Lecture 5

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Contents

1 Conditional Probability

The conditional probability of an event A occurring given an event B "i.e., knowing that B has already occurred first" is given by:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

2 Independent Events

In general, two events A and B are called independent if

$$P(A \cap B) = P(A) \times P(B)$$

So we can say that

$$P(A|B) = P(A)$$

$$P(B|A) = P(B)$$

Also

$$P(A \cap B \cap C) = P(A) \times P(B) \times P(C)$$

Example 2.1. Assume that we role a die once:

1. A is the event of getting an even number.

$$P(A) = \frac{1}{2}$$

2. B is the event of getting a number > 3.

$$P(B) = \frac{1}{2}$$

Are A & B Independent?

$$P(A \cap B) = \frac{1}{3}$$

$$P(A) \times P(B) = \frac{1}{4}$$

A & B are not independent

Example 2.2. In a box, there are 8 red and 3 green balls. We choose two balls randomly. Find the probability that:

1. Both balls are red.

$$P(2red) = P(r1)P(r2|r1) = \frac{8}{11} \times \frac{7}{10}$$

2. Both balls are green.

$$P(2green) = P(g1)P(g2|g1) = \frac{3}{11} \times \frac{2}{10}$$

3. First ball is red, and the second ball is

$$P(green, red) = P(r1) \times P(g2|r1) = \frac{8}{11} \times \frac{3}{10}$$

Example 2.3. One card is drawn from a standard deck, replaced, and then the second card is drawn. Find the probability that the first card is a king, and the second one is not.

$$P(E) = P(k) \times P(k') = \frac{4}{52} \times \frac{48}{52}$$

Note that the two events are independent.

Example 2.4. A box contains black chips and white chips. A person selects 2 chips. If the probability of selecting a black chip and a white chip is 15/56, and the probability of selecting a black chip on the first draw is 3/8.

Find the probability of selecting the white chip on the second draw, given that the first chip selected was a black chip.

$$P(W|B) = \frac{P(B \cap W)}{P(B)} = \frac{15/56}{3/8}$$

3 Total Probability

Let F be any event that is already occurred. Then, The sample space can be represented as $S = F \cup F'$ where $F \cap F' = \emptyset$

$$P(E) = P(E \cap S) = P(E \cap (F \cup F')) =$$

$$P((E \cap F) \cup (E \cap F')) = P(E \cap F) + P(E \cap F') =$$
$$P(F) \times P(E|F) + P(F') \times P(E|F')$$

It allows one to express the probability of any event in terms of conditional probabilities

$$P(E) = P(F).P(E|F) + P(F').P(E|F'); \quad F \cup F' = S; \quad F \cap F' = \emptyset$$

Example 3.1. There are two identical bottles. One bottle contains 2 green balls and 1 red ball, the other bottle contains 2 red balls. A bottle is selected at random, and a ball is drawn. What is the probability that the ball is red?

Let B1 and B2 stand for the events that the first bottle and the seconed bottle were selected. Hence $P(B_1) = P(B_2) = 0.5$ The chances to draw a red ball from b1 are $\frac{1}{3}$. for b2 the cahnces are 1.

$$P(r) = P(B_1)P(r|B_1) + P(B_2)P(r|B_2) =$$
$$\frac{1}{2} \times \frac{1}{3} + \frac{1}{2} \times \frac{2}{2}$$

4 Bayes' Rule

Suppose that S is represented as the union of finite number "n" of disjoint events F_j . Then the total probability rule can be generalized as follows:

$$P(E) = \sum_{k=1}^{n} P(F_k).P(E|F_k)$$

$$\bigcup_{k=1}^{n} F_k = S; \quad F_i \cap F_j = \emptyset : i \neq j.$$

Example 4.1. There are two identical bottles. One bottle contains 2 green balls and 1 red ball, the other bottle contains 2 red balls. A bottle is selected at random, and a ball is drawn.

Knowing that the ball is red. what is the probability that is drawn from the first bottle?

$$P(B1|red) = \frac{P(B1 \cap red)}{P(red)} = \frac{P(B1).P(red|B1)}{P(red)} = \frac{P(B_1).P(red|B_1)}{P(B_1).P(red|B_1) + P(B_2).P(red|B_2)}$$

$$P(B_1|red) = \frac{\frac{1}{2}.\frac{1}{3}}{\frac{1}{3}.\frac{1}{2} + 1.\frac{1}{2}} = \frac{1}{4}$$

4.1 Note

1. In the example, the event that a red ball was drawn from a bottle has affected the chances that it was a particular bottle.
2. Originally, both bottles were equally likely, i.e. P(B1) = P(B2) = 0.5.
3. With a red ball drawn, it is three times more likely that the second bottle was selected, as the respective conditional probability is 0.75 vs. 0.25 for Bottle 1.

5 Bayes' Theorem

Let A be an event of sample space S. Let B_i be n events such that:

1. one of them must occur
2. B_i and B_j cannot occur together.
3. S is the union of B_i

Then Bayes' Formula

$$P(B_n|A) = \frac{P(B_n)P(A|B_n)}{\sum_{i=1}^{n} P(B_i).P(A|B_i)}$$