

## 2012 年第三届全国大学生数学竞赛决赛 (非数学专业) 参考答案

### 一、计算题

1. 【参考解析】:  $\lim_{x \rightarrow 0} \frac{\sin^2 x - x^2 \cos^2 x}{x^2 \sin^2 x} = \lim_{x \rightarrow 0} \frac{\sin^2 x - x^2 \cos^2 x}{x^4}$

$$= \lim_{x \rightarrow 0} \frac{(\sin x - x \cos x)(\sin x + x \cos x)}{x^4}$$
$$= \lim_{x \rightarrow 0} \frac{\sin x + x \cos x}{x} \lim_{x \rightarrow 0} \frac{\sin x - x \cos x}{x^3}$$
$$= 2 \lim_{x \rightarrow 0} \frac{x \sin x}{3x^2} = \frac{2}{3} \lim_{x \rightarrow 0} \frac{\sin x}{x} = \frac{2}{3}.$$

2. 【参考解析】: 令  $t = \frac{1}{x}$ , 则

$$\begin{aligned} \text{原式} &= \lim_{t \rightarrow 0^+} \frac{1}{t^3} \left[ \left( 1 + \frac{t^2}{2} - t^3 \tan t \right) e^t - \sqrt{1 + t^6} \right] \\ &= \lim_{t \rightarrow 0^+} \frac{1}{t^3} \left[ \left( 1 + \frac{t^2}{2} \right) e^t - 1 \right] = \lim_{t \rightarrow 0^+} \frac{2 + 2t + t^2}{6t^2} \lim_{t \rightarrow 0^+} e^t = +\infty. \end{aligned}$$

3. 【参考解析】: 对  $z = f(x, y)$  两边对  $x$  求两次偏导, 分别得

$$0 = f_x + f_y \frac{\partial y}{\partial x}, 0 = f_{xx} + 2f_{xy} \frac{\partial y}{\partial x} + f_{yy} \left( \frac{\partial y}{\partial x} \right)^2 + f_y \frac{\partial^2 y}{\partial x^2}.$$

由前面的式子解出  $\frac{\partial y}{\partial x} = -\frac{f_x}{f_y}$ , 代入第二个式子并求解, 得

$$f_y \frac{\partial^2 y}{\partial x^2} = 0, \text{ 即 } \frac{\partial^2 y}{\partial x^2} = 0.$$

4. 【参考解析】:  $I = \int e^{x+\frac{1}{x}} dx + \int x \left( 1 - \frac{1}{x^2} \right) e^{x+\frac{1}{x}} dx$

$$= \int e^{x+\frac{1}{x}} dx + x e^{x+\frac{1}{x}} - \int e^{x+\frac{1}{x}} dx = x e^{x+\frac{1}{x}} + C.$$

5. 【参考解析】: 联立两个曲面方程, 解得交线所在平面  $z = a$  ( $z = 4a$  舍去), 它将表面积分为  $S_1, S_2$  两部分, 它们在  $xOy$  面上的投影为  $x^2 + y^2 \leq a^2$ .

$$\begin{aligned} S &= \iint_{x^2+y^2 \leq a^2} \left( \sqrt{1 + \frac{4x^2}{a^2} + \frac{4y^2}{a^2}} + \sqrt{2} \right) dx dy \\ &= \int_0^{2\pi} d\theta \int_0^a \frac{\sqrt{a^2 + 4r^2}}{a} r dr + \sqrt{2}\pi a^2 = \pi a^2 \left( \frac{5\sqrt{5}-1}{6} + \sqrt{2} \right). \end{aligned}$$

二、【参考证明】：记  $f(x) = \frac{x}{\cos^2 x + x^\alpha \sin^2 x}$ .

若  $\alpha \leq 0, f(x) \geq \frac{x}{2} (\forall x > 1)$ ;

若  $0 < \alpha \leq 2, \alpha - 1 \leq 1, f(x) \geq \frac{x^{1-\alpha}}{2} (\forall x > 1)$ ;

所以积分发散.

若  $\alpha > 2, a_n = \int_{n\pi}^{(n+1)\pi} f(x) dx$ , 考虑级数  $\sum_{n=1}^{\infty} a_n$  的收敛性即可.

当  $n\pi \leq x \leq (n+1)\pi$  时,

$$\frac{n\pi}{1 + (n+1)^\alpha \pi^\alpha \sin^2 x} \leq f(x) \leq \frac{(n+1)\pi}{1 + n^\alpha \pi^\alpha \sin^2 x},$$

对任何  $b > 0$ , 有

$$\begin{aligned} \int_{n\pi}^{(n+1)\pi} \frac{dx}{1 + b \sin^2 x} &= 2 \int_0^{\frac{\pi}{2}} \frac{dx}{1 + b \sin^2 x} \\ &= 2 \int_0^{\frac{\pi}{2}} \frac{d \cot x}{b + \csc^2 x} = 2 \int_0^{+\infty} \frac{dt}{b + 1 + t^2} = \frac{\pi}{\sqrt{b+1}} \end{aligned}$$

这样, 存在  $0 < A_1 \leq A_2$ , 使得  $\frac{A_1}{n^{\alpha/2-1}} \leq a_n \leq \frac{A_2}{n^{\alpha/2-1}}$ , 从而可知, 当  $\alpha > 4$  时, 所讨论的积分收敛; 否则发散.

三、【参考证明】：因为  $f(x)$  在  $(-\infty, +\infty)$  上无穷次可微, 且满足:  $|f^{(k)}(x)| \leq M (k = 1, 2, \dots)$ , 所以

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n. \quad (*)$$

由  $f\left(\frac{1}{2^n}\right) = 0, (n = 1, 2, \dots)$  得  $f(0) = \lim_{n \rightarrow \infty} f\left(\frac{1}{2^n}\right) = 0$ . 于是  $f'(0) = \lim_{n \rightarrow \infty} \frac{f\left(\frac{1}{2^n}\right) - f(0)}{\frac{1}{2^n}} = 0$ . 由

洛尔定理, 对于任意自然数  $n$ , 在  $\left[\frac{1}{2^{n+1}}, \frac{1}{2^n}\right]$  上, 存在  $\xi_n^{(1)} \in \left(\frac{1}{2^{n+1}}, \frac{1}{2^n}\right)$ , 使得  $f'(\xi_n^{(1)}) = 0$ , 从而

$\xi_n^{(1)} \rightarrow 0 (n \rightarrow \infty)$ , 这里

$$\xi_1^{(1)} > \xi_2^{(1)} > \xi_3^{(1)} > \dots > \xi_n^{(1)} > \xi_{n+1}^{(1)} \dots$$

在  $[\xi_{n+1}^{(1)}, \xi_n^{(1)}]$  上, 对  $f'(x)$  由洛尔定理, 存在  $\xi_n^{(2)} \in (\xi_{n+1}^{(1)}, \xi_n^{(1)})$ , 使得  $f''(\xi_n^{(2)}) = 0$  且

$\xi_n^{(2)} \rightarrow 0 (n \rightarrow \infty)$ , 于是

$$f''(0) = \lim_{n \rightarrow \infty} \frac{f'(\xi_n^{(2)}) - f'(0)}{\xi_n^{(2)}} = 0.$$

类似地, 对于任意的  $n$ , 有  $f^{(n)}(0) = 0$ . 由(\*), 有

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \equiv 0.$$

四、【参考解析】: (1)  $J = \iint_D \left( (c+x)^2 + y^2 \right) \rho \, dx \, dy$

$$= 2\rho \int_0^\pi d\varphi \int_0^1 (c^2 + 2act \cos \varphi + a^2 t^2 \cos^2 \varphi + b^2 t^2 \sin^2 \varphi) abt \, dt = \frac{ab\pi}{4} (5a^2 - 3b^2) \rho.$$

$$\int_0^\pi d\varphi \int_0^1 c^2 abt \, dt = abc^2 \frac{\pi}{2} = ab(a^2 - b^2) \frac{\pi}{2};$$

$$\int_0^\pi d\varphi \int_0^1 2act \cos \varphi abt \, dt = 0;$$

$$\int_0^\pi d\varphi \int_0^1 a^2 t^2 \cos^2 \varphi abt \, dt = \frac{a^3 b}{8} \int_0^\pi (1 + \cos 2\varphi) d\varphi = \frac{\pi a^3 b}{8};$$

$$\int_0^\pi d\varphi \int_0^1 b^2 t^2 \sin^2 \varphi abt \, dt = \frac{ab^3}{8} \int_0^\pi (1 - \cos 2\varphi) d\varphi = \frac{\pi ab^3}{8}.$$

(2) 设  $J$  固定,  $b(a)$  是  $J = \frac{ab\rho\pi}{4} (5a^2 - 3b^2)$  确定的隐函数, 则  $b'(a) = \frac{3b^3 - 15a^2b}{5a^3 - 9ab^2}$ . 对

$S = \pi ab(a)$  求导, 则有

$$S'(a) = \pi \left[ b(a) + ab'(a) \right] = \pi \left[ b + \frac{3b^3 - 15a^2b}{5a^2 - 9b^2} \right] = -2\pi b \left( \frac{3b^2 + 5a^2}{5a^2 - 9b^2} \right)$$

显然, 当  $b = \frac{\sqrt{5}}{3}a$  时,  $S'(a)$  不存在; 当  $b < \frac{\sqrt{5}}{3}a$  时,  $S'(a) < 0$ ; 当  $\frac{\sqrt{5}}{3}a < b \leq a$  时,  $S'(a) > 0$ .

由  $J = \frac{ab\rho\pi}{4} (5a^2 - 3b^2)$ , 当  $b = a$  时,

$$a = \left( \frac{2J}{\rho\pi} \right)^{1/4}, S = \left( \frac{2\pi J}{\rho} \right)^{1/2};$$

当  $b = \frac{\sqrt{5}}{3}a$  时,  $a = \left( \frac{18J}{5\sqrt{5}\rho\pi} \right)^{1/4}, S = \left( \frac{2\pi J}{\sqrt{5}\rho} \right)^{1/2};$

由  $\frac{\pi\rho}{2}a^3b \leq J = \frac{ab\rho\pi}{4}(5a^2 - 3b^2)$  可知, 当  $a \rightarrow +\infty$  时,  $b = O(a^{-3})$ , 所以  $\lim_{a \rightarrow +\infty} S = 0$ . 由

此可知, 椭圆的面积不存在最大值和最小值, 且  $0 < S < \left(\frac{2\pi J}{\rho}\right)^{1/2}$ .

五、【参考解析】: 令  $Q = xz^2 + 2yz, P = -(2xz + yz^2)$ , 则

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2(xz + y)\frac{\partial z}{\partial x} + 2(x + yz)\frac{\partial z}{\partial y} + 2z^2.$$

利用格林公式, 有

$$I = 2 \iint_{x^2+y^2 \leq 1} \left[ (xz + y)\frac{\partial z}{\partial x} + (x + yz)\frac{\partial z}{\partial y} + 2z^2 \right] dx dy$$

方程  $F = 0$  对  $x$  求导, 得到

$$\left( z + x \frac{\partial z}{\partial x} \right) F_u + \left( 1 - y \frac{\partial z}{\partial x} \right) F_v = 0 \quad \text{即} \quad \frac{\partial z}{\partial x} = -\frac{zF_u + F_v}{xF_u - yF_v}.$$

同样可得  $\frac{\partial z}{\partial y} = \frac{F_u + zF_v}{xF_u - yF_v}$ . 于是可得

$$\begin{aligned} x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} &= \frac{yF_u - xF_v}{xF_u - yF_v} - z, \quad y \frac{\partial z}{\partial x} + x \frac{\partial z}{\partial y} = 1 - \frac{z(yF_u - xF_v)}{xF_u - yF_v}. \\ (xz + y) \frac{\partial z}{\partial x} + (x + yz) \frac{\partial z}{\partial y} &= 1 - z^2. \end{aligned}$$

$$I = \oint_L (xz^2 + 2yz) dy - (2xz + yz^2) dx = 2 \iint_{x^2+y^2 \leq 1} dx dy = 2\pi.$$

六、【参考解析】: (1) 微分方程为一阶线性微分方程, 解方程的通解为

$$\begin{aligned} y &= e^{\int x dx} \left[ \int x e^{x^2} e^{-\int x dx} dx + C \right] = e^{\frac{x^2}{2}} \left[ \int x e^{x^2} e^{-\frac{x^2}{2}} dx + C \right] \\ &= e^{\frac{x^2}{2}} \left[ \int x e^{\frac{x^2}{2}} dx + C \right] = e^{\frac{x^2}{2}} \left[ e^{\frac{x^2}{2}} + C \right] = e^{x^2} + C e^{\frac{x^2}{2}} \end{aligned}$$

由  $y(0) = 1$ , 得  $C = 0$ , 所以  $y = e^{x^2}$ .

(2) 证明: 注意到

$$\lim_{n \rightarrow \infty} \int_0^1 \frac{n}{n^2 x^2 + 1} dx = \lim_{n \rightarrow \infty} \arctan n = \frac{\pi}{2},$$

$$\begin{aligned} \int_0^1 \frac{nf(x)}{n^2 x^2 + 1} dx &= \int_0^1 \frac{ne^{x^2}}{n^2 x^2 + 1} dx \\ &= \int_0^1 \frac{n}{n^2 x^2 + 1} (e^{x^2} - 1) dx + \int_0^1 \frac{n}{n^2 x^2 + 1} dx \end{aligned}$$

$\forall \varepsilon > 0$ , 由  $\lim_{x \rightarrow 0} (e^{x^2} - 1) = 0$  知  $\exists \delta > 0, \forall 0 < x < \delta$  时, 有  $|e^{x^2} - 1| < \varepsilon \frac{1}{\pi}$ , 因此有

$$\begin{aligned} \int_0^1 \frac{n}{n^2 x^2 + 1} (e^{x^2} - 1) dx &= \int_0^\delta \frac{n}{n^2 x^2 + 1} (e^{x^2} - 1) dx + \int_\delta^1 \frac{n}{n^2 x^2 + 1} (e^{x^2} - 1) dx \\ &\leq \frac{\varepsilon}{\pi} \int_0^\delta \frac{n}{n^2 x^2 + 1} dx + (e - 1) \int_\delta^1 \frac{n}{n^2 x^2 + 1} dx \\ &\leq \frac{\varepsilon}{2} + (e - 1) \frac{n}{n^2 \delta^2 + 1} (1 - \delta) \leq \frac{\varepsilon}{2} + (e - 1) \frac{n}{n^2 \delta^2 + 1} \end{aligned}$$

$\exists N, \forall n > N$  时,  $\frac{n}{n^2 \delta^2 + 1} \leq \frac{\varepsilon}{2(e - 1)}$ , 因此

$$\int_0^1 \frac{n}{n^2 x^2 + 1} (e^{x^2} - 1) dx < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

即  $\lim_{n \rightarrow \infty} \int_0^1 \frac{n}{n^2 x^2 + 1} (e^{x^2} - 1) dx = 0$ . 所以

$$\lim_{n \rightarrow \infty} \int_0^1 \frac{n}{n^2 x^2 + 1} f(x) dx = \lim_{n \rightarrow \infty} \int_0^1 \frac{n}{n^2 x^2 + 1} (e^{x^2} - 1) dx + \lim_{n \rightarrow \infty} \int_0^1 \frac{n}{n^2 x^2 + 1} dx = \frac{\pi}{2}.$$



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