H25

数学 4回目

[1] (1) $y=\sqrt{\dfrac{x^2+1}{x^2-1}}$

両辺に自然対数をとると、

$$\log y=\log\sqrt{\frac{x^2+1}{x^2-1}}=\frac{1}{2}\log(x^2+1)-\frac{1}{2}\log|x^2-1|$$

$$\frac{y'}{y}=\frac{1}{2}\frac{2x}{x^2+1}-\frac{1}{2}\frac{2x}{x^2-1}=\frac{x(x^2-1)-x(x^2+1)}{(x^2+1)(x^2-1)}=-\frac{2x}{(x^2+1)(x^2-1)}$$

$$\therefore\ y'=\boxed{-\frac{2x}{(x^2+1)(x^2-1)}\sqrt{\frac{x^2+1}{x^2-1}}}$$

(2) $\displaystyle\int_1^\infty\frac{\log x}{x^3}dx$

$$=\int_1^\infty\log x\left(-\frac{1}{2}x^{-2}\right)'dx=\left[-\frac{1}{2}x^{-2}\log x\right]_1^\infty+\frac{1}{2}\int_1^\infty\frac{1}{x^3}dx$$

$$=0-\frac{1}{4}\left[\frac{1}{x^2}\right]_1^\infty$$

$$=\boxed{\frac{1}{4}}$$

[2] (1) $y'-\dfrac{x}{x^2+1}y=0$

$$\frac{dy}{dx}=\frac{x}{x^2+1}y$$

$$\frac{dy}{y}=\frac{x}{x^2+1}dx$$

$$\log|y|=\frac{1}{2}\log|x^2+1|+C$$

$$\log\left|\frac{y}{\sqrt{x^2+1}}\right|=C$$

$$\frac{y}{\sqrt{x^2+1}}=C$$

$\therefore\ \boxed{y=C\sqrt{x^2+1}}$ (Cは任意定数)

(2) CをC(x)とすると、(1)より、

$$y'=C'(x)\sqrt{x^2+1}+\frac{1}{2}C(x)\frac{2x}{\sqrt{x^2+1}}=C'(x)\sqrt{x^2+1}+C(x)\frac{x}{\sqrt{x^2+1}}$$

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与式に代入して,

$$C'(x)\sqrt{x^2+1} + C(x)\frac{x}{\sqrt{x^2+1}} - \frac{x}{x^2+1}\cdot C(x)\sqrt{x^2+1} = x$$

$$C'(x)\sqrt{x^2+1} + C(x)\frac{x}{\sqrt{x^2+1}} - C(x)\frac{x}{\sqrt{x^2+1}} = x$$

$$C'(x)\sqrt{x^2+1} = x$$

$$C'(x) = \frac{x}{\sqrt{x^2+1}}$$

$$C(x) = \int \frac{x}{\sqrt{x^2+1}}dx$$

$$= \int \frac{x^2}{x\sqrt{x^2+1}}dx$$

$x^2 = t$ とおく

$2x\,dx = dt \to dx = dt/2x$

$$C(x) = \int \frac{t}{x\sqrt{t+1}}\cdot\frac{dt}{2x} = \frac{1}{2}\int\frac{t}{t\sqrt{t+1}}dt = \frac{1}{2}\int\frac{dt}{\sqrt{t+1}} = \sqrt{t+1} + C_1 = \sqrt{x^2+1} + C_1$$

$$\therefore y = (\sqrt{x^2+1} + C_1)\sqrt{x^2+1} = \boxed{x^2+1+C_1\sqrt{x^2+1}}$$ (C_1は積分定数)

[3]

$$|A| = \begin{vmatrix} 3 & -1 & -1 \\ 4 & 0 & 0 \\ -3 & 2 & 2 \end{vmatrix} \to -\begin{vmatrix} 4 & 0 & 0 \\ 3 & -1 & -1 \\ -3 & 2 & 2 \end{vmatrix} \to -4\begin{vmatrix} 1 & 0 & 0 \\ 3 & -1 & -1 \\ -3 & 2 & 2 \end{vmatrix} \to -4\begin{vmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 2 & 2 \end{vmatrix}$$

$$\to -4\begin{vmatrix} -1 & -1 \\ 2 & 2 \end{vmatrix} = -4(-2+2) = \boxed{0}$$

[4] (1) $\mathcal{L}\left[\int_0^t f(t-\tau)g(\tau)d\tau\right] = \mathcal{L}[f(t)]\,\mathcal{L}[g(t)]$

$$(\text{左辺}) = \mathcal{L}\left[\int_0^t f(t-\tau)g(\tau)d\tau\right]$$
$$= \int_0^\infty e^{-st}dt\int_0^t f(t-\tau)g(\tau)d\tau$$
$$= \int_0^\infty e^{-st}\int_0^\infty f(t-\tau)U(t-\tau)g(\tau)d\tau\,dt$$
$$= \int_0^\infty g(\tau)d\tau\int_0^\infty e^{-st}f(t-\tau)U(t-\tau)dt$$
$$= \int_0^\infty g(\tau)d\tau\int_\tau^\infty e^{-s(t-\tau)-s\tau}f(t-\tau)dt$$

$x = t-\tau$ とおく

$dx = dt$

t	$\tau \to \infty$
x	$0 \to \infty$

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$$(与式) = \int_0^\infty g(\tau)d\tau \int_0^\infty e^{-sx-s\tau} f(x)dx$$

$$= \int_0^\infty e^{-s\tau} g(\tau)d\tau \int_0^\infty e^{-sx} f(x)dx$$

$$= \mathcal{L}[f(t)]\,\mathcal{L}[g(t)]$$

$$= (右辺)$$

よって, $\underline{(左辺) = (右辺)}$ //

(2) $y(t) = \int_0^t e^{t-\tau} y(\tau)d\tau + t + 1$

ラプラス変換して, (1)で求めた公式を用いると,

$$Y(s) = \mathcal{L}[e^t]\,Y(s) + \frac{1}{s^2} + \frac{1}{s}$$

$$= \frac{1}{s-1}Y(s) + \frac{1}{s^2} + \frac{1}{s}$$

$$\frac{s-2}{s-1}Y(s) = \frac{1}{s^2} + \frac{1}{s} = \frac{s+1}{s^2}$$

$$Y(s) = \frac{(s+1)(s-1)}{s^2(s-2)}$$

$s-2$ の因子について, $(s+1)(s-1)/s^2 \big|_{s=2} = \frac{3}{4}$

s^2 の因子について, $(s+1)(s-1)/(s-2) \big|_{s=0} = \frac{1}{2}$

s の因子について, $\frac{d}{ds}\left\{\frac{(s+1)(s-1)}{s-2}\right\}\Big|_{s=0} = \frac{2s(s-2)-(s^2-1)}{(s-2)^2}\Big|_{s=0} = \frac{1}{4}$

$$Y(s) = \frac{3}{4}\frac{1}{s-2} + \frac{1}{2}\frac{1}{s^2} + \frac{1}{4}\frac{1}{s}$$

$$\therefore f(t) = \boxed{\frac{3}{4}e^{2t} + \frac{1}{2}t + \frac{1}{4}}$$