H27

数学 1回目

[1] $\left(x-\frac{1}{x}\right)y''-4y'=4$

$y'+P(x)y=Q(x)$ のとき、
$p=\exp\left[-\int P(x)dx\right]\times\left(\int Q(x)e^{\int P(x)dx}dx+C\right)$

クレローの微分方程式より、

$y'=p=e^{-\int P(x)dx}\left\{\int Q(x)e^{\int P(x)dx}dx+C\right\}$ より、

$$\left(x-\frac{1}{x}\right)p'-4p=4$$

$$p'-\frac{4x}{x^2-1}p=\frac{4x}{x^2-1}$$

$$p=e^{+\int\frac{4x}{x^2-1}dx}\cdot\left\{\int\frac{4x}{x^2-1}\cdot e^{-\int\frac{4x}{x^2-1}dx}dx+C\right\}$$

$$=e^{\int\frac{4x}{x^2-1}dx}\cdot\left\{-\int\left(e^{-\int\frac{4x}{x^2-1}dx}\right)'dx+C\right\}$$

$$=e^{\int\frac{4x}{x^2-1}dx}\left(-e^{-\int\frac{4x}{x^2-1}dx}+C\right)$$

$$=-1+Ce^{\int\frac{4x}{x^2-1}dx}$$

$\int\frac{4x}{x^2-1}dx=2\log|x^2-1|+C_1$ より、

$$p=-1+C\exp\left[2\log|x^2-1|+C_1\right]=-1+C(x^2-1)^2\cdot e^{C_1}=-1+C_2(x^2-1)^2$$

$(\because C_2=C\cdot e^{C_1})$

$$y=\int p\,dx=\int\left\{-1+C_2(x^2-1)^2\right\}dx$$

$$=\int\left(C_2x^4-2C_2x^2+C_2-1\right)dx$$

$$=\frac{1}{5}C_2x^5-\frac{2}{3}C_2x^3+C_2x-x+C_3$$

$$=\boxed{C_2\left(\frac{1}{5}x^5-\frac{2}{3}x^3+x\right)-x+C_3}$$

[2] $\begin{cases} x_1 \quad\quad -2x_3 - x_4 = -3 \\ -x_1 + x_2 \quad\quad +3x_4 = -1 \\ x_1 - x_2 + x_3 - 3x_4 = 5 \\ x_1 - 2x_2 \quad\quad -5x_4 = -3 \end{cases}$

$A = \begin{bmatrix} 1 & 0 & -2 & -1 \\ -1 & 1 & 0 & 3 \\ 1 & -1 & 1 & -3 \\ 1 & -2 & 0 & -5 \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$, $b = \begin{bmatrix} -3 \\ -1 \\ 5 \\ -3 \end{bmatrix}$ とおくと,

拡大係数行列 $[A \ b]$

$$\left[\begin{array}{cccc|c} 1 & 0 & -2 & -1 & -3 \\ -1 & 1 & 0 & 3 & -1 \\ 1 & -1 & 1 & -3 & 5 \\ 1 & -2 & 0 & -5 & -3 \end{array}\right] = B \text{ とおく}$$

$$P_{41}(-1)P_{31}(-1)P_{21}(1)B = \left[\begin{array}{cccc|c} 1 & 0 & -2 & -1 & -3 \\ 0 & 1 & -2 & 2 & -4 \\ 0 & -1 & 3 & -2 & 8 \\ 0 & -2 & 2 & -4 & 0 \end{array}\right] = B_1$$

$$P_{42}(2)P_{32}(1)B_1 = \left[\begin{array}{cccc|c} 1 & 0 & -2 & -1 & -3 \\ 0 & 1 & -2 & 2 & -4 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & -2 & 0 & -8 \end{array}\right] = B_2$$

$$P_{43}(2)P_{23}(2)P_{13}(2)B_2 = \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 5 \\ 0 & 1 & 0 & 2 & 4 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\begin{cases} x_1 - x_4 = 0 \\ x_2 + 2x_4 = 0 \end{cases}$

$x_4 = c$ とおく

$x_1 = c,\ x_2 = -2c$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \boxed{\begin{bmatrix} 5 \\ 4 \\ 4 \\ 0 \end{bmatrix} + c\begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix}}$$ (∵ c は任意定数)

[3] (1) $\lim_{x \to 0} \dfrac{a^x - 1}{x} \quad (a > 0)$

$g'(x) \neq 0$ かつ $\begin{cases} \lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty \\ \text{or} \\ \lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0 \end{cases}$

ロピタルの
~~フェルマーの最終~~定理より,

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f'(x)}{g'(x)}$$

$$\lim_{x\to 0} \frac{(a^x - 1)'}{(x)'} = \lim_{x\to 0} \frac{a^x \log a}{1} = \boxed{\log a}$$

(2) $\int_0^\infty t e^{-2t} dt$

$$= \int_0^\infty t \left(-\frac{1}{2} e^{-2t}\right)' dt$$

$$= \left[-\frac{1}{2} t e^{-2t}\right]_0^\infty + \frac{1}{2} \int_0^\infty e^{-2t} dt$$

$$= 0 - \frac{1}{4} \left[e^{-2t}\right]_0^\infty$$

$$= -\frac{1}{4}(0 - 1)$$

$$= \boxed{\frac{1}{4}}$$

[4] (1) $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}(1+2t)e^{at} dt \quad (t \geqq 0)$

$$= \int_0^\infty e^{-(s-a)t}(1+2t) dt$$

$$= \int_0^\infty \left(-\frac{1}{s-a} e^{-(s-a)t}\right)' (1+2t) dt$$

$$= \left[-\frac{1}{s-a} e^{-(s-a)t}(1+2t)\right]_0^\infty + \frac{1}{s-a} \int_0^\infty 2 \cdot e^{-(s-a)t} dt$$

$$= -\frac{1}{s-a}(0 - 1) - \frac{2}{(s-a)^2} \left[e^{-(s-a)t}\right]_0^\infty$$

$$= \boxed{\frac{1}{s-a} + \frac{2}{(s-a)^2}}$$

(2) $\mathcal{L}^{-1}\left\{\frac{s-1}{s(s+2)}\right\}$

sの因子について, $\left.\frac{s-1}{s+2}\right|_{s=0} = -\frac{1}{2}$

s+2の因子について, $\left.\frac{s-1}{s}\right|_{s=-2} = \frac{3}{2}$

$$\mathcal{L}^{-1}\left\{\frac{s-1}{s(s+2)}\right\} = \mathcal{L}^{-1}\left\{-\frac{1}{2}\frac{1}{s} + \frac{3}{2}\frac{1}{s+2}\right\}$$

$$= -\frac{1}{2} + \frac{3}{2}e^{-2t} = \boxed{\frac{1}{2}(3e^{-2t}-1)}$$

(3) $f'(t) - 2f(t) = e^{2t}$ $\quad f(0) = 0$

ラプラス変換して,

$$sF(s) - f(0) - 2F(s) = \frac{1}{s-2}$$

$$(s-2)F(s) = \frac{1}{s-2}$$

$$F(s) = \frac{1}{(s-2)^2}$$

$$\therefore f(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2}\right\} = \frac{1}{1!}\cdot e^{2t}\cdot t = \boxed{te^{2t}}$$