数学　4回目

[1] (1) $y = x^{\sin x}$ $(x > 0)$

$\log y = \log x^{\sin x} = \sin x \log x$

$\dfrac{y'}{y} = \cos x \log x + \dfrac{\sin x}{x} \rightarrow \therefore\ y' = \boxed{x^{\sin x}\left(\cos x \log x + \dfrac{\sin x}{x}\right)}$

(2) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(\sqrt{1+x^2})^3}$

$x = \tan\theta$ とおく

$dx = d\theta / \cos^2\theta$

x	$-\infty \to \infty$
θ	$-\frac{\pi}{2} \to \frac{\pi}{2}$

$$\Rightarrow \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{(\sqrt{1+\tan^2\theta})^3} \cdot \frac{d\theta}{\cos^2\theta} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos^3\theta}{\cos^2\theta} d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta\, d\theta$$

$$= \left[+\sin\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = +\{1-(-1)\} = \boxed{+2}$$

[2] $y = xy' + \sqrt{1+y'^2}$

(1) $y' = y' + xy'' + \dfrac{2y' \cdot y''}{2\sqrt{1+y'^2}}$

$xy'' + \dfrac{y' \cdot y''}{\sqrt{1+y'^2}} = 0 \rightarrow y''\left(x + \dfrac{y'}{\sqrt{1+y'^2}}\right) = 0$

$y'' = 0$ または $x + \dfrac{y'}{\sqrt{1+y'^2}} = 0$ //

(2) $y'' = 0$ より

$y' = \dfrac{dy}{dx} = C$

与式に代入

$\therefore y = \boxed{Cx + \sqrt{1+C^2}}$ (Cは任意定数)

(3) $x + \dfrac{y'}{\sqrt{1+y'^2}} = 0$

$y' = -x\sqrt{1+y'^2}$

$y'^2 = x^2(1+y'^2)$

$(1-x^2)y'^2 = x^2$

$y' = \dfrac{x}{\sqrt{1-x^2}}$

与式に代入

$$y = x\frac{x}{\sqrt{1-x^2}} + \sqrt{1+\frac{x^2}{1-x^2}} = \frac{x^2}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}} = \boxed{\frac{x^2+1}{\sqrt{1-x^2}}}$$

[3] $A = \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}$

$$\det(A-\alpha E) = \det\begin{bmatrix} 4-\alpha & -3 \\ -1 & 2-\alpha \end{bmatrix} = (4-\alpha)(2-\alpha)-3 = 8-6\alpha+\alpha^2-3$$

$$= \alpha^2-6\alpha+5 = 0$$

$(\alpha-1)(\alpha-5)=0$

固有値；$\alpha = 1, 5$

(i) $\alpha=1$ のとき，固有ベクトル；$x = {}^t[x_1, x_2]$ とおくと，

$$\begin{bmatrix} 3 & -3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

階段行列にして，

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$x_1 - x_2 = 0$

$x_2 = c_1$ とおくと，$x_1 = c_1$ → 固有ベクトル；${}^t[1, 1]$

(ii) $\alpha=5$ のとき，固有ベクトル；$x = {}^t[x_1, x_2]$ とおくと，

$$\begin{bmatrix} -1 & -3 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

階段行列にして，

$$\begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$x_1 + 3x_2 = 0$

$x_2 = c_2$ とおくと，$x_1 = -3c_2$ → 固有ベクトル；${}^t[-3, 1]$

(i)，(ii) より，

$$\therefore P = \boxed{\begin{bmatrix} 1 & -3 \\ 1 & 1 \end{bmatrix}}$$

$\mathcal{L}[(t-1)(t-3)U(t-3)] = e^{-3s}\mathcal{L}[\{(t+3)-1\}\{(t+3)-3\}]$
$= e^{-3s}\mathcal{L}[t(t+2)]$

[4] (1) $\mathcal{L}[f(t-a)u(t-a)] = e^{-as}\mathcal{L}[f(t)]$

$(左辺) = \mathcal{L}[f(t-a)u(t-a)] = \int_0^\infty e^{-st}dt\, f(t-a)\, u(t-a)$
$= \int_a^\infty e^{-s(t-a)-sa} f(t-a)\, dt$

$t-a = x$とおく
$dt = dx$

t	$a \to \infty$
x	$0 \to \infty$

$(与式) = \int_0^\infty e^{-sx-sa} f(x)\, dx = e^{-sa}\int_0^\infty f(x)\, dx = e^{-sa}\mathcal{L}[f(t)] = (右辺)$

よって、$(左辺) = (右辺)$ //

(2) $\mathcal{L}[f(t)u(t-a)] = e^{-as}\mathcal{L}[f(t+a)]$

$(左辺) = \mathcal{L}[f(t)u(t-a)]$
$F(t-a) = f(t)$とおく
$(与式) = \int_0^\infty e^{-st}dt\, F(t-a)\, u(t-a)$
$= e^{-sa}\mathcal{L}[F(t)]$
$t-a = t$とすると、
$F(t) = f(t+a)$
よって、$(与式) = e^{-sa}\mathcal{L}[f(t+a)] = (右辺)$

$(左辺) = (右辺)$ //

(3) $f(t) = \{U(t-1) - U(t-3)\}\{-(t^2-4t+3)\}$
$= (t-1)(t-3)U(t-3) - (t-1)(t-3)U(t-1)$
ラプラス変換して、
$F(s) = \mathcal{L}[(t-1)(t-3)U(t-3)] - \mathcal{L}[(t-1)(t-3)U(t-1)]$
$= e^{-3s}\mathcal{L}[\{(t+3)-1\}\{(t+3)-3\}] - e^{-s}\mathcal{L}[\{(t+1)-1\}\{(t+1)-3\}]$
$= e^{-3s}\mathcal{L}[t(t+2)] - e^{-s}\mathcal{L}[t(t-2)]$
$= -e^{-3s}\frac{d}{ds}\mathcal{L}[t+2] + e^{-s}\frac{d}{ds}\mathcal{L}[t-2]$

$\mathcal{L}[t+2] = \int_0^\infty (t+2)e^{-st}dt = \int_0^\infty te^{-st}dt + 2\int_0^\infty e^{-st}dt$
$= \int_0^\infty t\left(-\frac{1}{s}e^{-st}\right)'dt - \frac{2}{s}\left[e^{-st}\right]_0^\infty$
$= \left[-\frac{t}{s}e^{-st}\right]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}dt + \frac{2}{s}$
$= 0 - \frac{1}{s^2}\left[e^{-st}\right]_0^\infty + \frac{2}{s}$
$= \frac{1}{s^2} + \frac{2}{s}$

同様に，$\mathcal{L}[t-2] = \frac{1}{s^2} - \frac{2}{s}$

よって，

$$
\begin{aligned}
F(s) &= -e^{-3s}\frac{d}{ds}\left(\frac{1}{s^2}+\frac{2}{s}\right) + e^{-s}\frac{d}{ds}\left(\frac{1}{s^2}-\frac{2}{s}\right) \\
&= -e^{-3s}\left(-\frac{2}{s^3} - \frac{2}{s^2}\right) + e^{-s}\left(-\frac{2}{s^3} + \frac{2}{s^2}\right) \\
&= 2e^{-3s}\left(\frac{1}{s^3} + \frac{1}{s^2}\right) + 2e^{-s}\left(-\frac{1}{s^3} + \frac{1}{s^2}\right) \\
&= \boxed{2\left\{\frac{1}{s^3}\left(e^{-3s} - e^{-s}\right) + \frac{1}{s^2}\left(e^{-3s} + e^{-s}\right)\right\}}
\end{aligned}
$$