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LEC 14 ~~Mo~~ Non-independent variables

2025.1.14

Ex:

$f(P, V, T)$ where $PV = nRT$

find $f(x, y, z)$ where $g(x, y, z) = C$

If $g(x,y,z) = C$, then $z = z(x, y)$

how can we find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$?

Ex:

$x^2 + yz + z^3 = 8$ at $(2, 3, 1)$

take differential

because $g = 8$, $dg = 0$

$$\underbrace{2x\,dx}_{\frac{\partial f}{\partial x}} + \underbrace{z\,dy}_{\frac{\partial f}{\partial y}} + \underbrace{(y+3z^2)\,dz}_{\frac{\partial f}{\partial z}} = 0$$

$(2, 3, 1)$ substituting $\Rightarrow$ $4dx + dy + 6dz = 0$

$dz = -\frac{1}{6}(4dx + dy)$ we view $z = z(x, y)$

total differential

$\frac{\partial z}{\partial x} = -\frac{4}{6} = -\frac{2}{3}$; $\frac{\partial z}{\partial y} = -\frac{1}{6}$

y constant, set $dy = 0$ set x constant $\Rightarrow dx = 0$

In general $\boxed{g(x,y,z)=c}$

$$\boxed{dg = g_x dx + g_y dy + g_z dz = 0}$$

Example: $f(x,y) = x+y \qquad \frac{\partial f}{\partial x} = 1$

set $x=u,\ y=u+v \Rightarrow f(u,v) = 2u+v$

then $\frac{\partial f}{\partial u} = 2$

so — $x=u$, but $\frac{\partial f}{\partial u} \neq \frac{\partial f}{\partial x}$!

keep v constant   keep y constant

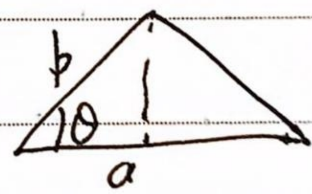
Need clearer notation.

$\left(\frac{\partial f}{\partial x}\right)_y$ keep y constant

$\neq$

$\left(\frac{\partial f}{\partial u}\right)_v$ keep v constant

Example: area of triangle



$A = \frac{1}{2} ab \sin\theta$

(constraint)

assume its a ~~tr~~ right triangle $\Rightarrow a = b\cos\theta$

Rate of change of A with respect to θ?

1) treat a, b, θ as independent: $\frac{\partial A}{\partial \theta} = \left(\frac{\partial A}{\partial \theta}\right)_{a,b}$

keep a,b fixed $\left(\frac{\partial A}{\partial \theta}\right)_{a,b} = \frac{1}{2} ab\cos\theta$

2) keep a constant, b will change $b = b(a,\theta)$,

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$(b = \frac{a}{\cos\theta})$, so that ~~right~~ we keep right angle

$(\frac{\partial A}{\partial \theta})_a$

3) keep b constant

$a = a(b, \theta)$ $\quad (\frac{\partial A}{\partial \theta})_b$

Compute $(\frac{\partial A}{\partial \theta})_a$?

Method 0: solve for B and substitute

$a = b\cos\theta \Rightarrow b = \frac{a}{\cos\theta} = a\sec\theta$;

$A = \frac{1}{2}a^2\sin\theta \cdot \sec\theta = \frac{1}{2}a^2\tan\theta$

So $(\frac{\partial A}{\partial \theta})_a = \frac{1}{2}a^2\sec^2\theta$

2 systematic methods:

$\boxed{\text{1) differentials}}$

· keep a fixed $da = 0$

(total differential)

· constrant $a = b\cos\theta \Rightarrow da = \cos\theta\, db - b\sin\theta\, d\theta$

$\Rightarrow 0 = da = \cos\theta \cdot db - b \cdot \sin\theta\, d\theta$

$\Rightarrow \cos\theta\, db = b\sin\theta\, d\theta \Rightarrow db = b \cdot \tan\theta \cdot d\theta$

· function $A = \frac{1}{2}ab\sin\theta$ $\quad dA = \frac{1}{2}b\sin\theta\, \underset{0}{da} + \frac{1}{2}a\sin\theta\, db$

$+ \frac{1}{2}ab\cos\theta \cdot d\theta$

so $dA = \frac{1}{2}ab(\tan\theta \cdot \sin\theta + \cos\theta) \cdot d\theta$

$= \frac{1}{2}ab\sec\theta \cdot d\theta$

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so $\left(\frac{\partial A}{\partial \theta}\right)_a = \frac{1}{2} ab \sec\theta$ $(a = b\cos\theta)$

$= \frac{1}{2} a^2 \sec^2\theta$

Summary: differential

(1) write dA in terms of $da, db, d\theta$

(2) a = constant ⇒ set $da = 0$

(3) differential constraint ⇒ ~~slove~~ solve for db in terms of $d\theta$ $b = \frac{a}{\cos\theta}$ sec

(4) plug into dA

2) Chain Rule use constraint

$$\boxed{\left(\frac{\partial A}{\partial \theta}\right)_a = A_\theta \cdot \overbrace{\left(\frac{\partial \theta}{\partial \theta}\right)_a}^{1} + A_a \overbrace{\cancel{\left(\frac{\partial a}{\partial \theta}\right)_a}}^{0} + A_b \left(\frac{\partial b}{\partial \theta}\right)_a}$$

$$= \frac{1}{2} ab\cos\theta + \frac{1}{2} a \cdot \sin\theta \cdot \sec\theta \cdot \tan\theta \cdot a$$

$$= \frac{1}{2} ab (\cos\theta + \sin\theta \cdot \tan\theta) = \frac{1}{2} ab\sec\theta$$

$\left(\frac{1}{2} a \cdot a \cdot \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{2} ab \cdot \tan\theta \cdot \sin\theta \right)$