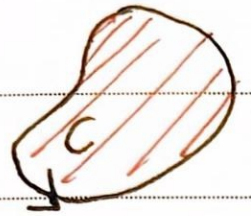
Mo Tu We Th Fr Sa Su

Memo No.

Date / /

LEC 22 Green's Theorem 2025.2.19



$\oint_C \vec{F} \cdot d\vec{r} = ?$

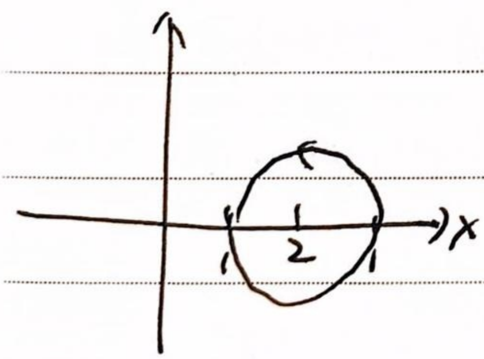
~~Grenen~~ Green's Theorem:

If C closed curve, enclosing a region R, counterclockwise, $\vec{F}$ vector field defined & differentiable in R then $\boxed{\oint_C \vec{F} \cdot d\vec{r} = \iint_R \operatorname{curl}(\vec{F})\, dA}$

$\oint_C M dx + N dy = \iint_R (N_x - M_y) dA$

Warning: only for closed curve

Example: Let C = circle of radius 1 centered at (2,0) counterclockwise



$\oint_C ye^{-x} dx + (\frac{1}{2}x^2 - e^{-x}) dy$

①: $x = 2 + \cos\theta$, $y = \sin\theta$

②: using green theorem:

compute instead $\iint_R \operatorname{curl}(\vec{F}) dA$

$\operatorname{curl}(\vec{F}) = N_x - M_y \Rightarrow \iint_R (x + e^{-x}) - e^{-x}\, dA = \iint_R x\, dA$

$\iint_R x\, dA = \int\int = \text{Area}(R) \cdot \underbrace{\bar{x}}_{2,\text{ by geommetry}} = 2\pi$

a Special case:

If curl $\vec{F} = 0$, then $\vec{F}$ is conservative?

Green's: $\oint_C \vec{F} \cdot d\vec{r} = \iint_R \text{curl} \vec{F} \cdot dA$

$= \iint_R 0 \, dA = 0$

Consequence: If $\vec{F}$ defined everywhere in the plane and curl$(\vec{F}) = 0$ everwhere, then $\vec{F}$ is conservative

Proof of Green's Therom: $\oint_C M dx + N dy = \iint_R (N_x - M_y) dA$

observe: $\oint_C M dx = \iint_R -M_y \, dA$ (where $N=0$)

β ~~&~~ similar argue-ment $\oint_C N dy = \iint_R N_x \, dA$

summing, get Green's therom

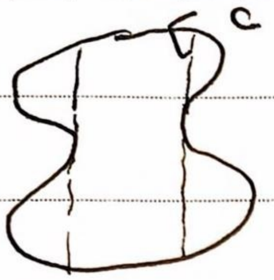
2) can decompose R into simpler regions.

if we prove $\oint_{C_1} M dx = \iint_{R_1} -M_y \, dA$

and $\oint_{C_2} M dx = \iint_{R_2} -M_y \, dA$

$\oint_C M dx = \oint_{C_1} + \oint_{C_2} = \iint_{R_1} + \iint_{R_2} = \iint_R -M_y \, dA$

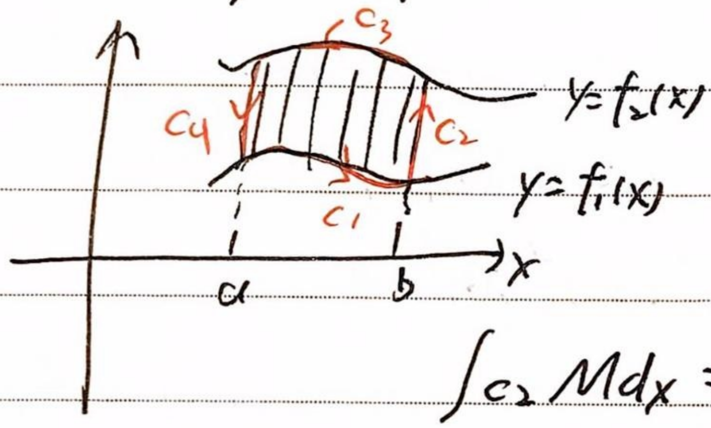
because we go twice ~~through~~ along boundary between R_1 and R_2 with opposit direction



cut R into "vertically simple" Reigons

$a < x < b$, $f_1(x) < y < f_2(x)$

Main step: prove $\oint_C M\,dx = \iint_R -M_y\,dA$ if R vertically simple C = boundary of R



$\int_{C_1} M\,dx = \int_a^b M(x, f_1(x))\,dx$

$y = f_1(x)$ x from a to b

$\int_{C_2} M\,dx = 0$, $x = b$, $dx = 0$

$\int_{C_4} M\,dx = 0$, $dx = 0$

$\int_{C_3} M\,dx$. $y = f_2(x)$. x from $b \to a$

$= \int_b^a M(x, f_2(x))\,dx = -\int_a^b M(x, f_2(x))\,dx$

sum together: $\oint_C M\,dx = \int_a^b M(x, f_1(x))\,dx - \int_a^b M(x, f_2(x))\,dx$

$\iint_R -M_y\,dA = -\int_a^b \int_{f_1(x)}^{f_2(x)} \frac{\partial M}{\partial y}\,dy\,dx$ $\cancel{= \int_a^b [M(x, f_2(x))]dx - \int_a^b M(x, f_2(x))dx}$

Inner : $\int_{f_1(x)}^{f_2(x)} \frac{\partial M}{\partial y}\,dy = M(x, f_2(x)) - M(x, f_1(x))$

$\Rightarrow \iint_R -M_y\,dA = -\int_a^b ([M(x_2, f_2(x)] + [M(x_1, f_1(x)])\,dx$