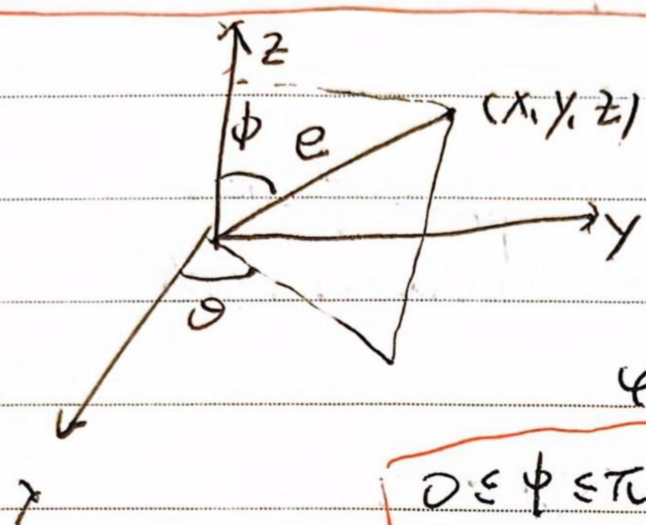
Mo | Tu | We | Th | Fr | Sa | Su

Memo No.

Date / /

LEC26. spherical coordinates (球坐标)

2025.2.22



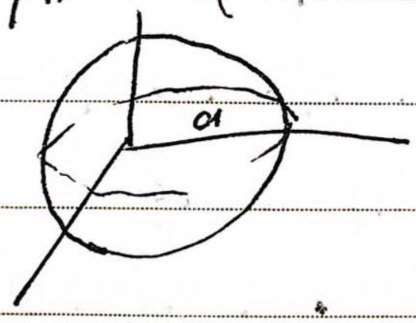
ρ = rho = distance from origin

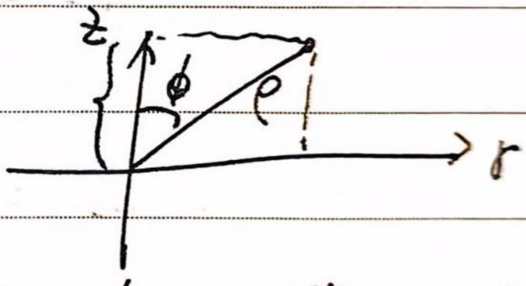
φ = ϕ = phi = angle go down from z-axis

$\boxed{0 \le \phi \le \pi}$

θ = same as before

On a sphere $\rho = a$





" ~~pole~~ polar coordinates in z-r

$$\begin{cases} z = \rho\cos\phi \\ r = \rho\sin\phi \end{cases}, \quad \begin{cases} x = r\cos\theta = \rho\sin\phi\cos\theta \\ y = r\sin\theta = \rho\sin\phi\sin\theta \end{cases}$$

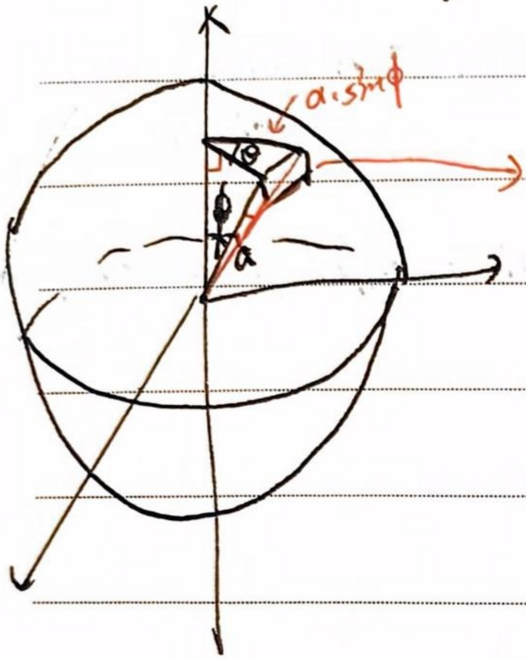
$$\rho = \sqrt{z^2 + r^2} = \sqrt{x^2 + y^2 + z^2}$$

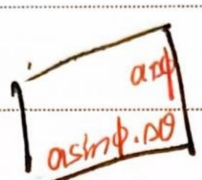
- $\rho = a$; sphere of radius a centered at O.
- $\phi = \frac{\pi}{4}$ a cone (圆锥)

Triple $\int$ in spherical coordinate:

$$dV = ??? \ d\rho \, d\phi \, d\theta$$

$$\Delta V = \underbrace{\quad} \ \Delta\rho \, \Delta\phi \, \Delta\theta$$





piece of circle of radius a, length $= a \cdot \Delta\phi$

$a(\sin\phi) \cdot \theta$ = length

$$\Delta S \approx (a \sin\phi \cdot \Delta\theta) \cdot (a\Delta\phi)$$

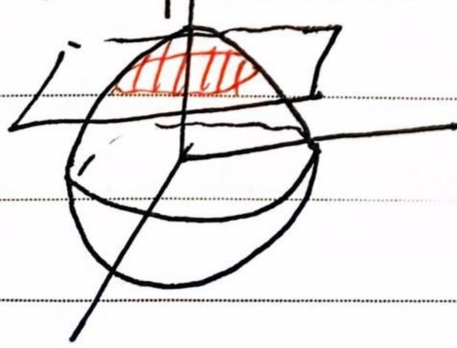
$$= a^2 \sin\phi \cdot \Delta\theta \Delta\phi$$

$$dS = a^2 \sin\phi \, d\theta \, d\phi$$

$$\Delta V \approx \Delta\rho \cdot \Delta S = \rho^2 \sin\phi \, \Delta\rho \, \Delta\phi \, \Delta\theta$$

$$\boxed{dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta} = dx\,dy\,dz$$

Example:



vol of portion of unit sphere above $z = \frac{1}{\sqrt{2}}$

$$\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_{\frac{1}{\sqrt{2}}\sec\phi}^{1} 1\ \rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

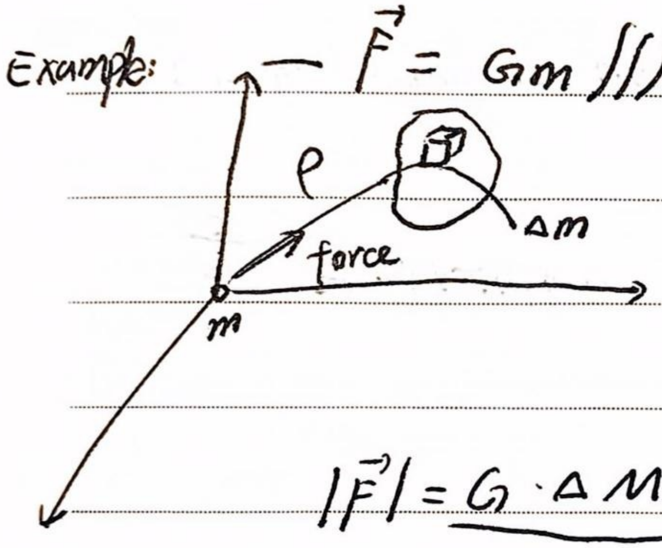
($\rho\cos\phi = \frac{1}{\sqrt{2}}$, $\rho = \frac{1}{\sqrt{2}\cos\phi}$, when $\rho = 1$, $\phi = \frac{\pi}{4}$)

$$\frac{2\pi}{3} - \frac{6\pi}{6\sqrt{2}}$$

~~Eval~~ Evaluation

Application : — find the avg distance to O

Example: — $\vec{F} = Gm\iiint \frac{\langle x, y, z\rangle}{\rho^3}\,\delta\, dV$



(gravitational force) 31p

exerted by ΔM at (x, y, z)

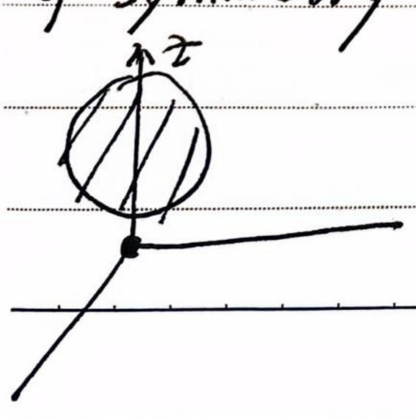
on a mass m at origin

$|\vec{F}| = \frac{G\cdot\Delta M\cdot m}{\rho^2}$; direction of force $\vec{F} = \frac{\langle x, y, z\rangle}{\rho}$ (unit v)

$$\vec{F} = \frac{G\Delta M m}{\rho^3}\cdot\langle x, y, z\rangle$$

So, integrating $\Delta M \approx \delta\Delta V$ $\quad\vec{F} = \iiint \frac{Gm\langle x, y, z\rangle}{\rho^3}\, dV$

Set up: • place solid so z axis is an axis of symmetry



Then $\vec{F} = \langle 0, 0, F\rangle$

z → component : $Gm\iiint \frac{z}{\rho^3}\, dV$