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LEc 12: Relations, Partial Orders, and Scheduling. 3.8

Relations

def A **Relation** from a set A to set B is a subset of $A \times B$ $R \subseteq A \times B$

Ex: $R = \{(a,b) : \text{student } a \text{ is taking class } b\}$

or $(a,b) \in R$; aRb, $a \sim_R b$

A **relation** on A is a subset $R \subseteq A \times A$

Ex: $A = \mathbb{Z}$: xRy iff $x \equiv y \pmod 5$

$A = \mathbb{N}$: xRy iff $x \mid y$ notation of relation

$A = \mathbb{N}$: xRy iff $x \leq y$

Set A together with R is a directed graph $G = (V, E)$ with $V = A$, $E = R$

$$\begin{tikzcd}
\text{Julie} \arrow[r] \arrow[d, shift left] & \text{Bill} \arrow[loop right] \\
\text{Ros} \arrow[u, shift left] &
\end{tikzcd}$$

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Properties:

A relation R on A is:

* reflxive : if xRx for all x in A
* symmetric : if ~~xRy~~ $xRy \Rightarrow yRx$ for all $x,y \in A$
* anti-symmetric: if $xRy \wedge yRx \Rightarrow x=y$
* transitive : if $xRy \wedge yRz \Rightarrow xRz$

		refl?	Symm?	AntiSy?	Trans?
Ex:	$x \equiv y \pmod 5$	T	T	F	T
	$x \mid y$	T	F	T	T
	$x \leq y$	T	F	T	T

equivalence relations | Partial Orders

Df An equivalence relation is reflexive, symm, Trans

Ex: equality (=) itself, $x \equiv y \pmod n$

The equivalence class of $x \in A$ is the set of all elements in A related to x by R: denoted ~~by~~ $[x]$

$$[x] = \{y : xRy\}$$

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Ex: $x \equiv y \pmod 5$

$[7] = \{\cdots, -3, 2, 7, 12, 17, 22, \cdots\}$

$[7] = [12] = [17] = [22] = \cdots$

(划分)

Def A **partition** of A is a collection of dis-joint, non-empty sets $A_1, \cdots, A_n \subseteq A$, whose union is A

Ex

partition:

$\{\cdots, -5, 0, 5, 10, \cdots\}$ equivalence classes

$\{\cdots, -4, 1, 6, 11, \cdots\}$

$\{\cdots, -3, 2, 7, 12, \cdots\}$

$\{\cdots, -2, 3, 8, 13, \cdots\}$

$\{\cdots, -1, 4, 9, 14, \cdots\}$

Theorem: The equivalence class of an equivalence relation on a set A form a partition of A

A relation is a (weak) partial order if it is reflexive, antisymmetric and transitive

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A partial order relation is denoted with $\boxed{\leq \text{ instead of } R}$

$R \Rightarrow \leq$

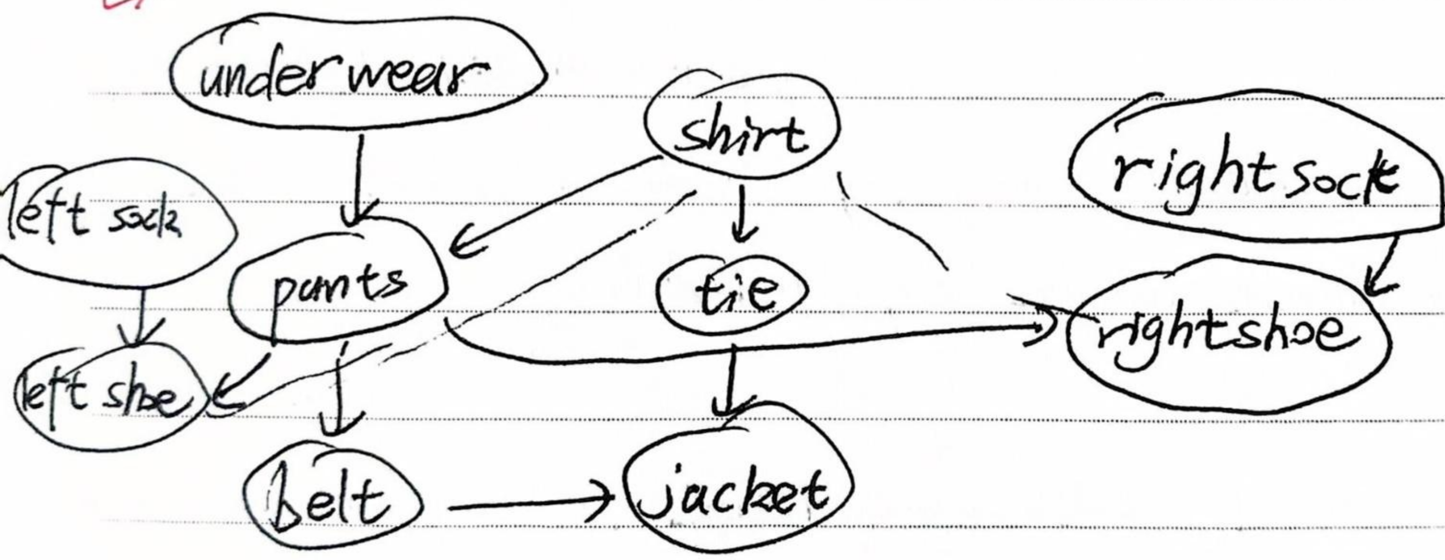
$xRy \Rightarrow x \leq y$

(vertex, edge)

$(A, \leq)$ is called a **partially order set** or poset

A poset is a **directed graph** with vertex set A and edge set $\leq$

Ex:



A **Hisse diagram** is for a poset $(A, \leq)$ is a directed graph which vertex set A and edge set $\leq$ minus all self-loops and edges impled by transisity

just one direction

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Theorem A poset has no directed cycle other than self loops

Pf (By contradiction)

Suppose $\exists\, n \geq 2$, distinct elements $a_1 \cdots a_n$

Sunch that $\underbrace{a_1 \leq a_2 \leq a_3}_{a_1 \leq a_3} \leq \cdots \leq a_{n-1} \leq a_n \leq a_1$

$a_1 \leq a_3$

$\vdots$

use induction $\Rightarrow a_1 \leq a_n$

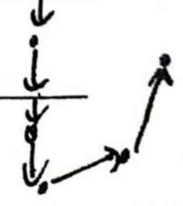
(anti-symmetric) $a_1 \neq a_n$ ✗

So, deleting self loop from a poset make a direet acyclic graph $\boxed{\text{(DAG)}}$

a and b are $\boxed{\text{incomparable}}$ if neither $a \leq b$ or $b \leq a$

—— $\boxed{\text{comparable}}$ if $a \leq b$ or $b \leq a$

A $\boxed{\text{total order}}$ is a partial order in which every pair of elements is comparable



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A total order consistent with a partial order called a $\boxed{\text{topological sort}}$. A top sort at a poset $(A, \leq)$ is a total order $(A, \preceq_T)$ such that

$\boxed{\preceq \subseteq \preceq_T}$

Theorem Every finite poset has a topological sort

$x \in A$ is called minimal if $\neg\exists\, y \in A,\ y \neq x$ s.t. $y \leq x$

$x \in A$ is called maximal if - - - $x \leq y$

$(\mathbb{Z}, \leq)$

Lemma: Every finite poset has a min elem

Def A chain is a sequence of distinct elements

$a_1 \leq a_2 \leq a_3 \cdots \leq a_t$ ← length

Pf