Basic Equation Solving

Fractals

October 6, 2021

Contents

1 Basic Equation Solving

1.1 Algebraic Manipulation

To introduce the topic of **algebraic manipulation**, let us start with a little known example: **egyptian fractions**

Theorem 1 (**Egyptian Fractions**)**.** For all a, b where $ab \neq 1$

$$\frac{a}{ab-1} = \frac{1}{b(ab-1)} + \frac{1}{b}.$$

From here, we can see that putting things together (factoring) is just as important as taking them apart (distributing). Now, let us turn the power of products:

Example 1.1. For positive real numbers a, b,

$$a + \frac{1}{b} = 4,$$

$$b + \frac{1}{a} = 5,$$

Find $ab + \dfrac{1}{ab}$.

Solution. It is very easy to get lost in the problem if we directly try to solve for a and b. Instead, let us multiply the equations:

$$(a + \frac{1}{b})(b + \frac{1}{a}) = 4(5) = 20.$$

$$ab + \frac{a}{a} + \frac{b}{b} + \frac{1}{ab} = ab + \frac{1}{ab} + 2 = 20,$$

$$\boxed{ab + \frac{1}{ab} = 18.}$$

Theorem 2. Let x, y be nonzero real numbers such that $x + y = a$ and $xy = b$ Then,

$$x^2 + y^2 = a^2 - 2b,$$

$$(x+1)(y+1) = a + b + 1,$$

$$x^2 + xy^2 = ab,$$

$$|x - y| = \sqrt{a^2 - 4b},$$

$$x^3 + y^3 = a^3 - 3ab,$$

$$\frac{1}{x} + \frac{1}{y} = \frac{a}{b},$$

1.2 Quadratic Equations

A polynomial is an equation of the following form:

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where

$$a_0, a_1, \ldots, a_n$$

are constants. A quadratic equation is a polynoimal with $n = 2$:

$$ax^2 + bx + c = 0,$$

A common way to solve a quadratic equation is to use the quadratic formula:

Theorem 3 (Quadratic Formula). For the equation $ax^2 + bx + c = 0$, the roots x_1, x_2 must be equal to

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a},$$

$$x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a},$$

Theorem 4. For the equation $ax^2 + bx + c = 0$, we have the following cases:

- If $b^2 - 4ac > 0$, we have **two real roots.**
- If $b^2 - 4ac = 0$, we have **one real root.**
- If $b^2 - 4ac < 0$, we have **no real roots.**

Using the Quadratic Formula, we can calculate the sum of roots and product of roots:

Theorem 5. For the equation $ax^2 + bx + c = 0$, the sum of roots is:

$$x_1 + x_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b + \sqrt{b^2 - 4ac}}{2a} = \frac{-2b}{2a} = -\frac{b}{a}$$

$$x_1 x_2 = \frac{-b^2 + 4ac}{2a} \times \frac{-b^2 + 4ac}{2a} = \frac{-(-b)^2 - (b^2 - 4ac)}{4a^2} = \frac{c}{a}$$

Theorem 6. For any polynoimal

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

The sum of roots is: $-\dfrac{\text{seconed coeffeicent}}{\text{first coeffeicent}} = -\dfrac{a_{n-1}}{a_n}$ and **the product of roots is:** $\dfrac{\text{last coeffeicent}}{\text{first coeffeicent}} = \dfrac{a_0}{a_n}$

For those who are looking for a more advanced and more powerful theorem, we can generalize this formula:

Theorem 7 (Vieta's Formulas). for any polynoimal

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

let $r_1, r_2, \dots r_n$ (an n-degree equation has n different roots). Vieta's formulas state that

$$a_n = a_n$$

$$a_{n-1} = -a_n(r_1 + r_2 + \cdots + r_n)$$

$$a_{n-2} = a_n(r_1 r_2 + r_1 r_3 + \cdots + r_{n-1} r_n)$$

$$\vdots$$

$$a_0 = (-1)^n a_n (r_1 r_2 \cdots r_n)$$

2 Problems

Problem 1. Let x be a real number such that $x + \frac{1}{x} = \sqrt{2020}$. What is $x^2 + \frac{1}{x^2}$?

Problem 2. Two non-zero real numbers, a and b, satisfy $ab = a - b$. Which of the following is a possible value of $\frac{a}{b} + \frac{b}{a} - ab$?

Problem 3. Let x, y be nonnegative real numbers such that $x + y = 5$ and $xy = 7$. Find $\frac{x}{y-1} + \frac{y}{p-1}$

Problem 4. Let a, b are real numbers such that

$$\frac{1}{a(b+1)} + \frac{1}{b(a+1)} = \frac{1}{(a+1)(b+1)}.$$