2019 年第十届全国大学生数学竞赛决赛
(非数学专业) 参考答案

一、填空题

(1) $a + b = -3$ **(2)** $I = \dfrac{\pi \ln a}{2a}$ **(3)** $I = \dfrac{9}{2}$ **(4)** $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{F_2^2 F_{11} - 2F_1 F_2 F_{12} + F_1^2 F_{22}}{F_2^3}$

(5) $y_1^2 + y_2^2 + \cdots + y_{n-1}^2$

二、【参考解析】：由于 $f(x)$ 在区间 $(-1,1)$ 内三阶可导，$f(x)$ 在 $x = 0$ 处有 Taylor 公式

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + o\left(x^3\right)$$

又 $f(0) = 0, f'(0) = 1, f''(0) = 0, f'''(0) = -1$，所以

$$f(x) = x - \frac{1}{6}x^3 + o\left(x^3\right)$$

由于 $a_1 \in (0,1)$，数列 $\left\{ a_n \right\}$ 严格单调且 $\lim\limits_{n \to \infty} a_n = 0$，则 $a_n > 0$，且 $\left\{ \dfrac{1}{a_n^2} \right\}$ 为严格单调增加趋于正无穷的数列，注意到 $a_{n+1} = f\left(a_n\right)$，故由 Stolz 定理及式，有

$$\lim_{n \to \infty} na_n^2 = \lim_{n \to \infty} \frac{n}{\dfrac{1}{a_n^2}} = \lim_{n \to \infty} \frac{1}{\dfrac{1}{a_{n+1}^2} - \dfrac{1}{a_n^2}} = \lim_{n \to \infty} \frac{a_n^2 a_{n+1}^2}{a_n^2 - a_{n+1}^2} = \lim_{n \to \infty} \frac{a_n^2 f^2\left(a_n\right)}{a_n^2 - f^2\left(a_n\right)}$$

$$= \lim_{n \to \infty} \frac{a_n^2 \left(a_n - \dfrac{1}{6}a_n^3 + o\left(a_n^3\right)\right)^2}{a_n^2 - \left(a_n - \dfrac{1}{6}a_n^3 + o\left(a_n^3\right)\right)^2} = \lim_{n \to \infty} \frac{a_n^4 - \dfrac{1}{3}a_n^6 + \dfrac{1}{36}a_n^8 + o\left(a_n^4\right)}{\dfrac{1}{3}a_n^4 - \dfrac{1}{36}a_n^6 + o\left(a_n^4\right)} = 3$$

三、【参考解析】：令 $y = x - \dfrac{1}{nx}$，则 $y' = 1 + \dfrac{1}{nx^2} > 0$ 故函数 $y(x)$ 在 $[\alpha, \beta]$ 上严格单调增加. 记 $y(x)$ 的反函数为 $x(y)$，则定义在 $\left[\alpha - \dfrac{1}{n\alpha}, \beta - \dfrac{1}{n\beta}\right]$ 上，且 $x'(y) = \dfrac{1}{y'(x)} = \dfrac{1}{1 + \dfrac{1}{nx^2}} > 0$. 于是

$$\int_\alpha^\beta f'\left(nx - \frac{1}{x}\right) \mathrm{d}x = \int_{\alpha - \frac{1}{n\alpha}}^{\beta - \frac{1}{n\beta}} f'(ny)x'(y) \mathrm{d}y.$$

根据积分中值定理，存在 $\xi_n \in \left[\alpha - \dfrac{1}{n\alpha}, \beta - \dfrac{1}{n\beta}\right]$，使得

$$\int_{\alpha\frac{1}{n\alpha}}^{\beta-\frac{1}{n\beta}} f'(ny)x'(y)\,\mathrm{d}\,y = x'(\xi_n)\int_{\alpha\frac{1}{n\alpha}}^{\beta-\frac{1}{n\beta}} f'(ny)\,\mathrm{d}\,y$$

$$= \frac{x'(\xi_n)}{n}\left[f\left(n\beta-\frac{1}{\beta}\right)-f\left(n\alpha-\frac{1}{\alpha}\right)\right]$$

因此 $\left|\int_a^\beta f'\left(nx-\frac{1}{x}\right)\mathrm{d}x\right| \leq \frac{\left|x'(\xi_n)\right|}{n}\left[\left|f\left(n\beta-\frac{1}{\beta}\right)\right|+\left|f\left(n\alpha-\frac{1}{\alpha}\right)\right|\right] \leq \frac{2\left|x'(\xi_n)\right|}{n}$.

注意到 $0 < x'(\xi_n) = \dfrac{1}{1+\dfrac{1}{n\xi_n^2}} < 1$, 则

$$\left|\int_\alpha^\beta f'\left(nx-\frac{1}{x}\right)\mathrm{d}\,x\right| \leq \frac{2}{n} , \text{ 即 } \lim_{n\to\infty}\int_\alpha^\beta f'\left(nx-\frac{1}{x}\right)\mathrm{d}\,x = 0$$

四、【参考解析】：采用"先二后一"法，并利用对称性，得

$$I = 2\int_0^1\mathrm{d}\,z\iint_D \frac{\mathrm{d}\,x\,\mathrm{d}\,y}{\left(1+x^2+y^2+z^2\right)^2} , \text{ 其中 } D: 0\leq x\leq 1, 0\leq y\leq x$$

交换积分次序，得

$$I = \int_0^{\frac{\pi}{4}}\mathrm{d}\,\theta\int_0^1\left(\frac{1}{1+z^2}-\frac{1}{1+\sec^2\theta+z^2}\right)\mathrm{d}\,z$$

$$= \frac{\pi^2}{16}-\int_0^{\frac{\pi}{4}}\mathrm{d}\,\theta\int_0^1\frac{1}{1+\sec^2\theta+z^2}\mathrm{d}\,z$$

作变量代换：$z = \tan t$ ，并利用对称性，得

$$\int_0^{\frac{\pi}{4}}\mathrm{d}\,\theta\int_0^1\frac{1}{1+\sec^2\theta+z^2}\mathrm{d}\,z = \int_0^{\frac{\pi}{4}}\mathrm{d}\,\theta\int_0^{\frac{\pi}{4}}\frac{\sec^2 t}{\sec^2\theta+\sec^2 t}\mathrm{d}\,t$$

$$= \int_0^{\frac{\pi}{4}}\mathrm{d}\,\theta\int_0^{\frac{\pi}{4}}\frac{\sec^2\theta}{\sec^2\theta+\sec^2 t}\mathrm{d}\,t = \frac{1}{2}\int_0^{\frac{\pi}{4}}\mathrm{d}\,\theta\int_0^{\frac{\pi}{4}}\frac{\sec^2\theta+\sec^2 t}{\sec^2\theta+\sec^2 t}\mathrm{d}\,t$$

$$= \frac{1}{2}\times\frac{\pi^2}{16} = \frac{\pi^2}{32}$$

所以，$I = \dfrac{\pi^2}{16}-\dfrac{1}{2}\dfrac{\pi^2}{16} = \dfrac{\pi^2}{32}$

五、【参考解析】：级数通项 $a_n = \dfrac{1}{3}\cdot\dfrac{2}{5}\cdot\dfrac{3}{7}\cdots\cdot\dfrac{n}{2n+1}\cdot\dfrac{1}{n+1} = \dfrac{2(2n)!!}{(2n+1)!(n+1)}\left(\dfrac{1}{\sqrt{2}}\right)^{2n+2}$ 令

$$f(x) = \sum_{n=0}^\infty \frac{(2n)!!}{(2n+1)!!(n+1)}x^{2n+2} ,$$

则收敛区间为 $(-1,1)$，则 $\displaystyle\sum_{n=1}^{\infty} a_n = 2\left[f\left(\frac{1}{\sqrt{2}}\right) - \frac{1}{2}\right]$，由逐项可导性质，得

$$f'(x) = 2\sum_{n=0}^{\infty} \frac{(2n)!!}{(2n+1)!!}x^{2n+1} = 2g(x)，$$

其中 $\displaystyle g(x) = \sum_{n=0}^{\infty} \frac{(2n)!!}{(2n+1)!!}x^{2n+1}$，因为

$$g'(x) = 1 + \sum_{n=1}^{\infty} \frac{(2n)!!}{(2n-1)!!}x^{2n} = 1 + x\sum_{n=1}^{\infty} \frac{(2n-2)!!}{(2n-1)!!}2nx^{2n-1}$$

$$= 1 + x\frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{n=1}^{\infty} \frac{(2n-2)!!}{(2n-1)!!}x^{2n}\right) = 1 + x\frac{\mathrm{d}}{\mathrm{d}x}[xg(x)]$$

所以 $g(x)$ 满足 $g(0) = 0$，$g'(x) - \dfrac{x}{1-x^2}g(x) = \dfrac{1}{1-x^2}$．解这个一阶线性方程，得

$$g(x) = e^{\int \frac{x}{1-x^2}\mathrm{d}x}\left(\int \frac{1}{1-x^2}e^{-\int \frac{x}{1-x^2}\mathrm{d}x}\mathrm{d}x + C\right) = \frac{\arcsin x}{\sqrt{1-x^2}} + \frac{C}{\sqrt{1-x^2}}$$

由 $g(0) = 0$ 得 $C = 0$，故 $g(x) = \dfrac{\arcsin x}{\sqrt{1-x^2}}$，所以

$$f(x) = (\arcsin x)^2,\quad f\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi^2}{16}$$

且 $\displaystyle\sum_{n=1}^{\infty} a_n = 2\left(\frac{\pi^2}{16} - \frac{1}{2}\right) = \frac{\pi^2 - 8}{8}$

六、【参考解析】：存在 n 阶可逆矩阵 H, Q，使得 $A = H\begin{pmatrix} I_r & O \\ O & O \end{pmatrix}Q$，因为 $A^2 = O$，所以

$$A^2 = H\begin{pmatrix} I_r & O \\ O & O \end{pmatrix}QH\begin{pmatrix} I_r & O \\ O & O \end{pmatrix}Q = O$$

对 QH 作相应分块为 $QH = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}$，则有

$$\begin{pmatrix} I_r & O \\ O & O \end{pmatrix}QH\begin{pmatrix} I_r & O \\ O & O \end{pmatrix} = \begin{pmatrix} I_r & O \\ O & O \end{pmatrix}\begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}\begin{pmatrix} I_r & O \\ O & O \end{pmatrix} = \begin{pmatrix} R_{11} & O \\ O & O \end{pmatrix} = O$$

因此，$R_{11} = O$．而 $Q = \begin{pmatrix} O & R_{12} \\ R_{21} & R_{22} \end{pmatrix}H^{-1}$，所以

$$A = H\begin{pmatrix} I_r & O \\ O & O \end{pmatrix}Q = H\begin{pmatrix} I_r & O \\ O & O \end{pmatrix}\begin{pmatrix} O & R_{12} \\ R_{21} & R_{22} \end{pmatrix}H^{-1} = H\begin{pmatrix} O & R_{12} \\ O & O \end{pmatrix}H^{-1}$$

显然，$r(A) = r(R_{12}) = r$，所以 R_{12} 为行满秩矩阵.

因为 $r < \dfrac{n}{2}$，所以存在可逆矩阵 S_1, S_2，使得 $S_1 R_{12} S_2 = (I_r, O)$，令 $P = H \begin{pmatrix} S_1^{-1} & O \\ O & S_2 \end{pmatrix}$，则有

$$P^{-1} A P = \begin{pmatrix} S_1 & O \\ O & S_2^{-1} \end{pmatrix} H^{-1} A H \begin{pmatrix} S_1^{-1} & O \\ O & S_2 \end{pmatrix} = \begin{pmatrix} O & I_r & O \\ O & O & O \end{pmatrix}$$

七、【参考解析】：$\displaystyle\sum_{n=1}^{\infty} a_n u_n$ 收敛，所以对任意给定 $\varepsilon > 0$，存在自然数 N_1，使得当 $n > N_1$ 时，有

$$-\frac{\varepsilon}{2} < \sum_{k=N_1}^{n} a_k u_k < \frac{\varepsilon}{2}.$$

因为 $\{u_n\}_{n=1}^{\infty}$ 单调递减的正数列，所以 $0 < \dfrac{1}{u_{N_1}} \le \dfrac{1}{u_{N_1+1}} \le \cdots \le \dfrac{1}{u_n}$. 注意到当 $m < n$ 时，有

$$\sum_{k=m}^{n} (A_k - A_{k-1}) b_k = A_n b_n - A_{m-1} b_m + \sum_{k=m}^{n-1} (b_k - b_{k+1}) A_k$$

令 $A_0 = 0$，$A_k = \displaystyle\sum_{i=1}^{k} a_i \quad (k = 1, 2, \cdots, n)$，得到

$$\sum_{k=1}^{n} a_k b_k = A_n b_n + \sum_{k=1}^{n-1} (b_k - b_{k+1}) A_k$$

下面证明：对于任意自然数 n，如果 $\{a_n\}, \{b_n\}$ 满足

$$b_1 \ge b_2 \ge \cdots \ge b_n \ge 0, \, m \le a_1 + a_2 + \cdots + a_n \le M$$

则有 $b_1 m \le \displaystyle\sum_{k=1}^{n} a_k b_k = b_1 M$.

事实上，$m \le A_k \le M$，$b_k - b_{k+1} \ge 0$，即得到

$$m b_1 = m b_n + \sum_{k=1}^{n-1} (b_k - b_{k+1}) m \le \sum_{k=1}^{n} a_k b_k \le M b_n + \sum_{k=1}^{n-1} (b_k - b_{k+1}) M = M b_1$$

令 $b_1 = \dfrac{1}{u_n}, b_2 = \dfrac{1}{u_{n-1}}, \cdots$ 可以得到

$$-\frac{\varepsilon}{2} u_n^{-1} < \sum_{k=N_1}^{n} a_k < \frac{\varepsilon}{2} u_n^{-1},$$

即 $\left| \displaystyle\sum_{k=N_1}^{n} a_k u_n \right| < \dfrac{\varepsilon}{2}$. 又由 $\lim\limits_{n \to \infty} u_n = 0$ 知，存在自然数 N_2，使得 $n > N_2$，

$$\left| (a_1 + a_2 + \cdots + a_{N_1-1}) u_n \right| < \frac{\varepsilon}{2}$$

取 $N = \max\{N_1, N_2\}$，则当 $n > N$ 时，有

$$\left|\left(a_1 + a_2 + \cdots + a_n\right)u_n\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

因此 $\lim\limits_{n\to\infty}\left(a_1 + a_2 + \ldots + a_n\right)u_n = 0$.