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LEC22 Diagonalization and Powers of A
2/1

diagonalization

$A-\lambda I$ singular $Ax=\lambda x$

Diagonaliging a matrix $S^{-1}AS=\Lambda$

Powers of A / equation $u_{k+1}=Au_k$

S is invertiable

Suppose n independent eigenvectors of A, put them in columns of S

$$AS = A[x_1\ x_2 \cdots x_n] = \begin{bmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \cdots & \lambda_n x_n \end{bmatrix}$$

the martix of eigenvectors

$$= \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ 0 & & 0 \\ \vdots & & \vdots \\ 0 & & \lambda_n \end{bmatrix} = S\Lambda$$

the

clumnx of eigenvectors

diagonal eigenvalue matrix Λ

$$\boxed{AS=S\Lambda} \Rightarrow S^{-1}AS=\Lambda, \quad A=S\Lambda S^{-1}$$

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if $Ax=\lambda x$, then $A^2x = A\lambda x = \lambda Ax = \lambda^2 x$

the eigenvalue of A^2 is λ^2, eigenvectors no change

$A = S\Lambda S^{-1}$

$\Rightarrow A^2 = S\Lambda S^{-1} \cdot S\Lambda S^{-1} = S\Lambda^2 S^{-1}$

it's telling me A^2's eigenvalue is Λ^2

$\Rightarrow A^k = S\Lambda^k S^{-1}$ and eigenvectors no change

$\boxed{\text{Theorem}}$: great fact

$A^k \to 0$ as $k \to \infty$

if all eigenvalue $|\lambda_i|$

how to justify the matrix is diagonalizable

A is sure to have n independent ~~ev~~ eigenvectors

(and be diagonalizable)

if all the λ's are different $\Rightarrow$ no reapted λ's

Repeated eigenvalues // may or may not have n independent eigenvectors

Suppose $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ $\quad \det(A - \lambda I) = (2-\lambda)^2$

$\Rightarrow \lambda = 2, 2$

1. $A - 2I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

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Equation $u_{k+1} = A u_k$

start with given vector u_0

$u_1 = A u_0$, $u_2 = A^2 u_0$ $\boxed{u_k = A^k u_0}$

$$\boxed{\frac{-b \pm \sqrt{b^2-4ac}}{2a}}$$

To really solve: write

$$u_0 = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n = Sc$$

$$A^{100} u_0 = c_1 \lambda_1^{100} x_1 + c_2 \lambda_2^{100} x_2 + \cdots + c_n \lambda_n^{100} x_n$$

$$= \cancel{\Lambda^{100} S c} \quad S \Lambda^{100} c$$

Fibonacci example: $0, 1, 1, 2, 3, 5, 8, 13$ ..

$$\boxed{F_{100} = ?}$$

$$\boxed{\begin{aligned} F_{k+2} &= F_{k+1} + F_k \\ F_{k+1} &= F_{k+1} + 0 F_k \end{aligned}} \xleftarrow{\text{TRICK}} u_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$$

$$u_{k+1} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}}_{u_k}$$

what means

$$\begin{bmatrix} F_{k+2} \\ F_{k+1} \end{bmatrix} = \begin{bmatrix} F_{k+1} + F_k \\ F_{k+1} + 0 \end{bmatrix}$$

$\Rightarrow u_{k+1} = A \cdot u_k$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix}$$

$$= \lambda^2 - \lambda - 1 = 0$$

$$\lambda = \frac{1 \pm \sqrt{1+4}}{2} \qquad \lambda_1 = \frac{1+\sqrt{5}}{2} \approx 1.618$$

$$\lambda_2 = \frac{1-\sqrt{5}}{2} \approx -.618$$

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So $F_{100} \approx c_1((1+\sqrt{5})/2)^{100}$

why?

$$A^{100}u_0 = c_1\lambda_1^{100}x_1 + c_2\lambda_2^{100}x_2$$

$$\begin{cases}\lambda_1 = \frac{1+\sqrt{5}}{2}\\ \lambda_2 = \frac{1-\sqrt{5}}{2}\end{cases}$$

$\lambda_2 = \frac{1-\sqrt{5}}{2}$, and this term goes to disappear! it tends to zero!

Amazing!

$$u_0 = c_1x_1 + c_2x_2$$

$$A - \lambda I = \begin{bmatrix}1-\lambda & 1\\ 1 & -\lambda\end{bmatrix}\begin{bmatrix}\lambda\\ 1\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

So $x_1 = \begin{bmatrix}\lambda_1\\ 1\end{bmatrix}$, $x_2 = \begin{bmatrix}\lambda_2\\ 1\end{bmatrix}$

$$u_0 = \begin{bmatrix}F_1\\ F_0\end{bmatrix} = \begin{bmatrix}1\\ 0\end{bmatrix}$$

$$\Rightarrow c_1x_1 + c_2x_2 = \begin{bmatrix}1\\ 0\end{bmatrix} \Rightarrow \begin{cases}c_1\cdot\frac{1+\sqrt{5}}{2} + \frac{1-\sqrt{5}}{2}c_2 = 1\\ c_1 + c_2 = 0\end{cases}$$

$$\boxed{u_k = A^k u_0}$$

Idea central:

when things are revolving in time by first order system staring with u_0, the key is finding eigenvalue and eigenvectors of A, and write u_0 as the linear combination of eigenvectors

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$$c_1\left(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right) = 1$$

$$c_1 = \frac{\sqrt{5}}{5}$$

So $F_{100} = \frac{\sqrt{5}}{5} \cdot \left(\left(1+\sqrt{5}\right)/2\right)^{100}$