

# Problem Set 1 Sloutions

## Fractals

October 2, 2021

1. Let  $A, B$  and  $C$  be digits, and assume  $A \neq 0$ , if  $AA + BB + CC = ABC$ , what are  $A, B$  and  $C$ ?

**Answer:**

$$AA + BB + CC = ABC \quad (1)$$

Note that (1) are digits can be rewritten as

$$10A + A + 10B + B + 10C + C = 100A + 10B + C$$

By Collecting like terms

$$B + 10C = 89A$$

Since  $B$  and  $C$  are digits.  $B + 10C$  is between 0 and 99 which implies  $89A$ , Hence  $A = 1$

Note in (1) Adding  $A, B, C$  give the digit  $C$  in the ones place, which means  $A + B$  adds to 10

Thus  $B = 9$ . This implies  $C = 8$

2. Find the value of  $\frac{1}{3^2+1} + \frac{1}{4^2+1} + \frac{1}{5^2+1} + \dots$

**Answer:**

Each term takes the form:

$$\frac{1}{n^2 + (n-2)} = \frac{1}{(n+2)(n-1)}$$

Using the method of partial fractions, we can rewrite :

$$\begin{aligned} \frac{1}{(n+2)(n-1)} &= \frac{A}{n+2} + \frac{B}{(n-1)} \\ \Rightarrow 1 &= A.(n-1) + B.(n+2) \end{aligned}$$

Setting  $n = 1$  we get  $B = \frac{1}{3}$  and smiliarly with  $n = -2$  we get  $A = \frac{-1}{3}$ .

$$\frac{-1/3}{(n+2)} + \frac{1/3}{(n-1)} = \frac{1}{3} \times \left( \frac{-1}{(n+2)} + \frac{1}{(n-1)} \right)$$

Hence the sum becomes

$$\frac{1}{3} \times \left[ \left( \frac{1}{2} - \frac{1}{5} \right) + \left( \frac{1}{3} - \frac{1}{6} \right) + \left( \frac{1}{4} - \frac{1}{7} \right) + \left( \frac{1}{5} - \frac{1}{8} \right) + \dots \right]$$

Thus, it telescopes, and the only terms that do not cancel produce a sum of  $\frac{1}{3} \times \left( \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \right) = \frac{13}{36}$

3. Let  $f(x) = 1 + x + x^2 + x^3 + \cdots + x^{100}$ . Find  $f'(1)$

**Answer:** Note that  $f'(x) = 1 + 2x + 3x^2 + \cdots + 100x^9$ , so  $f'(1) = 1 + 2 + \cdots + 100 = \frac{100 \cdot 101}{2} = 5050$

4. Two reals  $x$  and  $y$  are such that  $x - y = 4$  and  $x^3 - y^3 = 28$  compute  $xy$

**Answer:** We have

$$28 = x^3 - y^3 = (x - y)(x^2 + xy + y^2) = (x - y)((x - y)^2 + 3xy) = 4 \times (16 + 3xy)$$

from which  $xy = 3$ .

5. For each positive integer  $n$ , let  $f(n) = \frac{n}{n+1} + \frac{n+1}{n}$ . Then  $f(1) + f(2) + f(3) + \cdots + f(10)$  can be expressed as  $\frac{m}{n}$  where  $m$  and  $n$  are relatively prime positive integers. Compute  $m+n$ .

**Answer:**

$$\begin{aligned} f(n) &= \frac{n+1-1}{n+1} + \frac{n+1}{n} = \\ &= \left(\frac{n+1}{n+1} - \frac{1}{n+1}\right) + \left(\frac{n}{n} + \frac{1}{n}\right) = 2 + \frac{1}{n} - \frac{1}{n+1} \end{aligned}$$

So

$$f(1) + f(2) + f(3) + \cdots + f(10) = 2 \cdot 10 + \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{10} - \frac{1}{11}\right) = 20 + \frac{10}{11} = \frac{230}{11}.$$