LEC 23. Differential Equations and exp(At)

2.11

Differential Eqns $\frac{du}{dt} = Au$

Exponential exp(At) of a matrix

指数 exponential

Example:

e^{At}: the differential

$$\frac{du_1}{dt} = -u_1 + 2u_2$$

$u_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\frac{du_2}{dt} = u_1 - 2u_2$$

$\boxed{\text{find } u(t)}$

$\rightarrow A = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}$, $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $Au = \begin{bmatrix} -u_1+2u_2 \\ u_1-2u_2 \end{bmatrix}$

singular $\lambda_1 = 0$, $\lambda_2 = -3$ (from trace)

$\lambda_1 = 0$:

$x_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $Ax_1 = 0 \cdot x_1$

$\lambda_2 = -3$

$A - \lambda I = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$ $x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $Ax_2 = -3 \cdot x_2$

Solution will: $u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2$

Check: $\frac{du(t)}{dt} = Au$ Plug in $e^{\lambda_1 t} x_1 = u$

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~~left~~: $\lambda_1 e^{\lambda_1 t} x_1 = A e^{\lambda_1 t} x_1$

$(\frac{du}{dt})$ right (Au)

$\Rightarrow \lambda x_1 = A x_1$ ✓

$e^{\lambda_n t} x_n$ → need to remember

this one is which intereted in exponential

~~ttt~~ in last one: $u_k = c_1 \lambda^k x_1 + c_2 \lambda^k x_2 + \cdots$

this is for $u_{k+1} = A u_k$

(this is for differential eqns)

so $u(t) = c_1 \cdot e^{0t} x_1 + c_2 e^{-3t} x_2$

$= c_1 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

now use $u(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

at $t=0$, $c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

solution $c_1 = \frac{1}{3}$, $c_2 = \frac{1}{3}$ is $Sc = u_0$

so final $u(t) = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{1}{3} e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$t \to \infty$, the steady state $u(\infty) = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

① stability $u(t) \to 0$ / need $e^{\lambda t} \to 0$ / $\lambda < 0$

② steady state $\lambda_1 = 0$ and other $\mathrm{Re}\,\lambda < 0$

③ Blew up if any $\mathrm{Re}\,\lambda > 0$

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2 x 2 stability $\quad Re\,\lambda_1 < 0 \quad Re\,\lambda_2 < 0$

① $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$; so the trace $a+d = \lambda_1 + \lambda_2 < 0$

ex trace < 0, still blow up

$\begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$

② $\det(A) > 0 \quad \det(A) = \lambda_1 \cdot \lambda_2$

$\frac{du}{dt} = Au \quad$ Set $u = Sv$ (uncouples)

eigenvectors martix

$S\frac{dv}{dt} = ASv$

$\Downarrow \quad \frac{dv}{dt} = S^{-1}ASv = \Lambda v \quad \frac{dv_1}{dt} = \lambda_1 v_1 \cdots$

uncoupled

uncouple ⇒ diagonal matrix

~~v=c~~ $\quad v(t) = e^{\Lambda t} v(0)$

most important

$\boxed{u(t) = Se^{\Lambda t}S^{-1}u(0)} = e^{A(t)}u(0)$

$e^{At}u(0)$

$v = c_1 e^{\lambda_1 t}$

it's depend on $v(0)$

$e^{At} = Se^{\Lambda t}S^{-1}$

Matrix exponential e^{At}

$$e^{At} = I + At + \frac{(At)^2}{2} + \frac{(At)^3}{6} + \cdots + \frac{(At)^n}{n!} + \cdots$$

$$(I - At)^{-1} = I + At + (At)^2 + (At)^3 + \cdots$$

$$\frac{1}{1-x} = \sum_0^{\infty}(x)^n$$

we can do the same thing to Matrix exponential as a ordinary function

$Ax = \lambda x$

~~$AS = \lambda S$ $AS = \Lambda S$ $S^{-1}AS = \Lambda$~~

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$u(t) = e^{At} u(0)$

$$e^{At} = I + \underbrace{S\Lambda S^{-1}}_{A} t + \frac{S\Lambda^2 S^{-1}}{2} t^2 + \cdots$$

$S\Lambda^0 S^{-1}$

$$\boxed{= S e^{\Lambda t} S^{-1}}$$

the taylor's method

(A must have independent vectors)

$e^{\Lambda t}$ $\Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$

$$= \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix}$$

$Re\,\lambda < 0, \rightarrow u(t) \rightarrow 0$

how to find A from the differential equation?

$y'' + by' + ky = 0$ | 2nd order

$\rightarrow 2 \times 2$ matrix

$$u = \begin{bmatrix} y' \\ y \end{bmatrix} \quad u' = \begin{bmatrix} y'' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} -b & -k \\ 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} y' \\ y \end{bmatrix}$$

$\Rightarrow$ find y, use upper

$y = S e^{\Lambda t} S^{-1}$

!!!! just !!!!!

remember this

$u(t) = e^{At} \cdot u(0)$

$u(t) = S \cdot e^{\Lambda t} \cdot S^{-1} u(0)$

$$\boxed{\begin{aligned} &AS = S\Lambda \\ &A = S\Lambda S^{-1}, \quad \Lambda = S^{-1}AS \end{aligned}}$$