Functions

Fractals

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Contents

1 Introduction

1.1 Defintions

Defintion 1.1. A function f from a set X to a set Y is a relation that assigns to each element in set X exactly one element in set Y.

Defintion 1.2. The domain is the set of X (a.k.a. the input).

Defintion 1.3. The range is a subset of Y (a.k.a. the output).

1.2 Existence of a Function

Theorem 1 (Vertical Line Test). if you can draw a Vertical line that passes through more than one point of a relation on a grap, it's not a function, if you cannot, it's a function.

Example 1.1. what the domain and range of the function $f(x) = \sqrt{16 - x^2}$?

sloution Note that if $a < 0$, then $\sqrt{a}$ is undefined for reals, Thus, $16 - x^2 \geq 0 \Rightarrow \boxed{-4 \leq x \leq 4}$ since $x^2 \geq 0$, we have that $0 \leq 16 - x^2 \leq 16$, so the range is $\boxed{0 \leq y \leq 4}$

2 Combinations of Functions

Theorem 2 (common function Combinations). The following are some common combinations of functions:

- **Sum** $(f+g)(x) = f(x) + g(x)$
- **Differnce** $(f-g)(x) = f(x) - g(x)$
- **Product** $(fg)(x) = f(x)g(x)$
- **Quotient** $(\frac{f}{g})(x) = \frac{f(x)}{g(x)}$ where $g(x) \neq 0$
- **Compostion** $(f \circ g)(x) = f(g(x))$

Example 2.1. if $f(x) = 2x + 3$ and $g(x) = 2x - 3$, then what is $fg(4)$?

sloution $(fg)(x) = (2x+3)(2x-3)$ Thus $(fg)(4) = 55$.

2.1 Domain and Range of a Composite Function

The domain of a composite function is the intersection of domains of the starting and final function.
The range of a composite function is the range of the final function restricted by the starting function.

Example 2.2. let $f(x) = \frac{1}{x+2}$ and $\frac{x}{x-3}$. Then $g(x)$ is the starting function and $f(g(x))$ is the final function. Find the domain and range of $f(g(x))$.

sloution $f(g(x)) = \frac{1}{\frac{x}{x-3} + 2}$ so $x \neq 3$, impleing the domain is $\boxed{x \neq 2, 3}$.

Example 2.3. If $f(x) = \sqrt{x}$ and $g(x) = x - 1$, what is the domain and range of $(g \circ f)(x)$?

sloution $(g \circ f)(x) = \sqrt{x} - 1$ it's obivious that $x \geq 0$, and all other values work, so the domain is $\boxed{0 \leq x < \infty}$. Since $\sqrt{x} \geq 0$we have $(g \circ f)(x) \geq -1$, with no other restrictions, so the range is $\boxed{[-1, \infty]}$.

3 Types of Functions

3.1 Piecewise-Defined Function

A piecewise function is a function that is defined by two or more equations over a specified domain.

Example 3.1. let $f(x) = |x|$ Then

$$f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Example 3.2. What are the domain and range of the piecewise function as follows?

$$f(x) = \begin{cases} x^2 + 1 & x < 0 \\ x - 1 & x \geq 0 \end{cases}$$

sloution. The domain includes $x < 0$ and $x \geq 0$, which is all values, so the domain is $\boxed{(-\infty, \infty)}$. for $x \geq 0$, we have $f(x) = x - 1$, so the range is $y \geq -1$. For $x < 0$ we have $f(x) = x^2 + 1$, so the range is $y > 1$. Thus the range together is $\boxed{[-1, \infty)}$

4 Properties of Functions

4.1 Odd and Even Functions

A function f is even if $f(x) = f(-x)$

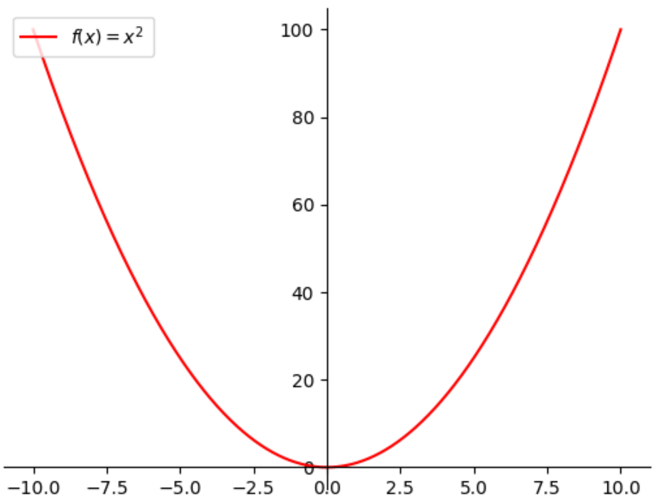

Figure 1: Graph of an even function.

A function f is odd if $f(x) = -f(-x)$

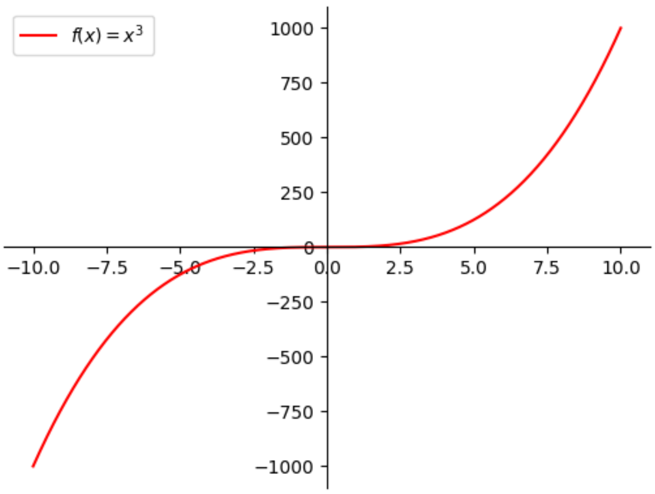

Figure 2: Graph of an odd function.

Note that

- Reflecting an even function across the y-axis yields the same function, and
- Rotating an odd function across the x-axis also yields the same function.

Theorem 3 (Parity of Functions Comes From Its Components)**.** if $h(x) = f(x)+g(x)$then $h(x)$ is even if $f(x)$ and $g(x)$ are both even, and $h(x)$ is odd if $f(x)$ and $g(x)$ are both odd.

Example 4.1. Is $f(x) = x^3 - 2x$ odd, even, neither?

sloution Note thr $f(-x) = -x^3 + 2x = -f(x)$ implying it's odd.

5 Inverse Functions

An inverse function is a function that reverses function f. If f is a function mapping x to y, then the inverse function of f maps y back to x. The inverse function of f is usually denoted by f^{-1}

- If f^{-1} exists, Then $f^{-1}(f(x)) = f(f^{-1}(x)) = x$.
- The graph of the inverse is the graph of the function reflected across $y = x$.2

Example 5.1. what's the inverse function of $f(x) = \dfrac{3x-5}{2}$?

sloution if we switch x and y we get $x = \dfrac{3y-5}{2} \Rightarrow \boxed{f^{-1}(x) = \dfrac{2x+5}{3}}$.

5.1 Existence of an Inverse Function

Defintion 5.1 (One to one)**.** If a function satisfies the property that each x-value corresponds to one y-value, and each y-value corresponds to one x-value, then the function is one-to-one.

Theorem 4 (Inverse Function Criterion)**.** If a function f is one-to-one, then its inverse is a function. More specifically, f is one-to-one if f is increasing/decreasing on its entire domain

Theorem 5 (Horizontal Line Test)**.** If you can draw a horizontal line passing through more than one point of a function on a graph, its inverse is not a function. If you cannot, it is a function.

This makes a lot of sense, since the graph of the inverse is just the function flipped across the $y = x$ line. This also makes it a lot easier to draw graphs.

Example 5.2. Does the function $f(x) = \sqrt{x-2} + 3$ have an inverse function?

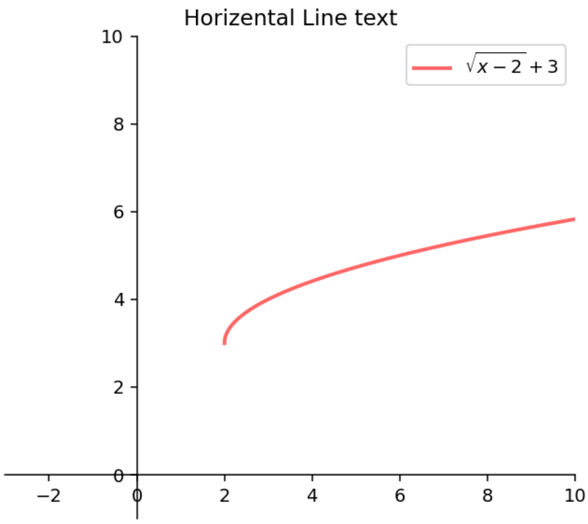

Figure 3: Graph of square root function.

6 problmes

TO BE ADDED