Nonlinear correlations in multifractals: Visibility graphs of magnitude and sign series

Cite as: Chaos **30**, 013151 (2020); https://doi.org/10.1063/1.5132614
Submitted: 18 October 2019 . Accepted: 20 January 2020 . Published Online: 29 January 2020

Pouya Manshour

Nonlinear correlations in multifractals: Visibility graphs of magnitude and sign series

Cite as: Chaos **30**, 013151 (2020); doi: 10.1063/1.5132614
Submitted: 18 October 2019 · Accepted: 20 January 2020 ·
Published Online: 29 January 2020

View Online Export Citation CrossMark

Pouya Manshour[a)]

AFFILIATIONS

Department of Physics, Faculty of Sciences, Persian Gulf University, 75169 Bushehr, Iran

[a)]**Author to whom correspondence should be addressed:** manshour@pgu.ac.ir

ABSTRACT

Correlations in a multifractal series have been investigated extensively. Almost all approaches try to find scaling features of a given time series. However, the scaling analysis has always been encountered with some difficulties. Of particular importance is finding a proper scaling region and removing the impact of the probability distribution function of the series on the correlation extraction methods. In this article, we apply the horizontal visibility graph algorithm to map a stochastic time series into networks. By investigating the magnitude and sign of a multifractal time series, we show that one can detect linear as well as nonlinear correlations, even for situations that have been considered as uncorrelated noises by typical approaches such as the multifractal detrended fluctuation analysis. Furthermore, we introduce a topological parameter that can well measure the strength of nonlinear correlations. This parameter is independent of the probability distribution function and calculated without the need to find any scaling region. Our findings may provide new insights about the multifractal analysis of a time series in a variety of complex systems.

An irregular time series is almost our only asset when we try to investigate a complex system. In spite of the irregular nature of such a series, it has been found that they usually evolve in some correlated patterns that observe in a huge number of phenomena with completely different microscopic elements. In fact, by studying such correlations, we hope to better understand the underlying dynamics of the system and consequently give us enough information that can be used for modeling, controlling, and predicting the system's future. Many approaches have been introduced to discover correlations in a time series, among which detrended fluctuation analysis techniques are of particular interest. However, it has been recently demonstrated that they can lead to wrong results in some important situations. To solve this, we apply the concept of the visibility graph algorithm and show that one can well detect linear and nonlinear correlations by using proper topological characteristics. In this respect, we also define a parameter that can measure the strength of nonlinear correlations, where previous methods are not able to discover such a nonlinearity.

I. INTRODUCTION

Many biological, physical, and social systems exhibit irregular behavior, which is a consequence of temporal or spatial interactions among their multitude of components. The emergence of scale invariant properties is almost a common output of such complex systems, a situation in which some properties of the system do not change if scales of the system are multiplied by a common factor.[1] Such concepts are usually described by the theory of critical phenomena, which investigates events occurring in a close vicinity of the critical points.[1] The fractal geometry[2] has proved itself as a significant framework to analyze such scaling behaviors in various fields of research such as physics,[3] chemistry,[4] biology,[5] geology,[6] neuroscience,[7] engineering,[8] finance,[9] meteorology,[10] and hydrology.[11]

In general, the fractal analysis helps us to better understand the underlying dynamics and to more precisely model various complex systems. If the fluctuations of all different magnitudes in a time series $x(t)$ scale with the same exponent, and some kind of homogeneity (linearity) exists in the behavior of the system over various scales from small to large ones, the system can be fully described by one parameter, called the Hurst exponent H.[2] This parameter characterizes the strength of the linear correlation, and in stationary series, it indicates how fast the second order correlation function, $C(s) = \langle x(t+s)x(t)\rangle - \langle x(t)\rangle^2$, decays as a function of scale s. Any higher order (>2) correlation function can also be obtained by the second order one. This situation is called *monofractality*.[12] However, this is not the whole story for many observed phenomena; i.e., often more than one exponent is needed to fully describe a complex system. The heterogeneity (nonlinearity) inherited in such systems results in the presence of different scaling behaviors for the fluctuations with different magnitudes, and higher order correlation functions are needed to describe system's features. This defines the concept of *multifractality*.[13] Mono- and multifractality have been explored in a huge number of phenomena such as stock markets,[14] turbulent flows,[15] earthquakes and seismic series,[16,17] human heartbeat dynamics,[18] music,[19] to name a few.

Among the numerous techniques that have been proposed to analyze multifractal features,[20] the multifractal detrended fluctuation analysis (MFDFA) has proved to be quite successful in one[21] or even higher dimensional spaces.[22] Indeed, MFDFA is a generalization of the detrended fluctuation analysis (DFA) designed for monofractals.[23] In the DFA method, after removing the local trend in boxes of size s by subtracting the original series from a polynomial of a certain order, the root mean square of the resulting series (the fluctuation function), F_2, is obtained in all scales s. For the power-law correlated series, we have $F_2(s) \sim s^{-\alpha}$, where α characterizes the strength of linear correlations in the series. In fact, for the stationary ($\alpha < 1$) and nonstationary ($\alpha > 1$) linearly correlated series, $\alpha = H$ and $H + 1$, respectively. In general, values of $\alpha < 1/2$ represent the anticorrelated series, and for $\alpha > 1/2$, the series is positively correlated. At $\alpha = 1/2$, the series is uncorrelated. It is worth mentioning here that for a stationary and positively correlated linear series (i.e., $1/2 < \alpha < 1$), the second order correlation function scales as $C(s) \sim s^{-\gamma}$, with $\gamma = 2 - 2\alpha$.

In the multifractal series, MFDFA generalizes the DFA method by analyzing the scaling of the fluctuations for all moments of order q, where $q = 2$ leads to DFA. For a power-law correlated series, the fluctuation function F_q scales as $F_q(s) \sim s^{-H(q)}$, where $H(q)$ is the generalized Hurst exponent, from which we can find the scaling function $\tau(q) = qH(q) - 1$. The multifractal spectrum $f(\alpha_q)$ indicates the distribution of scaling exponents $\alpha_q = d\tau/dq$ and can be obtained as $f(\alpha_q) = q\alpha_q - \tau(q)$. The width $\Delta\alpha_q$ of $f(\alpha_q)$ can be considered as a parameter for measuring the strength of multifractality.[21] Note that for a monofractal linear series, $H(q) = \alpha$ is independent of q, and the multifractal spectrum becomes a delta function $f(\alpha_q) = \delta(\alpha_q - \alpha)$; thus, $\Delta\alpha_q = 0$.

Another important approach for analyzing correlated processes that have been studied extensively is to decompose the series of increments ($x_i = X_{i+1} - X_i$) into magnitude ($x_i^{mag} = |x_i|$) and sign [$x_i^{sgn} = \text{sign}(x_i)$] series and then extract their scaling characteristics. For example, by using DFA and MFDFA methods, it has been shown[24,25] that the presence of correlation in the magnitude and sign series corresponds to the nonlinearity and linearity of the original series, respectively. This approach has been applied in various fields of study.[26–31]

In DFA and MFDFA techniques, one usually confronts with some challenges such as choosing an appropriate polynomial order for a detrending procedure, finding a proper scaling region, and detecting correct correlations that might be affected by the probability distribution function (PDF) of the series. Recently, it has been shown that in some conditions, DFA and MFDFA are not able to extract correct scaling behaviors of a time series.[31–33] The existence of crossovers in the scaling behavior at some particular scale s_c as well as the q dependency of that s_c are two examples of a possible inaccuracy in the multifractal spectrum estimation. On the other hand, it has been shown that in some situations, DFA (MFDFA) wrongly predicts the linearity in a nonlinear time series; i.e., they assign uncorrelated behavior to a correlated magnitude series of a nonlinear multifractal process, which occurs due to some technical issues.[31,32] For example, Carpena *et. al.* have proved analytically that for a correlated series with the DFA exponent of α, the second order fluctuation function of its magnitude, $F_2^{mag}(s)$, and sign $F_2^{sgn}(s)$ series scales as $(as + bs^{4\alpha-2})^{1/2}$ and $(cs + ds^{2\alpha})^{1/2}$, respectively, where a, b, c, and d are scale-independent coefficients.[32] Obviously, for values of $\alpha \leq 3/4$, the first term in $F_2^{mag}(s)$ dominates for large s, and thus, $F_2^{mag}(s) \sim s^{1/2}$. Similarly, for $\alpha \leq 1/2$, the first term in $F_2^{sgn}(s)$ dominates again for large s, and thus, we have $F_2^{sgn}(s) \sim s^{1/2}$. This means that DFA or MFDFA predicts uncorrelated behavior for magnitude and sign series when $\alpha \leq 3/4$ and $\alpha \leq 1/2$, respectively. However, they have shown that the magnitude and sign series of such correlated processes are actually linearly correlated by applying the second order correlation function, $C(s)$. Using this idea, Bernaola-Galván *et al.* proposed to find the deviation of the second order correlation function of the series magnitude $C_{|x|}(s) = \langle|x(t+s)||x(t)|\rangle - \langle|x(t)|\rangle^2$ from its expectation in a linear Gaussian series to correctly discover nonlinearities in a given time series.[31] They argued that this method can be used for a short series and can also be applied even for a series that do not show any scaling behavior. In a Gaussian multifractal model, they also showed that the nonlinearity implies the multifractality, but the reverse is not true.

In search of another possible way for extracting correlations in a time series, Lacasa *et al.* introduced an algorithm, called the visibility graph,[34,35] that maps a time series into a graph based on the ability of the data points to see each other. In this approach, time series features are believed to be inherited in the resulting graph. For example, they showed that the monofractal exponent H of a linear time series can be calculated from the degree distribution of the mapped graph. Thus, this algorithm may be considered as a novel method to analyze fractal and multifractal phenomena, along with other typical approaches such as DFA and MFDFA. In spite of exploring various aspects of the visibility graph algorithm in different systems and situations,[36–41] surprisingly, no general picture has emerged yet for the multifractal series with nonlinear correlations. In this article, we apply the horizontal visibility graph algorithm to map fractal and multifractal time series into graphs. By investigating the topological characteristics of the resulting graphs, we first show that this approach can well detect linear and nonlinear correlations, even for situations that DFA and MFDFA predict uncorrelatedness, due to technical issues,[31–33] mentioned above. On the other hand, we indicate that owing to the unique characteristic of the horizontal visibility graph algorithm, one can calculate linear or nonlinear correlations, without the need to eliminate the impact of non-Gaussianity of the original series. Finally, we introduce a parameter that can well measure the strength of the nonlinear correlation, where the multifractal spectrum width $\Delta\alpha_q$, that is, a typical and widely used measure for such an analysis, is not able to discover existing nonlinearities. Our results are in line with findings in recent studies.[31,32]

II. DEFINITIONS: VISIBILITY GRAPHS

In order to extract correlation information from a stochastic process, Lacasa *et al.* introduced the visibility graph algorithm[34] that maps a time series into a graph based on the visibility of the data points and allows us to apply the complex network theory for characterizing the time series. Afterward, they also defined another version of that algorithm, called the horizontal visibility graph[35] (HVG), which has some advantages when compared with the (normal) visibility graph (NVG). For example, HVG is a geometrically simpler and computationally faster algorithm than the original NVG. On the other hand, a unique characteristic of HVG is that this algorithm is independent of the PDF of the original series, in contrast to NVG.[41] It is worth mentioning here that since PDF can affect the correlation estimation in the time series, one usually needs to replace the original non-Gaussian series with a Gaussian one rank-wisely to eliminate such distributional effects.[42,43] However, it has been shown that this method performs well only where the data are linearly uncorrelated.[44] We note that HVG has no such limitations and thus has a special advantage in extracting correlation aspects of a time series, when compared with previous methods. In this respect, we apply the horizontal visibility algorithm here. Let x_i be a series of N data ($i = 1, 2, \ldots, N$). By assigning each data point to a node in the graph, one can map a time series of size N into a graph with N nodes. Two nodes i and j are connected if one can draw a horizontal line in the time series joining x_i and x_j that does not intersect any intermediate data height, i.e., two arbitrary points (t_i, x_i) and (t_j, x_j) become two connected nodes, if any other data point (t_q, x_q) placed between them satisfies $x_i, x_j > x_q$ for all q such that $t_i < t_q < t_j$. It has been shown that HVG is always connected by definition and is also invariant under affine transformations, due to the mapping method. On the other hand, ordered and random series are mapped into regular and random exponential graphs, respectively.

To analyze topological features of the mapped graphs, we first need to construct the corresponding adjacency matrix, A, so that $A_{ij} = 1$ if nodes i and j are connected and $A_{ij} = 0$, otherwise. The degree k_i of an arbitrary node i can be obtained via $k_i = \sum_j A_{ij}$. Afterward, the average degree $\langle k \rangle = \sum_k k p_k$ and standard deviation of the degree $\sigma_k^2 = \sum_k k^2 p_k - (\sum_k k p_k)^2$ can be calculated from the degree distribution p_k, defined as the probability of finding a node with degree k in the network. Also, the maximum eigenvalue of the adjacency matrix, e_{max}, can be obtained from A_{ij}, which measures the strength of the information flow in complex networks and is proportional to the largest degree.[45]

By definition, the time order of the original series, $x(t)$, is maintained in the resulting degree sequence, $k(t) = \{k_1, k_2, \ldots, k_N\}$. This leads us to find the presence of any possible correlation between $x(t)$ and $k(t)$. In this respect, we choose the Spearman correlation coefficient,[46] which measures the strength of a monotonic relationship, in comparison with the Pearson correlation coefficient, which can only find linear relationships among two quantities.[46] Taking into account x_r and k_r as the rank of $x(t)$ and $k(t)$, respectively, the Spearman correlation coefficient is defined as

$$S = \frac{\langle [x_r - \langle x_r \rangle][k_r - \langle k_r \rangle] \rangle}{\sigma_{x_r} \sigma_{k_r}}, \tag{1}$$

where $S \in [-1, 1]$ and also σ_{x_r} and σ_{k_r} are the standard deviations of x_r and k_r, respectively. We note that the Kendall coefficient, which is another widely used monotonic correlation measure,[46] yields similar results to the Spearman coefficient here.

Among a huge number of topological features, introduced to better characterize complex networks,[47] we choose two extensively studied quantities, the assortativity, r, and clustering coefficient, Cl, that can measure the presence of any correlation between nodes' degrees. The assortativity coefficient for mixing by a node degree in an undirected network is

$$r = \frac{\sum_{jl} jl(e_{jl} - q_j q_l)}{\sigma_q^2}, \tag{2}$$

where e_{jl} is the fraction of edges that connect nodes of degrees j and l, $q_k = (k+1)p_{k+1}/\langle k \rangle$ is the excess degree of a node defined as the number of edges leaving the node other than the one we arrived along, and σ_q is the standard deviation of the distribution q_k. In general, we have $-1 \le r \le 1$. Positive (negative) values of r indicate a correlation between nodes of a similar (different) degree. Indeed, the network is nonassortative when $r = 0$, and for $r = 1$ or $r = -1$, the network is said to be completely assortative or disassortative, respectively. The clustering coefficient, Cl, is an important (three-point) correlation measure in complex networks and can be considered as the density of connected triads of nodes in a network.[48] The local clustering of an arbitrary node i is defined as $c_i = n_e/n_p$, where n_e and n_p represent the number of existing triads and the total number of possible triads, respectively, that share node i. Thus, the average clustering coefficient of the network is

$$Cl = \frac{\sum c_i}{N}. \tag{3}$$

We should note here that one can certainly consider any other topological properties and do the same analysis in order to investigate the potential effects of the series characteristics on the resulting graphs.

III. DEFINITIONS: SIMULATED SERIES

We intend to investigate the impact of linear and nonlinear correlations as well as PDF of a series on the topological characteristics of the resulting visibility graph. In this respect, we generate series with the adjustable three features.

The fractional Brownian motion (fBm) was introduced to model a turbulent flow[49] and widely used in a variety of fields, including physics, statistics, hydrology, economics, biology, and many others.[50–52] A fBm is a Gaussian monofractal process with stationary increments called the *fractional Gaussian noise* (fGn) and has long memory, which depends on the Hurst index, H, with $0 < H < 1$.[2] In fact, $H = 1/2$ corresponds to the ordinary Brownian motion in which successive increments are statistically independent. For $H > 1/2$, the increments are positively correlated, and for $H < 1/2$, consecutive increments are more likely to have opposite signs and are anticorrelated. One can generate such correlated series by using an algorithm called the Fourier filtering method (FFM),[53] as follows: multiply the Fourier transform of a generated white noise by a power-law of the form $f^{-\beta}$ and then Fourier transform the resulting series again in order to come back to the time domain. Finally, we have a correlated series with a power spectrum of $S(f) \sim f^{-\beta}$. Note that $\beta = 2\alpha - 1$, where α is the DFA exponent.

Both fGn and fBm series have Gaussian PDF. However, a wide range of natural and social phenomena exhibit a heavy-tailed PDF with an infinite variance. To study such heavy tails, various models have been proposed. Among them, the *Lévy stable* distribution (LSD) has been considered extensively.[54] The LSD is a family of all attractors of normalized sums of independent and identically distributed random variables. A symmetric LSD is characterized by the stability parameter $\lambda \in (0, 2]$. For $0 < \lambda \leq 1$, the distribution has an indefinite mean and variance value, and for $1 < \lambda \leq 2$, it has a defined mean but infinite variance. The most well-known LSD functions are the Cauchy distribution with $\lambda = 1$ and the Gaussian distribution function with $\lambda = 2$. Thus, one can construct various uncorrelated series with non-Gaussian distributions, for different $\lambda < 2$.

All the time series defined above are linearly correlated (fBm and fGn) or completely uncorrelated (Lévy stable). To investigate the effect of nonlinear correlations, a *multiplicative* multifractal series has been proposed by Kalisky *et al.*[25] and can be generated by multiplying the magnitude and sign of two independent linear correlated time series with different DFA exponents of α_1 and α_2, as follows:

$$x_{mult} = |f_{\alpha_1}| sgn\left(f_{\alpha_2}\right), \tag{4}$$

where $|\cdots|$ and $sgn(\cdots)$ indicate the magnitude and sign operators. Also, f_α represents a correlated series generated by the Fourier filtering method described above with $\alpha = (\beta + 1)/2$. One can control the strength of linear and nonlinear correlations in x_{mult}, with parameters α_2 and α_1, respectively.[25]

IV. RESULTS

At first, to show that the horizontal visibility graph algorithm is independent of the series PDF, we plotted in Fig. 1 six topological properties described in Sec. II corresponding to the original, x, magnitude, $|x|$, and sign, $sgn(x)$, of a Lévy stable process, with $1 \leq \lambda \leq 2$. Figures 1(a)–1(f) represent the average degree $\langle k \rangle$, standard deviation σ_k, clustering coefficient Cl, assortativity r, Spearman's coefficient S, and maximum eigenvalue $e_{\max}$ of the adjacency matrix, respectively. We note that by increasing λ from 1 to 2, PDF

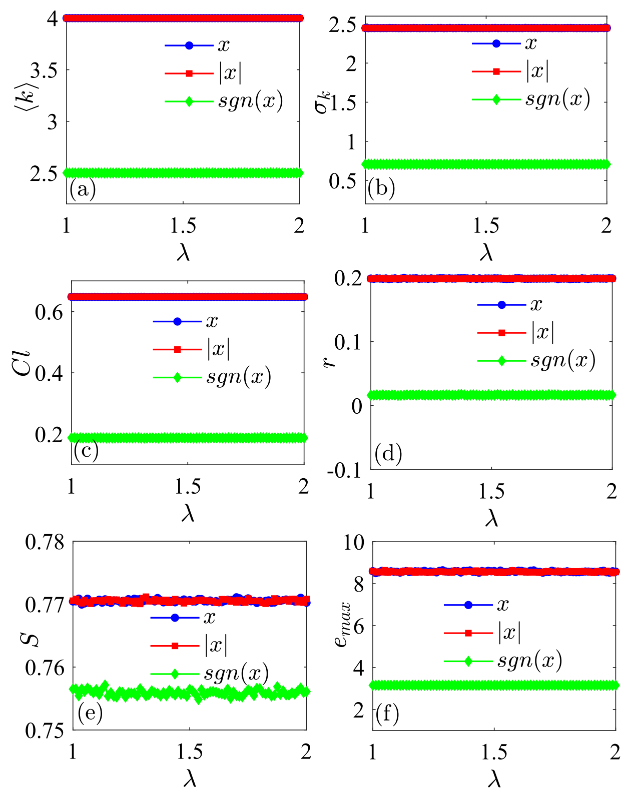

FIG. 1. The PDF dependency of (a) the average degree, $\langle k \rangle$; (b) standard deviation, σ_k; (c) clustering coefficient, Cl; (d) assortativity, r; (e) Spearman's coefficient, S; and (f) the maximum eigenvalue, e_{max}, of the adjacency matrix for the original, x; magnitude, $|x|$; and sign, $sgn(x)$, series of a Lévy stable process with $\lambda \in [1, 2]$. All features are completely independent of the non-Gaussianity of the series.

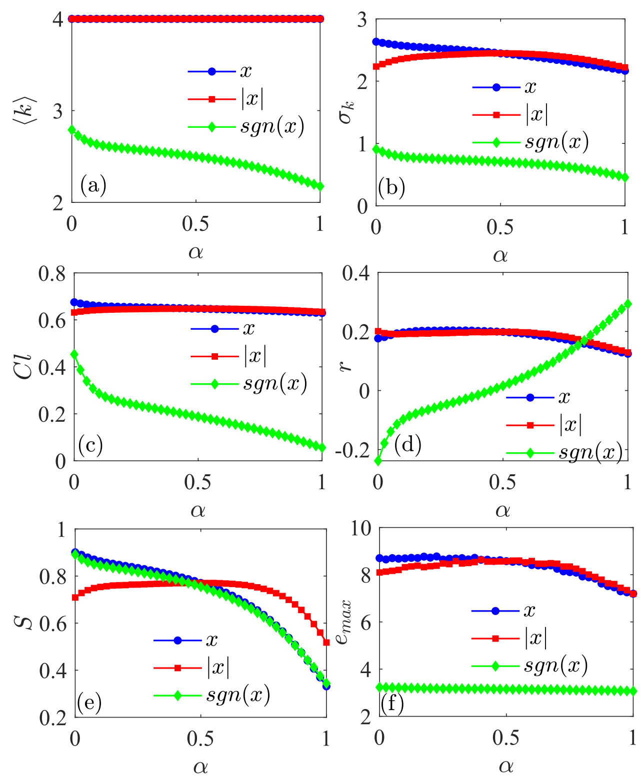

FIG. 2. The linear correlation dependency of (a) the average degree, $\langle k \rangle$; (b) standard deviation, σ_k; (c) clustering coefficient, Cl; (d) assortativity, r; (e) Spearman's coefficient, S; and (f) the maximum eigenvalue, e_{max}, of the adjacency matrix for the original, x; magnitude, $|x|$; and sign, $sgn(x)$, series of a fractional Gaussian noise with $\alpha \in [0, 1]$.

transits from a Cauchy (non-Gaussian) to a normal (Gaussian) functional form. As can be seen, non-Gaussianity cannot affect HVG, and all plots are constants for various λ values. We argue here that such a unique characteristic of the HVG algorithm shows that without any additional action, one can remove the impact of PDF on the correlation properties of a given time series. Thus, we can always apply this technique for situations where the exact estimation of correlations is the main goal.

In order to investigate the impact of linear correlations on the HVG, we map the fractal series with linear correlations generated by FFM methods for different $0 \le \alpha \le 1$ (see Sec. III). The resulting topological features are plotted in Fig. 2, similar to Fig. 1. We find that the average degree, $\langle k \rangle$, of the magnitude and original series is the same and cannot distinguish series with different linear correlations. However, some parameters such as σ_k, Cl, and e_{max} can well discriminate positive and negative correlations in the magnitude and original series. Also, the Spearman's coefficient for the sign and original series is the same, now with a high power of discrimination between all values of α. As we mentioned before, DFA cannot detect any correlation in the magnitude and sign series for $\alpha \le 3/4$ and $\alpha \le 1/2$, respectively,[33] and such regions are wrongly classified as uncorrelated white noises. This result leads to another wrong conclusion that the presence of correlation in the magnitude is an indicator of nonlinearity in the original series. This actually is a spurious result of DFA and MFDFA,[31,32] and our findings confirm such studies; i.e., the original series here is a linear stochastic process, while its magnitude series is also correlated.

Based on the results depicted in Fig. 2, it is interesting to find the functional form of $\sigma_k^{|x|}$ vs σ_k^x as well as $S^{sgn(x)}$ vs S^x. Figure 3 represents such relationships for various α values (color bars). In Fig. 3(a), we observe a symmetric behavior around $\alpha = 1/2$, which shows that the magnitude series of a negatively correlated series is also positively correlated. Recently, it has been shown[31] that for a linearly correlated Gaussian series, such a symmetric relationship is also observed in the behavior of the second order correlation function of the magnitude $C_{|x|}(s)$ vs original series $C(s)$. We argue that the method used in that work is strongly dependent on PDF of the original series x, and thus, before doing such an analysis, one needs

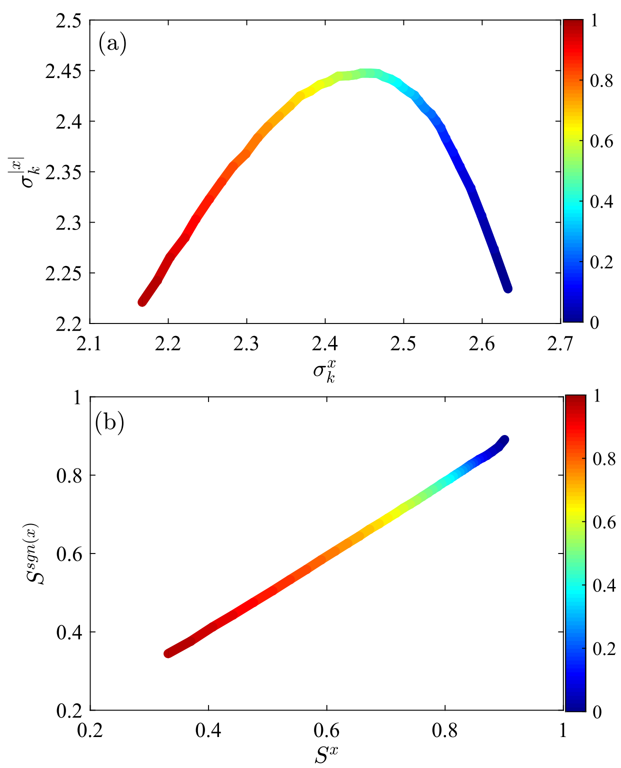

FIG. 3. The functionality of (a) $\sigma_k^{|x|}$ vs σ_k^x and (b) $S^{sgn(x)}$ vs S^x. Color bars show the corresponding values of the DFA exponent α. The quadratic and linear behaviors observed in (a) and (b) are in complete agreement with the recent studies.[31,33]

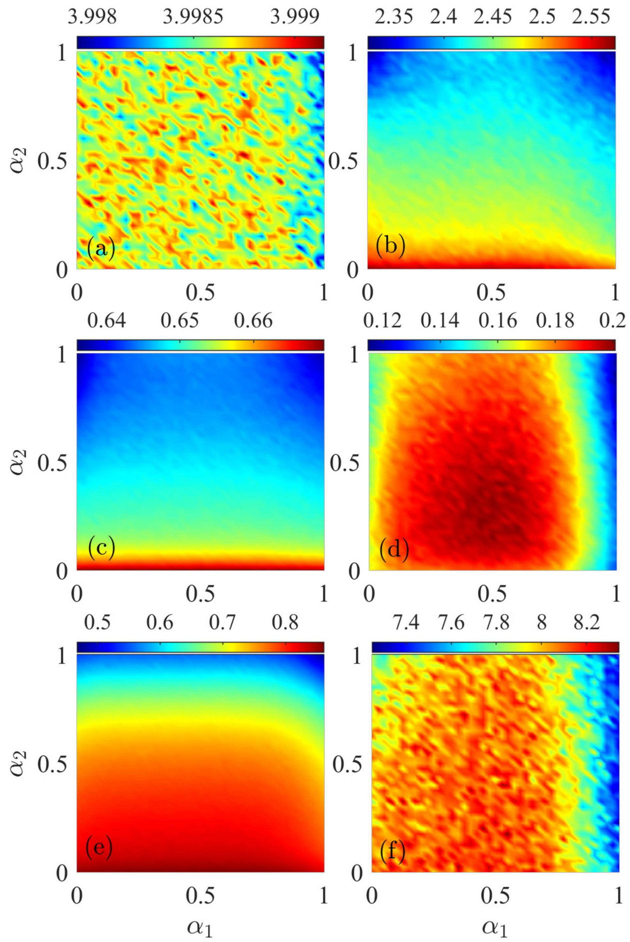

FIG. 4. The linear and nonlinear correlation dependency of (a) the average degree, $\langle k \rangle$; (b) standard deviation, σ_k; (c) clustering coefficient, Cl; (d) assortativity, r; (e) Spearman's coefficient, S; and (f) the maximum eigenvalue, e_{max}, of the adjacency matrix for a multiplicative series x_{mult} with $\alpha_1, \alpha_2 \in [0, 1]$.

to replace rank-wisely the series values with the Gaussian ones; however, in our approach, no such replacement is needed. Figure 3(b) also shows $S^{sgn(x)}$ vs S^{x} and indicates a nearly complete linear relationship, showing a one by one correspondence between linear correlations in the original and sign series. This result is in complete agreement with the recent studies.[33] We also note that the results of Fig. 3 indicate that for regions of $\alpha < 3/4$ and $\alpha < 1/2$, the magnitude and sign series are also correlated, respectively, where DFA or MFDFA is not able to discover any correlation, as mentioned in Sec. I.

To investigate the effects of nonlinearity on visibility graphs, we map a nonlinear multiplicative series, x_{mult}, described in Sec. III into HVGs for various values of $\alpha_1, \alpha_2 \in [0, 1]$. We calculate six topological features for various α_1 and α_2 in Fig. 4, similar to Figs. 1 and 2. Color bars in Figs. 4(a)–4(f) show the values of $\langle k\rangle$, σ_k, Cl, r, S, and $e_{\max}$, respectively. As can be seen, $\langle k\rangle$ and $e_{\max}$ do not depend on linear correlations and have a weak dependency on nonlinear correlations. On the other hand, S is strongly dependent on linearity with a very weak dependency on nonlinear correlations for large α_1. r is strongly dependent on nonlinear correlations but weakly dependent on linearity, and σ_k is dependent on linear as well as nonlinear correlations. Our results show that linear and nonlinear correlations are inherited in the HVG structure. To better understand this issue, we note that in the linear time series, Fourier phases are completely random. Thus, a straightforward method[42] to eliminate any nonlinear correlation is that after Fourier transforming a series x, one can shuffle its corresponding Fourier phases and then transform it back to the time domain to generate a phase randomized series x^{RP}. In this respect, we eliminate nonlinear correlations in the multiplicative series x_{mult} by shuffling its Fourier phases and then calculate topological characteristics of the mapped series x^{RP}_{mult} in Fig. 5. As expected, all features have lost their dependencies on nonlinear correlations (see Fig. 4 for a comparison). This result also confirms that the nonlinearity of x_{mult} is only determined by α_1.

Now, we seek to find a proper parameter that can well measure the strength of nonlinear correlations in a given series. Among these features, we choose σ_k since it can discover positive as well as

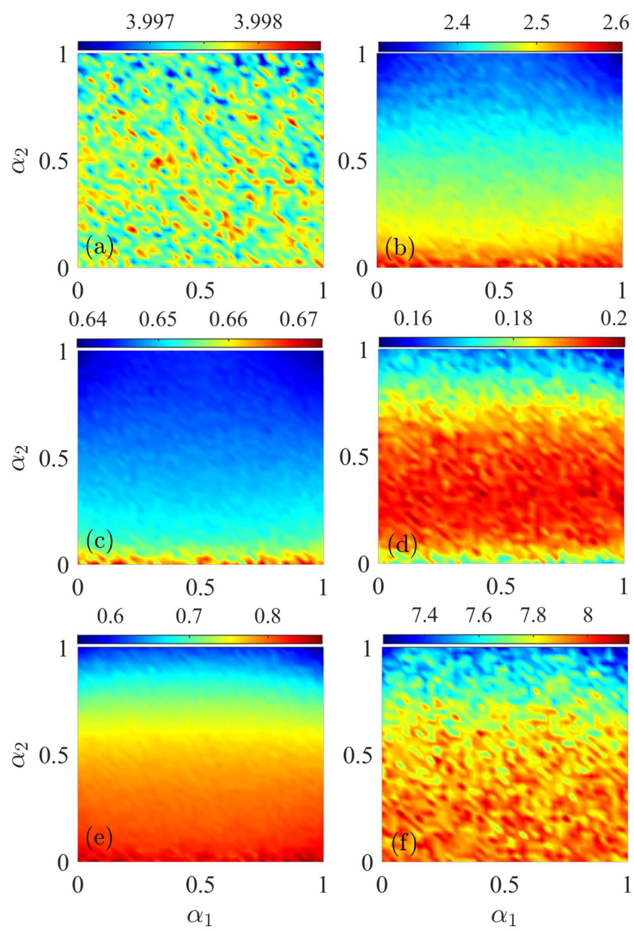

FIG. 5. The correlation dependency of (a) the average degree, $\langle k\rangle$; (b) standard deviation, σ_k; (c) clustering coefficient, Cl; (d) assortativity, r; (e) Spearman's coefficient, S; and (f) the maximum eigenvalue, $e_{\max}$, of the adjacency matrix for a random-phased (RP) multiplicative process, x^{RP}_{mult}, with $\alpha_1, \alpha_2 \in [0, 1]$.

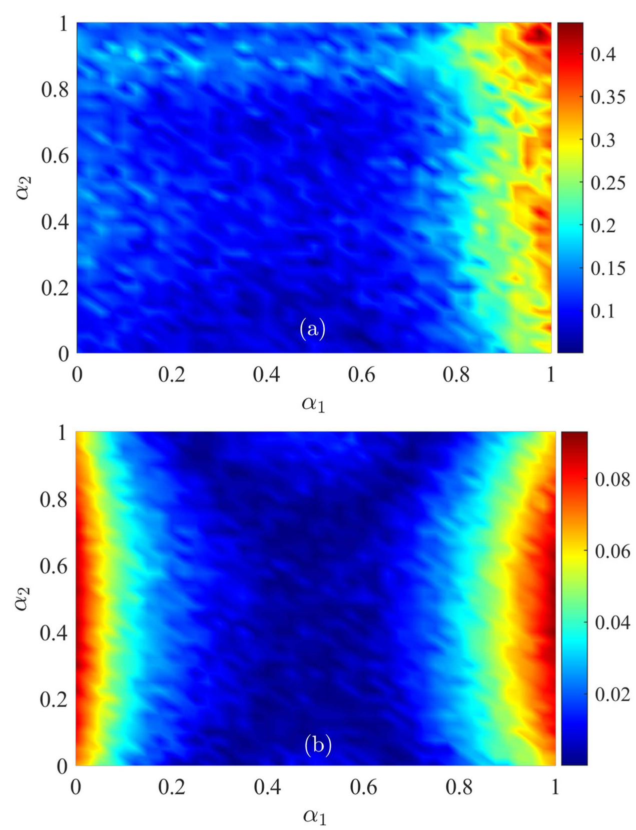

FIG. 6. (a) The multifractal width $\Delta\alpha_q$ calculated by the MFDFA method for multiplicative series x_{mult} with various $\alpha_1, \alpha_2 \in [0, 1]$ and (b) the nonlinearity measure $\Delta\sigma$ introduced in Eq. (5) for the same series similar to (a).

negative correlations in both the original and magnitude series (see Fig. 3). At first, we apply MFDFA to calculate the width $\Delta\alpha_q$ of the multifractal spectrum $f(\alpha_q)$. As we discussed in Sec. I, this quantity is a typical parameter used to measure the strength of nonlinearity in a series. In Fig. 6(a), we plotted $\Delta\alpha_q$ for a multiplicative series with various α_1 and α_2. We observe that the series is nonlinear only for values of $\alpha_1 > 3/4$ and this nonlinearity increases with increasing α_1. However, as we discussed before, this is a spurious result of MFDFA discussed in recent works[31,33] extensively. To solve this, we define a new topological parameter for measuring the strength of nonlinearity as follows:

$$\Delta\sigma = \frac{|\sigma_k^{|x|} - \sigma_k^{|x^{RP}|}|}{\sigma_k^{|x^{RP}|}}, \tag{5}$$

where x^{RP} shows the series x that its nonlinearity has been destroyed with the phase randomizing method. In fact, by subtracting the impact of linear correlations, $\Delta\sigma$ contains nonlinear information only.

We plot $\Delta\sigma$ for multiplicative series $x = x_{mult}$ with various α_1 and α_2 in Fig. 6(b). Interestingly, we observe that this measure can well discover nonlinear features of the multifractal series, especially for regions that MFDFA is unable to detect any correlations (i.e., for all $\alpha_1 \leq 3/4$). On the other hand, a nearly symmetric behavior is observed around $\alpha_1 = 1/2$, which means that below and above this point, the strength of nonlinear correlations in this multiplicative series is the same. Our results here are in line with the results found in recent works.[31,33]

V. CONCLUSION

The measurement of nonlinear correlations in a stochastic time series is generally a difficult task. Many methods have been proposed to extract such information, among which DFA and MFDFA are of practical importance. All such methods are engaged with some challenges such as finding a proper scaling region and eliminating the impacts of PDF for estimating the correlations correctly. On the other hand, DFA and MFDFA may wrongly predict that a given correlated series is uncorrelated, which is due to some technical issues. Therefore, the search for possible new methods that can better analyze the correlated time series is necessary. In this article, by using a recently proposed algorithm called the horizontal visibility graph (HVG) that maps a series into a graph, we investigated linear and nonlinear correlations in the fractal and multifractal stochastic time series. Since HVG does not depend on the series PDF, the resulting graphs contain only correlation information of the original series. We demonstrated that this unique feature can play the role of typical surrogate methods for eliminating the impacts of PDF, in which one usually replaces the series values with the Gaussian ones rank-wisely. We note that such a replacement method only works for the series with a zero linear correlation. However, there is no such restriction on the HVG algorithm, and thus, it indicates the supremacy of our approach when the main aim is to estimate correlations. Furthermore, we found that linear and nonlinear correlations are well inherited in the topological features of the resulting graphs. We noted that this occurs even for the time series, in which DFA and MFDFA cannot detect any correlations. We also represented that the presence of correlations in the series magnitude is not the indication of nonlinearity in the original series. At the end, we also introduced a topological parameter that can well measure the strength of nonlinearity. All such results are obtained without the need to find any scaling region and demonstrate the unique power of the correlation analysis via the HVG algorithm. Consequently, our approach may be considered as a novel and precise method to estimate nonlinear correlations in various complex systems.

ACKNOWLEDGMENTS

The support from the Persian Gulf University Research Council is kindly acknowledged.

REFERENCES

[1] K. G. Wilson, Phys. Rev. B **4**, 3174 (1971).
[2] B. B. Mandelbrot, *The Fractal Geometry of Nature* (Times Books, 1982).
[3] H. Takayasu, *Fractals in the Physical Sciences* (Manchester University Press, 1990).
[4] D. B. Hibbert, Chemom. Intell. Lab. Syst. **11**, 1 (1991).
[5] S. Havlin, S. V. Buldyrev, A. L. Goldberger, R. N. Mantegna, S. M. Ossadnik, C. K. Peng, and H. E. Stanley, Chaos Solitons Fractals **6**, 171 (1995).
[6] Q. Cheng, Math. Geol. **29**, 919 (1997).
[7] J. Z. Liu, L. D. Zhang, and G. H. Yue, Biophys. J. **85**, 4041 (2003).
[8] J. L. Véhel, E. Lutton, and C. Tricot, *Fractals in Engineering: From Theory to Industrial Applications* (Springer Science & Business Media, 2012).
[9] B. B. Mandelbrot and I. Stewart, Nature **391**, 758 (1998).
[10] S. Lovejoy and B. B. Mandelbrot, Tellus A **37**, 209 (1985).
[11] C. M. de Silva, J. Philip, K. Chauhan, C. Meneveau, and I. Marusic, Phys. Rev. Lett. **111**, 044501 (2013).
[12] B. Mandelbrot, *Gaussian Self-Affinity and Fractals* (Springer-Verlag, New York, 2002).
[13] B. B. Mandelbrot, *Multifractals and 1/f Noise: Wild Self-Affinity in Physics* (Springer, 1999).
[14] K. Matia, Y. Ashkenazy, and H. E. Stanley, Europhys. Lett. **61**, 422 (2003).
[15] C. Meneveau and K. R. Sreenivasan, J. Fluid Mech. **224**, 429 (1991).
[16] P. Manshour, S. Saberi, M. Sahimi, J. Peinke, A. F. Pacheco, and M. R. R. Tabar, Phys. Rev. Lett. **102**, 014101 (2009).
[17] P. Manshour, F. Ghasemi, T. Matsumoto, J. Gómez, M. Sahimi, J. Peinke, A. F. Pacheco, and M. R. R. Tabar, Phys. Rev. E **82**, 036105 (2010).
[18] J. Gierałtowski, J. J. Żebrowski, and R. Baranowski, Phys. Rev. E **85**, 021915 (2012).
[19] Z.-Y. Su and T. Wu, Physica D **221**, 188 (2006).
[20] R. Lopes and N. Betrouni, Med. Image Anal. **13**, 634 (2009).
[21] J. W. Kantelhardt, S. A. Zschiegner, E. Koscielny-Bunde, S. Havlin, A. Bunde, and H. Stanley, Physica A **316**, 87 (2002).
[22] G. F. Gu and W. X. Zhou, Phys. Rev. E **74**, 061104 (2006).
[23] C.-K. Peng, S. V. Buldyrev, S. Havlin, M. Simons, H. E. Stanley, and A. L. Goldberger, Phys. Rev. E **49**, 1685 (1994).
[24] Y. Ashkenazy, P. C. Ivanov, S. Havlin, C.-K. Peng, A. L. Goldberger, and H. E. Stanley, Phys. Rev. Lett. **86**, 1900 (2001).
[25] T. Kalisky, Y. Ashkenazy, and S. Havlin, Phys. Rev. E **72**, 011913 (2005).
[26] Y. Ashkenazy, D. R. Baker, H. Gildor, and S. Havlin, Geophys. Res. Lett. **30**, 11, https://doi.org/10.1029/2003gl018099 (2003).
[27] L. Zhu, N. D. Jin, Z. K. Gao, Y. B. Zong, L. S. Zhai, and Z. Y. Wang, J. Phys. Conf. **364**, 012067 (2012).
[28] I. Bartos and I. M. Jánosi, Nonlinear Proc. Geoph. **13**, 571 (2006).
[29] Q. Li, Z. Fu, N. Yuan, and F. Xie, Physica A **410**, 9 (2014).
[30] Y. Liu, P. Gopikrishnan, P. Cizeau, M. Meyer, C.-K. Peng, and H. E. Stanley, Phys. Rev. E **60**, 1390 (1999).
[31] P. A. Bernaola-Galván, M. Gómez-Extremera, A. R. Romance, and P. Carpena, Phys. Rev. E **96**, 032218 (2017).

[32] P. Carpena, M. Gómez-Extremera, C. Carretero-Campos, P. Bernaola-Galván, and A. V. Coronado, Entropy **19**, 261 (2017).
[33] M. Gómez-Extremera, P. Carpena, P. C. Ivanov, and P. A. Bernaola-Galván, Phys. Rev. E **93**, 042201 (2016).
[34] L. Lacasa, B. Luque, F. Ballesteros, J. Luque, and J. C. Nuño, Proc. Natl. Acad. Sci. U.S.A. **105**, 4972 (2008).
[35] B. Luque, L. Lacasa, F. Ballesteros, and J. Luque, Phys. Rev. E **80**, 046103 (2009).
[36] L. Lacasa and R. Toral, Phys. Rev. E **82**, 036120 (2010).
[37] Á. M. Núñez, B. Luque, L. Lacasa, J. P. Gómez, and A. Robledo, Phys. Rev. E **87**, 052801 (2013).
[38] L. Lacasa, Nonlinearity **27**, 2063 (2014).
[39] L. Lacasa and J. Iacovacci, Phys. Rev. E **96**, 012318 (2017).
[40] P. Manshour, M. R. R. Tabar, and J. Peinke, J. Stat. Mech. **2015**, P08031 (2015).
[41] P. Manshour, Chaos **25**, 103105 (2015).
[42] T. Schreiber and A. Schmitz, Physica D **142**, 346 (2000).
[43] P. Manshour, M. Anvari, N. Reinke, M. Sahimi, and M. R. R. Tabar, Sci. Rep. **6**, 27452 (2016).
[44] J. Ludescher, M. I. Bogachev, J. W. Kantelhardt, A. Y. Schumann, and A. Bunde, Physica A **390**, 2480 (2011).
[45] A. Banerjee, A. G. Chandrasekhar, E. Duflo, and M. O. Jackson, Science **341**, 1236498 (2013).
[46] M. Hollander, D. A. Wolfe, and E. Chicken, *Nonparametric Statistical Methods* (John Wiley & Sons, 2012).
[47] S. Boccaletti, V. Latora, Y. Moreno, M. Chavez, and D.-U. Hwang, Phys. Rep. **424**, 175 (2006).
[48] D. J. Watts and S. H. Strogatz, Nature **393**, 440 (1998).
[49] A. N. Kolmogorov, Dokl. Acad. Sci. USSR **26**, 115 (1940).
[50] *Time Series with Long Memory*, edited by P. M. Robinson (Oxford University Press, 2003).
[51] P. Embrechts and M. Maejima, *Self-Similar Processes* (Princeton University Press, Princeton, 2002).
[52] T. Karagiannis, M. Molle, and M. Faloutsos, IEEE Internet Comput. **8**, 57 (2004).
[53] H. A. Makse, S. Havlin, M. Schwartz, and H. E. Stanley, Phys. Rev. E **53**, 5445 (1996).
[54] R. N. Mantegna and H. E. Stanley, Nature **376**, 46 (1995).