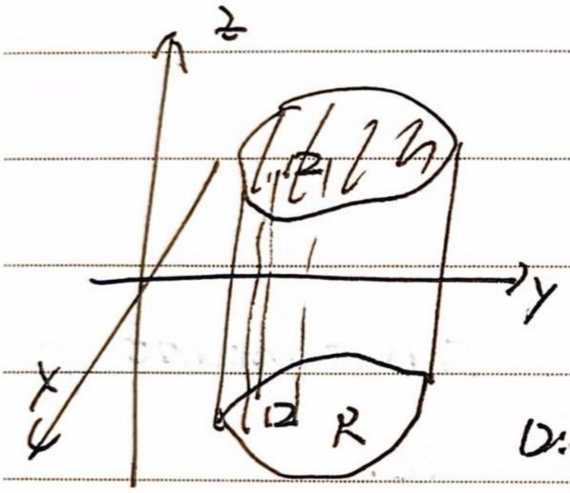
LEC 28. Divergence theorem

2025.1.23

Flux of $\vec{F}$ through surface S

$$\iint_S \vec{F}\cdot\hat{n}\, dS$$

* If S is graph of $z = f(x,y)$

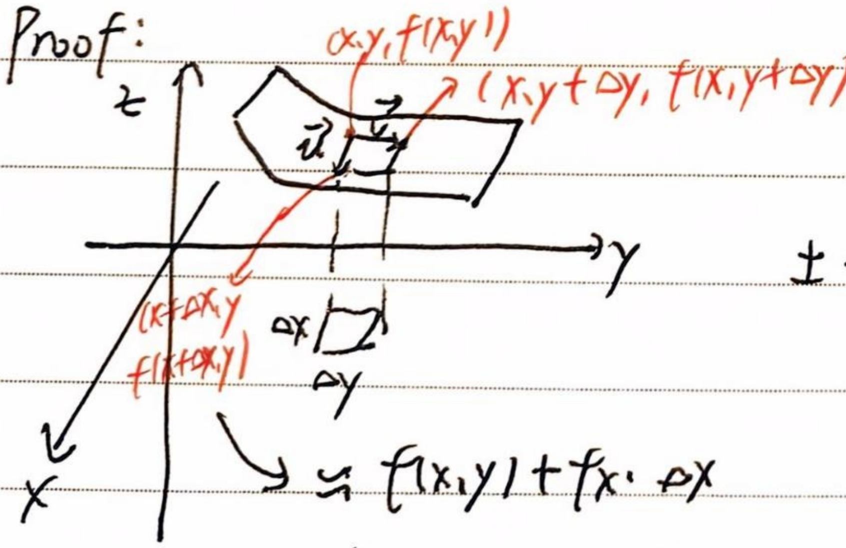


$$\boxed{\hat{n}\cdot dS = \pm\langle -f_x, -f_y, 1\rangle\, dx\,dy}$$

$\square$: $\Delta x \Delta y$

where this come from?

Proof:



$\rightarrow \approx f(x,y) + f_y\cdot\Delta y$

$\pm\vec{u}\times\Delta\vec{v} = \Delta S\cdot\hat{n}$

$\searrow \approx f(x,y) + f_x\cdot\Delta x$

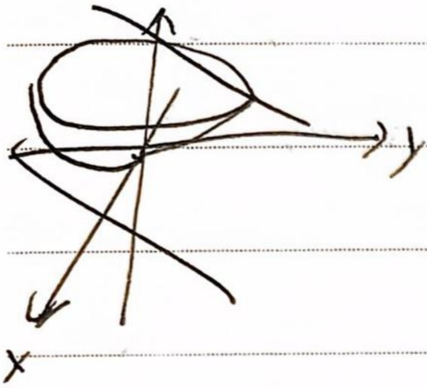
$\Rightarrow \vec{u} \approx \langle \Delta x, 0, f_x \Delta x\rangle = \langle 1, 0, f_x\rangle\cdot\Delta x$

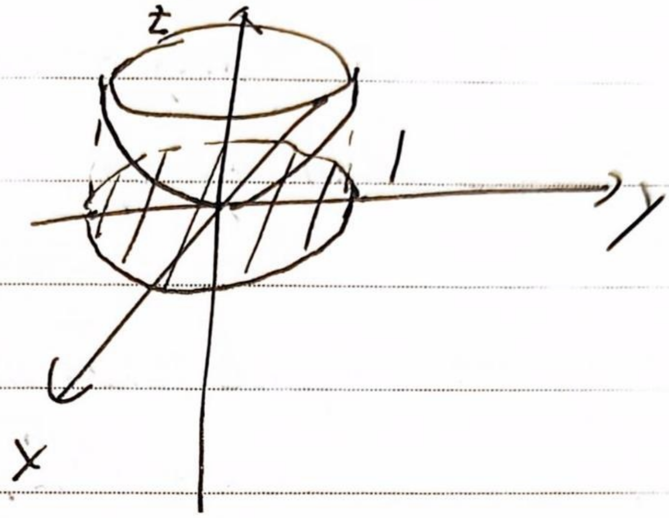
simily $\Rightarrow \vec{v} \approx \langle 0, \Delta y, f_y \Delta y\rangle = \langle 0, 1, f_y\rangle\cdot\Delta y$

$$\vec{u}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix}\cdot\Delta y\cdot\Delta x = \langle -f_x, -f_y, 1\rangle\cdot dx\,dy$$

Memo No. ______
Date / /

Example: $\vec{F} = z \cdot \hat{k}$ through portion of paraboloid $z = x^2 + y^2$ above unit disk.





$\iint_S \vec{F} \cdot \hat{n} \, dS \quad \hat{n} \cdot dS = \langle -2x, -2y, 1 \rangle \cdot dx\,dy$

$= \iint \langle 0, 0, z \rangle \cdot \langle -2x, -2y, 1 \rangle \, dx\,dy$

$= \iint_S z \, dx\,dy \quad = \iint (x^2+y^2) \, dx\,dy = \int_0^{2\pi} \int_0^1 r^2 \cdot r \, dr\,d\theta$

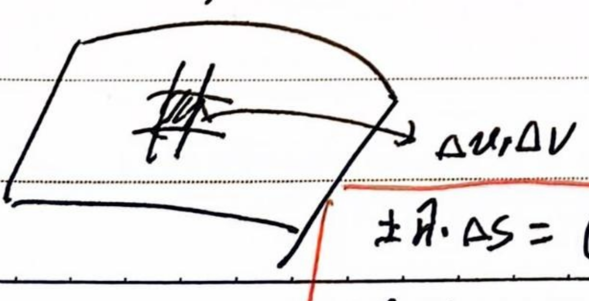
$= \frac{\pi}{2}$

More generally: (usually use special ones)

give parametric desciptions

$S = \begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{cases}$ final $du\,dv$ (example: $\begin{cases} z = \rho \cos\phi \\ x = \rho \sin\phi \cos\theta \\ y = \rho \sin\phi \sin\theta \end{cases}$

$\vec{r} = \langle x, y, z \rangle = \vec{r}(u,v)$



sides: $\frac{\partial \vec{r}}{\partial u} \cdot \Delta u = \langle \frac{\partial x}{\partial u} \cdot \Delta u, \frac{\partial y}{\partial u} \cdot \Delta u, \frac{\partial z}{\partial u} \cdot \Delta u \rangle$

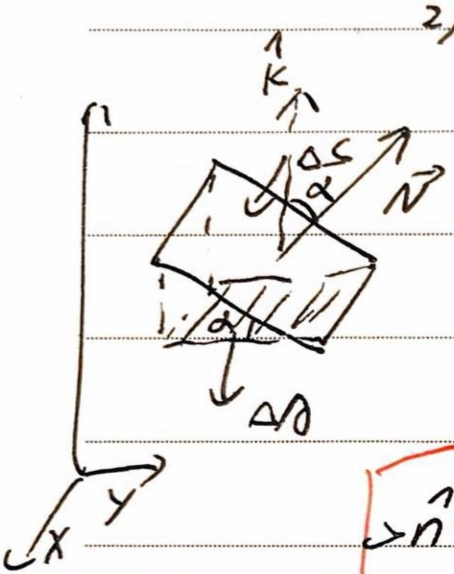
$\frac{\partial \vec{r}}{\partial v} \cdot \Delta v = \langle \frac{\partial x}{\partial v} \cdot \Delta v, \cdots \rangle$

$\boxed{\begin{aligned} \pm \hat{n} \cdot \Delta S &= (\frac{\partial \vec{r}}{\partial u} \cdot \Delta u) \times (\frac{\partial \vec{r}}{\partial v} \cdot \Delta v) \\ \pm \hat{n} \cdot dS &= du \cdot dv \cdot (\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}) \end{aligned}}$

• If we know a normal vector $\vec{N}$ to the surface

Example: 1) plane $ax+by+cz=d$, $\vec{N}=\langle a,b,c\rangle$

2) S given by eq $g(x,y,z)=0$, $\vec{N}=\nabla g$



surface element: $\Delta A = \Delta S \cdot \cos\alpha$

$\cos\alpha = \vec{N}\cdot\hat{k} / |\vec{N}|\cdot|\hat{k}|$

$\Rightarrow \Delta S = \frac{1}{\cos\alpha}\Delta A = \frac{|\vec{N}|}{\vec{N}\cdot\hat{k}}\cdot \Delta A$

$$\boxed{\begin{aligned} \Rightarrow \hat{n}\cdot\Delta S &= \frac{|\vec{N}|\cdot\hat{n}}{\vec{N}\cdot\hat{k}}\cdot\Delta A = \pm\frac{\vec{N}}{\vec{N}\cdot\hat{k}}\cdot dxdy \\ \text{same as reason: } &= \pm\frac{\vec{N}}{\vec{N}\cdot\hat{i}}\cdot dydz \\ &= \pm\frac{\vec{N}}{\vec{N}\cdot\hat{j}}\cdot dxdz \end{aligned}}$$

Example: S: $\underbrace{z-f(x,y)}_{g(x,y,z)}=0$ ← saw from front page

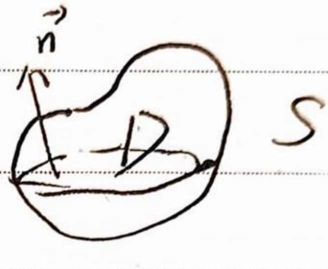
$\vec{N}=\nabla g=\langle -f_x, -f_y, 1\rangle$

$\hat{n}\cdot ds = \frac{\vec{N}}{\vec{N}\cdot\hat{k}}dxdy = \langle -f_x, -f_y, 1\rangle\cdot dxdy$

Divergence Theorem

(Green's Theorem) In 3D for flux

If S is a closed surface enclosing a region D with $\hat{n}$ outwards



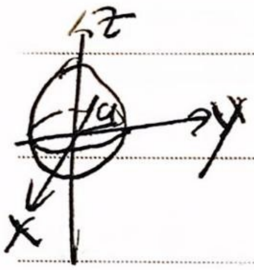
and $\vec{F}$ defined and differenable everywhere in D

then $\boxed{\oint\!\!\oint_S \vec{F}\cdot d\vec{s} = \iiint_D \operatorname{div}\vec{F}\, dV}$

where $\operatorname{div}(P\hat{i} + Q\hat{j} + R\hat{k}) = P_x + Q_y + R_z$

Examples

last time $\vec{F} = z\hat{k}$ of the sphere radius a, flux is $\frac{4}{3}\pi a^3$



$$\iint_S z\hat{k}\cdot\hat{n}\cdot ds = \iiint_D \operatorname{div}(z\hat{k})\cdot dV$$

$$= \iiint_D (0+0+1)\cdot dV = \frac{4}{3}\pi a^3$$