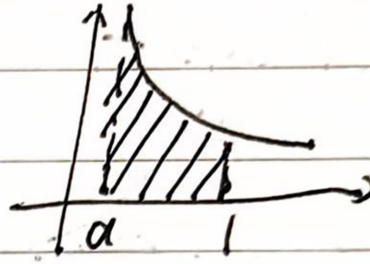
wrong, because $(-1,1)$, $\frac{1}{x^2}$ is diverges

LEC 37. 2025.1.3.

IMPROPER INTEGRALS (KIND 2)

$$\boxed{\int_0^1 f(x)\,dx = \lim_{a\to 0^+} \int_a^1 f(x)\cdot dx}$$




converge: the limit is exist $a \to 0^+$

diverge: if not

Ex1: $\int_0^1 \frac{dx}{\sqrt{x}} = \int_0^1 x^{-\frac{1}{2}}\,dx = 2\cdot x^{\frac{1}{2}}\Big|_0^1 = 2$

convergent

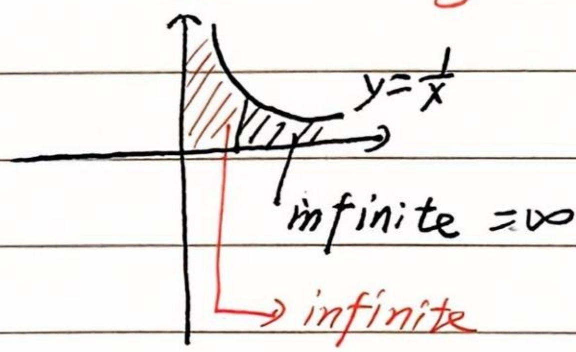
Ex2. $\int_0^1 \frac{dx}{x} = \ln x \Big|_0^1 = \ln 1 - \ln 0^+ = +\infty$ diverges

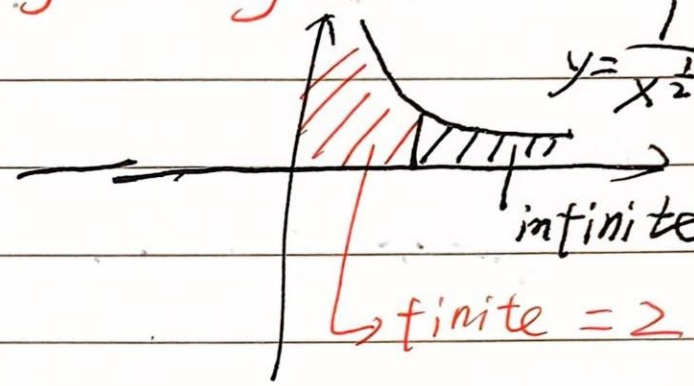
CONTRAST (对比)

$$\boxed{\frac{1}{x^{\frac{1}{2}}}} << \boxed{\frac{1}{x}} << \boxed{\frac{1}{x^2}}, \quad \text{as } x \to 0^+$$

$$\boxed{\frac{1}{x^{\frac{1}{2}}}} >> \boxed{\frac{1}{x}} >> \boxed{\frac{1}{x^2}}, \quad \text{as } x \to \infty$$

□ : diverges ~~integant~~ integral

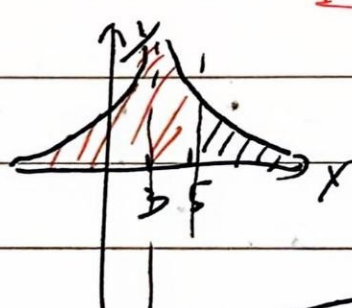



□ : converge, finite

EX: $\int_0^{\infty} \frac{1}{(x-3)^2} = \boxed{\int_0^5 \frac{1}{(x-3)^2}} + \boxed{\int_5^{\infty} \frac{1}{(x-3)^2}}$

↑ infinite ↑ finite

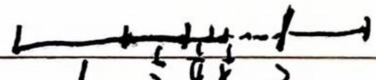



## INFINITE SERIES

~~Gem~~ Geometric series

$$1+a+a^2+\cdots+a^3+\cdots=\frac{1}{1-a}, \quad \text{as } |a|<1$$

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots=2$$




Date.

$a=1$, $1+1+\cdots+1 = \frac{1}{1-1} = \frac{1}{0} \to \infty$, diverges

$a=-1$, $1-1+1-1+1-1 = \frac{1}{1-(-1)} = \cancel{\frac{1}{2}}$ diverges

**Notation** (级数)

$S_N = \sum_{n=0}^{N} a_n$, partial sum

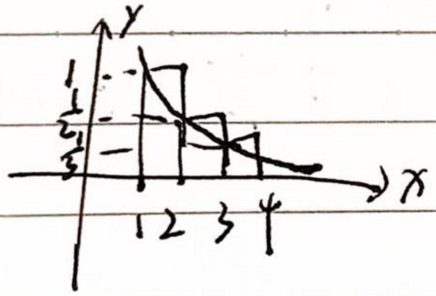
$S = N \to \infty = \lim_{N\to\infty} S_N \begin{cases} \text{limit exist (the series converge)} \\ \text{limit not exist (udisconverge)} \end{cases}$

Ex1. $\sum_{n=1}^{\infty} \frac{1}{n^2} \sim \int_1^{\infty} \frac{1}{x^2}dx$ (converge)

$\quad = \pi^2/6$ ↳ use Riemaa Sun as $\Delta x = 1$

Ex2. $\sum_{n=1}^{\infty} \frac{1}{n^3} \sim \int_1^{\infty} \frac{dx}{x^3} \sim (\frac{1}{2})$ converge

Ex3. $\sum_{n=1}^{\infty} \frac{1}{n} \longleftrightarrow \int_1^{\infty} \frac{1}{x}dx$ (diverges)



upper Riemann sum

$\int_1^N \frac{dx}{x} < 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{N-1} < S_N$

$S_N = 1+\frac{1}{2}+\cdots+\frac{1}{N-1}+\frac{1}{N}$

$\int_1^N \frac{dx}{x} < S_N$

$\downarrow$

$\ln N$

$\therefore \ln N < S_N$, $N\to\infty$, $S_N \to \infty$ $\therefore S_N$ diverges

The Lower Riemann Sum ($\Delta x = 1$):

$\int_1^N \frac{dx}{x} > \frac{1}{2}+\frac{1}{3}+\cdots \frac{1}{N} = S_N - 1$

$\boxed{(\ln N \le S_N < (\ln N)+1)}$

## INTEGRAL COMPARISON

if $f(x)$ is decreasing, $f(x) > 0$

Then $\left| \sum_{n=1}^{\infty} f(n) - \int_1^{\infty} f(x)\,dx \right| < f(1)$

and $\sum_{n=1}^{n} f(n)$ and $\int_1^{\infty} f(x)\,dx$ converge or diverge together

## LIMIT COMPARISON

if $f(x) \sim g(n)$, $\frac{f(n)}{g(n)} \to 1$ as $n \to \infty$

and $g(n) > 0$ then ~~same~~ $\sum f(n)$, $\sum g(n)$ either both converge or both diverge

Ex1. $\sum \frac{1}{\sqrt{1+n^2}}$

↳ $\sum \frac{1}{\sqrt{n^2}} = \sum \frac{1}{n}$ diverges.

Ex2: $\sum_{n=2}^{n} \frac{1}{\sqrt{n^5 - n^2}} \to \sum \frac{1}{\sqrt{n^5}} = \sum \frac{1}{n^{\frac{5}{2}}}$ converges

## Ratio Test:

$\sum_{n=1}^{n} \cancel{f(n)}\, a_n$,

with $a_n > 0$ for all $n$

$L = \lim_{n \to \infty} \frac{a_{n+1}}{a_n}$

if $L < 1$ ⇒ The series converges

$L > 1$ ⇒ The series diverges

$L = 1$ ⇒ No conclusion: the series ~~ca~~ both able to