LEC4. Factorization into $A=LU$ 2.3 (Text 2.6)

$$AA^{-1} = I = A^{-1}A$$

$$(AB)(B^{-1}A^{-1}) = I \longrightarrow A \cdot BB^{-1} A^{-1}$$

Transpot $\quad AA^{-1} = I$

$$(A^{-1})^T A^T = I$$

$$= (A^T)^{-1} \text{ inverse of } A^T$$

$E \quad A \qquad\qquad U$ (upper) $\quad A = LU$

$$\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \qquad \begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}\underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}}_{D}\begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}$$

3x3:

$$E_{32}E_{31}E_{21}A = U$$

$$A = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}U = LU$$

row exchanges: Permutations 3x3

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \qquad P^{-1} = P^T$$

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1.3 Matrices

Ax is a combination of the columns

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = Ax = x_1 \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + x_2 \cdot \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$$

2.2 The idea of Elimination

Elimination produces an upper triangular system U

Elimination:

$$\begin{aligned} 2x + 4y - 2z &= 2 \\ 4x + 9y - 3z &= 8 \\ -2x - 3y + 7z &= 10 \end{aligned} \quad Ax = b \Rightarrow \text{has became} \Rightarrow Ux = c \quad \begin{aligned} 2x + 4y - 2z &= 2 \\ 1y + 1z &= 4 \\ 4z &= 8 \end{aligned}$$

matrix U

U: a upper triangular system

2.6. Textbook

factorization that comes from elimination is $A = LU$

$\uparrow$ Lower, $\searrow$ ~~Upper~~ U

Example: $A = \begin{bmatrix} 2 & 1 \\ 6 & 8 \end{bmatrix}$

$$E_{21} A = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 5 \end{bmatrix} = U$$

$$E_{21}^{-1} \cdot U = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 6 & 8 \end{bmatrix} = A$$

$$LU = A$$

$n \times n$: $L = E_{21}^{-1} \cdot E_{31}^{-1} \cdot E_{32}^{-1} \cdots$

$$\boxed{3\times 3: \quad (E_{32} E_{31} E_{21}) A = U \Rightarrow \underbrace{(E_{21}^{-1} E_{31}^{-1} E_{32}^{-1})}_{L} \cdot U = A}$$

$\hookrightarrow A = L \cdot U$

1. every E^{-1} is lower triangular. its off-diagonal entry is l_{ij}, to undo the substraction produced by $-l_{ij}$

Examples:

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} = LU = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & \frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & \frac{3}{2} & 1 \\ 0 & 0 & \frac{4}{3} \end{bmatrix}$$

$(l_{21} = \frac{1}{2}, \ l_{32} = \frac{2}{3})$

$l_{31} = 0$

$\begin{cases} \text{when a row of } A \text{ starts with zeros, so does that row of } L \\ \text{when a column of } A \text{ starts with zeros, so does that column of } U \end{cases}$

Row 3 of U = (Row 3 of A) $- \ell_{31}$ (Row 1 of U) $- \ell_{32}$ (Row 2 of U)

Row 3 of A = ℓ_{31} (Row 1 of U) $+ \ell_{32}$ (Row 2 of U) $+ 1$(Row 3 of U)

split U: $\begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix} \begin{bmatrix} 1 & u_{12}/d_1 & u_{13}/d_1 & \\ & 1 & u_{23}/d_2 & \\ & & 1 & \\ & & & 1 & \\ & & & & \ddots \end{bmatrix}$

The factorization can be written $A=LU$ or $A=LDU$

$$\underset{L}{\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}} \underset{\cancel{D}\,U}{\begin{bmatrix} 2 & 8 \\ 0 & 5 \end{bmatrix}} \Rightarrow \underset{L}{\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}} \underset{D}{\begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}} \underset{U}{\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}}$$

L holds the numbers that multiplied the pivot rows

when do we need this record?

$\Rightarrow$ solving $Ax = b$

$\Rightarrow LUx = b, \quad x = L^{-1}U^{-1}b$

Solve $Lc = b$ and then solve $Ux = c$

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Example 3:

$$Ax=b \quad \begin{aligned} u+2v&=5 \\ 4u+9v&=21 \end{aligned} \qquad \left.\begin{aligned} u+2v&=5 \\ v&=1 \end{aligned}\right\} Ux=c$$

$$Lc=b \quad \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} c \end{bmatrix} = \begin{bmatrix} 5 \\ 21 \end{bmatrix} \Rightarrow c = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

$$Ux=c \quad \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix} \Rightarrow x = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$