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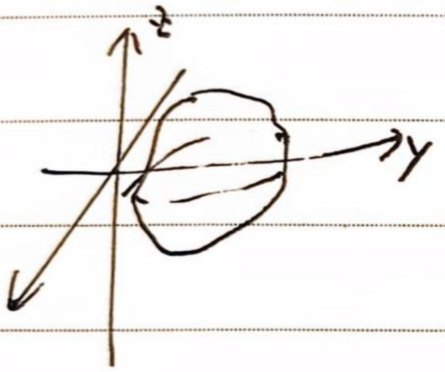
# Unit 4

## 2025.2.24. Triple Integrals: Rectangular and Cylindrical Coordinates

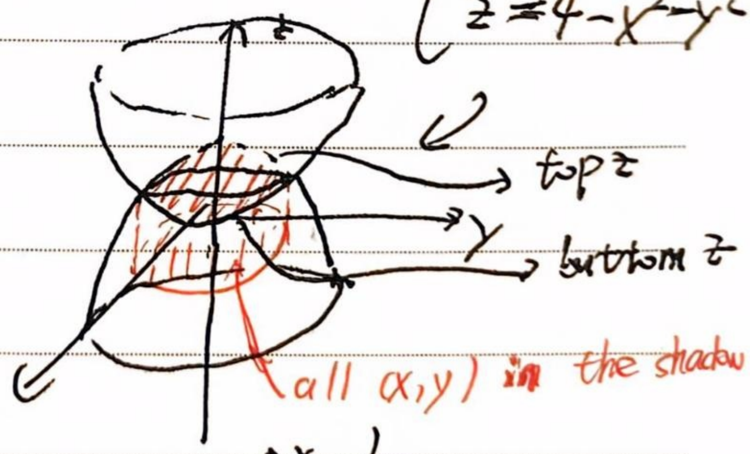
~~Trible~~ Triple Integral




$\iiint_R f \, dV$

↳ ex: $dx\,dy\,dz$

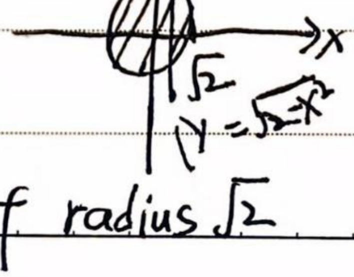
Example: region between paraboloids $\begin{cases} z = x^2+y^2 \\ z = 4-x^2-y^2 \end{cases}$




volume $\iiint 1 \, dV$

$\Rightarrow \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_{x^2+y^2}^{4-x^2-y^2} dz\,dy\,dx$

Find shadow in xy-plane?




Where $z$ buttom $< z$ top

$x^2+y^2 < 4-x^2+y^2$

how to find x-y plane shadow $\Rightarrow x^2+y^2<2$ disk of radius $\sqrt{2}$

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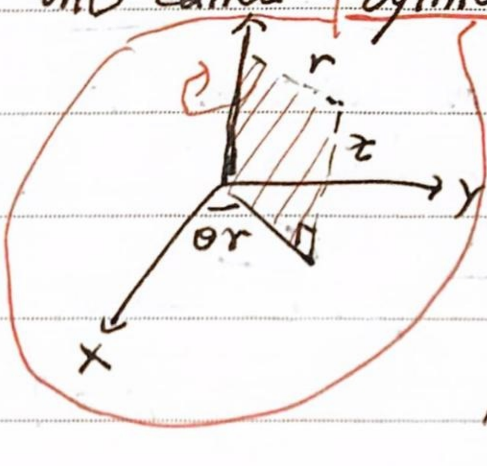
Better: use polar coordinates instead of $xy$

Inner: $\int_{x^2+y^2}^{4-x^2-y^2} dz = [z]_{x^2+y^2}^{4-x^2-y^2} = 4-2x^2-2y^2$

$\int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} 4-2x^2-2y^2 \, dy\,dx \Rightarrow$ switch to ~~pol~~ ~~pl~~ polar coordinates

$\Rightarrow \int_0^{2\pi} \int_0^{\sqrt{2}} \int_{r^2}^{4-r^2} dz \cdot r\,dr\,d\theta$ (easier to evaluate)

this called $\boxed{\text{cylindrical coordinates}}$

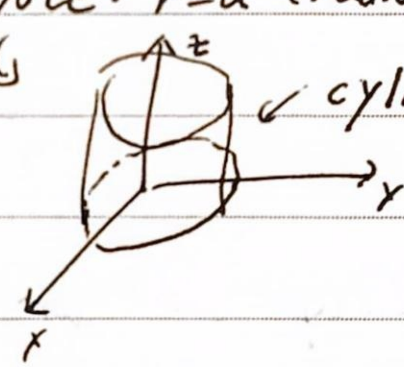



cylindrical

$(r, \theta, z)$

$x = r\cos\theta$, $y = r\sin\theta$

Note: $r = a$ (radius), $\boxed{r > 0}$




cylinder

Applications: - Mass ; density $\delta = \frac{\Delta m}{\Delta V}$

$dm = \delta \cdot \Delta V$

$\text{mass} = \iiint_R \delta \, dV$

- Average value of $f(x,y,z)$ in $R$:

$\bar{f} = \frac{1}{vol(R)} \iiint_R f \, dV$

or, with density (weighted avg) $\frac{1}{mass(R)} \iiint_R f \delta \, dV$

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- Center of mass: $(\bar{x}, \bar{y}, \bar{z})$

$$\bar{x} = \frac{1}{mass} \iiint_R x \delta \, dV \qquad y, z \text{ as same}$$

- Moment of inertia: / axis $(m \cdot r^2)$

$$\iiint_R (\text{distance to axis})^2 \delta \, dV$$

$$I_z = \iiint_R r^2 \delta \, dV = \iiint_R (x^2+y^2) \delta \, dV$$

$$I_x = \iiint_R y^2+z^2 \delta \, dV, \quad I_y = \iiint_R x^2+z^2 \delta \, dV$$

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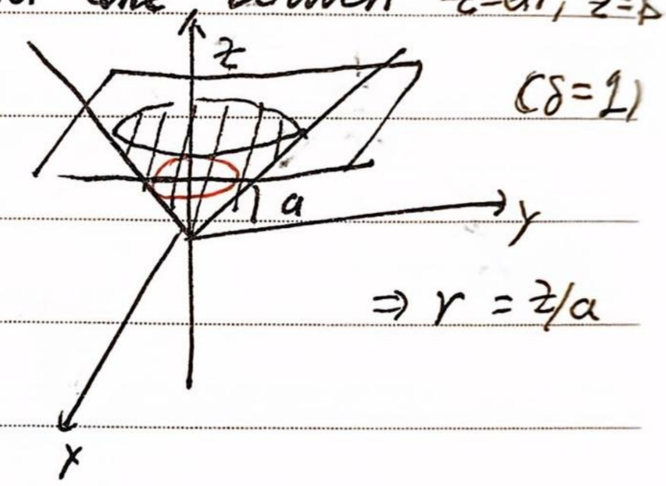
Example 2: $I_z$ of a solid cone between $z = ar$, $z = b$

$(\delta = 1)$

$$I_z = \iiint r^2 \cdot r \, dr \, d\theta \, dz$$

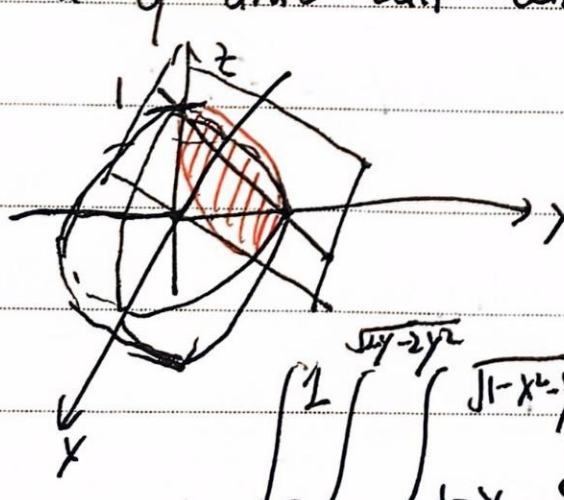
$$= \int_0^b \int_0^{2\pi} \int_0^{\frac{z}{a}} r^2 \cdot r \, dr \, d\theta \, dz$$

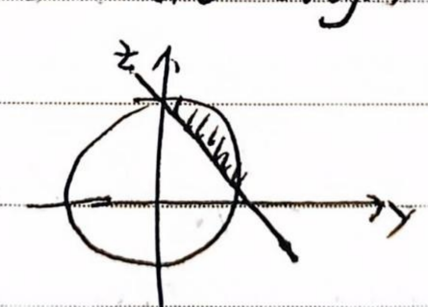
$$= \frac{\pi b}{10 a^4}$$

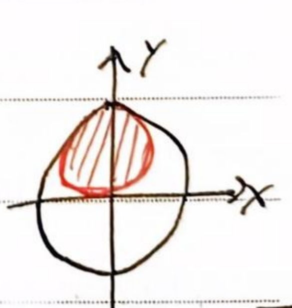



$\Rightarrow r = z/a$

Example 3: set up $\iiint$ for region $z > 1-y$ inside of unit ball centered at origin




$$\int_0^1 \int_{-\sqrt{2y-2y^2}}^{\sqrt{2y-2y^2}} \int_{1-y}^{\sqrt{1-x^2-y^2}} dz \, dx \, dy$$

$1-y < \sqrt{1-x^2-y^2}$

$\Downarrow$

$(1-y)^2 < 1-x^2-y^2$

$1-2y+y^2+y^2-1 < -x^2$

$x < \pm\sqrt{2y-2y^2}$ $x^2 < 2y-2y^2$