LEC 39. 2025.1.4.

Review of power series.

$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$

One Caution: there is a number R, $0 \le R \le \infty$

$|x| < R$, $f(x)$ is converge

$|x| > R$, $f(x)$ is diverge. R called

$\boxed{\text{radius of convergence}}$

for $|x| < R$, $f(x)$ has all derivative

& $\boxed{a_n = \frac{f^{(n)}(0)}{n!}}$ (Taloy's formula)

（只在收敛半径内可展开）

$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 \cdots$

Example: $f(x) = e^x$

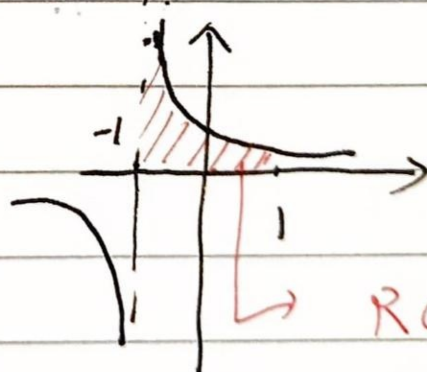
How to get the radius of convergence: $\boxed{\text{USE Ratio Test } \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1}$

$f'(x) = f''(x) = e^x \cdots$

$f(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 \cdots + \frac{1}{n!}x^n$

$(R = \infty)$

Ex2: geo series



$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$

$R = 1$

$R \in (0, 1)$

$0 < x < 1$

radius of convergence

~~Ex3~~

radio test: ~~talo~~ taylor: $\frac{1}{1+x} = \sum_{n=0}^{N} (-1)^n \cdot x^n$

$L = \lim_{n\to\infty} \left|\frac{(-1)^{n+1} x^{n+1}}{(-1)^n \cdot x^n}\right| = x$, $|L| < 1 \to$ converge

$\therefore$ radius of convergence: $|x| < 1$

$0 \quad 1 \quad 0 \quad -1$

EX3. $\sin(x)$ $\qquad \sin(x)$ $f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$

$f^{4}(x) = f^{(4)}(x) = \sin x$ 0

$$\therefore \sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 \cdots$$

$$= \boxed{\frac{x^{2n+1}}{(2n+1)!}} \qquad \boxed{R = \infty}$$

$\hookrightarrow n \to \infty$

$$\sum_{i=0}^{N} \frac{x^{2n+1}}{(2n+1)!} = \frac{x}{1} \cdot \frac{x}{2} \cdot \frac{x}{3} \cdot \frac{x}{4} \cdot \frac{x}{5} \cdots \frac{x}{(2n+1)}$$

$n \to \infty$, but x is fixed

For any x $\therefore$ x is inside of radius of convergence

New Power Series From Old

① Multiply

$x \cdot \sin x$ $\qquad x: a_n = 1$

$\hookrightarrow$ just like polynomials and multiply them

$$= x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} - \frac{x^8}{7!} + \cdots$$

R is smaller then two . $\boxed{R = \infty}$

because the power series can just give us the information of function where the function is converge.

② Differentiation

$$\cos(x) = \sin'(x)$$

$$= 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} \cdots$$

$$= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 \qquad \boxed{R = \infty}$$

③ Integrate

$$\ln(1+x) = \int_0^x \frac{dt}{1+t} = \int_0^x (1 - t + t^2 - t^3 + \cdots) dt$$

$(x > -1)$

$$= \left[t - \frac{1}{2}t^2 + \frac{1}{3}t^3 - \frac{1}{4}t^4 \cdots \right]_0^x$$

$$= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 \cdots$$

$(R = 1)$

Date.

④ Substitution

e^{-t^2} , $x = -t^2$ in e^x

$$e^{-t^2} = 1 + (-t^2) + (-t^2)^2/2! + (-t^2)^3/3! + \cdots$$

$$= 1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \cdots$$

$$= \sum_{i=0}^{N} \frac{(t)^{2i}}{i!} \cdot (-1)^i$$

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \cdot dt \rightarrow \left(\frac{\sqrt{\pi}}{2}\right)$$

(so that $\lim_{x \to \infty} \mathrm{erf}(x) = 1$)

$$= \frac{2}{\sqrt{\pi}} \left(x - \frac{1}{3} x^3 + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \cdots \right)$$