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LEC 28. Similar Matrices and Jordan Form 2.15

similar matrices A, B / JORDAN FORM

$B = M^{-1}AM$

importent: positive definite means:

$x^TAx > 0$ (except for $x=0$)

A^TA is positive definite!

If A & B are pos def, what about A+B?

$x^T(A+B)x > 0 \Leftarrow x^TAx > 0,\ x^TBx > 0$

so is A+B

Now A m by n

$A^TA \to n \times n$ square, symmetric, but is it pos def? Yes, A^TA is must be positive definite

x^TA^TAx why this can never be negative?

$\Rightarrow = (Ax)^T(Ax)$ it's length square of Ax

$= \|Ax\|^2 \geq 0$, only if $x=0$ it can be 0

if $\boxed{rank = n}$, ~~there is only~~ then the nullspace is only the $\{0\}$, so in this situation, $\|Ax\| \geq 0$ only $x=0$ can make $\|Ax\|^2 = 0$, then A^TA is positive definite

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n×n matrices

$\boxed{\text{A and B are similar}}$

that means: for some M, $B = M^{-1}AM$

Example:

$S^{-1}AS = \Lambda \Rightarrow A$ is similar to Λ

~~A = [~~ ↑ the best neat matrices in the family of A

there are a lot of others, (take a different S).

Suppose $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ $\Lambda = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$ $(\lambda_1 = 3, \lambda_2 = 1)$

similar

$$\underbrace{\begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}}_{M^{-1}} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}}_{M} = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 9 \\ 1 & 6 \end{bmatrix} = \underbrace{\begin{bmatrix} -2 & -15 \\ 1 & 6 \end{bmatrix}}_{B}$$

4
$-12 - (-15)$
det $= 3$

$\Rightarrow A, \Lambda, B$ are all similar matrices, they are all in a same family

$\Rightarrow$ they are all have the $\boxed{\text{same eigenvalues}}$!

great fact: all similar matrices have the same eigenvalues !!

, $\begin{bmatrix} 3 & 7 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 7 \\ 0 & 3 \end{bmatrix}$ example

$\boxed{\text{how?}}$

$\Rightarrow$ $Ax = \lambda x$ $(B = M^{-1}AM)$

$A \underbrace{MM^{-1}}_{I} x = \lambda x$

$\Rightarrow (M^{-1}AM)M^{-1}x = \lambda M^{-1}x$

↑ that's B

$\Rightarrow B M^{-1}x = \lambda M^{-1}x$ end of prove

so λ is also B's eigenvalue, and the eigenvector of B is $M^{-1}x$ ← (eigen vector of A)

$\Rightarrow$ similar Martices have ~~same~~ same eigenvalues and the eigenvectors just move around

but need to discuss the bad case: (not enough eigen vectors to be used to diagonize)

BAD CASE; $\lambda_1 = \lambda_2 = 4$ suppose

$\begin{cases} \text{one family has } \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \\ \text{big family includes } \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix} \text{ and others} \end{cases}$

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it means ~~all~~ the only matrix that's similar to $\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$ is it self.

$$\underbrace{M^{-1}}_{\text{any } M}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} M = 4 \cdot M^{-1}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} M = \underbrace{\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}}_{\text{it's back again!}}$$

$\begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$ is also ~~non undiago~~ diagonalizable, only one eigen~~value~~vector

the family is $\begin{bmatrix} 4 & x \\ 0 & 4 \end{bmatrix}$

$\begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$ is called $\boxed{\text{Jordan form}}$

$\boxed{\text{the climax of 18.06}}$

more members of family

$\begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$, $\begin{bmatrix} 5 & 1 \\ -1 & 3 \end{bmatrix}$, $\begin{bmatrix} 4 & 0 \\ 17 & 4 \end{bmatrix}$ similar

it's not diagonlizable

det = 16, trace = 8

$\Rightarrow \begin{bmatrix} a & m \\ m & 8-a \end{bmatrix}$ m: any number

they are all similar

see

$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ ← Jordan form

rank = 2

$\lambda = 0, 0, 0, 0$, | 2 eigenvectors, dim $N(A) = 2$ | 2 missing

$\begin{bmatrix} 0 & 1 & 7 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ similly

$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ not similly

jordan block is not same size

the $\boxed{\text{Jordan block}}$ 1 eigen vector only

$J_i = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & & \lambda_i \end{bmatrix}$

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Every square A is similar to a Jordan matrix J

$$J = \begin{bmatrix} \boxed{J_1} & & & \\ & \boxed{J_2} & & \\ & & \ddots & \\ & & & \boxed{J_d} \end{bmatrix}$$

the number blocks = # eigenvectors

the good case J is Λ (diagonalizable A)