LEC12 Bellman-Ford

3.30 https://github.com/GUMI-21/MIT6.006_note

Previously

- weighted graphs
  shortest-path weight, negative-weight cycles
- BFS positive weighted graphs
- DAG Relaxtion
  algorithm to solve SSSP on a weighted DAG in O(|V | + |E|) time
- SSSP for graph with negative weights
  -Compute $\delta(s, v)$ for all $v \in V$ ($-\infty$if v reachable via negative-weight cycle)
  -if a negative-weight cycle reachable from s, return one

for this lecture, we restrict our discussion to directed graphs

Restrictions		SSSP Algorithm		
Graph	Weights	Name	Running Time $O(\cdot)$	Lecture
General	Unweighted	BFS	$\|V\| + \|E\|$	L09
DAG	Any	DAG Relaxation	$\|V\| + \|E\|$	L11
General	Any	Bellman-Ford	$\|V\| \cdot \|E\|$	L12 (Today!)
General	Non-negative	Dijkstra	$\|V\| \log \|V\| + \|E\|$	L13

Warmup Exercise

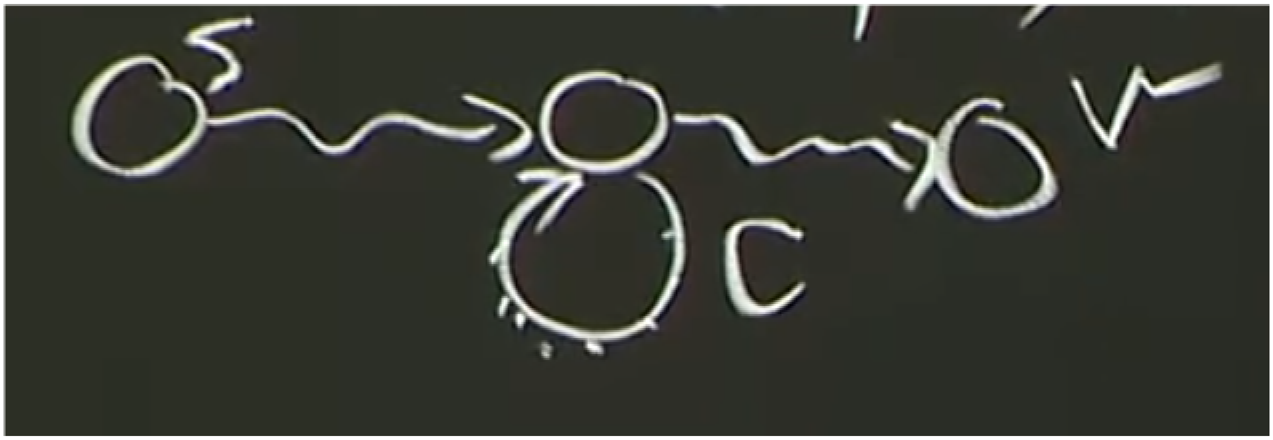
- Ex1: Given undirected graph G, return whether G contains a negative weight cycle
  every edge with negative edge can be a cycle.
- Ex2: If have Alg A solves SSSP in O(|V|(|V|+|E|))
  Show how to solve SSSP in O(|V|·|E|)
  Use BFS or DFS find all the things reachable from S, and throw ohters away, then I have a graph which V is asymptotically no bigger than E, means O(|V|(|V|+|E|) = O(|V||E|)

Simple Shortest Paths

If graph does not contain negative-weight cycles, shortest paths are simple!

- Claim 1: If δ(s, v) is finite, there exists a shortest path from s to v is *simple*
  PROOF: By contradiction

Assume there is a cycle in the path from s to v, two case: 1.the cycle weight is negative, contradiction, -> -infinite. 2.the cycle weight is 0 or negative, I can just erase the cycle to get the shortest path.



- simple paths cannot repeat vertices, finite shortest paths contain at most |V | − 1 edges
  So the Shortest path is simple, and *|E| <= |V| - 1*

Negative Cycle Witness

- then a Idea
  What if I limit the number of edges when find the shortest weighted path?
  this is called *k-Edges Distance $\delta_k(s, v)$: shortest s-v path using <= k edges*

- *A Statement*
  If $\delta_{|V|}(s, v) < \delta_{|V|-1}(s, v)$, then $\delta(s, v) = -\infty$
  ---If I can find a vertex that has this property, **this vertex is called a witness**.---
- *Claim 2* if $\delta(s, v) = -\infty$,then v is reachable from a witness.
  Proof: By contradiction
  Alenative: Prove every negative weighted cycle contains witness
  Think there is a Cycle with negative weighted, select a vetex v and his predeccessor v'.
  Then $\delta_{|v|}(s, v) <= \delta_{|v|-1}(s, v') + w(v', v)$
  then take this equation to all cycle vertices, sum of w(v',v) is the weight of cycle which is negative, then erase the w equation give $\delta_{|v|}(s, v) < \delta_{|v|-1}(s, v')$, which means if there isn't witness in cycle there will be a constrcution of the defination of witness. proved.

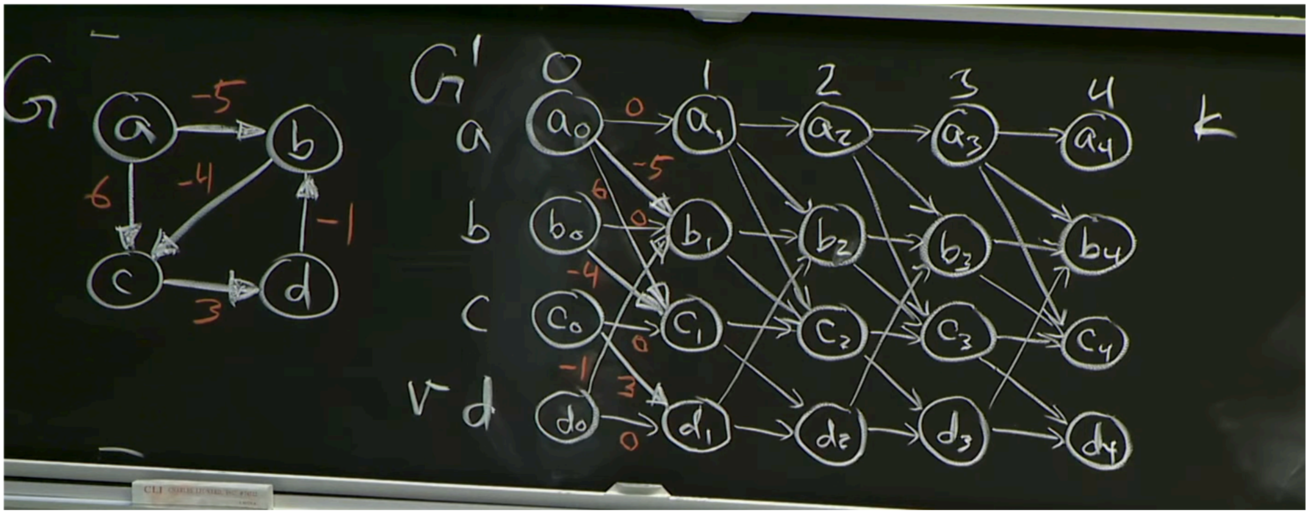
Bellman-Ford

see the R12 note whit the original Bellman-Ford, it's easier to understand!

A modified Bellman-Ford

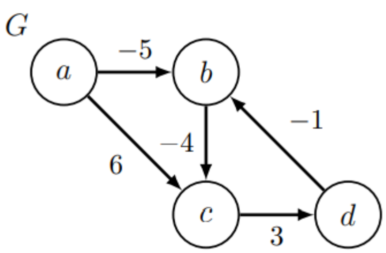
- Idea! Use graph duplication: make multiple copies (or levels) of the graph
  this is a very common technique.

=> Make |v| + 1 levels, V_k in level k represents reaching vertex v using at most k edges.
If we connect edges from one level to only higher levels, then the graph is DAG!



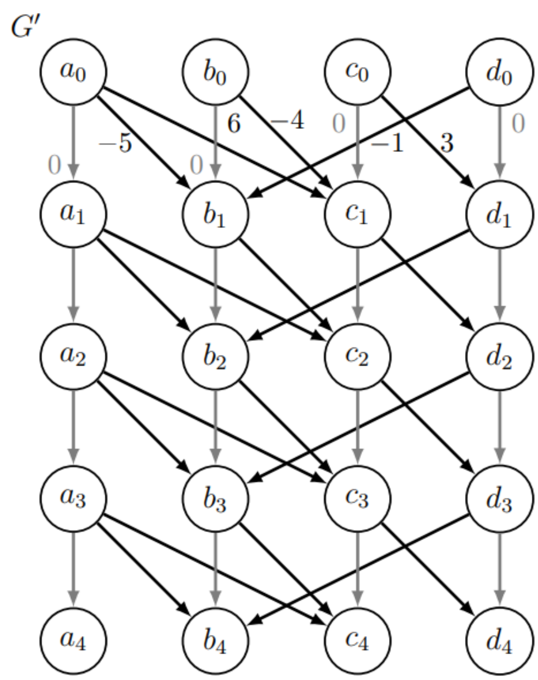
- ALG
  -Construct G' |V|(|V| + 1) vertices, |V||V| + |V||E| = |V|(|V| + |E|) edges
  -Run DAG Relaxation from S_0 compute $\delta(s_0, v_k)$ fro k = {0,...,|V|}
  -For each vertex V: set d(S,V) = $\delta(S_0, V_{|V|-1})$
- Claim: $\delta(S_0, V_k) = \delta(S, V)$
  -For each witness u in V, $\delta(s_o, u_{|v|}) < \delta(s_0, u_{|v|-1})$
  -for each v reachable from u, set d(s,v) = -infinty
- EXAMPLE from Note

Example



$\delta(a_0, v_k)$

$k \setminus v$	a	b	c	d
0	0	∞	∞	∞
1	0	$\mathbf{-5}$	$\mathbf{6}$	∞
2	0	-5	$\mathbf{-9}$	$\mathbf{9}$
3	0	-5	-9	$\mathbf{-6}$
4	0	$\mathbf{-7}$	-9	-6
$\delta(a, v)$	0	$-\infty$	$-\infty$	$-\infty$



Correctness

- Claim3: $\delta(s0, vk) = \delta k(s, v)$ for all $v \in V$ and $k \in \{0, . . . , |V|\}$