Fourier Series of CT signals

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Fourier series states that, any periodic signal can be expressed as summation of sines and cosines with one fundamental frequency and infinite number

1 Fourier Series Representation of CT Periodic Signals

The exponential form of Fourier series (complex representation of continuous time periodic signals)

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_o t}$$

Whrere:
a_k Fourier coefficients.
$x(t)$ is periodic with Period T whrere $\omega_o = \dfrac{2\pi}{T}$

$$x(t) = \dots + a_{-2}e^{-j2\omega_o t} + a_{-1}e^{-j\omega_o t} + a_o + a_1 e^{+j\omega_o t} + \dots$$

1.1 Notes

- The term a_o is constant (Dc coefficient)
- The terms a_1 & a_{-1} both have the fundemntal frequency as ω_o and called **first harmonic component**
- The terms a_2 & a_0 both have the fundemntal frequency as ω_o and called **seconed harmonic component**
- Generalally, The components for $k = -N$ & N are called **Nth order Component**.

2 Obtaining the Fourier Series Coefficients a_k

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_o t}$$

$$x(t)e^{-jn\omega_o t} = \sum_{k=-\infty}^{\infty} a_k e^{j(k-n)\omega_o t} \text{ integrating over the period}$$

$$\int_0^T x(t)e^{-jn\omega_o t} dt = \int_0^T \sum_{k=-\infty}^{\infty} a_k e^{j(k-n)\omega_o t} dt \; a_k \text{ are time-independent}$$

$$\int_0^T x(t)e^{-jn\omega_o t} dt = \sum_{k=-\infty}^{\infty} a_k \int_0^T e^{j(k-n)\omega_o t} dt$$

$$\int_0^T e^{j(k-n)\omega_o t} dt = \begin{cases} T, & k = n \\ 0, & k \neq n \end{cases} \text{ Integration over one period of cos and sin}$$

$$\underbrace{a_n = \frac{1}{T} \int_0^T x(t)e^{-jn\omega_o t} dt}_{\text{Fourier Series Coefficients}}$$

3 Fourier series of Periodic CT signal

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_o t}$$

$$a_k = \frac{1}{T} \int_0^T x(t)e^{-jk\omega_o t} dt \; , \; \omega_o = \frac{2\pi}{T}$$

3.1 Constant Value

This the average of $x(t)$ over one period.

$$a_o = \frac{1}{T} \int_T x(t) dt \Rightarrow \text{The area under the curve over one period}$$

$$a_o = \frac{\text{Area of one period}}{T}$$

4 Famous Signals

4.1 DC Signal

$$x(t) = A$$

$$x(t) = \dots + a_{-2}e^{-j2\omega_o t} + a_{-1}e^{-j\omega_o t} + a_o + a_1 e^{+j\omega_o t} + \dots$$

So by comparing

$$\boxed{a_o = A \;\; \& \;\; a_k = 0 \text{ for } k \neq 0}$$

4.2 Complex Exponential

$$x(t) = e^{j\omega_o t}$$

$$x(t) = \dots + a_{-2}e^{-j2\omega_o t} + a_{-1}e^{-j\omega_o t} + a_o + a_1 e^{+j\omega_o t} + \dots$$

So by comparing

$$\boxed{a_1 = 1 \;\; \& \;\; a_k = 0 \text{ for } k \neq 1}$$

Note if

$$x(t) = 2e^{-3j\omega_o t} \Rightarrow a_{-3} = 2 \;\; \& \;\; a_k = 0 \text{ for } k \neq -3$$

4.3 Sinusoidals

4.3.1 Cosine

$$x(t) = \cos(\omega_o t) = \underbrace{\frac{1}{2}e^{j\omega_o t} + \frac{1}{2}e^{-j\omega_o t}}_{\text{Euler's rule}}$$

$$x(t) = \dots + a_{-2}e^{-j2\omega_o t} + a_{-1}e^{-j\omega_o t} + a_o + a_1 e^{+j\omega_o t} + \dots$$

So by comparing

$$\boxed{a_1 = a_{-1} = \frac{1}{2} \;\; \& \;\; a_k = 0 \;\; \text{for} \;\; k \neq 1, -1}$$

4.3.2 Sine

$$x(t) = \sin(\omega_o t) = \underbrace{\frac{1}{2j}e^{j\omega_o t} + \frac{-1}{2j}e^{-j\omega_o t}}_{\text{Euler's rule}}$$

$$x(t) = \dots + a_{-2}e^{-j2\omega_o t} + a_{-1}e^{-j\omega_o t} + a_o + a_1 e^{+j\omega_o t} + \dots$$

So by comparing

$$\boxed{a_1 = \frac{1}{2j}, a_{-1} = \frac{-1}{2j} \;\; \& \;\; a_k = 0 \;\; \text{for} \;\; k \neq 1, -1}$$

4.3.3 Example

Find Fourier Series representation of the following Signal

$$x(t) = 1 + \sin(\omega_o t) + 2\cos(\omega_o t) + \cos(2\omega_o t + \frac{\pi}{4})$$

$$x(t) = 1 + \frac{1}{2j}[e^{j\omega_o t} - e^{-j\omega_o t}] + [e^{j\omega_o t} + e^{-j\omega_o t}] + \frac{1}{2}[e^{j(2\omega_o t + \frac{\pi}{4})} + e^{-j((2\omega_o t + \frac{\pi}{4}))}] =$$

Collecting like terms

$$x(t) = 1 + (1 + \frac{1}{2j})e^{j\omega_o t} + (1 - \frac{1}{2j})e^{-j\omega_o t} + (\frac{1}{2}e^{j\frac{\pi}{4}})e^{j2\omega_o t} + (\frac{1}{2}e^{-j\frac{\pi}{4}})e^{-j2\omega_o t}$$

$$x(t) = \underbrace{1}_{a_0} + \overbrace{(1 + \frac{1}{2j})}^{a_1} e^{j\omega_o t} + \underbrace{(1 - \frac{1}{2j})}_{a_{-1}} e^{-j\omega_o t} + \overbrace{(\frac{1}{2}e^{j\frac{\pi}{4}})}^{a_2} e^{j2\omega_o t} + \underbrace{(\frac{1}{2}e^{-j\frac{\pi}{4}})}_{a_{-2}} e^{-j2\omega_o t}$$

$$a_0 = 1,\ a_1 = (1 + \frac{1}{2j}),\ a_{-1} = (1 - \frac{1}{2j}),\ a_2 = \frac{1}{2}e^{j\frac{\pi}{4}},\ a_2 = \frac{1}{2}e^{-j\frac{\pi}{4}},\ a_k = 0 \text{ for } |k| > 2$$

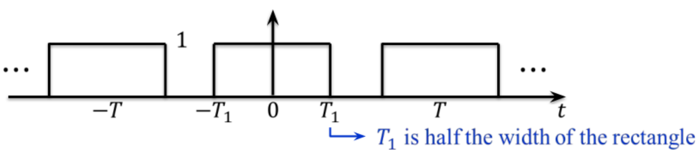
4.4 Periodic Impulse Train

Unit Impulse ($\delta(t)$) repating itself every period T

$$\boxed{a_k = \frac{1}{T}}$$

Note The periodic impulse train must be centered at 0 and has amplitude = 1 Otherwise, it will be a shifted and scaled periodic impulse train.

4.5 Periodic Rectangle Wave



The poof is letf to the reader.

$$a_o = \frac{\text{Area of one period}}{T} = \frac{2T_1}{T}.$$

$$a_k = \begin{cases} \frac{2T_1}{T} & , k = 0 \\ \frac{\sin(k\omega_o t T_1)}{k\pi} & , k \neq 0 \end{cases}$$

Note The periodic rectangle wave must be centered at 0 and has amplitude = 1 Otherwise, it will be a shifted and scaled periodic rectangle wave.

5 Convergence of The Fourier Series

The conditions for a periodic signal $x(t)$ to have a Fourier Series as follows :

5.1 Condition one

Over any period x(t) must be absolutely integrable

$$\int_T |x(t)|dt < \infty$$

and this guarantees that

$$|a_k| \leq \frac{1}{T}\int_T |x(t)e^{-j\omega_o t}|dt = \frac{1}{T}\int_T |x(t)|dt \Rightarrow |a_k| < \infty$$

5.2 Condition two

In any finite interval of time, $x(t)$ is of bounded variation; that is, there are no more than a finite number of maxima and minima during any single period of the signal.

5.3 Condition three

In any finite interval of time, there are only a finite number of discontinutous