Limit equilibrium method

1. Introduction

In this method of analysis an arbitrary failure surface is adopted (assumed) and equilibrium is applied to the failing soil mass, assuming that the failure criterion holds everywhere along the failure surface. This surface may be planar, curved or some combination of these. Only the global equilibrium of the blocks of soil between the failure surfaces and the boundaries of the problem is considered. The internal stress distribution within the blocks of soil is not considered.

For plane strain problems analysis can be carried out in two dimensions. Global equilibrium can be investigated by considering the sum of forces in two perpendicular directions and by taking moments. There are therefore three independent equations of global equilibrium.

To completely define a force, the following three pieces of information are required:

- its magnitude;
- its direction;
- its line of action (i.e. position in space through which it acts)

The method is best explained by considering some examples, which are presented below.

2. Planar failure surface

2.1. Critical height of a vertical cut

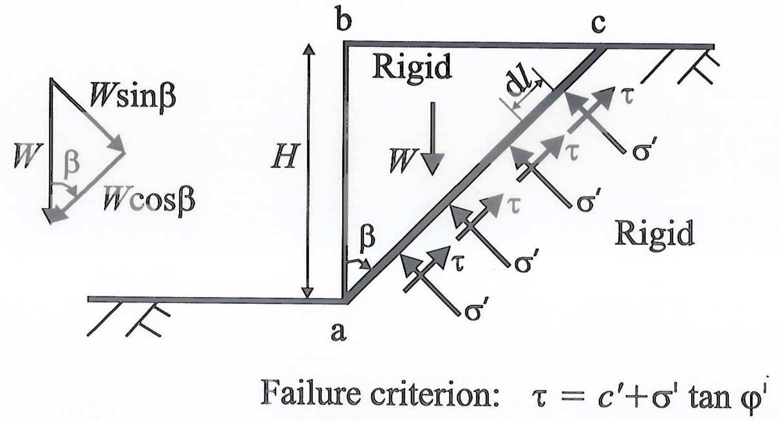

Figure 1: Failure mechanism for limit equilibrium solution

The vertical cut, presented schematically in Figure 1, is assumed to fail with the rigid block 'abc', sliding on the planar surface 'ac'. The component of the weight W acting parallel to the failure surface 'ac' acts as the *disturbing force* and it is resisted by the shear stress τ acting on 'ac'.On the other hand, the component of weight W acting perpendicular to the failure surface acts as the *stabilising force* and is supported by the direct stress σ acting on 'ac'. Coulomb failure (yield) condition is assumed to operate along 'ac'. The slope is assumed dry with no pore water pressures.

The actual distributions of σ and τ along the failure surface 'ac', presented in Figure 1, are unknown. However, if l is the length of the failure surface 'ac', then:

$$\int_0^l \tau \, \mathrm{d}l = \int_0^l c' \, \mathrm{d}l + \int_0^l \sigma' \tan\varphi' \, \mathrm{d}l = c'l + \tan\varphi' \int_0^l \sigma' \, \mathrm{d}l \tag{1}$$

where c' and ϕ' are the soil's cohesion and angle of shearing resistance respectively.

Applying equilibrium to the wedge 'abc', i.e. resolving forces normal and tangential to failure surface 'ac', gives:

$$\begin{aligned} \int_0^l \sigma' \mathrm{d}l &= W\sin\beta \\ \int_0^l \tau \, \mathrm{d}l &= W\cos\beta \end{aligned} \tag{2}$$

Noting that $W=\frac{1}{2}\gamma H^2\tan\beta$ and $l=H/\cos\beta$, Equations (1) and (2) can be combined to give:

$$H = \frac{2\,c'\cos\varphi'}{\gamma\cos(\beta+\varphi')\,\sin\beta} \tag{3}$$

The value of the angle β which produces the most conservative (lowest) value of H is obtained from $\partial H/\partial\beta=0$:

$$\frac{\partial H}{\partial \beta} = \frac{-2\,c'\cos\varphi'\cos(2\beta+\varphi')}{\gamma(\sin\beta\cos(\beta+\varphi'))^2} \tag{4}$$

Equation (4) equals zero if $\cos(2\beta+\phi')=0$. Therefore $\beta = \pi/4-\phi'/2$.

Substituting this angle into Equation (3) yields the *Limit equilibrium* value of H_{LE}:

$$H_{LE} = \frac{2\,c'\cos\varphi'}{\gamma\cos(\pi/4+\varphi'/2)\,\sin(\pi/4-\varphi'/2)} = \frac{4c'}{\gamma}\tan(\pi/4+\varphi'/2) \tag{5}$$

In terms of *total stress* (i.e. Tresca yield condition), the equation reduces to:

$$H_{LE} = \frac{4\,S_u}{\gamma} \tag{6}$$

where S_u is the undrained strength.

In Equation (2) the shear and normal stress integrals give the resultant normal, N′, and shear, S, forces acting on the failure surface 'ac'. The analysis essentially investigates the global equilibrium of these forces and the forces arising from the weight W. An instructive way of examining this force equilibrium is to construct the polygon of forces, see Figure 2. If the polygon of forces closes then the system of forces is in force equilibrium. In constructing the force polygon each line is drawn in the same direction as the corresponding force and the length is proportional to the magnitude of the force. The order of the forces in the polygon should be such that the directions of the forces are consistent, i.e. they are drawn in an anticlockwise manner. The condition shown in the top part of Figure 2 is appropriate to the vertical cut problem considered above. In the lower part of Figure 2 a horizontal force, P, is assumed to act on the vertical face of the failure wedge. There are now four forces acting and these are all included in the polygon of forces as shown. This situation is appropriate to the retaining wall problem considered in the next section. As a polygon of forces does not consider the line of action of the forces it gives no information on moment equilibrium.

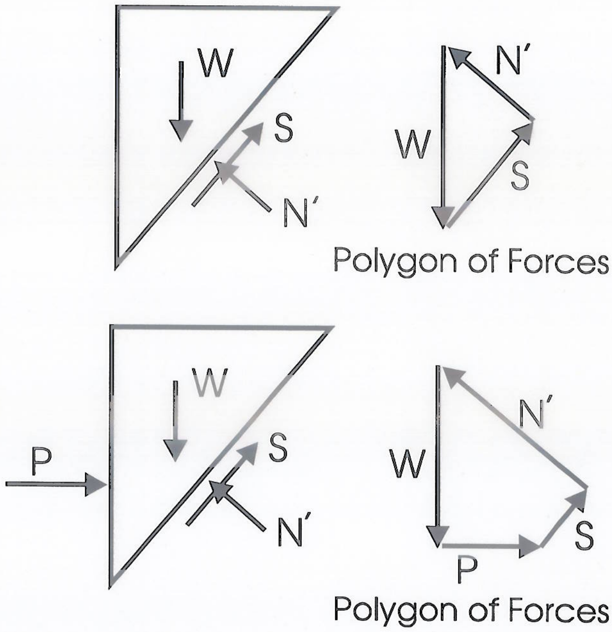

Figure 2: Polygon of forces

Only force equilibrium has been used to obtain the above solutions. For moment equilibrium to be satisfied the forces acting on the failing block must give no net moment. As the distribution of normal stress σ_n' along the slip surface 'ac' has not been considered, the point of action of N' is unknown. Application of moment equilibrium would enable this to be determined. However, it would not affect the solutions obtained above. These solutions are identical to the unsafe (upper) bound solutions obtained assuming a planar sliding surface (see later notes). The safe (lower) bound solution gives half the above value

2.2 Coulombs theory

Coulomb was concerned with determining the limiting forces acting on a retaining wall. In the situation where the soil moves away from the retained soil, see Figure 3, the force P exerted by the soil will reduce. If sufficient movement occurs, failure will be initiated in the soil and the force exerted on the wall will reach a limiting (minimum) value, P_a. In this condition the soil immediately adjacent to the wall is said to be in an *active state* and the force P_a is said to be the *active force* acting on the wall.

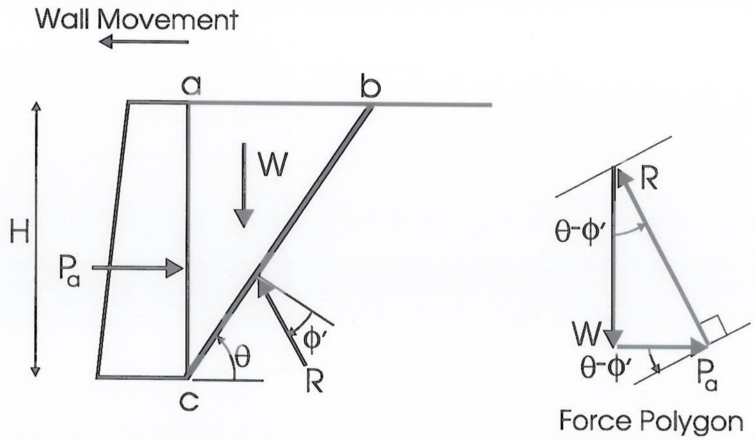

Figure 3: Coulomb active wedge

Considering the active zone behind the retaining wall the theory assumes that as the wall moves forward a wedge of soil 'abc' slides over a plane of failure 'bc'. The downward movement of the wedge is restricted by the force P_a between the soil and the wall and by the shear stresses along the failure surface. The limiting value of the force P_a is reached when the shear stress equals the shearing resistance along planes 'ac' and 'bc'.

On the failure surface 'bc' the Mohr-Coulomb failure criterion is assumed to operate with $c'=0$, i.e. $\tau=\sigma_n'.\tan\phi'$. As the failure surface is planar, the resultant force R is inclined at an angle ϕ' to the normal to the plane.

To reduce the complexity of the problem the back of the wall is assumed to be smooth such that no shear stresses are developed between the soil and the wall. As the back of the wall is also planar, the resultant force P_a acts normal to the back of the wall.

The method of Limit Equilibrium requires that the equilibrium of the wedge 'abc' is investigated.

Once the inclination of the failure surface, θ, has been set then the area of the wedge 'abc' can be calculated and the magnitude of the weight, W, and its line of action determined. As noted above the directions of R and P_a are also known. However neither the magnitude nor the line of action of R and P_a are known. There are therefore four unknown quantities. As there are only three equations of equilibrium (force equilibrium in two perpendicular directions and moment equilibrium), it is not possible to obtain a complete limit equilibrium solution and find values for all four unknowns. However it is possible to apply force equilibrium to obtain estimates of R and P_a.

Since the weight, W, is known in both magnitude and direction and the forces R and P_a are known in direction but unknown in magnitude, the system of forces can be solved by the force polygon (force equilibrium). Note that while this allows the magnitude of R and P_a to be found, it does not provide information on their lines of action.

Since the choice of θ is arbitrary the value that produces a maximum value of P_a is required.

This problem can be solved analytically by using statics (i.e. resolving forces) to find an expression for P_a as a function of θ. The resulting expression can then be differentiated with respect to θ to determine the critical value of θ.

Resolving perpendicular to the direction of R gives;

$$W\sin(\theta-\phi') = P_a\cos(\theta-\phi') \tag{7}$$

But the weight of wedge 'abc', $W = \frac{1}{2}\gamma H^2\cot(\theta)$, where γ is the bulk unit weight of the soil. Substituting in equation (7) and rearranging gives;

$$P_a = \tfrac{1}{2}\gamma H^2\cot(\theta)\tan(\theta-\phi') \tag{8}$$

The critical value of θ is that which produces a maximum value of P_a. This occurs when $dP_a/d\theta=0$. Differentiating equation 8 with respect to θ gives;

$$dP_a/d\theta = \tfrac{1}{2}\gamma H^2[\cot(\theta)(1+\tan^2(\theta-\phi')) - \tan(\theta-\phi')(1+\cot^2(\theta))] \tag{9}$$

which equals zero if

$$\cot(\theta)/(1+\cot^2(\theta)) = \tan(\theta-\phi')/(1+\tan^2(\theta-\phi')$$

Therefore:

$$\cot(\theta) = \tan(\theta-\phi')$$

Expanding this equation and rearranging gives;

$$2\cos(\theta)\sin(\theta)\sin(\phi') = \cos(\phi')[\sin^2(\theta)-\cos^2(\theta)]$$

Further rearrangement gives;

$$\tan(2\theta) = -\cot(\phi')$$

This above equation is satisfied if;

$$\theta = \tfrac{1}{4}\pi + \tfrac{1}{2}\phi' \qquad (10)$$

This gives the critical angle of the failure surface. Substituting this expression for θ into equation (8) gives the following expression for P_a:

$$P_a = \tfrac{1}{2}\gamma H^2[1-\sin(\phi')]/[1+\sin(\phi')] \qquad (11)$$

The magnitude of R could be determined by resolving parallel to the direction of R and using the above result.

In the situation where the soil moves into the retained soil, see Figure 4, the force P exerted by the soil will increase. If sufficient movement occurs failure will be initiated in the soil and the force exerted on the wall will reach a limiting (maximum) value, P_p. In this condition the soil immediately adjacent to the wall is said to be in a *passive state* and the force P_p is said to be the *passive force* acting on the wall.

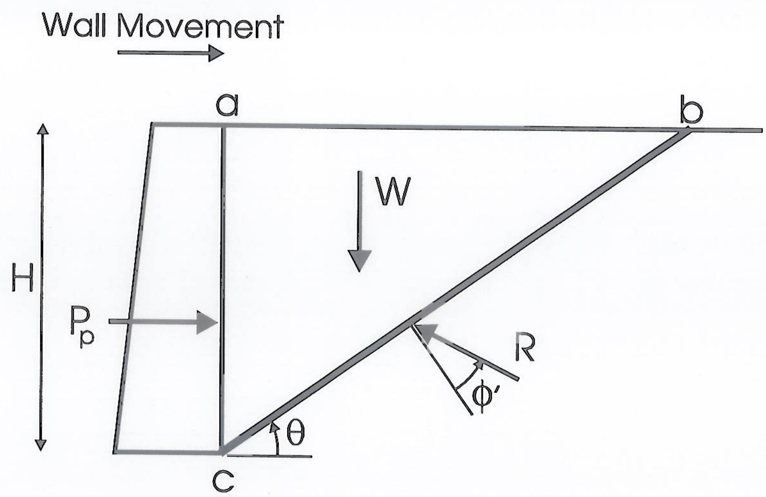

Figure 4: Coulomb passive wedge

Following a similar approach to that given above for the active case, the following expressions for the passive force can be obtained (see solutions to a tutorial sheet):

$$\theta = \tfrac{1}{4}\pi - \tfrac{1}{2}\phi' \tag{12}$$

$$P_p = \tfrac{1}{2}\gamma H^2[1+\sin(\phi')]/[1-\sin(\phi')] \tag{13}$$

As noted above the line of action of both P_a (or P_p) and R are unknown. As there is only the moment equilibrium equation unused, it is not possible to determine these values (i.e. one equation and two unknowns). This means the position on the wall at which P_a (or P_p) acts cannot be determined unless further assumptions are made.

The above analysis could be repeated with a full Mohr-Coulomb failure (yield) criterion considering both c' and ϕ' or with a Tresca criterion in which the soil is assumed to behave undrained with a strength S_u. They could also be repeated with an inclined wall and/or retained ground surface, with a water table, and/or with friction mobilised between the soil and the wall.

3. Circular failure surface

3.1. Undrained bearing capacity

Figure 5 shows a section of a surface foundation of width B and unit length into the page (i.e. plane strain condition). The foundation is loaded by a force Q and the soil is assumed to behave in an undrained manner with a Tresca failure condition. The mechanism of collapse is assumed to be a circular slip surface with centre O vertically above the edge of the foundation. The forces acting on the rotating block of soil are the footing load, Q, the normal force across the slip surface, N (not shown on the figure), the weight of soil above the slip surface, W (not shown on the figure) and the shear force arising from the shear stresses acting on the slip surface, S. The direction, line of action and magnitude of S can be determined from S_u and the geometry of the yield surface. All three properties can also be evaluated for W. The magnitude and direction of N are unknown, but as the resultant must pass through the centre of the slip surface, O, information on its line of action is known. The direction and line of action (assumed to be through the centre of the foundation) of Q are known, but its magnitude is unknown. There are therefore three unknown quantities and as we have three equations of equilibrium a complete limit equilibrium solution is possible. The magnitude of Q at failure can be determined by taking moments about O.

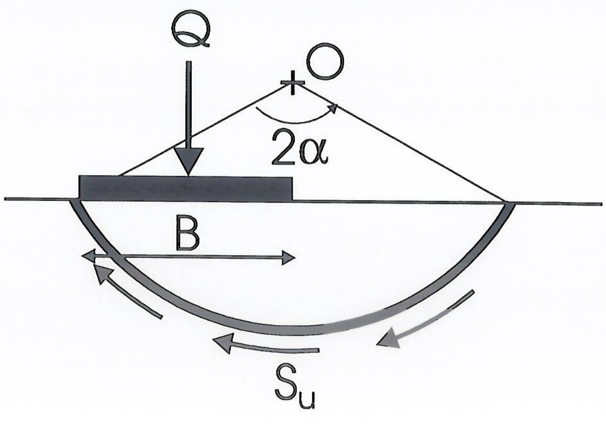

Figure 5: Circular failure surface for foundation failure

The value of Q will depend on the position of the slip surface, specified by the angle α in the Figure 5. The slip surface is required that provides the minimum value of Q. This problem forms one of the questions on tutorial sheet 2 and produces the following result:

The location of the critical slip surface occurs when α=67° and the associated value of Q is given by (see solutions to a tutorial):

$$Q = 5.52 \, B \, S_u \tag{14}$$

The other unknown quantities, the magnitude and direction of N, could be found by considering the two equations of force equilibrium. However as they have little practical significance they are usually not evaluated.

3.2 Undrained slope stability

The problem here is to investigate the stability of the slope shown in the Figure 6. The objective of the analysis is to determine either the height, H, for a fixed slope inclination β, or to determine β for a fixed height H at collapse. The forces acting on the slip surface are the weight, W, the normal force, N, across the slip surface and the shear force S arising from the shear stress mobilised around the slip surface. As for the footing problem discussed above there are three unknowns, the height of the slope, H, or its inclination, β, the direction of N and its magnitude. A full limit equilibrium solution is therefore possible.

To obtain an estimate for H (or β) the most expedient approach is to take moments about the centre of the slip circle:

Disturbing moment: $= Wx$

Resisting moment: $= S_u \, l \, R$
(where l is the arc length of the slip surface ab)

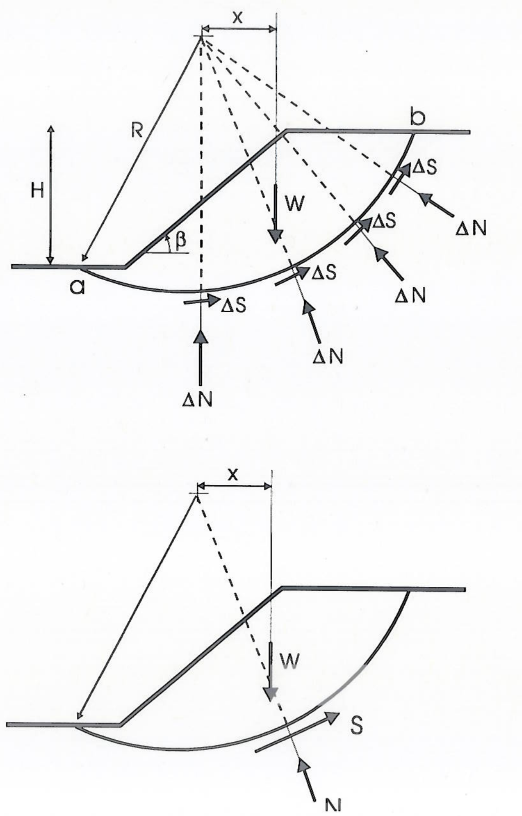

Figure 6: Circular failure surface for undrained slope stability problem

Noting that W depends on both H and β by equating the disturbing and resisting moments enables the required solution to be obtained.

A search must be made for the most critical slip surface (i.e. the one producing the minimum value of H (or β))

The analysis can be modified to account for layered soils or soils in which S_u varies spatially. It can also be adapted to account for any retained water.

For the case where S_u is constant with depth, there are no tension cracks and no water pressures acting on the slope, analysis have been performed to find the critical slip surface. The results are presented in the form of stability charts (Taylor, D.W., 1948).

3.3 Drained slope stability

The problem here is similar to that discussed above, except that the soil is assumed to be drained with a Mohr-Coulomb failure (yield) criterion.

As can be seen from Figure 7, there are now two force components, ΔC & ΔP, that act on the slip surface and resist sliding. ΔU are forces due to any pore water pressures acting on the slip surface. As with total stress analysis discussed above the ΔC forces act circumferentially at a constant radius R. They provide the following contribution to the resisting moment:

$$= R \sum \Delta C = R \sum c'dl$$

The ΔP terms are as follows:

$$\Delta P = dl.(\sigma_n - u)/\cos\phi'$$

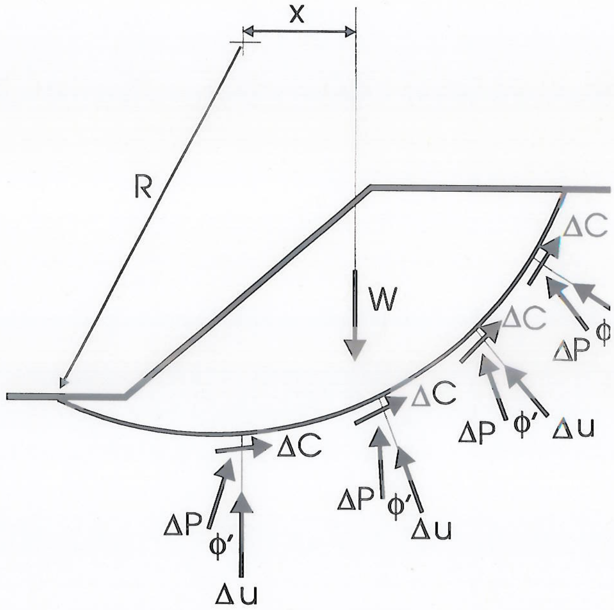

Figure 7: Circular failure surface in drained slope stability problems

Examination of the forces acting reveals that the magnitude, direction and line of action of the resultant force, P, are all unknown. (Note for the undrained case, information on the line of action of the equivalent force, N, was available). When combined with the unknown height, H (or inclination β) this results in a total of four unknowns. The problem is therefore statically indeterminate. A further approximation is therefore required if a solution is to be found. There are several different alternatives available, i.e. *friction circle method*, *log-spiral method* and *method of slices*.

This static indeterminacy is a common feature of limit equilibrium solutions adopting the Mohr- Coulomb failure criterion with non planar slip surfaces.

Method of slices

This is essentially a numerical approach in which the sliding soil mass is divided into 'n' slices, see Figure 8. For ease of calculation the sides of the slices are often assumed vertical. Equilibrium is then applied to each slice in turn.

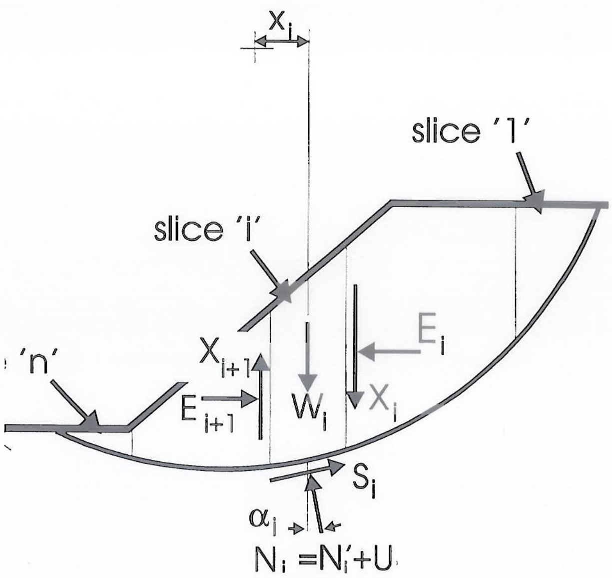

Figure 8: Method of slices in slope stability

For a particular failure surface with 'n' slices we have:

Knowns:
- Slope inclination, β (or height H)
- Bulk unit weight, γ, and strength parameters, c' and ϕ' (or S_u) for the soil
- Distribution of pore water pressures in the slope.

Unknowns:	number
- Magnitude of normal force, N (or N')	n
- Magnitude of shear force, S	n
- Magnitude of interslice force, E (orE')	n-1
- Magnitude of interslice force, X	n-1
- Point of application (i.e. line of action) of N (or N')	n
- Point of application (i.e. line of action) of E (or E')	n-1
- Slope height, H (or inclination β)	1
	$\sum = 6n-2$

Equations:	number
- Force equilibrium (2 per slice)	2n
- Moment equilibrium (1 per slice)	n
- Failure criterion, $\tau=c'+\sigma_n'\tan\phi'$ (1 per slice)	n
	$\sum = 4n$

The number of unknowns exceed the number of available equations by 2n-2. The problem is therefore *statically indeterminate*. To proceed with a limit equilibrium solution it is necessary to make 2n-2 additional independent assumptions. Clearly there are many (infinite ?) different combinations of assumptions that can be made. This has resulted in a large number of different solution methods based on slices. Some of these methods actually make too many assumptions (i.e. more than 2n-2) and consequently these methods either do not satisfy all three equations of equilibrium or have to iterate to obtain a consistent solution. This has resulted in much confusion in the literature. It should be noted that some methods are termed '*rigorous*', which implies they are theoretically correct. This is of course misleading as they are still only limit equilibrium calculations and therefore fail to satisfy the compatibility conditions. All that is actually meant by 'rigorous' is that the solutions, based on the appropriate 2n-2 assumptions, actually satisfy all three equations of equilibrium and are therefore true limit equilibrium solutions.

If the method is to be used to determine the slope height, H, for a given inclination, β, or to determine β for a given H, an iterative approach must be adopted. An initial value of H (or β) is assumed and the method applied. Inspection of the difference between the destabilising and stabilising forces indicates wether the value of H (or β) should be increased or decreased for the next iteration. The solution is obtained when the destabilising and stabilising forces are equal (or differ by only a small amount).

The accuracy of the solution depends on the number of slices. Layered soils or soils in which the strength parameters vary spatially can be dealt with. The method is clearly suited to solution by computer and there are currently many software programs available to carry out such calculations.

Example: Conventional Method of slices

Assumptions:	number
- Magnitude of interslice force, E (E=0)	n-1
- Magnitude of interslice force, X (X=0)	n-1
- Point of application (i.e. line of action) of N	n
	$\sum = 3n-2$

Consequently there are 'n' more assumptions than required. Only the moment and one force equilibrium equation are considered. Therefore force equilibrium is not fully satisfied.

For moment equilibrium assuming E=X=0 and N acts through centre of slip circle:

$\sum$Disturbing moments = $\sum$Stabilising moments

Therefore:

$$\Sigma W_i.x_i = \Sigma S_i.R = R.\Sigma(c_i'.l_i+(N_i-U_i).\tan\phi_i') \tag{15}$$

where l_i is the length of the base of slice 'i'. Resolving forces perpendicular to base of slice:

$N_i = W_i\cos\alpha_i$ (as E=X=0)

Noting that $x_i = R.\sin\alpha_i$ Equation (15) reduces to:

$$\Sigma W_i.\sin\alpha_i=\Sigma(c_i'.l_i+(W_i.\cos\alpha_i-U_i).\tan\phi_i') \tag{16}$$

Notes:
This method can be shown to be satisfactory only if the variation in α_i is not large (i.e. the central angle of the arc forming the slip circle is relatively small)

4. Non-circular Failure Surfaces

If the slip surface is curved and non circular there are always more unknowns than there are equilibrium equations. The problem is therefore statically indeterminate and further approximations are required to obtain a limit equilibrium solution. In some cases there are more than four unknowns (i.e. retaining wall situation).

4.1. Method of Slices for Slopes

For the analysis of slopes in which a non circular, but curved, failure surface is assumed, there are four unknown quantities. These are the same as those outlined above for drained stability using circular slip surfaces. However, for non circular slip surfaces there are four unknowns for both undrained analysis using the Tresca model and drained analysis using the Mohr-Coulomb model.

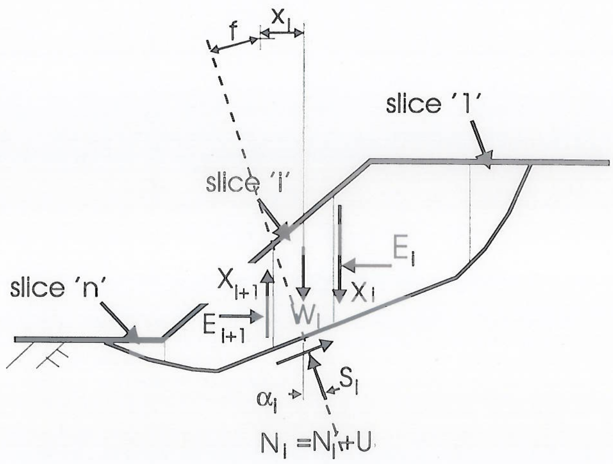

Figure 9: Method of slices for non-circular failure surface

If the slip surface is assumed to be of a general shape (non circular) as in Figure 9, then the method of slices can still be applied. However, the simple relationship:

$$x_i = R.\sin\alpha_i$$

no longer holds and the offset of the line of action of the normal force at the base of each slice, f_i must be established and account taken of it when taking moments. Clearly, this adds to the complexity of the analysis. Again 2n-2 assumptions are required for a full limit equilibrium solution to be obtained.

4.2. Multiple Wedge Analysis

An alternative approach sometimes adopted for non circular slip surfaces is to use the method of multiple wedges. Here the sliding soil mass is divided into a number of wedges as shown in the figure opposite. Force equilibrium is then applied to each wedge. Moment equilibrium is not considered.

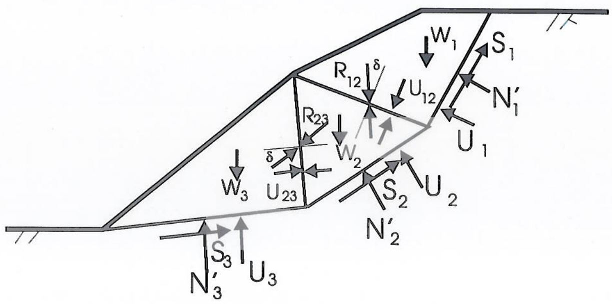

Figure 10: Method of wedges for non-circular failure surface

For a particular case with 'n' wedges we have:

Known:
- Slope inclination, β (or height H).
- Bulk unit weight, γ, and strength parameters, c' and ϕ' (or S_u) for the soil.
- Distribution of pore water pressure in the slope.

Unknown:	number
- Magnitude of forces, N'	n
- Magnitude of forces, S	n
- Magnitude of interwedge forces, R	n-1
- Direction of interwedge forces, R (i.e. δ)	n-1
- Slope height, H (or inclination β)	1

	$\sum = 4n-1$

Note:
The points of application of forces N' ('n' values) and R ('n-1' values) are also unknown. However they are not used when resolving forces to satisfy force equilibrium.

Equations:	number
- Force equilibrium (2 per wedge)	2n
- Failure criterion, $\tau = c' + \sigma'_n \tan\phi'$ (1 per wedge)	n
	$\sum = 3n$

Note: there are also 'n' equations of moment equilibrium but these are not used.

The number of unknowns exceeds the number of available equations by 'n-1'. A solution is possible if the directions of the interwedge forces, δ are specified. This provides n-1 extra assumptions and allows a solution which satisfies force equilibrium to be obtained. It should be noted that such a solution is unlikely to be in moment equilibrium. If this were to be considered, '2n-1' extra unknowns would be introduced while only 'n' extra equations would be available. Therefore an additional 'n-1' assumptions would be required.

As with the method of slices, an iterative approach has to be adopted if the method is used to either evaluate the slope height H for a given inclination β or evaluate β for a given H.

5. Concluding remarks

Although in the above discussion the friction circle, log-spiral, slices and multiple wedge methods have been explained in relation to slope stability, they can of course be used in other boundary value problems. For example, the friction circle method has been used in the past to examine retaining wall problems with curved failure surfaces.

It is clear from the above discussion that care must be exercised in choosing the shape of the failure surface to avoid static indeterminacy. However, in order to obtain realistic results it is often necessary to chose failure surfaces which lead to such indeterminacy. In such cases additional assumptions may be required in order for a solution to be possible. This may lead to failure to satisfy all the equations of equilibrium (e.g. some of the methods of slices).

Even if the solution does satisfy all the conditions of equilibrium it still does not satisfy all the requirements of a theoretical solution, as compatibility is not considered (see Table 1). The solutions are therefore only approximate.