Intermediate Analysis

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1 Sequence Convergence

1. Prove the convergence of
$$\lim_{n\to\infty}(\sqrt{n^2+n}-n)$$
   (a) First, notice that this equation approaches 0.5 as n increases. Thus, we will be proving:
   $$\forall\epsilon > 0, \exists N s.t. \forall n > N, \left|\sqrt{n^2+n}-n-\frac{1}{2}\right| < \epsilon$$
   (b) which is the same as proving both of the following:
   $$\sqrt{n^2+n}-n-\frac{1}{2} < \epsilon$$
   $$\sqrt{n^2+n}-n-\frac{1}{2} > -\epsilon$$
   (c) then add $n+1/2$ to both sides
   $$\sqrt{n^2+n} < \epsilon + n + \frac{1}{2}$$
   $$\sqrt{n^2+n} > -\epsilon + n + \frac{1}{2}$$
   (d) square both sides to get:
   $$n^2+n < \epsilon^2 + 2\epsilon n + 2\epsilon.5 + n^2 + 2n.5 + .25$$
   $$n^2+n > \epsilon^2 - 2\epsilon n - 2\epsilon.5 + n^2 + 2n.5 + .25$$
   (e) cancel the n^2+n from both sides to get
   $$0 < \epsilon^2 + 2\epsilon n + 2\epsilon.5 + 0.25$$
   $$0 > \epsilon^2 - 2\epsilon n - 2\epsilon.5 + 0.25$$
   (f) Bring the $2\epsilon n$ to the other side
   $$-2\epsilon n < \epsilon^2 + 2\epsilon.5 + 0.25$$
   $$2\epsilon n > \epsilon^2 - 2\epsilon.5 + 0.25$$
   (g) divide by 2ϵ on the second, and divide by -2ϵ on the first, which flips the direction of the $<$ to $>$.
   $$n > \frac{\epsilon^2 + 2\epsilon.5 + 0.25}{-2\epsilon}$$
   $$n > \frac{\epsilon^2 - 2\epsilon.5 + 0.25}{2\epsilon}$$
   (h) Given that both of these need to be true, and the second is bigger than the first because the second is positive and the first is negative, we can just use the second requirement. Also, notice that this is a decreasing function in ϵ, as desired.

2. Prove the convergence of

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^3$$

   (a) First, note that this sequence approaches 1. Thus, we need to prove that:

$$\forall\epsilon>0,\exists N s.t.\forall n>N,\left|\left(1+\frac{1}{n}\right)^3-1\right|<\epsilon$$

   (b) Noting that

$$\left(1+\frac{1}{n}\right)^3>1$$

   We can drop the absolute value and get

$$\left(1+\frac{1}{n}\right)^3-1<\epsilon$$

   (c) Move the 1 to the other side to get:

$$\left(1+\frac{1}{n}\right)^3<1+\epsilon$$

   (d) Take the cube root of both sides

$$1+\frac{1}{n}<(1+\epsilon)^{\frac{1}{3}}$$

   (e) subtract from to both sides

$$\frac{1}{n}<(1+\epsilon)^{\frac{1}{3}}-1$$

   (f) switch positions of the fractions

$$\frac{1}{(1+\epsilon)^{\frac{1}{3}}-1}<n$$

   (g) Switch sides:

$$n>\frac{1}{(1+\epsilon)^{\frac{1}{3}}-1}$$

   Which increases as ϵ decreases

3. Prove the convergence of:

$$\lim_{n\to\infty} \frac{\sin n}{n^2}$$

(a) First, note that this sequence approaches 0. Thus, we need to prove that:

$$\forall \epsilon > 0, \exists N s.t. \forall n > N, \left|\frac{\sin n}{n^2}\right| < \epsilon$$

(b) Noting that $n^2 > 0$, we can rewrite as:

$$\frac{|\sin n|}{n^2} < \epsilon$$

(c) Noting that

$$|\sin n| \leq 1 \to \frac{|\sin n|}{n^2} \leq \frac{1}{n^2} \leq \frac{1}{n}$$

(d) Which means that if we can prove convergence of $1/n$, then we have proven the convergence of the sequence in question. Thus, we want:

$$\frac{1}{n} < \epsilon$$

or

$$\frac{1}{\epsilon} < n$$

or

$$n > \frac{1}{\epsilon}$$

4. Prove that if $\{b_n\}$ is a sequence of positive terms and $b_n \to b > 0$, then there is a positive lower bound $m > 0$ such that $b_n \geq m$ for all n.

   - If $b_n \to b > 0$, then for any ϵ there exists an N such that for all $n > N$, $|b_n - b| < \epsilon$. In particular, if we let $\epsilon = b/2$, then for all $n > N$, we can ensure that $b_n > b/2$, and since $b > 0$, we know that $b/2 > 0$.
   - Next, since there are only a finite number of b_n values before N, we can take the minimum of those values. Call this m'. We know that each term is positive, so m' is positive.
   - Next, we take the minimum of $b/2$ and m', and we can take half of that value as well, and call it m: $\min\{b/2, m'\}/2 = m$. This m is one of infinitely many possible lower bounds.