German University in Cairo

Lectures 18

Math301

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Contents

1 Laplace Transform of Basic Functions

The Laplace Transform of f is the Functions $F(S)$ where F is denoted $\mathcal{L}\{f\}$ is

$$F(s) = \mathcal{L}\{f\} = \int_0^\infty e^{-st} f(t) dt \tag{1}$$

The **Laplace Transform** is $\mathcal{L}$. that accepts fountions $f(t)$ as input and outputs functions $F(s)$ in return, It changes the function friom space x to space S

$$Let \ f(t) = 1 \ when \ t \geq 0 \ Find \ \mathcal{L}\{f\}$$
$$\mathcal{L}\{f\} = \mathcal{L}\{1\} = \int_0^\infty e^{-st} dt$$
$$= -\frac{1}{s} e^{-st} \big|_0^\infty \tag{2}$$
$$= \frac{1}{s} \ (s > 0)$$

Remark (Laplace). Laplace Transforms have properties that makes it powerful to solve initial value problems

Remark (Laplace). if $F(s) = \mathcal{L}\{f\}$ then $f(t)$ is called the inverse Laplace Transform of $F(s)$ and the notation is $f = \mathcal{L}^{-1}(F)$ Thus,

$$\mathcal{L}^{-1}(\mathcal{L}(f)) = f \ and \ \mathcal{L}(\mathcal{L}^{-1}(F)) = F \tag{3}$$

1.1 Linearity

For any constants a and b
$$\mathcal{L}[af(x) + bg(t)] = a\mathcal{L}(f) + b\mathcal{L}(g) \tag{4}$$

A similar property holds for $\mathcal{L}^{-1}$. Namely,

$$\mathcal{L}^{-1}[aF(s) + bG(s)] = a\mathcal{L}^{-1}(F) + b\mathcal{L}^{-1}(G) \tag{5}$$

1.2 s- Shifiting Property

For any constant a, if $F(s) = \mathcal{L}(f)$ then

$$\mathcal{L}(e^{at} f(t)) = F(s - a)$$

(6)

or equivalently

$$e^{at} f(t) = \mathcal{L}^{-1}(F(s-a) \tag{7}$$

$$\textit{Find } \mathcal{L}(e^{at}coswt) \quad and \quad \mathcal{L}(e^{at}sinwt)$$

$$\mathcal{L}(e^{at}coswt) = \mathcal{L}(coswt)(s-a) = \frac{s-a}{(s-a)^2 + \omega^2}$$

$$\mathcal{L}(e^{at}\sin\omega t) = \mathcal{L}(\sin\omega t)\,(s-a) = \frac{\omega}{(s-a)^2 + \omega^2} \tag{8}$$

1.3 Laplace Table

	$f(t)$	$\mathcal{L}(f)$		$f(t)$	$\mathcal{L}(f)$
1	1	$1/s$	7	$\cos \omega t$	$\dfrac{s}{s^2 + \omega^2}$
2	t	$1/s^2$	8	$\sin \omega t$	$\dfrac{\omega}{s^2 + \omega^2}$
3	t^2	$2!/s^3$	9	$\cosh at$	$\dfrac{s}{s^2 - a^2}$
4	t^n $(n = 0, 1, \cdots)$	$\dfrac{n!}{s^{n+1}}$	10	$\sinh at$	$\dfrac{a}{s^2 - a^2}$
5	t^a (a positive)	$\dfrac{\Gamma(a+1)}{s^{a+1}}$	11	$e^{at} \cos \omega t$	$\dfrac{s-a}{(s-a)^2 + \omega^2}$
6	e^{at}	$\dfrac{1}{s-a}$	12	$e^{at} \sin \omega t$	$\dfrac{\omega}{(s-a)^2 + \omega^2}$

2 Transforms of Dervatives and Integrals

2.1 Laplace Transform of Dervatives

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$$
$$\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0) \tag{9}$$

More Generally, for all n $\geq$ 1 :

$$\mathcal{L}(f^{(n)}) = s^n\mathcal{L}(f) - s^{(n-1)}f(0) - s^{(n-2)}f'(0) - \ldots - f^{(n-1)}(0)$$