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$\boxed{\text{LEC 12 \quad SUM \quad 3.9}}$

Def: An n-year $\$m$-payment annuity pays $\$m$ at the start of each year for n years.

Assumption: Fixed interest rate p

$\$1$ today $= \$(1+p)$ in 1 year

" $= \$(1+p)^2$ in 2 years

" $= \$(1+p)^3$ in 3 years

$\$\frac{1}{1+p} = \$1$ in 1 year

$\$\frac{1}{(1+p)^2} = \$1$ in 2 year

$\boxed{\text{Current Value}}$ payments

$\$m = \$m$ now

$\$\frac{m}{1+p} = \$m$ in 1 year

$\vdots \qquad \vdots \qquad \vdots$

$\$\frac{m}{(1+p)^{n-1}} = \$m$ in $(n-1)$ years

$V = \sum_{i=0}^{n-1} \frac{m}{(1+p)^i}$ = Total Current Value

$= m \cdot \sum_{i=0}^{n-1} x^i, \quad x = \frac{1}{1+p} = \boxed{m \frac{1-x^n}{1-x}}$

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Thm $\forall n \geq 1,\ x \neq 1,\ \sum_{i=0}^{n-1} x^i = \frac{1-x^n}{1-x}$

① $\boxed{\text{Perturbation Method}}$

$S = 1 + x + x^2 + \cdots + x^{n-1}$

$-\ x \cdot S = \cancel{x} + x + x^2 + \cdots + x^{n-1} + x^n$

$(1-x) \cdot S = 1 - x^n$

$\Rightarrow S = \frac{1-x^n}{1-x}$

$$V = m\left(\frac{1-x^n}{1-x}\right)$$

$$= m\left(\frac{1-\left(\frac{1}{1+p}\right)^n}{1-\frac{1}{1+p}}\right)$$

$$= m\left(\frac{1+p-\frac{1}{(1+p)^{n-1}}}{p}\right)$$

ex:

For $m = \$50k,\ n = 20,\ p = 0.6$

$V = \$607,906$

Claim: If $n = \infty$, then $V = m\left(\frac{1+p}{p}\right)$

Pf: $\lim_{n\to\infty} \frac{1}{(1+p)^n} \to 0$ □

For $m = \$50k,\ p = .06,\ V = \$883,333$

Corollary

If $|x| < 1$, $\sum_{i=0}^{\infty} x^i = \frac{1}{1-x}$

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Pf: $\lim_{n\to\infty} x^n = 0$. $\square$

Ex: $1+\frac{1}{2}+\frac{1}{4}+\cdots = \frac{1}{1-\frac{1}{2}} = 2$

↑ geometric theory

Ex $\sum_{i=1}^{n} ix^i = x + 2x^2 + 3x^3 + \cdots + n\cdot x^n$

① $S = x + 2x^2 + \cdots + nx^n$

$$-xS = x^2 + 2x^3 + \cdots + (n-1)x^n + nx^{n+1}$$

$$(1-x)S = \underbrace{x + x^2 + \cdots + x^n}_{\frac{1-x^{n+1}}{1-x}-1} - nx^{n+1}$$

$$(1-x)S = \frac{1-x^{n+1}}{1-x} - 1 - nx^{n+1}$$

$$\Rightarrow S = \frac{x-(n+1)x^{n+1}+nx^{n+2}}{(1-x)^2}$$

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② $\boxed{\text{Derivative method}}$

For $x \neq 1$, $\sum_{i=0}^{n} x^i = \frac{1-x^{n+1}}{1-x}$

$$\Rightarrow \sum_{i=0}^{n} i\cdot x^{i-1} = \frac{-(1-x)(n+1)x^n - (-1)(1-x^{n+1})}{(1-x)^2}$$

$$= \frac{1-(n+1)x^n+nx^{n+1}}{(1-x)^2}$$

both side mutiply by x

$$\Rightarrow \sum_{i=0}^{n} ix = \frac{x-(n+1)x^{n+1}+nx^{n+2}}{(1-x)^2} \checkmark$$

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Thm: If $|x|<1$, $\sum_{i=1}^{\infty} i x^i = \frac{x}{(1-x)^2}$

Ex: An annuity that pays \$ i m at the end of year i ($i=1,2,3,\cdots$) is worth

$$m\left(\frac{\frac{1}{1+p}}{\left(1-\frac{1}{1+p}\right)^2}\right) = \frac{m(1+p)}{p^2}$$

$m=\$50k$, $p=.06$

$V=\$14,722,222$

ex!

$$\sum_{i=1}^{\infty} i\, 2^{-i} = \frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\cdots = \frac{1/2}{(1-1/2)^2} = 2$$

1. $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

2. $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

In this case

Guess: $\forall n \sum_{i=1}^{n} i^2 = an^3+bn^2+cn+D$

Plugin: $n=0 \Rightarrow 0=D$

$n=1 \Rightarrow a+b+c=1$; $n=2 \Rightarrow 5=8a+4b+2c+d$

$n=3 \Rightarrow 14=27a+9b+3c+d$

$\Rightarrow a=1/3,\ b=\frac{1}{2},\ c=\frac{1}{6},\ d=0$

$\sum_{i=1}^{n} \sqrt{i}$

$\boxed{\text{Integration Bounds}}$ for $\sum_{i=1}^{n} f(i)$ when f is positive increase function

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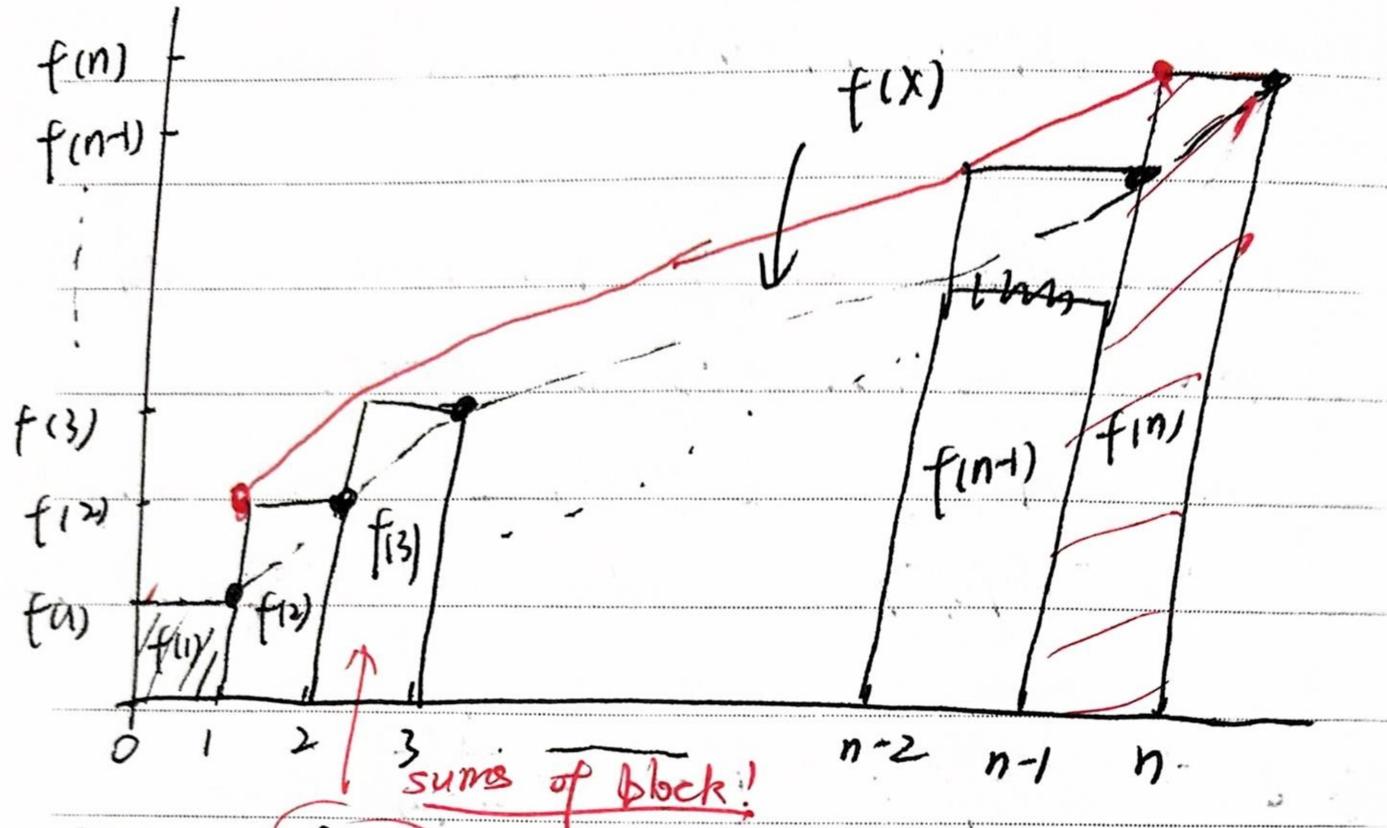

claim: $\sum_{i=1}^{n} f(i) \geq f(1) + \int_1^n f(x)\,dx$ (left Sum)

(Riemann Sum)

$\sum_{i=1}^{n} f(i) \leq f(n) + \int_1^n f(x)\,dx$ (right sum)

$\Rightarrow f(1) + \int_1^n f(x)dx \leq \sum_{i=1}^{n} f(i) \leq f(n) + \int_1^n f(x)dx$

Ex: $f(i) = \sqrt{i}$

$\int_1^n \sqrt{x}\,dx = \frac{x^{\frac{3}{2}}}{\frac{3}{2}}\Big|_1^n = \frac{2}{3}(n^{\frac{3}{2}} - 1)$

$\underset{\overset{\|}{1}}{f(1)} + \frac{2}{3}(n^{\frac{3}{2}} - 1) \leq \sum_{i=1}^{n} \sqrt{i} \leq \underset{\overset{\|}{\sqrt{n}}}{f(n)} + \frac{2}{3}(n^{3/2} - 1)$

$\Rightarrow \frac{2}{3} \cdot n^{\frac{3}{2}} + \frac{1}{3} \leq \sum_{i=1}^{n} \sqrt{i} \leq \frac{2}{3} n^{\frac{3}{2}} + \sqrt{n} - \frac{2}{3}$

Ex: $n = 100 \Rightarrow 667 \leq \sum_{i=1}^{n} \sqrt{i} \leq 676$

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$\sum_{i=1}^{n} \sqrt{i} = \frac{2}{3} n^{\frac{3}{2}} + \delta(n)$, $\frac{1}{3} \le \delta(n) \le \sqrt{n} - \frac{2}{3}$

$\Rightarrow \sum_{i=1}^{n} \sqrt{i} \sim \frac{2}{3} n^{\frac{3}{2}}$

<u>Def</u> $g(x) \sim h(x)$ means $\lim_{x \to \infty} \frac{g(x)}{h(x)} = 1$

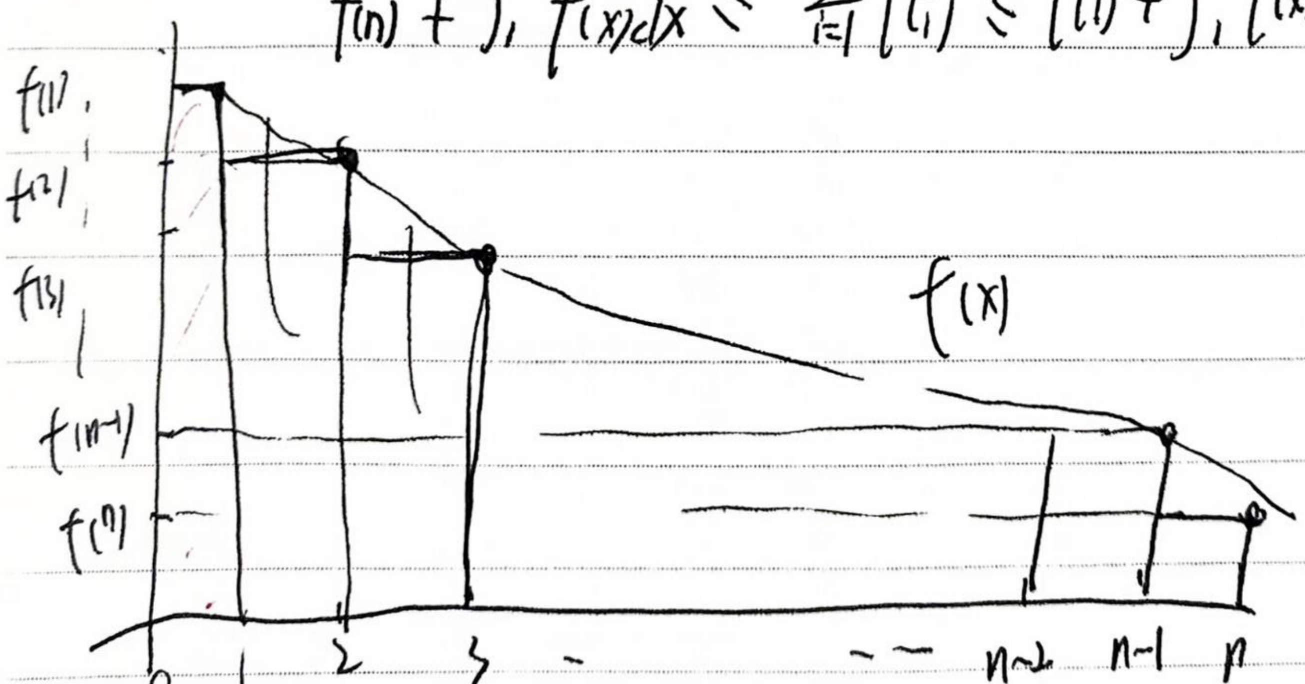
$$\Rightarrow \lim_{n \to \infty} \frac{\frac{2}{3} n^{\frac{3}{2}} + \delta(n)}{\frac{2}{3} n^{3/2}} = \lim_{n \to \infty} 1 + \frac{\delta(n)}{\frac{2}{3} n^{3/2}}$$

$$= 1 + \lim \cancel{\frac{\sqrt{n}}{\frac{2}{3} n^{\frac{3}{2}}}} \to 0$$

Integration bound when f is decreasing function

ex: $\sum_{i=1}^{n} 1/\sqrt{i}$

$$f(n) + \int_1^n f(x)dx \le \sum_{i=1}^{n} f(i) \le f(1) + \int_1^n f(x)dx$$



$f(i) = 1/\sqrt{i}$ $\int_1^n \frac{dx}{\sqrt{x}} = \frac{\sqrt{x}}{1/2}\Big|_1^n = 2(\sqrt{n} - 1) = 2\sqrt{n} - 2$

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$$\sum_{i=1}^{n} 1/\sqrt{i} \le f(1) + 2\sqrt{n} - 2$$

$$\le f(n) + 2\sqrt{n} - 2$$

$$\Rightarrow 2\sqrt{n} - 2 \le \qquad \le 2\sqrt{n} - 1$$

$$\sum_{i=1}^{n} 1/\sqrt{i} = 2\sqrt{n} - \delta(n) \qquad 1 \le \delta(n) < 2$$

$$\sum_{i=1}^{n} 1/\sqrt{i} \sim 2\sqrt{n}$$