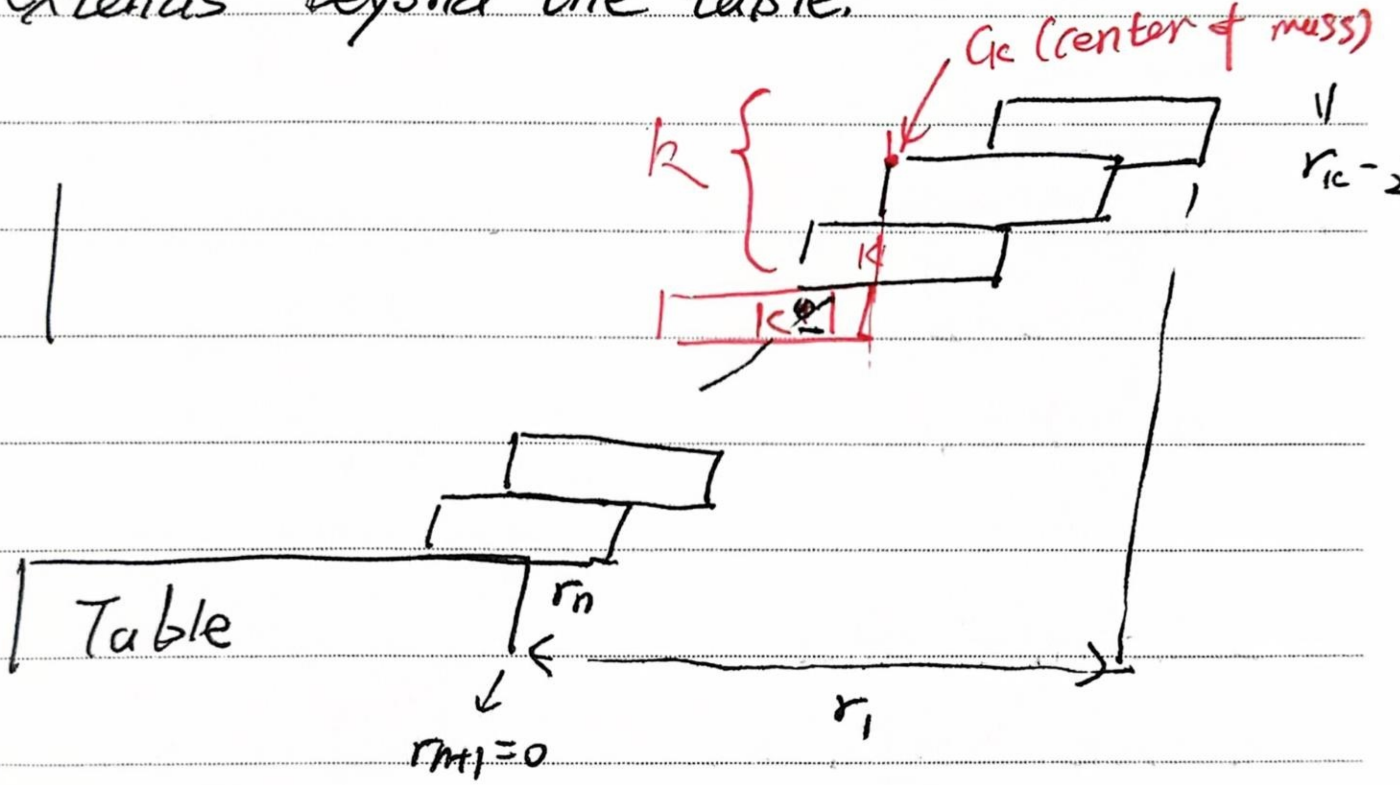
LEC13: SUMS AND ASYMPTOTICS 3.11

Greedy Strategy

Given n blocks of length 1.

Def: r_i = amount by which ith block extends beyond the table.



Stability Constraint: the center of mass C_k of the top k blocks must lie on the $(k+1)$st block (table = block $n+1$)

For Greedy Stacking $C_k = r_{k+1}$

The center of mass of kth block is at $r_k - \frac{1}{2}$

The center of mass of the top k blocks ~~C_k~~

$$C_k = \frac{(k-1)C_{k-1} + 1(r_k - \frac{1}{2})}{k-1+1}$$

$$= \frac{(k-1)\,C_{k-1} + r_k - \frac{1}{2}}{k}$$

$$\Rightarrow r_{k+1} = \frac{(k-1)r_k + r_k - \frac{1}{2}}{k} = \frac{k r_k - \frac{1}{2}}{k}$$

$$= r_k - \frac{1}{2k}$$

$$\boxed{r_{k+1} - r_k = \frac{1}{2k}}$$

$$r_1 - r_2 = \frac{1}{2}$$

$$r_2 - r_3 = \frac{1}{4}$$

$$\vdots$$

$$+\; r_n - r_{n+1} = \frac{1}{2n}$$

$$r_1 - \underbrace{r_{n+1}}_{\to 0} = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{2n} = \sum_{i=1}^{n} \frac{1}{2i}$$

$$\Rightarrow r_1 = \frac{1}{2} \sum_{i=1}^{n} \frac{1}{i}$$

↑ Harmonic Sum

nth Harmonic number $H_n = \sum_{i=1}^{n} \frac{1}{i}$

$$H_1 = 1,\quad H_2 = 1 + \frac{1}{2} = \frac{3}{2},\quad H_3 = \frac{3}{2} + \frac{1}{3} = \frac{11}{6}$$

$$H_4 = \frac{11}{6} + \frac{1}{4} = \frac{25}{12} > \frac{1}{2}$$

Mo | Tu | We | Th | Fr | Sa | Su

Memo No.
Date / /

Integration Bounds for decreasing sum

$$\Rightarrow f(n) + \int_1^n f(x)\,dx \le \sum_{i=1}^n f(i) \le f(1) + \int_1^n f(x)\,dx$$

$f(i) = \frac{1}{i}$, $\int_1^n \frac{1}{i}\,di = \ln(i)\Big|_1^n = \ln(n)$

so $\frac{1}{n} + \ln(n) \le \sum_{i=1}^n f(i) \le 1 + \ln(n)$

$\Rightarrow H(n) \sim \ln(n)$

$$H(n) = \ln(n) + \delta + \frac{1}{2n} + \frac{1}{12n^2} + \frac{\varepsilon(n)}{120n^2}$$

$\boxed{\text{Product}}$

$n! = \prod_{i=1}^n i \Rightarrow$ $\ln(n!) = \ln(1 \cdot \ldots \cdot n)$

$= \ln(1) + \ln(2) + \cdots + \ln(n)$

$= \sum_{i=1}^n \ln(i)$

$$\Rightarrow f(1) + \int_1^n f(x)\,dx \le \sum_{i=1}^n f(i) \le f(n) + \int_1^n f(x)\,dx$$

$$\int_1^n \ln(x)\,dx = (x\ln(x) - x)\Big|_1^n = n\ln(n) - n + 1$$

so $n \cdot \ln(n) - n + 1 \le \ln(n!) \le \ln(n) + n\ln(n) - n + 1$

$$\Rightarrow \frac{n^n}{e^{n-1}} \le n! \le \cancel{n+} \frac{n^{n+1}}{e^{n-1}}$$

Stirling's Formula : $n! = \left(\frac{n}{e}\right)^n \sqrt{2\pi n}\, e^{\varepsilon(n)}$

where $\frac{1}{12n+1} \le \varepsilon(n) \le \frac{1}{12n}$

$n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$

Memo No.
Date / /

Asymptotic Notation

tilde $f(x) \sim g(x)$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$

Oh. big-oh: $f(x) = O(g(x)) \Rightarrow$ $\boxed{\text{if } \lim_{x\to\infty} \left|\frac{f(x)}{g(x)}\right| < \infty}$

↑(finite)

ex: $f(x) \leq O(g(x))$, ← f grows slower than g

$=$ $f(x)$ is $O(g(x))$,

$f(x) \in O(g(x))$

EXAMPLES:

Thm: let $f(x) = x$, $g(x) = x^2$, Then $f(x) = O(g(x))$

Pf $\lim_{x\to\infty} \frac{x}{x^2} = 0 < \infty$

Thm: $x^2 \neq O(x)$

Pf: $\lim_{x\to\infty} \frac{x^2}{x} = \infty$

If $x^2 = O(10^6 x)$?

No

Is $10^6 x^2 = O(x^2)$?

YES

Thm $x^{10} = O(e^x)$

Pf: $\lim_{x\to\infty} \frac{x^{10}}{e^x} = 0 < \infty$

Time to multiply $n \times n$ matrices is $T(n) = O(n^3)$

Memo No. ______

Date / /

Is $4^x = O(2^x)$? No. $\lim_{x\to\infty} \frac{4^x}{2^x} = \lim_{x\to\infty} 2^x = \infty$

Is $10 = O(1)$? Yes! $\cdots = 10 < \infty$

$H_n = \ln(n) + \delta + O(1/n)$

it tells error term grows lower than $1/n$, $\Rightarrow H_n - \ln(n) - \delta = O(1/n)$

or $H_n \sim \ln(n) + \delta$

$f(x) \geq O(g(x))$ is meaningless

Omega notation do this $f(x) = \Omega(g(x))$, is $\lim_{x\to\infty} \left|\frac{f(x)}{g(x)}\right| > 0$

Thm, $f(x) = O(g(x))$ iff $g(x) = \Omega(f(x))$

$f(x) \geq \Omega(g(x))$ ✓ this is ok

Ex:

$x^2 = \Omega(x)$, $2^x = \Omega(x^2)$, $\frac{x}{100} = \Omega(100x + 25)$

$T(n) = \Omega(n^2)$

Theta $f(x) = \Theta(g(x))$, if $\lim_{x\to\infty} \left|\frac{f(x)}{g(x)}\right| > 0$ & $< \infty$

Thm: $f(x) = \Theta(g(x))$ iff $f(x) = O(g(x))$ & $f(x) = \Omega(g(x))$

Ex: ① $10x^3 - 20x + 1 = \Theta(x^3)$

② $\frac{x}{\ln x} \stackrel{?}{=} \Theta(x)$, $\lim_{x\to\infty} \frac{x/\ln(x)}{x} = \lim_{x\to\infty} \frac{1}{\ln(x)} = 0$, so No

$T(n) = \Theta(n^2)$, means T grows quadratically in n

Mo	Tu	We	Th	Fr	Sa	Su

Memo No.

Date / /

Summary

O means $\le$

Ω means $\ge$

Θ meas $=$

o $<$ ($\le$ not $=$)

ω $>$ ($\ge$ not $=$)

little oh $f(x) = o(g(x))$ if $\lim_{x\to\infty} \left|\frac{f(x)}{g(x)}\right| = 0$

little ω $f(x) = \omega(g(x))$ if $\lim_{x\to\infty} \left|\frac{f(x)}{g(x)}\right| = \infty$

Ex:

$\frac{x}{\ln(x)} = o(x)$

$\frac{x}{100} \stackrel{?}{=} o(x)$ X should be Θ or O

$x^2 = \omega(x)$

Thm (NOT!) Let $f(n) = \sum_{i=1}^{n} i$. Then $f(n) = O(n)$

False Proof: By induction on n.

I.H. $\boxed{P(n) : f(n) = O(n)}$ ← Bad. $f(n)$ this is not a function, n is a scala

Basecase: $f(1) = 1 = O(1)$ ✓

Inductive step: Assume $P(n)$ to prove $P(n+1)$

$P(n) \Rightarrow f(n) = O(n)$

r | Sa | Su

Memo No.

Date / /

$f(n+1) = f(n) + (n+1) = O(n) + O(n) = O(n).$ ☐