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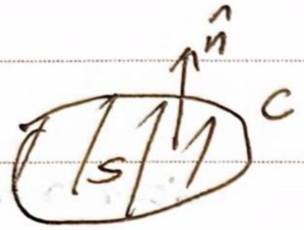
LEC 32 stokes' theorem cont 2025.1.25

Stokes:

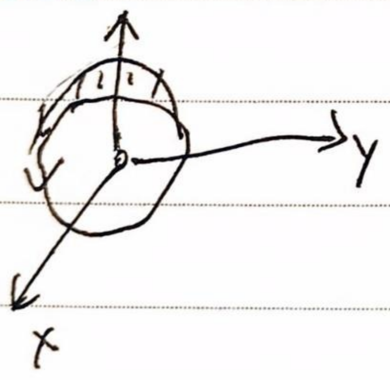


$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\text{curl}\,\vec{F})\cdot \hat{n}\,dS$$

STOKES AND PATH-INDEPENDENCE

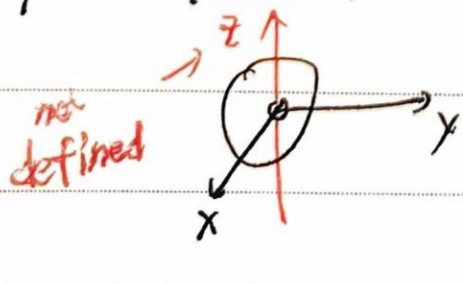
Defin: A region is simply-connected if every closed loop inside it bounds a surface inside it

Example:



space origin removed is a simply-connected (push the circle up a little bit)

space w/ z-axis removed, is not simply connected



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Recall: if $\vec{F} = \nabla f$ is a gradient, then $\text{curl} \vec{F} = 0$

Theorem: if $\text{curl} \vec{F} = 0$, and $\vec{F}$ defined in simply-connected region, then $\vec{F}$ is gradient field and ~~fa~~ $\int_C \vec{F} \cdot d\vec{r}$ is path-independent

Proof: assume $\text{curl} \vec{F} = 0$,

$$\boxed{\int_{C_1} \vec{F} \cdot d\vec{r} - \int_{C_2} \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot d\vec{r} \quad C = C_1 - C_2}$$

can find S because region is simply connected

$$\boxed{= \iint_S (\underbrace{\text{curl} \vec{F}}_{0}) \, d\vec{S} = 0}$$

Remark: Topology classifies surfaces in space:

Stokes and "surface independence" (Proof)

stokes:

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_{S_1} (\nabla \times \vec{F}) \cdot \hat{n} \, dS = \iint_{S_2} (\nabla \times \vec{F}) \cdot \hat{n} \, dS$$

$$\iint_{S_1} (\nabla \times \vec{F}) \cdot \hat{n} \, dS - \iint_{S_2} = \iint_{S = S_1 - S_2} (\nabla \times \vec{F}) \cdot \hat{n} \, dS$$

$$= \iiint_D \text{div} (\nabla \times \vec{F}) \, dV$$

by divergence thm

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Can check: $\operatorname{div}(\nabla \times \vec{F}) = 0$ always!

$\vec{F} = \langle P, Q, R \rangle$; $\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix} = \langle R_y - Q_z, \; P_z - R_x, \; Q_x - P_y \rangle$

$$\operatorname{div}(\nabla \times \vec{F}) = (R_y - Q_z)_x + (P_z - R_x)_y + (Q_x - P_y)_z$$

$$= \cancel{R_{yx}} - \cancel{Q_{zx}} + \cancel{P_{zy}} - \cancel{R_{xy}} + \cancel{Q_{xz}} - \cancel{P_{yz}}$$

$$= 0$$

$\nabla \cdot (\nabla \times \vec{F}) = 0$ always

[Note: for "real" vectors, $u \cdot (u \times v) = 0$

$\boxed{\text{TOPICS: For UNIT 4}}$

$\boxed{\iiint_R f\,dV}$ - rect: $dV = dx\,dy\,dz$

- cyl: $dV = dz\, r\, dr\, d\theta$

- spherical: $dV = \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$

$\iiint$ application: - Mass , - Avg value of f

- moment of inertia

- Gravitational attraction on mass at 0

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$\boxed{\iint_S \vec{F}\cdot\underbrace{\hat{n}\,dS}_{d\vec{S}}}$ ⇒ Formulus for $\hat{n}\cdot dS$

ex: $\hat{n}\,dS = \cdots dx\,dy$

(flux of S) becomes $\iint \cdots dx\,dy$

$\hat{n}\,dS$ · horizental plane   y-z plane $\hat{n} = \pm\hat{i}$

· sphere $\hat{n} = \frac{1}{a}\langle x, y, z\rangle \pm$, cylinder $\hat{n} = \pm\frac{\langle x,y\rangle}{a}$

$d\vec{S} = a^2 \sin\phi\, d\phi\, d\theta$ $d\vec{S} = a\,dz\,d\theta$

+ general case: · $z = z(x,y)$, $\hat{n}\,dS = \langle -z_x, -z_y, 1\rangle\,dx\,dy$

· $\vec{N}$ given normal vector: $\hat{n}\,dS = \pm\frac{\vec{N}}{\vec{N}\cdot\vec{k}}\,dx\,dy$

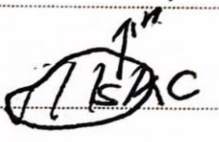
$\boxed{\int_C \vec{F}\cdot d\vec{r} = \int_C P\,\cancel{dt}\,dx + Q\,dy + R\,dz}$

parameteize C → express in single variable

$\iiint_R f\,dV \xleftrightarrow{\text{divergence thm}} \iint_S \vec{F}\cdot\hat{n}\,dS \xleftrightarrow{\text{stokes thm}} \int_C \vec{F}\cdot d\vec{r}$

divergence thm

$\iint_S \vec{F}\cdot\hat{n}\,dS = \iiint_D (\mathrm{div}\,\vec{F})\,dV$



$\oint \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot\hat{n}\,dS$ stokes' theorem

↑ any S boundary by C

$f(P_1) - f(P_0) = \int_C (\nabla f)\cdot d\vec{r}$

curl=0, find poential