Stress field method

1. Introduction

In this approach the soil is assumed to be at the point of failure everywhere and a solution is obtained by combining the failure criterion with the equilibrium equations. The lecture presents the main equations for two dimensional, plane strain, problems and shows a derivation of the equations for Rankine states of stress.

2. Stress field equations

For plane strain conditions and the Mohr-Coulomb model we have the following:

Equilibrium equations (see Figure 1):

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0$$
$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = \gamma \qquad (1)$$

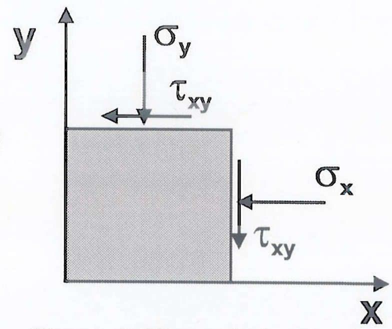

Figure 1: Element stresses in plane strain conditions

Mohr-Coulomb criterion (from Figure 2):

$$\sigma_1' - \sigma_3' = 2c'\cos\varphi' + (\sigma_1' + \sigma_3')\sin\varphi' \qquad (2)$$

Nothing that:

$$s = c'\cot\varphi' + \tfrac{1}{2}(\sigma_1' + \sigma_3') = c'\cot\varphi' + \tfrac{1}{2}(\sigma_x' + \sigma_y')$$
$$t = \tfrac{1}{2}(\sigma_1' - \sigma_3') = [\tfrac{1}{4}(\sigma_x' - \sigma_y')^2 + \tau_{xy}^2]^{0.5} \qquad (3)$$

and substituting into Equation (2) gives the following alternative equations for the Mohr-Coulomb criterion:

$$t = s\sin\varphi' \qquad (4)$$

$$[\tfrac{1}{4}(\sigma_x' - \sigma_y')^2 + \tau_{xy}^2]^{0.5} = [c'\cot\varphi' + \tfrac{1}{2}(\sigma_x' + \sigma_y')]\sin\varphi' \qquad (5)$$

The equilibrium equations (1) and the failure criterion (5) provide three equations in terms of three unknowns. It is therefore theoretically possible to obtain a solution.

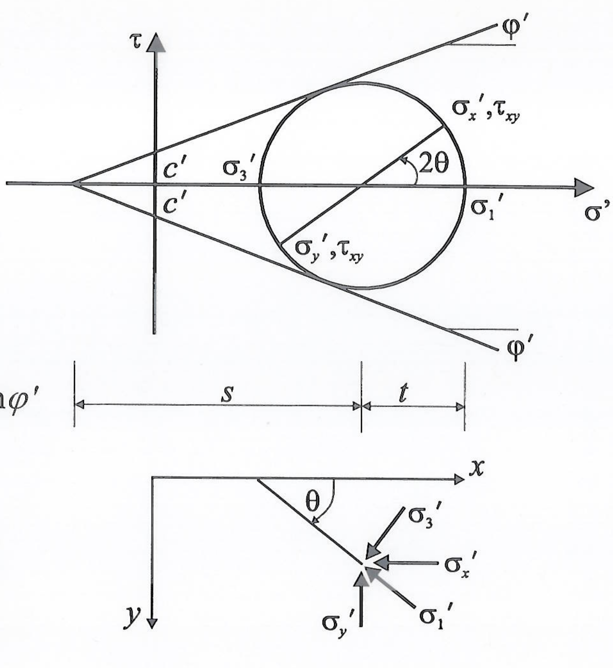

Figure 2: Mohr's circle of stress

Combining the above equations gives, after considerable manipulations, the following system of equations:

$$
\begin{aligned}
&(1+\sin\varphi'\cos2\theta)\frac{\partial s}{\partial x}+\sin\varphi'\sin2\theta\frac{\partial s}{\partial y}+2s\sin\varphi'(\cos2\theta\frac{\partial\theta}{\partial y}-\sin2\theta\frac{\partial\theta}{\partial x})=0\\
&\sin\varphi'\sin2\theta\frac{\partial s}{\partial x}+(1-\sin\varphi'\cos2\theta)\frac{\partial s}{\partial y}+2s\sin\varphi'(\sin2\theta\frac{\partial\theta}{\partial y}+\cos2\theta\frac{\partial\theta}{\partial x})=\gamma
\end{aligned}
\tag{6}
$$

These two partial differential equations can be shown to be of the hyperbolic type. A solution is obtained by considering the characteristic directions and obtaining equations for the stress variation along these characteristics (Atkinson & Potts, (1975). The differential equations of the stress characteristics are:

$$
\begin{aligned}
\frac{dy}{dx} &= \tan[\theta-(\pi/4-\varphi'/2)]\\
\frac{dy}{dx} &= \tan[\theta+(\pi/4-\varphi'/2)]
\end{aligned}
\tag{7}
$$

Along these characteristics the following equations hold:

$$
\begin{aligned}
ds - 2s\tan\varphi'\,d\theta &= \gamma(dy-\tan\varphi'\,dx)\\
ds + 2s\tan\varphi'\,d\theta &= \gamma(dy+\tan\varphi'\,dx)
\end{aligned}
\tag{8}
$$

Equations (7) and (8) provide four total differential equations with four unknowns x, y, s, and θ and in principle can be solved mathematically. However, to date, it has only been possible to obtain analytical solutions for very simple problems and/or if the soil is assumed to be weightless, i.e. $\gamma=0$. In general, they are solved numerically by adopting a finite difference approximation.

Solutions based on the above equations usually only provide a partial stress field which does not cover the whole soil mass but is restricted to the zone of interest. In general they are therefore not Lower Bound (Safe) solutions.

The above equations provide what appears to be and some times is static determinacy in the sense that there are the same number of equations as unknown stress components. In most practical problems, however, the boundary conditions involve both forces and displacements and the static determinacy is misleading. Compatibility is not considered in this approach.

The equivalent equations for a Tresca material can be obtained by substituting S_u and 0 for c' and ϕ' respectively in the above equations.

Rankine active and passive stress fields and the earth pressure tables obtained by Sokolvski (1960, 1965) and used in some codes of practice are examples of stress field solutions. Stress fields also form the basis of analytical solutions to the bearing capacity problems.

Examples of stress fields likely to be developed for a tunnel or a retaining wall problem are shown in Figure 3. The figure shows likely 'meshes' developed by characteristic directions, starting from problem boundaries.

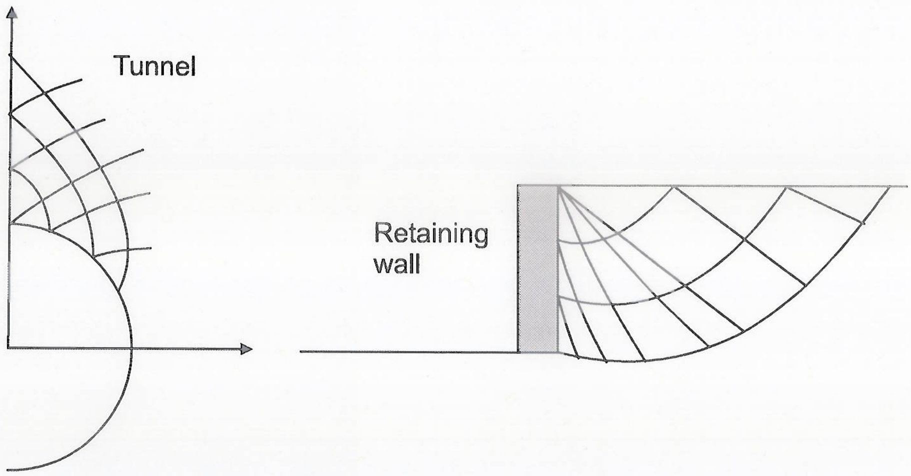

Figure 3: Stress fields for tunnel and retaining wall problems

Further to this, Figure 4 shows a task to be solved by applying the stress field analysis method. Namely, a circular tunnel of diameter R, at a depth 'a' below the ground surface is supported by the internal pressure σ_T=1kPa. What is the magnitude of the vertical stress σ_y, acting on the ground surface, that will cause the tunnel to collapse for different tunnel depths 'a'? A typical mesh of characteristic directions developed for this problem is shown in Figure 5. A solution for the case when a=R/2 is presented in Figure 6.

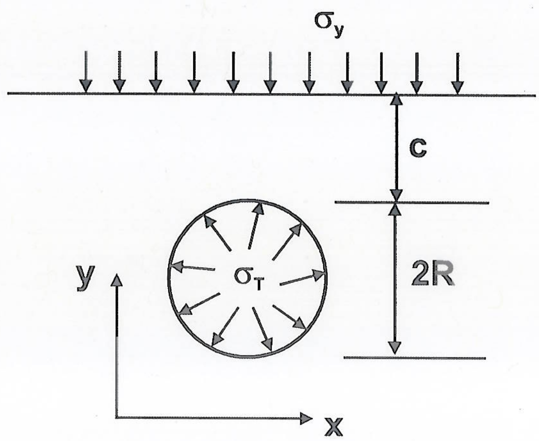

Figure 4: Geometry of a tunnel failure problem

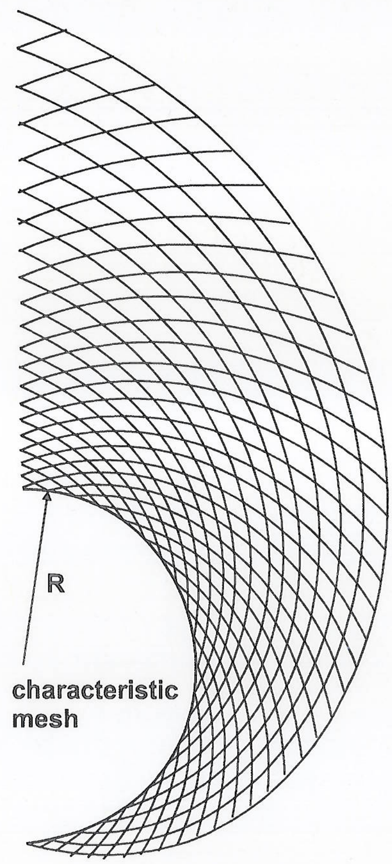

Figure 5: Mesh of characteristic directions for a tunnel

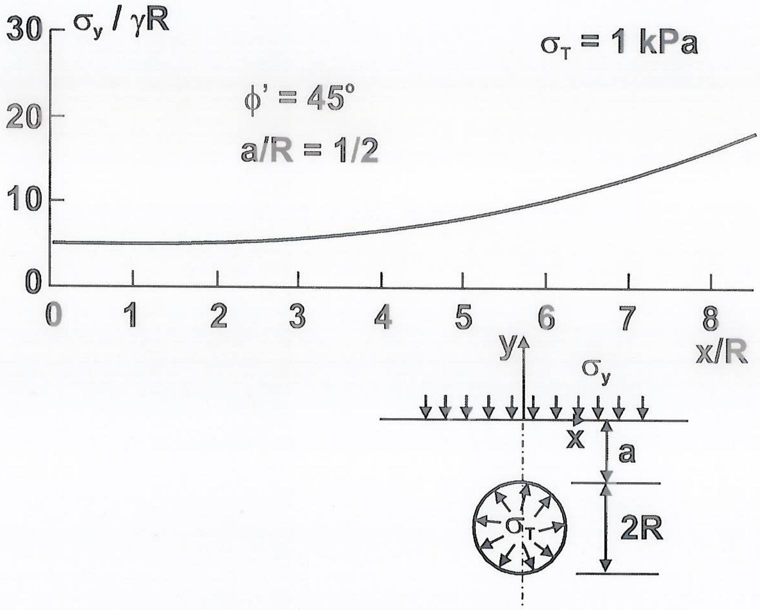

Figure 6: Solution for a = R/2

The failure criterion applied to this problem is that of Mohr-Coulomb, with $c'=0$ and $\phi'=45°$. The graph in Figure 6 shows the magnitude of σ_y, normalised by R and bulk unit weight of soil, γ, versus the normalised distance x from the tunnel axis. What can you discuss about this solution?

3. Rankine states of stress

As mentioned above, one of the most common applications of the stress field method in geotechnical engineering is Rankine's solution for the active and passive earth pressures in the ground. It will be shown that the expressions for K_a and K_p derived using this method are the same as those derived by Coulomb using the limit equilibrium approach.

Consider a semi infinite cohesionless soil mass in a K_o state of stress as shown in Figure 7.

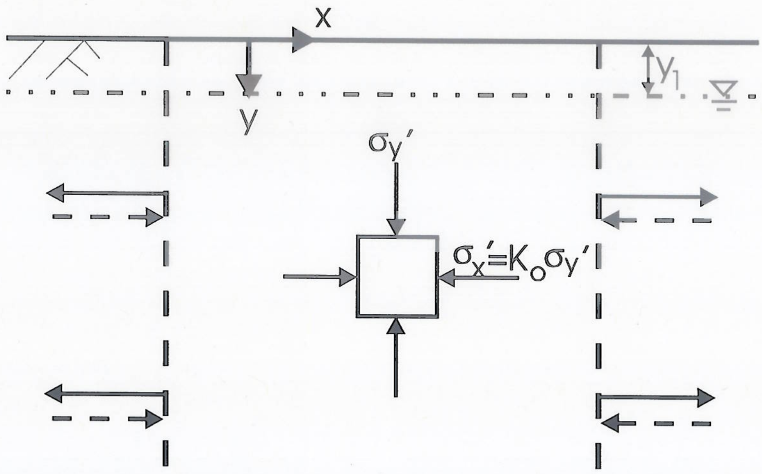

Figure 7: Schematic view of stress state in the ground

If the whole semi infinite soil mass is subjected to lateral extension (← →) the horizontal effective stress, σ_x', decreases from its initial value, $K_o.\sigma_y'$, while the vertical effective stress, σ_y', remains unchanged. As this occurs the shear stress increases until the shear strength of the soil is progressively mobilised. This continues until the shear strength is mobilised throughout the soil mass and a condition is reached at which there can be no further reduction in horizontal stress. This minimum value of horizontal stress is defined as the *active* earth pressure. The stress conditions in an element of soil at this limiting condition are:

$$\sigma_y' = \gamma_s.y_1 + \gamma'.(y-y_1); \quad \sigma x' = K_a.\sigma_y'$$

where K_a is the coefficient of active earth pressure.

If the whole semi infinite soil mass is subjected to lateral compression (→ ←) the horizontal effective stress, σ_x', increases from its initial value, $K_o.\sigma_y'$, while the vertical effective stress, σ_y', remains unchanged. As σ_x' increases above σ_y', the shear strength of the soil is mobilised. Again, a limiting condition is reached in which there is no further increase in the horizontal stress. This maximum value of horizontal stress is defined as the *passive* earth pressure. The stress conditions in an element of soil at this limiting condition are:

$$\sigma_y' = \gamma_s.y_1 + \gamma'(y-y_1); \qquad \sigma_x' = K_p.\sigma_y'$$

where K_p is the coefficient of passive earth pressure.

Because of the uniform lateral extension/compression of the whole soil mass in the active/passive situations, a uniform stress field exists. In both cases τ_{xy}=0. In the active case σ_y' is the major and σ_x' the minor principal effective stress. The reverse is true for the passive situation.

Considering an element of soil with strength parameters, c and ϕ', the following Mohr's circles can be drawn in Figure 8:

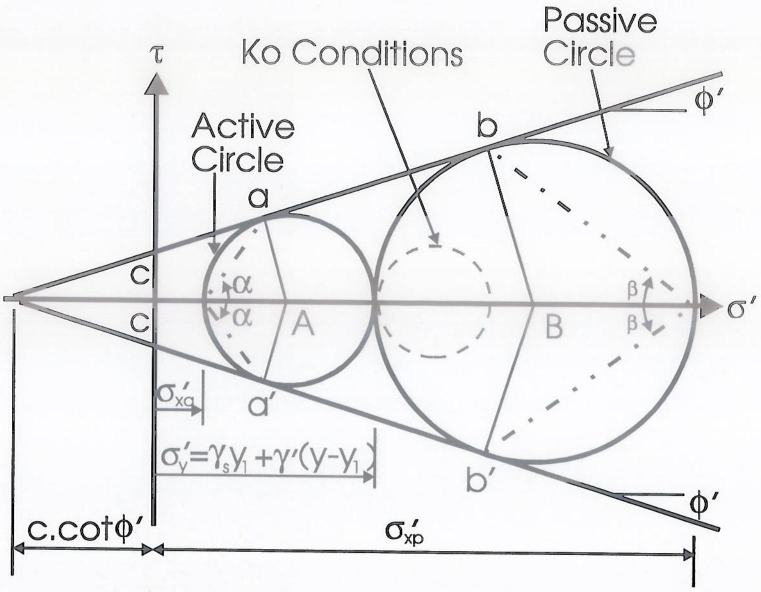

Figure 8: Mohr's circles for active and passive stress conditions in the ground

ACTIVE CASE	PASSIVE CASE
$\sigma_{xa}' = \sigma_y' - 2.Aa$	$\sigma_{xp}' = \sigma_y' + 2.Bb$
$\sin\phi' = Aa/(½(\sigma_{xa}' + \sigma_y') + c.\cot\phi')$	$\sin\phi' = Bb/(½(\sigma_{xp}' + \sigma_y') + c.\cot\phi')$
Combining above equations:-	Combining above equations:-
$\sigma_{xa}' = \sigma_y' - (\sigma_{xa}' + \sigma_y').\sin\phi' - 2c.\cos\phi'$	$\sigma_{xp}' = \sigma_y' + (\sigma_{xp}' + \sigma_y').\sin\phi' + 2c.\cos\phi'$
$\sigma_{xa}'(1+\sin\phi') = \sigma_y'(1-\sin\phi') - 2c.\cos\phi'$	$\sigma_{xp}'(1-\sin\phi') = \sigma_y'(1+\sin\phi') + 2c.\cos\phi'$
$\sigma_{xa}' = \sigma_y'(1-\sin\phi')/(1+\sin\phi') - 2c.\cos\phi'/(1+\sin\phi')$	$\sigma_{xp}' = \sigma_y'(1+\sin\phi')/(1-\sin\phi') + 2c.\cos\phi'/(1-\sin\phi')$
$\sigma_{xa}' = \sigma_y'.\tan^2(\pi/4-\phi'/2) - 2c.\tan(\pi/4-\phi'/2)$	$\sigma_{xp}' = \sigma_y'.\tan^2(\pi/4+\phi'/2) + 2c.\tan(\pi/4+\phi'/2)$
$\sigma_{xa}' = \sigma_y'K_a - 2c.K_{ac}$	$\sigma_{xp}' = \sigma_y'K_p + 2c.K_{pc}$
Planes of maximum stress obliquity (i.e planes on which the failure criterion is mobilised) are inclined at α to the x axis. From Mohr's circle:-	Planes of maximum stress obliquity (i.e planes on which the failure criterion is mobilised) are inclined at β to the x axis. From Mohr's circle:-
$\alpha = \pi/4+\phi'/2$	$\beta = \pi/4-\phi'/2$
Therefore planes of maximum stress obliquity are inclined at $\pm(\pi/4+\phi'/2)$ to the x axis OR $\pm(\pi/4-\phi'/2)$ to the direction of the major principal effective stress, σ_y'.	Therefore planes of maximum stress obliquity are inclined at $\pm(\pi/4-\phi'/2)$ to the x axis OR $\pm(\pi/4-\phi'/2)$ to the direction of the major principal effective stress, σ_{xp}'.

The above Rankine states of stress are simple stress field solutions. The planes of maximum stress obliquity (i.e. planes on which the failure condition is satisfied) are stress characteristics, as indicated in Figure 9 for active and passive case respectively.

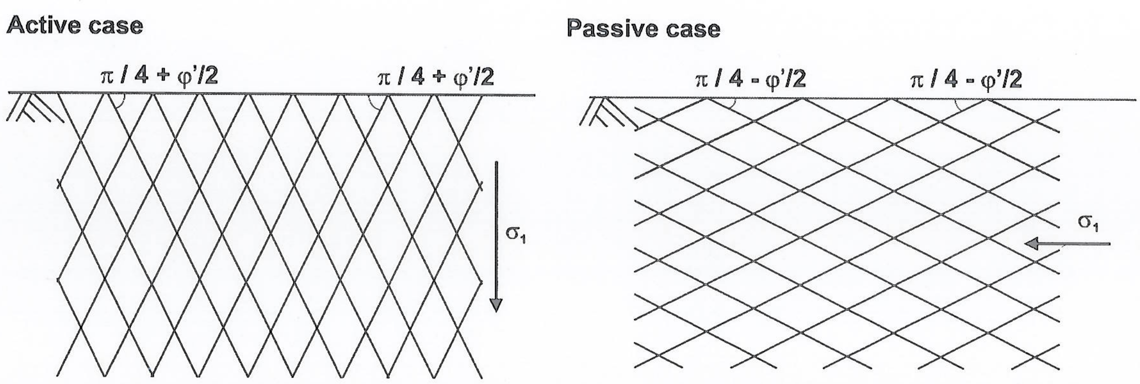

Figure 9: Characteristic directions for active and passive earth pressures

The above solutions can be extended to the situation of a cohesionless soil with a sloping ground surface:

ACTIVE CASE

$$\sigma_{xa}' = \sigma_y'.\cos\beta.\frac{(\cos\beta-(\cos^2\beta-\cos^2\phi')^{½})}{(\cos\beta+(\cos^2\beta-\cos^2\phi')^{½})}$$

PASSIVE CASE

$$\sigma_{xp}' = \sigma_y'.\cos\beta.\frac{(\cos\beta+(\cos^2\beta-\cos^2\phi')^{½})}{(\cos\beta-(\cos^2\beta-\cos^2\phi')^{½})}$$

Application of Rankine states of stress

As the solutions are based on the assumption that the whole soil mass is either subject to lateral extension or compression, further approximations are necessary when applying the results to boundary value problems. For example, consider the retaining wall problem shown in the Figure 10.

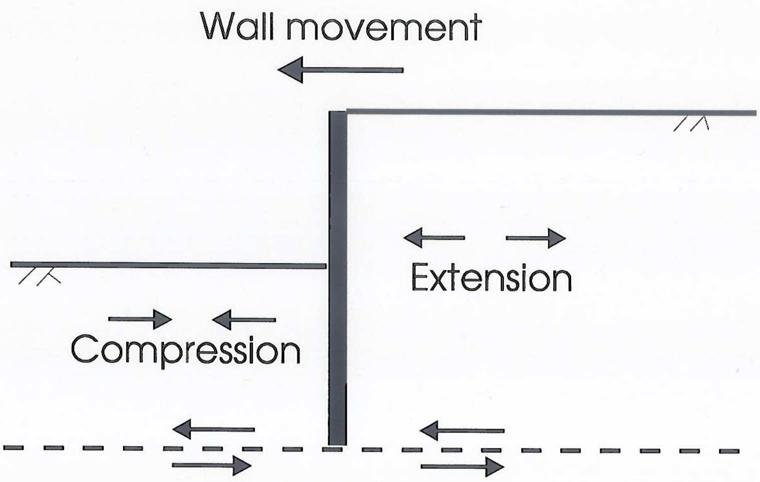

Figure 10: Rankine states of stress around a retaining wall

As shown schematically in Figure 10, the wall is assumed to displace to the left. The soil immediately to the right of the wall is subjected to lateral extension and the soil immediately to the left of the wall is subjected to lateral compression. If the wall is smooth so that no shear stresses τ_{xy} are induced at the interface between soil and wall, Rankine stress states are assumed to occur in the zones of soil immediately adjacent to the wall. The soil below the wall constrains movement of the soil above and thus imparts shear stresses, τ_{xy}. Although this violates the assumptions of the Rankine theory, the effects are likely to be small and are therefore ignored. Another consequence of the constraining effect of the soil beneath the wall is to laterally restrict the extent of the zones of compression and extension adjacent to the wall. Thus the Rankine assumption of the whole soil mass being subject to either compression or extension is violated. Again this effect is assumed to be small.

If the wall did not have a vertical front or back, or if the ground surface was not horizontal but had a slope, then for the Rankine theory to apply the shear stress mobilised between soil and wall must be a fixed value (>0) consistent with the assumptions of the theory. This can be obtained from the Mohr's circle of stress.

The resultant force on either side of the wall can be found by integrating the active and passive earth pressures over the appropriate wall length. The distribution of earth pressures enables the position of these resultant forces to be located.

The major limitation of the Rankine theory when applied to retaining structures is the implicit assumption as to the magnitude of the mobilised friction between soil and wall. In many real cases this assumption is not valid. For example, for a vertical retaining wall in horizontal ground significant wall friction is likely to be mobilised, whereas the Rankine theory implicitly assumes this to be zero. Solutions which account for this wall friction are therefore required and can be found from advanced stress field solutions which solve Equations (7) and (8) numerically. Alternatively Limit Equilibrium or Limit Analyses solutions can be found.