## 2015 年第六届全国大学生数学竞赛决赛 (非数学类)参考答案

## 一、填空题

## (1)【参考解答】:

原式 = 
$$\lim_{x \to \infty} rac{2e^{x^2}\left(\int_0^x e^{u^2} \,\mathrm{d}\,u
ight)}{e^{2x^2}} = \lim_{x \to \infty} rac{2\left(\int_0^x e^{u^2} \,\mathrm{d}\,u
ight)}{e^{x^2}} \ = \lim_{x \to \infty} rac{2e^{x^2}}{2xe^{x^2}} = 0.$$

(2)【参考解答】:令 p=y',则微分方程转换为  $p'-ap^2=0$ ,分离变量后有

$$\frac{\operatorname{d} p}{p^2} = a\operatorname{d} x \Rightarrow -\frac{1}{p} = ax + C_1\,.$$

由
$$p(0) = -1 \Rightarrow C_1 = 0$$
. 所以有 $y' = -\frac{1}{ax} \Rightarrow y = -\frac{1}{a} \ln \left( ax + C_2 \right)$ .

由 
$$y\left(0\right)=0\Rightarrow C_{2}=1$$
 ,所以解为  $y=-rac{1}{a}\ln\left(ax+1
ight).$ 

(3)【参考解答】:记
$$B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & 0 \end{pmatrix}$$
,则 $B^2$ 为零矩阵,故有

$$A^{50} = \left(\lambda E + B
ight)^{50} = \lambda^{50}E + 50\lambda^{49}B = egin{pmatrix} \lambda^{50} & 0 & 0 \ 0 & \lambda^{50} & 0 \ -50\lambda^{49} & 50\lambda^{49} & \lambda^{50} \end{pmatrix}.$$

(4) [参考解答]: 
$$I = \int \frac{1 + \frac{1}{x^2}}{x^2 + \frac{1}{x^2}} dx = \int \frac{1}{2 + \left(x - \frac{1}{x}\right)^2} d\left(x - \frac{1}{x}\right) = \frac{1}{\sqrt{2}} \arctan \frac{1}{\sqrt{2}} \left(x - \frac{1}{x}\right) + C.$$

或者 
$$I = \frac{1}{\sqrt{2}} \left[ \arctan \left( \sqrt{2}x - 1 \right) + \arctan \left( \sqrt{2}x + 1 \right) \right] + C.$$

(5)【参考解答】: 曲线 L 的方程为  $\left|x\right|+\left|y\right|=1$  ,记该曲线所围区域为 D .由格林公式,有

$$I = \oint\limits_L x \,\mathrm{d}\, y - y \,\mathrm{d}\, x = \iint\limits_D igl(1+1igr) \mathrm{d}\, \sigma = 2\sigmaigl(Digr) = 4.$$

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(6) 【参考解答】: 设 
$$F\left(t
ight)=rac{1}{A}\int_{D}f^{t}\left(x,y
ight)\mathrm{d}\,\sigma$$
,则

$$\lim_{n o +\infty} J_n = \lim_{t o 0+} igl( Figl(t) igr)^{\!1/t} = \lim_{t o 0+} \exp \left[ rac{\ln Figl(t)}{t} 
ight]$$

$$=\lim_{t\to0+}\frac{\ln F\left(t\right)-\ln F\left(0\right)}{t-0}=\left(\ln F(t)\right)'_{t=0}=\frac{F'\left(0\right)}{F\left(0\right)}=F'\left(0\right).$$

故有 
$$\lim_{n \to +\infty} J_n = \exp\Bigl(F'\bigl(0\bigr)\Bigr) = \exp\Biggl(rac{1}{A}\iint_D \ln f\bigl(x,y\bigr)\mathrm{d}\,\sigma\Biggr).$$

二、【参考证明】: 设  $\vec{l}_{j},j=1,2,\cdots,n$  都为单位向量,且设

$$egin{aligned} ec{l}_{j} &= \left(\cos\left( heta + rac{j2\pi}{n}
ight), \sin\left( heta + rac{j2\pi}{n}
ight)
ight), \ 
abla f\left(P_{0}
ight) &= \left(rac{\partial f\left(P_{0}
ight)}{\partial x}, rac{\partial f\left(P_{0}
ight)}{\partial y}
ight), \end{aligned}$$

则有
$$\dfrac{\partial f\left(P_{0}
ight)}{\partial ec{l}_{i}}=
abla f\left(P_{0}
ight)\cdot ec{l}_{j}$$
. 因此

$$\sum_{j=1}^n rac{\partial fig(P_0ig)}{\partial ec{l}_i} = \sum_{j=1}^n 
abla fig(P_0ig) \cdot ec{l}_j = 
abla fig(P_0ig) \cdot \sum_{j=1}^n ec{l}_j = 
abla fig(P_0ig) \cdot ec{0} = 0.$$

**三、【参考证明】**: 若存在可逆矩阵 P,Q 使得  $PA_iQ=B_i\left(i=1,2\right)$ ,则  $B_2^{-1}=Q^{-1}A_2^{-1}P^{-1}$ ,所以  $B_1B_2^{-1}=PA_1A_2^{-1}P^{-1}$ ,故  $A_1A_2^{-1}$ 和  $B_1B_2^{-1}$ 相似。反之,若  $A_1A_2^{-1}$ 和  $B_1B_2^{-1}$ 相似,则存在可逆矩阵 C ,使得  $C^{-1}A_1A_2^{-1}C=B_1B_2^{-1}$ . 于是  $C^{-1}A_1A_2^{-1}CB_2=B_1$ .令  $P=C^{-1}$  ,  $Q=A_2^{-1}CB_2$  ,则 P,Q可逆,且满足  $PA_iQ=B_i\left(i=1,2\right)$  .

**四、【参考证明】**: 记  $y_n=x_n^p$  ,则由题设,有  $y_{n+1}=y_n+y_n^2$  ,  $y_{n+1}-y_n=y_n^2\geq 0$  ,所以  $y_{n+1}\geq y_n$  、设  $y_n$  收敛,即有上界,记

$$A=\lim_{n o\infty} {y}_n \le iggl(rac{1}{4}iggr)^{\!p}>0$$
 ,

从而 $A=A+A^2$ ,所以A=0.矛盾.故 $y_n o +\infty$ . 由 $y_{n+1}=y_n ig(1+y_nig)$ ,即

$$\frac{1}{y_{n+1}} = \frac{1}{y_n + y_n^2} = \frac{1}{y_n} - \frac{1}{1 + y_n},$$

于是可得
$$\sum_{k=1}^n \frac{1}{1+y_k} = \sum_{k=1}^n \left( \frac{1}{y_k} - \frac{1}{y_{k+1}} \right) = \frac{1}{y_1} - \frac{1}{y_{n+1}} o \frac{1}{y_1} = 4^p.$$

五、【参考解答】: (1) f(x) 为偶函数,其傅里叶级数是余弦级数.  $a_0=rac{2}{\pi}\int_0^\pi x\,\mathrm{d}\,x=\pi$  .

$$a_n = \frac{2}{\pi} \int_0^\pi x \cos nx \, \mathrm{d} \, x = \frac{2}{\pi n^2} \Big( \cos n\pi - 1 \Big) = \begin{cases} -\frac{4}{\pi n^2}, n = 1, 3, \cdots \\ 0, n = 2, 4, \cdots \end{cases}$$

由于f(x)连续,所以当 $x \in [-\pi,\pi)$ 时,有

$$f\Big(x\Big)=\frac{\pi}{2}-\frac{4}{\pi}\bigg(\cos x+\frac{1}{3^2}\cos 3x+\frac{1}{5^2}\cos 5x+\cdots\bigg)$$
 令  $x=0$  得到  $\sum_{k=0}^{\infty}\frac{1}{\big(2k+1\big)^2}=\frac{\pi^2}{8}$ . 记

$$s_1 = \sum_{k=1}^{\infty} \frac{1}{k^2}, \, s_2 = \sum_{k=0}^{\infty} \frac{1}{\left(2k+1\right)^2}.$$

则 
$$s_1-s_2=rac{1}{4}s_1$$
. 故  $s_1=rac{4s_2}{3}=rac{\pi^2}{6}$  .

(2) 
$$\Leftrightarrow$$
  $g(u) = \frac{u}{1+e^u}$  ,则在 $[0,+\infty)$ 上成立

$$g(u) = \frac{ue^{-u}}{1 + e^{-u}} = ue^{-u} - ue^{-2u} + ue^{-3u} - \cdots$$

记该级数的前 n 项和为  $S_{n}\left(u\right)$  ,余项为  $r_{n}\left(u\right)=g\left(u\right)-S_{n}\left(u\right)$  ,则由交错(单调)级数的性质

$$\left|r_nig(u
ight| \leq ue^{-ig(n+1ig)u}$$
. 因为 $\int_0^{+\infty}ue^{-nu}\,\mathrm{d}\,u = rac{1}{n^2}$ ,就有

$$\int_0^{+\infty} \left| r_n \left( u 
ight) 
ight| \mathrm{d}\, u \leq rac{1}{(n+1)^2}$$
 ,

于是有

$$\int_0^{+\infty} gig(uig)\mathrm{d}\,u = \int_0^{+\infty} S_nig(uig)\mathrm{d}\,u + \int_0^{+\infty} r_nig(uig)\mathrm{d}\,u = \sum_{k=1}^n rac{ig(-1ig)^{k-1}}{k^2} + \int_0^{+\infty} r_nig(uig)\mathrm{d}\,u$$

由于 
$$\lim_{n \to \infty} \int_0^{+\infty} r_n \left( u \right) \mathrm{d} \, u = 0$$
 ,故  $I = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$  ,所以  $I + \frac{1}{2} s_1 = s_1$  .再由(1)所证明

的结果,得 $I = \frac{s_1}{2} = \frac{\pi^2}{12}$ .

六、【参考证明】: (1) 由于f(x,y)非负,所以

$$\iint\limits_{x^2+y^2\leq t^2} fig(x,yig)\mathrm{d}\,\sigma \leq \iint\limits_{-t\leq x,y\leq t} fig(x,yig)\mathrm{d}\,\sigma \leq \iint\limits_{x^2+y^2\leq 2t^2} fig(x,yig)\mathrm{d}\,\sigma$$

当  $t \to +\infty$  ,上式中左右两端极限都收敛于 I ,故结论成立.

(2) 
$$arprojling I\left(t
ight) = \iint\limits_{x^2+y^2 \leq t^2} e^{ax^2+2bxy+cy^2} \,\mathrm{d}\,\sigma$$
,则  $\lim\limits_{t o +\infty} I\left(t
ight) = I$ .记 $A = egin{pmatrix} a & b \\ b & c \end{pmatrix}$ ,则  $ax^2+2bxy+cy^2 = \left(x,y
ight)Aegin{pmatrix} x \\ y \end{pmatrix}.$ 

因 A 实对称,存在正交矩阵 P 使得  $P^TAP=egin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ ,其中  $\lambda_1,\lambda_2$  是 A 的特征值,也就是标准型的系数.

在变换
$$egin{aligned} x \ y \ \end{pmatrix} = Pigg(u \ v \ \end{pmatrix}$$
下 $ax^2 + 2bxy + cy^2 = \lambda_1 u^2 + \lambda_2 v^2$ . 又由于 $u^2 + v^2 = ig(u,v)igg(u \ v \ \end{pmatrix} = Pig(x,y)igg(x \ y \ \end{pmatrix} P^T = ig(x^2 + y^2ig)PP^T = x^2 + y^2,$ 

故变换把圆盘 $x^2+y^2 \le t^2$ 变为 $u^2+v^2 \le t^2$ ,且

$$egin{aligned} \left|rac{\partialig(x,yig)}{\partialig(u,vig)}
ight|=&|P|=1\,,\ I\left(t
ight)=\iint\limits_{u^2+v^2\leq t^2}e^{\lambda_1u^2+\lambda_2v^2}\left|rac{\partialig(x,yig)}{\partialig(u,vig)}
ight|\mathrm{d}\,u\,\mathrm{d}\,v=\iint\limits_{u^2+v^2\leq t^2}e^{\lambda_1u^2+\lambda_2v^2}\,\mathrm{d}\,u\,\mathrm{d}\,v. \end{aligned}$$

由 $\lim_{t \to +\infty} I(t) = I$ 和(1)所证的结果,得

$$\lim_{t o +\infty} \iint\limits_{-t\leq u,v\leq t} e^{\lambda_1 u^2 + \lambda_2 v^2} \,\mathrm{d}\,u\,\mathrm{d}\,v = I.$$

在矩形上分离积分变量得

$$\iint\limits_{-t < u,v < t} e^{\lambda_1 u^2 + \lambda_2 v^2} \,\mathrm{d}\,u \,\mathrm{d}\,v = \int_{-t}^t e^{\lambda_1 u^2} \,\mathrm{d}\,u \int_{-t}^t e^{\lambda_1 v^2} \,\mathrm{d}\,v = I_1 \big(t\big) I_2 \big(t\big).$$

因为 $I_1ig(tig),I_2ig(tig)$ 都严格单增,故 $\lim_{t o +\infty}\int_{-t}^t e^{\lambda_1 u^2}\,\mathrm{d}\,u$  收敛,所以有 $\lambda_1<0$ ;同理有 $\lambda_2<0$ .



## 考研竞赛数学(ID:xwmath)

个专注于大学数学公共基础课资源分享的微信公众平台高等数学,线性代数概率论与数理统计考研数学,竞赛数学数学文化,实验与建模大学学习、生活历程因为专业,所以精彩