LEC 27 2018.1.23

Vector Fields in space

work and flux

$\vec{F} = \langle x, y, z \rangle$

Example: force field (gravitational attraction of a solid mass at $(0,0,0)$ on a mass m at (x, y, z)

$\vec{F}$ direct towards origin, magnititude $\frac{c}{\rho^3}$

$\vec{F} = -c \frac{\langle x, y, z \rangle}{\rho^3}$

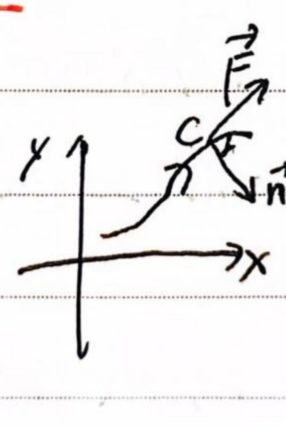
Electric field, ...

b) Velocity Field

c) gradient Fields $u = u(x, y, z)$ $\nabla u = \langle u_x, u_y, u_z \rangle$

Flux:

recall, in 2D



$\int_C (M\,dy - N\,dx)$

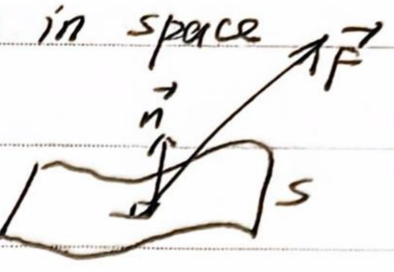
flux $= \int_C \vec{F} \cdot \hat{n}\, ds$

$(= \iint_R (M_x + N_y)\, dA$

In 3D, flux through a surface

Ex: $\vec{F}$ a vector field, S surface in space

$\hat{n}$ = unit normal vector



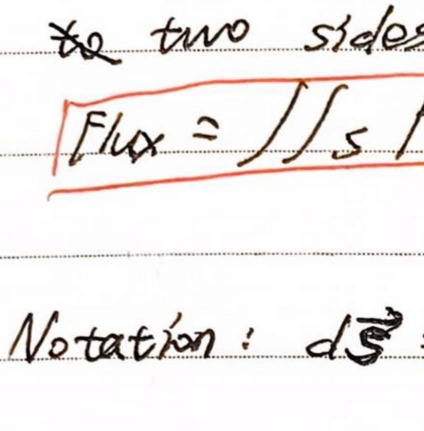
~~to~~ two sides of surface (often is up)

$$\boxed{\text{Flux} = \iint_S \vec{F} \cdot \hat{n} \cdot \underbrace{dS}_{\text{surface area element}}}$$

Notation: $d\vec{S} = \hat{n} \cdot dS$

Example: (1) Flux of $\vec{F} = \langle x, y, z \rangle$ through sphere of radius a at origin

1)



$$\iint_S \vec{F}\, d\vec{S} = \iint_S \vec{F} \cdot \hat{n}\, dS$$

$$\hat{n} = \frac{\langle x, y, z \rangle}{a} \qquad \vec{F} \,//\, \vec{n}$$

$(\sqrt{x^2+y^2+z^2} = a$ on sphere$)$

$$\vec{F} \cdot \hat{n} = |\vec{F}| = \sqrt{x^2+y^2+z^2} = a$$

$$= \iint_S a\, dS = a \iint_S dS = a \cdot 4\pi a^2 = 4\pi a^3$$

2) Same sphere, $\vec{H} = z\hat{k}$

$$\vec{H} \cdot \vec{n} = \langle 0, 0, z \rangle \cdot \frac{\langle x, y, z \rangle}{a} = \langle 0, 0, \frac{z^2}{a} \rangle$$

$\Rightarrow \iint_S \vec{H} \cdot \hat{n}\, dS = \iint_S \frac{z^2}{a} dS \qquad dS = a^2 \sin\phi \, d\phi\, d\theta$ (use spherical coordinates)

$z = a\cos\phi$

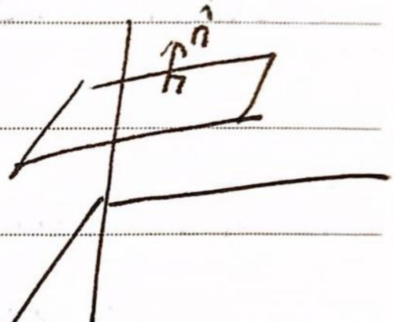
$$\Rightarrow \iint_S \frac{z^2}{a}\,dS = \int_0^{2\pi}\int_0^{\pi} \frac{a^2\cos^2\phi}{a}\cdot a^2\sin\phi\, d\phi\, d\theta$$

$$= \left(a^3\cdot\left[-\tfrac{1}{3}\cos^3\phi\right]_0^{\pi}\right)\cdot 2\pi$$

$$= \tfrac{4}{3}\pi a^3$$

Conclusions

① use geometry or need to set up $\iint_S \vec{F}\cdot\hat{n}\,dS$



0) S = Horizontal plane $z=a$

$\hat{n} = \pm\hat{k}$, $dS = dx\,dy$

0.1) vertical plane // y-z plane

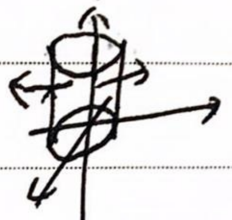
$x=a \Rightarrow$, $\hat{n} = \pm\hat{i}$, $\boxed{dS = dy\,dz}$

1) sphere of radius a centred origin

$\hat{n} = \langle x, y, z\rangle / a$

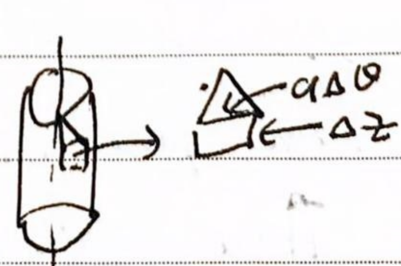
$\boxed{dS = a^2\sin\phi\, d\phi\, d\theta}$ $\quad \begin{cases} z = a\cos\phi \\ x = a\cos\theta \\ y = a\sin\theta \end{cases}$ ~~spherica~~ spherical coordinates

2) cylinder of radius a / z-axis



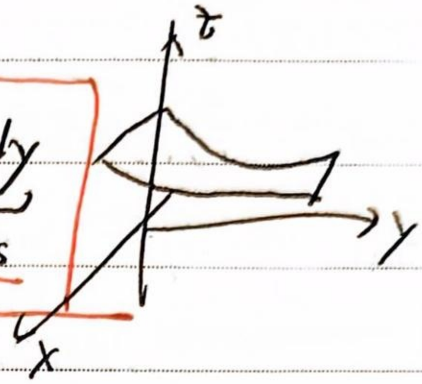
$\hat{n} = \frac{1}{a}\langle x, y, 0\rangle$

$\boxed{dS = a\,dz\,d\theta}$



(3) graph $z = f(x,y)$

$$\boxed{\hat{n}\,dS = \underbrace{\langle -f_x, -f_y, 1 \rangle}_{\text{not } \hat{n}}\ \underbrace{dx\,dy}_{\text{not } dS}}$$



To set up bounds $\iint \cdots dx\,dy$, look at shadow of S in xy-plane

Geometric interpretation:

If $\vec{F}$ is a velocity field, Flux = amount of matter through S per unit time.

ΔS, $v = \vec{F}$.