Session 28: Optimization Problems 2025.1.11

Critical point: $\frac{\partial f}{\partial x}(x_0, y_0) = 0$ and $\frac{\partial f}{\partial y}(x_0, y_0) = 0$

$\Rightarrow f_x = 0$ and $f_y = 0$

$\Rightarrow$ the tagent plane is horizontal $\Rightarrow f_x(x_0,y_0) = f_y(x_0,y_0) = 0$

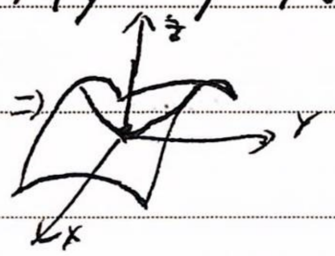
$z = z_0 + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$

$\Rightarrow z = z_0$

Not all critical point is maximum or minimum

Ex: $z = -x^2 + y^2$

$\frac{\partial z}{\partial x} = -2x$, $f_y = 2y$, thus critial point $\Rightarrow (0,0)$

but graph $\Rightarrow$ 

the point isn't maximun or minimum

ex: ~~a box~~

$w = \frac{12}{x} + \frac{16}{y} + 3xy \Rightarrow w_x = -\frac{12}{x^2} + 3y = 0$

$w_y = -\frac{6}{y^2} + 3x = 0$

$\Rightarrow x = 0, 2 \Rightarrow x = 2, y = 1$

$x = 0, y \Rightarrow$ undifined

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Problems:

1. Find all critical points

$$f(x,y) = x^6 + y^3 + 6x - 12y + 7$$

$f_x = 6x^5 + 6$, $f_y = 3y^2 - 12$

$f_x = 0 \Rightarrow x = -1$

$f_y = 0 \Rightarrow y = \pm \cancel{4}2$ so critical points:

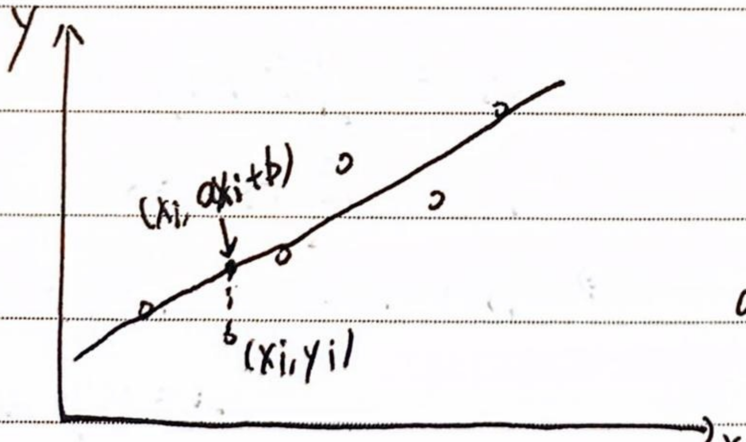
$(-1, -2)$, $(-1, 2)$

Session 29: Least Squares (最小二乘法)

2025.1.11

the lagrange interpolation formula

(拉格朗日插值公式)



we want to find a line "best" pass through the points

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$$\boxed{D = \sum_{i=1}^{n} (y_i - (ax_i + b))^2}$$ (Gaussian error distribution)

deviations ↑ only sum positive quantities

method of ~~least~~ least squares

$y = ax + b \Rightarrow$ the least squares line or the regression line

how to calculate a and b? $\Rightarrow$ make D a minimum

$$\Rightarrow \begin{cases} \frac{\partial D}{\partial a} = \sum_{i=1}^{n} 2(y_i - ax_i - b)\cdot(-x_i) = 0 \\ \frac{\partial D}{\partial b} = \sum_{i=1}^{n} 2(y_i - ax_i - b)\cdot(-1) = 0. \end{cases} \quad (3)$$

$$\Rightarrow \begin{cases} \left(\sum_{i=1}^{n} x_i^2\right) a + \left(\sum_{i=1}^{n} x_i\right) b = \sum_{i=1}^{n} x_i y_i \\ \left(\sum_{i=1}^{n} x_i\right) a + n\cdot b = \sum_{i=1}^{n} y_i \end{cases} \quad (4)$$

$\Rightarrow$ divided by n

$$\begin{cases} \underset{\text{avg of } x^2}{\bar{s}}\, a + \bar{x}\cdot b = \frac{1}{n}\sum x_i y_i \\ ~~x_i~~\ \bar{x}\cdot a + b = \frac{1}{n}\sum y_i = \bar{y} \end{cases} \quad (5)$$

2. Fitting curves by least squares. (通过最小二乘法拟合曲线) if the points seem to a curve rather than a line:

(6)

$$y = a_0 + a_1 x + a_2 x^2$$

$$D = \sum_{i=1}^{n} (y_i - (a_0 + a_1 x + a_2 x^2))^2$$

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use chain rule to get $\frac{\partial D}{\partial a_0}, \frac{\partial D}{\partial a_1}, \frac{\partial D}{\partial a_2}$

they can be solved by finding the inverse ~~matric~~ matrix or using a calculator on MatLab

In general, this method of least squares applies to a trial expression of the form

(9) $y = a_0 f_0(x) + a_1 f_1(x) + \cdots + a_r f_r(x)$

$f_i(x) \Rightarrow$ simple ones like $1, x, x^2, \frac{1}{x}, e^{kx}$ ...

sunch (9) called linear combination of the function $f_i(x)$

Examples:

Use the method of least squares to fit a line to the three data points.

$(0,0), (1,2), (2,1).$

$D = \Sigma (y_i - (ax_i + b))^2$

$D = (0 - (a\cdot 0 + b))^2 + (2 - (a+b))^2 + (1 - (a\cdot 2 + b))^2$

$= b^2 + (2-a-b)^2 + (1-2a-b)^2$

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$\Rightarrow \frac{\partial D}{\partial a} = -2(2-a-b) - 4(1-2a-b) = 0$

$\Rightarrow 5a+3b=4$

$\frac{\partial D}{\partial b} = 2b - 2(2-a-b) - 2(1-2a-b) = 0 \Rightarrow 6a+6b=6$

$\Rightarrow 3a+3b=3.$ so $a=\frac{1}{2}, b=\frac{1}{2} \Rightarrow y=\frac{1}{2}x+\frac{1}{2}$

Problems: Least Squares Interpolation

$(0,0)\quad (1,2)\quad (2,1),\quad (3,4)$

$y=ax+b$

$D=\Sigma\,(y_i-(ax_i+b))^2$

$D=(0-b)^2+(2-(a+b))^2+(1-(2a+b))^2$

$+(4-(3a+b))^2$

$\cancel{\partial D} = b^2+(2-a-b)^2+(1-2a-b)^2+(4-3a-b)^2$

$\cancel{\frac{\partial D}{\partial}}\ \frac{\partial D}{\partial a} = 2(2-a-b)\cdot -1 + 2(1-2a-b)\cdot -2 + 2(4-3a-b)\cdot -3$

$= -4+2a+2b\ \overset{-4+8a+4b}{\cancel{+(2-4a-2b)}}\ -24+18a+6b$

$= 28a+12b-32=0 \Rightarrow 7a+3b=8$

$\frac{\partial D}{\partial b} = 2b+2(2-a-b)\cdot -1 + 2(1-2a-b)\cdot -1 + 2(4-3a-b)\cdot -1$

$= 2b-4+2a+2b-2+4a+2b-8+6a+2b$

$= 12a+8b-14=0 \Rightarrow 6a+4b=7$

$b=\frac{9-7a}{3}\quad 6a+12-\frac{28a}{3}=7 \Rightarrow a=\cancel{\frac{3}{2}},\ b=\cancel{-\frac{1}{2}}\qquad a=\frac{11}{10},\ b=\frac{1}{10}$

$-\frac{10}{3}a=-5\qquad$ so $y=\frac{3}{2}x-\frac{1}{2}\qquad \Rightarrow y=\frac{11}{10}x+\frac{1}{10}$

$a=5\times\frac{3}{10}=\frac{3}{2}$