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Session 32: Total Differentials and the Chain Rule

Lec 11:

More tools to study Functions

2025.1.12

total differential $f(x, y, z)$

$$df = f_x dx + f_y dy + f_z dz$$

different things

$$\Delta f \approx f_x \Delta x + \Delta y \cdot f_y + f_z \Delta z$$

a number

Importent: df is NOT Δf

$$\frac{df}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} + f_z \frac{dz}{dt} \quad \text{(chain rule)}$$

proof:

$$\frac{\Delta f}{\Delta t} \approx \frac{f_x \Delta x + f_y \Delta y + f_z \cdot \Delta z}{\Delta t} \quad \text{in time } \Delta t$$

$\Rightarrow$

when $\Delta t \to 0$: $\frac{\Delta f}{\Delta t} \to \frac{df}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} + f_z \frac{dz}{dt}$

Example: $w = x^2 y + z$, $x = t$, $y = e^t$, $z = \sin t$

$$\Rightarrow \frac{dw}{dt} = 2xy \cdot \frac{dx}{dt} + x^2 \cdot \frac{dy}{dt} + \frac{dz}{dt}$$

$$= 2te^t \cdot 1 + t^2 e^t + \cos t$$

$$= 2te^t + t^2 e^t + \cos t$$

Application: Justify Product rule.

$$f = uv, \quad u = u(t), \quad v = v(t)$$

$$\frac{d(uv)}{dt} = f_u \cdot \frac{du}{dt} + f_v \cancel{\frac{du}{}}\frac{dv}{dt} = v\frac{du}{dt} + u \cdot \frac{dv}{dt}$$

Chain rule with more variables!

$$w = f(x, y), \quad \text{where } x = x(u,v),\ y = y(u,v)$$

$$= f(x(u,v), y(u,v))$$

$$dw = f_x\, dx + f_y\, dy$$

$$= f_x(x_u \cdot du + x_v \cdot dv) + f_y(y_u\, du + y_v\, dv)$$

$$= \underbrace{(f_x x_u + f_y y_u)}_{\partial f/\partial u} du + \underbrace{(f_x x_v + f_y y_y)}_{\partial f/\partial v} dv$$

$\Downarrow$

so $$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$$ chain rule!

can't simplify!

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial y}$$

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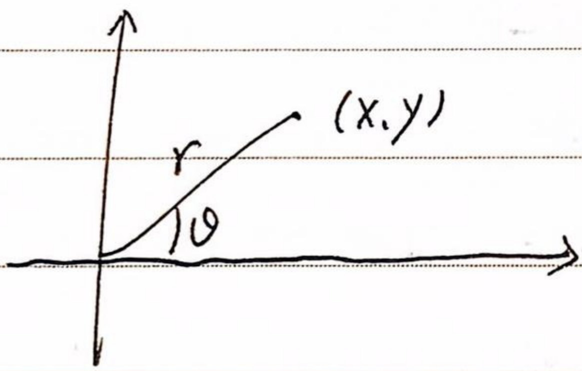
Example: polar coordinate



$x = r\cos\theta$

$y = r\sin\theta$

$f = f(x,y)$ $\quad \frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r}$

$= f_x \cos\theta + f_y \sin\theta$

$= \cdots$

GRADIENT VECTOR (梯度向量)

$\nabla f = \langle f_x, f_y, f_z \rangle$