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LEC 32 Quiz 3 Review 2.17

6.1–2 λ and x $Ax=\lambda x$

6.3 $du/dt = Au$ and e^{At}

6.4 $A=A^T \to \Lambda$ $=Q\Lambda Q^T$ 6.5 positive definite

6.6. Similar $B=M^{-1}AM$ 6.7 $A=U\Sigma V^T$

$\uparrow$ same eigenvalues SVD

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1. $$\frac{du}{dt} = Au = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} u$$

$u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2 + c_3 e^{\lambda_3 t} x_3$

A is ~~sign~~ singular, so $\lambda_1 = 0$

$$\begin{bmatrix} -\lambda & -1 & 0 \\ 1 & -\lambda & -1 \\ 0 & 1 & -\lambda \end{bmatrix} = -\lambda^3 - 2\lambda = 0$$

$\lambda(\lambda^2+2)=0$

$\Rightarrow \lambda_1 = \sqrt{2} i, \lambda_2 = -\sqrt{2} i$

$\Rightarrow u(t) = c_1 x_1 + c_2 e^{\sqrt{2}it} x_2 + c_3 e^{-\sqrt{2}it} x_3$

go around at unit circle

$t=0$ $u(0) = c_1 x_1 + c_2 x_2 + c_3 x_3$

what's the period?

$e^{2\pi i} = 1$ $\sqrt{2} i T = 2\pi i$

$(\cos\theta + \sin\theta\, i)$

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so the Periodic $T = \pi\sqrt{2}$

orthogonal eigenvectors $\Rightarrow AA^T = A^TA$

$\boxed{\text{symmetric, antisymmetric}}$

$\boxed{\text{orthogonal vectors}}$

back to $e^{At}\ \frac{du}{dt}$

e^{At} $\quad u(t) = e^{A(t)}u(0)$

e^{At} $\quad (\text{if } A = S\Lambda S^{-1}) \leftarrow$ eigenvectors are independent and $\# = n$

$= S e^{\Lambda t} S^{-1}$

2. A

$\lambda_1 = 0,\ \lambda_2 = c,\ \lambda_3 = 2$

$$x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad x_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad x_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

(a) diagonalizable? all c (we get enough eigenvectors)

(b) symmetric? all real c

(c) positive definite? no

semi " " $\quad c \geq 0$

(d) markov matrix? No, $\lambda_i \leq 1$

(e) $\frac{A}{2}$ projection matrix? $P^2 = P \Rightarrow \lambda^2 = \lambda$

so $\lambda = 0$ or 1, Need $c = 0$, $c = 2$

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$A^T = A$: $A\vec{v} = \lambda \vec{v}$ $\vec{v}$: eigenvector

why symmetric $\Rightarrow$ orthogonal eigenvectors

$v_1^T A v_2 = \lambda_2 v_1^T v_2$

$(A(v_1))^T v_2 = \lambda_1 v_1^T \cdot v_2$

$\Rightarrow \lambda_1 v_1^T v_2 = \lambda_2 v_1^T v_2$

$\lambda_1 \neq \lambda_2 \Rightarrow v_1^T v_2 = 0$

3. SVD Singular value decomposition

$A = (\text{orthogonal})(\text{diag})(\text{orthogonal}) = U \Sigma V^T$

↑ every A σ_i

$A^T A = (V \Sigma^T U^T)(U \cdot \Sigma V^T) = V(\Sigma^T \Sigma) V^T$

↑ symmetric $U = V = S$ $(S \Lambda S^{-1})$

$\Rightarrow V =$ ~~es~~ evector for $A^T A$

$\Rightarrow \sigma_i^2 = \lambda_i(A^T A)$

$x^T A^T A x = (Ax)^T \cdot Ax = \|Ax\|^2 > 0$

why $A^T A$ always positive definite? (exact $A = 0$)

$AA^T = (U \Sigma V^T)(V \Sigma^T U^T) = U \Sigma \Sigma^T U^T$

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they way to ~~decitive~~ decide sign of eigenvectors

Instead $\boxed{Av_i = \sigma_i u_i}$ $\boxed{AV = U\Sigma}$

$[u_1\ u_2]\begin{bmatrix}3 & 0\\ 0 & 2\end{bmatrix}[v_1\ v_2]^T$

$\cancel{\begin{bmatrix}3 & 0\\ 0 & 5\end{bmatrix}}$

$\begin{bmatrix}3 & 0\\ 0 & 0\end{bmatrix}$ $\Rightarrow$ A is singular, rank $=1$

$\Downarrow$ u_2 is the basis of $N(A)$

True / False

λ is real

Given A is symmetric and orthogonal — $|\lambda| = 1$

why? $Qx = \lambda x$

① eigenvalues can be 1 and -1

~~②~~ prof $\frac{1}{2}(A+I)$ is a projection matrix $2\cdot\|x\| = |\lambda|\,\|x\|$

($P^2 = P$ and symmetric)

$\frac{1}{4}(A^2 + 2AI + I^2) \overset{?}{=} \frac{1}{2}(A+I)$

what is A^2 ? $A = A^T = A^{-1}$ $AA^{-1} = I$

$\Rightarrow A^2 = I$ $= \frac{1}{4}(2(I+A)) = \frac{1}{2}(A+I)$

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The summar of unit 3. (by problems)

① orthogonal matrix

$Q \cdot Q^T = I \cdot c$ $\quad [x_1\, x_2\, x_3\, x_4] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1^2 & & & 0 \\ & x_2^2 & & \\ & & \ddots & \\ 0 & & & x_4^2 \end{bmatrix} = cI$

$\det|QQ^T| = \det(cI) \Rightarrow \det(Q) = c^{\frac{n}{2}}$

1. orthonormal: $\boxed{Q \cdot Q^T = I}$, $\det(Q) = 1$

2. $Q_1 Q_2 = Q_3$ two orthogonal matrix multiply

$Q_3^T \cdot Q_3 = Q_2^T \cdot Q_1^T \cdot Q_1 \cdot Q_2 = I$ $\Rightarrow$ orthogonal

3. $\|Qx\| = \|x\|$

4. $\boxed{|\lambda| = 1}$, eigenvectors are orthogonal

5. SVD: A always = U or V

② symmetric matrix

1. $A = A^T$

2. λ are all real

3. $\boxed{\text{eigenvectors orthogonal}}$

4. $A = Q \Lambda Q^T$

5. SVD $\Rightarrow$ $U = V$ $\Sigma = \Lambda$

$Av = \lambda_1 v$

$Av_1 = \lambda_1 v_1$

$Av_2 = \lambda_2 v_2$

$(Av_1)^T \cdot v_2 = (\lambda_1 v_1)^T \cdot v_2$

$A v_1^T v_2 =$

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③ Positive Definite Matrix

1. $\boxed{x^T A x > 0}$ $\boxed{\text{symmetric} + \lambda > 0}$

2. symmetric | $\lambda > 0$ det > 0

3. SVD ⇒ $\sigma = \lambda$

④ diagonal matrix

1. $\boxed{\lambda \text{ is the diagonals}}$

2. eigenvectors ⇒ orthonormal basis

3. SVD ⇒ $A = \Sigma$

⑤ SVD

$A = U \Sigma V^T$ U: column vector orthogonal

U: orthonormal basis of $C(A)$ V: ~~row~~ column vector orthogonal

V^T: orthonormal basis vectors of $C(A^T)$

V^T's row ⇒ A^TA's eigenvectors

V's column ⇒ A^TA's eigenvectors

the sort depend on the sort of singular values

⑥ AA^T

⑦ $\boxed{\text{1. symmetric}}$ 2. square