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LEC 8. Graph Theory II 3.5

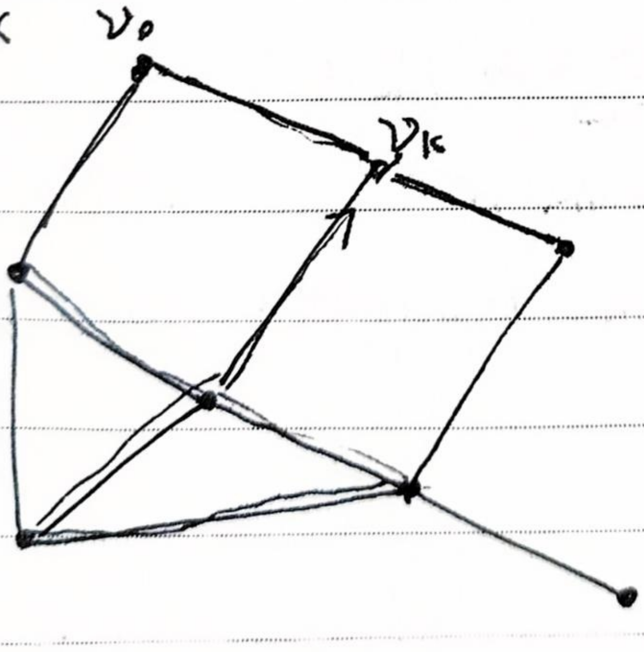
Tree!!

- Walks & Paths
- Connectivity
- Cycles & Closed walks
- Spanning Tree (ST) 生成树
- Minimun-weight spanning tree (MST)

1. Walks & Paths

Def A walk is a sequence of vertices connected by edges: $\left\{ \begin{matrix} v_0 - v_1 - \cdots - v_k \\ \text{start} \qquad\qquad \text{end} \end{matrix} \right\}$ length $= k$

EX



Def A path is a walk where all vertices are different

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① Lemma 1: If $\exists$ walk from u to v, then $\exists$ path from u to v

Pf $\exists$ walk from u to v

By well ordering principle: walk of mini length

$$u = v_0 - v_1 - \cdots - v_k = v$$

Case $k=0$ ✓

$k=1$ $u-v$ ✓

$k \geq 2$: Suppose walk is not a path

so $\exists$ $i \neq j$, $v_i = v_j$

$$u = v_0 - \cdots - \underline{v_i = v_j} - \cdots v_k = v$$

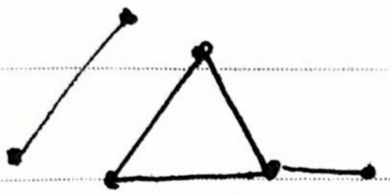
is a shorter walk ✗ □ ✓

↑ contradiction to suppose, it's a path

2. Connectivity

Def. u and v are connected if there is path from u to v

Def A graph is connected when every pair of vertices are connected



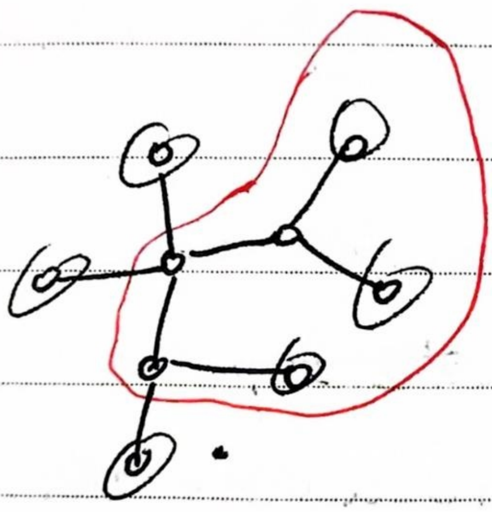
disconnected

3. Cycles & closed Walks

<u>Def</u> A closed walk is a walk which starts and ends at the same ~~vertex~~ vertex : $v_0 - v_1 ---- - v_k = v_0$ (loop)

<u>Def</u> ↓
If $k \geq 3$, and $v_0, v_1, \cdots v_{k-1}$ are different, then it is called a circle

4. Tree



<u>Def</u>. A connected and ~~a~~ no cyclic graph is called a tree

<u>Def</u> A leave is a node with degree 1 in the tree

leave: degree = 1 in tree

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Lemma Any connected subgraph of a tree is a tree

Pf By contradiction, Suppose the connected subgraph is not a tree: has a cycle, $\Rightarrow$ $\underbrace{\text{whole graph}}_{\text{tree}}$ has this circle

$\Rightarrow$ contradiction ✓

Lemma:

A tree with n vertices has $n-1$ edges

Pf By induction in n

$P(n)$ = "There are $n-1$ edges in any n vertex tree"

Base case: $P(1)$, 0 edges ✓

Inductive step: ~~Inductive step:~~

Suppose $P(n)$, Let T be a tree with $n+1$ vertices

$\Rightarrow$ let v be the leave of the tree, Delte v:

this created a connected subgraph $\rightarrow$ tree

By $P(n)$: it has $n-1$ edges, then reattach v: T has $(n-1)+1=n$ edges, $P(n+1)=n$ ✓

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4. Spanning tree

Def A spanning tree (ST) of a connective graph is actually a subgraph that is tree with the same vertices as the graph

Theorem: every connected graph has a spanning tree

Pf: By contradiction: Assume a connective graph G has no spanning tree (ST)

let T be a connected subgraph of G and assume with the same vertices as G and with the smallest ~~Eages a~~ number of eages possible

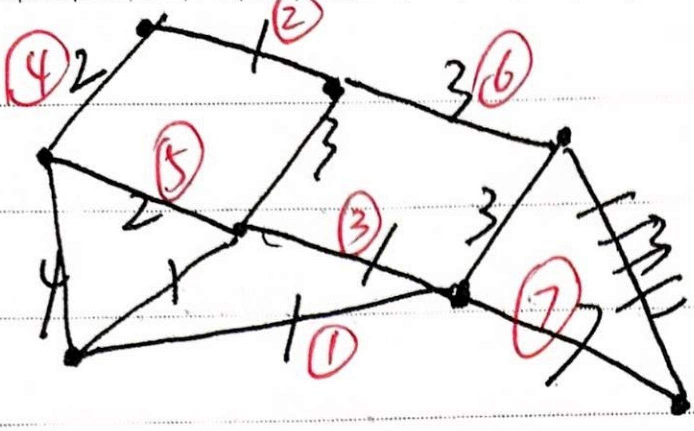
$\Rightarrow$ T is not a ST $\Rightarrow$ T must have a ~~di~~ cycle

ex

T

remove, we still have a conneeted graph

All vertices in G are still connected after removing e from cycle in T $\Rightarrow$ construct with T ~~is~~ has the smallest number of edges ✓



Weight of ST: 19 → Minimum weight spanning tree

<u>Def</u> The MST of an edge-weight graph G is the ST of G with the smallest possible sum of edge weights

<u>Alg</u>: Grow a subgraph ~~st~~ one edge at a time

at each step:

Add the minimum weight edge that keeps the graph no cycle

<u>Proof</u>:

<u>lemma</u>: Let S' consist of the first m edges selected by the Alg. Then $\exists$ MST $T=(V,E)$ for G such that $S' \subseteq E$

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the Theorem we want to show: $\boxed{\text{For any connected weight graph } G \text{, the Alg produce a MST}}$

Pf of theorem : $\# V = n$

1) If $< n-1$ edges are picked, then $\exists$ edges in $E-S$ that can be added without creating cycle

2) Once $m = n-1$, we know S' is an MST
$\Rightarrow$ theorem is true from Lemma

Pf of Lemma: (By induction)

$P(m)$ = "$\forall G \forall S'$ consisting of the first m selected edges, $\exists$ MST $T=(V,E)$ of G such that S' is Subgraph of G ($S' \subseteq E$)

Base case: $m=0$, $P(0) \Rightarrow S' = \phi$, $S' \subseteq E$ for any MST $T=(V,E)$ ✓

Inductive step: Assume $P(m)$

let e denote the $(m+1)$ st selected edges

let S' denote the first m selected edges

By $P(m)$: Let $T^* = (V, E^*)$ be a MST such that $S' \subseteq E^*$

1 case $e \in E^*$: $S' \cup e \subseteq E^* \rightarrow P(m+1)$ ✓ ✓

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2,

Case $e \notin E^*$: Alg $\Rightarrow$, $S' \cup \{e\}$ has no cycle

T^* is tree $\Rightarrow$ $(V, E^* \cup \{e\})$ has a cycle

$\Rightarrow$ this cycle has an edge $e' \in E^* - S'$

Alg would have selected e or e' $\Rightarrow$ weight of $e \leq$ weight of e'

So swap e and e' in T:

Let $T^{**} = (V, E^{**})$, $E^{**} = E^* - \{e'\} + \{e\}$

T^{**} is no cycle because removed e' from the only cycle into $E^* \cup \{e\}$

T^{**} is connected since e' was connected

$\Rightarrow T^{**}$ contains all vertices in G

$\Rightarrow$ T^{**} is a ST of G

$\left\{\begin{array}{l} \text{Weight } T^{**} \leq T^* \\ T^* \text{ is a MST} \end{array}\right. \Rightarrow \left\{\begin{array}{l} T^{**} \text{ is a MST} \\ \Rightarrow ✓ \end{array}\right.$