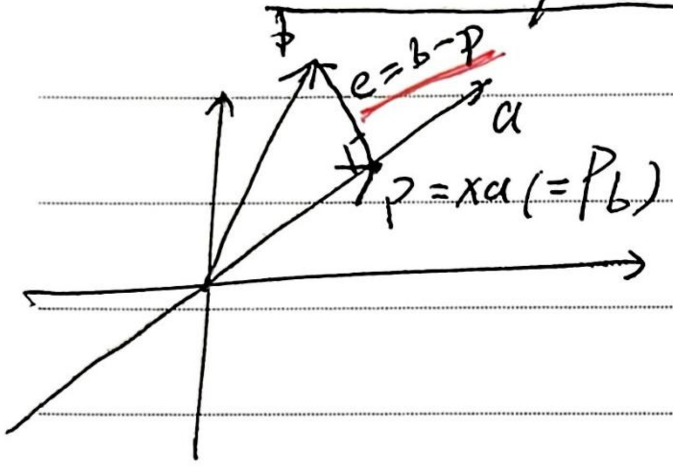
# LEC15. Projections onto Subspaces 2.8

Least squares ~~~~ projection MATRIX




$$a^T(b - xa) = 0$$

$$x a^T a = a^T b$$

$$x = \frac{a^T b}{a^T a} \qquad p = a x$$

↑ a scalar

$$\boxed{p = a\frac{a^T b}{a^T a}}$$

proj $p = Pb$ matrix $P = \frac{aa^T}{a^T a}$

↑ projection matrix

① $C(P)$ = line through $a$ : when I multiply any vectors by $P$ it will allways in the line a

② $\mathrm{rank}(P) = 1$

③* $P^T = P$ symmetric ~~orthogonal~~

④ $P^2 = P$

only orthonormal matrix project

how to sitify a matrix is a projection martix

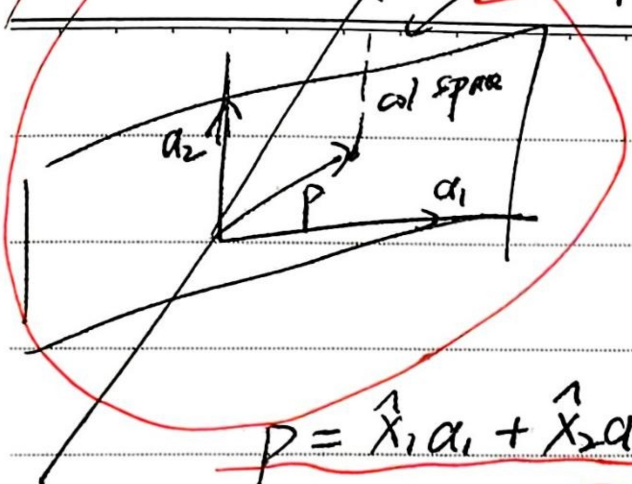
## Why projection

Because $Ax = b$ may have no solution

I'll solve $A\hat{x} = p$ instead $p = P\cdot b$

↘ projection of b onto col space

$b$ $\quad e = b - p$ is perp to the plane




plane of $a_1, a_2$ = col space of $A = \begin{bmatrix} | & | \\ a_1 & a_2 \\ | & | \end{bmatrix}$

$p = \hat{x}_1 a_1 + \hat{x}_2 a_2 = A\hat{x}$ $\quad$ ($\hat{x}$: must ~~ha~~ exist)

find $\hat{x}$, key: $b - A\hat{x}$ is perpendicular to the plane

$$\begin{cases} a_1^T(b - A\hat{x}) = 0 \\ a_2^T(b - A\hat{x}) = 0 \end{cases}$$

so it is also perp. to $a_1$ and $a_2$

$$A^T = \begin{bmatrix} a_1^T \\ a_2^T \end{bmatrix} (b - A\hat{x}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow A^T \underbrace{(b - A\hat{x})}_{e} = 0$$

so $e$ is $N(A^T) \Rightarrow e \perp C(A)$ YES!

$$\boxed{A^TA\hat{x} = A^Tb}$$

$\hat{x} = (A^TA)^{-1}A^Tb \quad \xrightarrow{1D} \frac{aa^T}{a^Ta}$

$p = A\hat{x} = A(A^TA)^{-1}A^Tb$

matrix $P = A(A^TA)^{-1}A^T$

if $A$ is not a square, $A^{-1}$ is not exist

($A$ is not intertiable)

if $A$ is a square $\Rightarrow$ $A$'s column space is $R^n$, $P = I$

$P^2 = P$

$P^T = P$

$$P = A(A^TA)^{-1}\underbrace{\cancel{A^T A}}_{I}(A^TA)^{-1}A^T$$

it means $N(A)$ is not zero space

all non-identity projection matrix are singular $\Rightarrow$ (all vectors to the $C(A)$ is projected to $0$)

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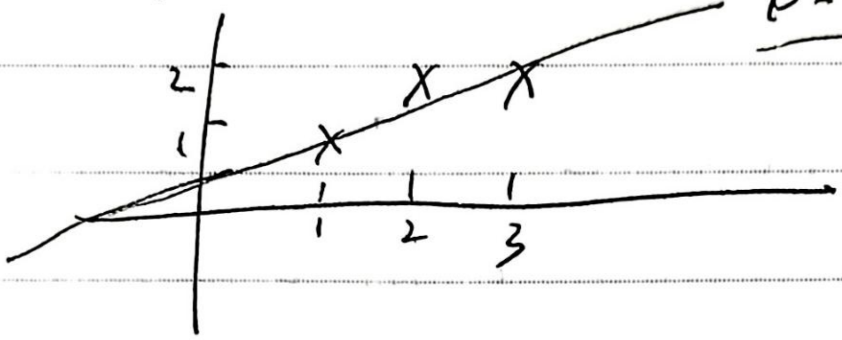
# Least Squares

Fitting by a Line




$b = C + Dt$

$$\begin{cases} C + D = 1 \\ C + 2D = 2 \\ C + 3D = 2 \end{cases}$$

three points

$(1,1),\ (2,2),\ (3,2)$

we would like to solve
but we can't

$$\underset{A}{\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}} \underset{x}{\begin{bmatrix} C \\ D \end{bmatrix}} = \underset{b}{\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}}$$

By solve

$\Rightarrow A^TA\hat{x} = A^Tb$

why not? $\Rightarrow A^T$ may be not square ~~It hasn't~~ is invertable

$A\hat{x} = (A^T)^{-1}A^T b$

The lec15's ~~consense~~ talk

$A\hat{x} = P \leftarrow \cancel{P =}\ P = x_1 a_1 + x_2 a_2 = A\hat{x}$

no equal to $x$

$A^T \cdot e = 0 \Rightarrow A^T(b - A\hat{x}) = 0$

$e \perp C(A)$

$A^TA\hat{x} = A^Tb$

martix $P = A \cdot (A^TA)^{-1}A^T$

$\hat{x} = A^{-1}A^{T^{-1}}A^Tb$

$\hat{x} = (A^TA)^{-1}A^Tb$

$p = A\hat{x} = A \cdot (A^TA)^{-1}A^Tb$