

[70240413 Statistical Machine Learning, Spring, 2019]

Supervised Learning

Classification

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Content

- ▶ **Supervised Learning**
- ▶ **Support Vector Machine**
- ▶ **Advanced Topics**

Supervised Learning

- ◆ Task: learn a predictive function

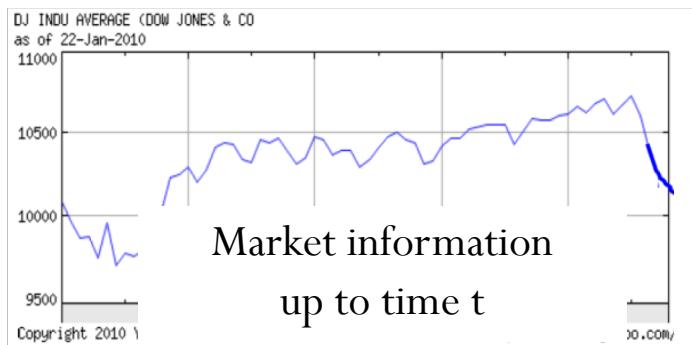
$$h : \mathcal{X} \rightarrow \mathcal{Y}$$

Feature space \mathcal{X}



Label space \mathcal{Y}

“Sports”
“News”
“Politics”
...



Share price
“\$ 20.50”

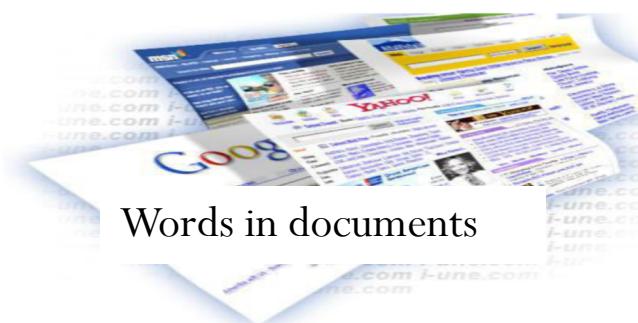


- ◆ “Experience” or training data:

$$\{\langle \mathbf{x}_d, y_d \rangle\}_{d=1}^D, \mathbf{x}_d \in \mathcal{X}, y_d \in \mathcal{Y}$$

Supervised Learning – classification

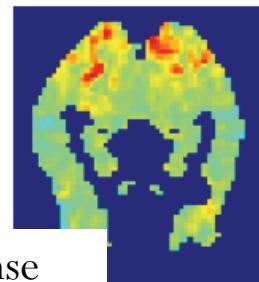
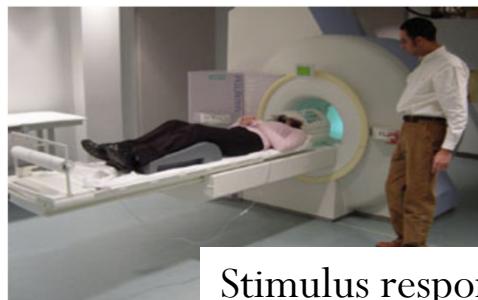
Feature space \mathcal{X}



Words in documents

Label space \mathcal{Y}

“Sports”
“News”
“Politics”
...



Stimulus response

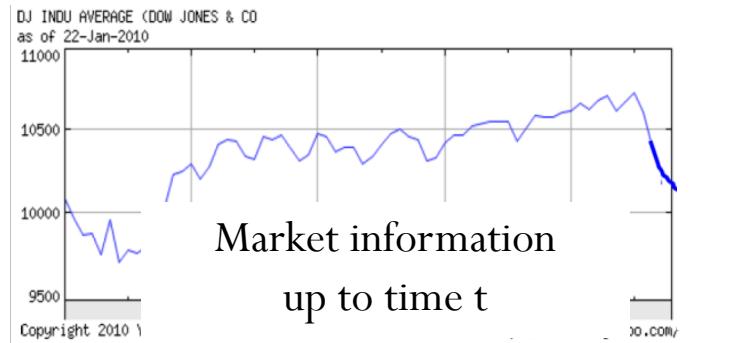
“Tool”
“Animal”
...



Discrete Labels

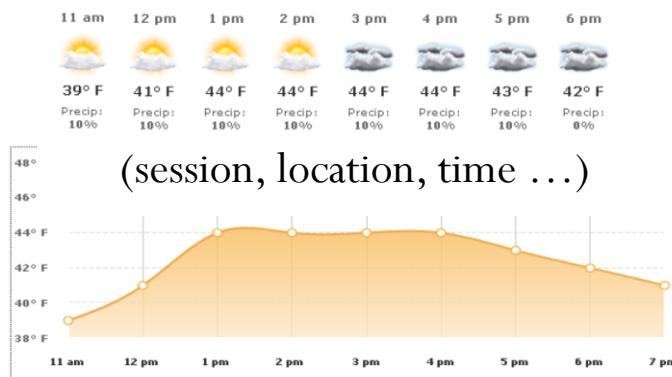
Supervised Learning – regression

Feature space \mathcal{X}



Label space \mathcal{Y}

Share price
“\$ 20.50”

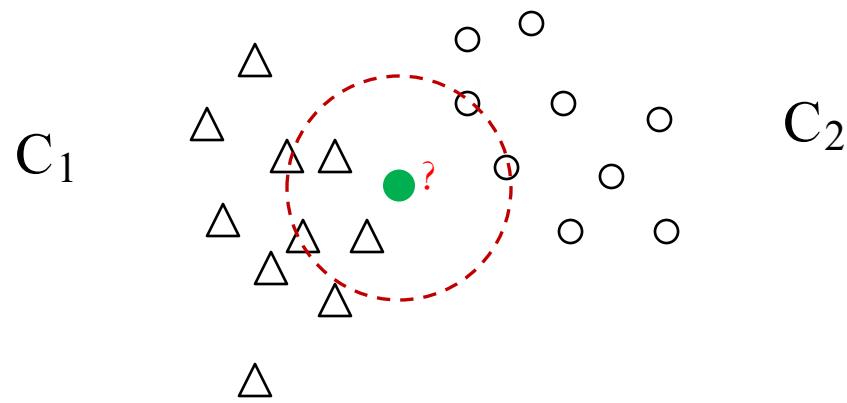


Temperature
“42° F”



Continuous Labels

How to learn a classifier?



K-NN: a Non-parametric approach

Properties of K-NN

- ◆ Simple
- ◆ Strong consistency results:
 - With infinite data, the error rate of K-NN is at most twice the optimal error rate (i.e., Bayes error rate)

$$R^* \leq R_\infty \leq 2R^*(1 - R^*) \rightarrow R^* \leq R_\infty \leq 2R^*$$

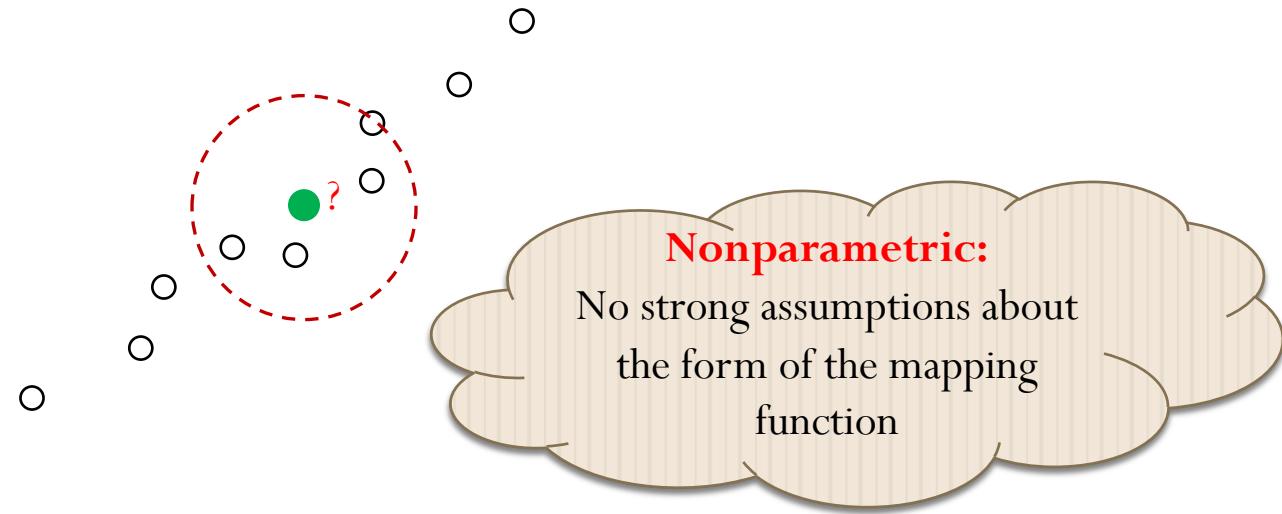
$$\Pr \{Y' = 0, Y = 1\} = \Pr \{Y' = 0\} \Pr \{Y = 1\} = (1 - \eta(\mathbf{X}))\eta(\mathbf{X})$$
$$\Pr \{Y' = 1, Y = 0\} = \Pr \{Y' = 1\} \Pr \{Y = 0\} = \eta(\mathbf{X})(1 - \eta(\mathbf{X}))$$

- ◆ **Note:** Bayes error rate – the minimum achievable error rate given the distribution of the data

Issues of K-NN

- ◆ Computationally intensive for large training sets
 - Clever nearest neighbor search helps
- ◆ Selection of K
- ◆ Distance metric matters a lot
 - Aware of the metric learning field

K-NN for regression

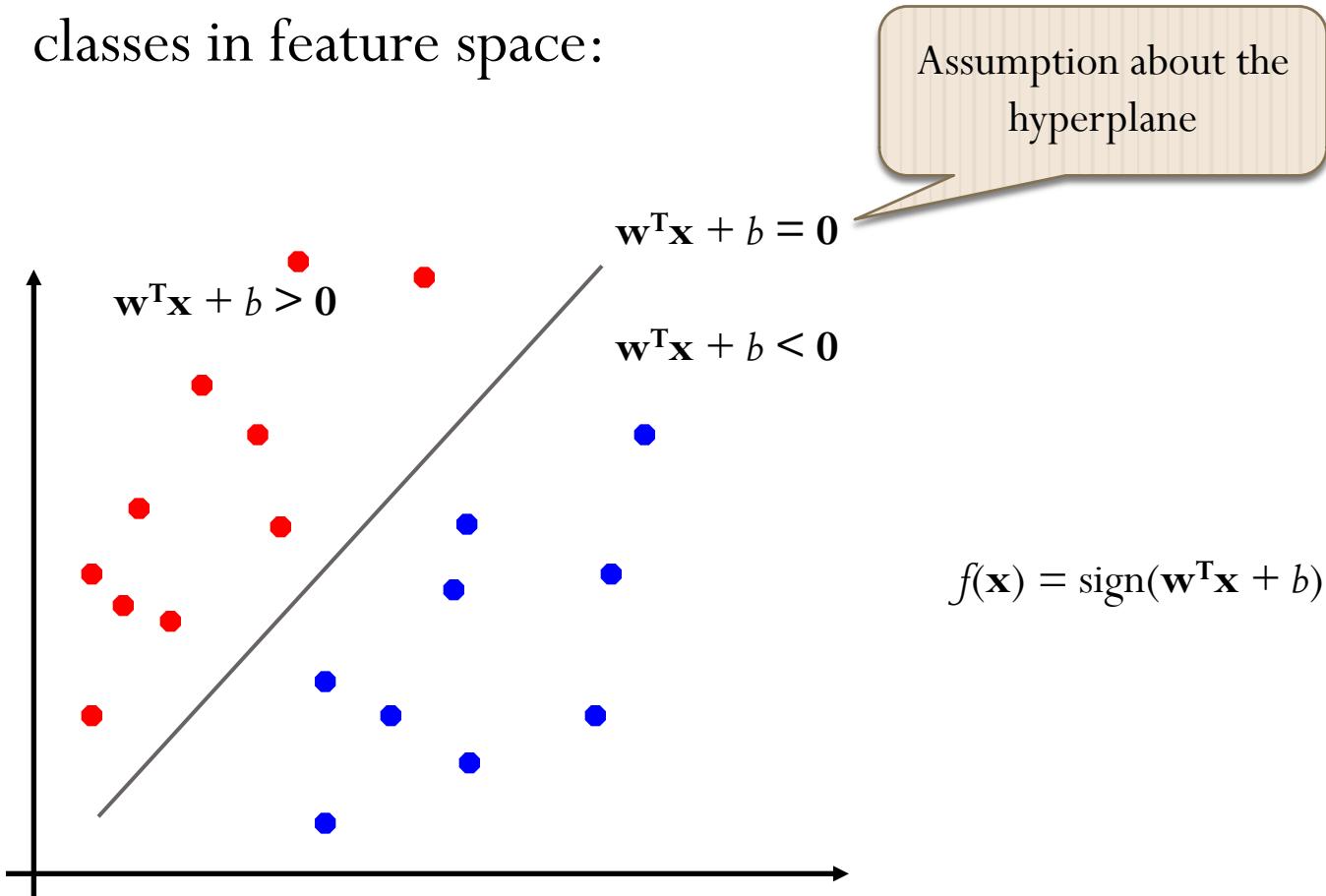


$$\hat{y} = \sum_{i \in \mathcal{N}_K} \frac{1}{dist(\mathbf{x}, \mathbf{x}_i)} y_i$$

A weighted average is an estimate; where the weight is the inverse distance

A Parametric Method

- ◆ Binary classification can be viewed as the task of separating classes in feature space:



A brief history

- ◆ Fisher's linear discriminator (1936)
 - Log-ratio of two Gaussians

$$F_{\text{sq}}(\mathbf{x}) = \text{sign} \left[\frac{1}{2} (\mathbf{x} - \mathbf{m}_1)^T \Sigma_1^{-1} (\mathbf{x} - \mathbf{m}_1) - \frac{1}{2} (\mathbf{x} - \mathbf{m}_2)^T \Sigma_2^{-1} (\mathbf{x} - \mathbf{m}_2) + \ln \frac{|\Sigma_2|}{|\Sigma_1|} \right]$$

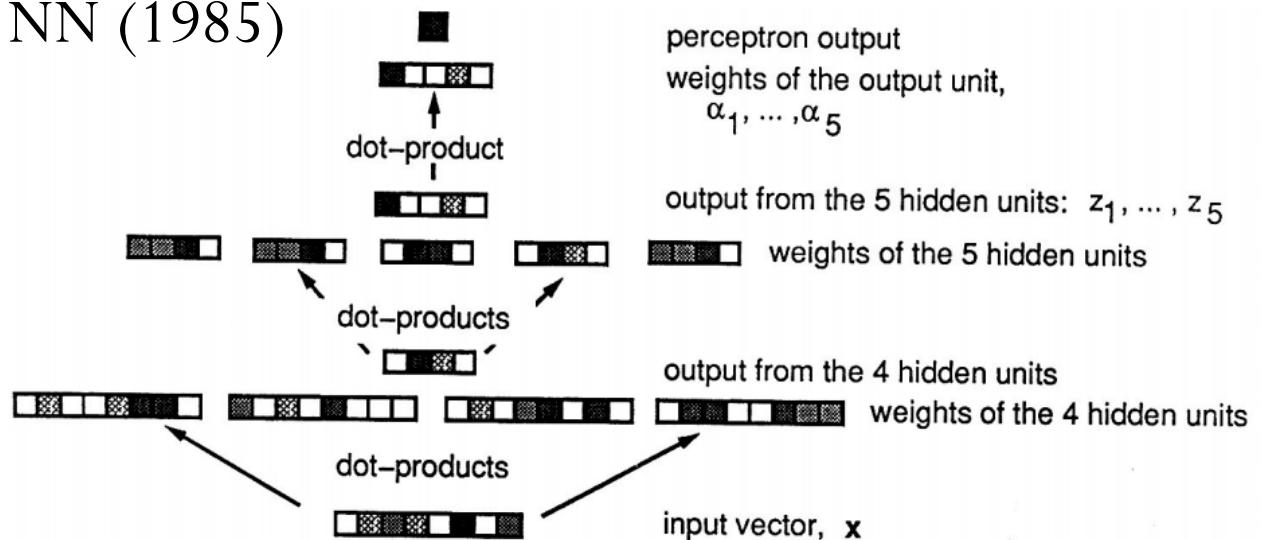
- ◆ Rosenblatt's perceptron (1962)

- Only top-layer is updated

- ◆ Back-prop for NN (1985)

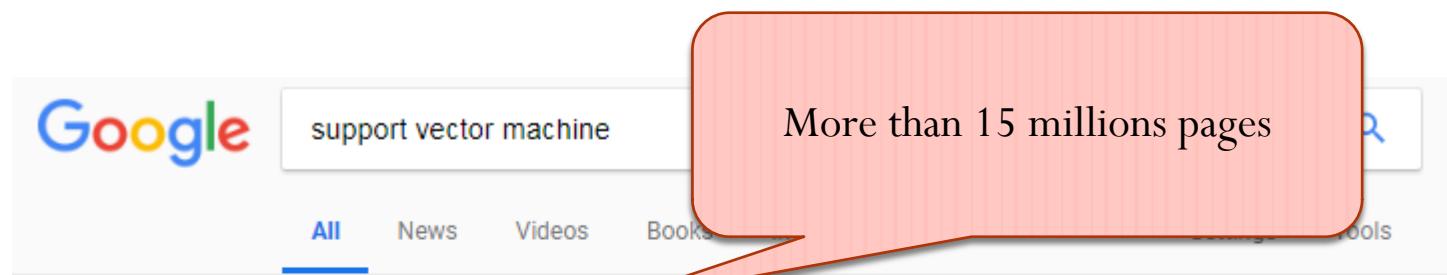
- ◆ SVM (1995)

- ◆ ...



Support Vector Machine

- ◆ The most popular supervised learning method



Scholarly articles for **support vector machine**

Least squares support vector machine classifiers - Suykens - Cited by 7052

Support vector machine - Adankon - Cited by 25

Support vector machine - Nugroho - Cited by 68

Support-vector networks

C Cortes, V Vapnik - Machine learning

Abstract The support-vector network is a new type of neural network, which
provides computational models for solving classification problems. The machine conceptually imple-
ments the following idea: input vectors are non-linearly mapped to a very high-dimension feature space. In this feature space a linear

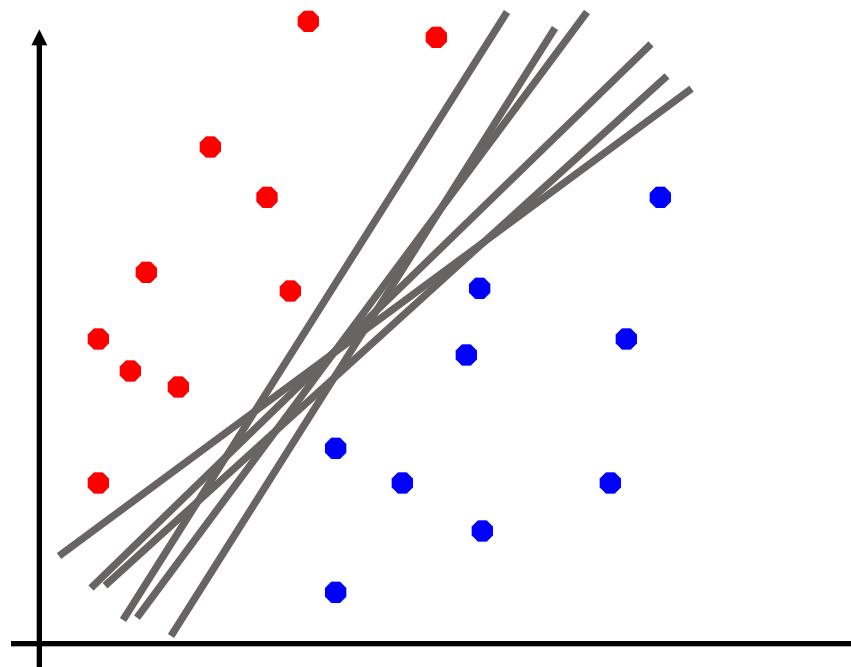
☆ 99 Cited by 29500 Related articles All 47 versions Web of Science: 12200

Nearly 30 thousand citations

[PDF] springer.com

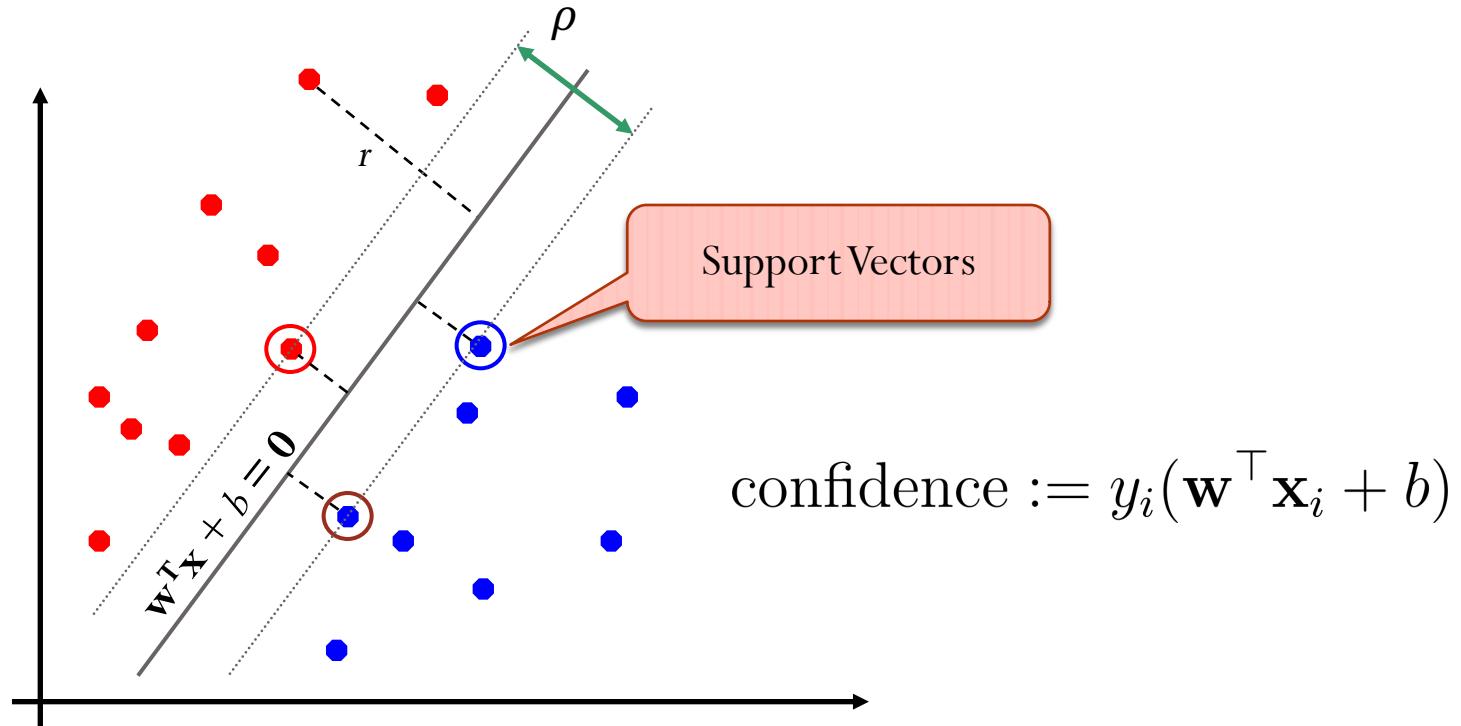
Linear Separators

- ◆ Which of the linear separators is optimal?



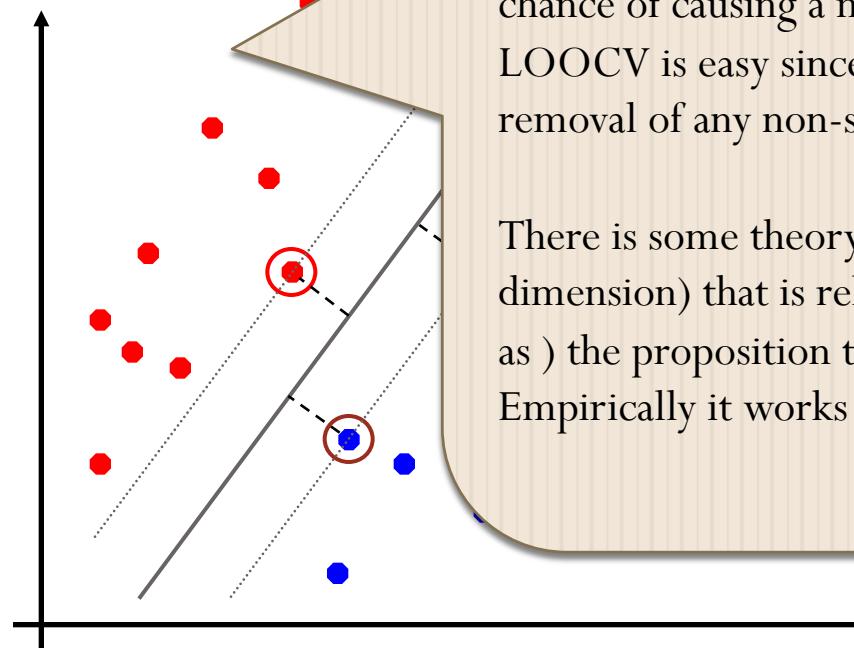
Classification Margin

- ◆ Distance from example \mathbf{x}_i to the separator is $r = \frac{|\mathbf{w}^\top \mathbf{x}_i + b|}{\|\mathbf{w}\|}$
- ◆ Examples closest to the hyperplane are *support vectors*.
- ◆ *Margin* ρ of the separator is the distance between supporting hyperplanes.



Max-margin Classification

- ◆ Maximizing the margin is good according to intuition and PAC theory.
- ◆ Implies that only support vectors matter; other training examples are ignorable.



Intuitively this feels safest.

If we've made a small error in the location of the boundary , this gives us least chance of causing a misclassification.
LOOCV is easy since the model is immune to removal of any non-support vector datapoints

There is some theory (using PAC theory or VC dimension) that is related to (but not the same as) the proposition that this is a good thing
Empirically it works very very well.

Linear SVM

- ◆ Let training set $\{(\mathbf{x}_i, y_i)\}_{i=1}^N$, $\mathbf{x}_i \in \mathbb{R}^d$, $y_i \in \{-1, +1\}$ be separated by a hyperplane with margin ρ . Then for each training example (\mathbf{x}_i, y_i) :

Constraint

$$\begin{aligned} \mathbf{w}^\top \mathbf{x}_i + b \leq -\rho/2 & \quad \text{if } y_i = -1 \\ \mathbf{w}^\top \mathbf{x}_i + b \geq \rho/2 & \quad \text{if } y_i = 1 \end{aligned} \Leftrightarrow y_i(\mathbf{w}^\top \mathbf{x}_i + b) \geq \rho/2$$

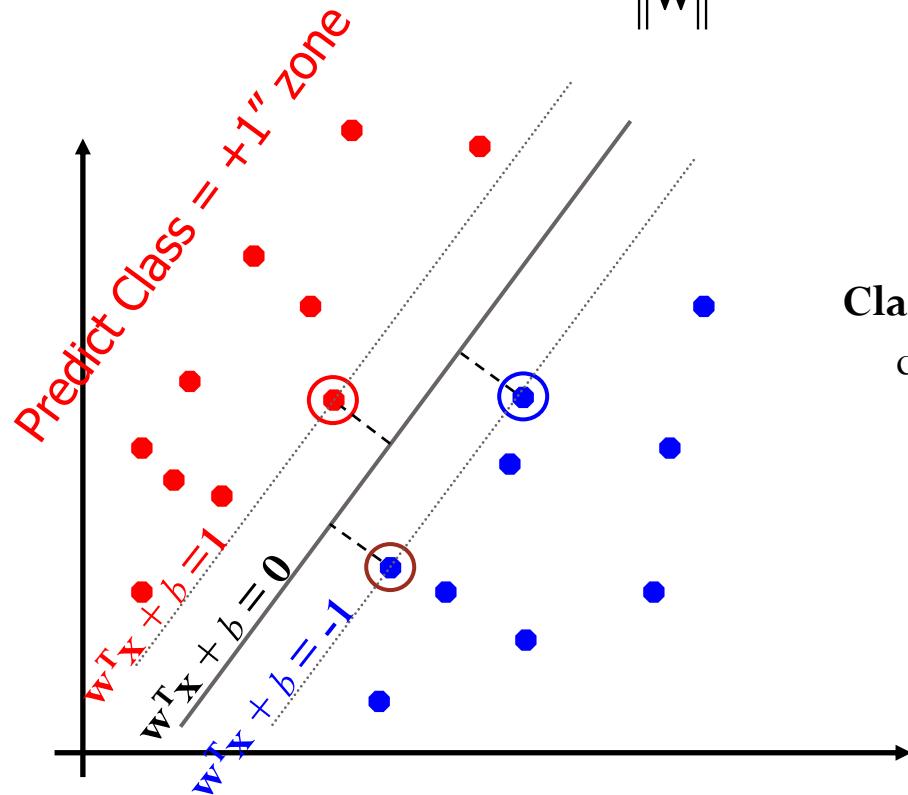
- ◆ For support vector \mathbf{x}_s the above inequality is an equality. After rescaling \mathbf{w} and b by $\rho/2$, we obtain that distance between each \mathbf{x}_s and the hyperplane is

$$r = \frac{y_s(\mathbf{w}^\top \mathbf{x}_s + b)}{\|\mathbf{w}\|} = \frac{1}{\|\mathbf{w}\|}$$

Linear SVM

- Then the margin can be expressed through (rescaled) \mathbf{w} and b as:

$$\rho = 2r = \frac{2}{\|\mathbf{w}\|} \quad \rightarrow \quad \hat{\mathbf{w}} = \arg \min_{\mathbf{w}} \frac{1}{2} \|\mathbf{w}\|^2$$



Classification rule:

classify as: +1 if $w^T x + b \geq 1$
-1 if $w^T x + b \leq -1$
universe explodes if $|w^T x + b| < 1$

Maximal Margin

Observations

- ◆ We can assume $b=0$

Classification rule:

$$\begin{aligned} \text{classify as: } & +1 & \text{if } & \mathbf{w}^\top \mathbf{x} + b \geq 1 \\ & -1 & \text{if } & \mathbf{w}^\top \mathbf{x} + b \leq -1 \\ & \text{universe } & \text{if } & |\mathbf{w}^\top \mathbf{x} + b| < 1 \\ & \text{explodes } & & \end{aligned}$$

- ◆ This is the same as:

$$y_i \mathbf{w}^\top \mathbf{x}_i \geq 1, \quad \forall i = 1, \dots, N$$

The Primal Hard SVM

- ◆ Given training dataset: $D = \{(\mathbf{x}_i, y_i)\}_{i=1}^N$
- ◆ Assume that D is linearly separable

$$\begin{aligned}\hat{\mathbf{w}} &= \arg \min_{\mathbf{w}} \frac{1}{2} \|\mathbf{w}\|^2 \\ \text{s.t.: } & y_i \mathbf{w}^\top \mathbf{x}_i \geq 1, \quad \forall i = 1, \dots, N\end{aligned}$$

- ◆ Prediction:

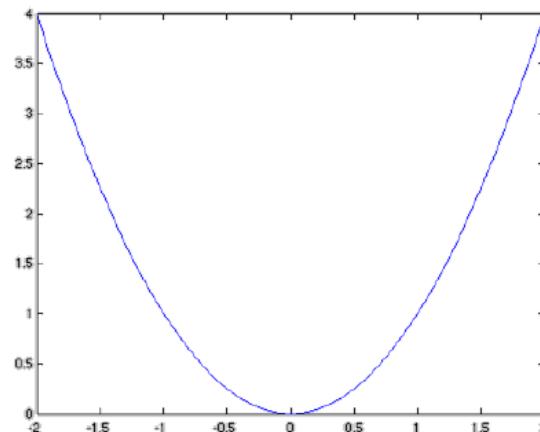
$$f(\mathbf{x}; \hat{\mathbf{w}}) = \text{sign}(\hat{\mathbf{w}}^\top \mathbf{x})$$

This is a QP problem (d-dimensional)
(Quadratic cost function, linear constraints)

Constrained Optimization

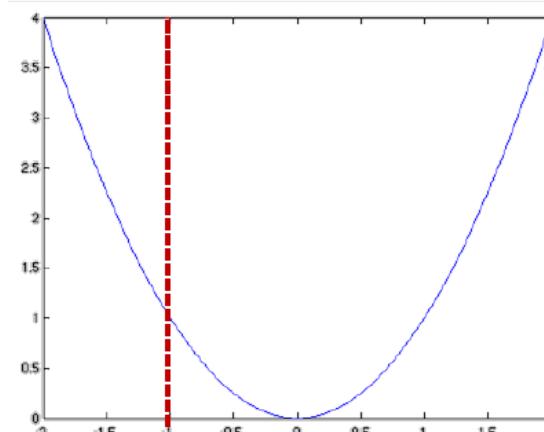
$$\begin{aligned} & \min_x \quad x^2 \\ \text{s.t.} \quad & x \geq b \end{aligned}$$

$$\min_x \quad x^2$$



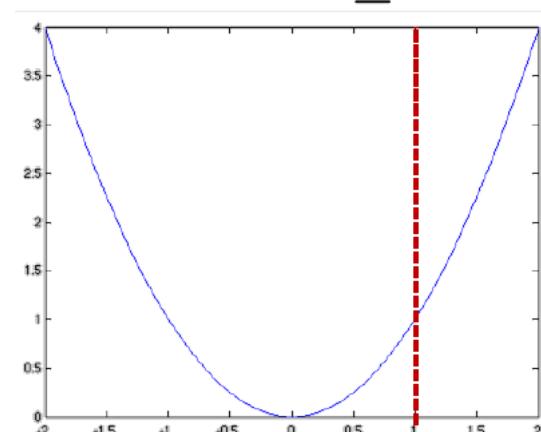
$$x^* = 0$$

$$\begin{aligned} & \min_x \quad x^2 \\ \text{s.t.} \quad & x \geq -1 \end{aligned}$$



$$x^* = 0$$

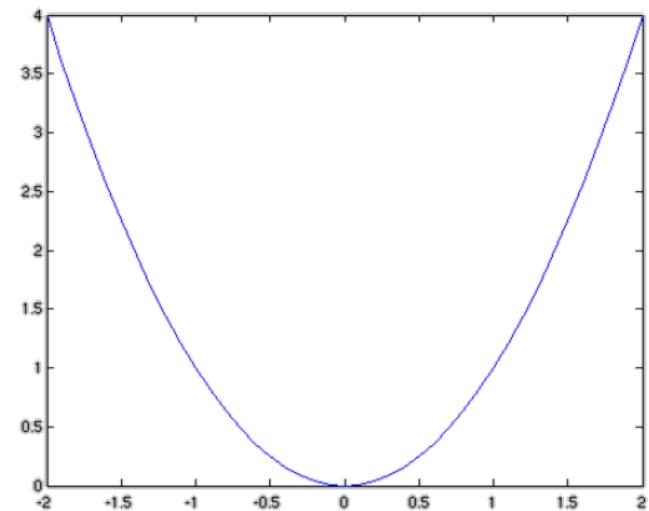
$$\begin{aligned} & \min_x \quad x^2 \\ \text{s.t.} \quad & x \geq 1 \end{aligned}$$



$$x^* = 1$$

Lagrange Multiplier

$$\begin{aligned} & \min_x \quad x^2 \\ \text{s.t.} \quad & x \geq b \end{aligned}$$



- ◆ Move the constraint to objective function – Lagrangian

$$L(x, \alpha) = x^2 - \alpha(x - b), \quad \text{s.t.: } \alpha \geq 0$$

- ◆ Solve:

$$\begin{aligned} & \min_x \max_{\alpha} \quad L(x, \alpha) \\ \text{s.t.:} \quad & \alpha \geq 0 \end{aligned}$$

Constraint is active when $\alpha > 0$

Lagrange Multiplier – dual variables

- ◆ Solving:

$$\begin{aligned} \min_x \max_{\alpha} \quad & L(x, \alpha) = x^2 - \alpha(x - b) \\ \text{s.t.:} \quad & \alpha \geq 0 \end{aligned}$$

- ◆ We get:

$$\frac{\partial L}{\partial x} = 0 \Rightarrow x^* = \frac{\alpha}{2}$$

$$\frac{\partial L}{\partial \alpha} = 0 \Rightarrow \alpha^* = \max(2b, 0)$$

When $\alpha > 0$, constraint is tight

From Primal to Dual

- ◆ Primal problem:

$$\begin{aligned}\hat{\mathbf{w}} &= \arg \min_{\mathbf{w}} \frac{1}{2} \|\mathbf{w}\|^2 \\ \text{s.t.: } & y_i \mathbf{w}^\top \mathbf{x}_i \geq 1, \quad \forall i = 1, \dots, N\end{aligned}$$

- ◆ Lagrangian function:

$$L(\mathbf{w}, \boldsymbol{\alpha}) = \frac{1}{2} \|\mathbf{w}\|^2 - \sum_{i=1}^N \alpha_i (y_i \mathbf{w}^\top \mathbf{x}_i - 1)$$

$$\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_N)^\top \geq 0$$

The Lagrange Problem

$$L(\mathbf{w}, \boldsymbol{\alpha}) = \frac{1}{2} \|\mathbf{w}\|^2 - \sum_{i=1}^N \alpha_i (y_i \mathbf{w}^\top \mathbf{x}_i - 1)$$

- ◆ The Lagrange problem:

$$(\hat{\mathbf{w}}, \hat{\boldsymbol{\alpha}}) = \arg \min_{\mathbf{w}} \max_{\boldsymbol{\alpha}} L(\mathbf{w}, \boldsymbol{\alpha})$$

$$0 = \frac{\partial L(\mathbf{w}, \boldsymbol{\alpha})}{\partial \mathbf{w}}|_{\hat{\mathbf{w}}} = \hat{\mathbf{w}} - \sum_i \alpha_i y_i \mathbf{x}_i$$

$$\Rightarrow \hat{\mathbf{w}} = \sum_{i=1}^N \alpha_i y_i \mathbf{x}_i$$

The Dual Problem

$$\begin{aligned} L(\mathbf{w}, \boldsymbol{\alpha}) &= \frac{1}{2} \|\mathbf{w}\|^2 - \sum_{i=1}^N \alpha_i (y_i \mathbf{w}^\top \mathbf{x}_i - 1) \\ \Rightarrow \hat{\mathbf{w}} &= \sum_{i=1}^N \alpha_i y_i \mathbf{x}_i \\ \Rightarrow L(\hat{\mathbf{w}}, \boldsymbol{\alpha}) &= \frac{1}{2} \|\hat{\mathbf{w}}\|^2 - \sum_i \alpha_i (y_i \hat{\mathbf{w}}^\top \mathbf{x}_i - 1) \\ &= \frac{1}{2} \left\| \sum_i \alpha_i y_i \mathbf{x}_i \right\|^2 + \boldsymbol{\alpha}^\top \mathbf{1} - \sum_i \alpha_i y_i \left(\sum_j \alpha_j y_j \mathbf{x}_j \right)^\top \mathbf{x}_i \\ &= \boldsymbol{\alpha}^\top \mathbf{1} - \frac{1}{2} \boldsymbol{\alpha}^\top \mathbf{Y} G \mathbf{Y} \boldsymbol{\alpha} \end{aligned}$$

$\mathbf{Y} := diag(y_1, \dots, y_N)$

$G \in \mathbb{R}^{N \times N}$, where $G_{ij} := \mathbf{x}_i^\top \mathbf{x}_j$ Gram matrix

The Dual Hard SVM

$$\mathbf{Y} := \text{diag}(y_1, \dots, y_N)$$

$G \in \mathbb{R}^{N \times N}$, where $G_{ij} := \mathbf{x}_i^\top \mathbf{x}_j$ Gram matrix

$$\hat{\boldsymbol{\alpha}} = \arg \max_{\boldsymbol{\alpha}} \quad \boldsymbol{\alpha}^\top \mathbf{1} - \frac{1}{2} \boldsymbol{\alpha}^\top \mathbf{Y} G \mathbf{Y} \boldsymbol{\alpha}$$

$$\text{s.t.: } \alpha_i \geq 0, \quad \forall i = 1, \dots, N$$

◆ Optimal solution:

$$\hat{\mathbf{w}} = \sum_{i=1}^N \hat{\alpha}_i y_i \mathbf{x}_i$$

◆ Prediction:

$$f(\mathbf{x}; \hat{\mathbf{w}}) = \text{sign} (\hat{\mathbf{w}}^\top \mathbf{x}) = \text{sign} \left(\sum_{i=1}^N \hat{\alpha}_i y_i \mathbf{x}_i^\top \mathbf{x} \right)$$

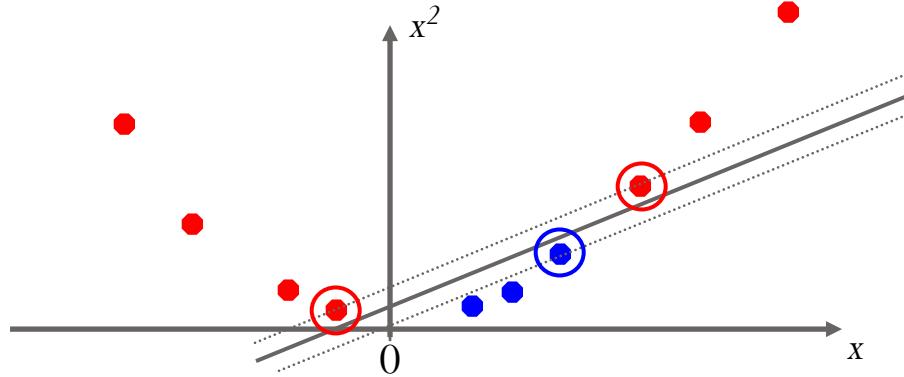
The Problem with Hard SVM

- ◆ It assumes samples are linearly separable ...
- ◆ How about if the data is not linearly separable?



The Problem with Hard SVM

- ◆ If the data is not linearly separable, adding new features might make it linearly separable



- Now drop this “augmented” data into our linear SVM!

The Problem with Hard SVM

- ◆ It assumes samples are linearly separable
- ◆ Solutions:
 - User feature transformation to a higher-dim space
 - Overfitting ☹
 - Soft margin SVM instead of hard SVM
 - Next slides

Hard SVM

- ◆ The hard SVM problem can be rewritten:

$$\begin{aligned}\hat{\mathbf{w}} &= \arg \min_{\mathbf{w}} \frac{1}{2} \|\mathbf{w}\|^2 \\ \text{s.t.: } & y_i \mathbf{w}^\top \mathbf{x}_i > 0, \quad \forall i = 1, \dots, N\end{aligned}$$



$$\hat{\mathbf{w}} = \arg \min_{\mathbf{w}} \frac{\lambda}{2} \|\mathbf{w}\|^2 + \sum_{i=1}^N \ell_{0-\infty}(y_i \mathbf{w}^\top \mathbf{x}_i)$$

$$\text{where } \ell_{0-\infty}(b) = \begin{cases} \infty & \text{if } b < 0 \\ 0 & \text{if } b \geq 0 \end{cases}$$

From Hard to Soft Constraints

- ◆ Instead of using hard constraints (linearly separable)

$$\hat{\mathbf{w}}_{hard} = \arg \min_{\mathbf{w}} \frac{\lambda}{2} \|\mathbf{w}\|^2 + \sum_{i=1}^N \ell_{0-\infty}(y_i \mathbf{w}^\top \mathbf{x}_i)$$

- ◆ We can try to solve the soft version of it:
 - The loss is only 1 instead of ∞ if misclassify an instance

$$\hat{\mathbf{w}}_{soft} = \arg \min_{\mathbf{w}} \frac{1}{2} \|\mathbf{w}\|^2 + \sum_{i=1}^N \ell_{0-1}(y_i \mathbf{w}^\top \mathbf{x}_i)$$

$$\text{where } \ell_{0-1}(b) = \begin{cases} 1 & \text{if } b < 0 \\ 0 & \text{if } b \geq 0 \end{cases}$$

Problems with 0/1 loss

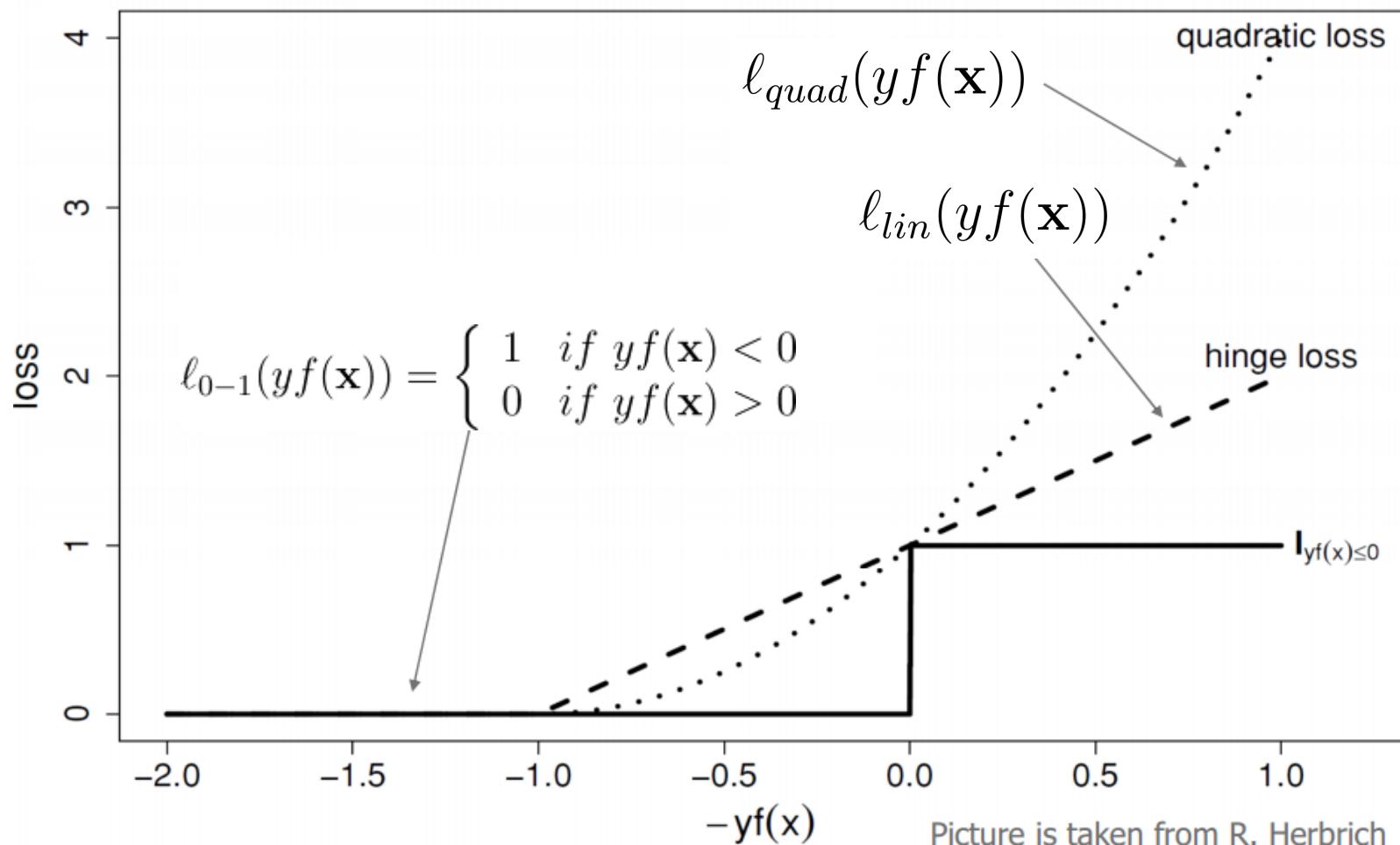
$$\hat{\mathbf{w}}_{soft} = \arg \min_{\mathbf{w}} \frac{\lambda}{2} \|\mathbf{w}\|^2 + \sum_{i=1}^N \ell_{0-1}(y_i \mathbf{w}^\top \mathbf{x}_i)$$

$$\text{where } \ell_{0-1}(b) = \begin{cases} 1 & \text{if } b < 0 \\ 0 & \text{if } b \geq 0 \end{cases}$$

- ◆ It is not convex in $y \mathbf{w}^\top \mathbf{x}$
 - It is not convex in \mathbf{w} , either
- ◆ We like convex functions ...

Approximation of the step function

$$f(\mathbf{x}) := \mathbf{w}^\top \mathbf{x}$$



Picture is taken from R. Herbrich

Approximation of 0/1 loss

- ◆ Piecewise linear approximation (hinge loss, convex, nonsmooth)

$$\ell_{lin}(yf(\mathbf{x})) = \max(0, 1 - yf(\mathbf{x}))$$

- we want $yf(\mathbf{x}) > 1$
- ◆ Quadratic approximation (square-loss, convex, smooth)

$$\ell_{quad}(yf(\mathbf{x})) = \max(0, 1 - yf(\mathbf{x}))^2$$

- ◆ Huber loss (combine the above two, convex, smooth)

$$\ell_{Huber}(yf(\mathbf{x})) = \begin{cases} 1 - yf(\mathbf{x}) & \text{if } yf(\mathbf{x}) < 0 \\ \max(0, 1 - yf(\mathbf{x}))^2 & \text{if } yf(\mathbf{x}) \geq 0 \end{cases}$$

The Hinge loss approximation of 0/1 loss

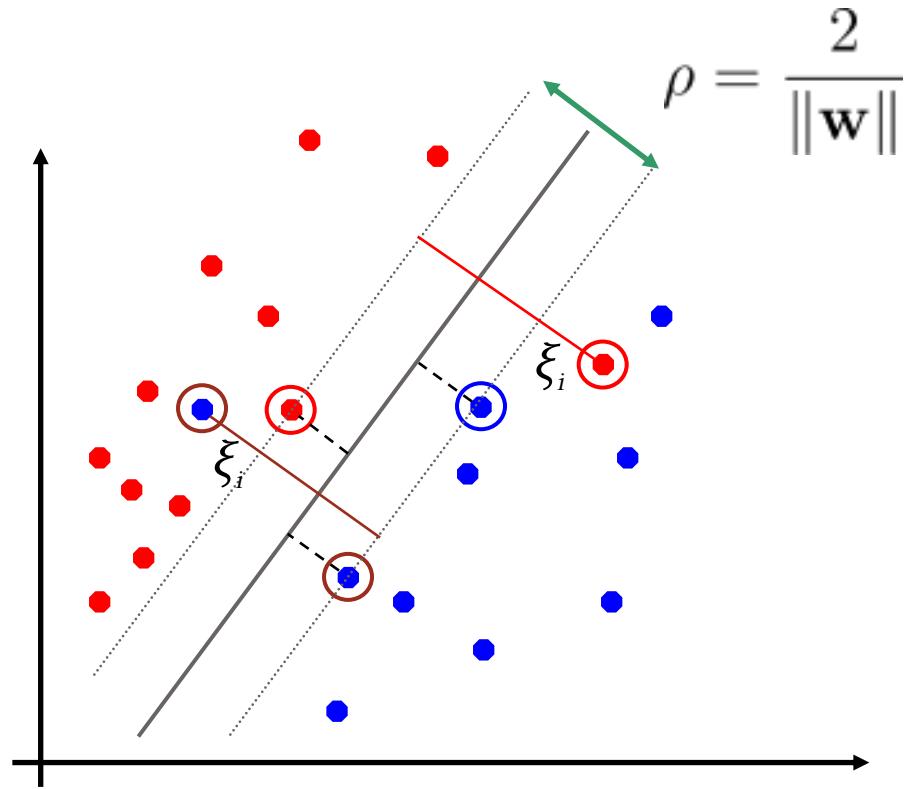
$$\hat{\mathbf{w}} = \arg \min_{\mathbf{w}} \frac{\lambda}{2} \|\mathbf{w}\|^2 + \sum_{i=1}^N \ell_{lin}(y_i \mathbf{w}^\top \mathbf{x}_i)$$

- where:

$$\begin{aligned}\ell_{lin}(y_i \mathbf{w}^\top \mathbf{x}_i) &= \max(0, 1 - y_i \mathbf{w}^\top \mathbf{x}_i) \\ &\geq \ell_{0-1}(y_i \mathbf{w}^\top \mathbf{x}_i)\end{aligned}$$

- The hinge loss upper bounds the 0/1 loss

Geometric interpretation: slack variables



$$\xi_i := \ell_{lin}(y_i \mathbf{w}^\top \mathbf{x}_i) = \max(0, 1 - y_i \mathbf{w}^\top \mathbf{x}_i)$$

The Primal Soft SVM problem

$$\hat{\mathbf{w}} = \arg \min_{\mathbf{w}} \frac{\lambda}{2} \|\mathbf{w}\|^2 + \sum_{i=1}^N \xi_i$$

where $\xi_i := \ell_{lin}(y_i \mathbf{w}^\top \mathbf{x}_i) = \max(0, 1 - y_i \mathbf{w}^\top \mathbf{x}_i)$

◆ Equivalently:

$$\hat{\mathbf{w}} = \arg \min_{\mathbf{w}, \xi} \frac{\lambda}{2} \|\mathbf{w}\|^2 + \sum_{i=1}^N \xi_i$$

s.t.: $y_i \mathbf{w}^\top \mathbf{x}_i \geq 1 - \xi_i, \forall i = 1, \dots, N$

$\xi_i \geq 0, \forall i = 1, \dots, N$

The Primal Soft SVM problem

$$\begin{aligned}\hat{\mathbf{w}} &= \arg \min_{\mathbf{w}, \xi} \frac{\lambda}{2} \|\mathbf{w}\|^2 + \sum_{i=1}^N \xi_i \\ \text{s.t.: } & y_i \mathbf{w}^\top \mathbf{x}_i \geq 1 - \xi_i, \quad \forall i = 1, \dots, N \\ & \xi_i \geq 0, \quad \forall i = 1, \dots, N\end{aligned}$$

◆ Equivalently:

$$\begin{aligned}\hat{\mathbf{w}} &= \arg \min_{\mathbf{w}, \xi} \frac{1}{2} \|\mathbf{w}\|^2 + C \sum_{i=1}^N \xi_i \\ C &= \frac{1}{\lambda}\end{aligned}$$

Dual Soft SVM (using hinge loss)

$$\begin{aligned}\hat{\mathbf{w}} &= \arg \min_{\mathbf{w}, \xi} \frac{1}{2} \|\mathbf{w}\|^2 + C \sum_{i=1}^N \xi_i \\ \text{s.t.: } &y_i \mathbf{w}^\top \mathbf{x}_i \geq 1 - \xi_i, \quad \forall i = 1, \dots, N \\ &\xi_i \geq 0, \quad \forall i = 1, \dots, N\end{aligned}$$

◆ Lagrange multipliers

$$\boldsymbol{\alpha} \geq 0, \quad \boldsymbol{\beta} \geq 0$$

◆ Lagrangian function

$$L(\mathbf{w}, \xi, \boldsymbol{\alpha}, \boldsymbol{\beta}) = \frac{1}{2} \|\mathbf{w}\|^2 + C \sum_i \xi_i - \sum_i \alpha_i (y_i \mathbf{w}^\top \mathbf{x}_i - 1 + \xi_i) - \sum_i \beta_i \xi_i$$

$$\min_{\mathbf{w}, \xi} \max_{\boldsymbol{\alpha}, \boldsymbol{\beta}} L(\mathbf{w}, \xi, \boldsymbol{\alpha}, \boldsymbol{\beta})$$

Dual Soft SVM (using hinge loss)

$$L(\mathbf{w}, \boldsymbol{\xi}, \boldsymbol{\alpha}, \boldsymbol{\beta}) = \frac{1}{2} \|\mathbf{w}\|^2 + C \boldsymbol{\xi}^\top \mathbf{1} - \sum_i \alpha_i y_i \mathbf{w}^\top \mathbf{x}_i + \boldsymbol{\alpha}^\top \mathbf{1} - \boldsymbol{\xi}^\top (\boldsymbol{\alpha} + \boldsymbol{\beta})$$

◆ We get:

$$0 = \frac{\partial L}{\partial \mathbf{w}}|_{\hat{\mathbf{w}}} \quad \Rightarrow \hat{\mathbf{w}} = \sum_{i=1}^N \alpha_i y_i \mathbf{x}_i$$

$$0 = \frac{\partial L}{\partial \boldsymbol{\xi}}|_{\hat{\boldsymbol{\xi}}} \quad \Rightarrow \boldsymbol{\beta} = C \mathbf{1} - \boldsymbol{\alpha} \geq 0 \\ \Rightarrow 0 \leq \boldsymbol{\alpha} \leq C \mathbf{1}$$

◆ Dual problem:

$$(\hat{\boldsymbol{\alpha}}, \hat{\boldsymbol{\beta}}) = \underset{0 \leq \boldsymbol{\alpha} \leq C \mathbf{1}; 0 \leq \boldsymbol{\beta}}{\operatorname{argmax}} L(\hat{\mathbf{w}}, \hat{\boldsymbol{\xi}}, \boldsymbol{\alpha}, \boldsymbol{\beta})$$

$$\hat{\boldsymbol{\alpha}} = \underset{0 \leq \boldsymbol{\alpha} \leq C \mathbf{1}}{\operatorname{argmax}} \boldsymbol{\alpha}^\top \mathbf{1} - \frac{1}{2} \boldsymbol{\alpha}^\top \mathbf{Y} G \mathbf{Y} \boldsymbol{\alpha}$$

Dual Soft SVM (using hinge loss)

$$\mathbf{Y} := \text{diag}(y_1, \dots, y_N)$$

$G \in \mathbb{R}^{N \times N}$, where $G_{ij} := \mathbf{x}_i^\top \mathbf{x}_j$ Gram matrix

$$\hat{\boldsymbol{\alpha}} = \underset{0 \leq \boldsymbol{\alpha} \leq C\mathbf{1}}{\operatorname{argmax}} \boldsymbol{\alpha}^\top \mathbf{1} - \frac{1}{2} \boldsymbol{\alpha}^\top \mathbf{Y} G \mathbf{Y} \boldsymbol{\alpha}$$

- ◆ This is the same as the dual hard SVM problem, except that we have additional constraints

SVM in the dual space

- ◆ Solve the dual problem

$$\hat{\boldsymbol{\alpha}} = \underset{0 \leq \boldsymbol{\alpha} \leq C\mathbf{1}}{\operatorname{argmax}} \boldsymbol{\alpha}^\top \mathbf{1} - \frac{1}{2} \boldsymbol{\alpha}^\top \mathbf{Y} G \mathbf{Y} \boldsymbol{\alpha}$$

- ◆ The primal solution

$$\Rightarrow \hat{\mathbf{w}} = \sum_{i=1}^N \hat{\alpha}_i y_i \mathbf{x}_i$$

- ◆ Prediction

$$f(\mathbf{x}; \hat{\mathbf{w}}) = \operatorname{sign} (\hat{\mathbf{w}}^\top \mathbf{x}) = \operatorname{sign} \left(\sum_{i=1}^N \hat{\alpha}_i y_i \mathbf{x}_i^\top \mathbf{x} \right)$$

Why it is called Support Vector Machines?

- ◆ Hard-SVM:

$$L(\mathbf{w}, \boldsymbol{\alpha}) = \frac{1}{2} \|\mathbf{w}\|^2 - \sum_{i=1}^N \alpha_i (y_i \mathbf{w}^\top \mathbf{x}_i - 1)$$

$$\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_N)^\top \geq 0$$

- ◆ KKT conditions (complementary slackness condition):

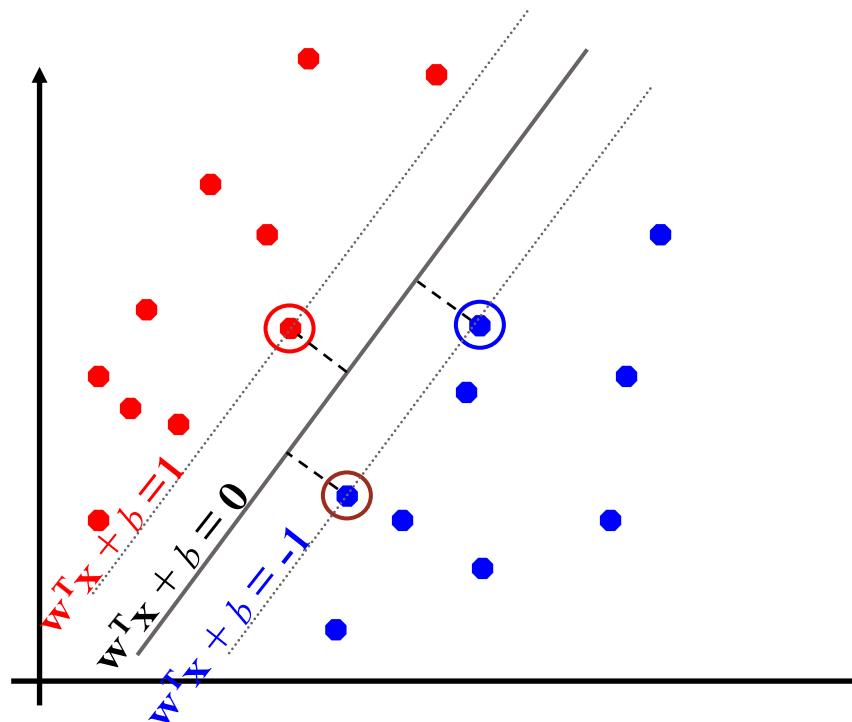
$$\forall i : \hat{\alpha}_i (y_i \hat{\mathbf{w}}^\top \mathbf{x}_i - 1) = 0$$

$$\hat{\alpha}_i = 0 \quad OR \quad \hat{\alpha}_i > 0 \Rightarrow y_i \hat{\mathbf{w}}^\top \mathbf{x}_i = 1$$

 \mathbf{x}_i is on the margin line! SUPPORT VECTORS

Why it is called Support Vector Machines?

- ◆ Hard SVM:



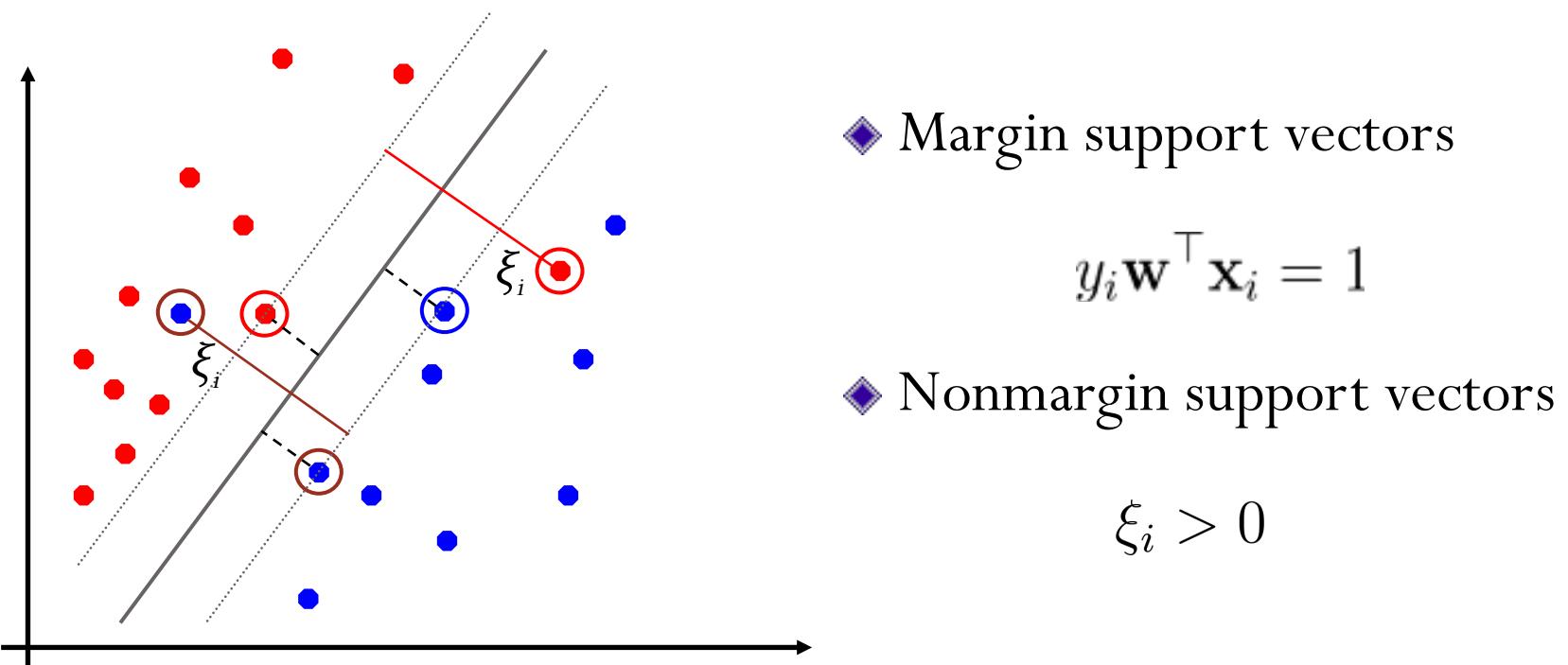
- Only need to store support vectors to predict labels of test data

Support vectors in Soft SVM

$$\hat{\mathbf{w}} = \arg \min_{\mathbf{w}, \xi} \frac{1}{2} \|\mathbf{w}\|^2 + C \sum_{i=1}^N \xi_i$$

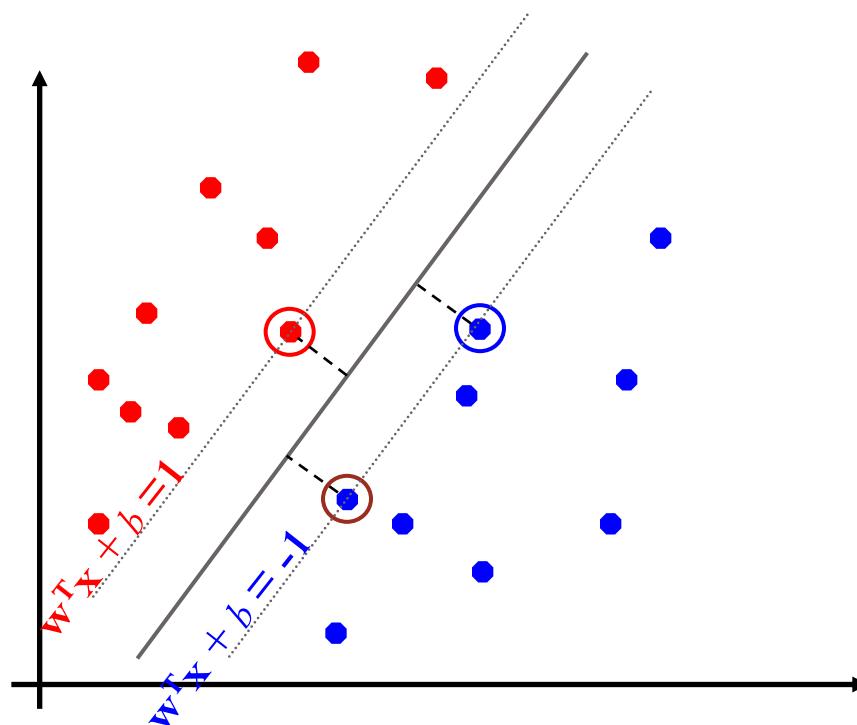
$$\text{s.t.: } y_i \mathbf{w}^\top \mathbf{x}_i \geq 1 - \xi_i, \quad \forall i = 1, \dots, N$$

$$\xi_i \geq 0, \quad \forall i = 1, \dots, N$$

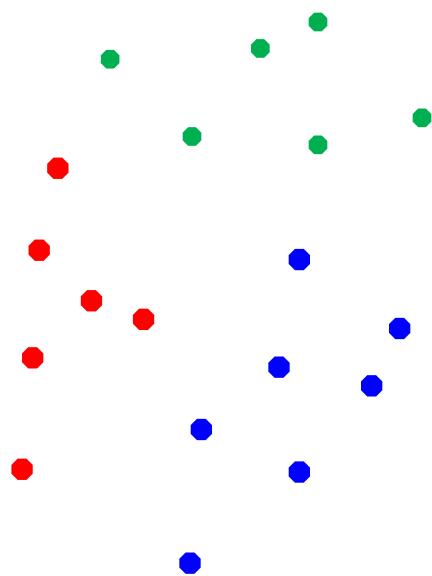


Dual Sparsity

- ◆ Only few Lagrange multipliers (dual variables) α_i can be non-zero

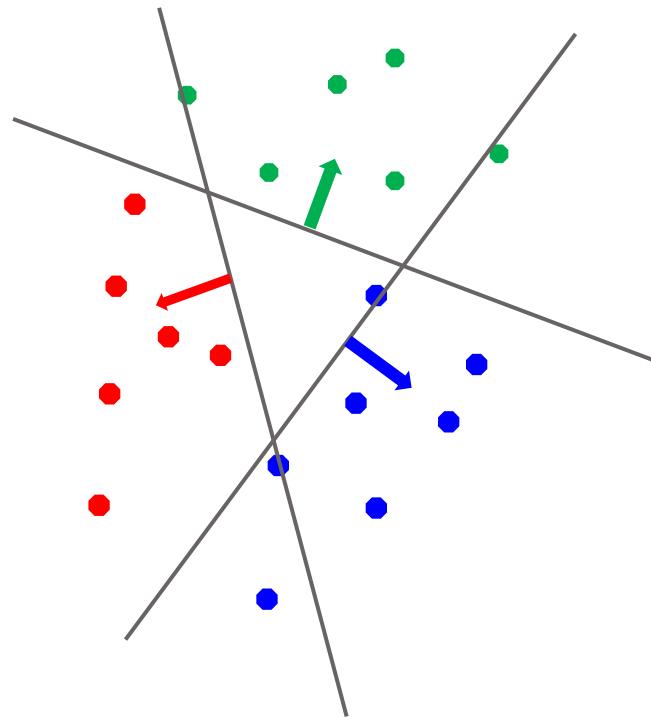


What about multiple classes?



One vs All

- ◆ Learn multiple binary classifiers separately:
 - class k vs. rest $(\mathbf{w}_k, b_k)_{k=1,2,3}$



- ◆ Prediction:
$$\hat{y} = \operatorname{argmax}_k (\mathbf{w}_k^\top \mathbf{x} + b_k)$$

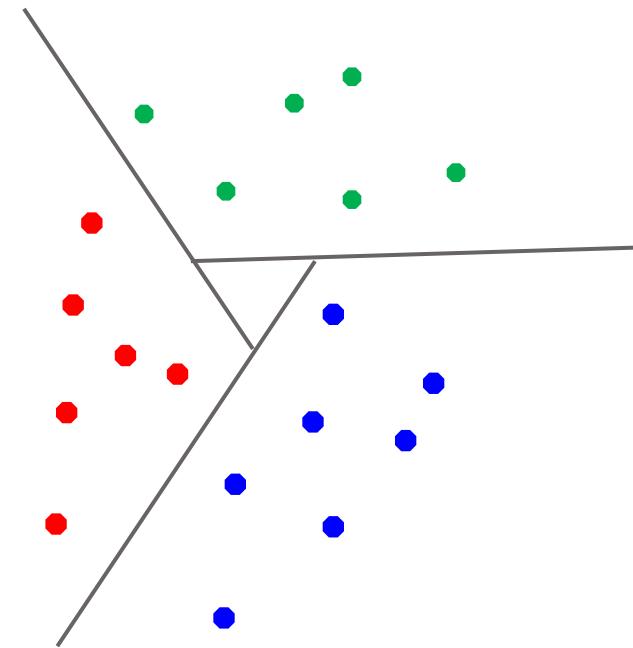
Problems with One vs All?

$$\hat{y} = \operatorname{argmax}_k (\mathbf{w}_k^\top \mathbf{x} + b_k)$$

- ◆ (1) The weights may not be based on the same scale
 - Note: $(a\mathbf{w}_k)^\top \mathbf{x} + (ab_k)$ is also a solution
- ◆ (2) Imbalance issue when learning each binary classifier
 - Much more negatives than positives

One vs One

- ◆ Learn $K(K-1)/2$ binary classifiers



- ◆ Prediction:
 - Majority voting
- ◆ Ambiguity issue!

Learning 1 Joint Classifier

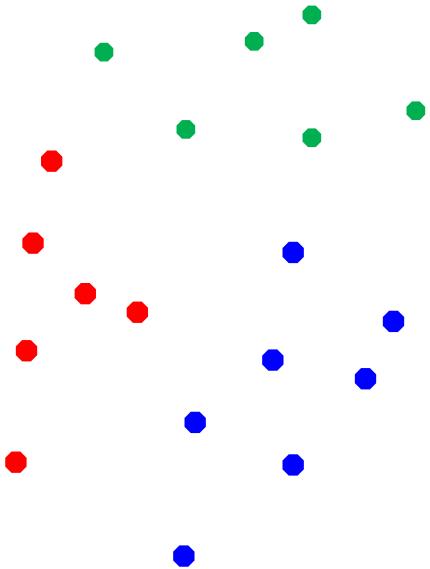
- ◆ Simultaneously learn 3 sets of weights $(\mathbf{w}_k, b_k)_{k=1,2,3}$

$$\hat{y} = \operatorname{argmax}_k (\mathbf{w}_k^\top \mathbf{x} + b_k)$$

$\forall i, \forall y \neq y_i :$

$$\mathbf{w}_{y_i}^\top \mathbf{x}_i + b_{y_i} \geq \mathbf{w}_y^\top \mathbf{x}_i + b_y + 1$$

Margin: gap between true class and nearest other class



Learning 1 Joint Classifier

- ◆ Simultaneously learn 3 sets of weights $(\mathbf{w}_k, b_k)_{k=1,2,3}$

Joint optimization:

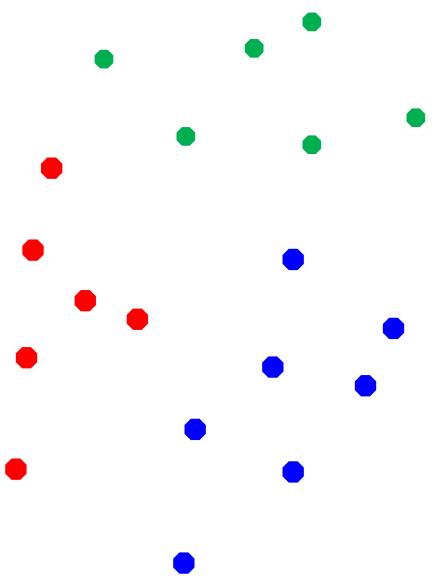
$$\min_{\mathbf{w}, b} \frac{1}{2} \sum_y \|\mathbf{w}_y\|^2 + C \sum_{i=1}^N \sum_{y \neq y_i} \xi_{iy}$$

$$\text{s.t.: } \mathbf{w}_{y_i}^\top \mathbf{x}_i + b_{y_i} \geq \mathbf{w}_y^\top \mathbf{x}_i + b_y + 1 - \xi_{iy} \quad \forall i, \quad \forall y \neq y_i$$

$$\xi_{iy} \geq 0 \quad \forall i, \quad \forall y \neq y_i$$

Prediction:

$$\hat{y} = \operatorname{argmax}_k (\mathbf{w}_k^\top \mathbf{x} + b_k)$$



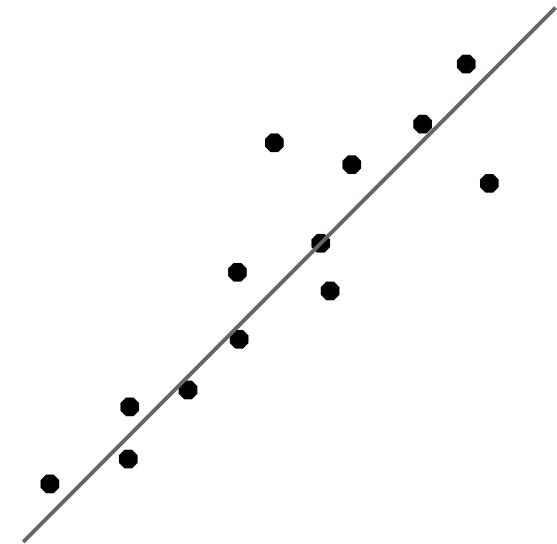
What you need to know

- ◆ Maximizing margin
- ◆ Derivation of SVM formulation
- ◆ Slack variables and hinge loss
- ◆ Relationship between
 - 0/1 loss
 - Hinge loss
- ◆ Tackling multiple class
 - One vs. All
 - Multiclass SVMs

SVM for Regression

- ◆ Training data (\mathbf{x}_i, y_i) , $\mathbf{x}_i \in \mathbb{R}^d$, $y_i \in \mathbb{R}$
- ◆ Still learn a hyper-plane (linear model)
- ◆ Squared error is the popular loss

$$\sum_{i=1}^N (y_i - \mathbf{w}^\top \mathbf{x}_i)^2$$



- a smooth function – no sparsity
- ◆ A piecewise linear approximation (ϵ -insensitive loss)

$$\sum_{i=1}^N \max(0, |y_i - \mathbf{w}^\top \mathbf{x}_i| - \epsilon)$$

SVM in the dual space

- ◆ Without offset b:

$$\hat{\boldsymbol{\alpha}} = \underset{\boldsymbol{\alpha}}{\operatorname{argmax}} \boldsymbol{\alpha}^\top \mathbf{1} - \frac{1}{2} \boldsymbol{\alpha}^\top \mathbf{Y} G \mathbf{Y} \boldsymbol{\alpha}$$

s.t.: $0 \leq \boldsymbol{\alpha} \leq C \mathbf{1}$

- ◆ With offset b:

$$\hat{\boldsymbol{\alpha}} = \underset{\boldsymbol{\alpha}}{\operatorname{argmax}} \boldsymbol{\alpha}^\top \mathbf{1} - \frac{1}{2} \boldsymbol{\alpha}^\top \mathbf{Y} G \mathbf{Y} \boldsymbol{\alpha}$$

s.t.: $0 \leq \boldsymbol{\alpha} \leq C \mathbf{1}$

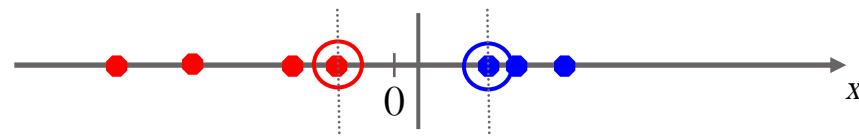
$$\sum_i \alpha_i y_i = 0$$

Why solve the dual SVM?

- ◆ The dual problem has simpler constraints
- ◆ There are some quadratic programming algorithms that can solve the dual fast, especially in high-dimensions ($d \gg N$)
 - See [Bottou & Lin, 2007] for a summary of dual SVM solvers
 - Be aware of the fast algorithms directly solving the primal problem, e.g., cutting-plane, stochastic subgradient, etc.
- ◆ More importantly, the **Kernel Trick**!!

Nonlinear SVM

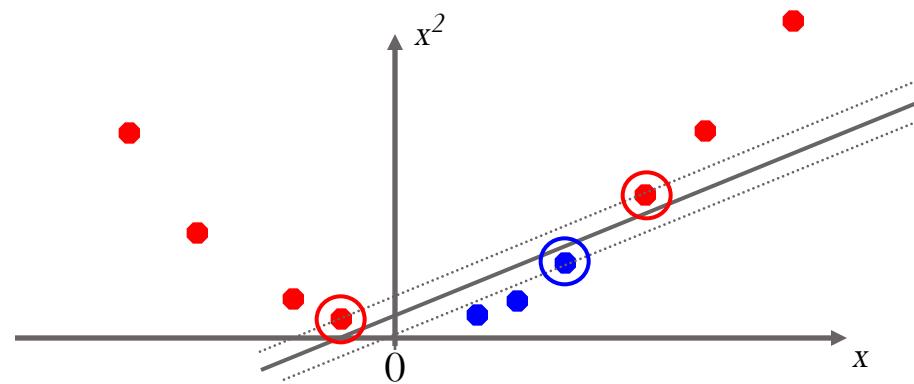
- ◆ Datasets that are linearly separable with some noise work out great:



- ◆ But what are we going to do if the dataset is just too hard?

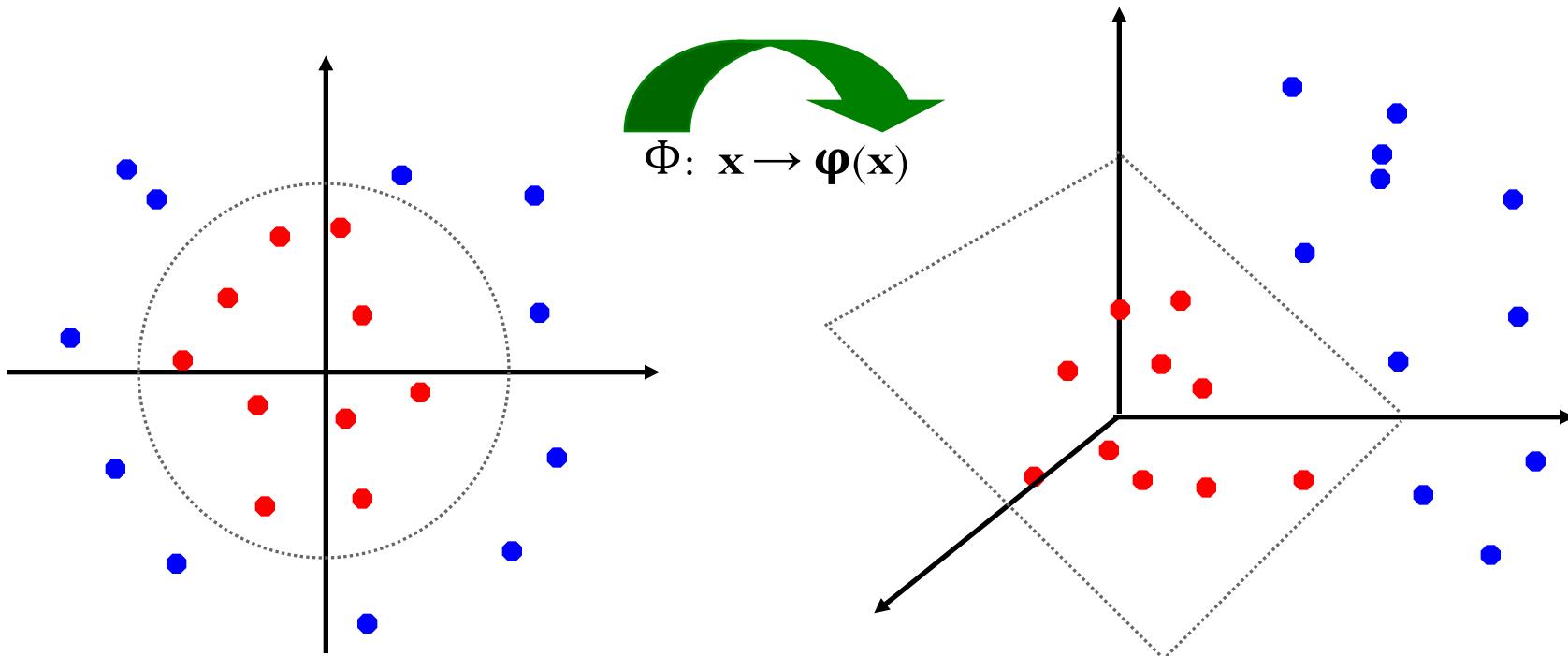


- ◆ How about... mapping data to a higher-dimensional space:



Non-linear SVMs: Feature Spaces

- ◆ **General idea:** the original feature space can always be mapped to some higher-dimensional feature space where the training set is separable:



Dot Product of Polynomials

- ◆ Polynomials of degree exactly d: $\Phi(\mathbf{x})$

$$\mathbf{x} = (x_1, x_2)^\top, \quad \mathbf{z} = (z_1, z_2)^\top$$

- ◆ d=1:

$$\Phi(\mathbf{x})^\top \Phi(\mathbf{z}) = \mathbf{x}^\top \mathbf{z}$$

- ◆ d=2: $\Phi(\mathbf{x}) = (x_1^2, \sqrt{2}x_1x_2, x_2^2)^\top$

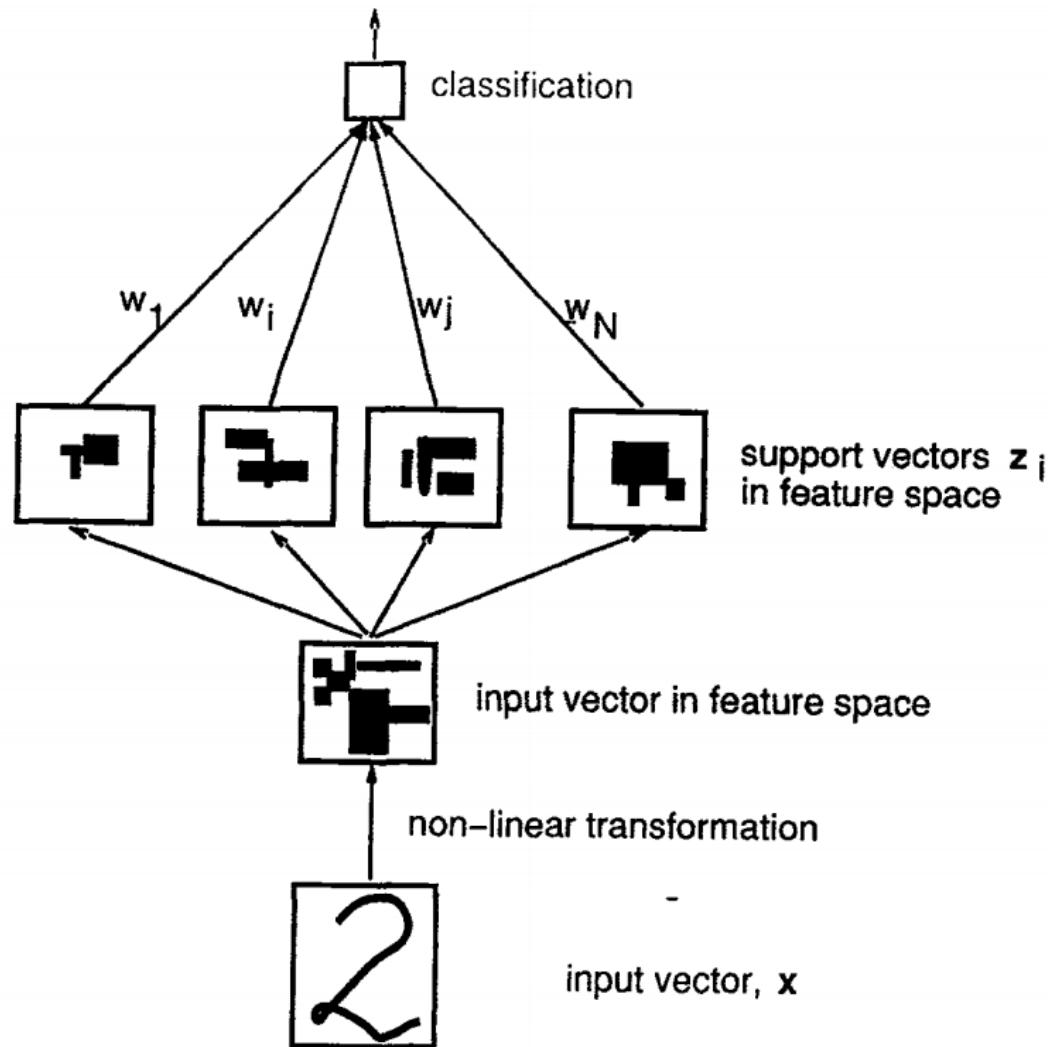
$$\Phi(\mathbf{x})^\top \Phi(\mathbf{z}) = (\mathbf{x}^\top \mathbf{z})^2$$

- ◆ In general:

$$\Phi(\mathbf{x})^\top \Phi(\mathbf{z}) = (\mathbf{x}^\top \mathbf{z})^d = K(\mathbf{x}, \mathbf{z})$$

Support Vector Nets

[Cortes & Vapnik, 1995]



The Kernel Trick

- ◆ Linear SVM relies on inner product between vectors

$$K(\mathbf{x}_i, \mathbf{x}_j) = \mathbf{x}_i^\top \mathbf{x}_j$$

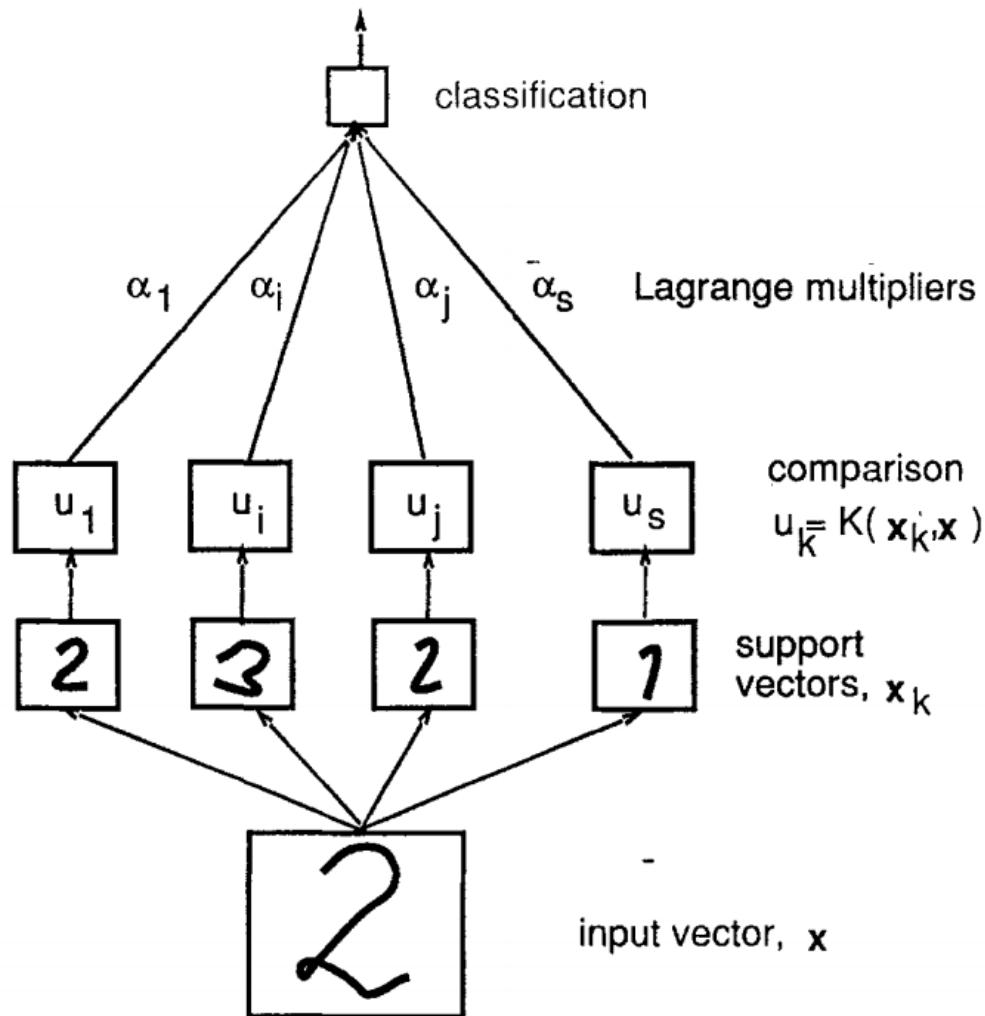
- ◆ If map every data point into high-dimensional space via $\Phi: \mathbf{x} \rightarrow \Phi(\mathbf{x})$, the inner product becomes:

$$K(\mathbf{x}_i, \mathbf{x}_j) = \Phi(\mathbf{x}_i)^\top \Phi(\mathbf{x}_j)$$

- ◆ A *kernel function* is a function that is equivalent to an inner product in some feature space.
- ◆ The feature mapping is not explicitly needed as long as we can compute the dot product using some Kernel K

Support Vector Nets

[Cortes & Vapnik, 1995]



What functions are kernels?

- ◆ For some function $K(\mathbf{x}_i, \mathbf{x}_j)$ checking that $K(\mathbf{x}_i, \mathbf{x}_j) = \varphi(\mathbf{x}_i)^T \varphi(\mathbf{x}_j)$ can be cumbersome.
- ◆ Mercer's theorem:

Every semi-positive definite symmetric function is a kernel

- ◆ Semi-positive definite symmetric functions correspond to a semi-positive definite symmetric Gram matrix:

K =	$K(\mathbf{x}_1, \mathbf{x}_1)$	$K(\mathbf{x}_1, \mathbf{x}_2)$	$K(\mathbf{x}_1, \mathbf{x}_3)$...	$K(\mathbf{x}_1, \mathbf{x}_n)$
	$K(\mathbf{x}_2, \mathbf{x}_1)$	$K(\mathbf{x}_2, \mathbf{x}_2)$	$K(\mathbf{x}_2, \mathbf{x}_3)$		$K(\mathbf{x}_2, \mathbf{x}_n)$

	$K(\mathbf{x}_n, \mathbf{x}_1)$	$K(\mathbf{x}_n, \mathbf{x}_2)$	$K(\mathbf{x}_n, \mathbf{x}_3)$...	$K(\mathbf{x}_n, \mathbf{x}_n)$

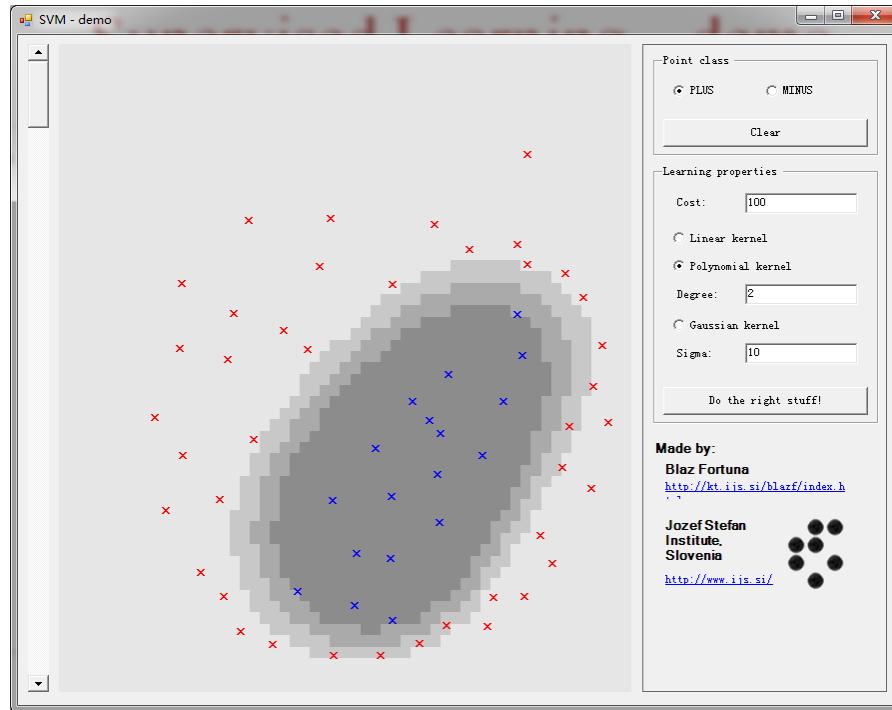
Example Kernel Functions

- ◆ Linear: $K(\mathbf{x}_i, \mathbf{x}_j) = \mathbf{x}_i^T \mathbf{x}_j$
 - Mapping $\Phi: \mathbf{x} \rightarrow \Phi(\mathbf{x})$, where $\Phi(\mathbf{x})$ is \mathbf{x} itself
- ◆ Polynomial of power p : $K(\mathbf{x}_i, \mathbf{x}_j) = (1 + \mathbf{x}_i^T \mathbf{x}_j)^p$
 - Mapping $\Phi: \mathbf{x} \rightarrow \Phi(\mathbf{x})$, where $\Phi(\mathbf{x})$ has $\binom{d+p}{p}$ dimensions
- ◆ Gaussian (radial-basis function):
$$K(\mathbf{x}_i, \mathbf{x}_j) = \exp\left(-\frac{\|\mathbf{x}_i - \mathbf{x}_j\|^2}{2\sigma^2}\right)$$
 - Mapping $\Phi: \mathbf{x} \rightarrow \Phi(\mathbf{x})$, where $\Phi(\mathbf{x})$ is *infinite-dimensional*: every point is mapped to a function; combination of functions for support vectors is the separator.
 - ◆ Higher-dimensional space still has *intrinsic* dimensionality d , but linear separators in it correspond to *non-linear* separators in original space.

Overfitting

- ◆ Huge feature space with kernels, what about overfitting??
 - Maximizing margin leads to a sparse set of support vectors
 - Some interesting theory says that SVMs search for simply hypothesis with a large margin
 - Often robust to overfitting

SVM – demo

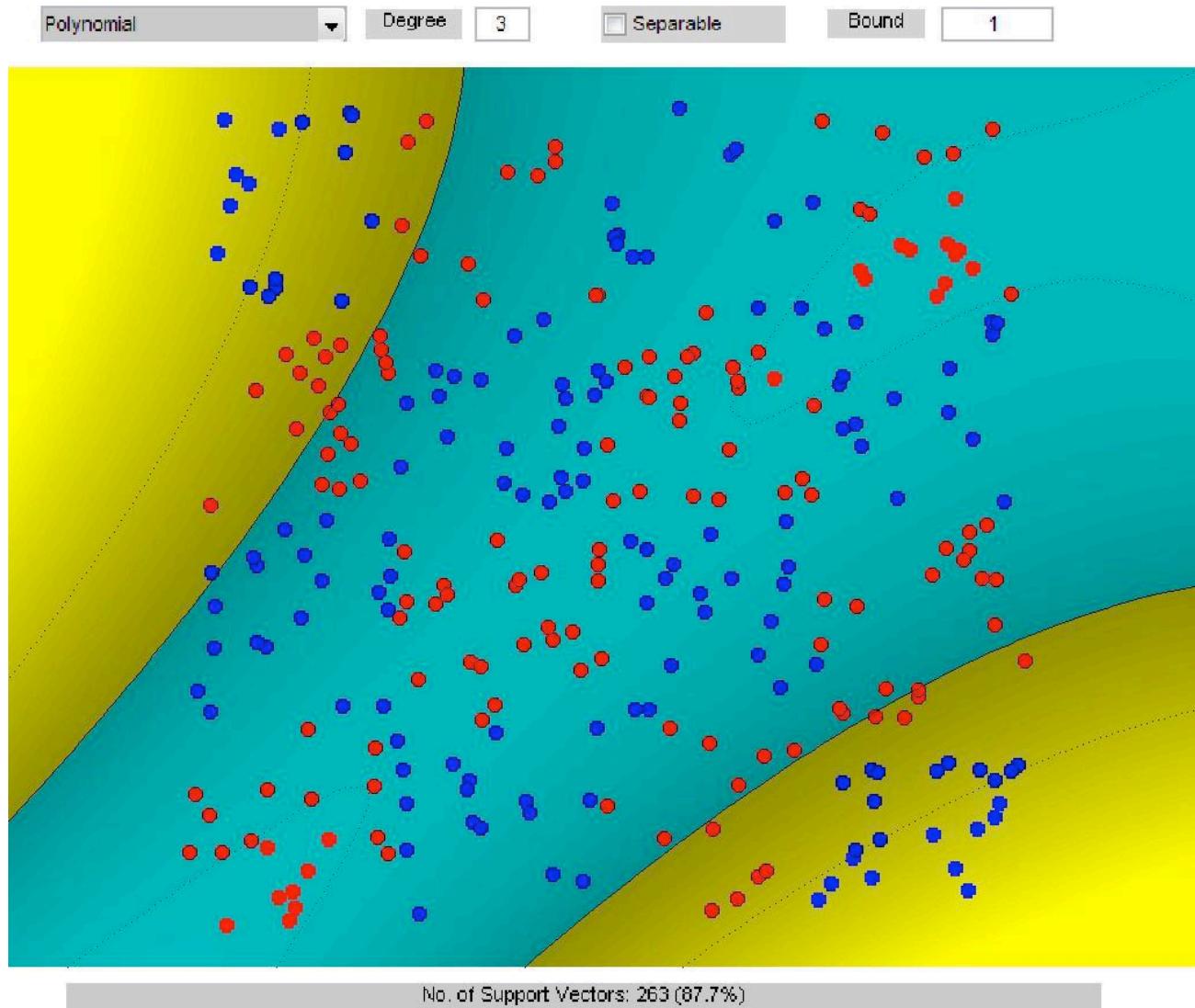


◆ <http://www.isis.ecs.soton.ac.uk/resources/svminfo/>

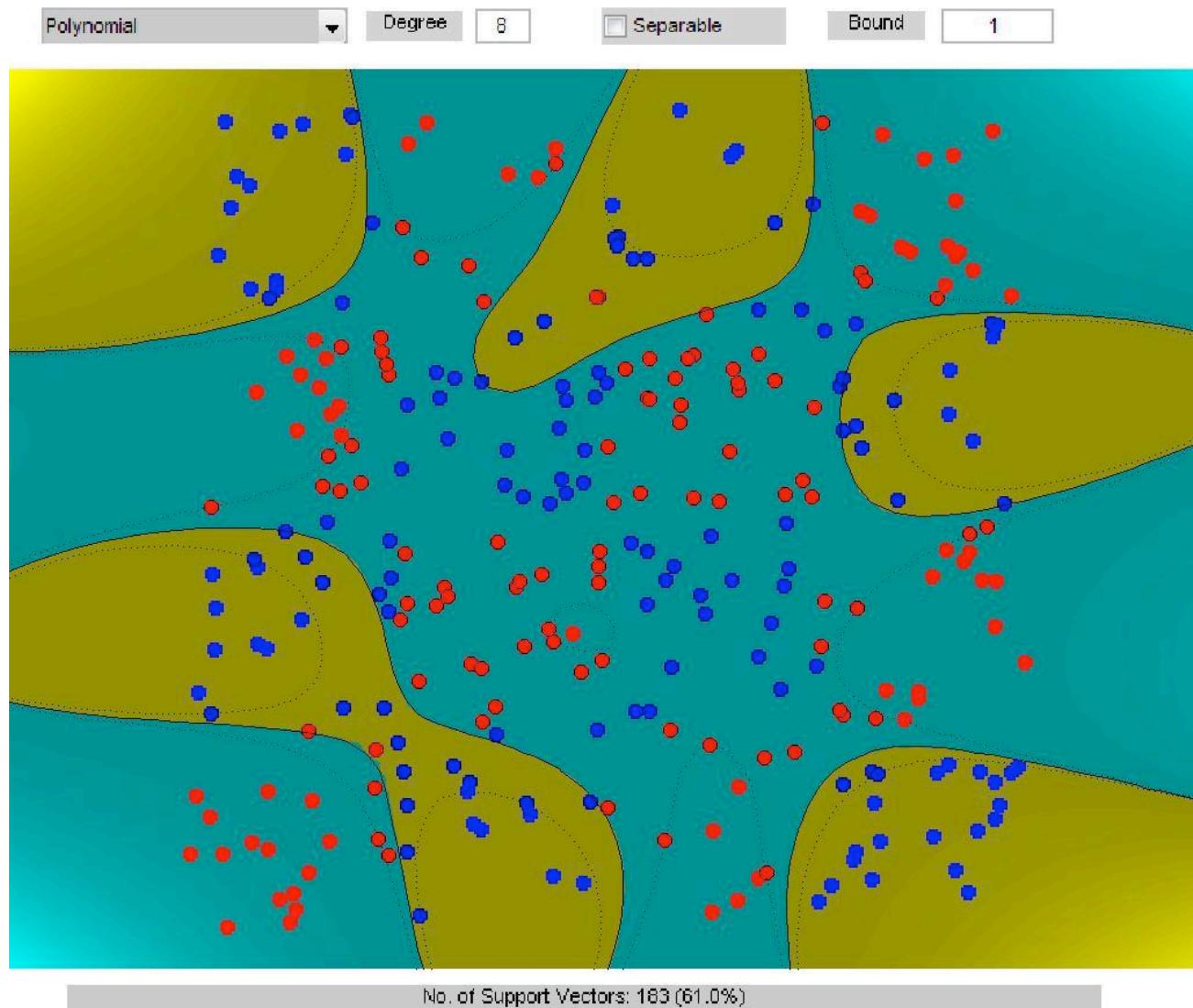
Good ToolKits: [1] SVM-Light: <http://svmlight.joachims.org/>

[2] LibSVM: <http://www.csie.ntu.edu.tw/~cjlin/libsvm/>

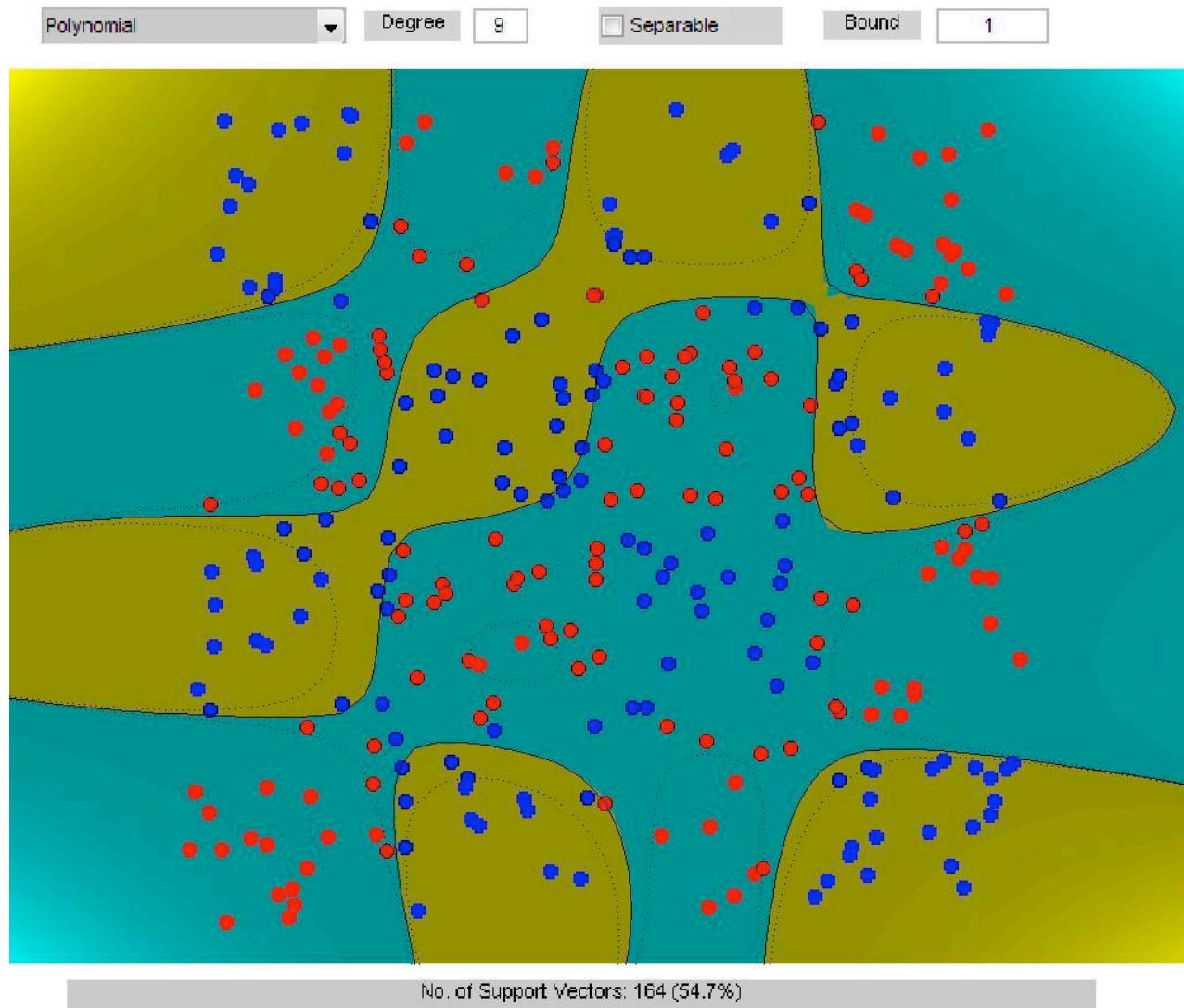
Chessboard dataset, Polynomial kernel



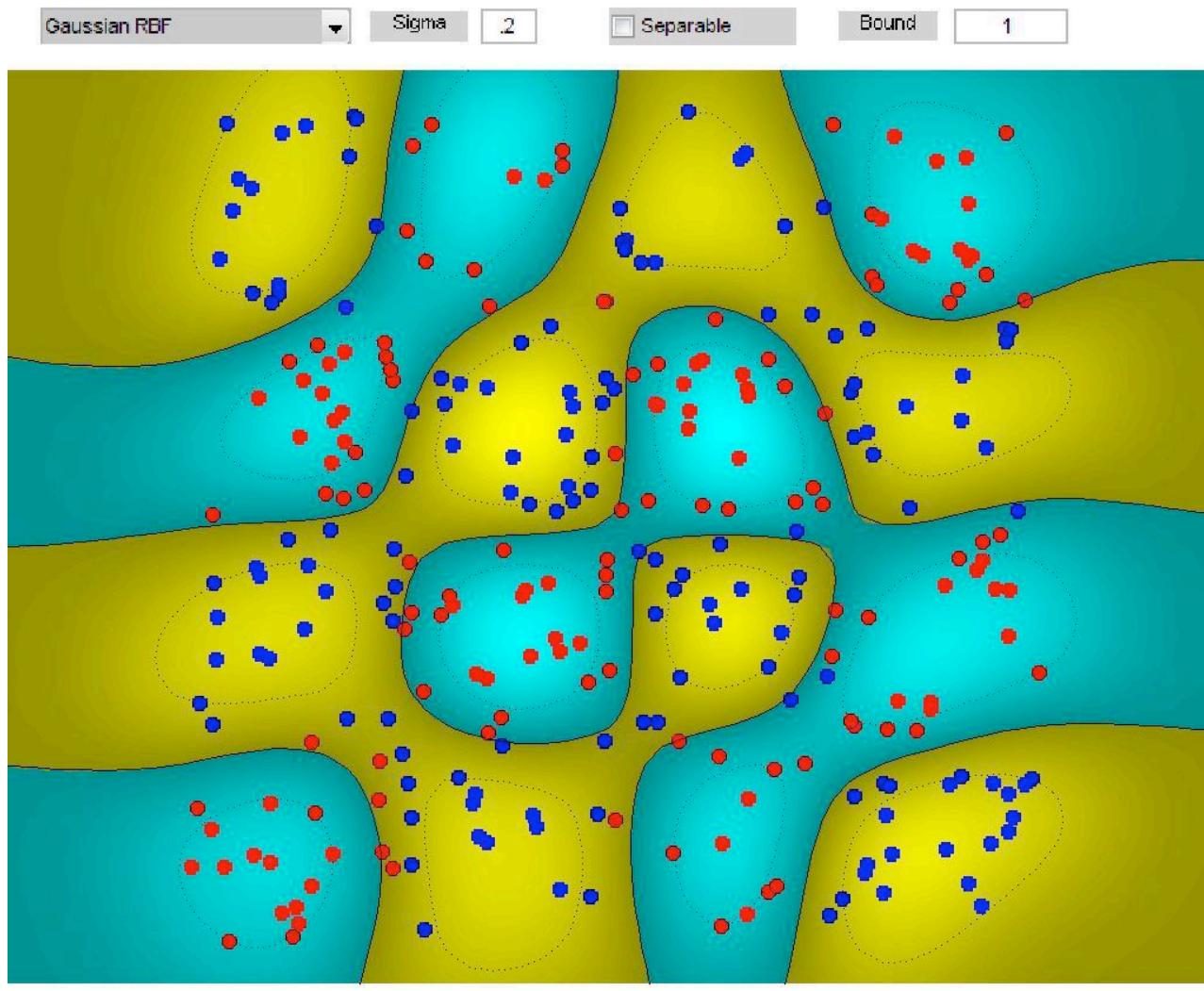
Chessboard dataset, Polynomial kernel



Chessboard dataset, Polynomial kernel



Chessboard dataset, RBF kernel



Advanced topics

- ◆ Scalable algorithms to learn SVMs
 - Linear SVMs
 - Linear algorithm, e.g., cutting-plane (2009)
 - Stochastic optimization, e.g., Pegasos (2007)
 - Distributed learning, e.g., divide-and-conquer (2013)
 - Non-linear SVMs
 - Kernel approximation, e.g., using low-rank or random features
- ◆ Structured output learning with SVMs
 - May cover later

◆ An incomplete list of SVM solvers [Menon, 2010]

Algorithm	Citation	SVM type	Optimization type	Style	Runtime
Core Vector Machine	SMO [Platt, 1999]	Kernel	Dual QP	Batch	$\Omega(n^2d)$
	SVM ^{light} [Joachims, 1999]	Kernel	Dual QP	Batch	$\Omega(n^2d)$
	[Tsang et al., 2005, 2007]	SL Kernel	Dual geometry	Batch	$O(s/\rho^4)$
	SVM ^{perf} [Joachims, 2006]	Linear	Dual QP	Batch	$O(ns/\lambda\rho^2)$
	NORMA [Kivinen et al., 2004]	Kernel	Primal SGD	Online(-style)	$\tilde{O}(s/\rho^2)$
	SVM-SGD [Bottou, 2007]	Linear	Primal SGD	Online-style	Unknown
	Pegasos [Shalev-Shwartz et al., 2007]	Kernel	Primal SGD/SGP	Online-style	$\tilde{O}(s/\lambda\rho)$
	LibLinear [Hsieh et al., 2008]	Linear	Dual coordinate descent	Batch	$O(nd \cdot \log(1/\rho))$
	SGD-QN [Bordes and Bottou, 2008]	Linear	Primal 2SGD	Online-style	Unknown
	FOLOS [Duchi and Singer, 2008]	Linear	Primal SGP	Online-style	$\tilde{O}(s/\lambda\rho)$
BMRM [Smola et al., 2007]		Linear	Dual QP	Batch	$O(d/\lambda\rho)$
OCAS [Franc and Sonnenburg, 2008]		Linear	Primal QP	Batch	$O(nd)$

Validation

◆ Model selection:

- Almost invariably, all ML methods have some free parameters
 - The number of neighbors in K-NN
 - The kernel parameters in SVMs

◆ Performance estimation:

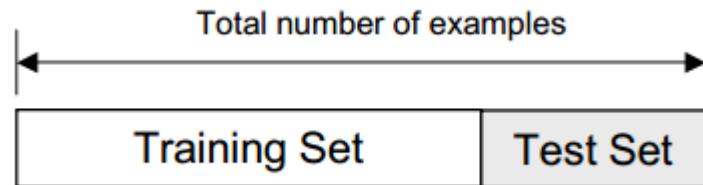
- Once we have chosen a model, how to estimate its performance?

Motivation

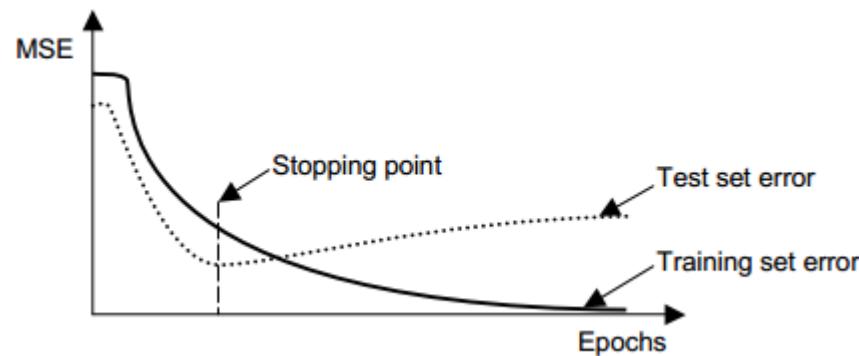
- ◆ If we had access to an unlimited number of examples, there is a straightforward answer
 - Choose the model with the lowest error rate on the entire population
 - The error rate is the true error rate
- ◆ In practice, we only access to a finite set of examples, usually smaller than we wanted
 - Use all training data to select model => too optimistic!
 - A better approach is to split the training set into disjoint subsets

Holdout Method

- ◆ Split dataset into two subsets
 - Training set: used to learn the classifier
 - Test set: used to estimate the error rate of the trained classifier



- ◆ E.g.: used to determine a stopping point of an iterative alg.:

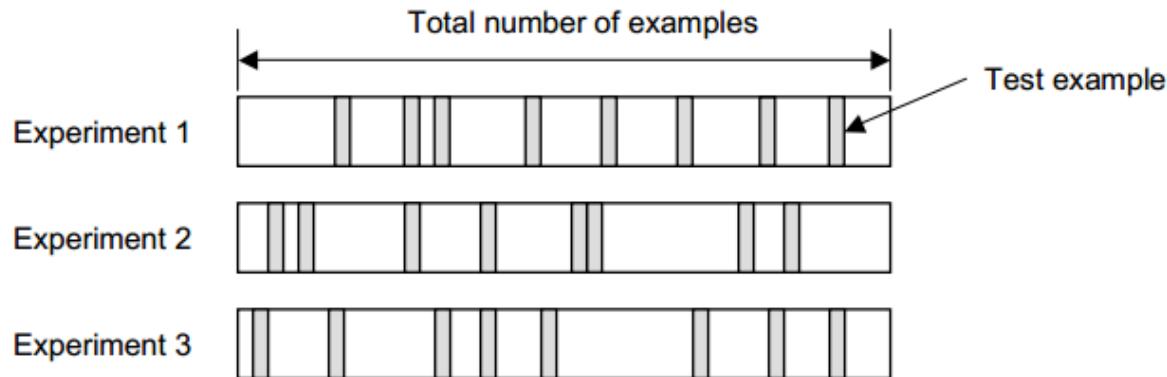


Holdout Method

- ◆ Two basic drawbacks
 - In problems with a sparse dataset, we may not be able to afford the “luxury” of setting aside a portion of data for testing
 - A single train-test split may lead to misleading results, e.g., if we happened to get an “unfortunate” split
- ◆ Resampling can overcome the limitations, but at the expense of more computations
 - Cross-validation
 - Random subsampling
 - K-fold cross-validation
 - Leave-one-out cross-validation
 - Bootstrap

Random Subsampling

- ◆ Performs K data splits of the entire dataset
 - Each split randomly selects a (fixed) no. examples
 - For each split, retrain the classifier with training data, and evaluate on test examples

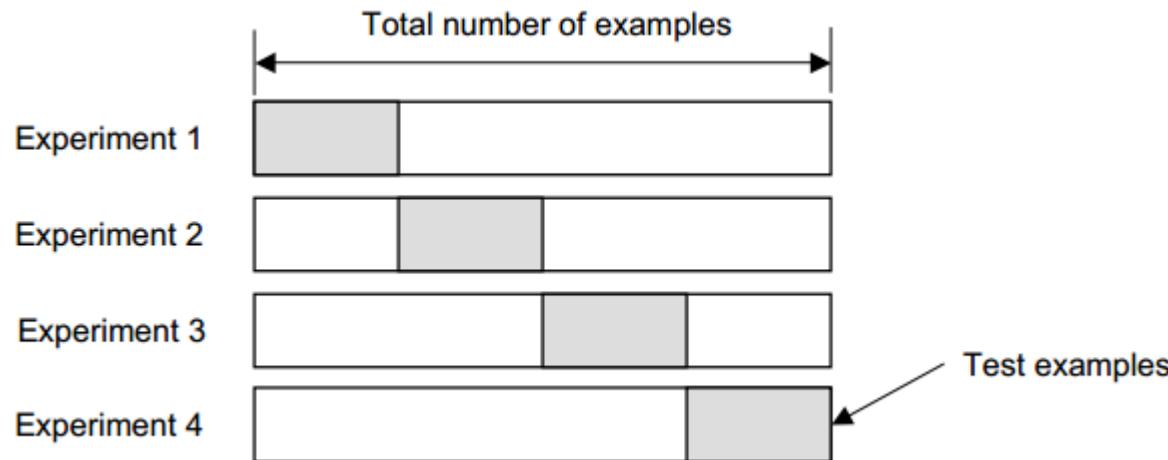


- ◆ The true error is estimated as the average

$$E = \frac{1}{K} \sum_{k=1}^K E_k$$

K-Fold Cross-validation

- ◆ Create a K-fold partition of the dataset
 - For each of K experiments, use K-1 folds for training and the remaining one for testing

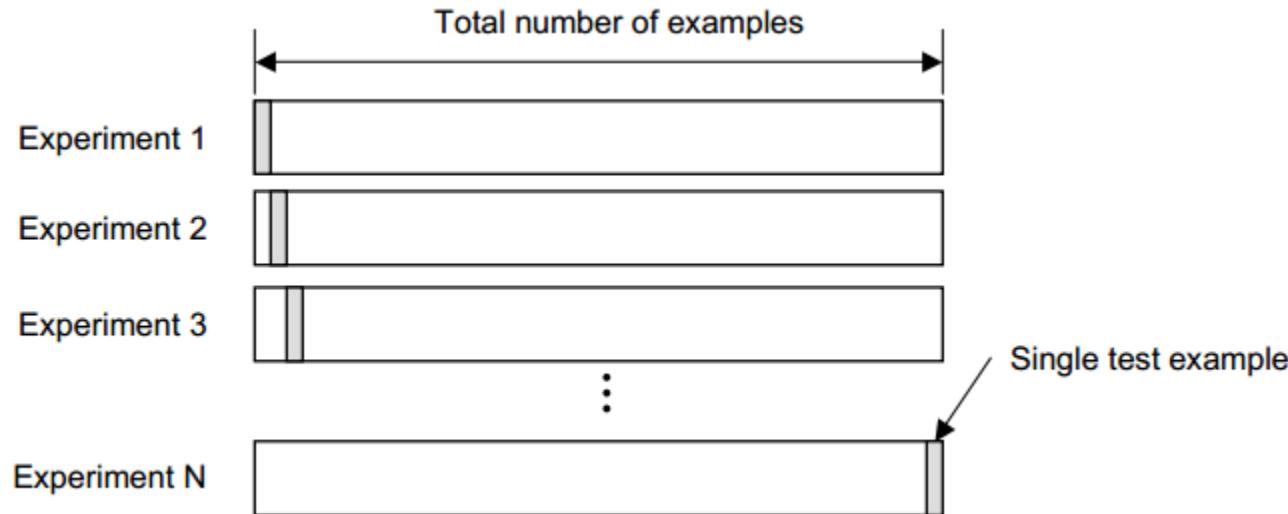


- ◆ K-fold
 - The advantage of K-fold CV is that all examples are eventually used for both training and testing
 - True error is estimated as the average

$$E = \frac{1}{K} \sum_{k=1}^K E_k$$

Leave-one-out Cross-Validation

- ◆ Leave-one-out CV is the extreme case of K-fold CV, where K=N



$$E = \frac{1}{N} \sum_{k=1}^N E_k$$

How many folds are needed?

- ◆ With a large number of folds
 - (+) The bias of true error estimate is small (i.e., accurate estimate)
 - (−) The variance of true error estimate is large – the K training sets are too similar to one another
 - (−) The computational time will be large (i.e., many experiments)
- ◆ With a small number of folds
 - (+) The computation time is reduced
 - (+) The variance of true error estimate is small
 - (−) The bias of the estimator is large, depending on the learning curve of the classifier
- ◆ In practice, a large dataset often needs a small K, while a very sparse dataset often needs a large K
- ◆ A common choice for K-fold CV is K=10

Three-way data splits

- ◆ If model selection and true error estimates are to be computed simultaneously, the data needs to be divided into 3 disjoint sets
 - **Training set:** used for learning – to fit the parameters of the classifier
 - **Validation set:** used to tune the parameters of a classifier
 - **Test set:** used only to assess the performance of a fully trained classifier

