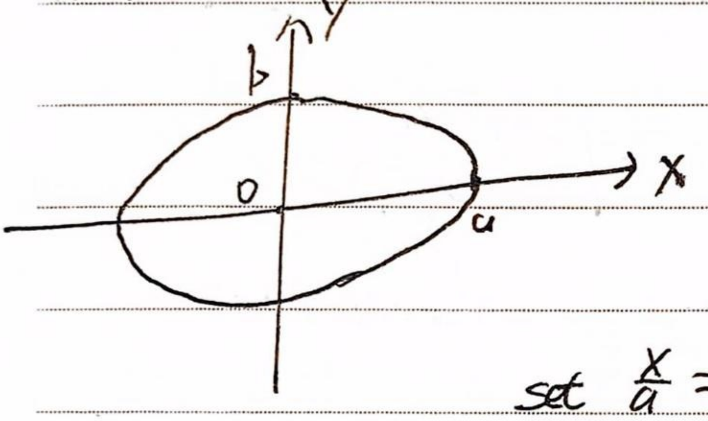
LEC18. 2025.2.16

Changing variable in $\iint$

Jacobin determinant

Example1: area of ellipse with semiaxes a, b



$$\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1$$

set $\frac{x}{a}=u$, $\frac{y}{b}=v \Rightarrow du=\frac{1}{a}dx$, $dv=\frac{1}{b}dy$

$dudv=\frac{1}{ab}dxdy$

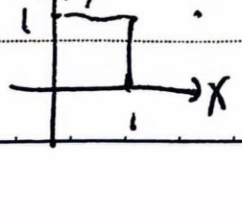
$$\iint_{\left|\frac{x}{a}\right|^2+\left|\frac{y}{b}\right|^2<1} dxdy = \iint_{u^2+v^2<1} ab\, dudv. = ab\iint_{u^2+v^2<1} dudv$$

$$= ab\cdot\pi$$

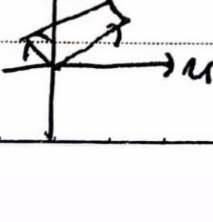
In general: find scaling factor (dx dy vs du dv)

Ex2: $u=3x-2y$, $v=x+y$

Relation betwen $dA=dxdy$ and $dA'=dudv$



$\Rightarrow$



$\Rightarrow A'=\left|\begin{vmatrix}3&1\\-2&1\end{vmatrix}\right|=5$, $A=1$

so $dA'=5dA$

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So $\iint \cdots dxdy = \iint \cdots \frac{1}{5} dudv$

General case:

$u = u(x,y) \qquad v = v(x,y)$

$\Delta u \approx u_x \cancel{\Delta u} \Delta x + u_y \Delta y$ $\qquad$ area' = det(⋯)ΔA

$\Delta v \approx v_x \Delta x + v_y \Delta y$ $\qquad \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \cdot \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}$

Jacobin: $\boxed{J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}}$

$dudv = |J| \cdot dxdy = \left|\frac{\partial(u,v)}{\partial(x,y)}\right| dxdy$

absolute value

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Ex: polar coordinates:

$x = r\cos\theta, \quad y = r\sin\theta$ $\qquad \left(\frac{\partial(x,y)}{\partial(r,\theta)}\right) = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix}$

$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$

$= r\cos^2\theta + r\sin^2\theta = r$

$\langle \Delta x, 0\rangle \rightsquigarrow \langle \Delta u, \Delta v\rangle \approx \langle u_x \Delta x, v_x \Delta x\rangle$

$\langle 0, \Delta y\rangle \rightsquigarrow \langle \Delta u, \Delta v\rangle \approx \langle u_y \Delta y, v_y \Delta y\rangle$ ← (S) vector

$dA = \Delta x \cdot \Delta y = \Delta A$ $\quad$ $\text{area}' = \det\left(\frac{\partial(u,v)}{\partial(x,y)}\right)\cdot \Delta x \Delta y$

why: (vector wed)

$\Delta A = \Delta x \cdot \Delta y \qquad \Delta A' = \begin{vmatrix} u_x\Delta x & v_x \Delta x \\ u_y \Delta y & v_y \cdot \Delta y\end{vmatrix} = \Delta x\cdot\Delta y\cdot\begin{vmatrix} u_x & v_x \\ u_y & v_y\end{vmatrix}$

$= \det\left(\frac{\partial(u,v)}{\partial(x,y)}\right)\cdot \Delta x \cdot \Delta y$

$\Delta \to d, \quad dA' = \det\left(\frac{\partial(u,v)}{\partial(x,y)}\right) dx\,dy$

Jacobian determinant

$\left|\frac{\partial(u,v)}{\partial(x,y)}\right| \cdot \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = 1$ (inverse matrix)

~~so we can~~

Ex2: compute $\int_0^1\int_0^1 x^2 y\, dx\, dy$ by chaning to $u = x$, $v = xy$

① area elements:

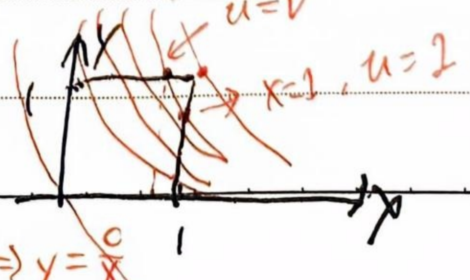
$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & 0 \\ y & x \end{vmatrix} = x \Rightarrow du\,dv = x\,dx\,dy$

② Integrand in terms of u, v:

$x^2 y\, dx\, dy = xy\, du\, dv = v\, du\, dv$

③ $\int_0^1 \boxed{\int_v^1 v\, du}\, dv$

↓ keep v constant, u change: → $xy = \text{constant} \Rightarrow y = \frac{c}{x}$



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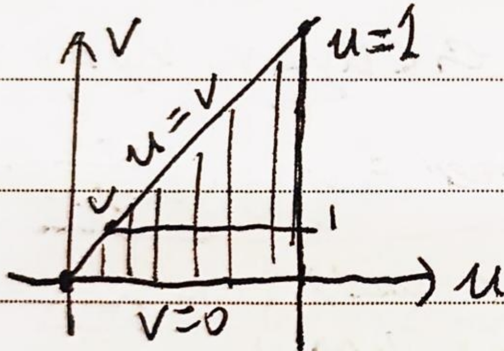
or switch to uv picture



u: $v \to 1$ (v constant

v: $0 \to 1$ (u constant)

Unit3 Part B: Vector Fields and Line Integrals

LEC19 2025.2.18

Vector fields

$\vec{F} = M\vec{i} + N\hat{j}$, M and N are function of x,y.

at each point, $\vec{F}$ a vector that depend on (x,y)