Lectures 15,16,17

Math301

Fall 2020

Contents

1 Homogeneous Linear ODEs of Second order

1.1 What is an ODE ordinary differential equation ?

it's a mathematical model that relates some independent variable X with a dependent variable Y and it's derivatives.

Note 1. the independent variable x can either be in the equation or not.

1.2 What is the order of a linear ODE ?

it's the **highest order derivative** in the equation.

1.3 Linear ODEs of Second Order

as we said before they are Linear ODEs with the highest order derivative is 2 and takes the following form:

$$F(x, y, y', y'') = 0 \tag{1}$$

1.4 Linear ?

A second order ODE can be called linear if the function F in equation (1) is **Linear** in y, y' and y''. So it can be put in the **Standard form**.

$$y'' + p(x)y' + q(x)y = r(x) \tag{2}$$

the following ODE is **NOT Linear**.

$$y''y + (y')^2 = 0 \tag{3}$$

1.5 Homogenous Linear ODEs of Second Order

same as we said before but $r(x) = 0$ in the equation (2).

$$y'' + p(x)y' + q(x)y = 0 \tag{4}$$

1.6 What is the solution of an ODE of Second order ?

the solution is a function $y = h(x)$ defined on $[a, b]$ (I) with some constraints:

- $h(x)$ is defined and twice differentiable on the interval I.
- Substituting with $h(x)$ and it's derivatives satisfies the equation.

$$F(x, h(x), h'(x), h''(x)) = 0 \tag{5}$$

1.7 IVPs initial value problems

they are ODEs which have initial values.

1.8 How to solve IVPs?

nothing new! you just solve the ODE and afterwards you substitute with the values given as initial values so that you remove any constant c in the solution $h(x)$ with it's original value.

$$\begin{aligned} y'' + p(x)y' + q(x)y &= r(x) \\ y'(x_0) &= k_1 \\ y(x_0) &= k_2 \end{aligned} \tag{6}$$

The solution of an **IVP** is called a **particular solution**.

1.9 Existence and Uniqueness Theorem for the Initial Value Problems

If the coefficients $p(x), q(x)$ and $r(x)$ in ODE (2) are continuous on an interval I and if $x_0 \in I$, the initial value problem (IVP) has a unique solution.

1.10 Superposition Principle

Rule 1. if $y_1(x), y_2(x)$ are solutions of the linear homogenous ODE (4) then: any **linear comination** of $y_1(x)$ and $y_2(x)$ is also a solution of the ODE.

so the general soltion of an ODE which has two solutions is:

$$y_G = c_1 y_1(x) + c_2 y_2(x) \tag{7}$$

1.11 Linearly independent solutions

two solutions are said to be linearly independent if y_1 and y_2 are not proportional on I, which means

$$y_1(x) \neq k y_2(x) \text{ where k is a constant} \tag{8}$$

Rule 2. if neither y_1 nor y_2 is equal to 0 then they're linearly independent if : $\frac{y_1(x)}{y_2(x)} \neq const.$

Exercise 1. Show that $y_1 = e^x$ and $y_2 = xe^x$ are two linearly independent solutions over $\Re$ of the ODE:

$$y'' - 2y' + y = 0. \tag{9}$$

Solution 1.

$$\begin{aligned} y_1''(x) - 2y_1'(x) + y_1(x) &= e^x - 2e^x + e^x = 0 \\ y_2''(x) - 2y_2'(x) + y_2(x) = xe^x + e^x + e^x &- 2(xe^x + e^x) + xe^x = 0 \end{aligned} \tag{10}$$

and since $\frac{y_1}{y_2} = \frac{1}{x} \neq const.$ Hence the two solytions are linearly independent solutions over $\Re$.

Rule 3. if the coeffecients p(x) and q(x) of the homogeneous equation

$$y'' + p(x)y' + q(x)y = 0 \tag{11}$$

are **continous** over the an interval I then every slution has the following form

$$y = c_1 y_1(x) + c_2 y_2(x) \tag{12}$$

where $y_1(x)$ and $y_2(x)$ are any linearly independent solutions the ODE.

Note 2. if two linearly independent solutions of (H) have been found, then any other particular solution can be found from the general solution. Thus, solving the ODE boils down to finding two linearly independent solutions.

1.12 Show that y_1, y_2 for a basis of the ODE

1. you differentiate and substitute in the ODE to prove it
2. show that the solutions are linearly independent

1.13 Reduction of Order Method of solving ODE

this method means simply finding the general solution of a linear second order ODE using merely one solution.

1. try to find the first solution of the ODE if not given (like assuming the solution is a power function and our goal is to find the power)
2. using the superposition principle we assume that the other solution is $y_2 = u(x)y_1$ so our goal is to find u(x)
3. differentiate y_2 twice and substitute.
4. the reduction of order comes when you substitute and simplify you will remove $u''(x)$ and put $w'(x)$ instead.
5. after finding w a simple integration would get us u

Exercise 2.

$$(x^2 - x)y'' - xy' + y = 0 \tag{13}$$

- Show that the ODE (13) has a solution y_1 that is a power function.
- find a solution y_2 using reduction of order method.
- deduce the general solution

Solution 2. First let $y_1 = x^p$

$$\begin{aligned} y_1' &= px^{p-1} \\ y_1'' &= p(p-1)x^{p-2} \\ (x^2 - x)(p(p-1)x^{p-2}) - x * px^{p-1} + x^p &= 0 \\ (p-1)[(p-1)x^p - px^{p-1}] &= 0 \\ p &= 1 \\ y_1 &= x \\ y_2 &= xu(x) \end{aligned} \tag{14}$$

$y_2' = u + xu'$ and $y''2 = 2u' + xu''$. y2 is a solution of (13) if it satisfies the equation. Substitution

$$x^2(x-1)u'' + x(x-2)u' = 0$$

and simplification imply that $\overset{(\text{let } v=u')}{\overbrace{\iff}}$ $x^2(x-1)v' + x(x-2)v = 0$

$$
\begin{aligned}
& x^2(x-1)v' + x(x-2)v = 0 \\
\Longrightarrow & \frac{v'}{v} = -\frac{x-2}{x(x-1)} \\
\Longrightarrow & \ln|v| = -\int \frac{x-2}{x(x-1)}\,dx \quad \text{(Decompose in partial fractions)} \\
\Longrightarrow & \ln|v| = \int \left(\frac{1}{x-1} - \frac{2}{x}\right) dx \\
\Longrightarrow & \ln|v| = \ln|x-1| - 2\ln|x| \quad \text{(We may drop the integration constant since any } y_2 \text{ would do)} \\
\Longrightarrow & \ln|v| = \ln\left|\frac{x-1}{x^2}\right|
\end{aligned}
$$

Thus, $v = \frac{1}{x} - \frac{1}{x^2} \Longrightarrow u' = \frac{1}{x} - \frac{1}{x^2} \Longrightarrow u = \ln|x| + \frac{1}{x}$. Therefore, $y_2 = xu = x\ln|x| + 1$.

(iii) Since $y_2/y_1 = u(x) \neq$ const, we conclude that y_1 and y_2 are linearly independent. Thus, the general solution of (E) is

$$
\begin{aligned}
y &= C_1 y_1 + C_2 y_2 \\
&= C_1 x + C_2(x\ln|x| + 1)
\end{aligned}
$$

2 Homogeneous Linear ODEs with Constant Coefficients

constant coeffecients means that $p(x)$ and $q(x)$ are constant in (4) :)
and to solve this kind of ODE we use the following table :

Roots of the **characteristic (or auxiliary)** equation	General Solution
λ_1, λ_2 are **Real** and **Distinct**	$y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$
λ_1, λ_2 are **Equal**. $\lambda_1 = \lambda_2 = \lambda$	$y = (C_1 + C_2 x)e^{\lambda x}$
λ_1, λ_2 are **Complex**: $\alpha \pm \beta i$	$y = e^{\alpha x}(C_1 cos(\beta x) + C_2 sin(\beta x))$

The previous table can be easily deduced if you suppose that the solution of (H) is $y = e^{\lambda x}$ so we notice that $y' = \lambda e^{\lambda} x$ and $y'' = \lambda^2 e^{\lambda x}$ so we see that the auxiliary equation can be extracted from the ODE as follows :

$$
y'' + ay' + by = 0 \qquad\qquad \lambda^2 + a\lambda + b = 0
$$

We can solve this system using the old known **polynomial** solution or using any other method like **Cramer's**.

$$
\omega = \sqrt{a^2 - 4 * b} \qquad\qquad \lambda = \frac{-a \pm \omega}{2}
$$

3 Nonhomogeneous Linear ODEs with constant coefficients

in order to solve this kind of linear ODEs we use the help of the following rule:

Rule 4. the general solution of a nonhomogeneous ODE is the sum of the general solution of its complementary equation and any particular solution: $y_H = y_G - y_p$ **Hence** $y_G = y_H + y_p$.

3.1 Where did this information come from ?!

Let's look at the following nonhomogeneous equation:

$$y'' + p(x)y' + q(x)y = r(x) \qquad \text{(NH)}$$

If y_G is the general solution of (NH) and y_p is just any particular solution, then their difference $y_H = y_G - y_p$ is the general solution of the homogeneous equation

$$y'' + p(x)y' + q(x)y = 0 \qquad \text{(H)}$$

where a and b are the same as in (NH).The ODE (H) is called the complementary equation of (NH). In fact,

$$\begin{aligned} y_H'' + ay_H' + by_H &= (y_G - y_p)'' + a(y_G - y_p)' + b(y_G - y_p) \\ &= (y_G'' + ay_G' + by_G) - (y_p'' + ay_p' + by_p) \\ &= r(x) - r(x) = 0. \end{aligned}$$

3.2 Solving NH Linear ODEs

Since we have a recipe for the general solution of the complementary equation (H), solving (NH) boils down to finding any particular solution yp.
We can do this using one of the following methods :

- Method of undetermined coeffecients
- Method of variation of parameters

3.3 Method of undetermined coeffecients

from its name: we have some coeffiects that are undetermined/unknown yet and we want to find them. Look at the following table

Term in $r(x)$	Choice for $y_p(x)$
$ke^{\gamma x}$	$Ce^{\gamma x}$
$kx^n \ (n = 0, 1, \cdots)$	$K_n x^n + K_{n-1}x^{n-1} + \cdots + K_1 x + K_0$
$k \cos \omega x$ $k \sin \omega x$	$K \cos \omega x + M \sin \omega x$
$ke^{\alpha x} \cos \omega x$ $ke^{\alpha x} \sin \omega x$	$e^{\alpha x}(K \cos \omega x + M \sin \omega x)$

So the method is simply as follows:

1. find y_H using the complementry equation.
2. use the suggested $y_p(x)$ from the table
3. check linear independence
   (a) if the suggested y_p is linearly independant with each term of y_H you do nothing
   (b) if not multiply y_p by x and check again
      i. if LI do nothing
      ii. else multiply by x
         A. ...
4. substitute in (NH) with y_p to find the coeffecients.
5. deduce y_G

Rule 5. Modification rule: is simply the 3rd step in the previous list, or you can follow this text:
If r(x) is one of the terms in the adjacent table but is actually a solution of the complementary equation $y'' + ay' + by = 0$, then

1. multiply the suggested yp by x in case $r(x)$ corresponds to a simple root of the complementary equation (H).
2. multiply the suggested y_p by x^2 in case $r(x)$ corresponds to a double root of the complementary equation (H).

Rule 6. Sum rule If $r(x)$ coressponds to the sum of two of the listed functions in the previous table then you can either

1. attempt y_p as the sum of the two suggested solutions .
2. attempt each y_p alone and then $y_p = y_{p1} + y_{p2}$

Rule 7. Product rule

1. If $r(x) = P(x)e^{mx}$ then attempt $y_p = Q(x)e^{mx}$ where P and Q are degree polynomials
2. If $r(x) = P(x)e^{mx}cos(kx)$ or $r(x) = P(x)e^{mx}sin(kx)$ attempt $y_p = Q(x)e^{mx}cos(kx) + R(x)e^{mx}sin(kx)$

3.4 Lecture Examples

Example
Write a trial solution for the method of undetermined coefficients.

1. $y'' - 3y' + 2y = e^x$
2. $y'' - 2y' + y = e^x$
3. $y'' - 3y' + 2y = e^{5x} + \sin x$
4. $y'' - y' - 2y = xe^x \cos x$
5. $y'' - 3y' + 2y = e^x + \sin x$

Solution

1. The complementary equation of the given ODE is $y'' - 3y' + 2y = 0$ whose characteristic equation is $\lambda^2 - 3\lambda + 2 = 0$ the roots of which are $\lambda_1 = 1$ and $\lambda_2 = 2$. Thus, $r(x)$ corresponds to the simple root $\lambda_1 = 1$ of the characteristic equation, $r(x) = e^{\lambda_1 x}$. Therefore, by the modification rule $y_p = Cxe^x$.

2. The complementary equation of the given ODE is $y'' - 2y' + y = 0$ whose characteristic equation is $\lambda^2 - 2\lambda + 1 = 0$ the only root of which is $\lambda = 1$. Thus, $r(x)$ corresponds to a double root of the characteristic equation, $r(x) = e^{\lambda x}$. Therefore, by the modification rule $y_p = Cx^2e^x$.

3. We apply the sum rule for finding a particular solution. $r_1 = e^{5x}$ contibutes $y_{p1} = Ce^{5x}$ to the particular solution. $r_2 = \sin x$ contibutes $y_{p2} = A \sin x + B \cos x$ to the particular solution. Therefore, $y_p = Ce^{5x} + A \sin x + B \cos x$.

4. By the product rule,
$$y_p = (Ax + B)e^x \cos x + (Cx + D)e^x \sin x.$$

5. We apply the sum rule for finding a particular solution. Notice that $r_1 = e^x$ is a slout ion of the comlementary ODE $y'' - 3y' + 2y = 0$ and it corresponds to the simple root $\lambda = 1$. Thus, it contibutes $y_{p1} = Cxe^x$ to the particular solution. $r_2 = \sin x$ contibutes $y_{p2} = A \sin x + B \cos x$ to the particular solution. Therefore, $y_p = Cxe^x + A \sin x + B \cos x$.

Example
Solve $y'' - 4y = xe^x + \cos 2x \qquad$ (E)

Solution
The complementary equation is $y'' - 4y = 0$ whose characteristic solution is $\lambda^2 - 4 = 0$ the roots of which are $\lambda = \pm 2$. Therefore, the solution of the complementary equation is $y_H = C_1e^{-2x} + C_2e^{2x}$.
We seek a particular solution of (E) of the form

$$y_p = y_{p1} + y_{p2}$$

where, $y_{p1} = (Ax + B)e^x$ and $y_{p2} = C \cos 2x + D \sin 2x$.Substituting y_{p1} into $y'' - 4y = xe^x$ and collecting like terms, one obtains

$$y_{p1}'' - 4y_{p1} = xe^x \Leftrightarrow (-3Ax + 2A - 3B)e^x = xe^x$$

Thus,$-3A = 1$ and $2A - 3B = 0$ which implies $y_{p1} = (-\frac{x}{3} - \frac{2}{9})e^x$.Substituting y_{p2} into $y'' - 4y = \cos 2x$ and collecting like terms, one obtains

$$y_{p2}'' - 4y_{p2} = \cos 2x \Leftrightarrow -8C \cos 2x - 8D \sin 2x = \cos 2x$$

Thus, $-8C = 1$ and $-8D = 0$ which implies that $y_{p2} = -\frac{1}{8} \cos 2x$.Therefore, a particular solution of (E) is $y_p = (-\frac{x}{3} - \frac{2}{9})e^x - \frac{1}{8} \cos 2x$ and the general solution of (E) is

$y = y_H + y_p = C_1e^{-2x} + C_2e^{2x} - (\frac{x}{3} + \frac{2}{9})e^x - \frac{1}{8} \cos 2x.$

3.5 Method of variation of parameters

this is the general method used to solve NH ODEs.
in very simple terms

1. find y_1, y_2 from y_H
2. find u_1, u_2 as follows :

$$u_1 = -\int \frac{r(x)y_2}{W}dx \qquad u_2 = \int \frac{r(x)y_1}{W}dx \qquad W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \qquad \text{W is called the Wronskian}$$

3.6 Where did this information come from ?

after finding $y_1$andy_2 we suppose that $y_p = u_1(x)y_1 + u_2(x)y_2$

Notice that $y_p' = (u_1'y_1 + u_2'y_2) + (u_1y_1' + u_2y_2')$

Require u_1 and u_2 to satify: $u_1'y_1 + u_2'y_2 = 0$. This requirement will simplify y_p' and the corresponding expression of y_p'' to the following:

$$y_p' = u_1y_1' + u_2y_2' \quad \text{and} \quad y_p'' = u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2''$$

Substiting y_p, y_p', and y_p'' into (NH) and using the assumption

$$y_1'' + ay_1' + by_1 = 0 \quad \text{and} \quad y_2'' + ay_2' + by_2 = 0,$$

the following equation is obtained: $u_1'y_1' + u_2'y_2' = r(x)$.

Thus, u_1 and u_2 must be obtained as to satisfy the following system of equations $\begin{cases} u_1'y_1 + u_2'y_2 &= 0 \\ u_1'y_1' + u_2'y_2' &= r(x) \end{cases}$ (S)

Let $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1'$ (called the Wronskian of y_1 and y_2).The system (S) may be solved for u_1' and u_2' using Cramer's rule to give

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ r(x) & y_2' \end{vmatrix}}{W} = -\frac{r(x)y_2}{W} \Longrightarrow u_1 = -\int \frac{r(x)y_2}{W}dx$$

$$u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & r(x) \end{vmatrix}}{W} = \frac{r(x)y_1}{W} \Longrightarrow u_2 = \int \frac{r(x)y_1}{W}dx.$$

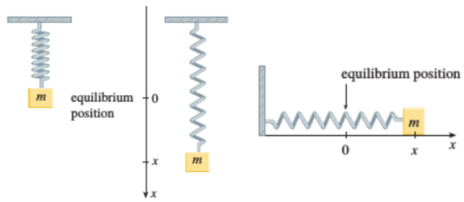
Therefore, the method of variation of parameters gives the particular solution

$$y_p = u_1y_1 + u_2y_2 = -y_1\int \frac{r(x)y_2}{W}dx + y_2\int \frac{r(x)y_1}{W}dx.$$

3.7 Vibrating Springs

Statement of the Problem

We disturb the equilibrium position of a mass-spring system, initially in equilibrium.



Knowing that the spring will vibrate along only one axis (shown as the $x-$axis), we wish to determine the behaviour of the system at any later time.

Mathematical Model

Let $x(t)$ denote the displcament at instant t of the mass m from its equilibrium position. At any instant t, the forces acting on the m are

- the restoring force of the spring, given by **Hooke's law** :
$$F_r = -kx$$
- the damping force (friction with a surface or damping caused by oscillation in a fluid):
$$F_d = -c\frac{dx}{dt} \qquad \text{(verified experimentally)}$$

Remark

The weight is ignored since it has been balanced out by an initial restoring force F_0 before the system's equilibrium was disturbed. $x(t)$ is the displacement from that equilibrium position.

To set up the model, we apply Newton's second law:

$$m\frac{d^2x}{dt^2} = -kx - c\frac{dx}{dt}$$
$$\iff m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$$

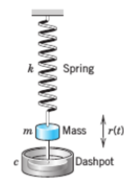
Therefore, the behaviour of the system is determined by solving a homogeneous linear ODE of order 2 and constant coefficients.

Remark

If there were an additional external **driving force** $r(t)$ acting on m, the model would become

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = r(t)$$

which is a nonhomogeneous linear ODE of order 2.



When no external driving force is present, the model for the system behaviour is the ODE

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0.$$

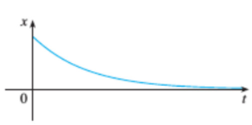
Lets analyse the behaviour of the system.

The characteristic equation is

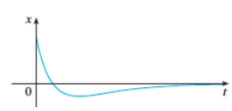
$$m\lambda^2 + c\lambda + k = 0$$

whose roots are $\lambda_1 = \frac{-c+\sqrt{c^2-4mk}}{2m}$ and $\lambda_2 = \frac{-c-\sqrt{c^2-4mk}}{2m}$.

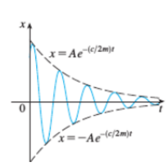
- If $c^2 - 4mk > 0$,
$$x(t) = C_1e^{\lambda_1 t} + C_2e^{\lambda_2 t},$$
where C_1 and C_2 are determined by the initial perturbation of the system (that is, $x(0)$ and $x'(0)$).Notice that both λ_1 and λ_2 are negative and, thus, $\lim_{t\to\infty} x(t) = 0$ (0 denotes the equilibrium position).Therefore, on the long run, the system returns to its equilibrium position before being perturbed. This is the **overdamping** case.The following graph depicts a typical solution $x(t)$ (notice that no oscillation occurs)



- If $c^2 - 4mk = 0$, $\lambda_1 = \lambda_2 = -c/(2m)$ and
$$x(t) = (C_1 + C_2t)e^{-ct/(2m)}.$$
Again $\lim_{t\to\infty} x(t) = 0$ and on the long run the system returns to its equilibrium position before being perturbed.This is the **critical damping** case. A typical solution $x(t)$ is shown (notice that no osciallation occurs)



- If $c^2 - 4mk < 0$,
$$\lambda_1 = -\frac{c}{2m} + \omega i \quad \text{and} \quad \lambda_2 = -\frac{c}{2m} - \omega i,$$
with $\omega = \frac{\sqrt{4mk-c^2}}{2m}$. The solution is
$$x(t) = e^{-(c/2m)t}(C_1\cos\omega t + c_2\sin\omega t),$$
This is the **underdamping** case.The following graph depicts typical solution $x(t)$ where the system oscillates back to equilibrium position



A commonly occuring type of external force is a periodic force function

$$r(t) = r_0 \cos \omega_0 t.$$

In the absence of damping, the model for system behaviour is

$$m \frac{d^2 x}{dt^2} + kx = r_0 \cos \omega_0 t.$$

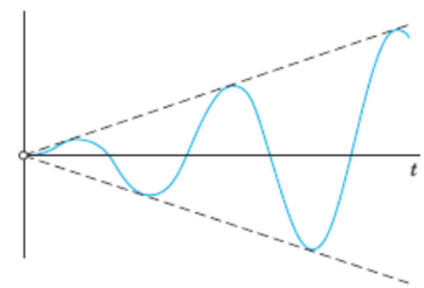
The ODE can be solved (using the method of undetermined coefficinets) to give

$$x(t) = C_1 \cos \omega t + C_2 \sin \omega t + \frac{r_0}{m(\omega^2 - \omega_0^2)} \cos \omega_0 t$$

where $\omega = \sqrt{k/m}$.

Remark
If $\omega_0 \to \omega$, then the applied frequency (of driving forced) reinforces the natural frequency (of the oscillating spring) and, on the long run, the result is vibrations of large amplitude. This is the phenomenon of **resonance**.

The graph depicts a typical $x(t)$, in the abscence of damping and for values of ω_0 very close to ω



4 Worksheet Notes

4.1 Worksheet 8