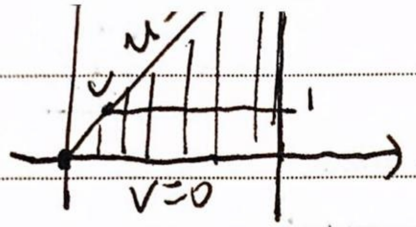


u: $v \to 1$ (v constant

v: $0 \to 1$ (u constant)

Unit 3 Part B: Vector Fields and Line Integrals

LEC 19 2025.2.18

Vector fields

$\vec{F} = M\vec{i} + N\hat{j}$, M and N are function of x, y.

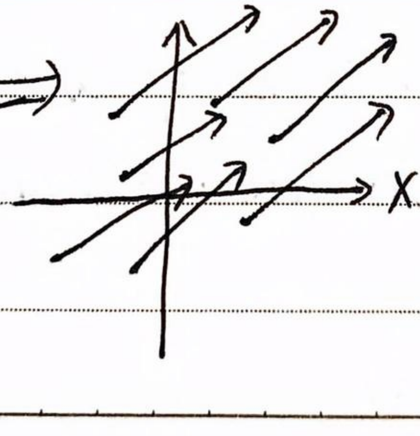
at each point, $\vec{F}$ a vector that depend on (x, y)

Example: velocity in a fluid $\vec{v}$

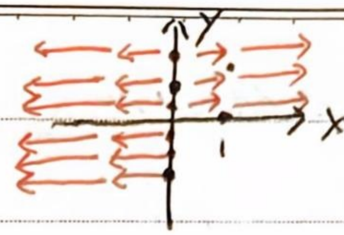
force field $\vec{F}$

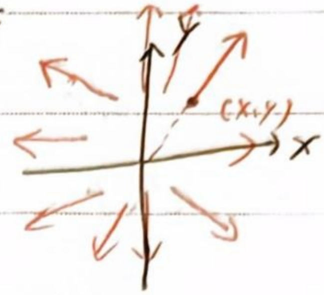
Ex:

$\vec{F} = 2\hat{i} + \hat{j} \Longrightarrow$



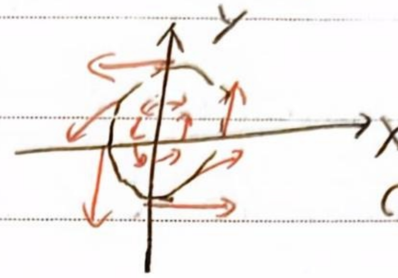
$\vec{F} = x\hat{i}$





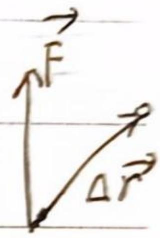
$\vec{F} = x\hat{i} + y\hat{j} \Longrightarrow$

$\vec{F} = -y\hat{i} + x\hat{j} \Rightarrow$



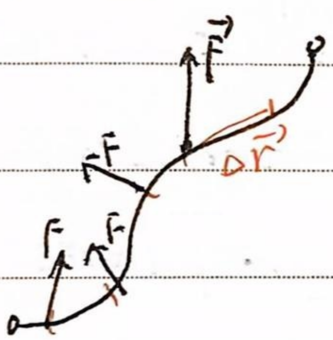
角速度
(unit angular velocity)

Work and line integral



$W =$ (force) $\cdot$ (distance) $= \vec{F}\cdot\Delta\vec{r}$

if:



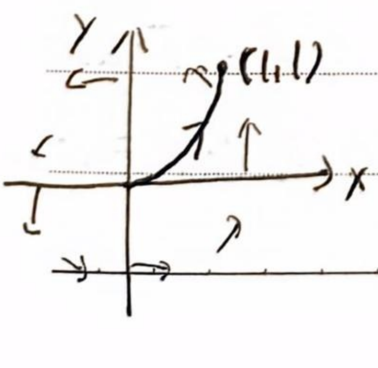
Along a ~~tr~~ trajectory for C (curve), work adds to $\boxed{W = \int_C \vec{F}\cdot d\vec{r}}$ $= \int_{t_1}^{t_2} \vec{F}\cdot\frac{d\vec{r}}{dt}\cdot dt$

$= \left(\lim_{\Delta r_i \to 0} \sum_i \vec{F}\cdot\Delta\vec{r}_i\right) = \sum_i \vec{F}\left(\underbrace{\frac{\Delta\vec{r}}{\Delta t}}_{\text{velocity vector}}\cdot\Delta t\right)$

$\Rightarrow \int_{t_1}^{t_2} \vec{F}\cdot\frac{d\vec{r}}{dt}\cdot dt \left(\lim_{\Delta t\to 0}\right)$ $\vec{v}$

Ex: $\vec{F} = -y\hat{i} + x\hat{j}$

C: $x = t,\ y = t^2,\ 0 \le t \le 1$



what the work had done?

$\vec{F} = \langle -y, x\rangle = \langle -t^2, t\rangle$

$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 \vec{F}\frac{d\vec{r}}{dt}dt = \int_0^1 \langle -t^2, t\rangle\cdot\langle 1, 2t\rangle\, dt$

$= \int_0^1 t^2\, dt = \frac{1}{3}$

Mo | Tu | We | Th | Fr | Sa | Su

Memo No. ______

Date / /

Another way: $\vec{F} = \langle M, N \rangle$

$d\vec{r} = \langle dx, dy \rangle$

$\vec{F} \cdot d\vec{r} = M dx + N dy$

$\int_C \vec{F} \cdot d\vec{r} = \int_C M dx + N dy$

Method to evaluate $\Rightarrow$ express x, y in terms of a single variable & substitute

$\boxed{\text{from the example before page}}$

$\Leftarrow \int_C \vec{F} \cdot d\vec{r} = \int_C -y dx + x dy$ (use in terms of t)

$x = t,\ y = t^2,\ dx = dt,\ dy = 2t\,dt$

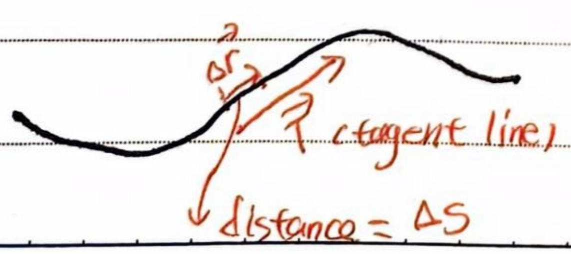
$\Rightarrow = \int_C -t^2 dt + 2t^2 dt = \int_0^1 t^2 dt = \frac{1}{3}$

Note: $\int_C \vec{F} \cdot d\vec{r}$ depends on the trajectory C but not on ~~paramention~~ parameterization

could do: $\begin{cases} x = \sin\theta \\ y = \sin^2\theta & \theta \in [0, \frac{\pi}{2}] \end{cases}$ but (NOT practical) in this example

$\boxed{\text{Geometric approach}}$

$d\vec{r} = \langle dx, dy \rangle = \hat{T} \cdot ds$



(Note, $\frac{d\vec{r}}{dt} = \langle \frac{dx}{dt}, \frac{dy}{dt} \rangle = \hat{T} \cdot \frac{ds}{dt}$

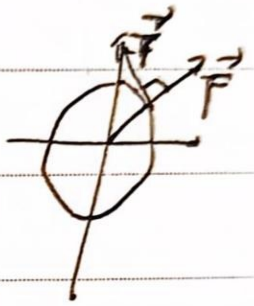
Memo No.
Date / /

So $\int_C \vec{F}\cdot d\vec{r} = \int_C M\,dx + N\,dy = \int_C \vec{F}\cdot\hat{T}\cdot ds$

Example:

C: circle of radius a at origin,

counterclockwise. $\vec{F} = x\hat{i} + \hat{j}\cdot y$



$\vec{F} \perp \hat{T} \Rightarrow \vec{F}\cdot\hat{T} = 0, \Rightarrow \int_C \vec{F}\hat{T}\cdot ds = 0$

2) Same C, $\vec{F} = -y\hat{i} + x\hat{j}$

$\vec{F} \parallel \vec{T} \Rightarrow \cancel{\vec{F}\cdot\vec{T}}$ $\vec{F}\cdot\hat{T} = |\vec{F}| = a$ (radius)

$\int_C a\,ds = a\int_C ds = a\cdot \text{length}(C) = 2\pi a\cdot a$

$= 2\pi a^2$

or: $\int -y\,dx + x\,dy$, $\Rightarrow x = a\cos\theta,\ y = a\sin\theta$

$= \int_0^{2\pi} a^2\sin^2\theta + a^2\cos\theta\cdot d\theta = 2\pi a^2$

So sometimes just think in ~~geometrictly~~ geometrically