2017 年第九届全国大学生数学竞赛初赛（数学类）参考答案

一、【参考解答】：设平面P上的抛物线C的顶点为$X_0=\left(x_0,y_0,z_0\right)$. 取平面$P$上$X_0$处相互正交的两单位向量$\alpha=\left(\alpha_1,\alpha_2,\alpha_3\right)$和$\beta=\left(\beta_1,\beta_2,\beta_3\right)$，使得$\beta$是抛物线$C$在平面$P$上的对称轴方向，则抛物线的参数方程为

$$X(t)=X_0+t\alpha+\lambda t^2\beta,t\in\mathrm{R}$$

λ为不等于 0 的常数.

记$X(t)=(x(t),\ y(t),\ z(t))$，则

$$x(t)=x_0+\alpha_1 t+\lambda\beta_1 t^2,y(t)=y_0+\alpha_2 t+\lambda\beta_2 t^2,z(t)=z_0+\alpha_3 t+\lambda\beta_3 t^2$$

因为$X\left(t\right)$落在单叶双曲面Γ上，代入方程$x^2+y^2-z^2=1$，得到对任意t要满足的方程

$$\lambda^2\left(\beta_1^2+\beta_2^2-\beta_3^2\right)t^4+2\lambda\left(\alpha_1\beta_1+\alpha_2\beta_2-\alpha_3\beta_3\right)t^3+A_1t^2+A_2t+A_3=0$$

其中A_1,A_2,A_3是与X_0,α,β相关的常数. 于是得到

$$\beta_1^2+\beta_2^2-\beta_3^2=0,\alpha_1\beta_1+\alpha_2\beta_2-\alpha_3\beta_3=0$$

因为$\{\alpha,\beta\}$是平面P上正交的两单位向量，则有

$$\alpha_1^2+\alpha_2^2+\alpha_3^2=1,\beta_1^2+\beta_2^2+\beta_3^2=1,\alpha_1\beta_1+\alpha_2\beta_2+\alpha_3\beta_3=0$$

于是得到

$$\beta_1^2+\beta_2^2=\beta_3^2=\frac{1}{2},\alpha_1\beta_1+\alpha_2\beta_2=0,\alpha_3=0,\alpha_1^2+\alpha_2^2=1$$

$$\alpha=\left(\alpha_1,\alpha_2,0\right),\beta=\left(-\frac{\varepsilon}{\sqrt2}\alpha_2,\frac{\varepsilon}{\sqrt2}\alpha_1,\beta_3\right),\quad\varepsilon=\pm1$$

于是得到平面P的法向量为$n=\alpha\times\beta=\left(A,B,\frac{\varepsilon}{\sqrt2}\right)$，它与$z-$轴方向$e=\left(0,0,1\right)$的夹角$\theta$满足$\cos\theta=n\cdot e=\pm\frac{1}{\sqrt2}$，所以夹角为$\frac{\pi}{4}$或$\frac{3\pi}{4}$.

二、【参考证明】：充分性：若$\left\{a_n\right\}$有界，则可设$a_n\le M$.

$$\sum_{n=1}^{m}\frac{a_{n+1}-a_n}{a_n\ln a_{n+1}}\le\sum_{n=1}^{m}\frac{a_{n+1}-a_n}{a_1\ln a_1}=\frac{a_{m+1}-a_1}{a_1\ln a_1}\le\frac{M}{a_1\ln a_1}$$

由此可知$\sum_{n=1}^{\infty}\frac{a_{n+1}-a_n}{a_n\ln a_{n+1}}$收敛.

必要性：设$\sum_{n=1}^{\infty}\frac{a_{n+1}-a_n}{a_n\ln a_{n+1}}$收敛. 由于

$$\ln\left(a_{n+1}\right)-\ln\left(a_n\right)=\ln\left(1+\frac{a_{n+1}-a_n}{a_n}\right)\le\frac{a_{n+1}-a_n}{a_n}$$

所以$\dfrac{b_{n+1}-b_n}{b_{n+1}} \le \dfrac{a_{n+1}-a_n}{a_n \ln a_{n+1}}$，其中$b_n = \ln a_n$. 因此，级数$\sum\limits_{n=0}^{\infty}\dfrac{b_{n+1}-b_n}{b_{n+1}}$收敛.

由 Cauchy 收敛准则，存在自然数m，使得对一切自然数p，有

$$\frac{1}{2} > \sum_{n=m}^{m+p}\frac{b_{n+1}-b_n}{b_{n+1}} \ge \sum_{n=m}^{m+p}\frac{b_{n+1}-b_n}{b_{m+p+1}} = \frac{b_{m+p+1}-b_m}{b_{m+p+1}} = 1-\frac{b_m}{b_{m+p+1}}$$

由此可知$\{b_n\}$有界，因为p是任意的，因而$\{a_n\}$有界.

题中级数的分母a_n不能换成a_{n+1}. 例如：$a_n = e^{n^2}$无界，但$\sum\limits_{n=1}^{\infty}\dfrac{a_{n+1}-a_n}{a_{n+1}\ln a_{n+1}}$收敛.

三、【参考证明】：必要性：由迹的性质直接得.

充分性：首先，对于可逆矩阵$W\in\Gamma$，有$WW_1,\dots,WW_r$各不相同. 故有

$$W\Gamma \equiv \{WW_1,WW_2,\dots,WW_r\} = \{W_1,W_2,\dots,W_r\}$$

即$W\Gamma=\Gamma,\forall W\in\Gamma$.

记$S=\Sigma_{i=1}^r W_i$，则$WS=S,\forall W\in\Gamma$. 进而$S^2=rS$，即$S^2-rS=0$. 若λ为S的特征值，则$\lambda^2-r\lambda=0$，即$\lambda=0$或r.

结合条件$\Sigma_{i=1}^r \operatorname{tr}(W_i)=0$知，$S$的特征值只能为 0. 因此有$S-rI$可逆（例如取$S$的约当分解就可以直接看出）.

再次注意到$S(S-rI)=S^2-rS=0$，此时右乘$(S-rI)^{-1}$，即得$S=0$. 证毕.

四、【参考证明】：反证：若$XN+Y^TM^T=0$，则$N^TX^T+MY=0$.

另外，由$(X,Y)\in T$得$XY+(XY)^T=2aI$，即$XY+Y^TX^T=2aI$.

类似有$MN+N^TM^T=2aI$. 因此

$$\begin{pmatrix} X & Y^T \\ M & N^T \end{pmatrix}\begin{pmatrix} Y & N \\ X^T & M^T \end{pmatrix} = 2a\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

进而$\dfrac{1}{2a}\begin{pmatrix} Y & N \\ X^T & M^T \end{pmatrix}\begin{pmatrix} X & Y^T \\ M & N^T \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$，得$YY^T+NN^T=0$，所以$Y=0,N=0$.

导致$XY=0$，与$XY=aI+A\ne 0$矛盾. 证毕.

五、【参考解答】：**【思路一】**由定积分的定义，有

$$\begin{aligned} A &= \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) = \int_0^1 f(x)\,\mathrm{d}x \\ &= x\arctan x\Big|_0^1 - \int_0^1 \frac{x}{1+x^2}\,\mathrm{d}x = \frac{\pi}{4}-\frac{\ln 2}{2} \end{aligned}$$

对于$x\in\left(\dfrac{k-1}{n},\dfrac{k}{n}\right),(1\le k\le n)$，由中值定理，存在$\xi_{n,k}\in\left(\dfrac{k-1}{n},\dfrac{k}{n}\right)$使得

$$f(x) = f\left(\frac{k}{n}\right) + f'\left(\frac{k}{n}\right)\left(x-\frac{k}{n}\right) + \frac{f''\left(\xi_{n,k}\right)}{2}\left(x-\frac{k}{n}\right)^2$$

于是

$$
\begin{aligned}
&\left|\sum_{k=1}^{n} f\left(\frac{k}{n}\right)-nA+\sum_{k=1}^{n} n\int_{\frac{k-1}{n}}^{\frac{k}{n}} f'\left(\frac{k}{n}\right)\left(x-\frac{k}{n}\right)\mathrm{d}x\right| \\
&=\left|\sum_{k=1}^{n} n\int_{\frac{k-1}{n}}^{\frac{k}{n}}\left[f\left(\frac{k}{n}\right)+f\left(\frac{k}{n}\right)\left(x-\frac{k}{n}\right)-f(x)\right]\mathrm{d}x\right| \\
&\leqslant M\sum_{k=1}^{n} n\int_{\frac{k-1}{n}}^{\frac{k}{n}}\left(x-\frac{k}{n}\right)^{2}\mathrm{d}x=\frac{M}{3n}
\end{aligned}
$$

其中$M=\frac{1}{2}\max_{x\in[0,1]}\left|f''(x)\right|$.　　因此,

$$
\begin{aligned}
&\lim_{n\to\infty}\left(\sum_{k=1}^{n} f\left(\frac{k}{n}\right)-An\right)=-\lim_{n\to\infty}\sum_{k=1}^{n} n\int_{\frac{k-1}{n}}^{\frac{k}{n}} f'\left(\frac{k}{n}\right)\left(x-\frac{k}{n}\right)\mathrm{d}x \\
&=\lim_{n\to\infty}\frac{1}{2n}\sum_{k=1}^{n} f'\left(\frac{k}{n}\right)=\frac{1}{2}\int_{0}^{1} f'(x)\,\mathrm{d}x=\frac{\pi}{8}
\end{aligned}
$$

【思路二】: 由定积分的定义，有

$$
\begin{aligned}
A&=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right)=\int_{0}^{1} f(x)\,\mathrm{d}x \\
&=x\arctan x\Big|_{0}^{1}-\int_{0}^{1}\frac{x}{1+x^{2}}\mathrm{d}x=\frac{\pi}{4}-\frac{\ln 2}{2}
\end{aligned}
$$

对于$x\in\left(\frac{k-1}{n},\frac{k}{n}\right),(1\leq k\leq n)$，由中值定理，存在$\xi_{n,k}\in\left(\frac{k-1}{n},\frac{k}{n}\right)$使得

$$
f\left(\frac{k}{n}\right)=f(x)+f'(x)\left(\frac{k}{n}-x\right)+\frac{f''\left(\xi_{n,k}\right)}{2}\left(\frac{k}{n}-x\right)^{2}
$$

于是

$$
\begin{aligned}
&\left|\sum_{k=1}^{n} f\left(\frac{k}{n}\right)-nA-\sum_{k=1}^{n} n\int_{\frac{k-1}{n}}^{\frac{k}{n}} f'(x)\left(\frac{k}{n}-x\right)\mathrm{d}x\right| \\
&=\left|\sum_{k=1}^{n} n\int_{\frac{k-1}{n}}^{\frac{k}{n}}\left[f\left(\frac{k}{n}\right)-f(x)-f'(x)\left(\frac{k}{n}-x\right)\right]\mathrm{d}x\right| \\
&\leq M\sum_{k=1}^{n} n\int_{\frac{k-1}{n}}^{\frac{k}{n}}\left(\frac{k}{n}-x\right)^{2}\mathrm{d}x=\frac{M}{3n}
\end{aligned}
$$

其中$M=\frac{1}{2}\max_{x\in[0,1]}\left|f''(x)\right|$. 因此,

$$
\lim_{n\to\infty}\left(\sum_{k=1}^{n} f\left(\frac{k}{n}\right)-An\right)=\lim_{n\to\infty}\sum_{k=1}^{n} n\int_{\frac{k-1}{n}}^{\frac{k}{n}} f'(x)\left(\frac{k}{n}-x\right)\mathrm{d}x
$$

$$
=\lim_{n\to\infty}\sum_{k=1}^{n} nf'\left(\eta_{n,k}\right)\int_{\frac{k-1}{n}}^{\frac{k}{n}}\left(\frac{k}{n}-x\right)\mathrm{d}x=\lim_{n\to\infty}\frac{1}{2n}\sum_{k=1}^{n} f'\left(\eta_{n,k}\right)=\frac{1}{2}\int_{0}^{1} f'(x)\,\mathrm{d}x=\frac{\pi}{8}
$$

其中$\eta_{n,k} \in \left(\frac{k-1}{n}, \frac{k}{n}\right)$.

【思路三】由定积分的定义，有

$$A=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}f\left(\frac{k}{n}\right)=\int_0^1 f(x)\,\mathrm{d}\,x$$
$$=x\arctan x\Big|_0^1-\int_0^1\frac{x}{1+x^2}\mathrm{d}\,x=\frac{\pi}{4}-\frac{\ln 2}{2}$$

对于$x\in\left(\frac{k-1/2}{n},\frac{k+1/2}{n}\right),(1\le k\le n)$，由中值定理，存在

$$\xi_{n,k}\in\left(\frac{k-1/2}{n},\frac{k+1/2}{n}\right)$$

使得$f(x)=f\left(\frac{k}{n}\right)+f'\left(\frac{k}{n}\right)\left(x-\frac{k}{n}\right)+\frac{f''\left(\xi_{n,k}\right)}{2}\left(x-\frac{k}{n}\right)^2$. 于是

$$\left|\sum_{k=1}^{n}f\left(\frac{k}{n}\right)-nA-n\int_1^{1+\frac{1}{2n}}f(x)\,\mathrm{d}\,x+n\int_0^{\frac{1}{2m}}f(x)\,\mathrm{d}\,x\right|$$
$$=\left|\sum_{k=1}^{n}n\int_{\frac{k-\frac{1}{2}}{n}}^{\frac{k+\frac{1}{2}}{n}}\left[f\left(\frac{k}{n}\right)-f(x)+f'\left(\frac{k}{n}\right)\left(\frac{k}{n}-x\right)\right]\mathrm{d}\,x\right|$$
$$\le M\sum_{k=1}^{n}n\int_{\frac{k-1}{n}}^{\frac{k}{n}}\left(\frac{k}{n}-x\right)^2\mathrm{d}\,x=\frac{M}{3n}$$

其中$M=\frac{1}{2}\max_{x\in[0,1]}\left|f''(x)\right|$. 因此

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n}f\left(\frac{k}{n}\right)-An\right)=\lim_{n\to\infty}n\int_1^{1+\frac{1}{2n}}f(x)\,\mathrm{d}\,x-\lim_{n\to\infty}n\int_0^{\frac{1}{2n}}f(x)\,\mathrm{d}\,x$$
$$=\frac{f(1)}{2}-\frac{f(0)}{2}=\frac{\pi}{8}.$$

六、【参考解答】: 容易知道$f\left(x\right)$连续，注意到$f(x)=1-x^2(1-x)$，于是有

$$0<f(x)<1=f(0)=f(1),\forall x\in(0,1)$$

任取$\delta\in\left(0,\frac{1}{2}\right)$，有$\eta=\eta_\delta\in(0,\delta)$使得

$$m_\eta\equiv\min_{x\in[0,\eta]}f(x)>M_\delta\equiv\max_{x\in[\delta,1-\delta]}f(x)$$

于是当$n\ge\frac{1}{\delta^2}$时,

$$
\begin{aligned}
0 &\le \frac{\int_\delta^1 f^n(x)\,\mathrm{d}x}{\int_0^\delta f^n(x)\,\mathrm{d}x} = \frac{\int_{1-\delta}^1 f^n(x)\,\mathrm{d}x}{\int_0^\delta f^n(x)\,\mathrm{d}x} + \frac{\int_\delta^{1-\delta} f^n(x)\,\mathrm{d}x}{\int_0^\delta f^n(x)\,\mathrm{d}x} \\
&= \frac{\int_0^\delta \left(1-x(1-x)^2\right)^n \mathrm{d}x}{\int_0^\delta \left(1-x^2(1-x)\right)^n \mathrm{d}x} + \frac{\int_\delta^{1-\delta} f^n(x)\,\mathrm{d}x}{\int_0^\delta f^n(x)\,\mathrm{d}x} \\
&\le \frac{\int_0^\delta \left(1-\frac{x}{4}\right)^n \mathrm{d}x}{\int_0^\delta \left(1-x^2\right)^n \mathrm{d}x} + \frac{\int_\delta^{1-\delta} f^n(x)\,\mathrm{d}x}{\int_0^\eta f^n(x)\,\mathrm{d}x} \le \frac{\int_0^\delta \left(1-\frac{x}{4}\right)^n \mathrm{d}x}{\int_0^{\frac{1}{\sqrt{n}}} \left(1-\frac{x}{\sqrt{n}}\right)^n \mathrm{d}x} + \frac{(1-2\delta)M_\delta^n}{\eta m_\eta^n} \\
&= \frac{\frac{4}{n+1}\left(1-\left(1-\frac{\delta}{4}\right)^{n+1}\right)}{\frac{\sqrt{n}}{n+1}\left(1-\left(1-\frac{1}{n}\right)^{n+1}\right)} + \frac{(1-\delta)}{\eta}\left(\frac{M_\delta}{m_\eta^n}\right)^n
\end{aligned}
$$

从而 $\lim\limits_{n\to\infty} \dfrac{\int_\delta^1 f^n(x)\,\mathrm{d}x}{\int_0^\delta f^n(x)\,\mathrm{d}x} = 0.$

对于$\varepsilon \in \left(0, \ln\frac{5}{4}\right)$，取$\delta = 2\left(e^\varepsilon - 1\right)$，则$\delta \in \left(0, \frac{1}{2}\right), \ln\frac{2+\delta}{2} = \varepsilon$.

另一方面，由前述结论，存在$N \ge 1$使得当$n \ge N$时，有$\dfrac{\int_\delta^1 f^n(x)\,\mathrm{d}x}{\int_0^\delta f^n(x)\,\mathrm{d}x} \le \varepsilon$.

从而又有

$$
\begin{aligned}
\left|\frac{\int_0^1 f^n(x)\ln(x+2)\,\mathrm{d}x}{\int_0^1 f^n(x)\,\mathrm{d}x} - \ln 2\right| &= \frac{\int_0^1 f^n(x)\ln\frac{x+2}{2}\,\mathrm{d}x}{\int_0^1 f^n(x)\,\mathrm{d}x} \\
&\le \frac{\int_0^\delta f^n(x)\ln\frac{x+2}{2}\,\mathrm{d}x}{\int_0^\delta f^n(x)\,\mathrm{d}x} + \frac{\int_\delta^1 f^n(x)\ln\frac{x+2}{2}\,\mathrm{d}x}{\int_0^\delta f^n(x)\,\mathrm{d}x} \\
&\le \ln\frac{\delta+2}{2} + \frac{\ln 2\int_\delta^1 f^n(x)\,\mathrm{d}x}{\int_0^\delta f^n(x)\,\mathrm{d}x} \le \varepsilon(1+\ln 2)
\end{aligned}
$$

因此 $\lim\limits_{n\to\infty} \dfrac{\int_0^1 f^n(x)\ln(x+2)\,\mathrm{d}x}{\int_0^1 f^n(x)\,\mathrm{d}x} = \ln 2.$