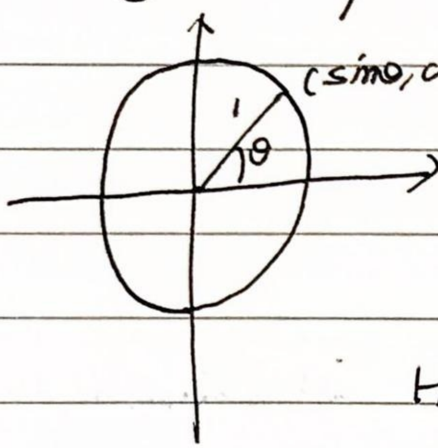
Unit 4 ~~Definite Integrals~~ 2024.12.30

Techniques of Integration

Lec 27.

trignometry (三角学)



$$\cos^2\theta + \sin^2\theta = 1$$

$$\boxed{\begin{aligned} \cos(2\theta) &= \cos^2\theta - \sin^2\theta \\ \sin 2\theta &= 2\sin\theta\cos\theta \end{aligned}}$$

Half-angle formule:

$$\cos(2\theta) = \cos^2\theta - (1-\cos^2\theta)$$

$$= 2\cos^2\theta - 1$$

$$\boxed{\begin{aligned} \cos^2\theta &= \frac{\cos 2\theta + 1}{2} \\ \& \quad \sin^2\theta &= \frac{1-\cos 2\theta}{2} \end{aligned}}$$

$1-\sin^2\theta = \cdots$

$\sin^2\theta = \frac{1-\cos 2\theta}{2}$

Kown:

$d\sin x = (\cos x)\,dx$: $\int \cos x\,dx = \sin x + C$

$d\cos x = (-\sin x)\,dx$: $\int (+\sin x)\,dx = -\cos x + C$

① $\int \sin^n(x)\cos^m(x)\cdot dx \qquad n, m = 0, 1, 2, 3, \sim$

Easy case: one is odd

Ex: $m=1 \quad = \int \sin^n(x)\cdot\cos x\cdot dx$

as $u = \sin x, du = \cos x\,dx$ $= \int u^n \cdot du = \frac{1}{n+1}u^{n+1} + C$

$= \frac{1}{n+1} (\sin x)^{n+1} + C$

talk the largest even power

Ex2: $\int \sin^3 x \cdot \cos^2 x \, dx$

USE $\sin^2 x = 1 - \cos^2 x$

$= \int (1 - \cos^2 x) \cdot \sin x \cos^2 x \cdot dx$

$= \int (-\cos^4 x + \cos^2 x) \cdot \sin x \cdot dx$

Substitution $u = \cos x$ $du = -\sin x \cdot dx$

$= \int (u^4 - u^2) \, du = \frac{1}{5} \cdot u^5 - \frac{1}{3} u^3 + C$

$= \frac{1}{5} \cos^5 x - \frac{1}{3} \cos^3 x$

Ex3: $\int \sin^3 x = \int (1 - \cos^2 x) \cdot \sin x \, dx$

$= \int (1 - u^2) \cdot (-du) = \frac{u^3}{3} - u = \frac{\cos^3 x}{3} + C - \cos x$

/指数

Hard case: only even exp's (exponents)

USE half angle formula

Ex 1. $\int \cos^2 x \, dx = \cancel{\int 1 + \frac{\cos}{2}} = \int \frac{1 + \cos(2x)}{2} dx$

$= \frac{x}{2} + \sin(2x) \cdot \frac{1}{4} + C$

Ex 2. $\int \sin^2 x \cdot \cos^2 x \, dx$

$= \int \frac{1 - \cos^2 2x}{4} dx$

$= \int (\frac{1}{8} - \frac{\cos 4x}{8}) \cdot dx$

$= \frac{1}{8} x - \frac{\sin 4x}{32} + C$

$\sin^2 x \cdot \cos^2 x = (\frac{1 - \cos 2x}{2}) \cdot (\frac{1 + \cos 2x}{2})$

$= \frac{1 - \cos^2 2x}{4}$

$= \frac{1}{4} - \frac{1}{4} \cdot \frac{1 + \cos 4x}{2}$

$= \frac{1}{4} - \frac{1 + \cos 4x}{8}$

$= \frac{1}{8} - \frac{\cos 4x}{8}$

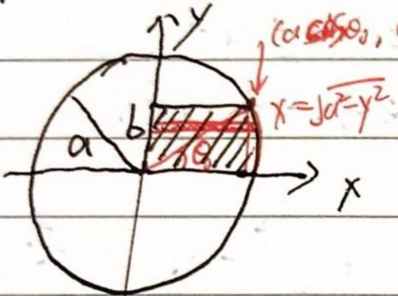
Alt method: $\sin^2 x \cdot \cos^2 x = (\sin x \cos x)^2$

$$= \left(\tfrac{1}{2}\sin 2x\right)^2$$

$$= \tfrac{1}{4}\cdot \sin^2 \cdot 2x$$

$$= \tfrac{1}{4}\cdot\left(\frac{1-\cos 4x}{2}\right)$$

Application: ② Trig substition



find the area

Area $= \int y \cdot dx$ oR $\int x \cdot dy$

$$= \int_0^b \sqrt{a^2-y^2}\cdot dy = \int_0^b a\cos\theta \cdot dy$$

$$\boxed{\begin{aligned} &y = a\cdot\sin\theta, \quad \theta = \arcsin\frac{y}{a} \\ &\sqrt{a^2-y^2} = \sqrt{a^2(1-\sin^2\theta)} = a\cos\theta = x \\ &dy = a\cdot\cos\theta\cdot d\theta \end{aligned}}$$

$$= \int a^2\cos^2\theta\, d\theta$$

$$= a^2\int \cos^2\theta \cdot d\theta$$

$$= a^2\cdot\left(\frac{\theta}{2} + \frac{\sin(2\theta)}{4}\right) + C$$

$$= a^2\cdot\left(\frac{\theta}{2} + \frac{\sin\theta\cos\theta}{2}\right) + C$$

$$= a^2\left(\frac{\theta}{2} + \frac{\sin\theta\cos\theta}{2}\right) + C$$

$$= a^2\left(\frac{\arcsin(y/a)}{2}\right) + \frac{a\sin\theta\cdot a\cos\theta}{2} + C$$

$$= \frac{a^2\arcsin(y/a)}{2} + \frac{y\sqrt{a^2-y^2}}{2} + C \Big|_0^b$$

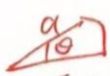
$$= \frac{a^2\arcsin(b/a)}{2} + \frac{b\sqrt{a^2-b^2}}{2}$$

$\arcsin(b/a) = \theta_0$

$$= \underbrace{\frac{a^2\theta_0}{2}}_{\text{area of sector}} + \frac{b\sqrt{a^2-b^2}}{2}$$

(扇形)

area of up tragle



$S = \frac{1}{2}\theta\cdot a^2$

$\theta = \arcsin\frac{b}{a}$

$S = \frac{1}{2}b\sqrt{a^2-b^2}$

Date.