PROBLEMS

The first seven problems for this chapter serve as a review of the topic of complex numbers, their representation, and several of their basic properties. As we will use complex numbers extensively in this book, it is important that readers familiarize themselves with the fundamental ideas considered and used in these problems.

The complex number z can be expressed in several ways. The *Cartesian* or *rectangular* form for z is given by

$$z = x + jy$$

where $j = \sqrt{-1}$ and x and y are real numbers referred to respectively as the *real part* and the *imaginary part* of z. As we indicated in the chapter, we will often use the notation

$$x = \mathcal{R}e\{z\}, \qquad y = \mathcal{I}m\{z\}$$

The complex number z can also be represented in *polar form* as

$$z = re^{j\theta}$$

where $r > 0$ is the *magnitude* of z and θ is the *angle* or *phase* of z. These quantities will often be written as

$$r = |z|, \qquad \theta = \sphericalangle z$$

The relationship between these two representations of complex numbers can be determined either from *Euler's relation*

$$e^{j\theta} = \cos\theta + j\sin\theta \tag{P2.0-1}$$

or by plotting z in the complex plane, as shown in Figure P2.0. Here the coordinate axes are $\mathcal{R}e\{z\}$ along the horizontal axis and $\mathcal{I}m\{z\}$ along the vertical axis. With respect to this graphical representation, x and y are the Cartesian coordinates of z, and r and θ are its polar coordinates.

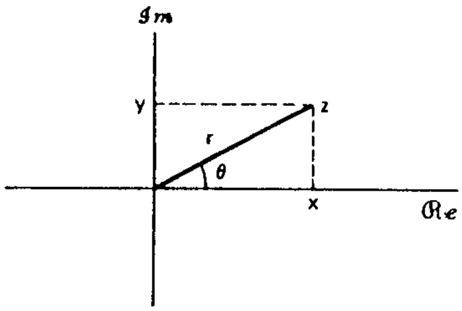

Figure P2.0

2.1. (a) Using Euler's relation or Figure P2.0, determine expressions for x and y in terms of r and θ.
(b) Determine expressions for r and θ in terms of x and y.
(c) If we are given only r and $\tan\theta$, can we uniquely determine x and y? Explain your answer.

2.2. Using Euler's relation, derive the following relationships.
(a) $\cos\theta = \frac{1}{2}(e^{j\theta} + e^{-j\theta})$
(b) $\sin\theta = \frac{1}{2j}(e^{j\theta} - e^{-j\theta})$
(c) $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$
(d) $(\sin\theta)(\sin\phi) = \frac{1}{2}\cos(\theta - \phi) - \frac{1}{2}\cos(\theta + \phi)$
(e) $\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$

2.3. Let z_0 be a complex number with polar coordinates r_0, θ_0 and Cartesian coordinates x_0, y_0. Determine expressions for the Cartesian coordinates of the following complex numbers in terms of x_0 and y_0. Plot the points z_0, z_1, z_2, z_3, z_4, and z_5 in the complex plane when $r_0 = 2$, $\theta_0 = \pi/4$ and when $r_0 = 2$, $\theta_0 = \pi/2$. Indicate on your plots the real and imaginary parts of each point.

(a) $z_1 = r_0 e^{-j\theta_0}$ (b) $z_2 = r_0$ (c) $z_3 = r_0 e^{j(\theta_0+\pi)}$

(d) $z_4 = r_0 e^{j(-\theta_0+\pi)}$ (e) $z_5 = r_0 e^{j(\theta_0+2\pi)}$

2.4. Let z denote a complex variable

$$z = x + jy = re^{j\theta}$$

The *complex conjugate* of z is denoted by z^* and is given by

$$z^* = x - jy = re^{-j\theta}$$

Derive each of the following relations, where z, z_1, and z_2 are arbitrary complex numbers.

(a) $zz^* = r^2$

(b) $\dfrac{z}{z^*} = e^{j2\theta}$

(c) $z + z^* = 2\,\mathcal{R}e\{z\}$

(d) $z - z^* = 2j\,\mathcal{I}m\{z\}$

(e) $(z_1 + z_2)^* = z_1^* + z_2^*$

(f) $(az_1z_2)^* = az_1^*z_2^*$, where a is any real number

(g) $\left(\dfrac{z_1}{z_2}\right)^* = \dfrac{z_1^*}{z_2^*}$

(h) $\mathcal{R}e\left\{\dfrac{z_1}{z_2}\right\} = \dfrac{1}{2}\left[\dfrac{z_1z_2^* + z_1^*z_2}{z_2z_2^*}\right]$

2.5. Express each of the following complex numbers in Cartesian form and plot them in the complex plane, indicating the real and imaginary parts of each number.

(a) $\dfrac{3 + 4j}{1 - 2j}$

(b) $\dfrac{j(2 + j)}{(1 + j)(2 - j)}$

(c) $2j\dfrac{(1 + j)^2}{(3 - j)}$

(d) $4e^{j(\pi/6)}$

(e) $\sqrt{2}\, e^{j(25\pi/4)}$

(f) $je^{j(11\pi/4)}$

(g) $3e^{j4\pi} + 2e^{j7\pi}$

(h) The complex number z whose magnitude is $|z| = \sqrt{2}$ and whose angle is $\measuredangle z = -\pi/4$

(i) $(1 - j)^9$

(j) $\dfrac{6e^{-j\pi/3}}{1 - j}$

2.6. Express each of the following complex numbers in polar form and plot them in the complex plane, indicating the magnitude and angle of each number.

(a) $1 + j\sqrt{3}$ (b) -5 (c) $-5 - 5j$

(d) $3 + 4j$ (e) $(1 - j\sqrt{3})^3$ (f) $(1 + j)^5$

(g) $(\sqrt{3} + j^3)(1 - j)$ (h) $\dfrac{2 - j(6/\sqrt{3})}{2 + j(6/\sqrt{3})}$ (i) $\dfrac{1 + j\sqrt{3}}{\sqrt{3} + j}$

(j) $j(1 + j)e^{j\pi/6}$ (k) $(\sqrt{3} + j)2\sqrt{2}\, e^{-j\pi/4}$ (l) $\dfrac{e^{j\pi/3} - 1}{1 + j\sqrt{3}}$

2.7. Derive the following relations, where z, z_1, and z_2 are arbitrary complex numbers.

(a) $(e^z)^* = e^{z^*}$

(b) $z_1 z_2^* + z_1^* z_2 = 2\,\mathcal{R}e\,\{z_1 z_2^*\} = 2\,\mathcal{R}e\,\{z_1^* z_2\}$

(c) $|z| = |z^*|$

(d) $|z_1 z_2| = |z_1||z_2|$

(e) $\mathcal{R}e\,\{z\} \leq |z|$, $\mathcal{I}m\,\{z\} \leq |z|$

(f) $|z_1 z_2^* + z_1^* z_2| \leq 2|z_1 z_2|$

(g) $(|z_1| - |z_2|)^2 \leq |z_1 + z_2|^2 \leq (|z_1| + |z_2|)^2$

2.8. The relations considered in this problem are used on many occasions throughout this book.

(a) Prove the validity of the following expression:

$$\sum_{n=0}^{N-1} \alpha^n = \begin{cases} N, & \alpha = 1 \\ \dfrac{1 - \alpha^N}{1 - \alpha}, & \text{for any complex number } \alpha \neq 1 \end{cases}$$

(b) Show that if $|\alpha| < 1$, then

$$\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1 - \alpha}$$