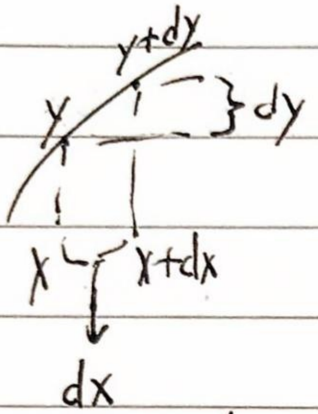
LEC15 2024.12.22

DIFFERENTIALS (微分)

$y=f(x)$

Differential of $y \Rightarrow \boxed{dy = f'(x)\cdot dx}$ $\frac{dy}{dx}=f'(x)$



just replace $\Delta x, \Delta y$

$f(x) \approx f(x_0) + f'(x_0)\cdot(x-x_0)$

Ex. $(64.1)^{\frac{1}{3}} = ?$ liner approx

$y = x^{\frac{1}{3}}$, $dy = \frac{1}{3}x^{-\frac{2}{3}}dx$

AT $x=64$, $y=64^{\frac{1}{3}}=4$, $dy = \frac{1}{3}(64)^{-\frac{2}{3}}dx$

$= \frac{1}{3}\cdot\frac{1}{16}dx$

$x+dx = 64.1$, $dx=\frac{1}{10}$ $= \frac{1}{48}\cdot dx$

~~$y+dy \approx y$~~ $(64.1)^{\frac{1}{3}} \approx y+dy = 4+\frac{1}{48}dx$

$= 4+\frac{1}{480} \approx 4.002$

same method
different notation

Mine	$(64.1)^{\frac{1}{3}} \approx 4 + \cancel{\frac{dy}{dx}}\,\frac{1}{3}(64)^{-\frac{2}{3}}\cdot\frac{1}{10}$
liner approx	$\approx 4+\frac{1}{48}\cdot\frac{1}{10}$
$f(x_0)=f(x_0)+f'(x_0)(x-x_0)$	$\approx 4+\frac{1}{480} \approx 4.002$

ANTIDERIVATIVES (不定积分)

$G(x) = \int g(x)dx$ $\int$: ~~integral~~ integral sign

↑ antiderivative of g $\parallel$ indefinite ~~integral~~ integral of g

不确定 (不定) → indefinite

ln ln ln

EX

1. $\int \sin x\,dx = -\cos x + C \leftarrow$ constant

2. $\int x^a\,dx = \frac{1}{a+1}x^{a+1} + C \quad , \quad a \neq -1$

$d(x^{a+1}) = (a+1)\cdot x^a\,dx$ (all a can)

3. $\int \frac{dx}{x} = (\ln|x|) + C$

$\frac{1}{\cos^2 x}$

$(\tan x)' = \frac{1}{\cos^2 x} = \sec^2 x$

$\int \sec^2 x\,dx = \tan x + C$

$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x + C$ $\quad\quad$ $\int \frac{dx}{1+x^2} = \tan^{-1}x + C$

$(\sin^{-1}x)' = \frac{1}{\sqrt{1-x^2}}$ $\quad\quad$ $(\tan^{-1}x)' = \frac{1}{1+x^2}$

Uniqueness of antiderivative up to a constant

Theorem if $F' = G'$, then $F(x) = G(x) + C$

proof: If $F' = G'$ then $(F-G)' = F' - G' = 0$

$F(x) - G(x) = C$, constant

$\Downarrow$ $F(x) = G(x) + C$

Ex. $\int x^3 (x^4+2)^5\,dx$

method of substitution

$u = x^4 + 2$, $\quad du = 4x^3\cdot dx \quad\quad x^3\cdot dx = \frac{du}{4}$

$= \int x^3 dx (x^4+2)^5 = \int \frac{du}{4}\cdot u^5 = \frac{1}{24}u^6 + C$

$= \frac{1}{24}(x^4+2)^6 + C$

Ex 2 $\int \frac{x\,dx}{\sqrt{1+x^2}}$

$u = 1+x^2 \quad du = 2x \cdot dx \quad \ldots\ldots$

a better method (recommand):

advance guessing: $\frac{d}{dx}(1+x^2)^{\frac{1}{2}} = \frac{1}{2}(1+x^2)^{-\frac{1}{2}} \cdot 2x$

$= \frac{x}{\sqrt{1+x^2}}$

$\therefore \int \frac{x\,dx}{\sqrt{1+x^2}} = \frac{x}{\sqrt{1+x^2}} + C$

E3 $\int e^{6x}\,dx$

guess: $e^{6x} \quad \frac{de^{6x}}{dx} = 6 e^{6x}$

$\therefore = \frac{1}{6} e^{6x} + C$

Also ok but slow to use substitution

E4 $\int x e^{-x^2} dx$

guess: $(e^{-x^2})' = -2x \cdot e^{-x^2}$

$\therefore = -\frac{1}{2} e^{-x^2} + C$

E5. $\int \sin x \cos x \, dx =$

guess: $(\sin^2 x)' = 2 \sin x \cdot \cos x$

$\therefore = \frac{1}{2} \sin^2 x + C_1$ ①

but $(\cos^2 x)' = -2 \cos x \sin x$

another: $= -\frac{1}{2} \cos^2 x + C_2$ ② $\quad \boxed{C_2 - C_1 = \frac{1}{2}}$

① − ② $= \frac{1}{2}$ (not C) $\quad \frac{1}{2}\sin^2 x + C_1$

$= -\frac{1}{2}\cos^2 x + C_2 + \underline{\underline{C}}$

E6 $\int \frac{dx}{x \ln x}$

$\ln(\ln x) \quad \frac{1}{\ln x} \cdot \frac{1}{x} \cancel{\cdot \frac{1}{x}} = \frac{1}{\ln x \cdot x}$

$= \cancel{\ln(\ln x)} + C$

sub: $\ln x = u, \quad du = \frac{1}{x} \cdot dx$

$= \int \frac{1}{u} \cdot du = \ln u$

$= \ln|\ln x| + C$