2016 年第八届全国大学生数学竞赛初赛（非数学类）
试卷及参考答案

一、填空题(满分 30 分，每小题 5 分)

1. 若 $f(x)$ 在点 $x=a$ 处可导，且 $f(a) \neq 0$，则 $\lim\limits_{n \to +\infty}\left[\dfrac{f(a+1/n)}{f(a)}\right]^n =$ ＿＿＿＿．

【参考解答】：由于 $\lim\limits_{x \to +\infty}\left(\dfrac{f\left(a+\dfrac{1}{x}\right)}{f(a)}\right)^x = \lim\limits_{x \to 0^+}\left(\dfrac{f(a+x)}{f(a)}\right)^{\frac{1}{x}}$，由已知条件：$f(x)$ 在点 $x=a$ 处

可导，且 $f(a) \neq 0$，由带皮亚诺余项的泰勒公式，有

$$f(x) = f(a) + f'(a)(x-a) + o(x-a)$$

可得 $f(a+x) = f(a) + f'(a)x + o(x)$，将其代入极限式，则有

$$\lim\limits_{x \to 0^+}\left(\dfrac{f(a+x)}{f(a)}\right)^{\frac{1}{x}} = \lim\limits_{n \to +\infty}\left(\dfrac{f(a) + f'(a)x + o(x)}{f(a)}\right)^{\frac{1}{x}} = \lim\limits_{n \to +\infty}\left[1 + \left(\dfrac{f'(a)}{f(a)}x + o(x)\right)\right]^{\frac{1}{x}}$$

$$= \lim\limits_{n \to +\infty}\left\{\left[1 + \left(\dfrac{f'(a)}{f(a)}x + o(x)\right)\right]^{\frac{1}{\frac{f'(a)}{f(a)}x + o(x)}}\right\}^{\frac{1}{x}\left[\frac{f'(a)}{f(a)}x + o(x)\right]} = e^{\lim\limits_{n \to +\infty}\frac{f'(a)}{f(a)}\left[1 + \frac{o(x)}{x}\right]} = e^{\frac{f'(a)}{f(a)}}.$$

2. 若 $f(1) = 0, f'(1)$ 存在，则极限 $I = \lim\limits_{x \to 0}\dfrac{f\left(\sin^2 x + \cos x\right)\tan 3x}{\left(e^{x^2} - 1\right)\sin x} =$ ＿＿＿＿＿＿＿．

【参考解答】：$I = \lim\limits_{x \to 0}\dfrac{f\left(\sin^2 x + \cos x\right) \cdot 3x}{x^2 \cdot x} = 3\lim\limits_{x \to 0}\dfrac{f\left(\sin^2 x + \cos x\right)}{x^2}$

$$= 3\lim\limits_{x \to 0}\dfrac{f\left(\sin^2 x + \cos x\right) - f(1)}{\sin^2 x + \cos x - 1} \cdot \dfrac{\sin^2 x + \cos x - 1}{x^2}$$

$$= 3f'(1) \cdot \lim\limits_{x \to 0}\dfrac{\sin^2 x + \cos x - 1}{x^2} = 3f'(1) \cdot \lim\limits_{x \to 0}\left(\dfrac{\sin^2 x}{x^2} + \dfrac{\cos x - 1}{x^2}\right)$$

$$= 3f'(1) \cdot \left(1 - \dfrac{1}{2}\right) = \dfrac{3}{2}f'(1).$$

3. 设 $f(x)$ 有连续导数，且 $f(1) = 2$．记 $z = f\left(e^x y^2\right)$，若 $\dfrac{\partial z}{\partial x} = z$，$f(x)$ 在 $x > 0$ 的

表达式为＿＿＿＿＿＿＿＿．

【参考解答】：由题设，得 $\dfrac{\partial z}{\partial x} = f'\left(e^x y^2\right)e^x y^2 = f\left(e^x y^2\right)$．令 $u = e^x y^2$，得到当 $u > 0$，有

$f'(u)u = f(u)$，即

$$\frac{f'(u)}{f(u)} = \frac{1}{u} \Rightarrow \big(\ln f(u)\big)' = \big(\ln u\big)'.$$

所以有 $\ln f(u) = \ln u + C_1$，$f(u) = Cu$．再由初值条件 $f(1) = 2$，可得 $C = 2$，即 $f(u) = 2u$．

所以当 $x > 0$ 时，有 $f(x) = 2x$．

4．设 $f(x) = e^x \sin 2x$，则 $f^{(4)}(0) = $ _____．

【参考解答】：由带皮亚诺余项余项的麦克劳林公式，有

$$f(x) = \left[1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + o(x^3)\right] \cdot \left[2x - \frac{1}{3!}(2x)^3 + o(x^4)\right]$$

所以 $f(x)$ 展开式的 4 次项为 $-\frac{1}{3!}(2x^3) \cdot x + \frac{2}{3!}x^4 = -x^4$，即有

$$\frac{f^{(4)}(0)}{4!} = -1，故 f^{(4)}(0) = -24.$$

5．曲面 $z = \dfrac{x^2}{2} + y^2$ 平行于平面 $2x + 2y - z = 0$ 的切平面方程为 _____．

【参考解答】：移项，曲面的一般式方程为 $F(x,y,z) = \dfrac{x^2}{2} + y^2 - z = 0$，有

$$\vec{n}(x,y,z) = \big(F_x', F_y', F_z'\big) = (x, 2y, -1).$$

$$\vec{n}(x,y,z) \,//\, \vec{n}_1 \Rightarrow (x, 2y, -1) \,//\, (2,2,-1),$$

可得 $\dfrac{x}{2} = \dfrac{2y}{2} = \dfrac{-1}{-1}$．由此可得 $x = 2, y = 1$，将它代入到曲面方程，可得 $z = 3$，即曲面上点 $(2,1,3)$ 处切平面与已知平面平行，所以由平面的点法式方程可得切平面方程为

$$2(x-2) + 2(y-1) - (z-3) = 0，即 2x + 2y - z = 3.$$

第二题：**(14 分)** 设 $f(x)$ 在 $[0,1]$ 上可导，$f(0) = 0$，且当 $x \in (0,1)$，$0 < f'(x) < 1$．试

证：当 $a \in (0,1)$ 时，有 $\left(\displaystyle\int_0^a f(x)\,\mathrm{d}x\right)^2 > \displaystyle\int_0^a f^3(x)\,\mathrm{d}x.$

【参考解答】：不等式的证明转换为证明不等式 $\left(\displaystyle\int_0^a f(x)dx\right)^2 - \displaystyle\int_0^a f^3(x)dx > 0$．于是对函数

求导，$F'(x) = 2f(x)\displaystyle\int_0^x f(t)dt - f^3(x) = 2f(x)\left(\displaystyle\int_0^x f(t)dt - f^2(x)\right)$

已知条件 $f(0) = 0$，可得 $F'(0) = 0$，并且由 $0 < f'(x) < 1$，所以函数 $f(x)$ 在 $(0,1)$ 内

单调增加，即 $f(x) > 0$，所以只要证明 $g(x) = 2\displaystyle\int_0^x f(t)dt - f^2(x) > 0$．

又 $g(0) = 0$，所以只要证明 $g'(x) > 0$，于是有

$$g'(x) = 2f(x) - 2f(x)f'(x) = 2f(x)\big[1 - f'(x)\big] > 0$$

所以 $g(x)$ 单调增加，所以 $g(x) > 0, x > 0$．所以也就有 $g(x) = 2\displaystyle\int_0^x f(t)dt - f^2(x) > 0$，即

$F'(x) > 0$，可得 $F(x) > 0$，因此 $F(x) = \left(\displaystyle\int_0^x f(t)dt\right)^2 - \displaystyle\int_0^x f^3(t)dt$ 单调增加，所以

$F(a) > F(0) = 0$，即有

$$F(a) = \left(\int_0^a f(t)dt\right)^2 - \int_0^a f^3(t)dt > 0 \Rightarrow \left(\int_0^a f(t)dt\right)^2 > \int_0^a f^3(t)dt.$$

第三题：(14 分)某物体所在的空间区域为 $\Omega : x^2 + y^2 + 2z^2 \leq x + y + 2z$，密度函数为 $x^2 + y^2 + z^2$，求质量 $M = \iiint\limits_{\Omega} \left(x^2 + y^2 + z^2\right) \mathrm{d}\,x\,\mathrm{d}\,y\,\mathrm{d}\,z.$

【参考解答】：令 $u = x - \dfrac{1}{2}, v = y - \dfrac{1}{2}, w = \sqrt{2}\left(z - \dfrac{1}{2}\right)$，即

$$x = u + \frac{1}{2}, y = v + \frac{1}{2}, z = \frac{w}{\sqrt{2}} + \frac{1}{2},$$

则椭球面转换为变量为 u, v, w 的单位球域，即 $\Omega_{uvw} : u^2 + v^2 + w^2 \leq 1$. 则由三重积分的换元法公式，即

$$M = \iiint\limits_{\Omega} \left(x^2 + y^2 + z^2\right) dxdydz = \iiint\limits_{\Omega_{uvw}} F(u,v,w) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dudvdw.$$

$$F(u,v,w) = \left(u + \frac{1}{2}\right)^2 + \left(v + \frac{1}{2}\right)^2 + \left(\frac{w}{\sqrt{2}} + \frac{1}{2}\right)^2 = u^2 + u + v^2 + v + \frac{w^2}{2} + \frac{w}{\sqrt{2}} + \frac{3}{4}$$

$$\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \left\|\begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \end{matrix}\right\| = \left\|\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{matrix}\right\| = \frac{1}{\sqrt{2}}$$

所以原积分就等于

$$M = \iiint\limits_{\Omega_{uvw}} \left(u^2 + u + v^2 + v + \frac{w^2}{2} + \frac{w}{\sqrt{2}} + \frac{3}{4}\right) \frac{1}{\sqrt{2}} dudvdw$$

由于单元圆域 $\Omega_{uvw} : u^2 + v^2 + w^2 \leq 1$ 关于三个坐标面都对称，所以积分也就等于

$$M = \frac{1}{\sqrt{2}} \iiint\limits_{\Omega_{uvw}} \left(u^2 + v^2 + \frac{w^2}{2}\right) dudvdw + \frac{3}{4\sqrt{2}} \iiint\limits_{\Omega_{uvw}} dudvdw$$

其中 $\dfrac{3}{4\sqrt{2}} \iiint\limits_{\Omega_{uvw}} dudvdw = \dfrac{3}{4\sqrt{2}} \dfrac{4\pi}{3} = \dfrac{\pi}{\sqrt{2}}$.

由于积分区域具有轮换对称性，所以有

$$\iiint\limits_{\Omega_{uvw}} u^2 dudvdw = \iiint\limits_{\Omega_{uvw}} v^2 dudvdw = \iiint\limits_{\Omega_{uvw}} w^2 dudvdw$$

$$\iiint\limits_{\Omega_{uvw}} \left(u^2 + v^2 + \frac{w^2}{2}\right) dudvdw = \frac{5}{2} \iiint\limits_{\Omega_{uvw}} u^2 dudvdw = \frac{5}{6} \iiint\limits_{\Omega_{uvw}} \left(u^2 + v^2 + w^2\right) dudvdw$$

所以

$$\frac{1}{\sqrt{2}} \iiint\limits_{\Omega_{uvw}} \left(u^2 + v^2 + \frac{w^2}{2}\right) dudvdw = \frac{5}{6\sqrt{2}} \iiint\limits_{\Omega_{uvw}} \left(u^2 + v^2 + w^2\right) dudvdw$$

$$= \frac{5}{6\sqrt{2}} \int_0^{2\pi} d\theta \int_0^{\pi} d\varphi \int_0^1 r^2 \cdot r^2 \sin\varphi dr = \frac{5}{6\sqrt{2}} \cdot 2\pi \cdot \left[-\cos\varphi\right]_0^{\pi} \cdot \left[\frac{r^5}{5}\right]_0^1 = \frac{5\pi}{3\sqrt{2}} \cdot 2 \cdot \frac{1}{5} = \frac{\sqrt{2}\pi}{3}$$

所以最终的结果就为 $M = \dfrac{\sqrt{2}\pi}{3} + \dfrac{\pi}{\sqrt{2}} = \dfrac{\sqrt{2}\pi}{3} + \dfrac{\sqrt{2}\pi}{2} = \dfrac{5\sqrt{2}\pi}{6}$.

第四题：(14 分) 设函数 $f(x)$ 在闭区间 $[0,1]$ 上具有连续导数，$f(0) = 0, f(1) = 1$. 证明：

$$\lim_{n \to \infty} n\left(\int_0^1 f(x)\mathrm{d}x - \frac{1}{n}\sum_{k=1}^n f\left(\frac{k}{n}\right) \right) = -\frac{1}{2}.$$

【参考解答】：将区间 $[0,1]$ n 等份，分点 $x_k = \dfrac{k}{n}$，则 $\Delta x_k = \dfrac{1}{n}$，且

$$\lim_{n \to \infty} n\left(\int_0^1 f(x)dx - \frac{1}{n}\sum_{k=1}^n f\left(\frac{k}{n}\right) \right) = \lim_{n \to \infty} n\left(\sum_{k=1}^n \int_{x_{k-1}}^{x_k} f(x)dx - \sum_{k=1}^n f(x_k)\Delta x_k \right)$$

$$= \lim_{n \to \infty} n\left(\sum_{k=1}^n \int_{x_{k-1}}^{x_k} \left[f(x) - f(x_k) \right]dx \right) = \lim_{n \to \infty} n\left(\sum_{k=1}^n \int_{x_{k-1}}^{x_k} \frac{f(x) - f(x_k)}{x - x_k}(x - x_k)dx \right)$$

$$= \lim_{n \to \infty} n\left(\sum_{k=1}^n \frac{f(\xi_k) - f(x_k)}{\xi_k - x_k} \int_{x_{k-1}}^{x_k}(x - x_k)dx \right), \xi_k \in (x_{k-1}, x_k)$$

$$= \lim_{n \to \infty} n\left(\sum_{k=1}^n f'(\eta_k) \int_{x_{k-1}}^{x_k}(x - x_k)dx \right) = \lim_{n \to \infty} n\left(\sum_{k=1}^n f'(\eta_k)\left[-\frac{1}{2}(x_k - x_{k-1})^2 \right] \right)$$

$$= -\frac{1}{2}\lim_{n \to \infty} \left(\sum_{k=1}^n f'(\eta_k)(x_k - x_{k-1}) \right) = -\frac{1}{2}\int_0^1 f'(x)dx = -\frac{1}{2}.$$

第五题：(14 分) 设函数 $f(x)$ 在区间 $[0,1]$ 上连续，且 $I = \displaystyle\int_0^1 f(x)\mathrm{d}x \neq 0$. 证明：在 $(0,1)$ 内存在不同的两点 x_1, x_2，使得 $\dfrac{1}{f(x_1)} + \dfrac{1}{f(x_2)} = \dfrac{2}{I}$.

【参考解答】：设 $F(x) = \dfrac{1}{I}\displaystyle\int_0^x f(t)dt$，则 $F(0) = 0, F(1) = 1$. 由介值定理，存在 $\xi \in (0,1)$，使得 $F(\xi) = \dfrac{1}{2}$. 在两个子区间 $(0,\xi), (\xi, 1)$ 上分别应用拉格朗日中值定理：

$$F'(x_1) = \frac{f(x_1)}{I} = \frac{F(\xi) - F(0)}{\xi - 0} = \frac{1/2}{\xi}, x_1 \in (0, \xi),$$

$$F'(x_2) = \frac{f(x_2)}{I} = \frac{F(1) - F(\xi)}{1 - \xi} = \frac{1/2}{1 - \xi}, x_2 \in (\xi, 1),$$

$$\frac{I}{f(x_1)} + \frac{I}{f(x_2)} = \frac{1}{F'(x_1)} + \frac{1}{F'(x_2)} = \frac{\xi}{1/2} + \frac{1 - \xi}{1/2} = 2.$$

第六题：(14 分) 设 $f(x)$ 在 $(-\infty, +\infty)$ 上可导，且 $f(x) = f(x+2) = f(x+\sqrt{3})$，用傅里叶(Fourier)级数理论证明 $f(x)$ 为常数。

【参考解答】：由 $f(x) = f(x+2) = f(x+\sqrt{3})$ 可知，f 是以 $2, \sqrt{3}$ 为周期的函数，所以它的傅里叶系数为

$$a_n = \int_{-1}^1 f(x)\cos n\pi x dx, b_n = \int_{-1}^1 f(x)\sin n\pi x dx$$

由于 $f(x) = f(x+\sqrt{3})$，所以

$$a_n = \int_{-1}^{1} f(x) \cos n\pi x \, dx = \int_{-1}^{1} f\left(x + \sqrt{3}\right) \cos n\pi x \, dx$$

$$= \int_{-1+\sqrt{3}}^{1+\sqrt{3}} f(t) \cos n\pi \left(t - \sqrt{3}\right) dt$$

$$= \int_{-1+\sqrt{3}}^{1+\sqrt{3}} f(t) \left[\cos n\pi t \cos \sqrt{3} n\pi + \sin n\pi t \sin \sqrt{3} n\pi \right] dt$$

$$= \cos \sqrt{3} n\pi \int_{-1+\sqrt{3}}^{1+\sqrt{3}} f(t) \cos n\pi t \, dt + \sin \sqrt{3} n\pi \int_{-1+\sqrt{3}}^{1+\sqrt{3}} f(t) \sin n\pi t \, dt$$

$$= \cos \sqrt{3} n\pi \int_{-1}^{1} f(t) \cos n\pi t \, dt + \sin \sqrt{3} n\pi \int_{-1}^{1} f(t) \sin n\pi t \, dt$$

所以 $a_n = a_n \cos \sqrt{3} n\pi + b_n \sin \sqrt{3} n\pi$ ；同理可得

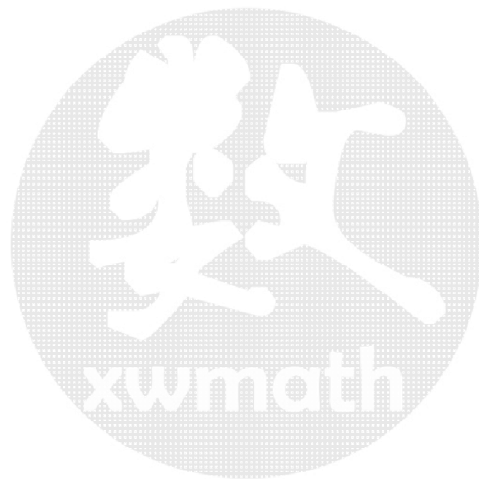$$b_n = b_n \cos \sqrt{3} n\pi - a_n \sin \sqrt{3} n\pi \, .$$

联立，有

$$\begin{cases} a_n = a_n \cos \sqrt{3} n\pi + b_n \sin \sqrt{3} n\pi \\ b_n = b_n \cos \sqrt{3} n\pi - a_n \sin \sqrt{3} n\pi \end{cases}$$

得 $a_n = b_n = 0 \left(n = 1, 2, \cdots\right)$. 而 f 可导，其 Fourier 级数处处收敛于 $f(x)$ ，所以有

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right) = \frac{a_0}{2} ,$$

其中 $a_0 = \int_{-1}^{1} f(x) \, dx$ 为常数.