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Section 30. Second Derivative Test

2025.1.11

from recalling one-variable

$f'(x_0)=0$

$f''(x_0)>0 \Rightarrow$ local ~~max~~ minimum point

$f''(x_0)<0 \Rightarrow$ local maximum point

geometric possibility

another critical point in $f(x,y)$ called saddle point (not max or min)

(partly above, partly below the tagent plane)

Second Derivative Test

1. Find x, y $\begin{cases} f_x(x,y)=0 \\ f_y(x,y)=0 \end{cases}$

2. notation: $(f)_0 = f(x_0, y_0)$

(1) $\boxed{A=(f_{xx})_0,\ B=(f_{xy})_0=\cancel{f(yx)}\ (f_{yx})_0,\ C=(f_{yy})_0}$

Then:

$AC-B^2>0$, $A>0$ or $C>0 \Rightarrow (f)_0$ is minimum

$AC-B^2>0$, $A<0$ or $C<0 \Rightarrow (f)_0$ is maximum

$AC-B^2<0 \Rightarrow (f)_0$ is a saddle point

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if $AC-B^2=0$, the test fail

Ex1:

$$w = 12x^2+y^3-12xy$$

$$\begin{cases} w_x = 24x-12y \\ w_y = 3y^2-12x \end{cases} \quad A = w_{xx} = 24,\ B = w_{xy} = -12, \quad C = w_{yy} = 6y$$

$\Downarrow$

$w_x, w_y = 0$

$\Downarrow$

$(x,y) = (0,0), (1,2)$ $\qquad AC-B^2 = 144y-144$

at $(0,0)$ $AC-B^2 = -144 \Rightarrow$ a saddle point

at $(1,2)$ $AC-B^2 = 144$, $A>0 \Rightarrow$ minimum point

Proof:

(3) $Ax^2+Bx+C=0 \Rightarrow x = \dfrac{-B\pm\sqrt{B^2-4AC}}{2A}$

(4) $Ax^2+2Bx+C=0 \Rightarrow x = \dfrac{-B\pm\sqrt{B^2-AC}}{A}$

(4) $\Rightarrow$ x no roots, so f(x) must all >0 or <0, it depends on A's single

$\Rightarrow$ (5) $AC-B^2>0$, $A>0$ or $C>0 \Rightarrow Ax^2+2Bx+C>0$ for all x

(6) $AC-B^2>0$, $A<0$ or $C<0 \Rightarrow Ax^2+2Bx+C<0$ for all x

(7) $AC-B^2<0 \Rightarrow Ax^2+2Bx+C>0, <0$ for some x

$\Rightarrow$ (5) so ~~f(x,y) for all~~, $f'(x,y)\uparrow$, the point is minimum

(6) $f'(x,y)\downarrow$, the point is maximum

(7) in point, $f'(x,y)\downarrow\uparrow$ or $\uparrow\downarrow$, is a saddle point

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Problems: Find and classify all the critical points of

$w = (x^3+1)(y^3+1) = x^3 \cdot y^3 + x^3 + y^3 + 1$

$$\cancel{\frac{\partial w}{\partial x}} \begin{cases} \dfrac{\partial w}{\partial x} = 3x^2y^3 + 3x^2 = 0 \Rightarrow y = -1 \quad (x \neq 0) \\ \dfrac{\partial w}{\partial y} = 3x^3y^2 + 3y^2 = 0 \Rightarrow x = -1 \end{cases}$$

so critial poins: $(0,0)$ and $(-1,-1)$

$(0,0)$:

$A = \left(\frac{\partial^2 w}{\partial x^2}\right|_0 = (w_{xx})_0 = 6x \cdot y^3 + 6x = 0$

$B = (w_{xy})_0 = 9x^2y^2 = 0$

$C = (w_{yy})_0 = 6x^3y + 6y = 0$

$AC - B^2 = 0$ test fill

$(-1,-1)$

$\Rightarrow A = 0,\ B = 9,\ C = 0$

$AC - B^2 = -81 < 0,\ A = 0,\ C = 0$

the point is a saddle point