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Session 11 Martix Inverse

2025.1.8

Inverse of $\vec{A}$: $AM = I$, $MA = I$

$M = A^{-1}$ $AX = B$, $X = B \cdot A^{-1}$

$$\boxed{\text{Formula: } A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)}$$

↑ adjoint (伴随)

Example:

$$A = \begin{bmatrix} 2 & 3 & 3 \\ 2 & 4 & 5 \\ 1 & 1 & 2 \end{bmatrix}$$

(1) minors:

$\begin{vmatrix} 4 & 5 \\ 1 & 2 \end{vmatrix}$ (余子式)

$$= \begin{bmatrix} (3) & -1 & -2 \\ 3 & 1 & -1 \\ 3 & 4 & (2) \end{bmatrix}$$

↑ $\begin{vmatrix} 2 & 3 \\ 2 & 4 \end{vmatrix}$

(2) Cofactors:

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix} \Rightarrow \begin{bmatrix} 3 & 1 & -2 \\ -3 & 1 & 1 \\ 3 & -4 & 2 \end{bmatrix}$$

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(3) Transpose:

switch rows & columns.

$$\begin{bmatrix} 3 & 1 & -2 \\ -3 & 1 & 1 \\ 3 & -4 & 2 \end{bmatrix} \xrightarrow{\text{transpose}} adj(A) = \begin{bmatrix} 3 & -3 & 3 \\ 1 & 1 & -4 \\ -2 & 1 & 2 \end{bmatrix}$$

(4) Divided by determinant of A

$$\begin{vmatrix} 2 & 3 & 3 \\ 2 & 4 & 5 \\ 1 & 1 & 5 \end{vmatrix} = 3, \quad A^{-1} = \frac{1}{3}\begin{bmatrix} 3 & -3 & 3 \\ 1 & 1 & -4 \\ -2 & 1 & 2 \end{bmatrix}$$

Reading:

Ex: $a_{11}x_1 + a_{12}x_2 = b_1$

$a_{21}x_1 + a_{22}x_2 = b_2$

$A\vec{x} = \vec{B}$, $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ $b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{23} \end{pmatrix}$

nxn system also can be written as:

$\vec{A} \cdot \vec{x} = \vec{b}$, $\vec{A}: (a_{ij})$

In Inverse Matrices:

$Ax = b$, set $MA = I$

$M \cdot (Ax) = M \cdot b$

$\underset{I \cdot x}{\shortparallel}$ $x = M \cdot b$

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Example 2.1

Let $A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$ and $M = \begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix}$

$MA = I \quad \begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

find the solution of systems, (2) the y_i in terms of x_i

(1) $x_1 + 2x_2 = -1$ (2) $x_1 = y_1 + 2y_2$

$2x_1 + 3x_2 = 4$ $\quad x_2 = 2y_1 + 3y_2$

(1): $\vec{b} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$

$A \cdot x = b$

$x = M \cdot b = \begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 11 \\ -6 \end{pmatrix}$

so the solution: $x_1 = 11$, $x_2 = -6$

(2) $x = Ay$

so $y = M \cdot x \Rightarrow \begin{cases} y_1 = -3x_1 + 2x_2 \\ y_2 = 2x_1 - x_2 \end{cases}$

How we can get $\vec{M}$?

first M exist $\Leftrightarrow$ $|\vec{A}| \neq 0$

because $MA = I \Rightarrow |M||A| = |I| = 1 \Rightarrow |A| \neq 0$

M's proper name A^{-1}

Defination:

$\vec{A}$: $n \times n$, $|\vec{A}| \neq 0$

the the inverse of $\vec{A}$ is an $n \times n$ matrix,

written A^{-1}, $A \cdot A^{-1} = I_n$, $A^{-1} A = I_n$

auxiliary matrix adj A called the adjoint

of A :

$$A^{-1} = \frac{1}{|A|} \operatorname{adj}(A) = \frac{1}{|A|} \cdot \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{34} \end{pmatrix}^T$$

steps:

1, calculate the matrix of minors

2, change the signs of the entries according to the checkboard rule $(-1)^{i+j}$

3, Transpose the resulting, this gives $\operatorname{adj}(A)$

4, Divided ~~by~~ every entry by $|A|$

The formula of 2x2

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \xrightarrow{T} \overbrace{\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}^{adj(A)} \rightarrow \frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Ex: a) $\begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix}$

$|A| = 2, \; A^{-1} = \frac{1}{2} \cdot \begin{pmatrix} 2 & 0 \\ -3 & 1 \end{pmatrix}$

$$A^{-1} = \frac{1}{|A|} \cdot adj(A)$$

Proof:

$A \cdot A^{-1} = \frac{1}{|A|} \, adj(A) \cdot A$

$|A| \cdot I = adj(A) \cdot A$

$\Rightarrow$ just prove $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \cdot \begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix} = \begin{pmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{pmatrix}$

$\Downarrow$

prove: $a_{11} A_{11} + a_{12} A_{12} + a_{13} A_{13} = |A|$

$a_{11} A_{21} + a_{12} A_{22} + a_{13} \cdot A_{23} = 0$

just $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = |A|$

$\Downarrow \; = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ \boxed{a_{11} \;\; a_{12} \;\; a_{13}} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

$\Downarrow$

$= 0$

notice we don't need to care what's actually in the second row, because A_{21}, A_{22}, A_{23} calculate don't need the second row