LEC 16. ~~Project~~ Projections Least squares and best stright line. 2.8.

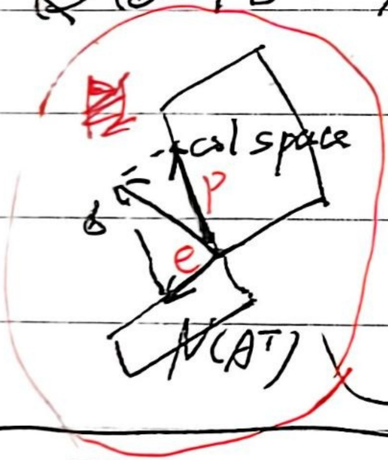
$$\begin{cases} Pb = b & \text{if } \underline{b \text{ in column } \overset{\overset{Ax}{\parallel}}{\text{Space}}} \ ① \\ Pb = 0 & \text{if } \underline{b \perp \text{column space}} \ ② \end{cases}$$

$$\Downarrow$$

$$A^T \cdot b = 0$$

$$② \ p = Pb = \boxed{A\cdot(A^TA)^{-1}A^T}\cdot \underline{b} = 0$$

$$\cancel{②} \ ① = Pb = A\cdot\underline{(A^TA)}^{-1}\,\underline{A^T\cdot A}x = Ax = b$$



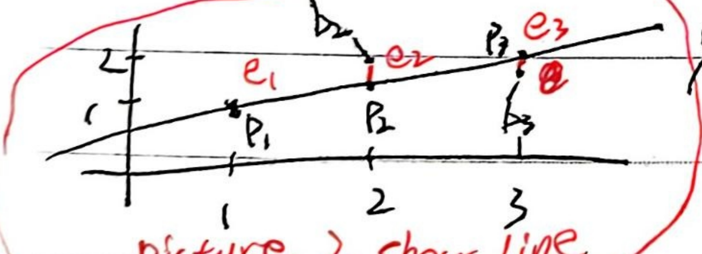
$p + e = b$ ↗ $(I-P)\cdot b$ ↑ projecton onto ⊥ space

↘ Pb

$(I-P)\cdot P = P - P = 0$

important

Find the best straight line



picture 2 show line

$$y = C + Dt$$

$$\begin{cases} C + D = 1 \\ C + 2D = 2 \\ C + 3D = 2 \end{cases}$$

$$Ax = b$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ \cancel{3}2 \end{bmatrix}$$

$$\text{Minimize } \|Ax - b\|^2 = \|e\|^2$$

$$= e_1^2 + e_2^2 + e_3^2$$

1 Find $\hat{x} = \begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix}$, P

$A = [a_1 \; a_2]$

why

$$\boxed{\begin{cases} P = A\hat{x} \\ A^T A \hat{x} = A^T b \end{cases}}$$

the most important equation

projection equation

$$\begin{cases} \cancel{e \cdot A\hat{x} = 0} & e \cdot A^T = 0 \\ \cancel{(b-p)A\hat{x} = 0} & (b-p)A^T = 0 \\ \cancel{(b-Ax} & (b-A\hat{x}) \cdot A^T = 0 \\ & \underline{A \cdot A^T \hat{x} = A^T \cdot b} \end{cases}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \left[\begin{array}{cc|c} 1 & 1 & \overset{b}{1} \\ 1 & 2 & 2 \\ 1 & 3 & 2 \end{array}\right] = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{A^T \cdot A} & A^T b \\ 3 & 6 & 5 \\ 6 & 14 & 11 \end{array}\right]$$

$$\Rightarrow \begin{cases} 3C + 6D = 5 \\ 6C + 14D = 11 \end{cases}$$

$$e_1^2 + e_2^2 + e_3^2 = ((C+D-1)^2 + (C+2D-2)^2 + (C+3D-2)^2)$$

this is the partial derivative $=0$ of this!

the square take detivative is a Linear equation!

$\Rightarrow \quad 2D = 1, \; D = \frac{1}{2}, \; C = \frac{2}{3}$

so the best line is $y = \frac{2}{3} + \frac{1}{2}D$

$$\left(\begin{array}{ll} P = \frac{AA^T}{A^TA}, & A\cdot\hat{x}\cdot A^TA = A\cdot A^T\cdot b \\ Pb = p = A\hat{x}, & \hat{x}A^TA = A^T\cdot b \end{array}\right)$$

$p_1 = \frac{7}{6}, e_1 = \frac{-1}{6}$ $p_2 = \frac{5}{3}, e_2 = +2/6$

$p_3 = \frac{13}{6}, e_3 = \frac{-1}{6}$

e ⊥ col space C(A), ⇒ in null space

$p + e = b$ $Pb + \boxed{(I-P)b} = b$

$$\begin{bmatrix}\frac{7}{6}\\ \frac{5}{3}\\ \frac{13}{6}\end{bmatrix} + \begin{bmatrix}\frac{-1}{6}\\ 2/6\\ -1/6\end{bmatrix} = \begin{bmatrix}1\\2\\2\end{bmatrix}$$

$p^Te = 0$! $p \perp e$

pictucal ⊥ for vectors

p is in the column space (b project to col space)

e is in the $N(A^T)$ (b project to $N(A^T)$)

$A = \begin{bmatrix}1&1\\1&2\\1&3\end{bmatrix} \Rightarrow e \perp \begin{bmatrix}1\\1\\1\end{bmatrix}$ two, $\Rightarrow e^T\cdot\begin{bmatrix}1\\1\\1\end{bmatrix} = [-\frac{1}{6}, \frac{2}{6}, -\frac{1}{6}]\begin{bmatrix}1\\1\\1\end{bmatrix} = 0$

c and D is the cofficient of A to p, is $\hat{x} = \begin{bmatrix}\hat{C}\\ \hat{D}\end{bmatrix}$

If A has independent columns, then A^TA is invertible

Why ? ⇒ next page

Proof:

Suppose $A^TAx = 0$ ⇒ prove x must be $\vec{0}$

it means $N(A^TA) = [0]$

1. TRICK $\underbrace{x^TA^T}_{}\underbrace{Ax}_{} = 0$ ⇒ $(Ax)^T(Ax) = 0$

⇒ Ax has to be zero

whe A ⇒ $Ax = 0$

has independent columns ⇒ $x = 0$

so x must be 0, showed

Columns definitely independent if they are perpendicular unit vectors

~~orogonal~~ / ~~normal~~

orthonormal

next LEC:

why orthonormal vectors are great and make ~~v~~ vectors orthonormal by picking the right basis