Mo | Tu | We | Th | Fr | Sa | Su

LEC ~~13~~ 14 Divide and Conquer Recurrences

3.11

H塔

Def : T_n = min # moves for n disks

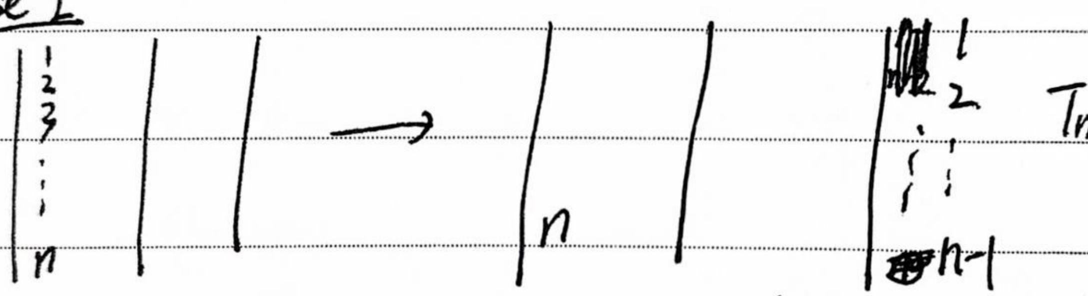
$T_1 = 1$

$T_2 = 3$

$T_3 \le 7$

Recursive Solution

Phase 1



T_{n-1} steps

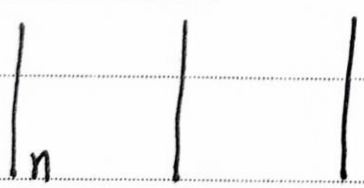
Phase 2: move the largest disk / Phase 3: move top $n-1$ disks

$\Rightarrow$ | n | 1 2 … $n-1$ | 1 step $\Rightarrow$ | | 1 2 … n | T_{n-1} steps

$\Rightarrow$ # moves $T_n \le 2T_{n-1} + 1$

$T_3 \le 2T_2 + 1 = 2 \cdot 3 + 1 = 7$

Lower Bound



$\ge T_{n-1}$ steps before moves

≥ 1 step for big ~~step~~ disk for move

$\ge T_{n-1}$ steps after last move of big disk

Memo No.
Date / /

$\Rightarrow T_n \geq 2T_{n-1} + 1 \Rightarrow T_n = 2T_{n-1} + 1$

Guess & Verify (Substitution) Method

$T_1 = 1, \; T_2 = 3, \; T_3 = 7, \; T_4 = 15$

Guess: $T_n = 2^n - 1$

Verify: By induction, $P(n) = 2^n - 1$

Base Case: $T_1 = 1 = 2^1 - 1 = 1$ ✓

Inductive Step: Assume $T_n = 2^n - 1$ to prove

$T_{n+1} = 2^{n+1} - 1$

$T_{n+1} = 2T_n + 1 = 2 \cdot (2^n - 1) + 1 = 2^{n+1} - 1$ ✓. □

Plug & Chug

Plug $T_{n+1} = 1 + 2T_n = 1 + 2(1 + 2T_{n-1})$

chug $= 1 + 2 + 4T_{n-1}$

$= 1 + 2 + 4(1 + 2T_{n-2})$

$= 1 + 2 + 4 + 8T_{n-2}$

$T_n = 1 + 2 + 4 + \cdots + 2^{i-1} + 2^i T_{n-i}$

$= 1 + 2 + 4 + \cdots + 2^{n-2} + 2^{n-1} T_1$

$= 2^n - 1$ $\quad \overset{\shortparallel}{T_1}$

Memo No.
Date / /

Merge Sort

To sort $n>1$ $x_1, x_2, \cdots, x_n$, (n = power of 2)

1. sort $x_1, x_2, \cdots, x_{n/2}$ & $x_{n/2+1}, \cdots, x_n$ recursively
2. merge

Ex: Sort {10, 7, 23, 5, 2, 4, 3, 9}

1. sort {10, 7, 23, 5} ⇒ {5, 7, 10, 23}

   ⇒ {2, 3, 4, 9}

merge: 2, 3, 4, 5, 7, 9, 10, 23

Let $T(n)$ = # comparisons used to sort

merging take $n-1$ comparisons (worse case)

$2T(n/2)$ comparisons for recursive sorting

$\Rightarrow T(n) = 2T(n/2) + n-1$

$T(1) = 0$, $T(2) = 1$, $T(4) = 5$

$T(8) = 2\cdot 5 + 8 - 1 = 17$, $T(16) = 2\cdot 17 + 16 - 1 = 49$

Plug & Chug

$$
\begin{aligned}
T(n) &= n-1 + 2T(n/2) \\
&= n-1 + 2\left(\frac{n}{2} - 1 + 2T\left(\frac{n}{4}\right)\right) \\
&= n-1 + n-2 + 4T\left(\frac{n}{4}\right) \\
&= n-1 + n-2 + 4\left(\frac{n}{4} - 1 + 2T\left(\frac{n}{8}\right)\right) \\
&= n-1 + n-2 + n-4 + 8T\left(\frac{n}{8}\right)
\end{aligned}
$$

Mo Tu We Th Fr Sa Su

Memo No.

Date /

$i = \log_2 n$

$= n-1 + n-2 + n-4 + \cdots + n - 2^{i-1} + 2^i T(n/2^i)$

$= n-1 + n-2 + \cdots n - 2^{\log n - 1} + 2^{\log n} T(1) \to = 0$

$= \sum_{i=0}^{\log n - 1} (n - 2^i) = \sum_{i=0}^{\log n - 1} n - \sum_{i=0}^{\log n - 1} 2^i = n \cdot \log n - (2^{\log_2 n} - 1)$

$= n\log n - n + 1$

$\sum_{i=0}^{\log n - 1} 2^i = 2^{\log n} - 1$

e $\boxed{\frac{a(1-q^n)}{1-q}} = \frac{1 \cdot (1 - 2^{\log n})}{1-2} = 2^{\log n} - 1$

$\in \begin{cases} S(n) = S(\lfloor n/2 \rfloor) + S(\lceil n/2 \rceil) + 1 \Rightarrow S(n) \sim n \\ T(n) = 2T(n-1) + 1 \Rightarrow T(n) \sim 2^n \\ T(n) = 2T(n/2) + n - 1 \Rightarrow T(n) \sim n\log n \quad \text{(tilde)} \end{cases}$

Define $\boxed{\text{Ex of Recursive}}$

$S(1) = 0, \quad S(n) = S(\lfloor \frac{n}{2} \rfloor) + S(\lceil \frac{n}{2} \rceil) + 1$ for $n \geq 2$

biggest integer $\leq \frac{n}{2}$ (smallest integer $\geq n/2$

$S(n) = 2S(\frac{n}{2}) + 1$ for n even

Guess

$S(1) = 0, \ S(2) = 1, \ S(3) = S(1) + S(2) + 1$

$= 2, \quad S(4) = 3$

guess: $S(n) = n - 1$

Verify: (by strong induction)

I.H. $P(n): S(n) = n - 1$. Base: $S(1) = 0 = 1 - 1$ ✓

Memo No.
Date / /

Induction step: Assume $P(1), P(2), \ldots, P(n)$ to Prove $P(n+1)$:

$$S(n+1) = S(\lfloor \tfrac{n+1}{2} \rfloor) + S(\lceil \tfrac{n+1}{2} \rceil) + 1$$

$= \lfloor \frac{n+1}{2} \rfloor - 1 + \lceil \frac{n+1}{2} \rceil - 1 + 1$ by Induction

$= n+1 - 1 - 1 + 1 = (n+1) - 1$ ✓. □

$$T(x) = \begin{cases} 2T(\frac{x}{2}) + \frac{8}{9}T(\frac{3x}{4}) + x^2 & \text{for } x \geq 1 \\ 0 & \text{for } x < 1 \end{cases}$$

Def: Divide & Conquer Recurrence has the form.

$T(x) = a_1 T(b_1 x + \varepsilon_1(x)) + a_2 T(b_2 x + \varepsilon_2(x))$

$+ \cdots + a_n T(b_n x + \varepsilon_k(x)) + g(x)$ for $x \geq x_0$

where, $a_i > 0$, $0 < b_i < 1$, k is fixed, $|\varepsilon_i(x)| \leq O(\frac{x}{\log_2 x})$

$|g'(x)| \leq x^c$ for $c \in R$

Memo No.

Date / /

Thm (Akra & Bazzi): Set p so that $\sum_{i=1}^{k} a_i b_i^p = 1$

Then $T(x) = \Theta\left(x^p + x^p \cdot \int_1^x \frac{g(u)}{u^{p+1}}\, du\right)$

Ex: $T(x) = 2T(x/2) + \underbrace{x-1}_{g(x)}$

$a_1 = 2, \; b_1 = \frac{1}{2}, \; k = 1$

$2\left(\frac{1}{2}\right)^p = 1 \Rightarrow p = 1$

$T(x) = \Theta\left(x + x\int_1^x \frac{u-1}{u^2}\, du\right)$

$= \Theta\left(x + x\int_1^x \left(\frac{1}{u} - \frac{1}{u^2}\right) du\right)$

$= \Theta\left(x + x\left(\ln(u) + \frac{1}{u}\right)\Big|_1^x\right)$

$= \Theta\left(x + x\left(\ln x + \frac{1}{x} - 1\right)\right)$

$= \Theta(x \ln x)$