LEC 29. 2025.1.24

Divergence theorem application & proof

$$\oiint_S \vec{F}\cdot d\vec{S} = \iiint_D \operatorname{div} F \, dV$$

$$\operatorname{div} F = (P_x + Q_y + R_z)$$

notation: ∇ "del" $= \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$

as $\nabla \cdot \vec{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \cdot \langle P, Q, R \rangle$

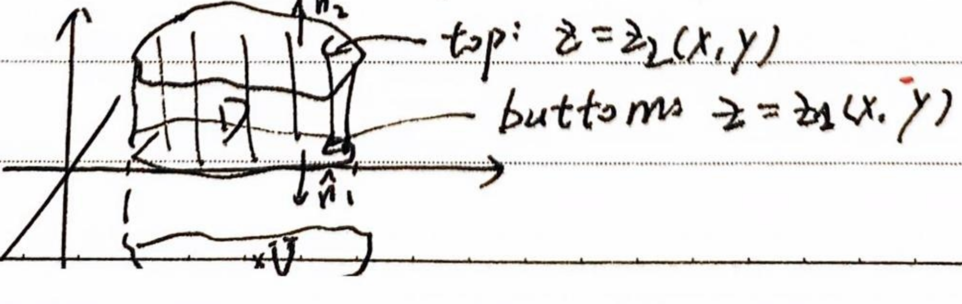
Physical interpretation:

$\operatorname{div}(\vec{F})$ = " source rate " = amount of flux generated per unit second

Proof of ~~et~~ $\oiint_S \langle 0, 0, R \rangle \cdot \hat{n} \, dS = \iiint_D R_z \, dV$

(then general case by summing three such ~~id~~ identities one)

- If given D is vertically simple reigon



right-hand side:

$$\iiint_D R_z\,dV = \iint_U \left(\int_{z_1(x,y)}^{z_2(x,y)} R_z\,dz \right) dx\,dy$$

$$= \iint_U [R(x,y,z_2(x,y)) - R(x,y,z_1(x,y))]\,dx\,dy$$

$$\oiint_{S = \text{buttom} + \text{top} + \text{sides}} \langle 0,0,R\rangle \cdot \hat{n}\, d\vec{S} = \iint_{top} + \iint_{buttom} + \iint_{sides}$$

Top: graph $z = z_2(x,y)$ $\quad \hat{n}\,dS = \langle -\frac{\partial z_2}{\partial x}, -\frac{\partial z_2}{\partial y}, 1\rangle\, dx\,dy$

$\langle 0,0,R\rangle \cdot \hat{n}\,dS = R\,dx\,dy$

$$\iint_{top} \langle 0,0,R\rangle \cdot \hat{n}\,dS = \iint_{top} R\,dx\,dy = \iint_U R(x,y,z_2(x,y))\,dx\,dy$$

Buttom: graph $z = z_1(x,y)$

$\hat{n}\,dS = \langle +\frac{\partial z}{\partial x}, +\frac{\partial z}{\partial y}, -1\rangle\, dx\,dy$

$\vec{F} \cdot \hat{n}\,dS = \langle 0,0,R\rangle \cdot \hat{n}\,dS = -R\,dx\,dy$

$$= \iint_{buttom} -R\,dx\,dy = \iint_U -R(x,y,z_1(x,y))\,dx\,dy$$

Sides: are vertical $\langle 0,0,R\rangle$ is tagent to sides,

Flux of sides $= 0$

So: $\iiint_D R_z\,dV = \oiint_{\text{buttom}+\text{top}+\text{sides}} \langle 0,0,R\rangle \cdot \hat{n}\,dS$

IF D is not vertically simple: cut it in to simple regions

Diffusion equation:

govens motion of smote in air

u = concentration at a given point $= u(x, y, z, t)$

(浓度)

$$\boxed{\begin{cases} \vec{F} = \nabla u \\ \operatorname{div}\vec{F} = -\frac{\partial u}{\partial t} \end{cases}}$$

$$\frac{\partial u}{\partial t} = k\,\nabla^2 u \leftarrow \text{(Laplacian)}$$

$$= k\,\nabla\cdot\nabla u \qquad \operatorname{div}(\nabla u)$$

$$\boxed{= k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)}$$

(the Heat equation) (also)

$\vec{F}$ = flow of smoke

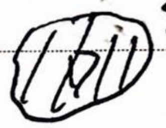
1) physics (+ common sense):

smoke flows from high concentration towards low concentration

so $\vec{F}$ directed along $(-\nabla u)$

In Fact: $\boxed{\vec{F} = -k\nabla u}$

2) Relate $\vec{F}$ and $\frac{\partial u}{\partial t}$? (divergence)



S

Flux out of D thorough S

$$\iiint_D \operatorname{div}\vec{F}\,dV = \oiint_S \vec{F}\cdot\hat{n}\,dS = \cancel{\text{amount of}}\ -\frac{d}{dt}\left(\iiint_D u\,dV\right)$$

$$\Rightarrow \iiint_D \operatorname{div} \vec{F} \, dV = -\frac{d}{dt}\left(\iiint_D u \, dV\right) = -\iiint_D \frac{\partial u}{\partial t} dV$$

$$\Rightarrow \boxed{\operatorname{div} \vec{F} = -\frac{\partial u}{\partial t}}$$

For any region D

avg of $(\operatorname{div} \vec{F})$ in D = avg of $\left(-\frac{\partial u}{\partial t}\right)$ in D

$$\Rightarrow \boxed{\operatorname{div} \vec{F} = -\frac{\partial u}{\partial t}}$$ ← just remember this

Diffusion equation:

u = concentration of substance

$$\frac{\partial u}{\partial t} = -\operatorname{div} \vec{F} = k \operatorname{div}(\nabla u) = k \nabla^2 u$$

(diffusion equation)