Problem Set 1 Sloutions

Fractals

October 2, 2021

1. Let A, B and C be digits, and assume $A \neq 0$, if $AA + BB + CC = ABC$, what are A, B and C?

   Answer:
   $$AA + BB + CC = ABC \tag{1}$$
   Note that (1) are digits can be rewritten as
   $$10A + A + 10B + B + 10C + C = 100A + 10B + C$$
   By Collecting like terms
   $$B + 10C = 89A$$
   Since B *and* C are digits. $B + 10C$ is between 0 and 99 which implies $89A$, Hence $A = 1$

   Note in (1) Adding A, B, C give the digit C in the ones place, which means $A + B$ adds to 10
   Thus $B = 9$. This implies $C = 8$

2. Find the value of $\dfrac{1}{3^2+1} + \dfrac{1}{4^2+1} + \dfrac{1}{5^2+1} + \ldots$

   Answer:
   Each term takes the form:
   $$\frac{1}{n^2 + (n-2)} = \frac{1}{(n+2)(n-1)}$$
   Using the method of partial fractions, we can rewrite :
   $$\frac{1}{(n+2)(n-1)} = \frac{A}{n+2} + \frac{B}{(n-1)}$$
   $$\Rightarrow 1 = A.(n-1) + B.(n+2)$$
   Setting $n = 1$ we get $B = \dfrac{1}{3}$ and smiliarly with $n = -2$ we get $A = \dfrac{-1}{3}$.
   $$\frac{-1/3}{(n+2)} + \frac{1/3}{(n-1)} = \frac{1}{3} \times \left(\frac{-1}{(n+2)} + \frac{1}{(n-1)}\right)$$
   Hence the sum becomes
   $$\frac{1}{3} \times [(\frac{1}{2} - \frac{1}{5}) + (\frac{1}{3} - \frac{1}{6}) + (\frac{1}{4} - \frac{1}{7}) + (\frac{1}{5} - \frac{1}{8}) + \ldots]$$
   Thus, it telescopes, and the only terms that do not cancel produce a sum of $\frac{1}{3} \times (\frac{1}{2} + \frac{1}{3} + \frac{1}{4}) = \frac{13}{36}$

3. Let $f(x) = 1 + x + x^2 + x^3 + \cdots + x^{100}$.Find $f'(1)$

   Answer: Note that $f'(x) = 1 + 2x + 3x^2 + \cdots + 100x^{99}$, so $f'(1) = 1 + 2 + \cdots + 100 = \dfrac{100.101}{2} = 5050$

4. Two reals x and y are such that $x - y = 4$ and $x^3 - y^3 = 28$ compute xy

   Answer: We have

$$28 = x^3 - y^3 = (x - y)(x^2 + xy + y^2) = (x - y)((x - y)^2 + 3xy) = 4 \times (16 + 3xy)$$

   from which $xy = -3$

5. For each positive integer n, let $f(n) = \dfrac{n}{n+1} + \dfrac{n+1}{n}$. Then $f(1)+f(2)+f(3)+\cdots+f(10)$ can be expressed as $\dfrac{m}{n}$ where m and n are relatively prime positive integers. Compute $m+n$.

   Answer:

$$f(n) = \frac{n+1-1}{n+1} + \frac{n+1}{n} =$$

$$(\frac{n+1}{n+1} - \frac{1}{n+1}) + (\frac{n}{n} + \frac{1}{n}) = 2 + \frac{1}{n} - \frac{1}{n+1}$$

   So

$$f(1)+f(2)+f(3)+\cdots+f(10) = 2.10+(\frac{1}{1}-\frac{1}{2})+(\frac{1}{2}-\frac{1}{3})+\cdots+(\frac{1}{10}-\frac{1}{11}) = 20+\frac{10}{11} = \frac{230}{11}.$$