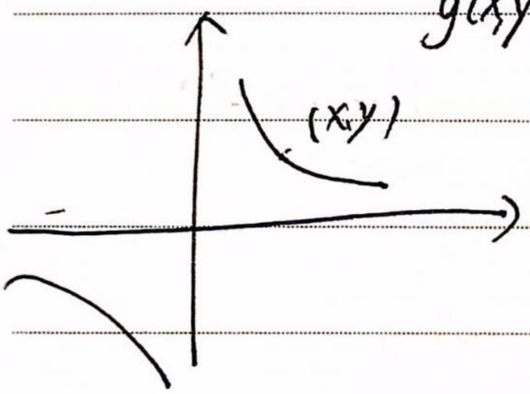
LEC13. Lagrange multipliers 2025.1.13

min/max a function

$f(x,y,z)$ $g(x,y,z)=C$

Example: point closest to the origin on hyperbola $xy=3$ ↓ $d=\sqrt{x^2+y^2}$

↓ $g(x,y)$



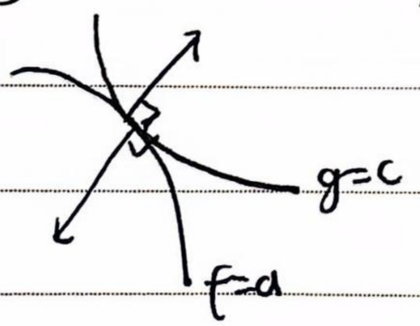
Minimize: ~~$f(x,y)=\sqrt{x^2+y^2}$~~

use square ⇒ $f(x,y)=x^2+y^2$

Observe: at the minimum the level curve of f is tagent to $g(x,y)=3$

⇒ how to find (x,y) where curve of f and g are tagent to each of other?

↓



$\boxed{\text{when this happens } \nabla f \parallel \nabla g \quad \text{so } \nabla f = \lambda \nabla g}$

lagrange multipliers → λ lambda

$g(x,y) = c \to$ system of equations

$$\nabla f = \lambda \nabla g \quad \begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \end{cases}$$

constrant $g = c$

$$f = x^2 + y^2 \quad g = xy \Rightarrow \begin{cases} 2x = \lambda y \\ 2y = \lambda x \\ xy = 3 \end{cases} \Rightarrow \begin{cases} 2x - \lambda y = 0 \\ \cancel{2y - \lambda x = 0} \ \ \lambda x - 2y = 0 \\ xy - 3 = 0 \end{cases}$$

$$M$$

$$\Rightarrow \begin{bmatrix} 2 & -\lambda \\ \lambda & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

trivial solution $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ cant solve $xy = 3$

other solutions exist if $\det(M) = 0$

so $(-4 + \lambda^2) = 0 \Rightarrow \lambda = \pm 2$

$\lambda = 2$ $\quad x = y$; $x^2 = 3 \Rightarrow x = \pm\sqrt{3}$ $\quad (\sqrt{3}, \sqrt{3})$ or $(-\sqrt{3}, -\sqrt{3})$

$\lambda = -2$ $\quad x = -y$, $-x^2 = 3$ ($\times$, no solutions here)

so the ~~closest~~ ~~pt~~ closest points: $(-\sqrt{3}, -\sqrt{3})$ or $(\sqrt{3}, \sqrt{3})$

Why is this method vaild?

At constrained min/max in any direction along level $g = c$, the rate of change of f must be zero (tagent)

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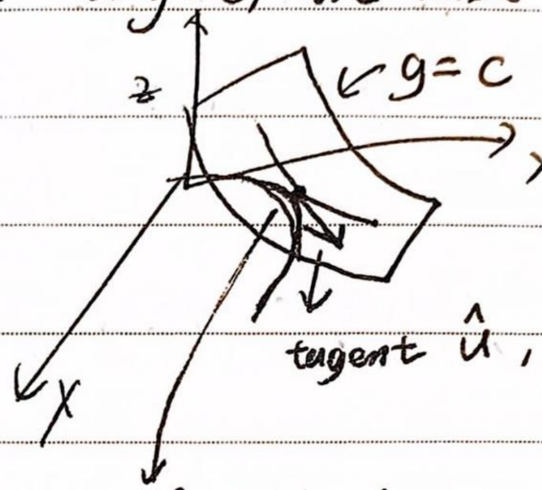
For any direction at tagent $\hat{u}$ to $g=c$, we must have $\frac{df}{ds}\Big|_{\hat{u}} = 0$

$$\frac{df}{ds}\Big|_{\hat{u}} = \nabla f \cdot \hat{u}$$



so $\nabla f \perp$ any sun $\hat{u}$

so $\nabla f \parallel \nabla g$ on the level surface

$\nabla f \perp$ level set of g

f should be just one point

so $\frac{df}{ds}\Big|_{\hat{u}} = 0$

Warnning: the Method doesn't tell the solution Is a minimum or a maximum.

And We CAN'T USE SECOND derivatives

To find min (or max)

We compare values of f at the various solutions to lagrange multiplier equations

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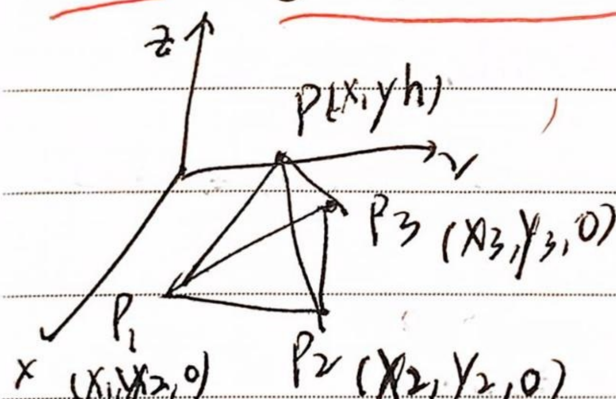
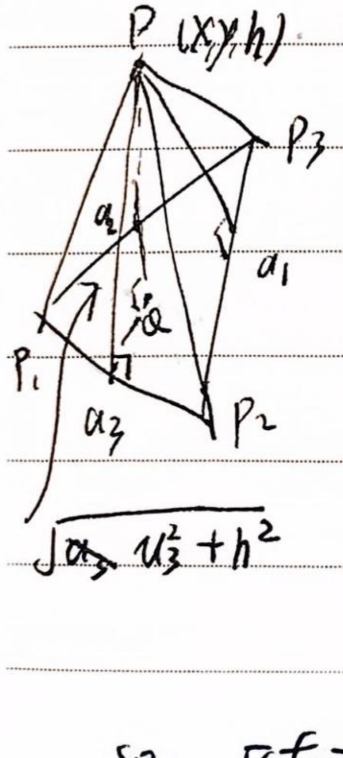
Advanced Example:

want to build a pyramid with given triangle base & given volume. minimize total ~~s~~ surface



$V = \frac{1}{3}\,\text{Area (base)} \cdot \text{height}\ (z)$

↑ fixed!



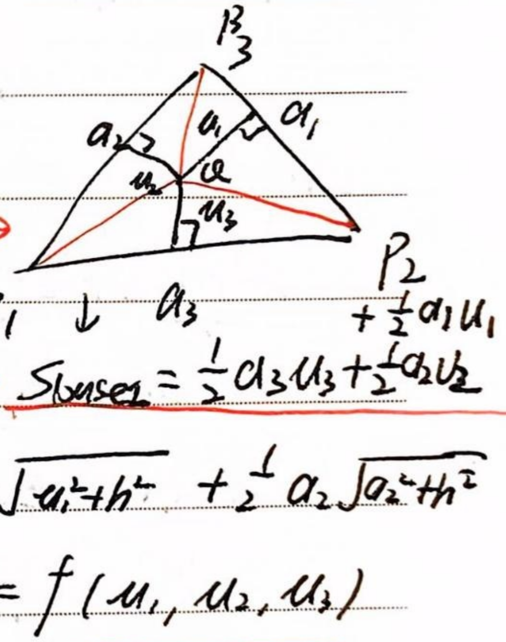
$\sqrt{a_3\, u_3^2 + h^2}$

$Q\ (x, y, 0)$

$\boxed{\text{find } f \text{ and } g}$

$f(u_1, u_2, u_3)$

$g(u_1, u_2, u_3)$ g



$S_{base} = \frac{1}{2}a_3 u_3 + \frac{1}{2}a_2 u_2 + \frac{1}{2}a_1 u_1$

so side area $= \frac{1}{2}a_1 \cdot \sqrt{u_1^2+h^2} + \frac{1}{2}a_2\sqrt{u_2^2+h^2} + \frac{1}{2}a_3\sqrt{u_3^2+h^2} = f(u_1, u_2, u_3)$

so $\nabla f = \lambda \nabla g$; $\frac{\partial f}{\partial u_1} = \frac{1}{2}a_1 \frac{u_1}{\sqrt{u_1^2+h^2}} = \frac{1}{2}a_1\lambda = \frac{\partial g}{\partial u_1}$

$\Downarrow$

$\frac{u_1}{\sqrt{u_1^2+h^2}} = \lambda$

$\frac{\partial f}{\partial u_2} = \lambda \cdot \frac{\partial g}{\partial u_2}$

$\Downarrow$

$\frac{u_2}{\sqrt{u_2^2+h^2}} = \lambda$, same: $\frac{u_3}{\sqrt{u_3^2+h^2}} = \lambda$,

$\Rightarrow u_1 = u_2 = u_3$

$f(u_1, u_2, u_3)$ can get critical point

so the Q should incenter

$\Rightarrow$ P should be incenter