- an exact differential (df)

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last time if $\vec{F} = \nabla f$ gradient field

then $\int_C$ "path independent $\int_C \vec{F} \cdot d\vec{r} = f(P_1) - f(P_0)$

If $\vec{F} = \nabla f$, $M = f_x$, $N = f_y$

then $f_{xy} = f_{yx} \Rightarrow$ ~~M_x~~ $M_y = N_x$

and if $M_y = N_x \Rightarrow \vec{F}$ is a gradient field and

$\vec{F} = \langle M, N \rangle$ defined, differentiable everwhere

Example $\vec{F} = \underbrace{-y}_{M}\hat{i} + \underbrace{x}_{N}\hat{j}$ $\frac{\partial M}{\partial y} = -1$, $\frac{\partial N}{\partial x} = 1$

$\Rightarrow \vec{F}$ is not agradient

Example: $\vec{F} = (4x^2 + axy)\hat{i} + (3y^2 + 4x^2)\hat{j}$

$M_y = ax$, ~~M_x~~ $N_x = 8x \Rightarrow a = 8$

Finding the potential? (only if $N_x = M_y$)

(1) Computing line integrals

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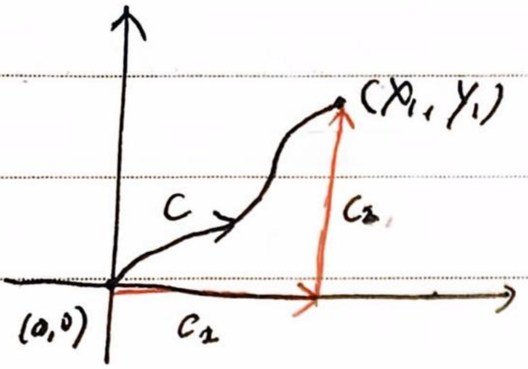

$$\int_C \vec{F}\cdot d\vec{r} = f(x_1, y_1) - f(0,0)$$

$$f(x_1, y_1) = \int_C \vec{F}\cdot d\vec{r} + \underbrace{f(0,0)}_{\text{constant}}$$

$$\vec{F} = \langle 4x^2 + 8xy,\ 3y^2 + 4x^2 \rangle$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C (4x^2 + 8xy)\,dx + (3y^2 + 4x^2)\,dy$$

C_1: x from 0 to X_1 ; $dy = 0$, $y = 0$

$$\Rightarrow \int_{C_1} \vec{F}\cdot d\vec{r} = \int_0^{X_1} 4x^2\,dx = \frac{4}{3}X_1^3$$

C_2: y from 0 to y_2; $X = X_2$, $dx = 0$

$$\Rightarrow \int_{C_2} \vec{F}\cdot d\vec{r} = \int_0^{y_1} (3y^2 + 4x_1^2)\,dy = [y^3 + 4x_1^2 y]_0^{y_1}$$

$$= y_1^3 + 4x_1^2 y_2$$

so

$$C_1 + C_2 = \frac{4}{3}X_1^3 + y_1^3 + 4x_1^2 y_1 + C$$ ← from begining $f(0,0)$

the potential: $= \frac{4}{3}x^3 + y^3 + 4xy + C$

(2) Use antiderivatives

Want to solve
$$\begin{cases} f_x = 4x^2 + 8xy & ① \\ f_y = 3y^2 + 4x^2 & ② \end{cases}$$

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(1) $\Rightarrow f = \frac{4}{3}x^3 + 4x^2y + g(y)$

$\Downarrow$

$f_y = 4x^2 + g'(y)$ — match this with (2)

$4x^2 + g'(y) = 3y^2 + 4x^2 \Rightarrow g(y) = y^3 + C$

$\Rightarrow f(x,y) = \frac{4}{3}x^3 + 4x^2y + y^3 \ (+C)$ $\Leftarrow$ potential

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$\vec{F}\langle M, N\rangle$ is a gradient field in a region of the plane $\Rightarrow$ conservative $\int_C \vec{F}\cdot d\vec{r} = 0$ for closed C, new notation: $\oint_C \vec{F}\cdot d\vec{r}$

$N_x = M_y$ at every point (where $\vec{F}$ definied)

$\searrow$ $\text{curl}(\vec{F}) = 0$

Defination:

$$\boxed{\text{curl}(\vec{F}) = N_x - M_y}$$

(旋度) (test for conservativeness: $\text{curl}(\vec{F}) = 0$)

1. For a velocity Field

curl measure rotation component of motion

$\vec{F} = \langle -y, x\rangle \Rightarrow \text{curl}(\vec{F}) = \frac{\partial(x)}{\partial x} - \frac{\partial(-y)}{\partial y} = 2$

Curl measure (2x) angular velocity of rotation component of velocity field

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(扭矩)

Force field measure torque exerted on a test object in the field

$$\frac{\text{torque}}{\text{moment of inertia}} = \frac{d}{dt}(\text{angular velocity})$$

$$\left(\frac{\text{force}}{\text{mass}}\right) = \frac{d}{dt}(\text{velocity})$$