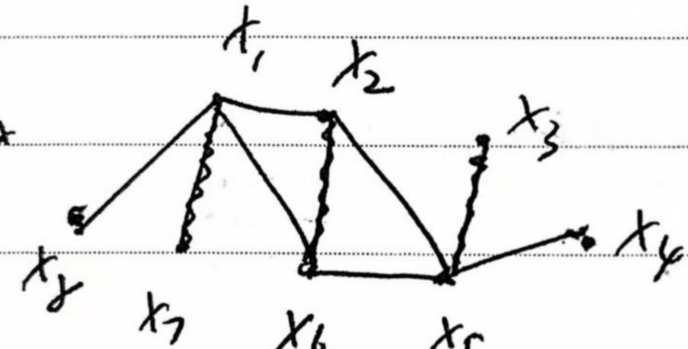
# LEC 7. Matching Problems

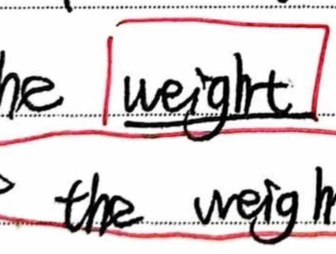
**Def**: Given graph $G=(V,E)$, a matching is a subgraph of $G$ where every node has degree 1

Ex




$\{x_1-x_6, x_2-x_5\}$ is a matching of size 2

**Def** A matching is perfect if it has sides $\frac{|V|}{2}$

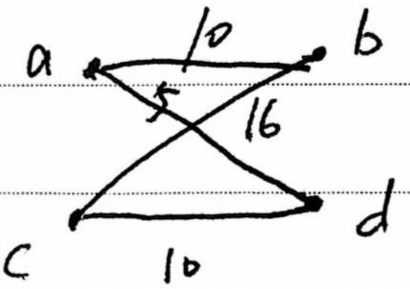



$(1,1), (2,3), (3,2), (4,4)$

**Def**: The weight of a matching $m$, is the sum of the weights on the edeges in $m$

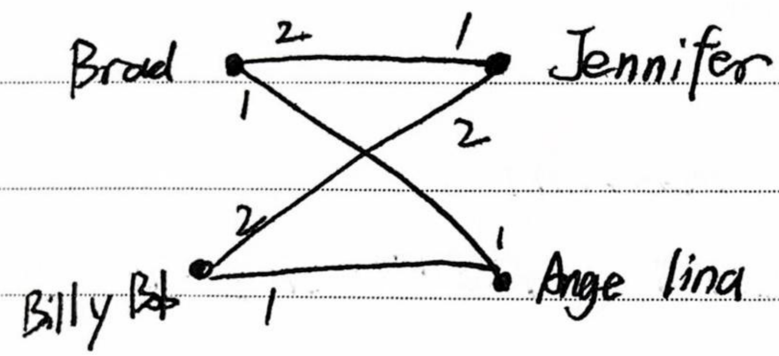
**Def** A min-weight matching for Graph $G$ is a perfect matching for $G$ with minimum weight

Ex




{a-b, c-d} weight 20

another version:




2: first choice

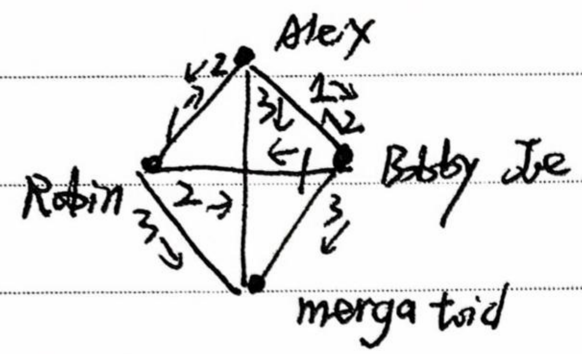
**Def** Given a matching M, X & Y form a rogue couple if they pr~~ef~~er each other ~~over~~ to their mates in M

{Brad-Jennifer, Billy Bob-Angelina} will form a rogue couple, becaute Brad and Angelina prefer eachother to their mates!

**Def** A matching is stable if there are no rogue couple

Goal: Find a perfect matching that is stable

{Brad-Angelina, Billy Bob-Jennifer} is stable a bad case:




$\underline{\text{Thm}}$ $\neg\exists$ stable matching

$\underline{\text{Pf}}$: By contradiction. Assume $\exists$ stable matching M. Then mergatroid matched with someone in M ~~with~~ (without loss of generality) WLOG (by symmetry), ~~Me~~ assume mergatroid matched to Alex

$\Rightarrow$ Alex and Robin formed a rogue couple for M

$\Rightarrow$ M not stable. ✓

Stable Marrange Problem

- N boys & N girls
- each boy has his own ranked list of all the girls
- each girl has — — — — ″ — — — — all the boys

Goal: find perfect matching without rogue couples

Mo | Tu | We | Th | Fr | Sa | Su

Memo No. ________
Date / /

| Boys | | Girls | |
|---|---|---|---|
| ① → | C, B, E, A, D | Ⓐ | 3, 5, 2, 1, 4 |
| ② → | A, B, E, C, D | Ⓑ | 5, 2, 1, 4, 3 |
| ③ | D, C, B, A, E | Ⓒ | 4, 3, 5, 1, 2 |
| ④ | A, C, D, B, E | Ⓓ | 1, 2, 3, 4, 5 |
| ⑤ | A, B, D, E, C | Ⓔ | 2, 3, 4, 1, 5 |

Try with Greedy:

①−C, ②−A, ③−D, ④−B, ⑤−E

④−C is a rogue couple

Mo | Tu | We | Th | Fr | Sa | Su

Memo No. ________
Date / /

# Samaders ~~Seranaders~~ ~~Serenades~~

| Girls | Day 2 | Day 2 | Day 3 | Day 4 |
|---|---|---|---|---|
| A | ~~2~~, ~~4~~, 5 | 5 | 5 | 5 |
| B | — | 2 | 2, ~~1~~ | 2 |
| C | 1 | ~~1~~, 4 | 4 | 4 |
| D | 3 | 3 | 3 | 3 |
| E | — | — | — | 1 |

| Boys | cross girls | | |
|---|---|---|---|
| 1 | | ~~C~~ | ~~B~~ |
| 2 | ~~A~~ | | |
| 3 | | | |
| 4 | ~~A~~ | | |
| 5 | | | |

↑ this case is stable matching

show this Alg is always stable matching
when # boys = # girls

Memo No.
Date / /

Need to show: TMA: (Traditional Marriage Algorithm)

- TMA terminates (quickly)
- everyone gets married
- No rogue couples
- fairness

Thm 1: TMA Terminates in $\leq N^2+1$ days

Pf: by contradiction. Suppose TMA doesn't not terminate in $N^2+1$ days

claim: if we don't terminate on a day, then some boy crosses a girl off his list ~~than~~ that night

$N$ list with $N$ names $\Rightarrow$ $\leq N^2$ cross outs

$\geq N^2+1$ cross outs ※ ✓

Let P = "If a girl G ever rejected a boy B, then G has a suitor who she prefers to B"

Lemma 1: P is an invariant for TMA

Pf: By induction on # days

base case: Day 0. No one rejected $\Rightarrow$ true

Inductive Step: Assume P holds at end of day d

Case 1: G rejected B on day d+1, Then there was a better boy ⇒ P true d+1.

Case 2: G rejected B before d+1,

P ⇒ G had ~~at least~~ a better suitor on day d ⇒ G has same or better suitor on d+1. ✓

Thm 2: Every man is married in TMA

Pf: by contradiction. Assume that some boy B is not married at end

⇒ B is rejected by every girl

⇒ every girl has better suitor (Lemma 1)

⇒ every girl married

⇒ every boy married, it's contradiction. ✓

Thm 3: TMA produces a stable matching

Pf: Let Bob & Gail be any pair that are not married.

case 2: Gail rejected Bob

⇒ Gail marries someone better than Bob

⇒ Gail & Bob are not Rouge couple

case 2: Gail didn't reject Bob

⇒ Bob never ~~serenaded~~ seranaded Gail

⇒ Gail is lower on Bob's list than Bob's mate ⇒ Gail & Bob is not a rouge couple

⇒ there is not rogue couple. ✓

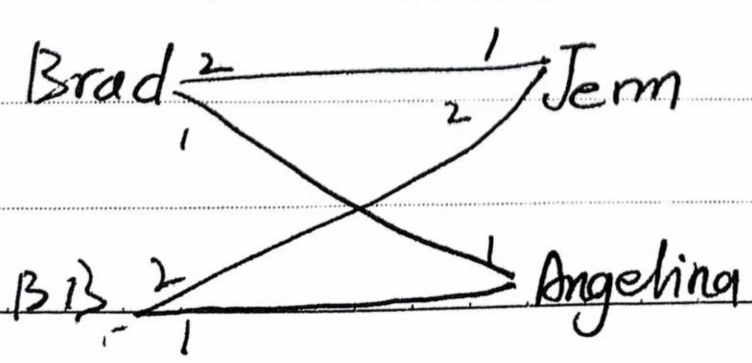
Fairness:

Let $S$: set of all stable matches $S \neq \emptyset$

For each person $P$, we define the realm of possibility for $P$ to be

$$\{Q \mid \exists M \in S, \{P, Q\} \in M\}$$

ex:



Df: A person's optimal mate is his or her favorite in the realm of possibility

Df: pessimal mate

lowest

Thm 4: TMA marries every boy with his optimal mate.

Thm5. TMA marries girl with her pessimal mate

Pf Thm5: By contradiction

Suppose that ∃ stable matching M where ∃ girl G who fares worse than in TMA

TMA