

Lecture 5

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1 Conditional Probability

The conditional probability of an event A occurring given an event B “i.e., knowing that B has already occurred first” is given by:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

2 Independent Events

In general, two events A and B are called independent if

$$P(A \cap B) = P(A) \times P(B)$$

So we can say that

$$P(A|B) = P(A)$$

$$P(B|A) = P(B)$$

Also

$$P(A \cap B \cap C) = P(A) \times P(B) \times P(C)$$

Example 2.1. Assume that we role a die once:

1. A is the event of getting an even number.

$$P(A) = \frac{1}{2}$$

2. B is the event of getting a number > 3 .

$$P(B) = \frac{1}{2}$$

Are A & B Independent?

$$P(A \cap B) = \frac{1}{3}$$

$$P(A) \times P(B) = \frac{1}{4}$$

A & B are not independent

Example 2.2. In a box, there are 8 red and 3 green balls. We choose two balls randomly. Find the probability that:

1. Both balls are red.

$$P(2red) = P(r1)P(r2|r1) = \frac{8}{11} \times \frac{7}{10}$$

2. Both balls are green.

$$P(2green) = P(g1)P(g2|g1) = \frac{3}{11} \times \frac{2}{10}$$

3. First ball is red, and the second ball is

$$P(\text{green, red}) = P(r1) \times P(g2|r1) = \frac{8}{11} \times \frac{3}{10}$$

Example 2.3. One card is drawn from a standard deck, replaced, and then the second card is drawn. Find the probability that the first card is a king, and the second one is not.

$$P(E) = P(k) \times P(k') = \frac{4}{52} \times \frac{48}{52}$$

Note that the two events are independent.

Example 2.4. A box contains black chips and white chips. A person selects 2 chips. If the probability of selecting a black chip and a white chip is $15/56$, and the probability of selecting a black chip on the first draw is $3/8$.

Find the probability of selecting the white chip on the second draw, given that the first chip selected was a black chip.

$$P(W|B) = \frac{P(B \cap W)}{P(B)} = \frac{15/56}{3/8}$$

3 Total Probability

Let F be any event that is already occurred. Then, The sample space can be represented as $S = F \cup F'$ where $F \cap F' = \emptyset$

$$\begin{aligned} P(E) &= P(E \cap S) = P(E \cap (F \cup F')) = \\ P((E \cap F) \cup (E \cap F')) &= P(E \cap F) + P(E \cap F') = \\ P(F) \times P(E|F) + P(F') \times P(E|F') & \end{aligned}$$

It allows one to express the probability of any event in terms of conditional probabilities

$$P(E) = P(F).P(E|F) + P(F').P(E|F'); \quad F \cup F' = S; \quad F \cap F' = \emptyset$$

Example 3.1. There are two identical bottles. One bottle contains 2 green balls and 1 red ball, the other bottle contains 2 red balls. A bottle is selected at random, and a ball is drawn. What is the probability that the ball is red?

Let B_1 and B_2 stand for the events that the first bottle and the second bottle were selected. Hence $P(B_1) = P(B_2) = 0.5$ The chances to draw a red ball from b_1 are $\frac{1}{3}$. for b_2 the chances are 1.

$$\begin{aligned} P(r) &= P(B_1)P(r|B_1) + P(B_2)P(r|B_2) = \\ &\frac{1}{2} \times \frac{1}{3} + \frac{1}{2} \times \frac{2}{2} \end{aligned}$$

4 Bayes' Rule

Suppose that S is represented as the union of finite number “ n ” of disjoint events F_j . Then the total probability rule can be generalized as follows:

$$P(E) = \sum_{k=1}^n P(F_k) \cdot P(E|F_k)$$

$$\bigcup_{k=1}^n F_k = S; \quad F_i \cap F_j = \emptyset : i \neq j.$$

Example 4.1. There are two identical bottles. One bottle contains 2 green balls and 1 red ball, the other bottle contains 2 red balls. A bottle is selected at random, and a ball is drawn.

Knowing that the ball is red. what is the probability that is drawn from the first bottle?

$$P(B_1|red) = \frac{P(B_1 \cap red)}{P(red)} = \frac{P(B_1) \cdot P(red|B_1)}{P(red)} = \frac{P(B_1) \cdot P(red|B_1)}{P(B_1) \cdot P(red|B_1) + P(B_2) \cdot P(red|B_2)}$$

$$P(B_1|red) = \frac{\frac{1}{2} \cdot \frac{1}{3}}{\frac{1}{3} \cdot \frac{1}{2} + 1 \cdot \frac{1}{2}} = \frac{1}{4}$$

4.1 Note

1. In the example, the event that a red ball was drawn from a bottle has affected the chances that it was a particular bottle.
2. Originally, both bottles were equally likely, i.e. $P(B_1) = P(B_2) = 0.5$.
3. With a red ball drawn, it is three times more likely that the second bottle was selected, as the respective conditional probability is 0.75 vs. 0.25 for Bottle 1.

5 Bayes' Theorem

Let A be an event of sample space S . Let B_i be n events such that:

1. one of them must occur
2. B_i and B_j cannot occur together.
3. S is the union of B_i

Then Bayes' Formula

$$P(B_n|A) = \frac{P(B_n)P(A|B_n)}{\sum_{i=1}^n P(B_i) \cdot P(A|B_i)}$$

Example 5.1. A manufacturer of DVD players purchases a microchip from 3 suppliers: hp, Sony, and Dell. 30% of the chips are purchased from hp, 20% from Sony, and 50% from Dell. The manufacturer has extensive histories on the 3 suppliers and knows that 3% of the chips from hp are defective, 5% of the chips from Sony are defective, and 4% of the chips from Dell are defective. When the chips arrive at the manufacturer, they are placed directly in a bin.

A worker selects a chip for installation in DVD player and finds it defective. What is the probability that it is Sony?

$$P(S|D) = \frac{P(S).P(D|S)}{P(h).P(D|h) + P(S).P(D|S) + P(d).P(D|d)}$$