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Final Review

LEC34 2025.1.26

Unit 1:

- vectors, dot product

$$\vec{A}\cdot\vec{B} = |\vec{A}|\cdot|\vec{B}|\cdot\cos\theta = \sum a_i b_i$$

cross product $\vec{A}\times\vec{B}$ ⇝ area in space

⇝ vector $\perp \vec{A}$ and $\vec{B}$

- equations of plane $\begin{cases} ax+by+cz=d \\ \langle a,b,c\rangle = \text{normal vector} \end{cases}$

- equation of lines (parametric $\begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases}$

point on L vector // L

- parametric equations of curve $\vec{r}(t)$

decompose vector to other vector's tor-

decompose position vector in to some simple vectors

- velocity $\vec{v} = \frac{d\vec{r}}{dt}$, speed $= |\vec{v}|$

$$\vec{v} = \hat{T}\cdot\frac{ds}{dt}$$

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Acceleration: $\vec{a} = \frac{d\vec{v}}{dt}$

- matrices, determinants, liner systems.

{ 3×3 liner system ⇔ $AX = B$ column vector

3×3 materix ↙ ↓ column vector

inverting (2×2 or 3×3) ~~mat~~

A [⋮ ⋮ ⋮] $\xrightarrow{\text{minors}}$ [entree are 2×2 determinant] ⇝ $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$ cofactors

⇝ transpose, $* \frac{1}{\det(A)}$

- A invertible ⇔ $\det A \neq 0$, $AX = B$, $X = A^{-1}B$ then

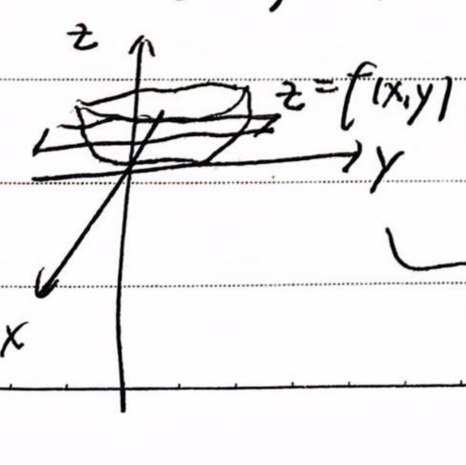
otherwise: $AX = B$ has either no solution or ∞ many solution

↳ $X \in$ intersection of ~~two~~ three planes

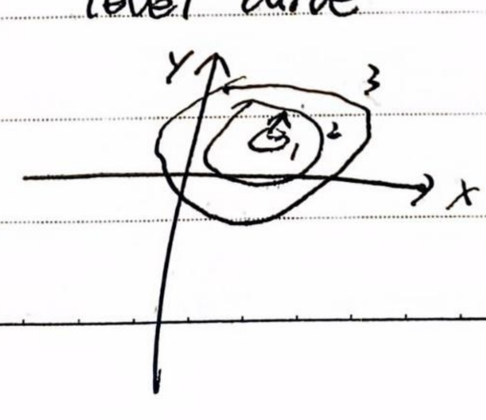
$AX = 0$, then 0 is always a solution ("trivial" solution)

Unit 2:

- viewing $f(x,y)$: graph, contour plot



level curve



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- partial derivative $f_x = \frac{\partial f}{\partial x}$, $f_y = \frac{\partial f}{\partial y}$

~~linear~~ linear approximation

$$\Delta f \approx f_x \Delta x + f_y \Delta y$$

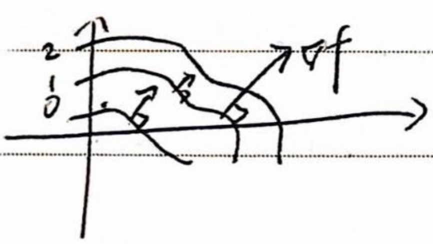
tagent plane to the graph of f

- differentials, chain rules

$$df = f_x dx + f_y dy$$

if $x = x(t)$, $y = y(t)$ $\Rightarrow \frac{df}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt}$

- gradient vector : $\nabla f = \langle f_x, f_y \rangle$



Directional derivative

for unit $\hat{u}$, $\left.\frac{df}{ds}\right|_{\hat{u}} = \nabla f \cdot \hat{u}$

- max / min problems

critical points $\Rightarrow \nabla f = 0 \rightarrow$ second derivative test $\rightarrow$ (local min), (local max), (saddle) $\rightarrow$ need check the boundary value of f or at ∞

$$\begin{cases} AC - B^2 > 0, & A > 0 \text{ min}, \ A < 0 \text{ max} \\ AC - B^2 < 0, & \text{saddle} \\ AC - B^2 = 0, & \text{fail test} \end{cases}$$

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- max/min f with non-indepent variables

$g=c \rightarrow$ $\boxed{\text{LAGRANGE MULTIPLIERS}}$

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = c \end{cases}$$

two curve tagent time

careful: second derivative does not apply this!

- constrained partial derivatives:

$g(x,y,z) = c$. $\frac{\partial f}{\partial x}$?

· $\left(\frac{\partial f}{\partial x}\right)_y$. y held constant

x varies, z depend on x, y

$\left(\frac{\partial f}{\partial x}\right)_z$, y depends on x, z

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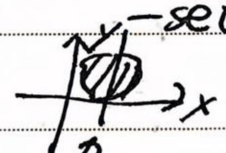
Unit 3 and 4

- Double integrals — setting up bounds

$\underbrace{\iint f(x,y)\, dy}\, dx$

↳ x is fixed

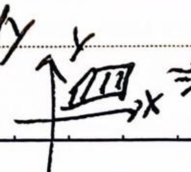


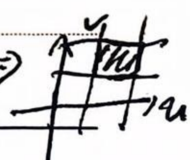
also in polar coordinates $dA = r\,dr\,d\theta$

also: changing to uv-coordinates

jacobin $du\,dv = \left|\frac{\partial(u,v)}{\partial(x,y)}\right| dx\,dy$

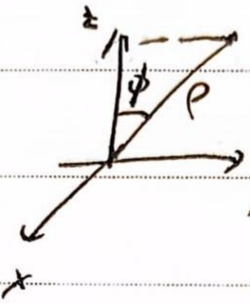
$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$





$\frac{\partial(u,v)}{\partial(x,y)}$

- Triple integrals, rectangular, cylindrical (z, r, θ), spherical (ϕ, θ, ρ)



$$\begin{cases} z = \rho\cos\phi \\ x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \end{cases}$$

$dV = dz\, r\,dr\,d\theta$, $dV = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$

Applications: - area / volume / mass

$\iint 1\,dA$ $\iiint 1\,dV$

-average value of a function f:

$\bar{f} = \frac{1}{\text{vol}}\iiint_D f\,dV$, or (weighted) $\frac{1}{\text{vol}}\iiint_D f\,\delta\,dV$

- center of mass $(\bar{x}, \bar{y}, \bar{z})$
- moments of inertia : $I_z = \iiint (x^2+y^2)\,\delta\,dV$ $(r^2 m)$
- gravitational attraction

$F = G\,m\cdot\iiint \delta\cos\phi/\rho^2\,dV$

- work and line integrals: (in plane & in space)

$\cdot\int_C \vec{F}\cdot d\vec{r} = \int_C M\,dx + N\,dy$, $\vec{F} = \langle M, N\rangle$

Express x & y in terms of ~~sign~~ single parametric (& z)

- gradient Field & path-independence:

  - if $\text{curl}\vec{F} = 0$ $\left[\begin{array}{l} \text{in 2D: } N_x - M_y \\ \text{in 3D: } \nabla \times \vec{F} \end{array}\right]$

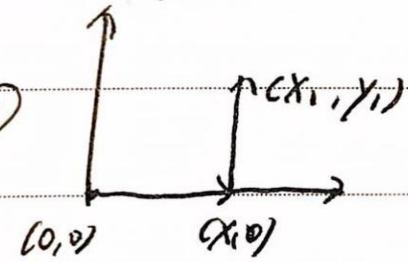
$\vec{F} = x\hat{i} + y\hat{j}$ $\text{curl}\vec{F} = \begin{vmatrix} \partial/\partial x & \partial/\partial y \\ x & y \end{vmatrix} = 0$

and $\vec{F}$ is defined in a simply-connected ~~region~~ region

then $\vec{F}$ is a gradient Field, $\vec{F} = \nabla f$ for some f

- how to find the potential

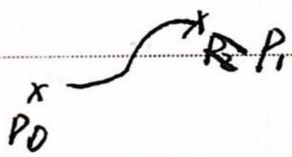
Ⓐ



$\int_{C_1 + C_2} \vec{F} \cdot d\vec{r}$ give. $f(x_1, y_1)$

Ⓑ start with $f_x = M \xrightarrow{\int dx} f = \sqcup + g(y, z)$ $(g(y))$

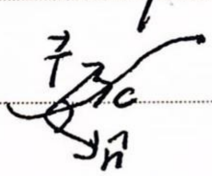
- once we have potential,



$\int_C \nabla f \cdot d\vec{r} = f(P_1) - f(P_0)$

- Flux in plane & space

  - in the plane



$\text{flux} = \int_C \vec{F} \cdot \hat{n}\, ds$

$\vec{F} = \langle P, Q \rangle$

$\hat{n} = \vec{T}$ rotated 90° clockwise

$= \int_C -Q\,dx + P\,dy$

$\hat{n}\,ds = \langle dy, -dx \rangle$

- in the space

$\iint_S \vec{F} \cdot \hat{n}\, dS \rightarrow$ set up: express $\hat{n}$ and dS

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or use $\hat{n}$

if S is given by $z = f(x,y)$, $\hat{n}\,dS = \langle -f_x, -f_y, 1\rangle\,dx\,dy$

but here $dx\,dy \neq dS$

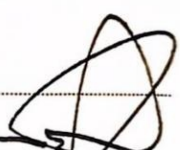
$\hat{n} = \pm\langle -f_x, f_y, 1\rangle / \sqrt{f_x^2 + f_y^2 + 1}$. $dS = \sqrt{f_x^2 + f_y^2 + 1}\,dx\,dy$

or: if know some normal vector to the surface $\vec{N}$

$$\hat{n}\,dS = \pm \frac{\vec{N}}{\vec{N}\cdot\hat{k}}\,dx\,dy$$

slanted plane.

S: $g(x,y,z) = 0$



2D

3D

work

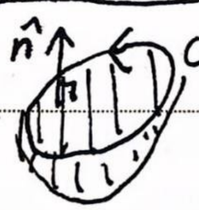
Green theorem

$\oint_C \vec{F}\cdot d\vec{r} = \iint_R \operatorname{curl}\vec{F}\,dA$

$\vec{F} = \langle M, N\rangle$ $\oint_C M\,dx + N\,dy$

$= \iint_R (\cancel{AA}\, N_x - M_y)\,dA$

Stokes' theorem

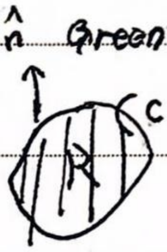


determinant

$\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot\hat{n}\,dS$

Flux

$\hat{n}$ Green for flux



$\oint_C \vec{F}\cdot\hat{n}\,ds = \iint_R \operatorname{div}\vec{F}\,dA$

divergence theorem

$\oiint_S \vec{F}\cdot\hat{n}\,dS$

$= \iiint_R \operatorname{div}\vec{F}\,dV$

$\operatorname{div}\langle P, Q, R\rangle = P_x + Q_y + R_z$