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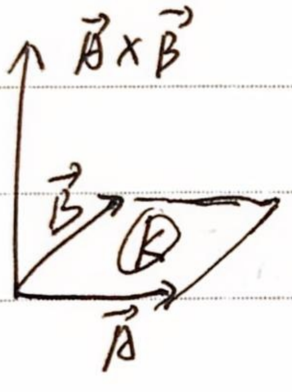
session 1: $\boxed{\text{Cross Products 1}}$ 2025.1.6.

$\boxed{\text{cross-product}}$ of 2vectors in 3-space

$$\vec{A}\times\vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\vec{k}$$

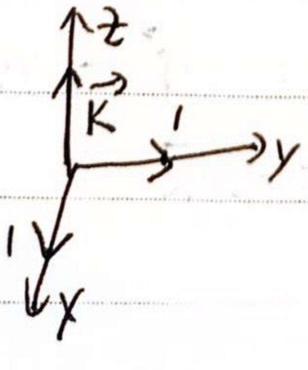
definition

is a vector



- $|\vec{A}\times\vec{B}|$ is the area of parallelogram K
- dir$(\vec{A}\times\vec{B})$ = $\perp$ to plane of the parallelogram with right hand rule.

Ex: $\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 0\vec{i} - 0\vec{j} + \vec{k} = \vec{k}$



Another look at volume

$$V = \text{base} \cdot \text{height}$$

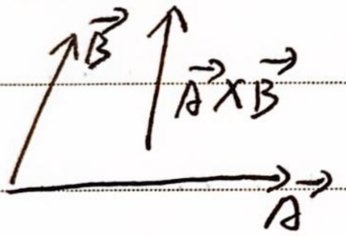
the direction $\perp$ to $\vec{B}$ and $\vec{C}$

$$= \underbrace{|\vec{B}\times\vec{C}|}_{\text{base}} \cdot \underbrace{(\vec{A}\cdot\hat{n})}_{\text{hight}} \qquad \hat{n} = \frac{(\vec{B}\times\vec{C})}{|\vec{B}\times\vec{C}|}$$

$$= |\vec{B}\times\vec{C}| \cdot \left(\frac{\vec{A}\cdot(\vec{B}\times\vec{C})}{|\vec{B}\times\vec{C}|}\right)$$

$$\therefore \boxed{\det(\vec{A},\vec{B},\vec{C}) = \vec{A}\cdot(\vec{B}\times\vec{C})} = V$$

Direction $\perp$ to $\vec{A}$ and $\vec{B}$



$$\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$$

$$\vec{A}\times\vec{A} = 0$$

Reading: Cross Product $\times$

also called vector product

Example:

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & 0 \\ 3 & -2 & 0 \end{vmatrix} = 0\cdot\vec{i} - 0\cdot\vec{j} + (-4-9)\cdot\vec{k}$$

$$= -13\vec{k}$$

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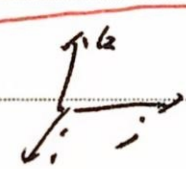
Algebraic facts:

1. $\vec{A}\times\vec{A}=0$
2. $\vec{A}\times\vec{B}=-\vec{B}\times\vec{A}$
3. $\vec{A}\times(\vec{B}+\vec{C})=\vec{A}\times\vec{B}+\vec{A}\times\vec{C}$
4. $(\vec{A}\times\vec{B})\times\vec{C}\neq\vec{A}\times(\vec{B}\times\vec{C})$
5. $\vec{i}\times\vec{j}=\vec{k},\quad \vec{j}\times\vec{k}=\vec{i},\quad \vec{k}\times\vec{i}=\vec{j}$



Example:

$\cancel{(2\vec{i}+3\vec{j})\times(3\vec{j}-2\vec{j})=}$

$$(2\vec{i}+3\vec{j})\times(3\vec{i}-2\vec{j})=(2\vec{i}\times3\vec{i})-(2\vec{i}\times2\vec{j})$$
$$+(3\vec{j}\times3\vec{i})-(3\vec{j}\times2\vec{j})$$
$$=5\cdot0-4\cdot\vec{k}+9(-\vec{k})-6\cdot0$$
$$=-13\vec{k}$$

Geometric description:

the magnitude of $\vec{A}\times\vec{B}$ is

$$|\vec{A}\times\vec{B}|=|\vec{A}|\cdot|\vec{B}|\cdot\sin\theta$$

= area of the sparallelogram spanned by $\vec{A}$ and $\vec{B}$

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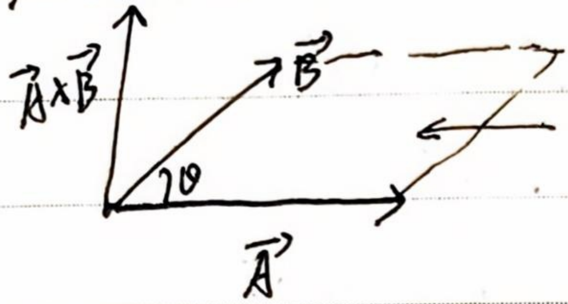
The direction:

$\vec{A} \times \vec{B}$ is perpendicular to the plane of $\vec{A}$ and $\vec{B}$, the direction is decided by right-hand rule



$|\vec{A} \times \vec{B}| = area = |\vec{A}| \cdot |\vec{B}| \cdot \sin\theta$

we will not go through the prof, it make use of Lagrange identity

$$\boxed{|\vec{A} \times \vec{B}|^2 = |\vec{A}|^2 |\vec{B}|^2 - (\vec{A} \cdot \vec{B})^2}$$

Example: Find the area of triangle

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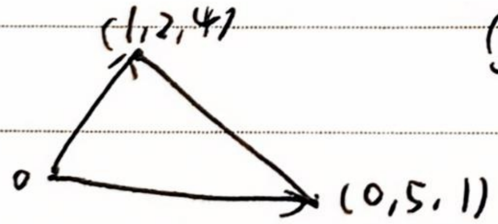

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①

$area = \frac{1}{2} | \langle 1,2,4 \rangle \times \langle 0,5,1 \rangle |$

$= \frac{1}{2} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 4 \\ 0 & 5 & 1 \end{vmatrix} = -18\vec{i} - \vec{j} + 5\vec{k}$

$\Rightarrow area = \frac{1}{2} \cdot \sqrt{18^2 + 1^2 + 25} = \frac{1}{2}\sqrt{350}$

②

$|\vec{A} \times \vec{B}| = \sqrt{21 \cdot 26 - 18^2 + 25}$

$= \sqrt{350}$

$area = \frac{1}{2}\sqrt{350}$

$1+4+16$

$\sqrt{21} \cdot 26$

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