## 2019 年第十届全国大学生数学竞赛决赛 (非数学专业)参考答案

## 一、填空题

$$\text{(1)} \ a+b=-3 \qquad \text{(2)} \ I=\frac{\pi \ln a}{2a} \qquad \text{(3)} \ I=\frac{9}{2} \qquad \text{(4)} \ \frac{\partial^2 z}{\partial x \partial y}=\frac{F_2^2 F_{11}-2 F_1 F_2 F_{12}+F_1^2 F_{22}}{F_2^3}$$

(5) 
$$y_1^2 + y_2^2 + \cdots + y_{n-1}^2$$

二、【参考解析】: 由于 f(x) 在区间 (-1,1) 内三阶可导, f(x) 在 x=0 处有 Taylor 公式

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + o(x^3)$$

又f(0) = 0, f'(0) = 1, f''(0) = 0, f'''(0) = -1,所以

$$f(x) = x - \frac{1}{6}x^3 + o\left(x^3\right)$$

由于  $a_1\in(0,1)$  ,数列  $\left\{a_n\right\}$  严格单调且  $\lim_{n\to\infty}a_n=0$  ,则  $a_n>0$  ,且  $\left\{\dfrac{1}{a_n^2}\right\}$  为严格单调增加趋于正无

穷的数列,注意到 $a_{n+1}=f\left(a_{n}
ight)$ ,故由 $\operatorname{Stolz}$ 定理及式,有

$$\lim_{n o \infty} n a_n^2 = \lim_{n o \infty} rac{n}{rac{1}{a_n^2}} = \lim_{n o \infty} rac{1}{rac{1}{a_{n+1}^2} - rac{1}{a_n^2}} = \lim_{n o \infty} rac{a_n^2 a_{n+1}^2}{a_n^2 - a_{n+1}^2} = \lim_{n o \infty} rac{a_n^2 f^2 ig(a_nig)}{a_n^2 - f^2 ig(a_nig)}$$

$$= \lim_{n \to \infty} \frac{a_n^2 \left(a_n - \frac{1}{6} a_n^3 + o\left(a_n^3\right)\right)^2}{a_n^2 - \left(a_n - \frac{1}{6} a_n^3 + o\left(a_n^3\right)\right)^2} = \lim_{n \to \infty} \frac{a_n^4 - \frac{1}{3} a_n^6 + \frac{1}{36} a_n^8 + o\left(a_n^4\right)}{\frac{1}{3} a_n^4 - \frac{1}{36} a_n^6 + o\left(a_n^4\right)} = 3$$

三、【参考解析】: 令  $y=x-\frac{1}{nx}$ ,则  $y'=1+\frac{1}{nx^2}>0$  故函数 y(x) 在  $[\alpha,\beta]$  上严格单调增加.记 y(x)

的反函数为 
$$x(y)$$
 ,则定义在 $\left[ lpha - rac{1}{nlpha}, eta - rac{1}{neta} 
ight]$ 上,且  $x'(y) = rac{1}{y'(x)} = rac{1}{1 + rac{1}{nx^2}} > 0$  .于是

$$\int_{lpha}^{eta} f'iggl(nx-rac{1}{x}iggr)\mathrm{d}\,x = \int_{lpharac{1}{nlpha}}^{etarac{1}{neta}} f'(ny)x'(y)\,\mathrm{d}\,y\,.$$

根据积分中值定理,存在  $\xi_n \in \left[\alpha - \frac{1}{n\alpha}, \beta - \frac{1}{n\beta}\right]$ ,使得

$$\begin{split} &\int_{\alpha\frac{1}{n\alpha}}^{\beta-\frac{1}{n\beta}}f'(ny)x'(y)\,\mathrm{d}\,y = x'\Big(\xi_n\Big)\int_{\alpha\frac{1}{n\alpha}}^{\beta-\frac{1}{n\beta}}f'(ny)\,\mathrm{d}\,y \\ &= \frac{x'\Big(\xi_n\Big)}{n}\bigg[f\bigg(n\beta-\frac{1}{\beta}\bigg)-f\bigg(n\alpha-\frac{1}{\alpha}\bigg)\bigg] \end{split}$$

$$\left|\operatorname{Sut}\left|\int_a^\beta f'\!\left(nx-\frac{1}{x}\right)\!\mathrm{d}x\right| \leq \frac{\left|x'\left(\xi_n\right)\right|}{n} \left\|f\!\left(n\beta-\frac{1}{\beta}\right)| + \left|f\!\left(n\alpha-\frac{1}{\alpha}\right)\right| \leq \frac{2\left|x'\left(\xi_n\right)\right|}{n}.$$

注意到 
$$0 < x'ig( \xi_n ig) = rac{1}{1+rac{1}{n extstyle ^2}} < 1$$
,则

$$\left|\int_{lpha}^{eta}f'iggl(nx-rac{1}{x}iggr)\mathrm{d}\,x
ight|\leqrac{2}{n}\,,\;\;eta\lim_{n o\infty}\int_{lpha}^{eta}f'iggl(nx-rac{1}{x}iggr)\mathrm{d}\,x=0$$

四、【参考解析】: 采用"先二后一"法,并利用对称性,得

$$I = 2 \int_0^1 \mathrm{d}\,z \iint_D rac{\mathrm{d}\,x\,\mathrm{d}\,y}{\left(1+x^2+y^2+z^2
ight)^2}$$
,其中 $D: 0 \le x \le 1, 0 \le y \le x$ 

交换积分次序,得

$$I = \int_0^{\frac{\pi}{4}} \mathrm{d}\, heta \int_0^1 \left( \frac{1}{1+z^2} - \frac{1}{1+\sec^2 heta + z^2} \right) \mathrm{d}\, z$$
 $= \frac{\pi^2}{16} - \int_0^{\frac{\pi}{4}} \mathrm{d}\, heta \int_0^1 \frac{1}{1+\sec^2 heta + z^2} \mathrm{d}\, z$ 

作变量代换:  $z = \tan t$ , 并利用对称性, 得

$$\int_{0}^{\frac{\pi}{4}} d\theta \int_{0}^{1} \frac{1}{1 + \sec^{2}\theta + z^{2}} dz = \int_{0}^{\frac{\pi}{4}} d\theta \int_{0}^{\frac{\pi}{4}} \frac{\sec^{2}t}{\sec^{2}\theta + \sec^{2}t} dt$$

$$= \int_{0}^{\frac{\pi}{4}} d\theta \int_{0}^{\frac{\pi}{4}} \frac{\sec^{2}\theta}{\sec^{2}\theta + \sec^{2}t} dt = \frac{1}{2} \int_{0}^{\frac{\pi}{4}} d\theta \int_{0}^{\frac{\pi}{4}} \frac{\sec^{2}\theta + \sec^{2}t}{\sec^{2}\theta + \sec^{2}t} dt$$

$$= \frac{1}{2} \times \frac{\pi^{2}}{16} = \frac{\pi^{2}}{32}$$

所以, 
$$I = \frac{\pi^2}{16} - \frac{1}{2} \frac{\pi^2}{16} = \frac{\pi^2}{32}$$

五、【参考解析】: 级数通项 
$$a_n=\frac{1}{3}\cdot\frac{2}{5}\cdot\frac{3}{7}\cdots\cdot\frac{n}{2n+1}\cdot\frac{1}{n+1}=\frac{2(2n)!!}{(2n+1)!(n+1)}\bigg(\frac{1}{\sqrt{2}}\bigg)^{2n+2}$$
 令 
$$f(x)=\sum_{n=0}^{\infty}\frac{(2n)!!}{(2n+1)!!(n+1)}x^{2n+2}\,,$$

则收敛区间为
$$(-1,1)$$
,则 $\sum_{n=1}^{\infty}a_n=2igg[figg(rac{1}{\sqrt{2}}igg)-rac{1}{2}igg]$ ,由逐项可导性质,得

$$f'(x) = 2 \sum_{n=0}^{\infty} rac{(2n)!!}{(2n+1)!!} x^{2n+1} = 2g(x)$$
 ,

其中
$$g(x) = \sum_{n=0}^{\infty} \frac{(2n)!!}{(2n+1)!!} x^{2n+1}$$
,因为

$$g'(x) = 1 + \sum_{n=1}^{\infty} \frac{(2n)!!}{(2n-1)!!} x^{2n} = 1 + x \sum_{n=1}^{\infty} \frac{(2n-2)!!}{(2n-1)!!} 2nx^{2n-1}$$

$$=1+x\frac{\mathrm{d}}{\mathrm{d}x}\!\left(\!\sum_{n=1}^{\infty}\!\frac{(2n-2)!!}{(2n-1)!!}\!x^{2n}\right)\!=1+x\frac{\mathrm{d}}{\mathrm{d}x}[xg(x)]$$

所以 
$$g(x)$$
 满足  $g(0) = 0$ ,  $g'(x) - \frac{x}{1-x^2}g(x) = \frac{1}{1-x^2}$ . 解这个一阶线性方程,得

$$g(x) = e^{\int rac{x}{1-x^2} \mathrm{d}x} \Biggl( \int rac{1}{1-x^2} e^{-\int rac{x}{1-x^2} \mathrm{d}x} \mathrm{d}x + C \Biggr) = rac{rcsin x}{\sqrt{1-x^2}} + rac{C}{\sqrt{1-x^2}}$$

由
$$g(0)=0$$
得 $C=0$ ,故 $g(x)=rac{rcsin x}{\sqrt{1-x^2}}$ ,所以

$$f(x) = (\arcsin x)^2, \quad f\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi^2}{16}$$

六、【参考解析】: 存在 n 阶可逆矩阵 H,Q ,使得  $A=Hegin{pmatrix}I_r&O\\O&O\end{pmatrix}$  Q ,因为  $A^2=O$  ,所以

$$A^2 = Hegin{pmatrix} I_r & O \ O & O \end{pmatrix}\!QHegin{pmatrix} I_r & O \ O & O \end{pmatrix}\!Q = O$$

对QH作相应分块为 $QH=egin{pmatrix} R_{11} & R_{12} \ R_{21} & R_{22} \end{pmatrix}$ ,则有

$$\begin{pmatrix} I_r & O \\ O & O \end{pmatrix} Q H \begin{pmatrix} I_r & O \\ O & O \end{pmatrix} = \begin{pmatrix} I_r & O \\ O & O \end{pmatrix} \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} \begin{pmatrix} I_r & O \\ O & O \end{pmatrix} = \begin{pmatrix} R_{11} & O \\ O & O \end{pmatrix} = O$$

因此,
$$R_{11}=O$$
.而 $Q=egin{pmatrix}O&R_{12}\\R_{21}&R_{22}\end{pmatrix}H^{-1}$ ,所以

$$A = H \begin{bmatrix} I_r & O \\ O & O \end{bmatrix} Q = H \begin{bmatrix} I_r & O \\ O & O \end{bmatrix} \begin{bmatrix} O & R_{12} \\ R_{21} & R_{22} \end{bmatrix} H^{-1} = H \begin{bmatrix} O & R_{12} \\ O & O \end{bmatrix} H^{-1}$$

显然,  $r(A) = r(R_{12}) = r$ , 所以 $R_{12}$ 为行满秩矩阵.

因为 $r<rac{n}{2}$ ,所以存在可逆矩阵 $S_1,S_2$ ,使得 $S_1R_{12}S_2=\begin{pmatrix}I_r,O\end{pmatrix}$ ,令 $P=Hegin{pmatrix}S_1^{-1}&O\\O&S_2\end{pmatrix}$ ,则有

$$P^{-1}AP = \begin{pmatrix} S_1 & O \\ O & S_2^{-1} \end{pmatrix} H^{-1}AH \begin{pmatrix} S_1^{-1} & O \\ O & S_2 \end{pmatrix} = \begin{pmatrix} O & I_r & O \\ O & O & O \end{pmatrix}$$

七、【参考解析】:  $\sum_{n=1}^{\infty}a_nu_n$  收敛,所以对任意给定arepsilon>0,存在自然数 $N_1$ ,使得当 $n>N_1$ 时,有

$$-rac{arepsilon}{2} < \sum_{k=N_1}^n a_k u_k < rac{arepsilon}{2}$$
 .

因为 $\left\{u_n^{}\right\}_{n=1}^\infty$ 单调递减的正数列,所以 $0<rac{1}{u_N^{}}\leqrac{1}{u_{N_{+}+1}}\leq\cdots\leqrac{1}{u_n^{}}$ .注意到当m< n时,有

$$\sum_{k=m}^{n} \left(A_{k} - A_{k-1}\right) \! b_{k} = A_{n} b_{n} - A_{m-1} b_{m} + \sum_{k=m}^{n-1} \left(b_{k} - b_{k+1}\right) \! A_{k}$$

$$\sum_{k=1}^{n} a_k^{} b_k^{} = A_n^{} b_n^{} + \sum_{k=1}^{n-1} \! \left( b_k^{} - b_{k+1}^{} \right) \! A_k^{}$$

下面证明:对于任意自然数 n ,如果  $\left\{a_{n}\right\} ,\left\{b_{n}\right\}$  满足

$$b_1 \geq b_2 \geq \cdots \geq b_n \geq 0, m \leq a_1 + a_2 + \cdots + a_n \leq M$$

则有 $b_1 m \leq \sum_{k=1}^n a_k b_k = b_1 M$  .

事实上, $m \leq A_k \leq M, \quad b_k - b_{k+1} \geq 0$ ,即得到

$$mb_1 = mb_n + \sum_{k=1}^{n-1} (b_k - b_{k+1}) m \leq \sum_{k=1}^{n} a_k b_k \leq Mb_n + \sum_{k=1}^{n-1} (b_k - b_{k+1}) M = Mb_1$$

令 
$$b_1=rac{1}{u_n}, b_2=rac{1}{u_{n-1}}, \cdots$$
可以得到

$$-rac{arepsilon}{2}u_n^{-1}<\sum_{k=N_{\star}}^n a_k<rac{arepsilon}{2}u_n^{-1}$$
 ,

即 $\left|\sum_{k=N_1}^n a_k^{} u_n^{}\right| < rac{arepsilon}{2}$ .又由 $\lim_{n o\infty} u_n^{} = 0$ 知,存在自然数 $N_2^{}$ ,使得 $n>N_2^{}$ ,

$$\left|\left(a_1+a_2+\cdots+a_{N_1-1}\right)u_n\right|<\frac{\varepsilon}{2}$$

取 $N = \max\left\{N_1, N_2\right\}$ , 则当n > N时, 有

$$\left|\left(a_1+a_2+\dots+a_n\right)u_n\right|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$$
 因此  $\lim_{n\to\infty}\left(a_1+a_2+\dots+a_n\right)u_n=0$  .



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