Properties of CT Fourier Series

Ibrahim Abou Elenein

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1 Linearity

Let

$$x(t) \xrightarrow{\mathcal{FS}} a_k$$
$$y(t) \xrightarrow{\mathcal{FS}} b_k$$

two periodic signals with period T, then

$$z(t) = Ax(t) + By(t) \xrightarrow{\mathcal{FS}} c_k = Aa_k + Bb_k$$

Proof

$$\begin{aligned} c_k = \frac{1}{T}\int_T z(t)e^{-jk\omega_o t}dt &= \frac{1}{T}\int_T [Ax(t) + By(t)]e^{-jk\omega_o t}dt \\ &= A\frac{1}{T}\int_T x(t)e^{-jk\omega_o t}dt + B\frac{1}{T}\int_T y(t)e^{-jk\omega_o t}dt \\ &= Aa_k + Ba_k \end{aligned}$$

2 Time Shifting

Let

$$x(t) \xrightarrow{\mathcal{FS}} a_k$$

Then

$$x(t - t_o) \xrightarrow{\mathcal{FS}} b_k = e^{-jk\omega_o t_o} a_k = e^{jk(2\frac{\pi}{T})t_0} a_k$$

Note $|b_k| = |a_k|$

Proof

$$\begin{aligned} b_k = \frac{1}{T}\int_T x(t - t_0)e^{-jk\omega_o t}dt &\overset{\tau = t - t_o}{=} \frac{1}{T}\int_T x(\tau)e^{-jk\omega_o(\tau + t_o)}d\tau \\ &= e^{-jk\omega_o t_o} \underbrace{\frac{1}{T}\int_T x(\tau)e^{-jk\omega_o \tau}d\tau}_{a_k} = e^{-jk\omega_o t_o} a_k \end{aligned}$$

3 Time Scaling

$$t \Rightarrow \alpha t, \quad \alpha > 0$$

1. period changes $T \rightarrow \frac{T}{\alpha}$
2. fundamental frequency changes $\omega_o \rightarrow \alpha\omega_o$

Fourier Series representation

$$x(\alpha t) = \sum_{k=-\infty}^{\infty} a_k e^{jk(\frac{\alpha 2\pi}{T})t} = \sum_{k=-\infty}^{\infty} a_k e^{jk(\alpha\omega_o)t}$$

4 Differentiation

Let

$$x(t) \xrightarrow{\mathcal{FS}} a_k$$

Then

$$\frac{dx(t)}{dt} = jk\omega_o a_k$$

5 Multiplication

Let

$$x(t) \xrightarrow{\mathcal{FS}} a_k$$

$$y(t) \xrightarrow{\mathcal{FS}} b_k$$

$$z(t) = x(t)y(t) \xrightarrow{\mathcal{FS}} c_k = \sum_{i=-\infty}^{\infty} a_i b_{k-i} \Rightarrow \text{discrete-time convloution of} a_k \text{ and } b_k$$

$$c_k = a_k * b_k$$

6 Parseval's Relation

Let

$$x(t) \xrightarrow{\mathcal{FS}} a_k$$

Then

$$\boxed{\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2}$$

Hence Average total power of the signal is sum of the average powers in all of harmonic components.

7 Time Reversal

Let

$$x(t) \xrightarrow{\mathcal{FS}} a_k$$

Then

$$x(-t) \xrightarrow{\mathcal{FS}} b_k = a_{-k}$$

Proof

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_o t}, \Longrightarrow x(-t) = \sum_{k=-\infty}^{\infty} a_k e^{-jk\omega_o t}$$

$$\stackrel{k=-m}{=} \sum_{m=-\infty}^{\infty} a_{-m} e^{jm\omega_o t}$$

8 Conjugation and Conjugate Symmetry

Let

$$x(t) \xrightarrow{\mathcal{FS}} a_k$$

Then

$$x^*(t) \xrightarrow{\mathcal{FS}} a^*_{-k}$$

- If $x(t) = x^*(t) \Rightarrow a_k = a^*_{-k} \Longrightarrow x(t)$ real
- If $x(t) = -x^*(t) \Rightarrow a_k = -a^*_{-k} \Longrightarrow x(t)$ img

9 Summary

Let

$$x(t) \xrightarrow{\mathcal{FS}} a_k$$

$$y(t) \xrightarrow{\mathcal{FS}} b_k$$

1. Linearity

$$Ax(t) + By(t) \xrightarrow{\mathcal{FS}} Aa_k + Bb_k$$

2. Time Shift

$$x(t - t_o) \xrightarrow{\mathcal{FS}} a_k e^{-jk\omega_o t_o}$$

3. Time Scale

$$x(\alpha t) \xrightarrow{\mathcal{FS}} a_k$$

4. Differentiation

$$\frac{dx(t)}{dt} \xrightarrow{\mathcal{FS}} jk\omega_o a_k$$

5. Integration

$$\int_{-\infty}^{t} x(\tau)d\tau \xrightarrow{\mathcal{FS}} \frac{a_k}{jk\omega_o}$$

6. Convolution

$$x(t) * y(t) \xrightarrow{\mathcal{FS}} Ta_k b_k$$

7. Multiplication

$$x(t)y(t) \xrightarrow{\mathcal{FS}} a_k * b_k$$

10 Notes

- if $x(t)$ is real & even $\Rightarrow a_k$ is real & even
- if $x(t)$ is real & odd $\Rightarrow a_k$ is pure & odd.