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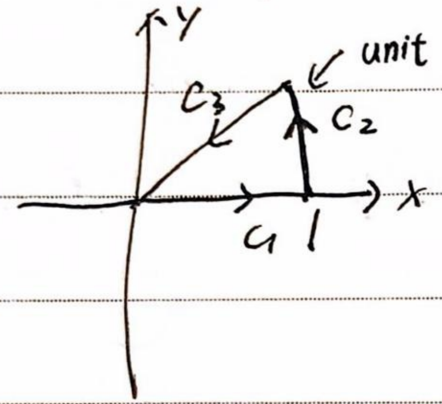
LEC 20. 2025.1.18

gradient field

$\vec{F} = \langle M, N \rangle$

Example:



Ex : $\vec{F} = \langle y, x \rangle$

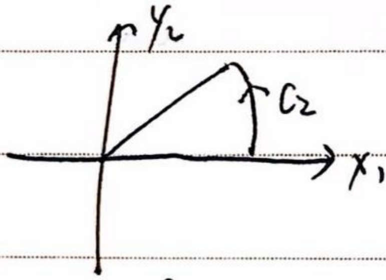
$\int_C \vec{F} \cdot d\vec{r}$, $C = C_1 + C_2 + C_3$

enclosing a sector of a unit disk $\theta \in [0, \frac{\pi}{4}]$

Need $\int_{C_i} y\,dx + x\,dy$

1) $\int_{C_1} y\,dx + x\,dy = \int_{C_1} 0\,dx + 0 = 0$

2) C_2: portion of unit circle



$x = \cos\theta$ $\quad \theta \in [0, \frac{\pi}{4}]$

$y = \sin\theta$ $\quad dx = -\sin\theta\, d\theta$

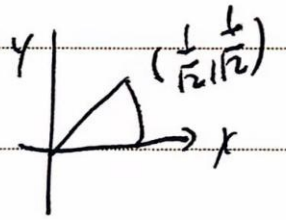
$dy = \cos\theta\, d\theta$

$$\int_{C_2} y\,dx + x\,dy = \int_0^{\frac{\pi}{4}} \sin\theta(-\sin\theta)\,d\theta + \cos\theta \cdot \cos\theta\, d\theta = \int_0^{\frac{\pi}{4}} \cos 2\theta\, d\theta$$

$$= \left[\frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{2}$$

3) $\int_{C_3} y\,dx + x\,dy$



Could do: $x = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} t$ $\quad (t \in [0, 1])$

$y = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} t$

either: $x = t,\ y = t,\quad t \in [0, \frac{1}{\sqrt{2}}] \Rightarrow -C_3$

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$$\int_{-C_3} y\,dx + x\,dy = \int_0^{\frac{1}{\sqrt{2}}} t\,dt + t\,dt = [t^2]_0^{\frac{1}{\sqrt{2}}} = \frac{1}{2}$$

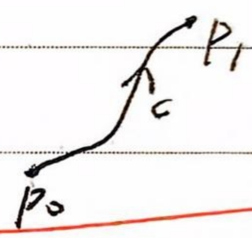
$$\Rightarrow \int_{C_3} y\,dx + x\,dy = -\frac{1}{2} \Rightarrow \int_C y\,dx + x\,dy = 0 + \frac{1}{2} - \frac{1}{2} = 0$$

Special case: $\vec{F} = \nabla f$

Then we can ~~sa~~ say

Fundamental theorem of calculus for line integrals

$$\int_C \nabla f \cdot d\vec{r} = f(P_1) - f(P_0)$$



Proof: $\int_C f_x\,dx + f_y\,dy = \int_C df = f(P_1) - f(P_0)$

$t \in [t_0, t_1]$

$x = x(t)$

$y = y(t)$

$$\int_C \nabla f \cdot d\vec{r} = \int_C f_x\,dx + f_y\,dy,$$

$$\cancel{= \int_C \nabla f \cdot d\vec{r}} = \int_C \left(f_x \frac{dx}{dt} + f_y \frac{dy}{dt}\right) dt$$

$$\Rightarrow = \int_C \frac{df}{dt}\,dt = \int_{t_0}^{t_1} \frac{df}{dt}\,dt = \cancel{f(t_1) - f(t_0)}$$

$$= [f(x(t), y(t))]_{t_0}^{t_1}$$

$$\Rightarrow = f(P_1) - f(P_0)$$

Example:

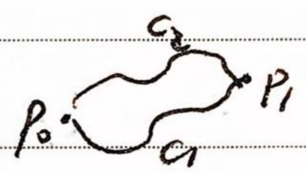
$\vec{F}\langle y, x\rangle = \nabla f, \quad f(x,y) = xy$

so $\int_{C_2} \vec{F}\cdot d\vec{r} = f(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) - f(1,0) = \frac{1}{2} - 0 = \frac{1}{2}$

WARNING: Everything today only apply if $\vec{F}$ is a gradient field! Not true Otherwise!

Consequence of fund: IF $\vec{F}$ is a gradient field then

1. • Path-independence:



$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r}$

2. $\vec{F} = \nabla f$ is conservative (保守)

closed curve $\Rightarrow \int_C \vec{F}\cdot d\vec{r} = 0$

$(\int_{C \text{ closed}} \nabla f \cdot d\vec{r} = f(\text{end}) - f(\text{start}) = 0$

Remark: $\vec{F} = \langle -y, x\rangle$, $\cancel{f(x,y) = -xy}$

↑ not conservative (not gradient field)

3. $\vec{F}$ is a gradient field $\Leftrightarrow$ 1 and 2

$\Leftarrow$ how we find the potential (势能)

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4. $Mdx + Ndy$ is an exact differential (df)

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last time if $\vec{F} = \nabla f$ gradient field

then $\int_C$ "path independent" $\int_C \vec{F} \cdot d\vec{r} = f(P_1) - f($

If $\vec{F} = \nabla f$, $M = f_x$, $N = f_y$

then $f_{xy} = f_{yx} \Rightarrow \cancel{M_x}$ $M_y = N_x$

and if $M_y = N_x \Rightarrow \vec{F}$ is a gradient field and

$\vec{F} = \langle M, N \rangle$ defined, differentiable everwhere

Example $\vec{F} = \underbrace{-y}_{M} \hat{i} + \underbrace{x}_{N} \hat{j}$ $\qquad \frac{\partial M}{\partial y} = -1$, $\frac{\partial N}{\partial x} = 1$

$\Rightarrow \vec{F}$ is not a gradie

$\hat{i} + (-3y^2 + 4x^2)\hat{j}$