FINITE ELEMENT METHODS FOR STOKES EQUATIONS

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1. Stokes Equations

In this section, we shall study the well posedness of the weak formulation of the steady-state Stokes equations

$$-\mu\Delta \boldsymbol{u} + \nabla p = \boldsymbol{f}, \tag{1}$$

$$-\operatorname{div} \boldsymbol{u} = 0, \tag{2}$$

where u can be interpreted as the velocity field of an incompressible fluid motion, and p is then the associated pressure, the constant μ is the viscosity coefficient of the fluid. For simplicity, we consider homogenous Dirichlet boundary condition for the velocity, i.e. $\boldsymbol{u}|_{\partial\Omega} = 0$ and $\mu = 1$.

Multiplying test function $\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega)$ to the momentum equation (1) and $q \in L^2(\Omega)$ to the mass equation (2), and applying integration by part for the momentum equation, we obtain the weak formulation of the Stokes equations: Find $\boldsymbol{u} \in \boldsymbol{H}_0^1(\Omega)$ and a pressure $p \in L^2(\Omega)$ such that

$$(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) - (p, \operatorname{div} \boldsymbol{v}) = \langle \boldsymbol{f}, \boldsymbol{v}\rangle, \qquad \text{for all } \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) \tag{3}$$

$$-(\operatorname{div} \boldsymbol{u}, q) = 0 \qquad \text{for all } q \in L^2(\Omega). \tag{4}$$

The conditions for the well posedness of a saddle point system is known as inf-sup conditions or Ladyzhenskaya-Babuška-Breezi (LBB) condition; see *Chapter: Inf-sup conditions for operator equations* for details.

The setting for the Stokes equations:

- Spaces:

$$\mathbb{V} = \boldsymbol{H}_0^1(\Omega), \text{ with norm } |\boldsymbol{v}|_1 = \|\nabla \boldsymbol{v}\|,$$

$$\mathbb{P} = L_0^2(\Omega) = \{q \in L^2(\Omega), \int_\Omega q = 0.\}, \text{ with norm } \|p\|.$$

- Bilinear form:

$$a(\boldsymbol{u}, \boldsymbol{v}) = \mu \int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{v}, \qquad b(\boldsymbol{v}, q) = -\int_\Omega (\operatorname{div} \boldsymbol{v})\, q.$$

- Operator:

$$A = -\Delta : \boldsymbol{H}_0^1(\Omega) \mapsto \boldsymbol{H}^{-1}(\Omega), \qquad \langle Au, v\rangle = a(u,v) = \mu(\nabla u, \nabla v),$$

$$B = -\operatorname{div} : \boldsymbol{H}_0^1(\Omega) \mapsto L_0^2(\Omega), \qquad \langle Bv, q\rangle = b(v,q) = -(\operatorname{div} v, q),$$

$$B' = \operatorname{grad} : L_0^2(\Omega) \mapsto \boldsymbol{H}^{-1}(\Omega), \qquad \langle \operatorname{grad} q, v\rangle = b(v,q) = -(\operatorname{div} v, q).$$

Recall that we need to verify the following assumptions

(A)
$$\inf_{u\in\mathbb{Z}}\sup_{v\in\mathbb{Z}}\frac{a(u,v)}{|u|_1|v|_1}=\inf_{v\in\mathbb{Z}}\sup_{u\in\mathbb{Z}}\frac{a(u,v)}{|u|_1|v|_1}=\alpha>0.$$
(B)
$$\inf_{q\in\mathbb{P}}\sup_{v\in\mathbb{V}}\frac{b(v,q)}{|v|_1\|q\|}=\beta>0$$
(C)
$$a(u,v)\le C_a|u|_1|v|_1,\quad \text{for all } u,v\in\mathbb{V},$$
$$b(v,q)\le C_b|v|_1\|q\|,\quad \text{for all } v\in\mathbb{V},q\in\mathbb{P}.$$

Remark 1.1. A natural choice of the pressure space is $L^2(\Omega)$. Note that
$$\int_\Omega \operatorname{div}\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}=\int_{\partial\Omega}\boldsymbol{v}\cdot\boldsymbol{n}\,dS=0$$
due to the boundary condition. Thus div operator will map $\boldsymbol{H}_0^1(\Omega)$ into the subspace $L_0^2(\Omega)$, in which the pressure satisfying the Stokes equations is unique. But in $L^2(\Omega)$, it is unique only up to a constant.

Remark 1.2. By the same reason, for Stokes equations with non-homogenous Dirichlet boundary condition $\boldsymbol{u}|_{\partial\Omega}=\boldsymbol{g}$, the data $\boldsymbol{g}$ should satisfy the compatible condition
$$\int_{\partial\Omega}\boldsymbol{g}\cdot\boldsymbol{n}\,dS=\int_{\partial\Omega}\operatorname{div}\boldsymbol{u}\,\mathrm{d}\boldsymbol{x}=0.$$

Exercise 1.3. Prove
$$-\Delta=-\operatorname{grad}\operatorname{div}+\operatorname{curl}\operatorname{curl}$$
holds as an operator from $\boldsymbol{H}_0^1\to\boldsymbol{H}^{-1}$. Namely for all $\boldsymbol{u},\boldsymbol{v}\in\boldsymbol{H}_0^1$
$$(\nabla\boldsymbol{u},\nabla\boldsymbol{v})=(\operatorname{div}\boldsymbol{u},\operatorname{div}\boldsymbol{v})+(\operatorname{curl}\boldsymbol{u},\operatorname{curl}\boldsymbol{v}).$$
Therefore $\|\operatorname{div}\boldsymbol{u}\|\le\|\nabla\boldsymbol{u}\|$ for all $\boldsymbol{u}\in\boldsymbol{H}_0^1(\Omega)$.

Conditions (A) and (C) are easy to verify (the readers are encouraged to verify them). The key is the inf-sup condition (B) which is equivalent to either

- $\operatorname{div}:\boldsymbol{H}_0^1(\Omega)\to L_0^2(\Omega)$ is surjective, or
- $\operatorname{grad}:L_0^2(\Omega)\to\boldsymbol{H}^{-1}(\Omega)$ is injective and bounded below.

We shall construct a suitable function to verify the inf-sup condition (B).

Lemma 1.4. *For any $q\in L_0^2(\Omega)$, there exists a $\boldsymbol{v}\in\boldsymbol{H}_0^1(\Omega)$ such that*
$$\operatorname{div}\boldsymbol{v}=q,\quad \textit{and}\ \ \|\boldsymbol{v}\|_1\lesssim\|q\|_0.$$
Consequently the inf-sup condition (B) holds.

Proof. We consider a simpler case when Ω is smooth or convex and in two dimensions. We can solve the Poisson equation
$$\Delta\psi=q\ \text{ in }\ \Omega$$
$$\frac{\partial\psi}{\partial n}=0\ \text{ on }\partial\Omega.$$
The equation is well posed since $q\in L_0^2(\Omega)$. If we set $\boldsymbol{v}=\nabla\psi$, then $\operatorname{div}\boldsymbol{v}=\Delta\psi=q$ and $\|\boldsymbol{v}\|_1=\|\psi\|_2\lesssim\|p\|_0$ by the regularity result.

The remaining part is to verify the boundary condition. First $\boldsymbol{v}\cdot\boldsymbol{n}=\nabla\psi\cdot\boldsymbol{n}=0$ by the construction. To take care of the tangential component $\boldsymbol{v}\cdot\boldsymbol{t}$, we invoke the trace theorem for $H^2(\Omega)$ to conclude that: there exist $\phi \in H^2(\Omega)$ such that $\phi|_{\partial\Omega} = 0$ and $\nabla\phi \cdot \boldsymbol{n} = \boldsymbol{v}\cdot\boldsymbol{t}$ and $\|\phi\|_2 \lesssim \|\boldsymbol{v}\|_1$. Let $\tilde{\boldsymbol{v}} = \operatorname{curl}\phi$. We have

$$\begin{aligned} &\operatorname{div}\tilde{\boldsymbol{v}} = 0, \\ &\tilde{\boldsymbol{v}}\cdot\boldsymbol{n} = \operatorname{curl}\phi\cdot n = \operatorname{grad}\phi\cdot\boldsymbol{t} = 0, \\ \text{and}\ \ &\tilde{\boldsymbol{v}}\cdot\boldsymbol{t} = -\operatorname{grad}\psi\cdot\boldsymbol{n} = -\boldsymbol{v}\cdot\boldsymbol{t}. \end{aligned}$$

Then we set $\boldsymbol{v}_q = \boldsymbol{v} + \tilde{\boldsymbol{v}}$ to obtain the desired result.

If the domain is not smooth, we can still construct such ψ; see [2, 8, 5]. □

Remark 1.5. Since

$$(\operatorname{div}\boldsymbol{v}, q) \le \|\operatorname{div}\boldsymbol{v}\|\|q\| \le \|\nabla\boldsymbol{v}\|\|q\|,$$

we have a upper bound on the inf-sup constant

$$\beta = \inf_{q\in\mathbb{P}}\sup_{\boldsymbol{v}\in\mathbb{V}} \frac{(\operatorname{div}\boldsymbol{v}, q)}{\|\nabla\boldsymbol{v}\|\|q\|} \le 1.$$

We shall also sketch the other approach to prove the operator grad is injective and bounded below which can formulated as the generalized Poincare inequality

$$\|\operatorname{grad} p\|_{-1} \ge \beta\|p\| \quad \text{for any } p \in L_0^2(\Omega). \tag{5}$$

The natural domain of the gradient operator is $H^1(\Omega)$, i.e., $\operatorname{grad}\,: H^1(\Omega) \to L^2(\Omega)$. We can continuously extend the domain of the gradient operator from $H^1(\Omega)$ to $L^2(\Omega)$, i.e., $\operatorname{grad}\,: H^1(\Omega) \to L^2(\Omega)$ and prove the range $\operatorname{grad}(L^2)$ is a closed subspace of $\boldsymbol{H}^{-1}$. The most difficult part is the following norm equivalence.

Theorem 1.6. *Let $X(\Omega) = \{v \mid v \in H^{-1}(\Omega), \operatorname{grad} v \in (H^{-1}(\Omega))^n\}$ endowed with the norm $\|v\|_X^2 = \|v\|_{-1}^2 + \|\operatorname{grad} v\|_{-1}^2$. Then for Lipschitz domains, $X(\Omega) = L^2(\Omega)$.*

Proof. A proof $\|v\|_X \lesssim \|v\|$, consequently $L^2(\Omega) \subseteq X(\Omega)$, is trivial (using the definition of the dual norm). The non-trival part is to prove the inequality

$$\|v\|^2 \lesssim \|v\|_{-1}^2 + \|\operatorname{grad} v\|_{-1}^2 = \|v\|_{-1}^2 + \sum_{i=1}^{d} \|\frac{\partial v}{\partial x_i}\|_{-1}^2. \tag{6}$$

The difficulty is associated to the non-computable dual norm. We only present a special case $\Omega = \mathbb{R}^n$ and refer to [9, 4] for general cases.

We use the characterization of H^{-1} norm using Fourier transform. Let $\hat{u}(\xi) = \mathscr{F}(u)$ be the Fourier transform of u. Then

$$\|u\|_{\mathbb{R}^n}^2 = \|\hat{u}\|_{\mathbb{R}^n}^2 = \left\|1/(\sqrt{1+|\xi|^2})\hat{u}\right\|_{\mathbb{R}^n}^2 + \sum_{i=1}^{d}\left\|\xi_i/(\sqrt{1+|\xi|^2})\hat{u}\right\|_{\mathbb{R}^n}^2 = \|u\|_X^2.$$

□

Exercise 1.7. Use the fact L^2 is compactly embedded into H^{-1} and the inequality (6) to prove the Poincaré inequality (5).

Exercise 1.8. For Stokes equations, we can solve $\boldsymbol{u} = A^{-1}(f - B'p)$ and substitute into the second equation to get the Schur complement equation

$$BA^{-1}B'p = BA^{-1}f - g. \tag{7}$$

Define a bilinear form on $\mathbb{P}\times\mathbb{P}$ as

$$s(p,q) = \langle A^{-1}B'p, B'q\rangle.$$

Prove the well-posedness of (7) by showing:

- the continuity of $s(\cdot,\cdot)$ on $L_0^2 \times L_0^2$;
- the coercivity $s(p,p) \geq c\|p\|^2$ for any $p \in L_0^2$.
- relate the constants in the continuity and coercivity of $s(\cdot,\cdot)$ to the inf-sup condition of A and B.

In summary, we have established the well-posedness of Stokes equations.

Theorem 1.9. *For a given $\boldsymbol{f} \in \boldsymbol{H}^{-1}(\Omega)$, there exists a unique solution $(\boldsymbol{u}, p) \in \boldsymbol{H}_0^1(\Omega) \times L_0^2(\Omega)$ to the weak formulation of the Stokes equations* (3)-(4) *and*

$$\|\boldsymbol{u}\|_1 + \|p\| \lesssim \|\boldsymbol{f}\|_{-1}.$$

2. Fortin operator

Verification of the discrete inf-sup condition for the bilinear form $a(\cdot,\cdot)$ is relatively easy. Again the difficult part is the verification of the inf-sup condition for the bilinear form $b(\cdot,\cdot)$ or simply called *div-stability* for Stokes equations.

Note that the inf-sup condition (B) in the continuous level implies: for any $q_h \in \mathbb{P}_h$, there exists $v \in \mathbb{V}$ such that $b(v, q_h) \geq \beta\|v\|_{\mathbb{V}}\|q_h\|_{\mathbb{P}}$ and $\|v\| \leq C\|q_h\|$. For the discrete inf-sup condition, we need a $v_h \in \mathbb{V}_h$ satisfying such property. One approach is to use the so-called Fortin operator [10] to get such a v_h from v.

Definition 2.1 (Fortin operator)**.** *A linear operator $\Pi_h : \mathbb{V} \to \mathbb{V}_h$ is called a* Fortin operator *if*

(1) $b(\Pi_h v, q_h) = b(v, q_h)$ *for all* $q_h \in \mathbb{P}_h$
(2) $\|\Pi_h v\|_{\mathbb{V}} \leq C\|v\|_{\mathbb{V}}$.

Namely the following commutating diagram holds

$$\begin{array}{ccc}
\mathbb{V} & \xrightarrow{\operatorname{div}} & \mathbb{P} \\
\Big\downarrow \Pi_h & & \Big\downarrow Q_h \\
\mathbb{V}_h & \xrightarrow{\operatorname{div}_h} & \mathbb{P}_h
\end{array}$$

with a stable projection Π_h.

Theorem 2.2. *Assume the continuous inf-sup condition (B) holds and there exists a Fortin operator Π_h, then the discrete inf-sup condition (B_h) holds.*

Proof. The inf-sup condition (B) in the continuous level implies: for any $q_h \in \mathbb{P}_h$, there exists $v \in \mathbb{V}$ such that $b(v, q_h) \geq \beta\|v\|\|q_h\|$ and $\|v\| \leq C\|q_h\|$. We choose $v_h = \Pi_h v$.

By the definition of Fortin operator

$$b(v_h, q_h) = b(v, q_h) \geq \beta\|v\|_{\mathbb{V}}\|q_h\|_{\mathbb{P}} \geq \beta C\|v_h\|_{\mathbb{V}}\|q_h\|_{\mathbb{P}}.$$

The discrete inf-sup condition then follows. ☐

In the application to Stokes equations, $\mathbb{P} = L_0^2(\Omega)$ endowed with L^2-norm $\|\cdot\|$ and $\mathbb{V} = \boldsymbol{H}_0^1(\Omega)$ with norm $|v|_1 := \|\nabla v\|$. In the definition of Fortin operator, we require the operator is stable in $|\cdot|_1$-norm and call it the H^1-stability of the operator Π_h. With a slightly abuse of names, we shall call any operator satisfying (1) in Def 2.1 a Fortin operator which could be stable in other norms, i.e. (2) in Def 2.1 may not hold.

When velocity spaces containing the linear finite element space, it suffices to construct a Fortin operator stable in a weaker norm. Let us define a mesh dependent norm

$$\|v\|_h = \|v\| + h|v|_1.$$

For $v \in \mathbb{V}_h$, by the inverse inequality $\|v\|_h \eqsim \|v\|$. The idea is to apply a weaker stable Fortin operator to a high frequency. For high frequency functions, a weaker stability will imply the stronger H^1 stability.

Theorem 2.3. *Suppose the velocity space $\mathbb{V}_h$ contains the piecewise linear and continuous function space. Suppose there exists a Fortin operator $\Pi_B : \boldsymbol{H}_0^1(\Omega) \to \mathbb{V}_h$ and stable in $\|\cdot\|_h$ norm which is equivalent to*

$$\|\Pi_B u\| \lesssim \|u\| + h|u|_1, \quad \text{for all } u \in \boldsymbol{H}_0^1(\Omega), \tag{8}$$

then there exists a Fortin operator $\Pi_h : \boldsymbol{H}_0^1(\Omega) \to \mathbb{V}_h$ and stable in H^1 norm.

Proof. Let $\Pi_1 : \boldsymbol{H}_0^1(\Omega) \to P^1$ be the Scott-Zhang quasi-interpolation [12] which satisfies

$$|\Pi_1 u|_1 + h^{-1}\|u - \Pi_1 u\| \lesssim |u|_1, \quad \|\Pi_1 u\| \lesssim \|u\|. \tag{9}$$

We define the Fortin operator as

$$\Pi_h u = \Pi_1 u + \Pi_B(u - \Pi_1 u).$$

Then $(\operatorname{div} u - \operatorname{div} \Pi_h u, q_h) = 0$ for all $q_h \in \mathbb{P}_h$ by definition.

Next we prove the H^1-stability of Π_h. By the triangle inequality, inverse inequality, stability of Π_B, and the property (9) of Π_1, we get the desired inequality

$$|\Pi_h u|_1 \le |\Pi_1 u|_1 + |\Pi_B(u - \Pi_1 u)|_1 \lesssim |\Pi_1 u|_1 + h^{-1}\|\Pi_B(u - \Pi_1 u)\| \lesssim |u|_1.$$

□

3. Finite Element Spaces for Stokes Equations

Given a triangulation $\mathcal{T}$ of the domain Ω, we shall use the following piecewise polynomial spaces

$$\mathcal{P}_k(\mathcal{T}) = \{v \in C(\Omega) : v|_\tau \in \mathcal{P}_k, \text{for all } \tau \in \mathcal{T}\}, \quad \text{for } k \ge 1$$
$$\mathcal{P}_k^{-1}(\mathcal{T}) = \{v \in L^2(\Omega) : v|_\tau \in \mathcal{P}_k, \text{for all } \tau \in \mathcal{T}\}, \quad \text{for } k \ge 0.$$

Here the superscript $^{-1}$ means the space is discontinuous. Finite element spaces will be chosen as $\mathbb{V}_h = (\mathcal{P}_k(\mathcal{T}))^n \cap \boldsymbol{H}_0^1(\Omega)$ and $\mathbb{P}_h = \mathcal{P}_l(\mathcal{T}) \cap L_0^2(\Omega)$ or $P_l^{-1}(\mathcal{T}) \cap L_0^2(\Omega)$ for careful chosen integers k and l. To simplify the notation, we simply write the space as $(\mathcal{P}_k, \mathcal{P}_l^{-1})$ or $(\mathcal{P}_k, \mathcal{P}_l)$. And we use $\mathcal{P}_0$ for $\mathcal{P}_0^{-1}$ since piecewise constant function is obviously discontinuous.

Here is a list of stable spaces pairs for Stokes equations.

- $(\mathcal{P}_2, \mathcal{P}_0)$: A simple element. Local mass conservation.
- $(\mathcal{P}_1^{CR}, \mathcal{P}_0)$: Non-conforming velocity. Local divergence free.
- $(\mathcal{P}_{1,h/2}^0, \mathcal{P}_{0,h})$ and $(\mathcal{P}_{1,h/2}^0, \mathcal{P}_{1,h}^0)$: Easy to code.
- $(\mathcal{P}_k, \mathcal{P}_{k-1}^{-1})$: stable if $k \ge 4$ in $\mathbb{R}^2$ and for meshes without singular-vertex. Exact divergence free. Scott Vogelius element.
- $(\mathcal{P}_k, \mathcal{P}_{k-1})$: Taylor-Hood element. Optimal convergent rate. Lowest order: $(\mathcal{P}_2, \mathcal{P}_1)$.
- $(\mathcal{P}_1 + \mathcal{B}_3, \mathcal{P}_1)$: Mini element. Most economic element.
- $(\mathcal{P}_k + \mathcal{B}_{k+1}, \mathcal{P}_{k-1}^{-1})$: stabilization using bubble functions. Lowest order: $(\mathcal{P}_2 + \mathcal{B}_3, P_1^{-1})$.

Before we discuss these pairs in detail, we emphasis several considerations:

- Since the inf-sup condition for Stokes equations holds in continuous level, for a fixed pressure space, the velocity space can be enlarged to get discrete inf-sup condition. The enlargement can be done by increasing the polynomial order or refining the mesh.
- We use Fortin operator approach to verify the div stability. This approach is relatively simple but has its own limitation. There are other methods to verify the inf-sup condition for Stokes equations: Verfurth [14], Boland and Nicolaides [3], and Stenberg [13].
- The equation $\operatorname{div} \boldsymbol{u}_h = 0$ holds in a weak topology and in general $\operatorname{div} \boldsymbol{u}_h \neq 0$ point-wise. To enforce $\operatorname{div} \boldsymbol{u}_h = 0$ pointwise, it is better to use $(\mathcal{P}_k, \mathcal{P}_{k-1}^{-1})$ since $\operatorname{div} \mathcal{P}_k \subset \mathcal{P}_{k-1}^{-1}$.
- Due to the coupling of $\boldsymbol{u}_h$ and p_h, it is efficient to equilibrate the rates of convergence. Note the error measured in H^1 norm is usually one order lower than in L^2 norm. To balance the approximation order, it is better to use $(\mathcal{P}_k, \mathcal{P}_{k-1}^{-1})$ or $(\mathcal{P}_k, \mathcal{P}_{k-1})$.
- The trade-off between the increased accuracy of high-order elements and the increased complexity of those elements should be taken into account. Piecewise linear or constant function spaces will be much simpler to programming in practice.

3.1. $(\mathcal{P}_1, \mathcal{P}_0)$**.** The simplest and straightforward pair is $(\mathcal{P}_1, \mathcal{P}_0)$, i.e. using piecewise linear and continuous space for velocity and piecewise constant space for pressure. The continuity of the velocity space is due to the requirement $\mathbb{V}_h \subset \boldsymbol{H}_0^1(\Omega)$. Recall that a piecewise smooth function to be in $H^1(\Omega)$ is equivalent to be globally continuous. The space for pressure is not necessary continuous since only L^2 integrable is required.

Unfortunately this simple pair is not suitable for the Stokes equations. The velocity space is not big enough to provide meaningful approximation. The discrete inf-sup condition cannot be true. The rectangular matrix representation B of the divergence operator is of dimension $NT \times 2N$, where N is the number of interior nodes and NT is the number of triangles. Counting the angles nodal-wise and element-wise, we obtain the inequality $2\pi N < \pi NT$. Note that the inf-sup condition for B is equivalent to asking B is onto. So $\operatorname{rank}(B) = NT$. But it is impossible since $2N < NT$.

In other words, the gradient operator B^t contains kernel more than a global constant function. For the stable pair, $B^t p = 0$ implies $p =$constant. For $(\mathcal{P}_1, \mathcal{P}_0)$ pair, there exists non-constant pressure p s.t. $B^t p = 0$ which is called spurious pressure modes. One way to stabilize the $(\mathcal{P}_1, \mathcal{P}_0)$ pair is to remove the spurious pressure modes. But this process is highly mesh dependent.

3.2. $(\mathcal{P}_2, \mathcal{P}_0)$**.** We enlarge the space of velocity to quadratic polynomials to get a stable pair.

We prove the discrete inf-sup condition by constructing a Fortin operator. By Theorem 2.3, we need only a L^2-stable Fortin operator. Apply the integration by parts element by element, we obtain

$$\sum_{\tau\in\mathcal{T}} \int_\tau \operatorname{div}(\boldsymbol{v} - \Pi_h \boldsymbol{v}) q_h = \sum_{\tau\in\mathcal{T}} \int_{\partial\tau} (\boldsymbol{v} - \Pi_h \boldsymbol{v}) \cdot \boldsymbol{n}\, q_h.$$

Since q_h is piecewise constant, it is sufficient to construct a stable operator $\Pi_h \boldsymbol{v}$

$$\int_e \boldsymbol{v}\, \mathrm{d}s = \int_e \Pi_h \boldsymbol{v}\, \mathrm{d}s \quad \text{for all edges } e \text{ of } \mathcal{T}_h, \tag{10}$$

and that $\|\Pi_h \boldsymbol{v}\| \leq \|\boldsymbol{v}\|_h$.

Let us write $\mathcal{P}_2 = \mathcal{P}_1 \bigoplus \mathcal{B}_E$, where $\mathcal{B}_E$ is the quadratic bubble functions associated to edges. Then (10) is indeed define a function in $\mathcal{B}_E$. More specifically, let e be an edge with vertices v_i, v_j. Denoted by $b_e = 6\phi_i\phi_j/|e|$ where ϕ_i is standard hat basis for $\mathcal{P}_1$. By Simpson rule, the integral $\int_e b_e = 1$. Then the operator

$$\Pi_h^B \boldsymbol{v} := \sum_{e\in E}\left(\int_e \boldsymbol{v}\, \mathrm{d}s\right) b_e$$

satisfies (10). Now we check the stability. For bubble function spaces, since b_e are finite overlapping,

$$\|\Pi_h^B \boldsymbol{v}\|^2 \lesssim \sum_{e\in E}\left(\int_e \boldsymbol{v}\, dt\right)^2 \|b_e\|^2 \lesssim \sum_T \left(\int_T |\boldsymbol{v}|^2 + h^2|\nabla \boldsymbol{v}|^2\, \mathrm{d}x\right) = \|\boldsymbol{v}\|^2 + h^2\|\nabla \boldsymbol{v}\|^2.$$

In the second step, we have used Cauchy-Schwarz inequality and the scaled trace theorem for integral on edges: for any function $g \in H^1(T)$

$$\|g\|_e^2 \leq C\left(h_T^{-1}\|g\|_T^2 + h_T\|\nabla g\|_T^2\right). \tag{11}$$

The drawback of this stable pair is that:

- $\mathbb{Z}^h \not\subset \mathbb{Z}$ since $\operatorname{div}\mathcal{P}_2 \subset \mathcal{P}_1^{-1}$. The velocity approximation u_h is thus not pointwise divergence free. Nevertheless the mass conservation holds in each element.
- the approximation is only first order since $\|p - p_h\| \leq Ch$ although the velocity space could provide one order higher approximation.

3.3. $(\mathcal{P}_k, \mathcal{P}_{k-1}^{-1})$**.** Scott and Vogelius [11] showed that the inf-sup condition holds for $(\mathcal{P}_k, \mathcal{P}_{k-1}^{-1})$ pairs in 2D if $k \geq 4$ provided the meshes are singular-vertex free. An internal vertex in 2D is said to be singular if edges meeting at the point fall into two straight lines. Note that one can perturb the singular vertex to easily get singular-vertex free triangulations.

The relation $\operatorname{div}\mathcal{P}_k \subset \mathcal{P}_{k-1}^{-1}$ implies that the pointwise divergence free for the approximated velocity $\boldsymbol{u}_h$ which is a desirable property (since the conservation of mass everywhere.) The convergent rate is optimal

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_1 + \|p - p_h\| \lesssim h^k, \tag{12}$$

provided the solution $(\boldsymbol{u}, p)$ are smooth enough, say $\boldsymbol{u} \in \boldsymbol{H}^{k+1}(\Omega), p \in H^k(\Omega)$ which is not likely to hold in practice.

The drawback is the complication of programming. There are a lot of unknowns for high order polynomials for vector functions and for discontinuous polynomials. For example, locally for one triangle, the lowest order element $(\mathcal{P}_4, \mathcal{P}_3^{-1})$ contains 30 d.o.f for velocity and 10 for pressure. Globally the dimension of the velocity space is $2(N+3NE+3NT) \approx 32N$ and the dimension of the pressure space is $10NT \approx 20N$. The stability of this type of pair in 3D is not clear and partial results can be found in [15].

3.4. $(\mathcal{P}_k, \mathcal{P}_{k-1})$**.** If we use continuous space for the pressure, then the degree of freedom for pressure can be saved a lot. For example, the dimension of $\mathcal{P}_1^{-1}$ is $3NT$ which is almost 6 times larger than N, the dimension of $\mathcal{P}_1$.

Going from a discontinuous space to a continuous one, the dimension of pressure space is reduced. Then it is optimistic that the velocity space becomes big enough to have the div-stability. Indeed one can show the pair $(\mathcal{P}_k, \mathcal{P}_{k-1})$ for $k \geq 2$ satisfy the div stability. This is known as Taylor-Hood (or Hood-Taylor) elements. The proof of the div stability is delicate. We shall skip it here and refer to, for example, [5, 6]. sketch a proof on $P_2 - P_1$ using edge element.

For this stable pair, we still maintain the optimal convergent order; see (12). The pair is stable for $k \geq 2$. The simplest case $k = 2$ (not $k = 1$ since $(\mathcal{P}_1, \mathcal{P}_0)$ is unstable), $(\mathcal{P}_2, \mathcal{P}_1)$ is very popular. It uses less degree of freedom than the stable pair $(\mathcal{P}_2, \mathcal{P}_0)$ but provide one order higher approximation.

The drawback of Hood-Taylor elements is: First it is still not point-wise divergence free. Second since continuous pressure space is used, there is no element-wise mass conservation. A simple fix is adding the piecewise constant into the pressure space, i.e., $(\mathcal{P}_k, \mathcal{P}_{k-1} + \mathcal{P}_0)$. The div stability of the modified Hood-Taylor elements can be found in xxx.

3.5. $(\mathcal{P}_1 \bigoplus \mathcal{B}_{\mathcal{T}}, \mathcal{P}_1)$**.** We can further reduce the degree of freedom of velocity space to get a stable pair. One well known element is the so-called mini-element developed by Arnold, Brezzi, and Fortin [1].

The idea is to add bubble functions to the velocity space

$$\mathcal{B}_{\mathcal{T}} = \bigoplus_{\tau \in \mathcal{T}} \mathcal{B}_\tau, \quad \mathcal{B}_\tau = \operatorname{span}\{\lambda_1 \lambda_2 \lambda_3\},$$

to stabilize the unstable pair $(\mathcal{P}_1, \mathcal{P}_1)$.

To construct a Fortin operator Π_h, we apply the integration by parts element by element to obtain

$$\begin{aligned}
\sum_{\tau \in \mathcal{T}} \int_\tau \operatorname{div}(\boldsymbol{v} - \Pi_h \boldsymbol{v}) q_h &= \sum_{\tau \in \mathcal{T}} \int_{\partial \tau} (\boldsymbol{v} - \Pi_h \boldsymbol{v}) \cdot \boldsymbol{n}\, q_h - \sum_{\tau \in \mathcal{T}} \int_\tau (\boldsymbol{v} - \Pi_h \boldsymbol{v}) \cdot \nabla q_h \\
&= - \sum_{\tau \in \mathcal{T}} \int_\tau (\boldsymbol{v} - \Pi_h \boldsymbol{v}) \cdot \nabla q_h.
\end{aligned}$$

Since ∇q_h is constant, it suffices to get a stable operator such that $\int_\tau \boldsymbol{v} \,\mathrm{d}x = \int_\tau \Pi_h \boldsymbol{v} \,\mathrm{d}x$ for all $\tau \in \mathcal{T}$. The bubble functions for each element is introduced for this purpose. Let us define $\Pi_B v \in \mathcal{B}_{\mathcal{T}}$ by

$$\int_\tau \Pi_B v \,\mathrm{d}x = \int_\tau v \,\mathrm{d}x, \quad \text{for all } \tau \in \mathcal{T}.$$

It is trivial to show $\|\Pi_h^B\|$ is stable in L^2 norm and then the desirable Fortin operator can be constructed using Theorem 2.3.

3.6. $(\mathcal{P}_1^{CR}, \mathcal{P}_0)$**.** An easy fix of the div-stability is through the sacrifices of conformity of the velocity space. From the proof of the stability of $(\mathcal{P}_2, \mathcal{P}_0)$ (see (10)), the degree of freedom on edges is important. We then introduce the following piecewise linear finite element space

$$\mathcal{P}_1^{CR} = \{v \in L^2(\Omega), v|_\tau \in \mathcal{P}_1(\tau), \int_e v \text{ is continuous for all } e\}.$$

The superscript ^{CR} is named after Crouzeix and Raviart who introduced this space in [7]. To impose the boundary condition, one can require $\int_e v = 0$ for $e \in \partial\Omega$. That is the boundary condition is not imposed pointwise but in a weak sense. One can easily show functions in $\mathcal{P}_1^{CR}$ is continuous at middle points of edges but not on vertices and thus $\mathcal{P}_1^{CR} \not\subset H^1(\Omega)$.

Follow the proof of the stability of $(\mathcal{P}_2, \mathcal{P}_0)$ (see (10)), one can also easily proof the stability of $(\mathcal{P}_1^{CR}, \mathcal{P}_0^{-1})$. This is the simplest stable element for Stokes equations. sketch a proof here.

The sacrifice is that $\mathcal{P}_1^{CR} \not\subset H^1(\Omega)$. One needs to show the violation is get controlled by estimating the consistency error carefully.

3.7. $(\mathcal{P}_{1,h/2}, \mathcal{P}_{0,h})$ **and** $(\mathcal{P}_{1,h/2}, \mathcal{P}_{1,h})$**.** Another way to enrich the velocity space is through the mesh refinement. We denoted by $\mathcal{T}_{h/2}$ a fine triangulation obtained by regular uniform refinement of $\mathcal{T}_h$, i.e., each triangle in $\mathcal{T}_h$ is divided into 4 similar triangles by connecting middle points of edges. $\mathcal{P}_{1,h/2}$ is piecewise linear and continuous finite element space on $\mathcal{T}_{h/2}$. Comparing with $\mathcal{P}_{1,h}$, new degree of freedoms are created on edges. Then $\mathcal{P}_{1,h/2}$ can be used to replace $\mathcal{P}_2$ in the stable pair $(\mathcal{P}_2, \mathcal{P}_0)$ and $(\mathcal{P}_2, \mathcal{P}_1)$. The benefit of replacing a better approximation space by a less accurate one is the simplify of programming.

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