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Lec 30. Line Integral in Space

2025.1.24

LINE INTEGRALS

vector field $\vec{F} = P\hat{i} + Q\hat{j} + R\hat{k}$

(force) $d\vec{r} = \langle dx, dy, dz \rangle$

Curve C in space work $= \int_C \vec{F} \cdot d\vec{r}$

$= \int_C P\,dx + Q\,dy + R\,dz$

①

Evaluate: parameterize C, express x, y, z & dx, dy, dz in terms of param

Ex: $\vec{F} = \langle yz, xz, xy \rangle$

C: $x = t^3,\ y = t^2,\ z = t,\quad t \in [0, 1]$

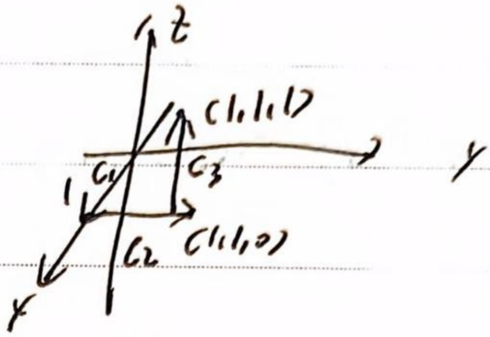
$dx = 3t^2 dt,\ dy = 2t\,dt,\ dz = dt$

$\int_C \vec{F} \cdot d\vec{r} = \int_C yz\,dx + xz\,dy + xy\,dz$

$= \int_0^1 t^3 \cdot 3t^2 dt + t^4 \cdot 2t\,dt + t^5 dt$

$= \int_0^1 6t^5 dt = 1$

Ex 2. same $\vec{F}$ (0: from O to C(1,0,0)) C_1



$\int_{C'} = \int_{C_1} + \int_{C_2} + \int_{C_3}$

$C_{1,2}$: $z = 0,\ dz = 0,\ \int_{C_1} 0 = 0$

~~C_3~~ $\int_{C_2} = 0$

C_3: $x = 1, y = 1, dx = dy = 0$, dz from $[0,1]$

$$\int_{C_3} \vec{F}\cdot d\vec{r} = \int_{C_3} xy\,dz = \int_0^1 dz = 1$$

$\int_{C'} = 1$

In fact $\vec{F}$ is conservative Ex 1, 2: (0,0,0) → (1,1,1)

$\int_C \vec{F}\cdot d\vec{r} = \int_{C'} \vec{F}\cdot d\vec{r}$

$$\vec{F} = \nabla(xyz) = \langle yz, xz, xy \rangle$$

knowing the ~~Flu~~ fundemental thm (gradient field)

$$\int_C \nabla f\cdot d\vec{r} = f(P_1) - f(P_0)$$

how to verify $\vec{F}$ is a gradient field

Test for gradient field

$\vec{F} = \langle P, Q, R \rangle = \langle f_x, f_y, f_z \rangle$

If so then $P_y = f_{xy} = f_{yx} = Q_x$, $P_z = f_{xz} = f_{zx} = R_x$

$Q_z = f_{yz} = f_{zy} = R_y$

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$\Rightarrow$ criterion: $\vec{F} = \langle P, Q, R \rangle$ (defined in a simpley connected region) is a gradient field

$\Rightarrow$ $\boxed{P_y = Q_x, \ P_z = R_x, \ Q_z = R_y}$

Example:

($P\,dx + Q\,dy + R\,dz$ is exact) ($= df$)

for which a and b is $axy\,dx + (x^2 + z^3)\,dy + (byz^2 - 4z^3)\,dz = df$ exact?

$P_y \quad Q_x \Rightarrow \quad ax = 2x \quad \Rightarrow \quad a = 2$

$P_z \quad R_x \Rightarrow \quad 0 = 0 \quad$ AND

$Q_z \quad R_y \Rightarrow \quad 3z^2 = bz^2 \quad \Rightarrow b = 3$

Find potential?

① $f(x_1, y_1, z_1) = \int_C \vec{F} \cdot d\vec{r}$ (+constant)

where C from $(0,0,0)$ to (x, y, z)

② Antiderivatives:

want: $f_x = 2xy, \quad f_y = x^2 + z^3, \quad f_z = 3yz^2 - 4z^3$

(dx) $f = x^2 y + \cancel{g(y,z)} \; g(y,z)$ $\quad$ ~~$f_y = x^2 + g'(y,z)$~~

(dy) $f_y = x^2 + z^3 = x^2 + g_y$ $\quad$ $f_z = 3yz^2 - 4z^3$

so $g_y = z^3 \Rightarrow g = yz^3 + h(z)$ $\quad \int dy$ $\quad \Rightarrow f = x^2 y + yz^3 + h(z)$

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$f_z = 3yz^2 + h'(z) \Rightarrow h'(z) = -4z^3 \Rightarrow h = -z^4 + c$

$\Rightarrow f = x^2y + yz^3 + (-z^4) + C$ (the ~~pon~~ potential)

Curl IN 3D

Stokes' Theorem: if $\vec{F} = P\hat{i} + Q\hat{j} + R\hat{k}$

$$\text{curl}\,\vec{F} = (R_y - Q_z)\hat{i} + (P_z - R_x)\hat{j} + (Q_x - P_y)\hat{k}$$

IF $\vec{F}$ is defined in a simply-conn region,

$\vec{F}$ conservative $\Leftrightarrow$ curl $\vec{F} = 0$.

how to remember?

$\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \qquad \nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle$

$$\boxed{\nabla \cdot \vec{f} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \text{div}\,\vec{F}}$$

$$\boxed{\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{i} - \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right)\hat{j} + \left(\frac{\partial}{\partial x}Q - \frac{\partial}{\partial y}P\right)\hat{k} = \text{curl}\,\vec{F}}$$

what's geomertically? "curl measure rotation component in a ~~volt~~ velocity field"

$\vec{v} = \langle -\omega y, \omega x, 0 \rangle$; curl $\vec{v} = 2\omega\vec{k}$