Limit analysis

1. Introduction

If the soil can be modelled as an ideal elasto-plastic material (i.e. Tresca or Mohr-Coulomb) then it can be shown that for most boundary value problems unique conditions exist at collapse. For example consider the surface footing shown in the Figure 1.

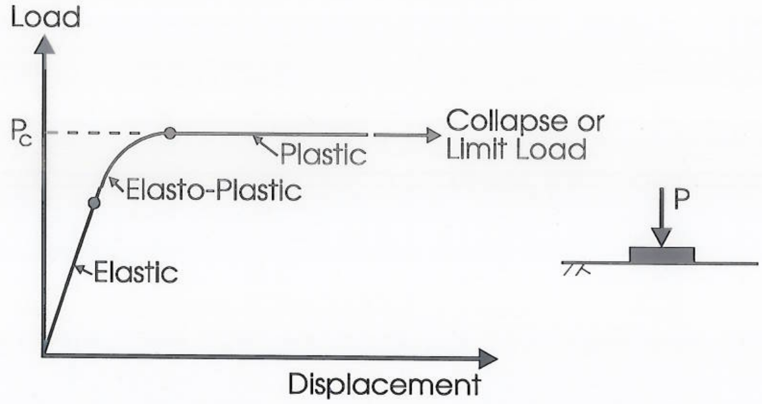

Figure 1: Load-displacement curve of a footing

As the load on the footing is increased the soil initially behaves elastically. If the elastic properties are constant (i.e. linear elastic) this results in a linear load-displacement response of the footing. With further increase in load, elements of soil in highly stressed regions yield plastically. If the footing is stiff compared to the soil this will first occur in the soil immediately under the corners of the footing. As these elements of plastic soil are surrounded by elastic material they are contained and failure of the footing does not occur. It is therefore possible to increase the footing load. With additional load on the footing further elements of soil become plastic and the plastic zone grows. During this stage the load-displacement curve becomes nonlinear. Eventually, when sufficient soil becomes plastic a failure mechanism forms, no further load can be added to the footing. The footing then displaces at constant load. The load, P_c, when this occurs is known as the plastic collapse load. This collapse load is independent of the initial state of stress in the soil before application of load to the footing.

A similar pattern of behaviour occurs in the tunnel problem shown in Figure 2. Here the problem of reducing tunnel support during construction is examined. Initially the support pressure in the cylindrical cavity is set equal to the overburden stress of the soil above. As the pressure is reduced the soil initially behaves elastically, then some elements yield and finally collapse occurs when the support pressure is reduced to the plastic collapse pressure, p_{tc}.

As noted previously, closed form solutions satisfying all the basic theoretical requirements (i.e. equilibrium, compatibility, constitutive behaviour and boundary conditions) are not possible for the majority of boundary value problems. However, the methods of Limit Analysis provide bounding estimates to the ultimate loads that cause collapse. In the above examples they would provide estimates of P_c and p_{tc}. They do this by ignoring some of the conditions of equilibrium or compatibility. It turns out that by ignoring equilibrium it is possible to obtain an *unsafe* estimate of the ultimate loads. Alternatively, by ignoring compatibility it is possible to obtain a *safe* estimate. The true collapse loads (for the ideal soil behaviour) are between these bounds.

The methods of limit analysis are based on two theorems. The assumptions on which these theorems are based and their proofs are given in the Appendix.

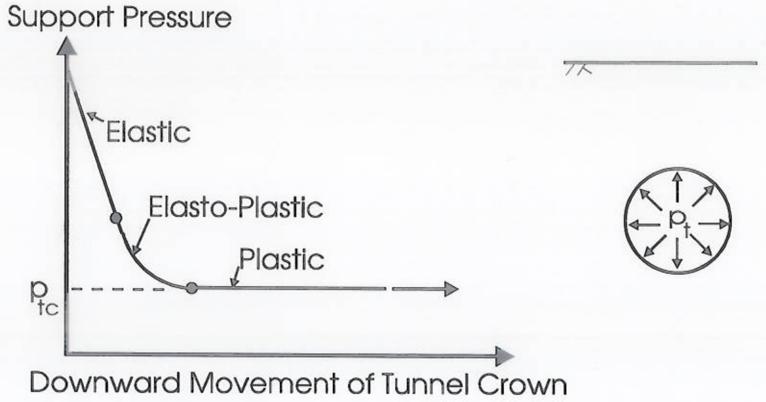

Figure 2: Pressure-displacement curve in a tunnel

2. Theorems of Limit Analysis

Unsafe theorem

An unsafe solution to the true collapse loads (for the ideal plastic material) can be found by selecting any kinematically possible failure mechanism and performing an appropriate work rate calculation. The loads so determined will either be on the unsafe side or equal to the true collapse loads.

This theorem is often referred to as the 'Upper Bound' theorem. A *kinematically possible* failure mechanism is one that satisfies the compatibility conditions. A *work rate calculation* consists of equating the work done by the soil stresses to the work done by the external loads during an increment of movement of the failure mechanism. As equilibrium is not considered there will be an infinite number of solutions which can be found. The accuracy of the solution will depend on the closeness of the assumed failure mechanism to the real one.

Safe theorem

If a statically admissible stress field covering the whole soil mass can be found which nowhere violates the yield condition, the loads in equilibrium with the stress field will be on the safe side or equal to the true collapse loads.

This theorem is often referred to as the 'Lower Bound' theorem. A *statically admissible* stress field consists of an equilibrium distribution of stress which balances the applied loads and body forces. As compatibility is not considered there will be an infinite number of solutions which can be found. The accuracy of the solution will depend on the closeness of the assumed stress field to the real one.

If safe and unsafe solutions can be found which give the same estimates of collapse loads, then this is the correct solution for the ideal plastic material. It should be noted that in such a case all the fundamental solution requirements will have been satisfied.

The terms 'Lower Bound' and 'Upper Bound' can be misleading when the theorems are applied to geotechnical engineering. For example, consider the footing and tunnel problems discussed above. If analyses are performed satisfying the unsafe theorem they will produce unsafe estimates of the collapse loads. For the footing problem this means that the estimate for the footing load P_c^{unsafe} will be numerically larger than the correct value, P_c. On the other hand, for the tunnel problem an unsafe estimate implies that p_{tc}^{unsafe} will be numerically lower than the correct value, p_{tc}. If analyses are performed satisfying the safe theorem then for the footing problem $P_c^{safe} \le P_c$, whereas for the tunnel problem $p_{tc}^{safe} \ge p_{tc}$. The term 'Upper Bound' gives the impression that the numerical value of the collapse load from an *unsafe* calculation will be higher than the true value. Likewise the term 'Lower Bound' implies that an analysis based on the *safe* theorem will provide a numerical value of the collapse load lower than the true value. For the footing problem such an implication would be correct, but not for the tunnel problem where the reverse is true.

3. Unsafe analysis

To perform an analysis which satisfies the unsafe theorem a failure mechanism must be postulated (guessed) and then the work done by the external forces (surface tractions and body forces) equated to the work done by the stresses in the soil mass.

3.1. Failure mechanisms

The only restraint on the selection of a failure mechanism is that it satisfies the conditions of compatibility. It is often convenient to consider the failure mechanism in terms of rigid blocks of soil moving on thin plastic shear zones, see Figure 3.

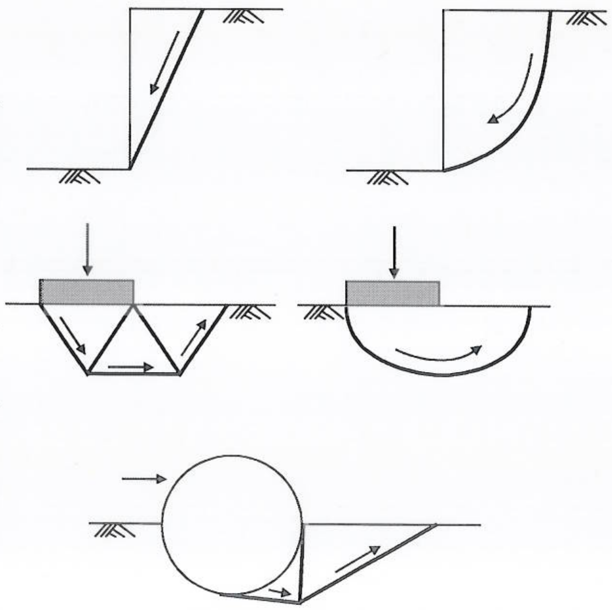

Figure 3: Failure mechanisms

Although this is clearly an approximation of real soil behaviour it considerably simplifies calculations and makes the selection of mechanisms that are compatible easier. It should be noted the assumption of rigid blocks sliding on thin plastic zones is not a restriction of the method. It is possible to replace the rigid blocks with zones of soil undergoing plastic deformation. However, as this considerably increases the complexity of any calculations it will not be considered in these lectures.

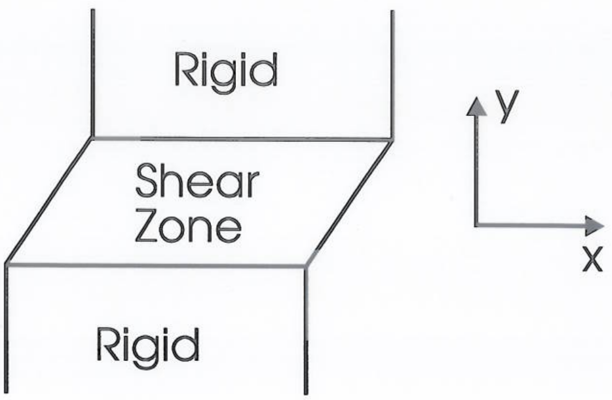

Figure 4: Element of a shear zone

In Figure 3 the failure mechanisms involve translational movements along planar shear zones or rotational movements along curved shear zones. In some cases the mechanism involves several planar shear surfaces. For the failure mechanisms to satisfy the compatibility conditions restrictions are imposed on the orientation of the planar surfaces and on the shapes of the curved surfaces. This arises because in the plastic shear zones the strains must satisfy the plastic constitutive equations and in particular the flow rule (normality condition).

Consider the small element of a typical shear zone shown in the Figure 4. If the plane strain conditions are assumed, the out of plane plastic strains at collapse $\epsilon_z^{pl}=\gamma_{xz}^{pl}=\gamma_{yz}^{pl}=0$. If the in plane axes x and y, local to the shear zone, are chosen with x acting along the shear zone, then $\epsilon_x^{pl}=0$ because of the rigid nature of the soil on either side of the shear zone. These geometric restrictions therefore imply that there can be only two non zero strains, namely ϵ_y^{pl} and γ_{xy}^{pl}. The relative magnitude between these two strains is controlled by the plastic constitutive model and are considered separately for the Tresca and Mohr-Coulomb models below.

Tresca model

As the only two potentially non zero strains are the direct strain normal to the shear zone (ϵ_y^{pl}) and the shear strain parallel to the shear zone (γ_{xy}^{pl}) it is convenient to consider the constitutive behaviour in terms of normal total stress, σ_n, and tangential shear stress, τ. The yield surface can be written as $\tau=S_u$ and in τ-σ_n space produces a horizontal line as shown in Figure 5. During plastic deformation the shear stress τ, acting in the shear zone, will be equal to S_u. Because of the assumption of coincidence of principal stress and plastic strain increment directions, the incremental plastic strains $\delta\epsilon_y^{pl}$ and $\delta\gamma_{xy}^{pl}$ can be plotted on the same axes. The normality (associated plastic flow) condition incorporated in the Tresca model results in the strain increment vector being normal to the yield surface. As the yield surface is horizontal, the plastic strain increment vector is vertical. The increment of plastic normal strain $\delta\epsilon_y^{pl}$ must therefore be zero. γ_{xy}^{pl} is therefore the only non zero plastic strain.

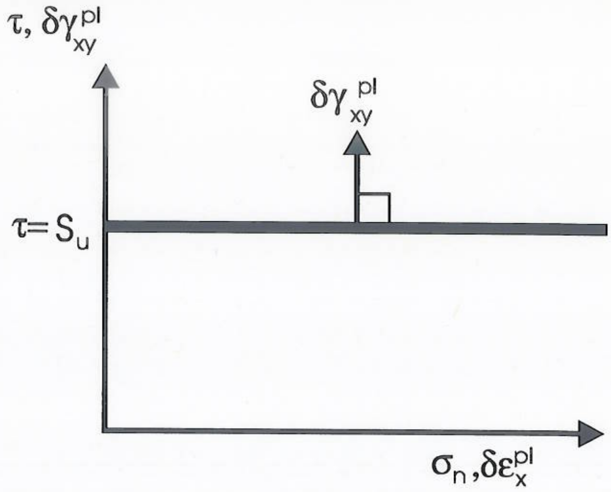

Figure 5: Plastic strain components in Tresca model

As the normal direct strain is zero, the thickness of the shear zone remains constant during plastic deformation. The relative movement between the two rigid blocks either side of the shear zone is therefore in the direction of the shear zone.

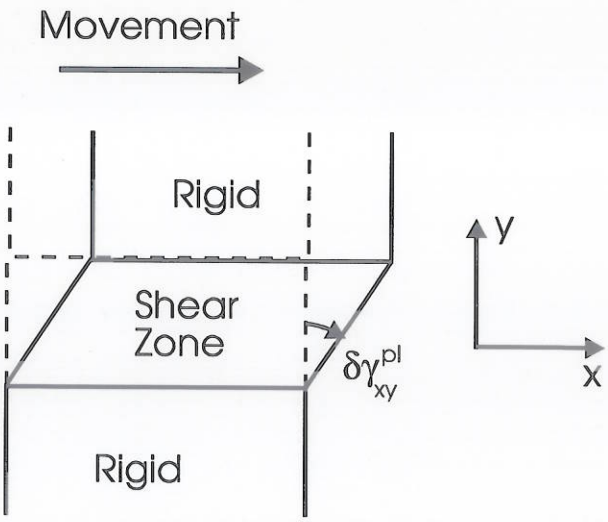

Figure 6: Deformation of the shear zone in Tresca conditions

If the failure mechanism involves rotational movement about some fixed point, this compatibility requirement restricts the curvature of the shear zone. Consider the two rigid blocks shown in Figure 7. Block 'B' remains stationary while block 'A' rotates about point 'O'. At any point in block 'A' movement is in a direction perpendicular to the radius from 'O'. As block 'B' is stationary then the relative movement between blocks 'A' and 'B' is also in a direction normal to a radius through 'O'. As this relative movement must be along the shear zone (see above) the shear zone must trace a curve normal to a radius through 'O' and must therefore be an arc of a *circle*. Clearly if the radius of rotation becomes very large the curvature of the slip zone reduces and eventually will degenerate into a planar surface.

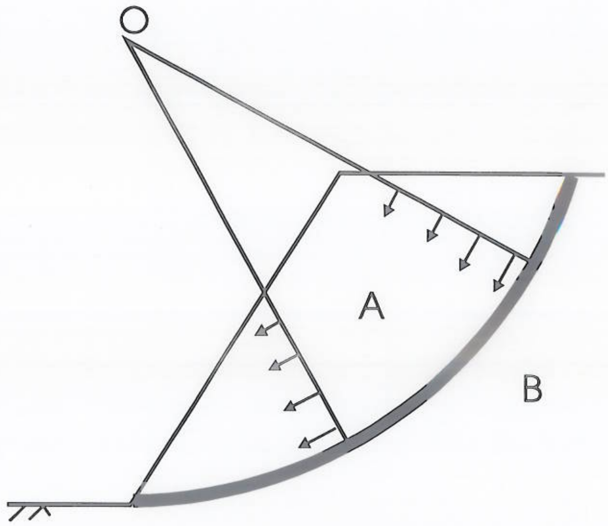

Figure 7: Shape of the curved shear zone in Tresca soil conditions

Mohr-Coulomb model

For similar reasons to those given above for the Tresca model it is convenient to consider the constitutive behaviour in terms of normal total stress, σ_n', and tangential shear stress, τ. The yield surface can be written as $\tau = c' + \sigma_n' \tan\phi'$ and in τ-σ_n' space produces an inclined line as shown in Figure 8. During plastic deformation the shear stress, τ acting in the shear zone will satisfy this yield equation. Because of the assumption of coincidence of principal stress and plastic strain increment directions, the incremental plastic strains $\delta\epsilon_y^{pl}$ and $\delta\gamma_{xy}^{pl}$ can be plotted on the same axes. The normality (associated plastic flow) condition results in the strain increment vector being perpendicular to the yield surface. Consequently both $\delta\epsilon_y^{pl}$ and $\delta\gamma_{xy}^{pl}$ are non zero and are related to each other by the following equation:

$$\delta\epsilon_y^{pl} = - \delta\gamma_{xy}^{pl} \tan\phi' \qquad (1)$$

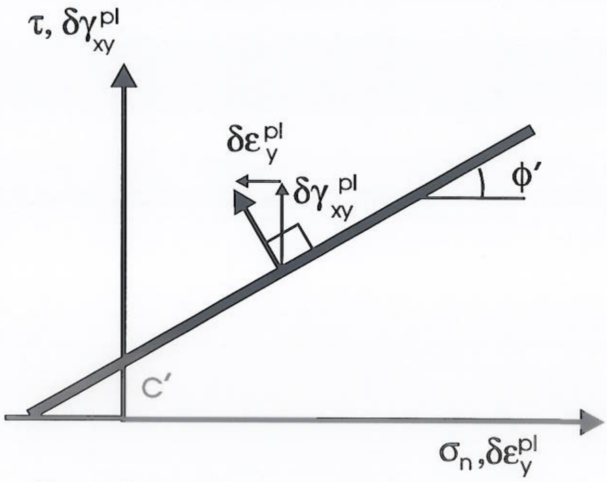

Figure 8: Plastic strain components in Mohr-Coulomb model

This indicates that plastic shear strains are accompanied by negative (dilation) direct strains normal to the shear zone. Consequently as plastic deformation occurs the thickness of the shear zone increases.

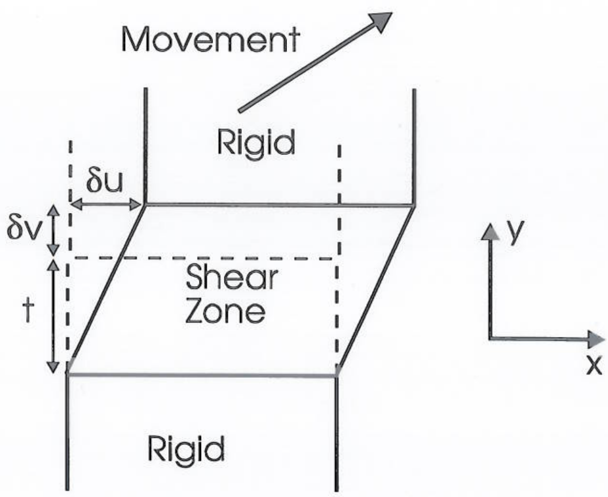

Figure 9: Deformation of the shear zone in Mohr-Coulomb conditions

Consider the element of a shear zone, of thickness 't', shown in the Figure 9. An increment of plastic deformation will result in displacements δu and δv tangential and normal to the shear zone respectively (as shown in the figure). Assuming that the shear zone is thin, and that uniform conditions of stress and strain occur within it, we can write:

$$\delta\epsilon_y^{pl} = -\delta v/t \quad \text{and} \quad \delta\gamma_{xy}^{pl} = \delta u/t \tag{2}$$

Substituting in Equation (1) gives:

$$\delta v/\delta u = \tan\phi' \tag{3}$$

The relative movement between the two rigid blocks either side of the shear zone therefore is at an angle ϕ' to the direction of the shear zone. When using failure mechanisms involving several rigid blocks this compatibility condition must be considered and it will impose restrictions on the critical orientations of the shear zones.

If the failure mechanism involves rotational movement about some fixed point this compatibility requirement controls the curvature of the shear zone. Consider the two rigid blocks shown in the Figure 10. Block 'B' remains stationary while block 'A' rotates about point 'O'. At any point in block 'A' movement is in a direction perpendicular to the radius from 'O'. As block 'B' is stationary then the relative movement between blocks 'A' and 'B' is also in a direction normal to a radius through 'O'. As this relative movement must be at an angle ϕ' to the shear zone (see above) we obtain, in the limit as $\Delta\theta \rightarrow 0$;

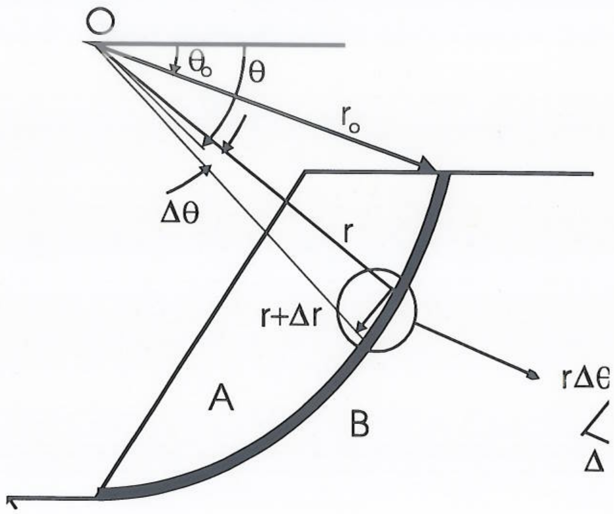

Figure 10: Shape of the curved shear zone in Mohr-Coulomb conditions

$$dr = r \tan\phi' \, d\theta$$

which on integrating between the limits of θ_o, r_o and θ, r gives;

$$r = r_o e^{(\theta-\theta_o)\tan\phi'} \tag{4}$$

This is the equation of a *log-spiral*. As the radius of rotation becomes large (i.e $r_o \rightarrow$ infinity) this degenerates into a straight line.

3.2. Work done by external forces

The work done by external forces includes the work done by surface tractions (line loads and surcharges) and body forces (self weight). For an increment of deformation of the failure mechanism the work done is simply the sum of each force multiplied by the increment of displacement in the direction of the resultant of that force. It is therefore necessary to calculate the relative magnitudes and directions of the displacements of the rigid blocks in the failure mechanism. This is most conveniently achieved by constructing a hodograph.

Consider the problem of a footing resting on a Tresca soil shown in the Figure 11. The failure mechanism consists of two rigid triangular blocks, A and B. At failure the footing moves vertically down, block 'A' moves downwards at angle of 45° to the horizontal and block 'B' moves upward at an angle of 45° to the horizontal. Each block is assumed to be rigid and the soil below the failure mechanism, which is assumed to be stationary, is labelled as 'O'. The hodograph is constructed from the directions of the relative displacements of the rigid blocks. Once the magnitude of one of the displacement components is fixed it is then possible to determine all the others from this diagram. In particular it is possible to calculate the relative displacement between the two blocks A and B, δab, and between the footing and block A, δaf.

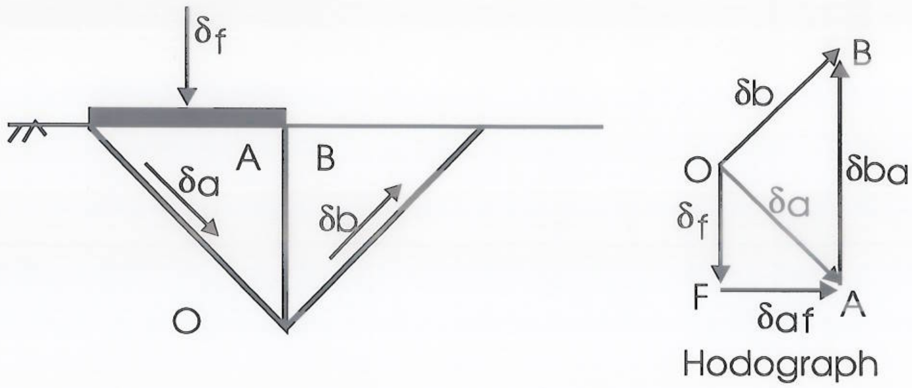

Figure 11: Failure mechanism for a strip footing and hodograph of its displacements

3.3. Work done by internal stresses

For an increment of deformation of the failure mechanism the work down by the internal stresses is equal to:

$$\int^{Volume} \sigma_{ij}^{pl}.\delta\varepsilon_{ij}^{pl}.dVol$$

where σ_{ij}^{pl} are the stresses throughout the soil and $\delta\epsilon_{ij}^{pl}$ are the plastic strain increments arising from the incremental movement of the failure mechanism. As there are no plastic strains within the rigid blocks the integral reduces to the volume of the shear zones themselves. As noted above the plastic strain increments $\delta\epsilon_z^{pl}=\delta\gamma_{xz}^{pl}=\delta\gamma_{yz}^{pl}=0$ due to the assumption of plane strain and there is no direct strain in the direction of the shear zone due to the restraining effects of the rigid blocks on either side. This leaves the possibility of only two non zero plastic strain increments, namely the direct strain normal to, and the shear strain tangential to, the shear zones. As discussed above the relative magnitudes of these two strain components depend on the constitutive model.

Tresca model

As discussed above the combination of the yield surface and normality (plastic flow) condition results in a zero direct plastic strain increment normal to the yield surface. The plastic increment of shear strain tangential the shear zone is therefore the only non-zero strain component. During plastic deformation the shear stress in the shear zone must satisfy the yield condition and therefore $\tau=S_u$.

Considering the element of a typical shear zone of thickness t and of unit length shown in the Figure 12, the work done by the internal stresses is given by:

$$WD = \tau\, \delta\gamma^{pl}\, t$$

per unit length of the shear zone. Noting that $\tau = S_u$ and $\delta\gamma^{pl} = \delta u/t$, where δu is the relative displacement across the shear zone, and substituting in the above equation gives:

$$WD = \delta u\, S_u \tag{5}$$

per unit length of the shear zone

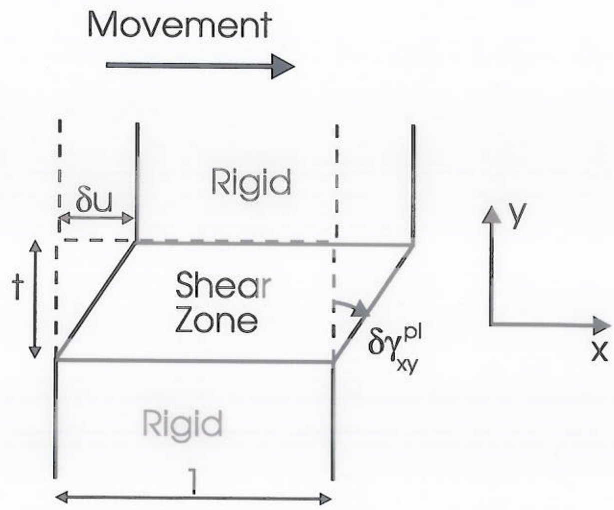

Figure 12: Internal work of Tresca shear zone

The total work done by the internal stresses in the failure mechanism is therefore simply the integral of WD over the length of the shear zones. It should be noted that the thickness 't' of the shear zone does not appear in the above equation and therefore the shear zones may be conveniently viewed as being of zero thickness.

Mohr-Coulomb

As noted previously both the increments of the direct strain normal to, and the shear strain tangential to, the shear zone are non zero. Their relative magnitudes are related by Equation (1).

Considering the element of a typical shear zone of thickness t and of unit length shown in the Figure 13, the work done by the internal stresses is given by:

$$WD = t(\tau \delta\gamma_{xy}{}^{pl} + \sigma_n' \delta\epsilon_y{}^{pl})$$

per unit length of the shear zone. Using Equation (1) to eliminate $\delta\epsilon_y{}^{pl}$ gives

$$WD = t\, \delta\gamma^{pl}\, (\tau - \sigma_n' \tan\phi')$$

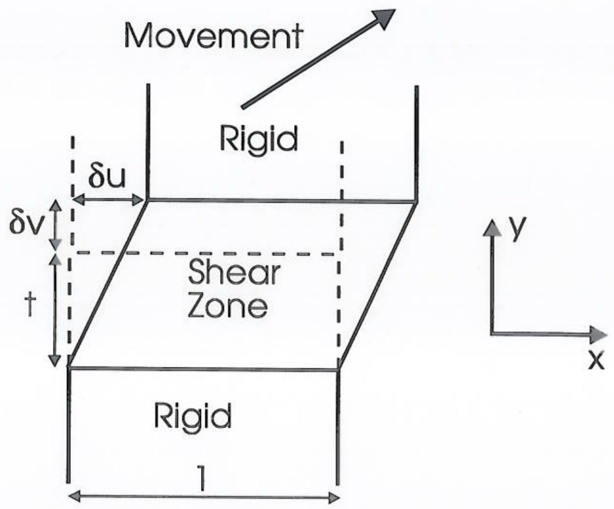

Figure 13: Internal work of Mohr-Coulomb shear zone

per unit length of the shear zone. Noting that during plastic deformation the stresses in the shear zone must satisfy the yield condition ($\tau = c' + \sigma_n' \tan\phi'$) and that $\delta\gamma_{xy}{}^{pl} = \delta u/t$, where δu is the tangential component of the relative displacement across the shear zone, the above equation reduces to;

$$WD = \delta u\, c' \qquad \text{per unit length of the shear zone} \tag{6}$$

The total work done by the internal stresses in the failure mechanism is simply the integral of WD over the length of the shear zones. It should be noted that the thickness 't' of the shear zone does not appear in the above equation and therefore the shear zones may again be conveniently viewed as being of zero thickness. If the soil is cohesionless (c'=0) the work done by the internal stresses is zero.

3.4. Examples

Critical height of a vertical cut in Tresca soil (undrained clay)

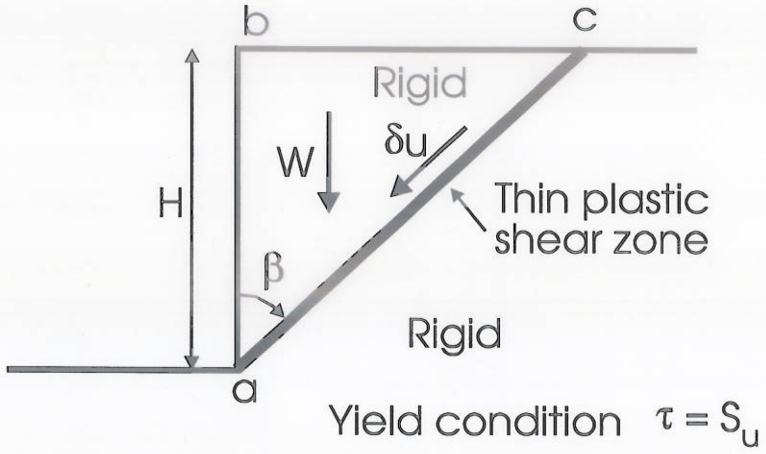

Figure 14: Schematic presentation of the problem in Tresca conditions

Rigid block abc moves with respect to the rigid base along the thin plastic shear zone ac. Let the relative displacement between the two rigid blocks be δu. Then:

Using Equation (5) the internal rate of energy dissipation:

$$= \delta u . S_u . H/\cos\beta \qquad (7)$$

Rate of work done by external forces:

$$= \tfrac{1}{2} H^2 . \delta u . \gamma . \sin\beta \qquad (8)$$

Equating Equations (7) and (8):

$$H = 4.S_u/(\gamma.\sin 2\beta) \qquad (9)$$

As this is an unsafe estimate the value of β which produces the smallest value of H is required. Therefore:

$$\frac{\partial H}{\partial \beta} = -\frac{8.S_u.\cos 2\beta}{\gamma.\sin^2 2\beta}$$

Which = 0 if $\cos 2\beta = 0$. Therefore $\beta = \pi/4$ which when substituted in Equation (9) gives:

$$H^{Unsafe} = 4.S_u/\gamma$$

Critical height of a vertical cut in dry Mohr-Coulomb soil

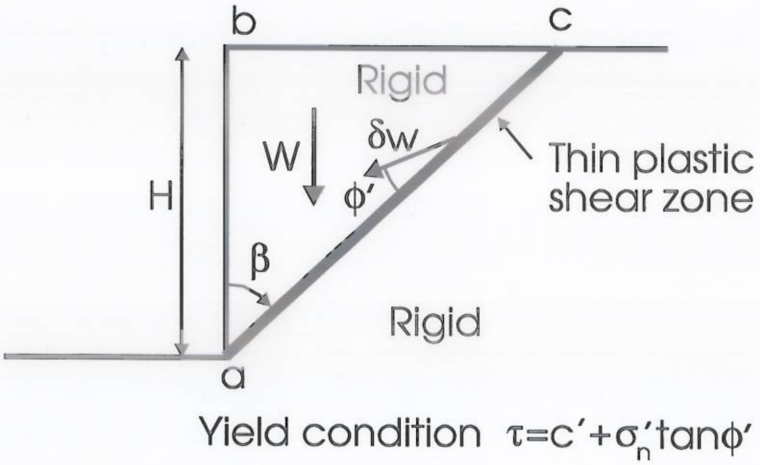

Figure 15: Schematic presentation of the problem in Mohr-Coulomb conditions

Rigid block abc moves with respect to the rigid base along the thin plastic shear zone ac. Let the relative displacement between the two rigid blocks be δw. The component of this displacement tangential to the plastic shear zone δu=δwcosφ'.

Using Equation (6) the internal rate of energy dissipation:

$$= \delta w.\cos\phi'.c'.H/\cos\beta \tag{10}$$

Rate of work done by external forces:

$$= \tfrac{1}{2}.\gamma.H^2.\tan\beta.\delta w.\cos(\beta+\phi') \tag{11}$$

Equating Equations (10) and (11):

$$H = 2.c'.\cos\phi'/(\gamma.\sin\beta.\cos(\beta+\phi')) \tag{12}$$

As this is an unsafe estimate the value of β which produces the smallest value of H is required. Therefore:

$$\frac{\partial H}{\partial \beta} = -\frac{2.c'.\cos(2\beta+\phi').\cos\phi'}{\gamma.(\sin\beta.\cos(\beta+\phi'))^2}$$

which = 0 if cos(2β+φ')=0. Therefore β=π/4-φ'/2 which when substituted in Equation (12) and rearranging gives:

$$H^{Unsafe} = 4.c'.\tan(\pi/4+\phi'/2)/\gamma$$

Footing on Tresca soil (undrained clay)

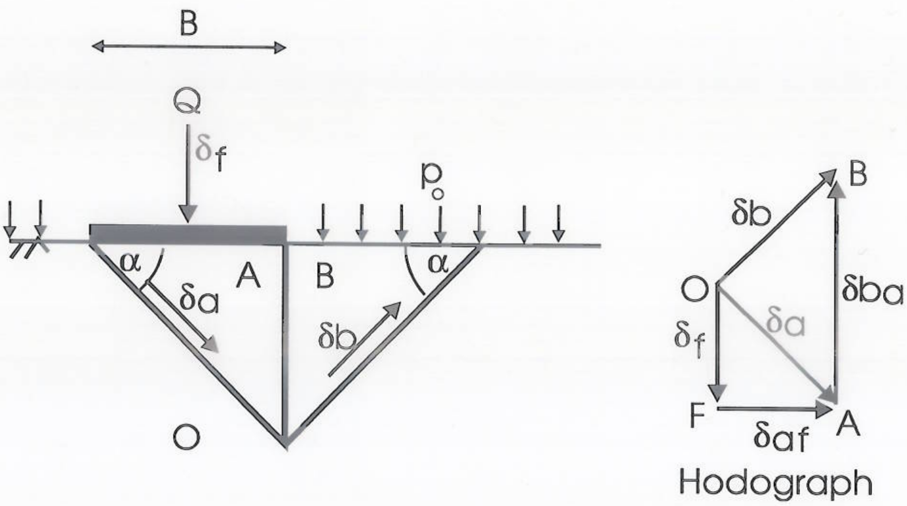

Figure 16: Failure mechanism for footing analysis

Work done by external forces:

$$= Q.\delta_f - p_o.B.\delta_f \tag{13}$$

Internal energy dissipation:

Interface	Length	Relative Disp.	Internal Work
OA	$B/\cos\alpha$	$\delta_f/\sin\alpha$	$S_u.B.\delta_f/(\sin\alpha.\cos\alpha)$
OB	$B/\cos\alpha$	$\delta_f/\sin\alpha$	$S_u.B.\delta_f/(\sin\alpha.\cos\alpha)$
AB	$B.\tan\alpha$	$2\delta_f$	$S_u.B.2\delta_f.\tan\alpha$

$$= 2.S_u.B.\delta_f.(\tan\alpha + 1/(\sin\alpha.\cos\alpha)) \tag{14}$$

Equating Equations (13) and (14) gives:

$$Q = 2.S_u.B.(\tan\alpha + 1/(\sin\alpha.\cos\alpha)) + p_o.B$$

As this is an unsafe estimate the value of α which produces the smallest value of Q is required. This occurs when α=35.3°. Therefore:

$$Q^{Unsafe} = 5.66S_u.B + p_o.B$$

4. Safe analysis

To perform an analysis which satisfies the safe theorem a stress field covering the complete soil mass must be postulated (guessed). The stress field must satisfy the equations of equilibrium, not exceed yield at any point, and be in equilibrium with the applied tractions and/or body forces.

4.1. Stress discontinuities

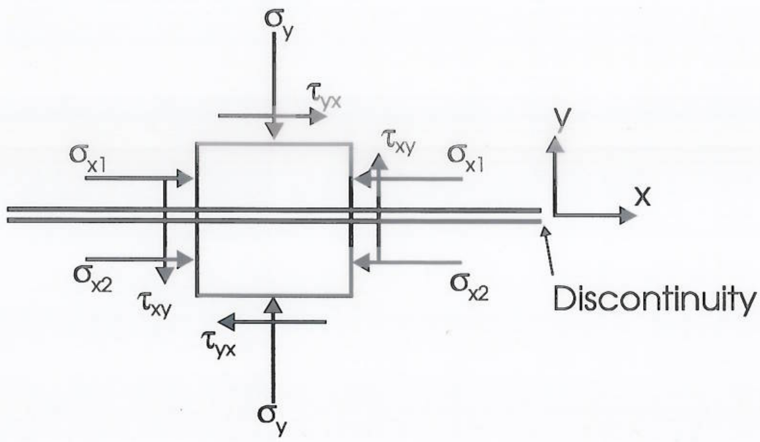

Figure 17: Stress discontinuity

As it is difficult to establish continuously varying stress fields that satisfy the restrictions of the safe theorem considerable use is made of zones of uniform stress separated by stress discontinuities. A stress discontinuity is shown in Figure 17. Considering the equilibrium of the element of soil that straddles the discontinuity it can be shown that for equilibrium only the direct stress (σ_y) normal to the direction of the discontinuity and the associated shear stress (τ_{xy}) must be continuous. In particular there can be a jump in the magnitude of the direct stress in the direction of the discontinuity from one side of the discontinuity to the other.

4.2. Examples

Critical height of a vertical cut in Tresca soil (undrained clay)

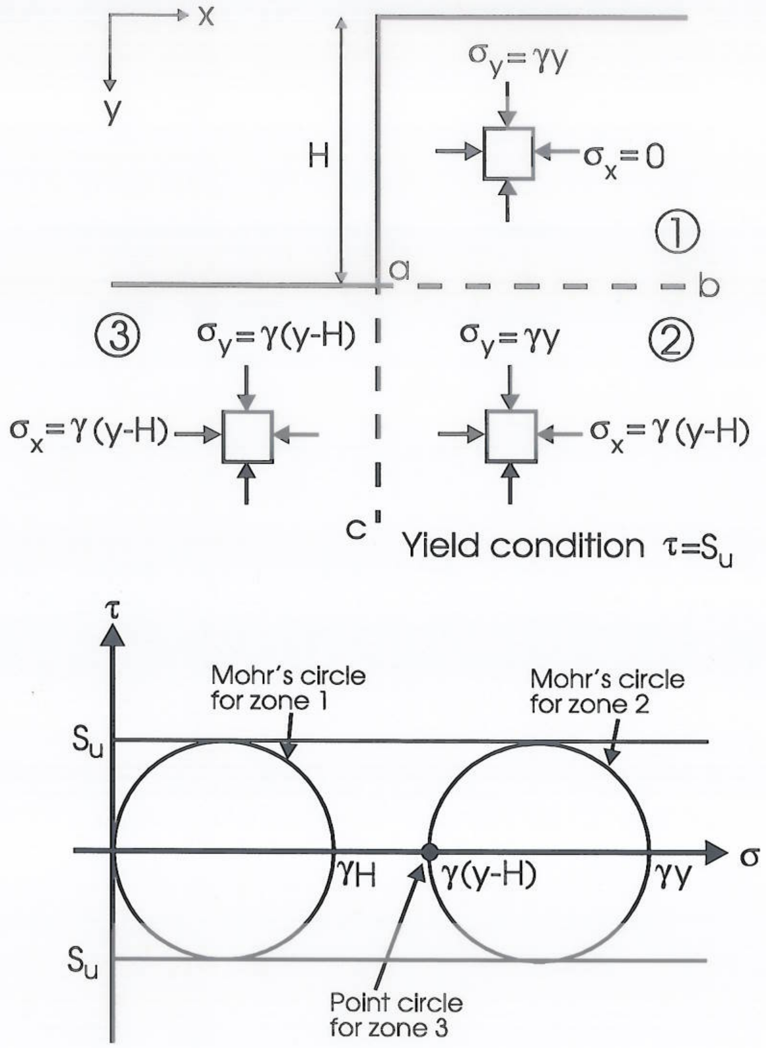

Figure 18: Postulated stress field and equilibrium in Tresca conditions

Assume stress discontinuities ab and ac. From Mohr diagrams the stresses in regions 1 and 2 approach yield simultaneously as H is increased. As this is a safe solution we require the maximum value of H. This occurs when the Mohr's circles for zone 1 and 2 just reach the yield condition. Therefore:

$$H^{Safe} = 2.S_u/\gamma$$

Critical height of a vertical cut in dry Mohr-Coulomb soil

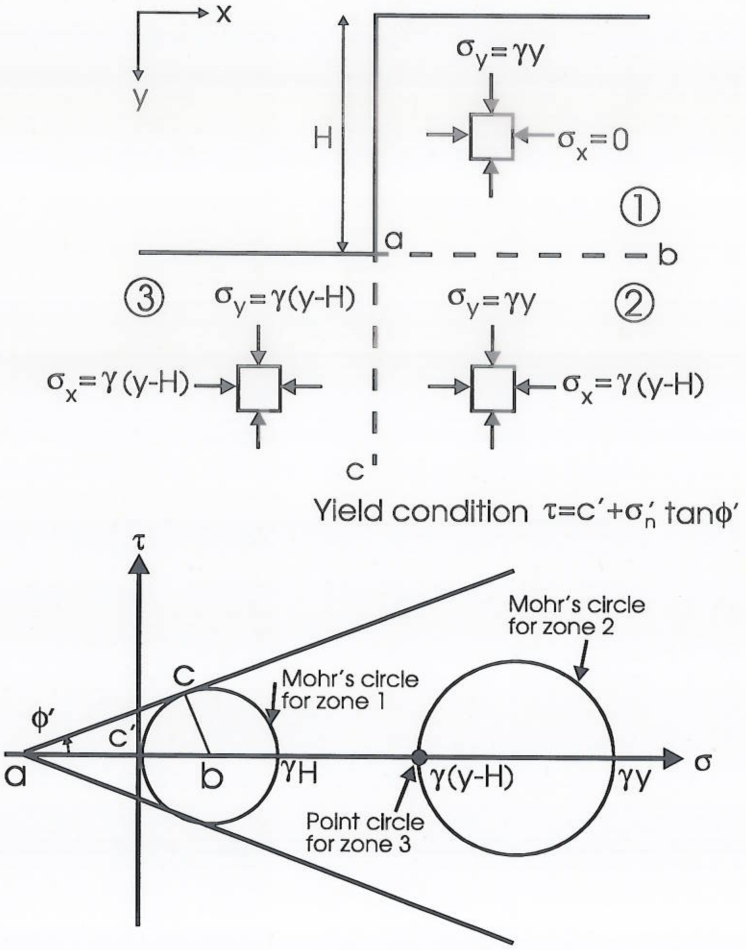

Figure 19: Postulated stress filed and equilibrium in Mohr-Coulomb conditions

Assume stress discontinuities ab and ac. From Mohr diagrams the stresses in region 1 approach yield first as H is increased. As this is a safe solution we require the maximum value of H. This occurs when the Mohr's circle for zone 1 just reaches the yield condition. Therefore:

$$\sin\phi' = L_{cb}/L_{ab} = \tfrac{1}{2}\gamma H/(\tfrac{1}{2}\gamma H + c'.\cot\phi')$$

Rearranging gives:

$$H^{Safe} = 2.c'.\cos\phi'/(\gamma(1-\sin\phi')) = 2.c'.\tan(\tfrac{1}{4}\pi + \tfrac{1}{2}\phi')/\gamma$$

Footing on Tresca soil (undrained clay)

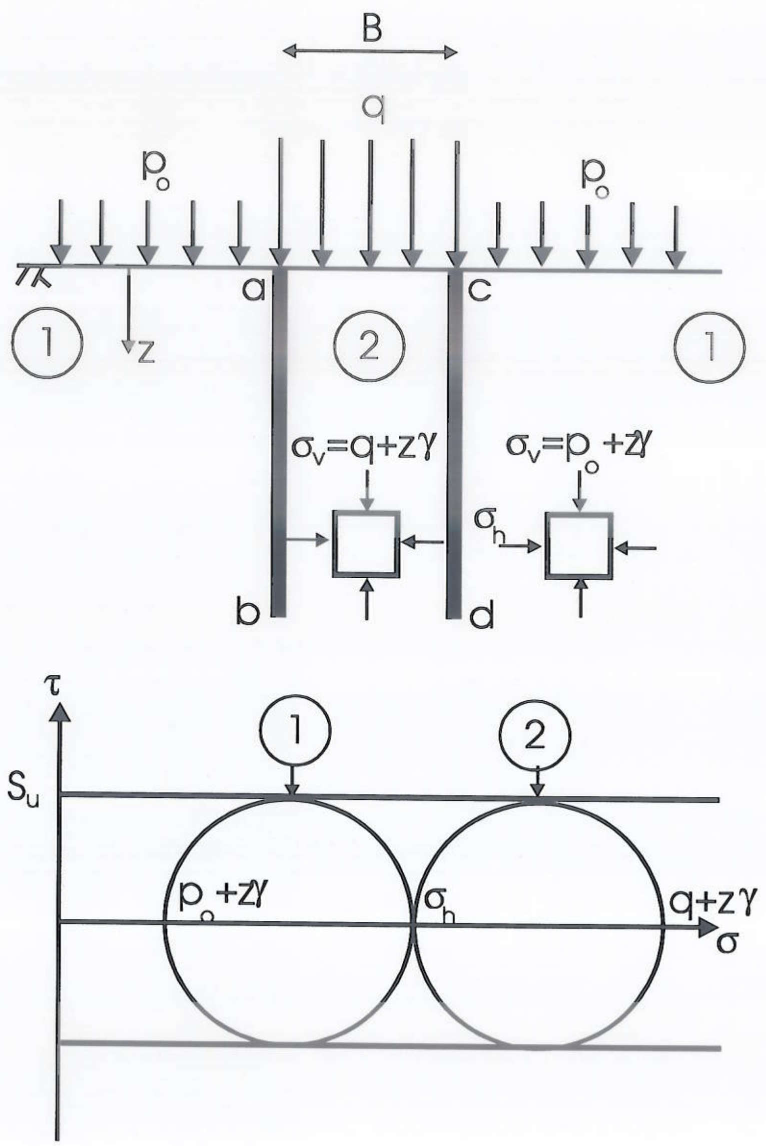

Figure 20: Postulated stress field and equilibrium for Tresca conditions

Assume vertical stress discontinuities ab and cd. From Mohr diagrams the stresses in regions 1 and 2 approach yield simultaneously as q is increased. As this is a safe solution we require the maximum value of q. This occurs when the Mohr's circles for zones 1 and 2 just reach the yield condition. Therefore:

$$q^{Safe} = 4S_u + p_o$$

$$Q^{Safe} = 4.B.S_u + B.p_o$$