Discrete Random variable

Ibrahim Abou Elenein

May 17, 2021

Contents

1 Random Variable

A random variable is a quantity "X " resulting from an experiment, by chance, that can assume different values.

A random variable is a variable "X" that has a single numerical value determined by chance, for each outcome of a procedure.

If a sample space S is discrete, then every R.V. defined on S is also discrete, i.e., its range is countable (think of random counts for examples).

2 Discrete Random Variable

A Discrete Random Variable is a variable that can assume only certain clearly separated values.

A Discrete Random Variable has either a finite or countable number of values, where "countable" refers to the fact that there might be infinitely many values, but they can be associated with a counting process.

2.1 Examples of Discrete Random Variables

- The outcome of rolling a single die.
- The number of boys in a family with three children.
- The number of heads that appear when a coin is flipped nine times.
- The sum of the numbers on the dice, when k dice are rolled.
- The number of bits received in error when n bits are received.
- The number of bits received until the r-th error.

3 Discrete Probability Distributions

A discrete probability distribution is a listing of all possible values of a random variable along with their probabilities.

$$\frac{X}{P(X)}\frac{|x_1|}{|p_1|}\frac{|x_2|}{|p_2|}\frac{|x_2|}{|p_2|}\frac{|\dots|}{|\dots|}\frac{|x_k|}{|p_k|}; \quad \sum_{k\geq 1} p_k = 1. \tag{1}$$

1. The sum of all probabilities must be 1in any probability distribution
2. All probability values must be in [0,1]

Example 3.1. x $\Rightarrow$ The number of heads appearing when a coin is flipped three times.

X	0	1	2	3
P(X)	1/8	3/8	3/8	1/8

3.1 Bernoulli

A Bernoulli trail is an experiment with only two outcomes Success and Failure

1. $P(S) = p =$ Probability of a success
2. $P(F) = q = 1 - p =$ Probability of a failure

If a Bernoulli random variable, X, denotes No. of successes, then

1. $X = 1$ if the outcome is success
2. $X = 0$ if the outcome is failure

Example 3.2. A bit is transmitted, it is received in error with probability "0.1". Assume that 8 bits are transmitted independently.

1. How many bits can be received in error?

$$^8P_2$$

2. What is the probability that 2 bits are received in error

$$P(X = 2) = {}^8P_2 \times (0.1)^2(0.9)^6$$

3.2 Binomial Distribution

In general, let X stand for the number of bits received in error, when n bits are transmitted, with the probability of a single bit in error being p

$$P(X = k) = {}^nC_k \times p^k \times (1-p)^{n-k} \tag{2}$$

$$\sum_{k=0}^{n} p_k = {}^nC_k \times p^k \times (1-p)^{n-k} = [(p) + (1-p)]^n = 1^n = 1 \tag{3}$$

Example 3.3. A fair coin is tossed 10 times, what is the probability of getting:

1. Exactly 6 heads.

$$n = 10; \quad p = 0.5; q = 0.5$$

$$P(X = 6) = {}^{10}C_6 \times (0.5)^6 \times (0.5)^4$$

2. At least 6 heads.

$$P(X \geq 6) = P(X = 6) + P(X = 7) + P(X = 8) + P(X = 9) + P(X = 10)$$

3.3 Geometric Distribution

A bit is transmitted, it is received in error with probability "0.1".
Assume that bits are transmitted independently, until the first bit is received in error.

1. How many bits can be received?

   Infinity many many

2. What is the probability that the 5-th bit is received in error?

$$P(X = 5) = 0.1 \times 0.9^4$$

3. What is the probability that at least 5 (i.e. 5 or more) bits are received until the first error

$$P(X \geq 5) = P(X = 5) + P(X = 6) + \dots =$$

$$\sum_{k=5}^{\infty} P(X = K) = \sum_{k=5}^{\infty} (0.1)(0.9)^{k-1} = 0.656$$

So The General case is

$$P(X = k) = p \times (1 - p)^{k-1}; \quad k \geq 1 \tag{4}$$

$$\sum_{k=1}^{\infty} P(X = k) = \sum_{k=1}^{\infty} p \times (1 - p)^{k-1} = p \sum_{k=1}^{\infty} (1 - p)^{k-1} = p \times \frac{1}{p} = 1 \tag{5}$$

Example 3.4. Given that the first k trials were Failures. Find the probability that $(k+1)$-th trial will be a Success.

we need to find that $P(X = k + 1 | X > k)$

$$P(X = k + 1 | X > k) = \frac{P(X = k + 1 \cap X > k)}{P(X > k)} = \frac{p \times (1 - p)^k}{(1 - p)^k} = p = p(X = 1) \text{ this called}$$
lack of memory where the probability of k + 1 is the same as the first

3.4 Negative Binomial

When a bit is transmitted, it is received in error with probability "0.1". Assume that bits are transmitted independently, until FOUR bits are received in error.

1. At least, how many bits can be received ?

   at least 4 bits.

2. What is the probability that exactly 10 bits will be received

$$P(X=10) = (0.1)^9 \times {}^9C_3 \times (0.9)^6 \times (0.1)^3$$

In General, $P(X=k) = {}^{k-1}C_3 \times (1-p)^{k-4} \times p^4$

Definition 3.1. In general, let X stand for the number of bits received until r bits are received in error, with the probability of a single bit in error being p.

$$P(X=k) = \underbrace{p^r}_{\text{last trial}} \quad \underbrace{{}^{k-1}C_{r-1}(1-p)^{k-r}}_{(r-1)\text{success in}(k-1)\text{trial}} \; ; \quad k \geq r. \tag{6}$$

$$\sum_{k=r}^{\infty} {}^{k-1}C_{r-1} p^r (1-p)^{k-r} = \sum_{k'=0}^{\infty} {}^{k'+r-1}C_{k'} \; p^r (1-p)^{k'} \tag{7}$$

given that ${}^nC_k = {}^nC_{n-k}$

$$\sum_{k'=0}^{\infty} {}^{k'+r-1}C_{k'} \; p^r (1-p)^{k'} = p^r \sum_{k'=0}^{\infty} {}^{k'+r-1}C_{r-1}(1-p)^{k'} \tag{8}$$

given that $\dfrac{1}{(1-x)^r} = \displaystyle\sum_{k=0}^{\infty} {}^{k+r-1}C_{r-1} x^r$

$$p^r \sum_{k'=0}^{\infty} {}^{k'+r-1}C_{r-1}(1-p)^{k'} = p^r * \frac{1}{(1-(1-p))^r} = 1 \tag{9}$$

4 Notes

1. Recall that a binomial random variable is a count of the number of successes in n Bernoulli trials. That is, the number of trials n is predetermined, and the number of successes represents the random variable X.

2. A negative binomial random variable is a count of the number of trials required to obtain r successes. That is, the number of successes r is predetermined, and the number of trials (n or k) represents the random variable X.

5 Probability Mass Function

Using the probability distribution "PD" of a discrete RV, X, with possible values $x_1, x_2 \ldots, x_n$. A probability mass function is a function defined such that:

1. $$f(X_k) = P(X = x_k) = p_k,$$

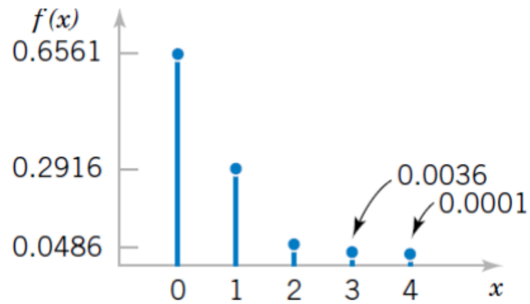
2. $$f(x_k) \geq 0,$$

3.
$$\sum_{k=1}^{n} f(x_k) = 1.$$

Example :

$$f(x) = \begin{cases} p_k, & x = x_k \\ 0, & otherwise \end{cases}$$

f is a function that is non-trivial only on the range of X.

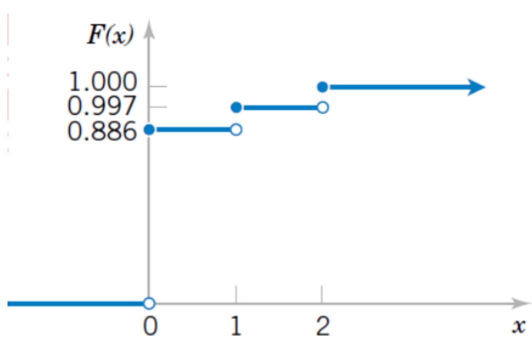


6 Cumulative Probability Function

The following function provides an alternative definition for X. The cumulative probability function of a DRV X is defined by:

$$F(x) = P(X \leq x) = \sum_{x_k \leq x} f(x_k)$$

$$F(x) = \begin{cases} 0, & x < x_1 \\ p_1, & x_1 \leq x < x_2 \\ p_1 + p_2, & x_2 \leq x < x_3 \\ \ldots \end{cases}$$



6.1 Notes

1. F is a step function
2. $0 \leq F(X) \leq 1$
3. if $x \leq y \Rightarrow F(x) \leq F(y)$

7 Average of a Discrete Random Variable

- Each time a random experiment is carried out, an associated RV takes on a value. Is it possible find an "average" value of the RV ?
- For instance, on average, how many bits are received in error when 8 bits are transmitted?
- Or, on average, how many bits must be received until the first (the r-th) error occurs?
  - Note that in both cases, you can take a sample, i.e. repeat the respective experiment several times and take the value of the RV each time.
  - An average computed this way is, however, dependent on the samle!
  - Is it possible to get a sample independent value?

8 Expectation

The expected value, or expectation, of a discrete random variable X is given by

$$\mu = E(X) = \sum X * P(X) = \sum_k x_k p_k$$

The formula gives a weighted average of values of a random variable, with the probabilities being the weights of the respective values. (One can see that the expected value as a weighted probability).

9 Variance and Standard Deviation

The average spread of a DRV around its expectation is called the standard deviation, and is computed via the so-called VARIANCE

$$\sigma^2 = V(X) = \sum_k [(x_k - \mu)^2 . p_k] \quad \text{or} =$$

$$[\sum_k x_k^2 . p_k] - \mu^2 = E(X^2) - [E(X)]^2$$

Standard Deviation is $\sigma = \sqrt{\sigma^2}$

Example 9.1. Compute the mean and the standard deviation of the number of girls in a family with 3 children.

X	$P(X)$	$X.P(X)$	$X^2.P(X)$
0	1/8	0	0
1	3/8	3/8	3/8
2	3/8	6/8	12/8
3	1/8	3/8	9/8
		$\mu = 1.5$	$\sum[X^2.P(X)] = 3$

$$\sigma = \sqrt{\sigma^2} = \sqrt{[\sum X^2.P(X)] - \mu^2} = \sqrt{3 - (1.5)^2}$$

9.1 Expectation and Variance of Bernoulli Random Variable