Memo No.

Date / /

LEC 8 solving $Ax=b$: Row Reduced ~~from~~ form R

2.5

Compelte solve equation of $Ax=b$

Rank r

$$x_1+2x_2+2x_3+2x_4=b_1$$
$$2x_1+4x_2+6x_3+8x_4=b_2$$
$$3x_1+6x_2+8x_3+10x_4=b_3$$

elimination

$b_3=b_1+b_2$

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 2 & 4 & 6 & 8 & b_2 \\ 3 & 6 & 8 & 10 & b_3 \end{array}\right]$$

Augmented matrix

$[A \vdots b]$

$$\downarrow$$

$$\begin{bmatrix} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2-2b_1 \\ 0 & 0 & 2 & 4 & b_3-3b_1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2-2b_1 \\ 0 & 0 & 0 & 0 & b_3-b_2-b_1 \end{bmatrix}$$

pivot columns

$$0=b_3-b_2-b_1$$

let $b=\begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix} \Rightarrow \begin{cases} b_1=1 \\ b_2-2b_1=3 \\ b_3-b_2-b_1=0 \end{cases}$

Solvability Condition on b

$\Rightarrow Ax=b$ is Solvable (b has in the column space of A) (C(A)). If a combination of rows of A gives zero row, then the same combination of entries of b must give 0.

$\boxed{\text{To find the complete solution to } Ax=b}$

① X particular: Set all free variables to 0

solve $Ax=b$ for pivot variables

in this case: $x_2 = x_4 = 0 \Rightarrow \begin{cases} x_1 + 2x_3 = 1 \\ 2x_3 = 3 \end{cases} \Rightarrow \begin{cases} x_1 = -2 \\ x_3 = \frac{3}{2} \end{cases}$

So $x_p = \begin{bmatrix} -2 \\ 0 \\ \frac{3}{2} \\ 0 \end{bmatrix}$

② X nullspaces

$x_{complete} = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix} + c\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d\begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$

① + ② $x = x_p + x_n \Rightarrow Ax_p = b$, $Ax_n = 0$ $\boxed{\text{so } A(x_p + x_n) = b}$

x_n: Xnullspace

so, it is the complete Solution

$$x_{complete} = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix} + \underbrace{c\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d\begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}}_{\text{null space}}$$

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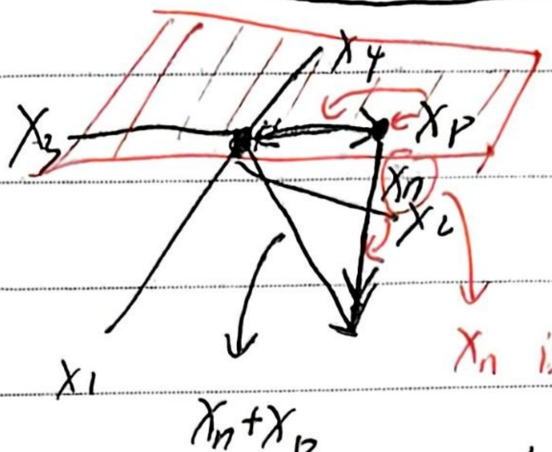
Date / /

Plot all solutions x in $\mathbb{R}^4$



x_n: two dimensional subspace in $\mathbb{R}^4$

x_n is anywhere in the subspace

so it is like a subspace shifted from the origin (a plane not go through the origin)

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① m by n matrix A of rank r

(know $r \le m$ and $r \le n$)

Full column rank means $\boxed{r = n}$: No free variable

$N(A)$ = zero vector ← only $x = [0]$

Solution to $Ax = b$: $x = x_{particular}$, Unique solution if it exist

(0 or 1 solution)

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 6 & 1 \\ 5 & 1 \end{bmatrix} \qquad R(A) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$Ax = b$, $x_{null} = [0]$

full column rank

Mo Tu We Th Fr Sa Su

Memo No. ______

Date / /

② $\boxed{\text{Full row rank means } r = m}$

can solve $Ax = b$ for which b? $\Rightarrow$ $\boxed{\text{for every } b}$

~~because~~ left hand no zero. Exist

Left with $\boxed{n-r}$ free variables

$$n - r = n - m$$

$$A = \begin{bmatrix} 1 & 2 & 6 & 5 \\ 3 & 1 & 1 & 1 \end{bmatrix} \Rightarrow R = \begin{bmatrix} 1 & 0 & - & - \\ 0 & 1 & F & \\ & & - & - \end{bmatrix}$$

The rank ~~r=~~ tell us about the solutions ⇓

③ $\boxed{r = m = n}$ $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$ ⇐ $\boxed{\text{full rank matrix}}$

$R = I$ $X_n = [0]$ is invertible

1 solution for every b

⇐① $\boxed{r = \cancel{m < n}}$ $\boxed{n < m}$ ⇐ full column rank matrix

$R = \begin{bmatrix} I \\ 0 \end{bmatrix}$ (0 or one solution) to $Ax = b$

⇑

② $\boxed{r = \cancel{n < m}}$ $\boxed{m < n}$ ⇐ full row rank matrix

$R = \begin{bmatrix} I & F \end{bmatrix}$ (~~1 or~~ ∞ solutions) null space

↑ no requirements for b

③ $\boxed{r < m,\ r < n}$

$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$ (0 or ∞ solutions)

↑ $\boxed{\text{if } b \text{ not in the } C(A)}$