H30 数学 解答（途中あり）

[1] (1) $y = \dfrac{x^{x^x}}{\sqrt{\cos x}} = x^{x^x} \cdot (\cos x)^{-\frac{1}{2}}$

$y' = (x^{x^x})' \cdot (\cos x)^{-\frac{1}{2}} + x^{x^x} \cdot \{(\cos x)^{-\frac{1}{2}}\}'$ — ①

$a_1 = x^{x^x}$ とすると

$\log a_1 = x^x \log x$

$\dfrac{a_1'}{a_1} = (x^x)' \log x + x^x \cdot (\log x)'$ — ②

ここで $a_2 = x^x$ とすると

$\log a_2 = x \log x$

$\dfrac{a_2'}{a_2} = \log x + 1$

$a_2' = x^x(\log x + 1)$

これを②に代入すると

$\dfrac{a_1'}{a_1} = x^x(\log x + 1)\log x + \dfrac{x^x}{x}$

$a_1' = x^{x^x}\left[x^x\left\{(\log x + 1)\log x + \dfrac{1}{x}\right\}\right]$ — ③

また $b = \dfrac{1}{\sqrt{\cos x}}$ とすると

$b' = -\dfrac{1}{2}\dfrac{-\sin x}{\cos x\sqrt{\cos x}} = \dfrac{1}{2}\dfrac{\tan x}{\sqrt{\cos x}}$ — ④

③,④を①に代入すると

$y' = \dfrac{x^{x^x}}{\sqrt{\cos x}}\left[x^x\{(\log x + 1)\log x\}\right] + \dfrac{x^{x^x}\tan x}{2\sqrt{\cos x}}$

$= \dfrac{x^{x^x}}{\sqrt{\cos x}}\left[x^x\{(\log x + 1)\log x\} + \dfrac{\tan x}{2}\right]$

(2) $\displaystyle\int_0^\infty \frac{4}{e^x + e^{-x}}\,dx = 4\int_0^\infty \frac{e^x}{e^{2x}+1}\,dx$

$e^x = t$ とおくと $e^x dx = dt$

x	$0 \to \infty$	
t	$1 \to \infty$	

$(\text{与式}) = 4\displaystyle\int_1^\infty \frac{1}{t^2+1}\,dt$

$= 4\left[\tan^{-1} t\right]_1^\infty$

$= 4(\tan^{-1}\infty - \tan^{-1} 1)$

$= 4\left(\dfrac{\pi}{2} - \dfrac{\pi}{4}\right)$

$= \pi$

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[2]
$$\begin{cases} x + z = 0 \\ x + y + 3z + 2au = 0 \\ 2x + y + 5z + au = 0 \\ 3x - y + z - au = b \end{cases}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 3 & 2a & 0 \\ 2 & 1 & 5 & a & 0 \\ 3 & -1 & 1 & -a & b \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2a & 0 \\ 0 & 1 & 3 & a & 0 \\ 0 & -1 & -2 & -a & b \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2a & 0 \\ 0 & 0 & 1 & -a & 0 \\ 0 & 0 & 0 & a & b \end{array}\right]$$

$$= \left[\begin{array}{cccc|c} 1 & 0 & 0 & a & 0 \\ 0 & 1 & 0 & 4a & 0 \\ 0 & 0 & 1 & -a & 0 \\ 0 & 0 & 0 & a & b \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -b \\ 0 & 1 & 0 & 0 & -4b \\ 0 & 0 & 1 & 0 & b \\ 0 & 0 & 0 & 1 & b/a \end{array}\right]$$

(1) 一意解をもつためには
上式の結果より $a \neq 0$

(2) 無数の解をもつ条件
上式の結果より $a = b = 0$

(3) 解が存在しない条件
上式の結果より $a = 0,\ b \neq 0$

(4) $A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 3 & 2a \\ 2 & 1 & 5 & a \\ 3 & -1 & 1 & -a \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 3 & 2 \\ 2 & 1 & 5 & 1 \\ 3 & -1 & 1 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \\ -2 \end{bmatrix}$

まず

$$A_{11} = (-1)^{1+1}\begin{vmatrix} 1 & 3 & 2 \\ 1 & 5 & 1 \\ -1 & 1 & -1 \end{vmatrix} = (-5-3+2) - (-10-3+1) = 6$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix} 0 & 1 & 0 \\ 1 & 5 & 1 \\ -1 & 1 & -1 \end{vmatrix} = (0-1+0) - (0+0-1) = 0$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} 0 & 1 & 0 \\ 1 & 3 & 2 \\ -1 & 1 & -1 \end{vmatrix} = (-2+0+0) - (-1+0+0) = -1$$

$$A_{41} = (-1)^{4+1}\begin{vmatrix} 0 & 1 & 0 \\ 1 & 3 & 2 \\ 1 & 5 & 1 \end{vmatrix} = \{(0+2+0) - (0+0+1)\} = -1$$

$$A_{12} = (-1)^{3}\begin{vmatrix} 1 & 3 & 2 \\ 2 & 5 & 1 \\ 3 & 1 & -1 \end{vmatrix} = \{(-5+9+4) - (30+1-6)\} = 17$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 1 & 0 \\ 2 & 5 & 1 \\ 3 & 1 & -1 \end{vmatrix} = (-5+3+0) - (0+1-2) = -1$$

$$A_{23} = (-1)^{2+3} \cdot \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 1 \\ 3 & -1 & -1 \end{vmatrix} = -\{(-1)-(-1)\} = 0$$

$$A_{24} = (-1)^{2+4} \cdot 2 \begin{vmatrix} 1 & 0 & 1 \\ 2 & 1 & 5 \\ 3 & -1 & 1 \end{vmatrix} = 2\{(1-2)-(3-5)\} = 2$$

$$A_{13} = (-1)^{1+3} \begin{vmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 3 & -1 & -1 \end{vmatrix} = (-1+3-4)-(6-1-2) = -5$$

$$A_{14} = (-1)^5 \begin{vmatrix} 1 & 1 & 3 \\ 2 & 1 & 5 \\ 3 & -1 & 1 \end{vmatrix} = -\{(1-6+15)-(9-5+2)\} = -4$$

$$A_{32} = (-1)^5 \cdot \begin{vmatrix} 1 & 1 & 0 \\ 1 & 3 & 2 \\ 3 & 1 & -1 \end{vmatrix} = -\{(-3+6+0)-(0+2-1)\} = -2$$

$$A_{33} = (-1)^6 \cdot \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 2 \\ 3 & -1 & -1 \end{vmatrix} = \{(-1+0+0)-(0+0-2)\} = 1$$

$$A_{34} = (-1)^7 \cdot \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 3 \\ 3 & -1 & 1 \end{vmatrix} = -\{(1-1+0)-(3+0-3)\} = 0$$

$$A_{42} = (-1)^6 \begin{vmatrix} 1 & 1 & 0 \\ 1 & 3 & 2 \\ 2 & 5 & 1 \end{vmatrix} = \{(3+4+0)-(0+10+1)\} = -4$$

$$A_{43} = (-1)^7 \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 2 \\ 2 & 1 & 1 \end{vmatrix} = -1\{(1+0+0)-(0+0+2)\} = 1$$

$$A_{44} = (-1)^8 \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 3 \\ 2 & 1 & 5 \end{vmatrix} = (5+0+1)-(2+0+3) = 1$$

よって

$$A^{-1} = \begin{bmatrix} 6 & 0 & -1 & -1 \\ 17 & -1 & -2 & -4 \\ -5 & 0 & 1 & 1 \\ -1 & 2 & 0 & 1 \end{bmatrix}$$

したがって

$$A^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ -2 \\ -2 \end{bmatrix}$$

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[別解]

$$A = \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 3 & 2 & 0 & 1 & 0 & 0 \\ 2 & 1 & 5 & 1 & 0 & 0 & 1 & 0 \\ 3 & -1 & 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & -2 & 0 & 1 & 0 \\ 0 & -1 & -2 & -1 & -3 & 0 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 2 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -4 & 1 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 2 & 1 & -1 & 0 \\ 0 & 1 & 0 & 4 & 1 & 3 & -2 & 0 \\ 0 & 0 & 1 & -1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -4 & 1 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 6 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 & 17 & -1 & -2 & -4 \\ 0 & 0 & 1 & 0 & -5 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -4 & 1 & 0 & 1 \end{array}\right]$$

よって

$$A^{-1} = \begin{bmatrix} 6 & 0 & -1 & -1 \\ 17 & -1 & -2 & -4 \\ -5 & 0 & 1 & 1 \\ -4 & 1 & 0 & 1 \end{bmatrix}$$

$$A^{-1}\begin{bmatrix} 0 \\ 0 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ -2 \\ -2 \end{bmatrix}$$

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[3] (1) $y'' + 3y' - 4y = 2x^2$

$y = e^{\lambda x}$ とおくと 特性方程式は

$\lambda^2 + 3\lambda - 4 = 0$

$(\lambda + 4)(\lambda - 1) = 0$

$\lambda = 1, -4$

判別式Dは $9 - 4\cdot(-4) = 25 > 0$ よって 異なる2つの実数解をもつ

よって

$y = C_1 e^{x} + C_2 e^{-4x}$

(2) 特殊解 $y_0 = Ax^2 + Bx + C$ とすると

$y_0' = 2Ax + B$

$y_0'' = 2A$

これらを与式に代入すると

$2A + 3(2Ax + B) - 4(Ax^2 + Bx + C) = 2x^2$

$-4Ax^2 + (6A - 4B)x + 2A + 3B - 4C = 2x^2$

$$\begin{cases} -4A = 2 \\ 6A - 4B = 0 \\ 2A + 3B - 4C = 0 \end{cases}$$

$A = -\frac{1}{2},\ B = -\frac{3}{4},\ C = -\frac{13}{16}$

よって

$y_0 = -\frac{1}{2}x^2 - \frac{3}{4}x - \frac{13}{16}$

(3)

$y = C_1 e^{x} + C_2 e^{-4x} - \frac{1}{2}\left(x^2 + \frac{3}{2}x + \frac{13}{8}\right)$

[4] (1) $2\sin 5t \cos 4t + \cos 5t = \sin 9t + \sin t + \cos 5t$

$\mathcal{L}\{2\sin 5t\cos 4t + \cos 5t\} = \mathcal{L}\{\sin 9t + \sin t + \cos 5t\}$

$= \frac{9}{s^2+81} + \frac{1}{s^2+1} + \frac{s}{s^2+25}$

(2) $\frac{se^{-as}}{s^2+b^2}$ $(a>0)$

$\mathcal{L}^{-1}\left\{\frac{se^{-as}}{s^2+b^2}\right\} = U(t-a)\cos\{b(t-a)\}$

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(3) $f(t) + 4\int_0^t (t-\lambda) f(\lambda) d\lambda = t$

$f(t) + 4t\int_0^t f(\lambda) d\lambda - 4\int_0^t \lambda f(\lambda) d\lambda = t$

$f(t) + 4t\int_0^t f(\lambda) d\lambda - 4[\lambda F(\lambda)]_0^t + 4\int_0^t F(\lambda) d\lambda = t$

$f'(t) + 4\int_0^t f(\lambda) d\lambda + 4tf(t) - 4F(t) - 4tf(t) + 4F(t) = 1$

$f''(t) + 4f(t) = 0$

$s^2\mathcal{L}\{f(t)\} - f(0) - f'(0) + 4\mathcal{L}\{f(t)\} = 0$ —①

ここで

$f(0) + 4\int_0^0 (t-\lambda) f(\lambda) d\lambda = 0$

$f(0) = 0$

$f'(0) + 4\int_0^0 f(\lambda) d\lambda = 1$

$f'(0) = 1$

よって

①に代入すると

$s^2\mathcal{L}\{f(t)\} - 0 - 1 + 4\mathcal{L}\{f(t)\} = 0$

$(s^2+4)\mathcal{L}\{f(t)\} = 1$

$\mathcal{L}\{f(t)\} = \frac{1}{s^2+4}$

したがって

$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}$

$= \frac{1}{2}\sin 2t$