LEC17 Orthogonal Matrices and Gram-Schmidt orthogonal basis 2.9

Orthonormal vectors

~~$q_i q_j$~~ $q_i q_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$

vector

q

~~$q\ q\ q\ q$~~

$$Q = \begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix} \qquad \underline{Q^T Q} = \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \begin{bmatrix} q_1 & \cdots & q_n \end{bmatrix}$$

$$= I$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix}$$

orthogonal Matrices Q: only the square

what means $Q^T Q = I$, $Q^T = Q^{-1}$

Examples: permutation $Q = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, Q is orthogonal matrix

$$QQ^T = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = I$$

ex2.

$$Q = \begin{array}{cc} v1 & v2 \\ \end{array}$$
$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \qquad \left(\theta = \frac{\pi}{4}\right)$$

$$Q = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \qquad Q = \begin{bmatrix} 1 & -2 \\ 2 & 1 \\ 2 & 2 \end{bmatrix}\frac{1}{3}$$

Suppose Q has orthogonal columns

project onto its column space

$$P = \cancel{\frac{Q^TQ}{QQ^T}} = Q \underbrace{(Q^TQ)^{-1}}_{I} Q^T$$

$$= QQ^T \quad \} = I \text{ if } Q \text{ is square}$$

because if Q is a square, and Q's column is independent

$C(Q)$ = whole space, which means $P = I$

if Q is not a square

① $(QQ^T)(QQ^T) = QQ^T$, ② symmetric

$A^TA\hat{x} = A^Tb$, now A is Q

$\Rightarrow \underbrace{Q^TQ}_{I}\hat{x} = Q^Tb \Rightarrow \hat{x} = Q^Tb \Rightarrow \boxed{\hat{x}_i = q_i^Tb}$

Gram-Schmidt

Independent vectors a, b My goal: orthogonal A, B

$\downarrow$ orthonormal $q_1 = \frac{A}{\|A\|}$

$q_2 = \frac{B}{\|B\|}$

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only $A^T \cdot b$ is a scalar can be written as this

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1 $(\dim(A) = 1)$

$$B = b - p = b - \frac{A^T b}{A^T A} A = b - \frac{A A^T}{A^T A} \cdot b$$

$$A^T B = A^T \left(b - A \frac{A^T b}{A^T A}\right) = A^T b - A^T A \cdot \frac{A^T b}{A^T A} = 0$$

$\Rightarrow A \perp B$

if there is a third vector C, $C \perp A$, $C \perp B$

$$q_3 = \frac{c}{\|C\|} = c - \frac{A^T c}{A^T A} A - \frac{B^T c}{B^T B} B$$

$\dim(A) = 1$

c - the components in A and B

$\Rightarrow C \perp A \perp B$

$$a^T a \hat{x} = a^T b$$

$$\hat{x} = a^T b$$

Ex) $a = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \frac{A^T b}{A^T A} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\underset{A}{\Downarrow}$ $\quad = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \frac{3}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\Rightarrow q_1\ q_2 \quad a = \begin{bmatrix} q_1 & q_2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & -1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{2} \end{bmatrix}$ $\quad = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

$B \perp A$

coming from gram-Schmidt

how to get A, B | $q_1 = \frac{A}{|A|}$, $q_2 = \frac{B}{|B|}$

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$A = LU \Rightarrow \boxed{A = QR}$

$A = [a_1 \ a_2] = [q_1 \ q_2] \underbrace{\begin{bmatrix} & \\ 0 & \end{bmatrix}}_{R}$