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LEC 7 solving $Ax=b$: Pivot Variables 2nS special solutions | $Ax=0$ null space

$$A=\begin{bmatrix}1&2&2&2\\2&4&6&8\\3&6&8&10\end{bmatrix}$$

pivot columns

pivot

elimination ⟹

$$\begin{bmatrix}\textcircled{1}&2&2&2\\0&0&\textcircled{2}&4\\0&0&2&4\end{bmatrix}\longrightarrow\begin{bmatrix}1&2&2&2\\0&0&2&4\\0&0&0&0\end{bmatrix}=U$$

the rank of A = number of pivots

free columns

in this case = 2

2 pivot columns, and 2 free columns

solve $Ux=0$

$$x=\begin{bmatrix}\textcircled{1}\\0\\\textcircled{0}\end{bmatrix}$$

① → free clumn, can be any number

⓪ → free column

$Ux=0$

$$\begin{cases}\textcircled{x_1}+2x_2+2x_3+2x_4=0\\ \textcircled{2}x_3+4x_4=0\end{cases}$$

let $x_2=1,\ x_4=0$

⇒ then $x_3=0,\ x_1=-2$

$$\Rightarrow x=\begin{bmatrix}\textcircled{-2}\\1\\\textcircled{0}\\0\end{bmatrix}\cdot c$$

in null space, it's a line

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if $Ax=0$, we do the elimination, x always in the null space, right always be 0

if $x_2=0, x_4=1$ $\quad x = \begin{bmatrix} 0 \\ -\frac{1}{2} \\ 1 \end{bmatrix}$ d· $\begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$ → free, → free

What all the solution to $Ax=0$?

$$x = c\cdot\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d\cdot\begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} \qquad Ax=0 \Rightarrow Ux=0$$

take all combination of the special solutions

in this case, U called row Echelon form

$n-r = 4-2 = 2$ free variables

R = reduced row echelon form ⇒ next step

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

elimination knocked out that row3 is the combination of row1 & 2

zero & above & below pivots = 1

$$\begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R = \text{rref}(A)$$

↑ matlab

↑ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ in pivot rows/cols

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row echelon form R $\quad x = c\begin{bmatrix}2\\1\\0\\0\end{bmatrix} + d\begin{bmatrix}2\\0\\-2\\1\end{bmatrix}$

$$\Rightarrow \begin{cases} x_1 + 2x_2 - 2x_4 = 0 \\ x_3 + 2x_4 = 0 \end{cases} \qquad Rx = 0 \quad \text{(final way)}$$

Pivot cols: $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \to I$

free cols $\begin{bmatrix}2 & -2\\ 0 & 2\end{bmatrix} \to F$ of matrix

$b = \begin{matrix}0 & 0\end{matrix}$ $\qquad$ $\begin{matrix}0 & 0\end{matrix}$

$$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$$ reduce echelon form

r pivot col

n-r free colum

$Rx = 0$ $\qquad$ null space Matrix (columns = special solutions)

$RN = 0$

$$N = \begin{bmatrix} -F \\ I \end{bmatrix}$$

$$Rx, \quad [I\ F]\begin{bmatrix} x_{pivot} \\ x_{free} \end{bmatrix} = 0$$

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Another example

$$A=\begin{bmatrix}1&2&3\\2&4&6\\2&6&8\\2&8&10\end{bmatrix}\rightarrow\begin{bmatrix}1&2&3\\0&0&0\\0&2&2\\0&4&4\end{bmatrix}\rightarrow\begin{bmatrix}\textcircled{1}&2&3\\0&\textcircled{2}&2\\0&0&0\\0&0&0\end{bmatrix}$$

↓ ↓ ↙ 1 free

pivot column

U

rank = 2 again! 3 − 2 = 1 free column

whole null space

$$x=\begin{bmatrix}-1\\-1\\1\end{bmatrix}\cdot C \iff \begin{cases}x_1+2x_2+3x_3=0\\2x_2+2x_3=0\\x_3=1\end{cases}$$

this answer

↑ free variable

and if free variable = 0, x will be 0, it is the origin!

1. find U and Rank
2. take the free variables
3. write X, the null space of A

keep going to R

$$\rightarrow\begin{bmatrix}1&0&1\\0&1&1\\0&0&0\\0&0&0\end{bmatrix}=R=\begin{bmatrix}I&F\\0&0\end{bmatrix}\Rightarrow x=\begin{bmatrix}-1\\-1\\1\end{bmatrix}\cdot C$$

pivot variables

I, free variables

$$\begin{cases}x_1+x_3=0\\x_2+x_3=0\end{cases}$$

$$=\begin{bmatrix}-F\\I\end{bmatrix}\cdot C$$

↑ N