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Memo No.
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LEC 26. Complex Matrices; Fast Fourier Transform
224

$z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$ length in $\mathbb{C}^n$ not in $\mathbb{R}^n$

FFT n^2 mults to $n \log n$ mults

the point is $z^T z$ is no good : doesn't give length

what I want is $\bar{z}^T z$ $[\bar{z}_1\ \bar{z}_2 \cdots \bar{z}_n]\begin{bmatrix} \bar{z}_1 \\ \bar{z}_2 \\ \vdots \\ \bar{z}_n \end{bmatrix}$

ex $z = \begin{bmatrix} 1 \\ i \end{bmatrix}$ $\bar{z}^T z = [1\ -i]\begin{bmatrix} 1 \\ i \end{bmatrix} = 1+1 = 2$

there is a simple who do ~~this~~ :

$z^H z$ H: Hermit give the Hermition

$H = \bar{\ }^T$

inner product : for real matrix: $y^T x$

for complex matrix : $y^H x = \bar{y}^T x$

$z^H z = |z_1|^2 + |z_2|^2 + \cdots + |z_n|^2$ the length square

Symmetric : $A^T = A$ no good if A is complex

So we insteade : $\bar{A}^T = A$, the complex version

$\bar{A}^T = A = \begin{bmatrix} 2 & 3+i \\ 3-i & 5 \end{bmatrix}$ this is Hermition $A^H = A$

go on to perpendicular:

$q_1, q_2, \cdots, q_n$ $\qquad Q = \begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix}$

$q_i^T q_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

orthogonal matrix

$Q^T Q = \boxed{I = Q^H Q}$

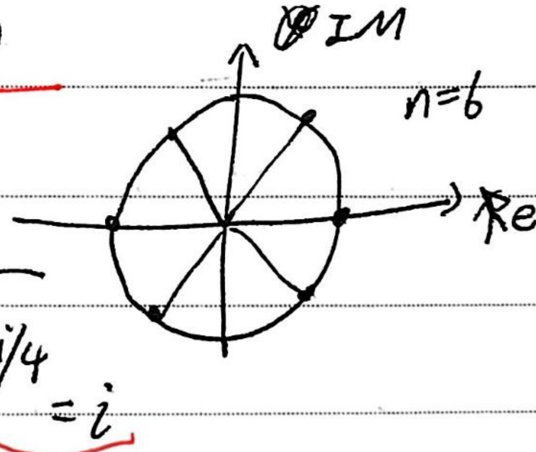
FFT:

$$F_n = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & w & w^2 & \cdots & w^{n-1} \\ 1 & w^2 & w^4 & \cdots & w^{2(n-1)} \\ \vdots & & \vdots & & \vdots \\ 1 & w^{n-1} & w^{2(n-1)} & \cdots & w^{?} \end{bmatrix}$$

$(F_n)_{ij} = w^{ij}, \quad i,j = 0, \cdots, n-1$

$w^n = 1 \qquad w = e^{i2\pi/n}$

$= \cos\frac{2\pi}{n} + i \sin\frac{2\pi}{n}$



$n=4 \quad w^4 = 1 \quad w = e^{2\pi i/4} = i$

$w = i$

$i, \; i^2 = -1, \; i^3 = -i, \; i^4 = 1$

$$F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}$$

$w^{3\times 3} = w^9 = \cancel{(e^{i\frac{\pi}{2}})^9}$

Inner product of col2 and col4

$$1\cdot 1 + i\cdot(-i) + 1 + (-i)\cdot i = 4$$

but they are complex vector! so we

$$\text{col}2^{H}\cdot \text{col}4 = 1 + i\cdot i + 1 + (-i)\cdot(-i) = 0$$

$\Rightarrow$ divided by 2 $\Rightarrow$ cols orthonormal

$$\frac{1}{2}F_4^H \frac{1}{2}F_4 = I \quad , \quad \frac{1}{4}F_4^H\cdot F_4 = I$$

$$(w_{64})^2 = w_{32} \quad , \quad \left(e^{\frac{2\pi}{64}i}\right)^2 = e^{\frac{2\pi}{32}}$$

$\begin{bmatrix} F_{64} \end{bmatrix}$ and $\begin{bmatrix} F_{32} & 0 \\ 0 & F_{32} \end{bmatrix}$ connected

$$\begin{bmatrix} F_{64} \end{bmatrix} = \begin{bmatrix} I & D \\ I & -D \end{bmatrix} \begin{bmatrix} F_{32} & 0 \\ 0 & F_{32} \end{bmatrix} \underbrace{\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \\ & & 1 & \end{bmatrix}}_{P}$$

$\rightarrow \cancel{x} \longrightarrow 64^2$

$$D = \begin{bmatrix} 1 & & & \\ & w & & \\ & & w^2 & \\ & & & \ddots \\ & & & & w^{31} \end{bmatrix}$$

$2(32)^2 + \text{fix}$

$= 2(32)^2 + 32$

$= 2\left[2[16]^2 + 16\right] + 32$

$\rightarrow 6\times 32 \quad \log_2 64$

final count $\rightarrow \boxed{\frac{1}{2}n\log_2 n}$