Sample Contest

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1. Define The operation $a@b$ to be $3+ab+a+2b$. There exists a number x such that $x@b=1$ for all b. Find x.

   Sol.

   $$3+xb+x+2b=1$$

   $$xb+x+2b+2=0 \Rightarrow (x+2)(b+1)=0$$

   We see that if $x=-2$, then this expression is always true. Indeed plugging -2 for x yeilds $3-2b-2+2b=1$ so The answer is -2

2. Let $y=x^2+bx+c$ be a quadratic function. it has only one real root. if b is postive, find $\dfrac{b+2}{\sqrt{c}+1}$.

   sol.
   since it has only one real root, we know that the discriminant b^2-4ac is 0 since $a=1$, we have $b^2=4c$ thus $\dfrac{b+2}{\sqrt{c}+1}=\dfrac{2\sqrt{c}+2}{\sqrt{c}+1}=2$

3. A circle of nonzero radius r has a circumference numerically equal to $\dfrac{1}{3}$ of its area. What is its area?

   $\frac{1}{3}\text{Area}=\frac{1}{3}\pi r^2=2\pi r$

   $$\Rightarrow r=6 \Rightarrow \text{Area}=\pi r^2=\pi 6^2=36\pi$$

4. Let set $\mathcal{A}$ be a 90-element subset of $\{1,2,3,\ldots,100\}$, and let S be the sum of the elements of $\mathcal{A}$. Find the number of possible values of S.

   The smallest S is $1+2+\ldots+90=91\cdot 45=4095$. The largest S is $11+12+\ldots+100=111\cdot 45=4995$. All numbers between 4095 and 4995 are possible values of S, so the number of possible values of S is $4995-4095+1=901$.

5. A *gorgeous* sequence is a sequence of 1's and 0's such that there are no consecutive 1's. For instance, the set of all gorgeous sequences of length 3 is [1, 0, 0], [1, 0, 1], [0, 1, 0], [0, 0, 1], [0, 0, 0]. Determine the number of gorgeous sequences of length 7.

> Let S_n be the number of *gorgeous* sequences of length n. Looking at aribtary sequence of length n, we see that it either strats with 1 or 0. If starts with 0, Then we can simply take all sequences of length $n-1$ and append it to the 0. If it starts with 1, Then the next value must be. After that we can take all sequences of length $n-2$ and append it to the $[0,1]$. So $S_n = S_{n-1} + S_{n-2}$ The first two terms are $S_0 = 1$ and $S_1 = 2$ so S_n is just fibonacci numbers. so recuresivly we have $S_7 = 34$

6. A 8×8 chessboard with the northeast and southwest corner unit squares removed is given. Is it possible to partition such a a board into thirty-one unit dominoes(where a domino is a 1×2 rectangle)? Show your work.

> For such board, we can color the unit squares alternatively. in black and white. Each domino will cover two adjacent squares one with color balck and one with color white. So if 31 dominoes can over the board there should be 31 squares with black color and 31 squares with white color. However, in the board we have 32 squares of color balck and 30 of white, So the task is impossible.

7. The function f satifies

$$f(x) + f(2x+y) + 5xy = f(3x-y) + 2x^2 + 1$$

for all real numbers x, y. Determine the vlaue of $f(10)$

> Setting $x = 10$ and $y = 5$ gives
>
> $$f(10) + f(25) + 250 = f(25) + 200 + 1$$
>
> from which gives $f(10) = -49$

8. Let

$$a = \underbrace{19191919191\ldots1919}_{\text{19 is repeted 3838 times}}$$

What is the reminder of a when divided by 13?

> Not that $13|191919$. Thus $13|\underbrace{1919191\ldots1919100}_{\text{19 is repeated 3837 times}}$ since $3|3837$. Howeveer $a = \underbrace{19191919\ldots191900}_{\text{19 is repeated 3837}} + 19$ so the reminder whwn a is divided by 13 is the same as the reminder when 19 is divided by 13 which is 6.