LEC 35. 2025.1.2

L' Hospital's rule

a convenient way to calculate limits including new ones.

$x \ln x$, $x \to 0^+$

$x e^{-x}$, $x \to \infty$

$\frac{\ln x}{x}$, $x \to \infty$

Ex1. $\lim_{x \to 1} \frac{x^{10}-1}{x^2-1}$

$\lim_{x \to a} \frac{f(x)}{g(x)}$ in $f(a) = g(a) = 0$ $f(a)=0$

$$\lim_{x \to a} \frac{f(x)/(x-a)}{g(x)/(x-a)} = \boxed{\frac{\lim_{x \to a} \frac{f(x)-f(a)}{x-a}}{\lim_{x \to a} \frac{g(x)-g(a)}{x-a}}} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

(work in $g'(a) \neq 0$)

Date.

L'Hôpital's Rule (version 1)

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$ provided $f(a)=g(a)=0$ and right hand limit exist

Ex 2 $\lim_{x\to 0}\frac{\sin 5x}{\sin 2x} \overset{\text{L'Hôp}}{=} \frac{5\cos 5x}{2\cos 2x} = \frac{5}{2}$

Ex 3 $\lim_{x\to 0}\frac{\cos x - 1}{x^2} = \lim_{x\to 0}\frac{-\sin x}{2x}$ second Time $\underset{x\to 0}{\sim}$ $\frac{-\cos x}{2} = -\frac{1}{2}$

$$\boxed{f(x) \simeq f(x_0) + f'(x_0)\cdot(x-x_0),\ x\to x_0}$$

$\sin 5(x) \simeq f(0) + f'(0)(x-0) = 5x$

$\sin 2x \simeq 2x$ $\quad\therefore \lim_{x\to 0}\frac{\sim}{\sim} = \frac{5}{2}$

Other Case $a=\pm\infty$ allowed

$f(a), g(a) = \pm\infty$, ok (∞/∞), right hand exist or $\pm\infty$

Ex 4. $\lim_{x\to 0^+} x\cdot\ln x \sim \frac{\ln x}{\frac{1}{x}} = \frac{-\infty}{\infty}$

$0\cdot(-\infty)$

winner

$\to \frac{\frac{1}{x}}{-\frac{1}{x^2}} \overset{x\to 0^+}{\sim} -x = 0$

Ex 5. $\lim_{x\to\infty} x\cdot e^{-xp},\ p>0$

$= \frac{x}{e^{px}} \sim \frac{1}{pe^{px}} \sim 0$

E5' $\lim_{x\to\infty}\frac{e^{px}}{x^{100}}$

$\sim \left(\frac{e^{px/100}}{x}\right)^{100}$

$\sim \left(\frac{\frac{p}{100}\cdot e^{px/100}}{1}\right)^{100} \to \infty$

Ex6. $\frac{\ln x}{x^{\frac{1}{3}}} \overset{x\to\infty}{\backsim} \frac{\frac{1}{x}}{\frac{1}{3}x^{-\frac{2}{3}}} = 3x^{\frac{2}{3}-1}$

$= 3x^{-\frac{1}{3}} \backsim 0$

Another form: 0^0

$\lim_{x\to 0} 0^0 : x^x$

$\because x^x = e^{\ln x^x} = e^{x\ln x} \to e^0 = 1$

$x\ln x \backsim \frac{\ln x}{\frac{1}{x}} \backsim -x \backsim 0 \quad \therefore \lim_{x\to 0} x^x = 1$

$\therefore$

Fishy

wrong, because $\cos 0 = 1$, $\neq 0$

$\frac{0}{1} \Rightarrow \frac{1}{0}$, だめ

$\frac{\sin x}{x^2} \overset{x\to 0}{\backsim} \frac{\cos x}{2x} \backsim \frac{-\sin x}{2} \xrightarrow[x\to 0]{} 0$ ✗

Liner approx: $\sin x \fallingdotseq x \to x/x^2 = \frac{1}{x} \to \infty \; (x\to 0^+)$ ✓

Don't use L'Hospital as a crutch (拐杖)

$\frac{\infty}{\infty} \xrightarrow{L'H} \frac{\infty}{\infty}$ or $\frac{0}{0} \xrightarrow{L'H} \frac{0}{0}$

Don't forget basic algebra when you doing this