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An extension of Bishop's simplified method of slope stability analysis to three dimensions

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A new algorithm for three-dimensional limit equilibrium slope stability analysis is presented, based on a direct extension of Bishop's simplified method of slices. This new variant of the method of columns retains the conceptual and mathematical simplicity of Bishop's original simplified method. Existing computer programs can be easily modified for three-dimensional analysis. Comparisons with two other computer-based methods indicate somewhat higher factors of safety, due to the inclusion of intercolumn forces. Comparisons with closed form wedge analyses derived from rock mechanics theory show excellent agreement for both frictional and cohesive materials.

INTRODUCTION

The assumption of two-dimensional (plane strain) geometry is common to all routine limit equilibrium slope stability calculations, with the exception of wedge analyses of rock slopes, yet no landslide occurs in plane strain. End effects, lateral curvature of the sliding surface, plan curvature of the slope and lateral non-homogeneity are all routinely neglected to compress a slope stability problem into the limited framework of a two-dimensional model.

The effect of such simplifications on the precision of the stability index is in many cases negligible. There are cases, however, where the error introduced by neglecting the three-dimensional character of a potential landslide cannot reasonably be brushed aside without a quantitative assessment of its influence. Examples that can be given include shafts and other deep, narrow excavations, corners and re-entrants of natural or excavated slopes, or slopes which fail by a narrow localized failure—possibly due to a concentrated load at the crest. In all such cases a factor of safety that is greater than that predicted by a two-dimensional analysis of the central cross-section would be expected. Examples can also be encountered where the two-dimensional analysis produces non-conservative results, such as ends and outside corners of embankments, ridges or conical fills (e.g. artificial islands).

Despite this, only a few attempts at three-dimensional slope stability analysis have been published. The most general approach is the method of columns, which is analogous to the two-dimensional method of slices. The sliding body is divided into a series of vertical columns of a rectangular cross-section (Figs 1 and 2). Hovland (1977) developed a limiting equilibrium stability algorithm for an arbitrary three-dimensional curved sliding surface, assuming zero stresses on all the vertical intercolumn surfaces. This corresponds to the ordinary method of slices in two dimensions (Bishop, 1955). Chen & Chameau (1983) extended the method and took intercolumn forces into account using a set of original and conceptually rather complicated assumptions.

The method described here is based on a direct extension of the assumptions of Bishop (1955) into three dimensions. The method was developed independently by the Author in December 1983. It is perhaps not surprising that the same algorithm was developed earlier by at least two researchers and is described both by Hutchison (1981) and by Humphrey & Dunne (1982).

In their original plane strain application, Bishop's assumptions led to the derivation of Bishop's widely used simplified method of slices. This yields factor of safety estimates that are often significantly greater than those resulting from the ordinary method and proves therefore to be less conservative as it accounts for the positive effect of interslice forces.

Bishop's simplified method has also been shown to produce remarkably accurate results when compared with more recent and sophisticated techniques (e.g. Spencer (1967)). As will be seen in the following section, its extension into three dimensions involves no additional assumptions and is fully physically analogous to Bishop's original simplified method. The three-dimensional method would therefore be intuitively expected to exhibit as good a performance as the original method. The method neglects vertical inter-

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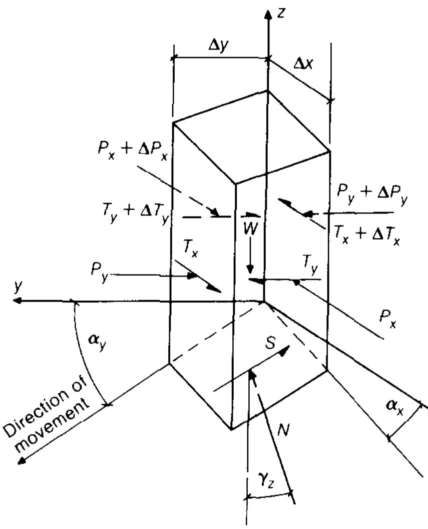

Fig. 1. Forces acting on a single column: vertical intercolumn shear forces, which are neglected in the analysis, are not shown; force application points are approximate; the pore pressure resultant is not shown but would act in the line of the normal force N

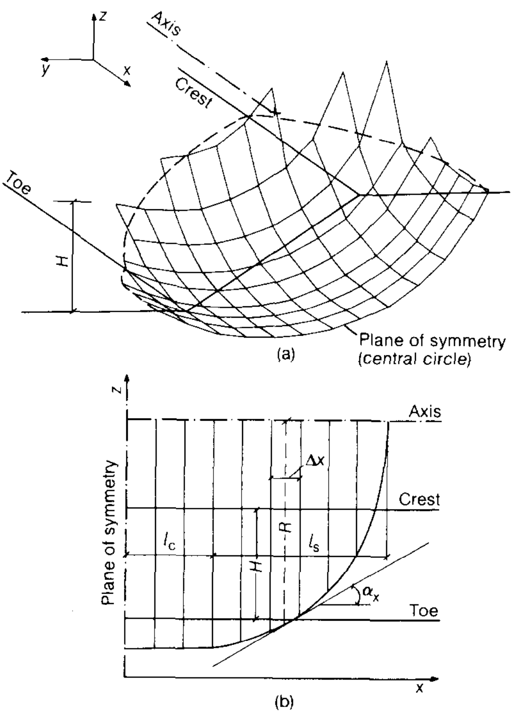

Fig. 2. (a) Isometry of a rotational sliding body, symmetrical with respect to a central vertical plane, divided into a series of columns (only the bases of the 'active' columns are shown); (b) vertical cross-section of the sliding body in the plane of the axis of rotation (each figure represents only one half of the body)

column shear, but not intercolumn normal forces and horizontal shear forces.

The algorithm has been implemented in a microcomputer program (CLARA-3) and calculations have been carried out to provide a comparison with some earlier results by Chen & Chameau (1983) and Hovland (1977).

DERIVATION OF ALGORITHM

The key formulae of the algorithm are derived by adopting literally the two assumptions of Bishop (1955)

(*a*) vertical shear forces acting on both the longitudinal and the lateral vertical faces of each column can be neglected in the equilibrium equations

(*b*) the vertical force equilibrium equation of each column and the summary moment equilibrium equation of the entire assemblage of columns are sufficient conditions to determine all the unknown forces.

It is implicit in the second assumption that both the lateral and the longitudinal horizontal force equilibrium conditions are neglected, as they are in the two-dimensional model.

With reference to Fig. 1, the total normal force N acting on the base of a column can be derived from the vertical force equilibrium equation

$$W = N_z + S_z = N \cos \gamma_z + \left[\frac{(N - uA) \tan \phi}{F} + \frac{cA}{F}\right] \sin \alpha_y \quad (1)$$

where W is the total weight of the column, u is the pore pressure acting in the centre of the column base, A is the true base area, c is the cohesion, ϕ is the friction angle and F is the factor of safety. From this

$$N = \frac{W - cA \sin \alpha_y/F + uA \tan \phi \sin \alpha_y/F}{m_\alpha} \quad (2)$$

where

$$m_\alpha = \cos \gamma_z\left(1 + \frac{\sin \alpha_y \tan \phi}{F \cos \gamma_z}\right) \quad (3)$$

The true area of the column base, A, has been derived by Hovland (1977)

$$A = \Delta x \, \Delta y \, \frac{(1 - \sin^2 \alpha_x \sin^2 \alpha_y)^{1/2}}{\cos \alpha_x \cos \alpha_y} \quad (4)$$

The angle γ_z between the direction of the normal force N and the vertical axis can be derived from geometrical considerations as

$$\cos \gamma_z = \left(\frac{1}{\tan^2 \alpha_y + \tan^2 \alpha_x + 1} \right)^{1/2} \quad (5)$$

The plan area of the sliding body is now divided into a series of columns arranged in rows of uniform width, as indicated in Fig. 2, their bases lying on a rotational surface related to a unique axis of rotation.

A moment equilibrium equation for an assemblage of j columns can be written as follows, since all the intercolumn forces cancel out against their respective reactions

$$\sum_{i=1}^{j} (N - uA) \frac{\tan \phi}{F} + \frac{cA}{F} = \sum_{i=1}^{j} W \sin \alpha_y \quad (6)$$

From this, with the substitution of N from equation (2)

$$F = \textstyle\sum_{i=1}^{j} [(W - uA \cos \gamma_z) \tan \phi + cA \cos \gamma_z]/m_\alpha \times (\sum_{i=1}^{j} W \sin \alpha_y)^{-1} \quad (7)$$

The normal intercolumn forces P and horizontal shear forces T are not neglected in the analysis, although they do not enter the equations and neither their magnitudes nor the position of their points of application need be known. This is an advantage inherited from Bishop's original simplified method.

Equation (7) is implicit in F and yields the required factor of safety when solved by an iterative procedure. For $\alpha_x = 0$ (i.e. a cylindrical failure surface), equation (7) will reduce to the well-known formula of Bishop's simplified method.

The similarity of the proposed algorithm to the original two-dimensional equivalent provides an opportunity for simple conversions of computer programs to three dimensions.

COMPARISON WITH OTHER METHODS

To demonstrate typical results of the proposed slope stability analysis method, a parametric study represented by fig. 9 of Chen & Chameau (1983) was repeated using CLARA-3.

The subject of the analysis is a homogeneous, fully drained slope 6·1 m high, with a horizontal-to-vertical inclination of 2·5:1. Each half of the symmetrical sliding body has a compound shape, consisting of a cylindrical part l_c wide and a semi-ellipsoidal part centred on the axis of rotation l_s wide (Fig. 2(b)). The cross-section of the cylindrical part corresponds to the 'critical' two-dimensional sliding circle as determined by Chen & Chameau (1983) for each of three different materials. The radii and the centre co-ordinates, y_c and z_c, of each circle are given in Table 1, together with the assumed Coulomb strength parameters of each material. To facilitate comparison, the circle geometries were scaled from fig. 8 of Chen & Chameau (1983) and no independent search for the critical circles was done. The unit weight of the material, unspecified by Chen and Chameau, was assumed to be 20 kN/m^3. The two-dimensional factors of safety, obtained by Bishop's simplified method, are shown in Table 1.

The narrowness of the sliding body is a function of the two parameters l_s and l_c defined earlier. Fig. 3(a), reproduced from Chen & Chameau (1983), shows the ratio between the three-dimensional factor of safety F_3 calculated by the program LEMIX, and the corresponding two-dimensional result for the central cross-section, F_2. This diagram was criticized by Hutchinson & Sarma (1985), who pointed out that the ratio F_3/F_2 can approach, but should not fall below, 1·0.

A corresponding diagram obtained by CLARA-3 based on Bishop's three-dimensional simplified method is shown in Fig. 3(b). The factors of safety were obtained with a column assembly that was similar to that shown in Fig. 2, the number of columns being in the range 120–150 in each half of the sliding body. The factor F_2 was calculated using Bishop's simplified method of slices. The new results generally indicate a somewhat strong-

Table 1. Input parameters for the parametric study*

Circle	Cohesion: kPa	Friction angle: deg	Radius: m	y_c:† m	z_c:† m	Plane strain factor of safety
1	0	40	14·10	4·38	13·43	2·713
3	14·4	25	14·49	6·17	12·20	2·763
5	28·7	15	15·33	5·56	11·47	2·872

* Dimensions scaled from fig. 8 of Chen & Chameau (1983).
† Cartesian co-ordinates of the sliding circle's centre, relative to an origin placed at the toe of the slope.

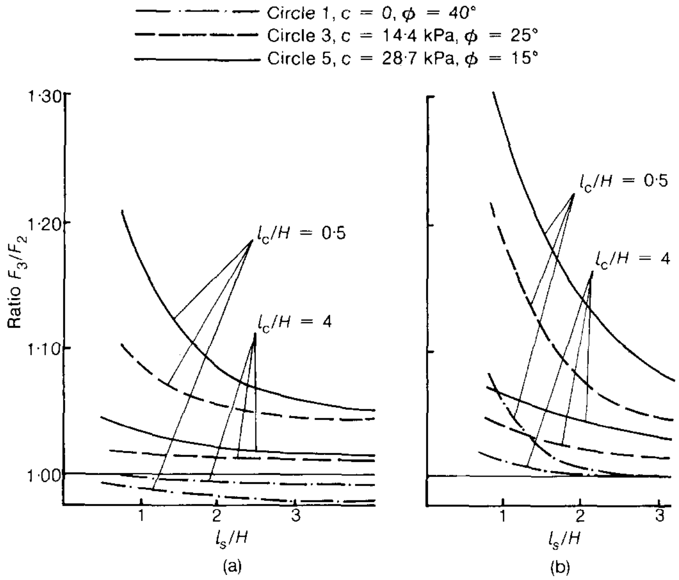

Fig. 3. (a) Ratio of three- and two-dimensional factors of safety for a 2·5 horizontal in 1 vertical slope, calculated by the program LEMIX (after Chen & Chameau (1983)); (b) the same ratio calculated by the present method using the program CLARA-3

er influence of the three-dimensional geometry than shown by Fig. 3(a). Of particular interest are the curves obtained for circle 1 with zero cohesion, which confirm the opinion of Hutchinson & Sarma (1985).

The LEMIX results (Chen & Chameau, 1983) produce greater factors of safety than those due to the ordinary method designated here as OM (Hovland, 1977). As an example, for circle 1, $l_c = 0{\cdot}5H$ and $l_s = H$, the percentage difference (OM − LEMIX)/LEMIX equals −14·9% (table 1 in Chen & Chameau (1983)). For circle 5, $l_c = 4H$ and $l_s = 2H$ it is −4·8%. The corresponding percentage differences between the ordinary method and Bishop's simplified method are −20% and −7%.

The new program was also tested by analysing several examples of wedge stability, amenable to closed form solution using the routine methods of rock mechanics (e.g. Hoek & Bray (1977)). Since the present version of CLARA-3 cannot accept other than rotational geometry, the wedge geometry was approximated by a rotational body with a very large radius and a V-shaped cross-section.

Both symmetrical and asymmetrical fully drained wedges have been analysed, with material properties including both friction and cohesion. Approximately 100 columns have been used in each computer solution. The factors of safety computed by Bishop's simplified method approached within 3% of the closed form solutions in all cases investigated.

CONCLUSIONS

A method of three-dimensional limit equilibrium slope stability analysis has been presented, based on a direct extension of Bishop's well-known simplified method. Preliminary results indicate higher factors of safety than estimated by earlier variants of the method of columns. In particular, the method described removes the apparent paradox of three-dimensional factors of safety that are smaller in magnitude than those corresponding to the plane strain analysis of the central cross-section. The trend of these results is consistent with the critical comments of Hutchinson & Sarma (1985) and also with the concept of the wedge factor established in rock mechanics. The wedge factor may approach, but cannot fall below, 1·0 (Hoek & Bray (1977), fig. 96).

The method has also been shown to be capable of duplicating factors of safety predicted by the analytical methods of wedge stability analysis for both frictional and cohesive materials. It is felt that no further confirmation of the new algorithm can be supplied, until it is possible to back-calculate the stability of landslides or centrifuge models.

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