_{xy})^{1/2}} \tag{21}$$

because $a_{yz} = 0$.

They then reason that 'driving moments produced by the weight of the soil columns with unit thickness and the same height are the same in both cases. However, N_{3D} is less than N_{2D} and thus the resisting moment is smaller in the three-dimensional case', i.e. this is why the three-dimensional factor of safety may sometimes be smaller than the two-dimensional factor.

This reasoning is based on an incorrect premise. N_{2D} is by definition the normal force at the base of a column (slice) on the critical cross-section for the two-dimensional analysis which corresponds to $F_{2\min}$. In any other non-critical cross-section of a slope any soil column will have a force N acting normally to its base, but such forces N are not relevant to a two-dimensional analysis, i.e. N_{2D}, as defined, does not correspond to any cross-section but only to the 'worst' cross-section: for Chen and Chameau's geometry this is any cross-section of the cylindrical part. In that part $a_{yz} = 0$ and thus $N_{3D} = N_{2D}$ numerically for unit thickness. In this case $W = W_0$. When, however, $a_{yz} \neq 0$ there is no N_{2D} as defined, because N_{2D} corresponds to the *worst* cross-section and not to any cross-section, i.e. the comparison of factors of safety should not be between the minimum three-dimensional factor and a two-dimensional factor corresponding to, for example, a cross-section of the ellipsoidal ends, which is obviously not a critical cross-section in two dimensions.

In other cross-sections i, not on the cylindrical part or, more generally, not on the critical cross-section for two-dimensional analysis, N_{3Di} may be less than N_{3D0} but at the same time W_i must be less than W_0. Chen and Chameau's statement, quoted earlier, does not hold because if it did then there would be cross-sections i ($i \neq 0$) where

$$R_i/D_i < R_0/D_0$$

but such a cross-section i then would be the critical two-dimensional cross-section and not cross-section 0, which is contrary to the initial assumption.

In the cases where $F_3 < F_2$ the error should be traced to the particular assumptions of the method that produces such a result. Simplifications concerning interslice side forces or lateral forces on the slip surface (on the base of the columns), as seems to be the case discussed in Chen & Chameau (1983), may lead to erroneous decreased values of the resisting moment when the base of a column is laterally inclined.

CONCLUSIONS

The three-dimensional factor of safety of a slope is always greater than the two-dimensional factor of the same slope. The three-dimensional factor of safety corresponding to a surface which includes the critical two-dimensional line is even greater than the two-dimensional factor since it, in general, is not even the minimum three-dimensional factor.

These conclusions were derived using simple algebra and hold in general for the minimum two- and three-dimensional factors of safety. A comparison between other than the minimum factors may produce incorrect impressions concerning the ratio of factors of safety.

Methods that give F_3/F_2 ratios which are smaller than unity either compare inappropriate factors or, more probably, contain simplifying assumptions which neglect important aspects of the problem.

Finally simplifying assumptions should be used as a (tested) simplification of a more rigorous method (see for example Bishop's (1955) modified and rigorous method) and not as a way to solve a problem without having any basis for evaluating the effects of the simplifications.

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