# LEC 10. The Four Fundamental Subspaces

2.6

① correct error in LEC 9

② four Fundemental subspace (for matrix A)

Today: connect the column space with row space

## 4 fundemental subspaces

① the column space $\boxed{C(A)}$ in $\mathbb{R}^m$ (pivot) ($r \le m$)

② the nullspace $\boxed{N(A)}$ in $\mathbb{R}^n$ (free $\le n$)

③ the row space = all combinations of rows

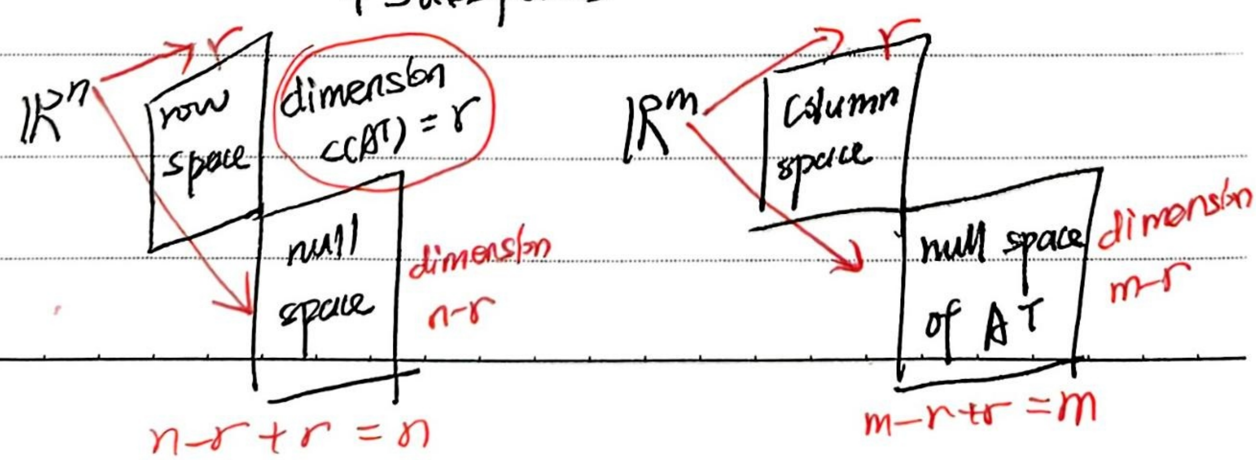
= all combinations of columns of $A^T$ = $\boxed{C(A^T)}$ in $\mathbb{R}^n$ ($r \le n$)

④ the nullspace of $A^T$, $\boxed{N(A^T)}$ ⇒ usual name: $\boxed{\text{left null space of } A}$

in $\mathbb{R}^m$ (free $\le m$)

A is $m \times n$

4 subspaces



basis? dimension?

dimension of $C(A)$ = rank $r$

① $C(A)$ basis: pivot columns, dimension $= r$

② $N(A)$ basis: special solutions, dimension $= n-r$

Example: ③ basis of $C(A^T)$ dimension $= r$

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} \overbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}^{I} & \overbrace{\begin{matrix} 1 & 1 \\ 1 & 0 \end{matrix}}^{F} \\ \begin{matrix} 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \end{matrix} \end{bmatrix} = R$$

$C(R) \neq C(A)$

different column space

but they have same row space

> the basis for row space
>
> is first $r$ rows of $R$

(why? when I took the operation, the row always stay in the row space)

4th space: $N(A^T)$, $\boxed{A^T y = 0}$, $[\quad][\quad] = [0]$

$y^T A = 0^T$, $[\ y^T\ ][A] = [0]$

↑ why called left null space

basis of $C(A^T)$

$E$

rref $[A_{m\times n} \ \vdots\ I_{m\times m}] \rightarrow [R_{m\times n} \ \vdots\ E_{m\times m}]$

$EA = R$ in chapter 2, $R$ was $I$, then $E$ was $A^{-1}$

in this case $E = \begin{bmatrix} -1 & 2 & 0 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} R$

how to find basis of $N(A^T)$

> I am finding the combination gives a zero row

④ basis of left null space $N(A^T)$

dimension $= m-r$

I don't need to transpot the matrix!

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$\boxed{\text{new vector space!}}$

All $3 \times 3$ matrices!!

$\underline{M}$ $A+B$ and $cA$

subspaces of $M$ : all upper triangulars / all symmetric matrices / diagonal matrices

→ the dimension of this is three

$[\,{}^{1}{}_{1}\,]\ [\,{}^{1}\,]\ [\,{}_{1}\,]$

stretching the idea from $R^n$ to $R^{n \times n}$!!