Imperial College London

CI 250 Soil Mechanics & Engineering Geology

- Geotechnical Analysis -

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Geotechnical analysis

1. Synopsis

This lecture is an introduction to the analysis of geotechnical structures. Design requirements are presented, fundamental theoretical considerations are discussed and the various methods of analysis categorised. The main objective of the lecture is to provide a framework in which different methods of analysis may be compared. This will provide an insight into the potential advantages of numerical analysis over the more 'conventional' approaches currently in use.

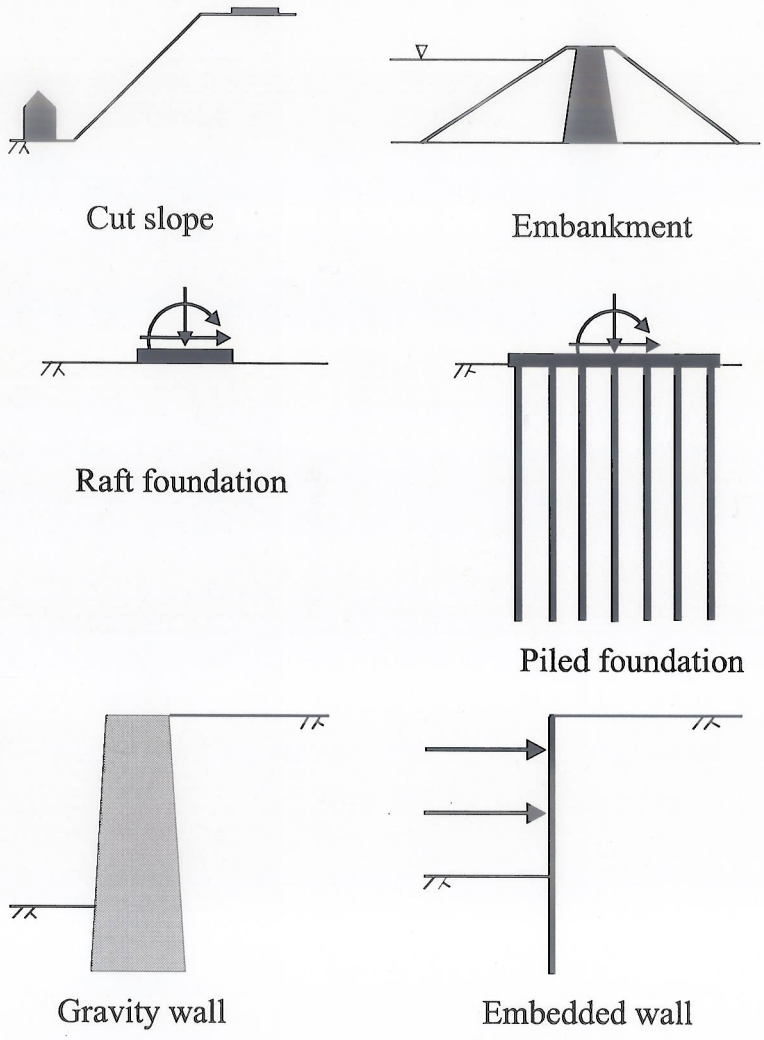

Figure 1: Examples of geotechnical structures

2. Introduction

Nearly all civil engineering structures involve the ground in some way. Cut slopes, earth and rockfill embankments, see Figure 1, are made from geological materials. The soil (or rock) provides both the destabilising and stabilising forces which maintain equilibrium of the structure. Raft and piled foundations transfer loads from buildings, bridges and offshore structures to be resisted by the ground. Retaining walls enable vertical excavations to be made. In most situations the soil provides both the activating and resisting forces, with the wall and its structural support providing a transfer mechanism. Geotechnical engineering, therefore, plays a major role in the

design of civil engineering structures.

The design engineer must assess the forces imposed in the soil and structural members, and the potential movements of both the structure and the surrounding soil. Usually these have to be determined under both working and ultimate load conditions.

Traditionally geotechnical design has been carried out using simplified analyses or empirical approaches. Most design codes or advice manuals are based on such approaches. The introduction of inexpensive, but sophisticated, computer hardware and software has resulted in considerable advances in the analysis and design of geotechnical structures. Much progress has been made in attempting to model the behaviour of geotechnical structures in service and to investigate the mechanisms of soil-structure interaction.

At present, there are many different methods of calculation available for analysing geotechnical structures. This can be very confusing to an inexperienced geotechnical engineer. This lecture introduces *geotechnical analysis*. The basic theoretical considerations are discussed and the various methods of analysis categorised. The main objectives are to describe the analysis procedures that are in current use and to provide a framework in which the different methods of analysis may be compared. Having established the place of numerical analysis in this overall framework, it is then possible to identify its potential advantages.

3. Design objectives

When designing any geotechnical structure, the engineer must ensure that it is stable. Stability can take several forms.

Firstly, the structure and support system must be stable as a whole. There must be no danger of rotational, vertical or translational failure, see Figure 2.

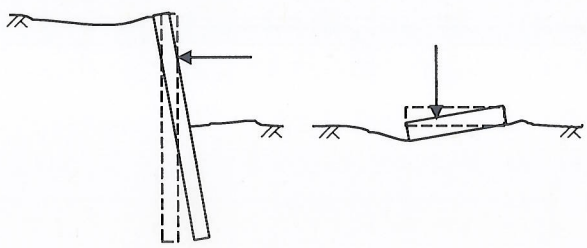

Figure 2: Local stability

Secondly, overall stability must be established. For example, if a retaining structure supports sloping ground, the possibility of the construction promoting an overall slope failure should be investigated, see Figure 3.

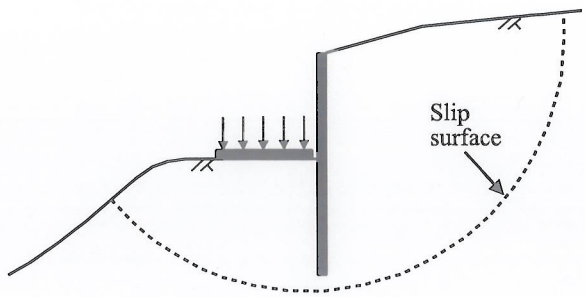

Figure 3: Overall stability

The loads on any structural elements involved in the construction must also be calculated, so that these may be designed to carry them safely, see Figure 4.

Movements must be estimated, both of the structure and of the ground. This is particularly important if there are adjacent buildings and/or sensitive services. For example, if an excavation is to be made in an urban area close to existing

services and buildings, see Figure 5, one of the key design constraints is the effect that the excavation has on the adjacent structures and services. It may be necessary to predict any structural forces induced in these existing structures and/or services.

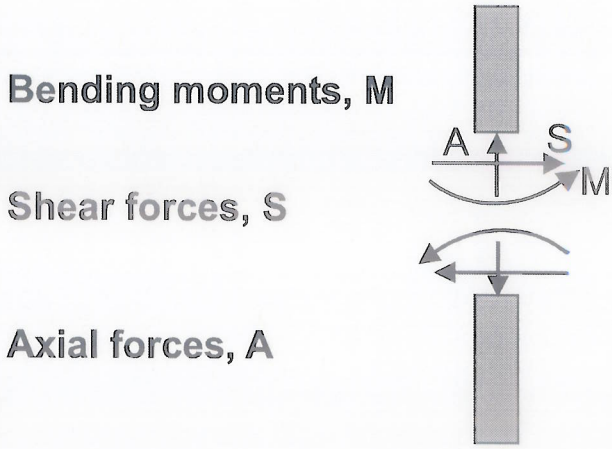

Figure 4: Structural forces

As part of the design process, it is necessary for an engineer to perform calculations to provide estimates of the above quantities. *Analysis* provides the mathematical framework for such calculations. A good analysis, which simulates real behaviour, allows the engineer to understand problems better. While an important part of the design process, analysis only provides the engineer with a tool to quantify effects once material properties and loading conditions have been set. The design process involves considerably more than *analysis*.

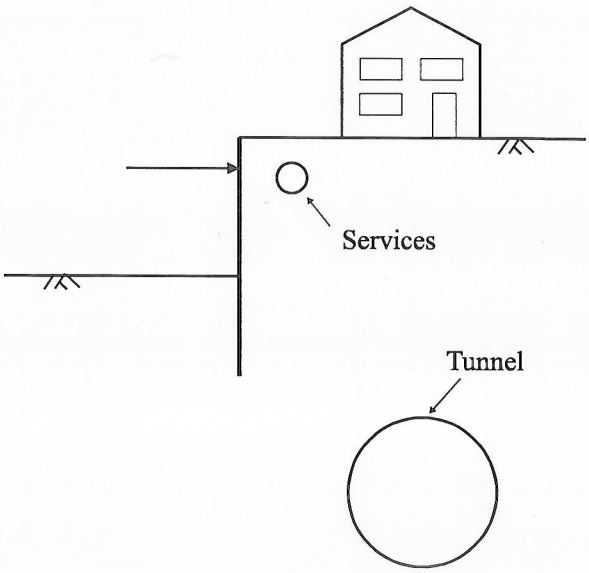

Figure 5: Interaction of structures

1.4 Design requirements

Before the design process can begin, a considerable amount of information must be assembled. The basic geometry and loading conditions must be established. These are usually defined by the nature of the engineering project.

A geotechnical site investigation is then required to establish the ground conditions. Both the soil stratigraphy and soil properties should be determined. In this respect it will be necessary to determine the strength of the soil and, if ground movements are important, to evaluate its stiffness too. The position of the ground water table and whether or not there is underdrainage or artesian conditions must also be established. The possibility of any changes to these water conditions should be investigated. For example, in many major cities around the world the ground water level is rising.

The site investigation should also establish the location of any services (gas, water, electricity, telecommunications, sewers and/or tunnels) that are in the vicinity of the proposed construction. The type (strip, raft and/or piled) and depth of the foundations of any adjacent buildings should also be determined. The allowable movements of these services and foundations should then be established.

Any restrictions on the performance of the new geotechnical structure must be identified. Such restrictions can take many different forms. For example, due to the close proximity of adjacent services and structures there may be restrictions imposed on ground movements.

Once the above information has been collected, the design constraints on the geotechnical structure can be established. These should cover the construction period and the design life of the structure. This process also implicitly identifies which types of structure are and are not appropriate. For example, when designing an excavation, if there is a restriction on the movement of the retained ground, propped or anchored embedded retaining walls are likely to be more appropriate than gravity or reinforced earth walls. The design constraints also determine the type of design analysis that needs to be undertaken.

5. Theoretical considerations

5.1. Requirements for a general solution

In general, a theoretical solution must satisfy *Equilibrium, Compatibility,* the material *Constitutive behaviour* and *Boundary conditions* (both *force* and *displacement*). Each of these conditions is considered separately below.

5.2. Equilibrium

To quantify how forces are transmitted through a continuum engineers use the concept of stress (force/unit area). The magnitude and direction of a stress and the manner in which it varies spatially indicates how the forces are transferred. However, these stresses cannot vary randomly but must obey certain rules.

Before considering the concept of stresses, an analogous example of the problem of water flowing through a tank full of sand is presented in Figure 6. The tank full of sand has one inlet and two outlets. This figure indicates vectors of water velocity at discrete points within the tank. The size of the arrows represents the magnitude of the flow velocity, while their orientation shows the direction of flow. Due to the closer proximity of the left hand outlet to the inlet, more water flows in this direction than to the right hand outlet. As would be expected, the flows are very small in regions A, B and C. Such a result could be observed by using a transparent tank and injecting dye into the flow.

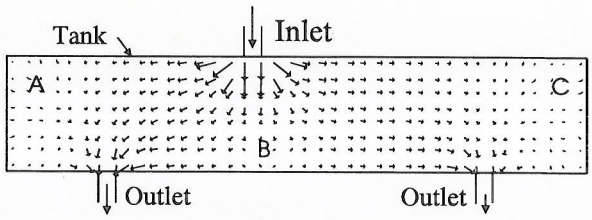

Figure 6: Flow trajectories

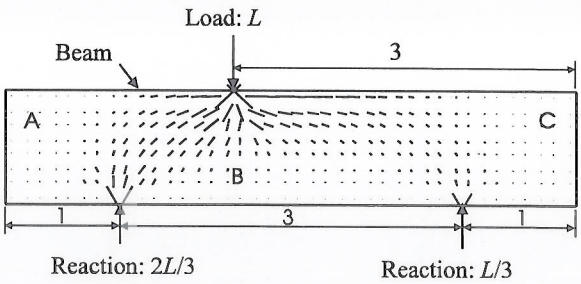

Figure 7: Stress trajectories

Similarly, a concrete beam, supported by two reactions on its lower surface and loaded by a load L on its upper surface, is presented in Figure 7. Clearly, for overall equilibrium the reactions must be $2L/3$ and $L/3$. What is not so clear, however, is how the load is transferred through the beam. It is not possible to see how the load is transmitted in this case. As noted above, engineers use the concept of stress to investigate

the load transfer. Stresses are essentially fictitious quantities. For example, the manner in which the major principal stress varies through the beam is given in Figure 7. The length of the trajectories represents the magnitude of the stress and their orientation its direction.

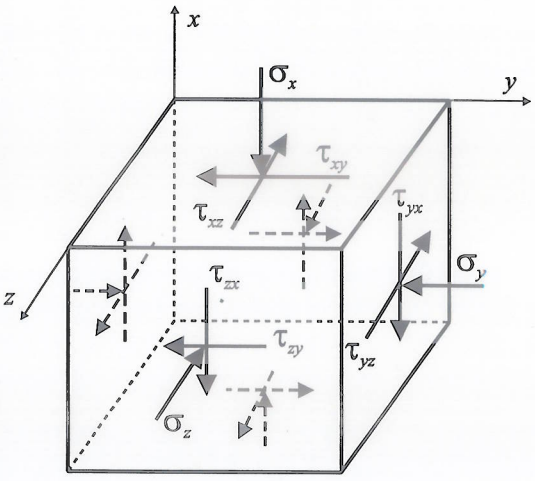

Figure 8: Stresses on a typical element

Whereas the velocity of flow is a vector with essentially three components, one in each of the Cartesian coordinate directions, stress is a tensor consisting of six components, see Figure 8. In the same way as there are rules which govern the flow of water through the tank, there are also rules which control the manner in which the stress components vary throughout the concrete beam. Neglecting inertia effects and all body forces, except self weight, stresses *in* a soil mass must satisfy the following three equations (Timoshenko and Goodier (1951)):

$$\begin{aligned} &\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \gamma = 0 \\ &\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = 0 \\ &\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0 \end{aligned} \qquad (1)$$

The following should be noted:

- self weight, γ, acts in the x direction;
- compressive stresses are assumed positive;
- the equilibrium Equations (1) are in terms of total stresses;
- stresses must satisfy the boundary conditions (i.e. at the boundaries the stresses must be in equilibrium with the applied surface traction forces).

5.3. Compatibility

Physical compatibility

Compatible deformation involves no overlapping of material and no generation of holes. The physical meaning of compatibility can be explained by considering a plate composed of smaller plate elements, as shown in Figure 9a. After straining, the plate elements may be so distorted that they form the array shown in Figure 9b. This condition might represent failure by rupture. Alternatively, deformation might be such that the various plate elements fit together (i.e. no holes created or overlapping) as shown in Figure 9c. This condition represents a compatible deformation.

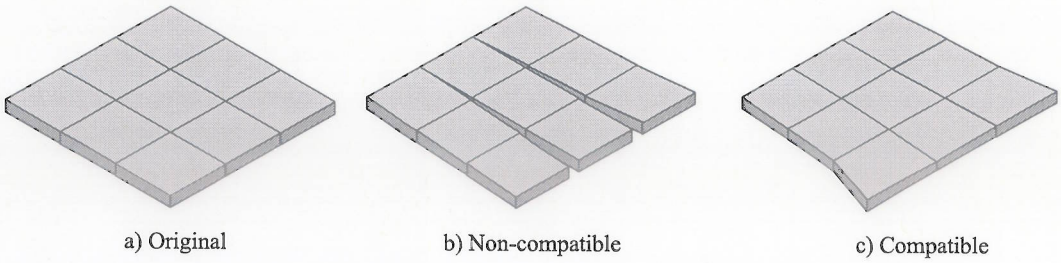

Figure 9: Modes of deformation

Mathematical compatibility

The above physical interpretation of compatibility can be expressed mathematically, by considering the definition of strains. If deformations are defined by continuous functions u, v and w in the x, y and z directions respectively, the strains (assuming small strain theory and a compression positive sign convention) are defined as (Timoshenko and Goodier (1951)):

$$\varepsilon_x = \frac{\partial u}{\partial x} \; ; \; \varepsilon_y = \frac{\partial v}{\partial y} \; ; \; \varepsilon_z = \frac{\partial w}{\partial z}$$

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \; ; \; \gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \; ; \; \gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \qquad (2)$$

As the six strains are a function of only three displacements, they are not independent. It can be shown mathematically that for a compatible displacement field to exist, all the above components of strain and their derivatives must exist (are bounded) and be continuous to at least the second order. The displacement field must satisfy any specified displacements or restraints imposed on the boundary.

5.4. Equilibrium and compatibility conditions

Combining the Equilibrium (Equations (1)) and Compatibility conditions (Equations (2)), gives:

Unknowns:	6 stresses + 6 strains + 3 displacements	= 15
Equations:	3 equilibrium + 6 compatibility	= 9

To obtain a solution therefore requires 6 more equations. These come from the constitutive relationships.

5.5. Constitutive behaviour

This is a description of material behaviour. In simple terms it is the stress - strain behaviour of the soil. It usually takes the form of a relationship between stresses and strains and therefore provides a link between equilibrium and compatibility.

For calculation purposes the constitutive behaviour has to be expressed mathematically:

$$\begin{Bmatrix} \Delta\sigma_x \\ \Delta\sigma_y \\ \Delta\sigma_z \\ \Delta\tau_{xy} \\ \Delta\tau_{xz} \\ \Delta\tau_{zy} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} & D_{15} & D_{16} \\ D_{21} & D_{22} & D_{23} & D_{24} & D_{25} & D_{26} \\ D_{31} & D_{32} & D_{33} & D_{34} & D_{35} & D_{36} \\ D_{41} & D_{42} & D_{43} & D_{44} & D_{45} & D_{46} \\ D_{51} & D_{52} & D_{53} & D_{54} & D_{55} & D_{56} \\ D_{61} & D_{62} & D_{63} & D_{64} & D_{65} & D_{66} \end{bmatrix} \begin{Bmatrix} \Delta\varepsilon_x \\ \Delta\varepsilon_y \\ \Delta\varepsilon_z \\ \Delta\gamma_{xy} \\ \Delta\gamma_{xz} \\ \Delta\gamma_{zy} \end{Bmatrix} \tag{3}$$

or

$$\Delta\sigma = [D]\ \Delta\epsilon$$

For a linear elastic material, for example, the $[D]$ matrix takes the following form:

$$\frac{E}{(1+\mu)} \begin{bmatrix} (1-\mu) & \mu & \mu & 0 & 0 & 0 \\ \mu & (1-\mu) & \mu & 0 & 0 & 0 \\ \mu & \mu & (1-\mu) & 0 & 0 & 0 \\ 0 & 0 & 0 & (1/2-\mu) & 0 & 0 \\ 0 & 0 & 0 & 0 & (1/2-\mu) & 0 \\ 0 & 0 & 0 & 0 & 0 & (1/2-\mu) \end{bmatrix} \tag{4}$$

where E and μ are the Young's Modulus and Poisson's ratio respectively.

However, because soil usually behaves in a nonlinear manner, it is more realistic for the constitutive equations to relate increments of stress and strain, as indicated in Equation (3), and for the $[D]$ matrix to depend on the current and past stress history.

The constitutive behaviour can either be expressed in terms of total or effective stresses. If specified in terms of effective stresses, the principle of effective stress ($\sigma = \sigma' + \sigma_f$) may be invoked to obtain total stresses required for use with the equilibrium equations:

$$\Delta\sigma' = [D']\ \Delta\epsilon; \quad \Delta\sigma_f = [D_f]\ \Delta\epsilon; \quad \text{therefore } \Delta\sigma = ([D'] + [D_f])\ \Delta\epsilon \tag{5}$$

where $[D_f]$ is a constitutive relationship relating the change in pore fluid pressure to the change in strain. For undrained behaviour, the change in pore fluid pressure is related to the volumetric strain (which is small) via the bulk compressibility of the pore fluid (which is large.

6. Geometric idealisation

In order to apply the above concepts to a real geotechnical problem, certain assumptions and idealisations must be made. In particular, it is necessary to specify soil behaviour in the form of a mathematical constitutive relationship. It may also be necessary to simplify and/or idealise the geometry and/or boundary conditions of the problem.

6.1. Plane strain

Due to the special geometric characteristics of many of the physical problems treated in soil mechanics, additional simplifications of considerable magnitude can be applied. Problems, such as the analysis of retaining walls, continuous footings, and the stability of slopes, generally have one dimension very large in comparison with the other two, see Figure 10. Hence, if the force and/or applied displacement boundary conditions are perpendicular to, and independent of, this dimension, all cross sections will be the same. If the z dimension of the problem is large, and it can be assumed that the state existing in the x-y plane holds for all planes parallel to it, the displacement of any x-y cross section, relative to any parallel x-y cross section, is zero. This means that w=0, and the displacements u and v are independent of the z coordinate. The conditions consistent with these approximations are said to define the very important case of *plane strain*:

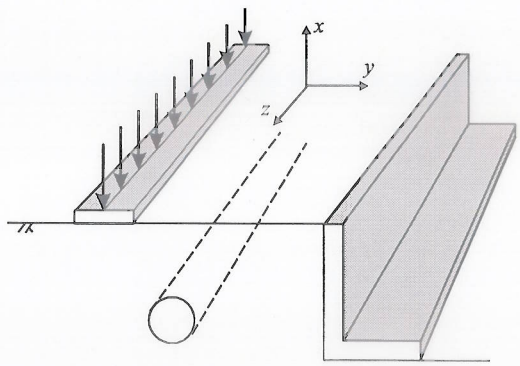

Figure 10: Examples of plane strain

$$\varepsilon_z = \frac{\partial w}{\partial z} = 0\ ;\ \gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = 0\ ;\ \gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = 0 \tag{6}$$

The constitutive relationship then reduces to:

$$\begin{Bmatrix} \Delta\sigma_x \\ \Delta\sigma_y \\ \Delta\sigma_z \\ \Delta\tau_{xy} \\ \Delta\tau_{xz} \\ \Delta\tau_{zy} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{14} \\ D_{21} & D_{22} & D_{24} \\ D_{31} & D_{32} & D_{34} \\ D_{41} & D_{42} & D_{44} \\ D_{51} & D_{52} & D_{54} \\ D_{61} & D_{62} & D_{64} \end{bmatrix} \begin{Bmatrix} \Delta\varepsilon_x \\ \Delta\varepsilon_y \\ \Delta\gamma_{xy} \end{Bmatrix} \tag{7}$$

However, for elastic and the majority of material idealisations currently used to represent soil behaviour $D_{52}=D_{51}=D_{54}=D_{61}=D_{62}=D_{64}=0$, and consequently $\Delta\tau_{xz}=\Delta\tau_{zy}=0$. This results in four non-zero stress changes, $\Delta\sigma_x$, $\Delta\sigma_y$, $\Delta\sigma_z$ and $\Delta\tau_{xy}$.

It is common to consider only the stresses σ_x, σ_y and τ_{xy} when performing analysis for plane strain problems. This is acceptable if D_{11}, D_{12}, D_{14}, D_{21}, D_{22}, D_{24}, D_{41}, D_{42} and D_{44} are not dependent on σ_z. This condition is satisfied if the soil is assumed to be elastic. It is also true if the Tresca or Mohr-Coulomb failure condition is adopted and it is assumed that the intermediate stress $\sigma_2=\sigma_z$. *Such an assumption is usually adopted for the simple analysis of geotechnical problems* . It should be noted, however, that these are special cases.

6.2. Axi-symmetry

Some problems possess rotational symmetry. For example, a uniform or centrally loaded circular

footing, acting on a homogeneous or horizontally layered foundation, has rotational symmetry about a vertical axis through the centre of the foundation. Cylindrical triaxial samples, single piles and caissons are other examples where such symmetry may exist, see Figure 11.

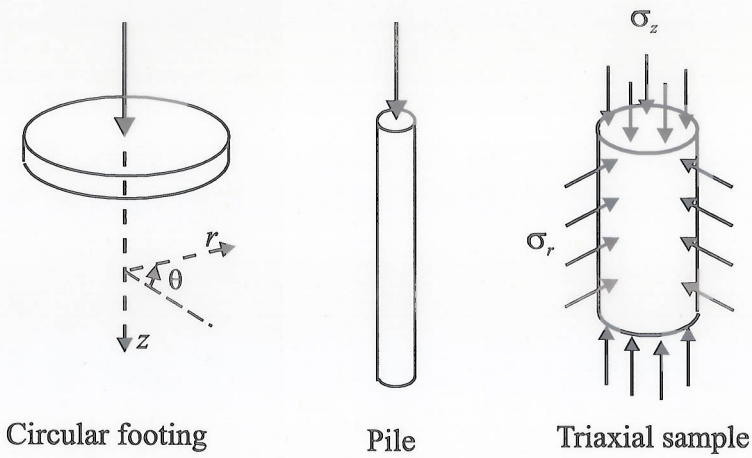

Figure 11: Examples of axi-symmetry

In this type of problem it is usual to carry out analyses using cylindrical coordinates r (radial direction), z (vertical direction) and θ (circumferential direction). The strain components in this coordinate system are calculated as follows:

$$\varepsilon_r = \frac{\partial u}{\partial r} \;;\; \varepsilon_z = \frac{\partial v}{\partial z} \;;\; \varepsilon_\theta = \frac{u}{r} + \frac{1}{r}\frac{\partial w}{\partial \theta}$$
$$\gamma_{rz} = \frac{\partial u}{\partial z} + \frac{\partial v}{\partial r} \;;\; \gamma_{r\theta} = \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial w}{\partial r} - \frac{w}{r} \;;\; \gamma_{z\theta} = \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} \tag{8}$$

where u, v and w are the displacements in the r, z and θ are directions respectively.

Due to the symmetry, there is no displacement in the θ direction and the displacements in the r and z directions are independent of θ and therefore the strains reduce to (Timoshenko and Goodier (1951)):

$$\varepsilon_r = \frac{\partial u}{\partial r} \;;\; \varepsilon_z = \frac{\partial v}{\partial z} \;;\; \varepsilon_\theta = \frac{u}{r} \;;\; \gamma_{rz} = \frac{\partial v}{\partial r} + \frac{\partial u}{\partial z} \;;\; \gamma_{r\theta} = \gamma_{z\theta} = 0 \tag{9}$$

This is similar to the plane strain situation discussed above and, consequently, the same arguments concerning the $[D]$ matrix apply here too. As for plane strain, there are four non-zero stress changes, $\Delta\sigma_r$, $\Delta\sigma_z$, $\Delta\sigma_\theta$ and $\Delta\tau_{rz}$.

7. Methods of analysis

As noted above, fundamental considerations assert that for an exact theoretical solution the requirements of *equilibrium, compatibility, material behaviour* and *boundary conditions*, both force and displacement, must all be satisfied. It is therefore useful to review the broad categories of analysis currently in use against these theoretical requirements.

Current methods of analysis can be conveniently grouped into the following categories: closed form, simple and numerical analysis. Each of these categories is considered separately. The

ability of each method to satisfy the fundamental theoretical requirements and provide design information are summarised in Tables 1 and 2 respectively.

Table 1: Basic solution requirements satisfied by the various methods of analysis

METHOD OF ANALYSIS		SOLUTION REQUIREMENTS				
		Equilibrium	Compatibility	Constitutive behaviour	Boundary conditions	
					Force	Disp
Closed form		S	S	Linear elastic	S	S
Limit equilibrium		S	NS	Rigid with a failure criterion	S	NS
Stress field		S	NS	Rigid with a failure criterion	S	NS
Limit analysis	Lower bound	S	NS	Ideal plasticity with associated flow rule	S	NS
	Upper bound	NS	S		NS	S
Full Numerical analysis		S	S	Any	S	S

S - Satisfied; NS - Not Satisfied

Table 2: Design requirements satisfied by the various methods of analysis

METHODS OF ANALYSIS		DESIGN REQUIREMENTS		
		Stability	Movements	Adjacent structures
Closed form (Linear elastic)		NO	YES	YES
Limit Equilibrium		YES	NO	NO
Stress Field		YES	NO	NO
Limit Analysis	Lower Bound	YES	NO	NO
	Upper Bound	YES	CRUDE ESTIMATE	NO
Full Numerical Analysis		YES	YES	YES

8. Idealisation of soil behaviour

8.1. Elasticity

The simplest assumption for the soil behaviour is that it is a linear elastic isotropic material, characterised by only two parameters: a Young's modulus E and Poisson's ratio μ. The constitutive matrix [D] then takes the form given in Equation (4). In geotechnical engineering, however, it is more common to use the parameters G and K (shear and bulk modulus respectively) to express the elastic behaviour. It may be noted that G and K are related to E and μ by the following expressions:

$$G = \frac{E}{2(1+\mu)} \quad ; \quad K = \frac{E}{3(1-2\mu)} \tag{10}$$

The [D] matrix then takes the following form:

$$\begin{bmatrix} K+4G/3 & K-2G/3 & K-2G/3 & 0 & 0 & 0 \\ K-2G/3 & K+4G/3 & K-2G/3 & 0 & 0 & 0 \\ K-2G/3 & K-2G/3 & K+4G/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & G & 0 & 0 \\ 0 & 0 & 0 & 0 & G & 0 \\ 0 & 0 & 0 & 0 & 0 & G \end{bmatrix} \tag{11}$$

Although very simple and requires only two parameters to describe soil behaviour, the elastic framework has serious limitations in describing real soil behaviour.

- Firstly, application of shear strains $\Delta\gamma_{xy}$, $\Delta\gamma_{xz}$ and $\Delta\gamma_{yz}$ results only in a corresponding change in the associated shear stress.
- Secondly, application of an isotropic volumetric strain, $\Delta\varepsilon_x=\Delta\varepsilon_y=\Delta\varepsilon_z$, results only in a change of mean stress $\Delta p'=\frac{1}{3}(\Delta\sigma_x+\Delta\sigma_y+\Delta\sigma\varepsilon_z) = K(\Delta\varepsilon_x+\Delta\varepsilon_y+\Delta\varepsilon_z)$. This leads to the following statement: For a linear elastic isotropic material, spherical (mean) and deviatoric(shear) behaviour are uncoupled. This implies that application of deviatoric (shear) stresses will only give rise to deviatoric (shear) strains and not give volumetric strains, which is not the case in real soil behaviour.
- A more severe limitation of the isotropic linear elastic model is that failure cannot be simulated.
- In addition, elastic materials are conservative, which means that all the work done by external stresses during an increment of deformation is stored and recovered on unloading.

8.2. Plasticity

An improvement in modelling soil behaviour is to assume it to be elasto-plastic. An elasto-plastic constitutive model requires the following essential ingredients:

- Assumption that the principal directions of stress and plastic strain increment coincide.
- A *yield surface*, expressed as $F(\sigma, \mathbf{k})=0$, which separates purely elastic from elasto-plastic behaviour. In general, the surface is a function of the stress state σ and its size also changes as a function of the hardening/softening parameters $\mathbf{k}$.
- A direction of plastic strain increments. This is usually specified as the gradient of the *plastic potential surface*, $P(\sigma, \mathbf{k})=0$, multiplied by an unknown scalar, λ:

$$\Delta\varepsilon^p = \lambda\frac{\partial P}{\partial\sigma} \tag{12}$$

- A *hardening/softening law* which describes how the parameters $\mathbf{k}$ vary with plastic straining. This enables the scalar λ to be quantified.