LC9 Graph Theory 3   3.6

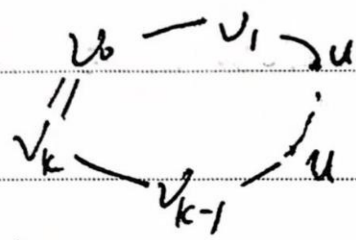
$\boxed{\text{Euler TOUR}}$

Def An Euler ~~to~~ tour is a walk that traverses every Edge eacthy once and starts and Finishes at the same vertex

Thm A connected graph ~~as~~ is an Euler tour iff every vertex has $\boxed{\text{even degree}}$

Proof

$\Rightarrow$ Assume $G=(V,E)$ has an Euler tour



Since Every edge is in E ~~trave~~ traversed once:

the degree of u = # times u appears in tour v_0 to ... v_k, times 2, (every edge occur 1 time, go into u + go out from u)

so the degree(u) must even.

$\Leftarrow$ For $G=(V,E)$, assume deg(v) is even for all vertex. Let $W: v_0 - v_0 - \cdots - v_k$ be the

Memo No.
Date / /

longest walk that traverses no edge more than once.

~~② $v_k = v_0$ show, otherwise~~

① $v_k - u$ where this edge not in W : I can lengthen walk $\Rightarrow v_0 - v_1 \cdots v_k - u$, it's contradict to W is the longest walk

$\Rightarrow$ All edges ~~incdtc~~ incident to v_k are used in W

② want to show $v_k = v_0$, otherwise ($v_k \neq v_0$) v_k has odd degree in $W \Rightarrow$ by ① we v_k has odd degree in G $\boxed{\text{X contradiction}} \Rightarrow v_k = v_0$

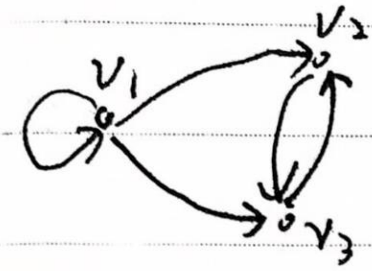
Suppose W is not an Euler tour
G is connected, $\exists$ edge is not in W, but incident to some vertex in W. Let $u - v_i$ be this edge

$u - v_i - \cdots - v_k = v_0 - v_1 \cdots v_i$

a longer walk X $\Rightarrow$ W is a Euler tour

(but not a Euler tour)

Directed ~~path~~ Graphs



also called (digraphs)

$v_2 \longrightarrow v_3$

tail head

indegree$(v_2) = 2$

outdegree$(v_2) = 1$

Thm Let $G = (V, E)$, be an n nodes graph $V = \{v_1, \cdots v_n\}$, Let $A = \{a_{ij}\}$ denote the adjency matrix for G. That is

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \to v_j \\ 0 & \text{if other case} \end{cases}$$

Let $p_{ij}^{(k)}$ = # directed walks of length k from v_i to v_j

Then $A^k = \{p_{ij}^{(k)}\}$

$$\begin{array}{c|ccc} & v_1 & v_2 & v_3 \\ \hline v_1 & & & \\ v_2 & & \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} & \\ v_3 & & & \end{array} = A$$

$$\Rightarrow A^2 = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad A^3 = \begin{pmatrix} 1 & 3 & 3 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Pf $a_{ij}^{(k)}$ denote the (i,j)th entry in ~~A~~ P^k

By induction $P(k)$ = "The thm is true for k"

Mo	Tu	We	Th	Fr	Sa	Su

Memo No. ______

Date / /

$P(k) =$ " $\forall i j \quad a_{ij}^{(k)} = P_{ij}^{(k)}$ "

Base Case : $k=1$, Edge $v_i \to v_j$: $P_{ij}^{(1)} = 1 = a_{ij}$ ✓

No Edge . $P_{ij}^{(1)} = 0 = a_{ij}^{(1)}$ ✓ ↑definition

$P(k)$

Assume $P(k)$ $\quad P_{ij}^{(k+1)} = \sum_{h: v_h \to v_j} P_{ih}^{(k)} = \sum_{h=1}^{n} P_{ih}^{(k)} \cdot a_{hj}$

/ is edge in G

$v_i \xrightarrow{k} v_h \to v_j$

(use $P(k)$) $= \sum_{h=1}^{n} a_{ih}^{k} \cdot a_{hj}$

↑ $h \to j$

$\begin{cases} v_h \to v_j = 1 \\ v_h \not\to v_j = 0 \end{cases}$

length +1

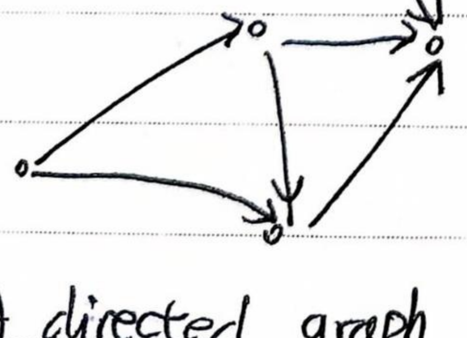
$v_i \to \cdots \to v_j$ $\quad = a_{\boxed{ij}}^{\boxed{(k+1)}}$, ✓

↑

matrix multiple ✓

Σ a expence of matrix. K : length of walk

Def A digraph $G = (V, E)$ is strongly connected if for all $u, v \in V$, $\exists$ directed path from u to v in G



directed acyclic graph

Def A directed graph is called a directed acyclic graph (DAG) if it does not contain

Mo Tu We Th Fr Sa Su

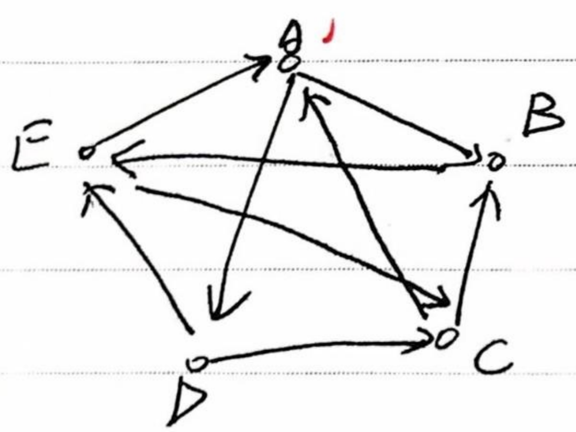
Memo No. ____

Date / /

any directed cycles

~~property~~ property of tournament graph

$\boxed{\text{Tournament graph}}$



tournament

who is the best player?

$\boxed{\text{either } u \text{ beats } v}$: $u \to v$

or v " u: $v \to u$

Ex: $A \to B \to D \to E \to C$

wait, $C \to A$

Ex: $C \to B \to D \to E \to A$

Def: A directed Hamiltonian path is a directed walk that $\boxed{\text{visits every vertex}}$ exactly once

Thm: every tournament graph exactly contains a directed Hamiltonian path.

Pf (By induction) on n

$P(n)$ = "Every tournament graph on n nodes actually contains a directed Hamiltonian path"

Base case: $n=1$ ✓

Mo Tu We Th Fr Sa Su

Memo No.

Date / /

Inductive Step: Assume $P(n)$

Consider a tournament graph on $n+1$ nodes

$\Rightarrow$ Take out one node v. This gives a tournament graph on n nodes

By $P(n)$: $v_1 \to v_2 \to \cdots \to v_n$

Case 1 $v \to v_1 \cdots \to v_n$ ✓

Case 2 $v_1 \to v$

smallest i, such that $v \to v_i \Rightarrow v \not\to v_{i-1}$

$v_1 \to \cdots \to v_{i-1} \to v_i \to \cdots v_n$

$v_{i-1} \to v \to v_i$

largest i, such that $v_i \to v$, so $v_i \to v \to v_{i+1}$

or $v_n \to v$

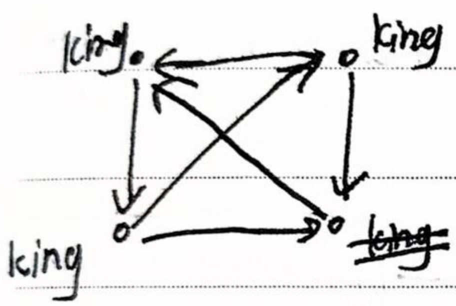
if every $v_i \to v$, then we can set v to the head of v_n, $v_1 \to \cdots \to v_n \to v$ is a directed Hamiltonian path ✓ case 3 application

either chicken u pecks chicken v : $u \to v$

or v pecks u : $v \to u$

- u ~~virtually pecks~~ v if
  - $u \to v$. OR
  - $\exists w \; u \to w \to v$ (just 2 edges)

A chicken that virtually pecks every other chicken is called king ~~chit~~ chicken



Thm The chicken with highest outdegree is a king

Pf (By contradiction)

Let u have highest outdegree

Suppose u is not king. ~~∃u : ∀v→u~~

$\Rightarrow \exists v : v \to u$, and ~~&~~ $\forall w \; \underbrace{\neg u \to w}_{w \to u}$ OR $\underbrace{\neg w \to v}_{v \to w}$

$\Rightarrow$ if $u \to w$ ~~and~~ ~~then~~ then $v \to w$, but $v \to u$

it means outdegree $(v) \geq$ outdegree $(w) + 1$

$\Rightarrow$ contradiction ✗ □