流体力学　4回目

[1] (1) ベルヌーイの定理より，z軸上では流速0より，

$$p_0(z) + \rho g z = p_a \rightarrow \therefore p_0(z) = \boxed{p_a - \rho g z}$$

(2) $F = mr\omega^2 = \rho V r\omega^2 = \rho \cdot r\,dr\,d\theta\,dz\; r\Omega^2 = \boxed{\rho r^2 \Omega^2 dr\,d\theta\,dz}$

(3) $dp\; r\,d\theta\,dz = \rho r^2 \Omega^2 dr\,d\theta\,dz$

$dp = \rho \Omega^2 r\,dr$

$p(r) = \frac{1}{2}\rho\Omega^2 r^2 + p_0$　　　　$\rightarrow p_0 = p_0(z)$

~~水面上では大気圧と等しいため，~~　　$p(0) = p_0 = p_a - \rho g z$

$\cancel{p(0) = 0 + p_0 = p_a \rightarrow p_0 = p_a}$　　$z = 0$ より，$p_0 = p_a$

$\cancel{\therefore p(r) = \boxed{\frac{1}{2}\rho\Omega^2 r^2 + p_a}}$　　$\rightarrow p(r) = \boxed{\frac{1}{2}\rho\Omega^2 r^2 + p_a}$

(4) 水面では，$p(r) = p_a$ より，

$p(r) = \frac{1}{2}\rho\Omega^2 r^2 + p_0(z) = \frac{1}{2}\rho\Omega^2 r^2 + p_a - \rho g z = p_a$

$\therefore z = \boxed{\frac{\Omega^2 r^2}{2g}}$

[2] (1) $w = i\left(Uz - \frac{\mu}{z}\right)$

$\frac{dw}{dz} = i\left(U + \frac{\mu}{z^2}\right) = 0 \rightarrow z = \pm\sqrt{-\frac{\mu}{U}} = \pm i\sqrt{\frac{\mu}{U}} = x + iy$

$\therefore$ よどみ点：$(x, y) = \boxed{\left(0, \sqrt{\frac{\mu}{U}}\right),\ \left(0, -\sqrt{\frac{\mu}{U}}\right)}$

(2) $w = i\left(Uz - \frac{\mu}{z}\right)$

$z = re^{i\theta}$ より，

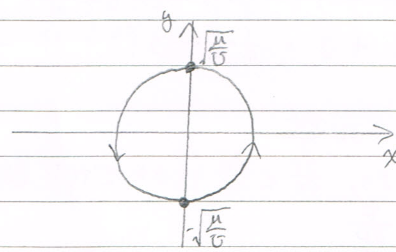
$w = i\left(Ure^{i\theta} - \frac{\mu}{r}e^{-i\theta}\right) = i\left\{Ur(\cos\theta + i\sin\theta) - \frac{\mu}{r}(\cos\theta - i\sin\theta)\right\}$

$= -Ur\sin\theta - \frac{\mu}{r}\sin\theta + i\left(Ur\cos\theta - \frac{\mu}{r}\cos\theta\right) = \phi + i\psi$

$\therefore \psi = \boxed{\cos\theta\left(Ur - \frac{\mu}{r}\right)}$

よどみ点を通るとき，$\psi = 0$ より

$$\begin{cases} \cos\theta = 0 \rightarrow \theta = \pm\frac{\pi}{2} \\ Ur - \frac{\mu}{r} = 0 \rightarrow r^2 = \frac{\mu}{U} = x^2 + y^2 \end{cases}$$



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$$u_r = \frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$$
$$u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r}$$

(3) $w = i\left(Uz - \frac{\mu}{z} - \gamma \log z\right)$

$z = re^{i\theta}$ より,

$$w = i\left(Ure^{i\theta} - \frac{\mu}{r}e^{-i\theta} - \gamma \log re^{i\theta}\right)$$
$$= iUr(\cos\theta + i\sin\theta) - i\frac{\mu}{r}(\cos\theta - i\sin\theta) - i\gamma\log r + \gamma\theta$$
$$= \left(-Ur\sin\theta - \frac{\mu}{r}\sin\theta + \gamma\theta\right) + i\left(Ur\cos\theta - \frac{\mu}{r}\cos\theta - \gamma\log r\right) = \phi + i\psi$$
$$\phi = -\left(Ur + \frac{\mu}{r}\right)\sin\theta + \gamma\theta, \quad \psi = \left(Ur - \frac{\mu}{r}\right)\cos\theta - \gamma\log r$$

$$u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r}$$

$$\Gamma = \int_0^{2\pi} u_\theta \cdot r d\theta = \int_0^{2\pi} \frac{1}{r}\frac{\partial \phi}{\partial \theta} \cdot r d\theta = \left[-\left(Ur + \frac{\mu}{r}\right)\sin\theta + \gamma\theta\right]_0^{2\pi} = \boxed{2\pi\gamma}$$

(4) $\frac{dw}{dz} = i\left(U + \frac{\mu}{z^2} - \frac{\gamma}{z}\right) = 0 \longrightarrow U + \frac{\mu}{z^2} - \frac{\gamma}{z} = 0$

$Uz^2 - \gamma z + \mu = 0$ —①

よどみ点が一点のみとなるのは, z の解が一つのみのときなので,

$D = \gamma^2 - 4U\mu = 0$

$\gamma > 0$ より, $\therefore \gamma = \boxed{2\sqrt{\mu U}}$

①に代入

$Uz^2 - 2\sqrt{\mu U}\, z + \mu = 0$

$(\sqrt{U}z - \sqrt{\mu})^2 = 0$

$z = \sqrt{\frac{\mu}{U}} = x + iy \longrightarrow \therefore$ よどみ点; $(x, y) = \boxed{\left(0, \sqrt{\frac{\mu}{U}}\right)}$

3 (1) $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$

$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$

$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

層流より, $v = 0$, 定常より, $\frac{\partial}{\partial t} = 0 \longrightarrow \frac{\partial u}{\partial x} = 0 \Rightarrow$ uはyのみの関数である —①

ナビエ・ストークス方程式を書き直すと,

$0 + 0 + 0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(0 + \frac{\partial^2 u}{\partial y^2}\right)$

$0 + 0 + 0 = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu(0 + 0) \longrightarrow \frac{\partial p}{\partial y} = 0 \longrightarrow$ pはxのみの関数である —②

$\frac{\partial p}{\partial x} = +\rho\nu\frac{\partial^2 u}{\partial y^2}$

①, ②より, $\frac{\partial p}{\partial x}$ と $\frac{\partial^2 u}{\partial y^2}$ はそれぞれ定数とおくことができる. $\frac{\partial p}{\partial x} = F$ とすると,

$\frac{\partial^2 u}{\partial y^2} = \frac{F}{\rho\nu}$

$\frac{\partial u}{\partial y} = \frac{F}{\rho\nu}y + C_1$, $u = \frac{F}{2\rho\nu}y^2 + C_1 y + C_2$

境界条件より，$u(h) = U$，$u(0) = 0$

$u(0) = 0 + 0 + C_2 = 0 \rightarrow C_2 = 0$

$u(h) = \frac{F}{2\rho\nu} h^2 + C_1 h = U \longrightarrow C_1 = \frac{U}{h} - \frac{F}{2\rho\nu} h$

$\therefore u(y) = \boxed{\frac{F}{2\mu} y^2 + \left(\frac{U}{h} - \frac{Fh}{2\mu} \right) y}$

(2) $u_m = \frac{1}{A} \int u(y)\, dA = \frac{1}{h} \int_0^h \left\{ \frac{F}{2\rho\nu} y^2 + \left(\frac{U}{h} - \frac{Fh}{2\rho\nu} \right) y \right\} dy$

$= \frac{1}{h} \left\{ \left[\frac{F}{6\rho\nu} y^3 + \frac{1}{2} \left(\frac{U}{h} - \frac{Fh}{2\rho\nu} \right) y^2 \right]_0^h \right\}$

$= \frac{1}{h} \left\{ \frac{Fh^3}{6\rho\nu} + \frac{1}{2} \left(\frac{U}{h} - \frac{Fh}{2\rho\nu} \right) h^2 \right\} = \frac{Fh^2}{3\rho\nu} + U - \frac{Fh^2}{2\rho\nu} = U - \frac{Fh^2}{6\rho\nu}$

$u_m = 0$ のとき，

$U - \frac{Fh^2}{6\rho\nu} = 0 \rightarrow \therefore F = \frac{6\rho\nu U}{h^2} = \boxed{\frac{6\mu U}{h^2}}$

(3) $\tau_h = \mu \left(\frac{\partial u}{\partial y} \right)_{y=h} = \mu \left(\frac{Fy}{\mu} + \frac{U}{h} - \frac{Fh}{2\mu} \right)_{y=h} = \mu \left(\frac{y}{\mu} \cdot \frac{6\mu U}{h^2} + \frac{U}{h} - \frac{h}{2\mu} \cdot \frac{6\mu U}{h^2} \right)_{y=h}$

$= \mu \left(\frac{6U}{h^2} y + \frac{U}{h} - \frac{3U}{h} \right)_{y=h} = \mu \left(\frac{6U}{h^2} y - \frac{2U}{h} \right)_{y=h} = \mu \left(\frac{6U}{h} - \frac{2U}{h} \right) = \frac{4\mu U}{h}$

$\tau_0 = \mu \left(\frac{\partial u}{\partial y} \right)_{y=0} = \mu \left(\frac{6U}{h^2} y - \frac{2U}{h} \right)_{y=0} = -\frac{2\mu U}{h}$

$\therefore \tau = \tau_h - \tau_0 = \boxed{\frac{6\mu U}{h}}$