LEC27 Positive Definite Matrices and Minima

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☆ ① Test Positive definite $x^T A x > 0$

② tests for Minumum

start with 2 by 2 $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ symmetric

① $\lambda_1 > 0$, $\lambda_2 > 0$?

② $a > 0$, $ac - b^2 > 0$?

③ pivots $a > 0$, $\frac{ac - b^2}{a} > 0$ ← elimination

☆ ④ $x^T A x > 0$

Examples

$\begin{bmatrix} 2 & 6 \\ 6 & \ \end{bmatrix}$ what number make positive?

③ pivot: need $\geq$ 19

if =18, pos semidefinite (半定义), $\lambda_1 = 0$, $\lambda_2 = 20$

↑ $\lambda_1 = 0$, pivots: 2, singular, only 1

$$\underset{A}{\begin{bmatrix} 2 & 6 \\ 6 & 18 \end{bmatrix}} \underset{x}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} \qquad \underset{x^T}{[x_1\ x_2]} \underset{A}{\begin{bmatrix} 2 & 6 \\ 6 & 18 \end{bmatrix}} \underset{x}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}$$

$$= [x_1\ x_2] \begin{bmatrix} 2x_1 + 6x_2 \\ 6x_1 + 18x_2 \end{bmatrix} = 2x_1^2 + 12x_1x_2 + 18x_2^2$$

$\uparrow ax^2 + 2bxy$, $\uparrow cy^2$

(二次方程)
quadratic form

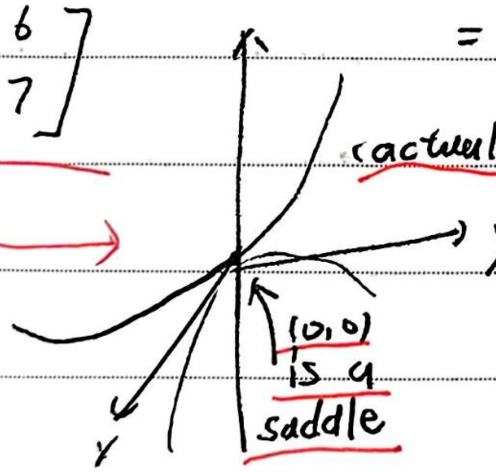
its not linear any more

is $ax^2 + 2bxy + cy^2 > 0$?

$2x^2 + 12x_1x_2 + 18x_2^2 > 0$?

$\boxed{\text{Graphs of } f(x,y) = \vec{x}^T A \vec{x}}$

if $\begin{bmatrix} 2 & 6 \\ 6 & 7 \end{bmatrix}$ $\quad = ax^2 + 2bxy + cy^2$



critical point
(actually, the perfect directions are the (eigenvectors))

$2x^2 + 12xy + 7y^2 = z$

there's a saddle point

so $2x^2+12xy+7y^2$ can <0

why:

$$\begin{cases} f_x = 4x + 12y = 0 \\ f_y = 12x + 14y = 0 \end{cases} \Rightarrow \text{critical point } (0,0)$$

$\underset{A}{f_{xx} = 4}$, $\underset{C}{f_{yy} = 14}$ $\underset{B}{f_{xy} = 12}$ $\Rightarrow AC - B^2 = 56 - 144 < 0$

so the critical point is a saddle point !

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Let's back to 2D $\boxed{X^TAX}$

$$\underset{X^T}{[x_1 \;\; x_2]} \overset{A=}{\begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix}} \underset{X}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = [x_1 \;\; x_2] \begin{bmatrix} 2x_1 + 6x_2 \\ 6x_1 + 20x_2 \end{bmatrix}$$

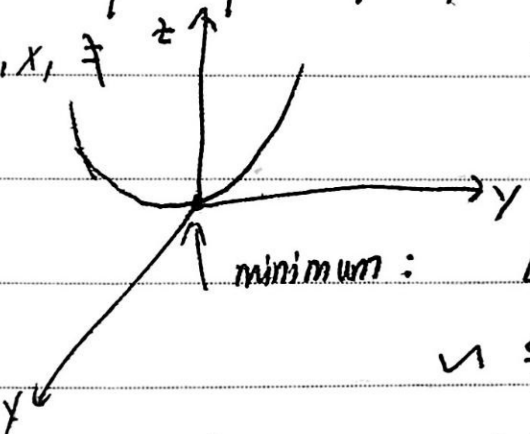
$$= 2x_1^2 + 12x_1x_2 + 20x_2^2 > 0 ?$$

✓

$\Rightarrow X^TAx > 0$ except at $x = 0$

Graph of $f(x,y) = 2x^2 + 12xy + 20y^2$

$f\,x_1 x_1\,z$



$$= \underbrace{2(x+3y)^2 + 2y^2}_{\text{what's number go out side?}} \geq 0$$

$\Rightarrow$ its pivots!

minimum : 1st derivatives $= 0$

$\wedge \frac{d^2u}{dx^2} > 0$

in 18.06 MIN $f(x_1, x_2 \cdots x_n)$

when Matrix of 2nd Derivatives is positive definite

$$\begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix} \xrightarrow{\text{elimination}} \begin{bmatrix} 2 & 6 \\ 0 & 2 \end{bmatrix}$$

$A \quad L = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \quad U$

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whats the matrix of 2nd derivatives?

$$\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} \Rightarrow \text{symmetric}$$

the second derivatives

$\Rightarrow AC-B^2$

$\Rightarrow f_{xx}\cdot f_{yy} - f_{xy}\cdot f_{yx}$

$<0 \Rightarrow$ saddle

$\begin{cases} ac-b^2>0 \\ a>0 \end{cases} \Rightarrow \min$

$\cancel{b^2 \quad ac <0}$

minimum $\begin{cases} ac-b^2>0 \\ a>0 \end{cases}$ $\qquad a<0 \Rightarrow$ maximum

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3x3 example

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

① det 2, 3, 4

② pivots $2, \frac{3}{2}, \frac{4}{3}$

③ eigenvalues: $2-\sqrt{2}, 2, 2+\sqrt{2}$

trace = 6

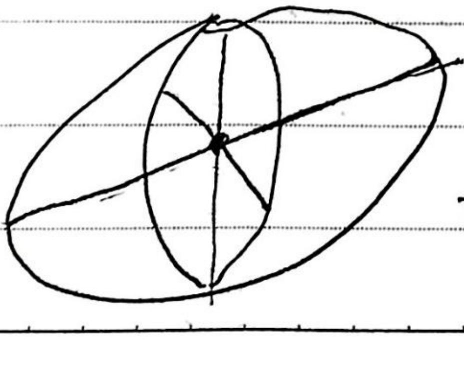
$x^T A x = 2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_2x_3 > 0$

$[x_1 \; x_2 \; x_3][A]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

suppose cut through at height 1 $\qquad = 1$

$\boxed{A = Q\Lambda Q^{-1}}$

diagonalization for symmetric Matrix



three eigenvalues all different