Mo Tu We Th Fr Sa Su

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LEC 29 : Singular Value Decomposition SVD

2.15

SVD

$A = U \Sigma V^T$

Σ: diagonal

U, V: orthogonal

the good family:

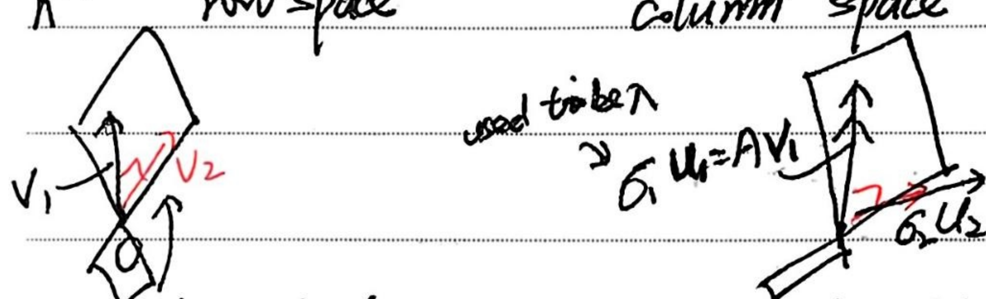
$\begin{cases} A = Q \Lambda Q^T & \text{sym pos def} \\ \cancel{A = S \Lambda S^{-1}} \end{cases}$

this is what I look

(not orthogonal) for

SVD: we goal: find the orthonormal basis in rowspace, and find A make $\sigma_i u_i = A v_i$, u in column space

R^n row space column space R^m



orthogonal basis (gram-shimit)

it means $A [v_1\ v_2 \cdots v_r] = [u_1\ u_2 \cdots u_r] \begin{bmatrix} \sigma_1 & & 0 \\ & \sigma_2 & \\ 0 & & \ddots \sigma_r \end{bmatrix}$

orthonormal basis

$\Rightarrow \boxed{AV = U\Sigma}$

$\boxed{\Rightarrow A = U \Sigma V^{-1} = U \Sigma V^T}$

Example:

$A = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}$

$v_1\ v_2$ in row ~~space~~ R^2

$u_1\ u_2$ in col space R^2

$\sigma_1 > 0 \quad \sigma_2 > 0$

$\begin{cases} A v_1 = \sigma_1 u_1 \\ A v_2 = \sigma_2 u_2 \end{cases}$ (1)

(symm, pos defin ~~positive definite~~)

$\Rightarrow A^T A = (V \Sigma^T U^T) U \Sigma V^T \Rightarrow (U^T U = I)$ (orthogonal)

good

$= \cancel{VV}\ V \cdot \begin{bmatrix} \sigma_1^2 & & \\ & \sigma_2^2 & \\ & & \ddots \sigma_r^2 \end{bmatrix} V^T$

$\Sigma^T \cdot \Sigma$

$$\Rightarrow A^TA = V \begin{bmatrix} \sigma_1^2 & & & \\ & \sigma_2^2 & & \\ & & \ddots & \\ & & & \sigma_n^2 \end{bmatrix} V^T$$

① find v's and $\Sigma\Sigma^T$

$$A^TA = \begin{bmatrix} 4 & -3 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} = \begin{bmatrix} 25 & 7 \\ 7 & 25 \end{bmatrix}$$

eigenvector

$$\boxed{\frac{1}{\sqrt{2}} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}}}$$

need to be unit

$$V = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -\frac{1}{\sqrt{2}} \end{bmatrix} = V^T$$

λ_2 $\sqrt{\frac{1}{2}}$

$$A^TA \cdot x_1 = \boxed{32} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \qquad A^TA \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 18 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \frac{1}{\sqrt{2}}$$

$\frac{1}{\sqrt{2}}$ $\uparrow \lambda_1$

from A^TA $= V\Sigma\Sigma^T V$

$$\text{so} \quad A = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} = \underset{U}{\begin{bmatrix} & \\ & \end{bmatrix}} \underset{\Sigma}{\begin{bmatrix} \sqrt{32} & 0 \\ 0 & \sqrt{18} \end{bmatrix}} \underset{V^T}{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}}$$

$\uparrow \sqrt{\sigma^2}$

② find u's u_1 u_2

$$AA^T = U\Sigma \cancel{V^T V}\Sigma^T U^T = \underset{\text{eigenvectors}}{U} \underset{\text{eigenvalues}}{\Sigma\Sigma^T} U^T$$

symmetric ~~and positive def~~

$$AA^T = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} 4 & -3 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 32 & 0 \\ 0 & 18 \end{bmatrix}$$

$$\begin{cases} AA^T \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 32 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ AA^T \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 18 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{cases} \Rightarrow \quad U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \Sigma = \begin{bmatrix} \sqrt{32} & 0 \\ 0 & \sqrt{18} \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \sqrt{32} & 0 \\ 0 & \sqrt{18} \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

but: $= \begin{bmatrix} 4 & 4 \\ 3 & -3 \end{bmatrix}$

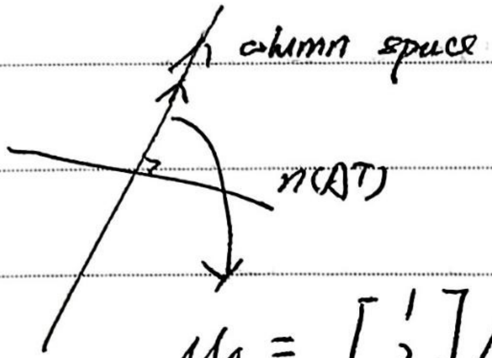
$\boxed{\text{To Finish}}$

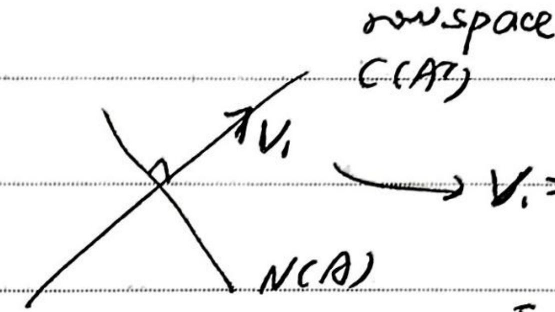
the sign is wrong!

Example 2. one dimension

$$A = \begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix} \qquad \text{rank} = 2, \ 2\times 2$$

$$C(A) = c \cdot \begin{bmatrix} 4 \\ 8 \end{bmatrix} \qquad C(A^T) = c \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$





$$v_1 = \begin{bmatrix} 4 \\ 3 \end{bmatrix} / 5$$

$$v_2 = \begin{bmatrix} \frac{3}{5} \\ -\frac{8}{5} \end{bmatrix}$$

$$u_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} / \sqrt{5}$$

SVD: decomposition:

rank = 1

$$\begin{array}{cccc} A & U & \Sigma & V^T \\ \begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix} = & \begin{bmatrix} \frac{1}{\sqrt5} & \frac{2}{\sqrt5} \\ \frac{2}{\sqrt5} & -\frac{1}{\sqrt5} \end{bmatrix} & \begin{bmatrix} \sqrt{125} & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} .8 & .6 \\ .6 & -.8 \end{bmatrix} \end{array}$$

$$A^T A = \begin{bmatrix} 4 & 8 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix} = \begin{bmatrix} 80 & 60 \\ 60 & 45 \end{bmatrix}$$

3 240 240 = 480 160

4 120 120 = 360

$$\lambda = 0, 125$$

just find the orthonormal vector in column space directly

in $n(A^T)$

$$\begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix} = \underbrace{\frac{1}{\sqrt5} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} \sqrt{125} & 0 \\ 0 & 0 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} .8 & .6 \\ .6 & -.8 \end{bmatrix}}_{V^T}$$

in $n(A)$

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summary:

$$\overset{m\times n}{A} = U \Sigma V^T$$

U: $m\times m$ orthonormal vectors of column space (basis)
column is AA^T's eigenvectors
Σ: $m\times n$ diagonal matrix, σ is $\sqrt{\lambda}$ of A
V^T: $n\times n$ orthonormal basis vectors
row is A^TA's eigenvectors

$v_1 \cdots v_r$ orthonormal basis for row space ($C(A^T)$)
$u_1, \cdots, u_r$ " " " column space ($C(A)$)
$v_{r+1} \cdots v_n$ " " " null space
$u_{r+1} \cdots u_m$ " " " null space of A^T

and

$$A v_i = \sigma_{v_i} u_i$$