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LEC 5. Transposes, Permutations, Space R^n

textbook 2.7 + 3.1 2.3.

textbook 2.7. transpose and permutation

Transpose:

the columns of A^T are the rows of A.

$$(A^T)_{ij} = (A)_{ji}$$

$$\begin{cases} \text{Sum: } (A+B)^T = A^T + B^T \\ \text{Product: } (AB)^T = B^T \cdot A^T \quad \text{why} \\ \text{Inverse: } (A^{-1})^T = (A^T)^{-1} \end{cases}$$

$$\begin{cases} Ax \text{ combines the column of } A \\ x^T A^T \text{ combines the rows of } A^T \end{cases}$$

If $A = LDU$ the $A^T = U^T D^T L^T$, $D = D^T$

$A \cdot A^{-1} = I$, $\cancel{A^T (A^{-1})^T = I}$

$\Rightarrow (A^{-1})^T \cdot A^T = I$

dot product $x \cdot y \Rightarrow$ the sum of numbers $x_i y_i$

$= x^T y$ $(1 \times n)(n \times 1) \rightarrow$ a number

^{T} is outside, the rank one product or outer product is xy^T $(n\times 1)(1\times n) \rightarrow$ a matrix

$$(Ax)^T y = x^T (A^T y)$$

Definition A symmetric matrix has $S^T = S$

This means $S_{ji} = S_{ij}$

1. and the inverse of symmetric matrix is also symmetric
2. A^TA is always symmetric, because $(A^TA)^T = A^T \cdot A$ again

$A\cdot A^T$ also, but $A^TA \neq A\cdot A^T$

all squares

3. if S is symmetric $S = LDU \Rightarrow U = L^T$

$$\begin{bmatrix} 1 & 2 \\ 2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$\boxed{S = LDL^T}$$

Permutation Matrices

Definition: A permutation matrix P has the rows of the identity I in any order

Ex

$$I = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}, \quad P_{21} = \begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix}, \quad P_{32}P_{21} = \begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix}$$

$$P_{31} = \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix} \quad P_{32} = \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix} \quad P_{21}P_{32} = \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix}$$

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P^{-1} is always equal to P^T $\boxed{P^{-1} = P^T}$

$\Rightarrow$ because $PP^T = I$

The $\boxed{PA = LU}$ Factorization with Row Exchange

then $A = P^{-1}LU$

concentrate on this form

when $A = LU$, $P = I$

Examples

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 7 & 9 \end{bmatrix} \qquad PA = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 2 & 7 & 9 \end{bmatrix}$$

$l_{31} = 2$, $l_{32} = 3$

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad PA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix} = LU$$

LECTURE 5: some important points

- Permutations P : row exchanges
- $P^{-1} = P^T$

transpose:

- $(A^T)_{ij} = A_{ji}$
- Symmetric matrix $A^T = A$

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R^TR is always symmetric
why? ⇒ take the transpose $(R^TR)^T = R^TR$

textbook 3.1 Spaces of Vectors.
"Fundemental theorem of Linear Algebra"
The space $\mathbb{R}^n$ consists of all column vectors v with n components.

Example.
$\begin{bmatrix} 4 \\ \pi \end{bmatrix}$ in $\mathbb{R}^2$, $(1,1,0,1,1)$ in $\mathbb{R}^5$
$\begin{bmatrix} 1+i \\ 1-i \end{bmatrix}$ is in $\mathbb{C}^2$

Subspace
in $\mathbb{R}^3$ a plane is a vector space inside $\mathbb{R}^3$ (look like $\mathbb{R}^2$)
→ the plane going through $(0,0,0)$ is a subspace of the full vector space $\mathbb{R}^3$

Definition.
how to justy subspace
↓
A subspace of a vector space is a set of vectors (including 0) that satisfies two requirements: if v and w are vectors in the subspace and c is any scalar, then (i) $v+w$ is in the space ii) cv is in the space

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The Column Space of A

The system $Ax=b$ is solvable if and only if b is in the column space of A.

LEC5 continues

$\boxed{\text{Vector Spaces.}}$

Examples: $\mathbb{R}^2$ = all 2-dim real vectors

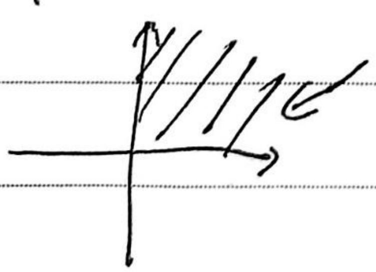
$\begin{bmatrix}3\\2\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}\pi\\e\end{bmatrix}$ || x-y plane

$\mathbb{R}^3$ = all vectors with 3 components

ex $\begin{bmatrix}3\\2\\0\end{bmatrix}$

$\mathbb{R}^n$ = all column vectors with n components.

example not a vector space:



$\frac{1}{4}$

it is not ~~col~~ closed by ~~mutipl~~ mulitple

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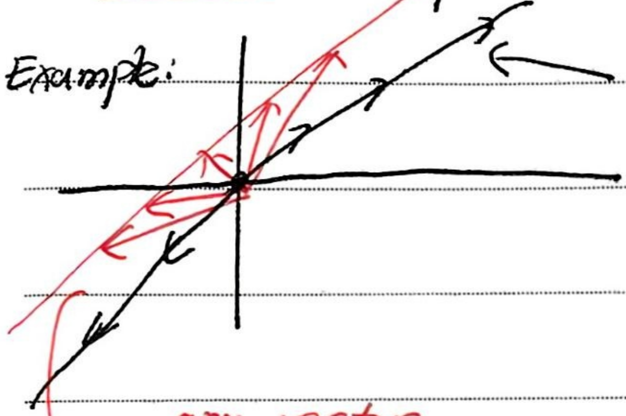
a vector space inside R^2 ⇒ $\boxed{\text{Subspace of } R^2}$

Example:



a whole line

line in R^2 must go through the zero vector

Subspace of R^2:

Plane L line L

① the whole space ② lines through the zero $\begin{bmatrix}0\\0\end{bmatrix}$

③ zero vector alone $Z = \begin{bmatrix}0\\0\end{bmatrix}$ look likes R^2 but not R^2

subspace: adding and multiply always in the same ~~space~~ subspace

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$$A = \begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}$$

~~Tow~~ Columns are in R3

the column space from the matrix

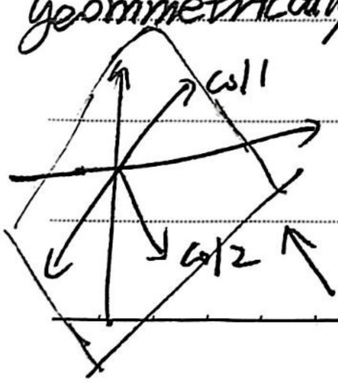
add, multiply, take all the linear combinations

form a subspace

$\boxed{\text{called column space } C(A)}$

how to create subspace from a matrix

geommetrically:



C(A) is getting a plane thorough (0,0,0)