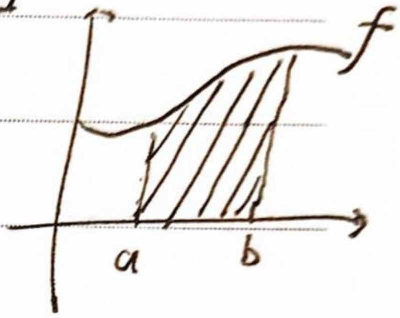
Lec16. Integrals.

2025.1.15

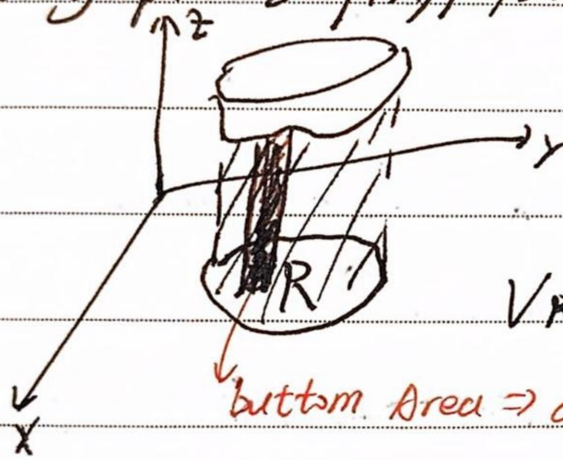
Remember: function of 1 variable $\int_a^b f(x)\,dx$

= area below graph of f over $[a,b]$



Double integral = volume

below graph $z=f(x,y)$, over a reigon

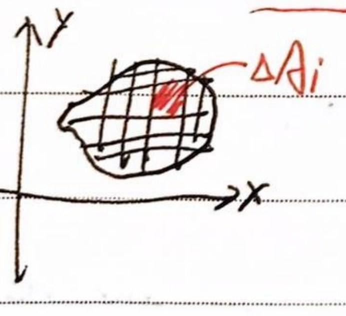


buttom Area ⇒ dA

R on x-y plane

$$V_R = \iint_R f(x,y)\,dA$$

Definition: cut R to small pieces of are ΔA



$$\boxed{V=\sum_i f(x_i,y_i)\cdot A_i}$$

↳ take limit as $\Delta A \to 0$ get $\iint$

To compute $\iint_R f(x,y)\,dA$: take slices let

$S(x)$ = area of slices by // y-x plane

The volume $= \int_{x_{min}}^{x_{max}} S(x)\,dx$

for given x, ~~f(x~~ $S(x) = \int_{y_{min}(x)}^{y_{max}(x)} f(x,y)\,dy$

$$\Rightarrow \boxed{\iint_R f(x,y)\,dA = \int_{x_{min}}^{x_{max}} \left[\int_{y_{min}(x)}^{y_{max}} f(x,y)\,dy\right] dx}$$

ITERATED INTEGRAL

Example 1:

$z = 1 - x^2 - y^2$ region $0 \le x \le 1$, $0 \le y \le 1$

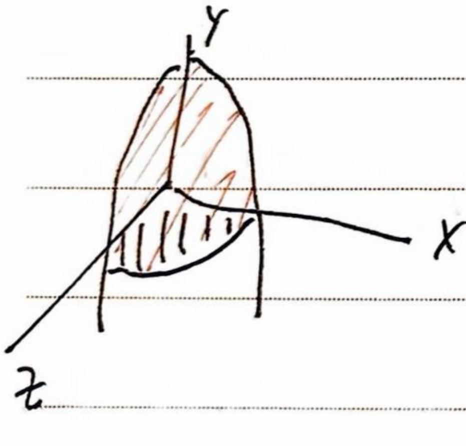
$$\Rightarrow \int_0^1 \boxed{\int_0^1 1 - x^2 - y^2\,dy}\,dx$$

1) inner Integral

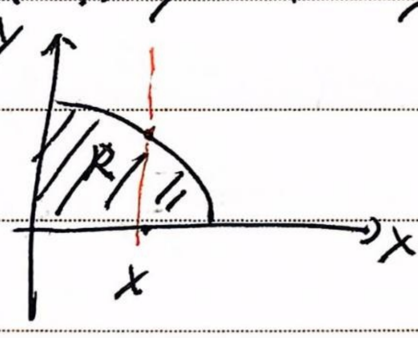
$$\int_0^1 1 - x^2 - y^2\,dy = \left[y - x^2 y - \tfrac{1}{3}y^3\right]_0^1 = (1 - x^2 - \tfrac{1}{3}) - (0) = \tfrac{2}{3} - x^2$$

2) Outers: $\int_0^1 (\frac{2}{3} - x^2)\,dx = [\frac{2}{3}x - \frac{1}{3}x^3]_0^1 = \frac{1}{3}$

Example 2: same function ↑



R: $x^2 + y^2 = 1$, $x \ge 0$, $y \ge 0$



for given x, the range of y should be

0 to $\sqrt{1-x^2}$

$$\iint_R f(x,y)\,dA = \int_0^1 \int_0^{\sqrt{1-x^2}} (1 - x^2 - y^2)\,dy\,dx$$

Inner: $\int_0^{\sqrt{1-x^2}} (1-x^2-y^2)\,dy = \left[y - x^2 y - \frac{y^3}{3}\right]_0^{\sqrt{1-x^2}}$

$= \sqrt{1-x^2} - x^2\cdot\sqrt{1-x^2} - \frac{1}{3}\cdot(1-x^2)^{\frac{3}{2}}$

$= \frac{2}{3}(1-x^2)^{\frac{3}{2}}$

out $\cancel{\int_0} \int_0^1 \frac{2}{3}(1-x^2)^{\frac{3}{2}}\,dx = \cdots = \frac{\pi}{8}$

$\Rightarrow x=\sin\theta,\ (1-x^2)^{\frac{1}{2}} = \cos\theta \Rightarrow \frac{\pi}{8}$

------ $\longrightarrow$ will be easier in polar coordinate

~~Exchag.~~ $\boxed{\text{Exchanging of integration.}}$

Ex 1:

$\int_0^1\int_0^2 dx\,dy = \int_0^2\int_0^1 dy\,dx$

$(uv)' = u'v + v'u$

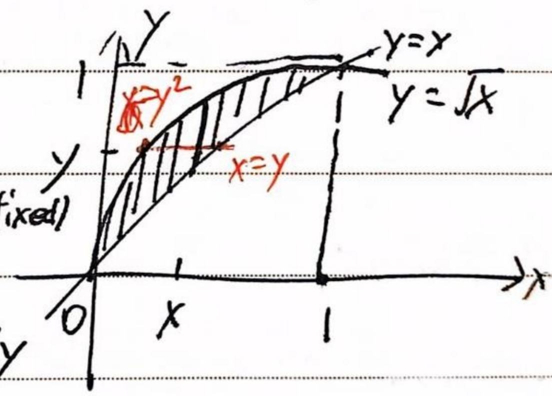
$v'u = (uv)' + u'v$

$= uv|_0^1 + \int u'v \cdots$

Ex 2:

$\int_0^1\int_x^{\sqrt{x}} \frac{e^y}{y}\,dy\,dx$

$= \int_0^1\int_{y^2}^{y} \frac{e^y}{y}\,dx\,dy \longleftrightarrow$ (fixed)



Inner: $\left[\frac{e^y}{y}x\right]_{x=y^2}^{y} = e^y - e^y y$

Outer $\int_0^1 (e^y - y\cdot e^y)\,dy = [\cancel{e^y} - ye^y + 2e^y]_0^1$

$= e-2$

$(-y\cdot e^y)' = \cancel{y} - e^y - ye^y$

so $(-ye^y + 2e^y)'$

$= e^y - ye^y$