German University in Cairo

Lectures 18

Math301

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Contents

1 Laplace Transform of Basic Functions

The Laplace Transform of f is the Functions $F(S)$ where F is denoted $\mathcal{L}\{f\}$ is

$$F(s) = \mathcal{L}\{f\} = \int_0^\infty e^{-st} f(t) dt \tag{1}$$

The **Laplace Transform** is $\mathcal{L}$. that accepts fountions $f(t)$ as input and outputs functions $F(s)$ in return, It changes the function friom space x to space S

$$\begin{aligned} Let\ f(t) = 1\ when\ t \geq 0\ Find\ \mathcal{L}\{f\} \\ \mathcal{L}\{f\} = \mathcal{L}\{1\} = \int_0^\infty e^{-st} dt \\ = -\frac{1}{s} e^{-st}|_0^\infty \\ = \frac{1}{s} \quad (s > 0) \end{aligned} \tag{2}$$

Remark (Laplace)**.** Laplace Transforms have properties that makes it powerful to solve initial value problems

Remark (Laplace)**.** if $F(s) = \mathcal{L}\{f\}$ then $f(t)$ is called the inverse Laplace Transform of $F(s)$ and the notation is $f = \mathcal{L}^{-1}(F)$ Thus,

$$\mathcal{L}^{-1}(\mathcal{L}(f)) = f\ and\ \ \mathcal{L}(\mathcal{L}^{-1}(F)) = F \tag{3}$$

1.1 Linearity

For any constants a and b

$$\mathcal{L}[af(x) + bg(t)] = a\mathcal{L}(f) + b\mathcal{L}(g) \tag{4}$$

A similar property holds for $\mathcal{L}^{-1}$. Namely,

$$\mathcal{L}^{-1}[aF(s) + bG(s)] = a\mathcal{L}^{-1}(F) + b\mathcal{L}^{-1}(G) \tag{5}$$

1.2 s- Shifiting Property

For any constant a, if $F(s) = \mathcal{L}(f)$ then

$$\mathcal{L}(e^{at} f(t)) = F(s - a)$$

(6)

or equivalently

$$e^{at} f(t) = \mathcal{L}^{-1}(F(s-a) \tag{7}$$

$$\begin{gathered}
Find\ \mathcal{L}(e^{at} coswt) \quad and \quad \mathcal{L}(e^{at} sinwt) \\
\mathcal{L}(e^{at} coswt) = \mathcal{L}(coswt)(s-a) = \frac{s-a}{(s-a)^2+\omega^2} \\
\mathcal{L}(e^{at} \sin \omega t) = \mathcal{L}(\sin \omega t)\,(s-a) = \frac{\omega}{(s-a)^2+\omega^2}
\end{gathered} \tag{8}$$

1.3 Laplace Table

	$f(t)$	$\mathcal{L}(f)$
1	1	$1/s$
2	t	$1/s^2$
3	t^2	$2!/s^3$
4	t^n $(n = 0, 1, \cdots)$	$\frac{n!}{s^{n+1}}$
5	t^a (a positive)	$\frac{\Gamma(a+1)}{s^{a+1}}$
6	e^{at}	$\frac{1}{s-a}$

	$f(t)$	$\mathcal{L}(f)$
7	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
8	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
9	$\cosh at$	$\frac{s}{s^2-a^2}$
10	$\sinh at$	$\frac{a}{s^2-a^2}$
11	$e^{at} \cos \omega t$	$\frac{s-a}{(s-a)^2+\omega^2}$
12	$e^{at} \sin \omega t$	$\frac{\omega}{(s-a)^2+\omega^2}$

2 Transforms of Dervatives and Integrals

2.1 Laplace Transform of Dervatives

$$\begin{aligned}
\mathcal{L}(f') &= s\mathcal{L}(f) - f(0) \\
\mathcal{L}(f'') &= s^2\mathcal{L}(f) - sf(0) - f'(0)
\end{aligned} \tag{9}$$

More Generally, for all n ≥ 1 :

$$\mathcal{L}(f^{(n)}) = s^n \mathcal{L}(f) - s^{(n-)} f(0) - s^{(n-2)} f'(0) - \ldots - f^{(n-1)}(0)$$