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LEC15: Linear Recurrences 3.12

Graduate Student Job Problem

- Total #jobs = M (fixed overtime)
- each prof generates 1 grad(new prof)/year
- except 1st year profs. who produce 0
- No retirements

Q: when are all m jobs filled?

Boundary Condition: 1st prof hired in year 1

Solution

Let $f(n)$ = # proofs during year n

$f(0) = 0$, $f(1) = 1$, $f(2) = 1$, $f(3) = 2$,

$f(4) = 3$, $f(5) = 5$

1+1, 3+2

For $n \geq 2$, $f(n) = f(n-1) + f(n-2)$

previous new

Def: A recurrence is linear if it is of the form $f(n) = a_1 f(n-1) + a_2 f(n-2) + \cdots + a_d f(n-d)$

$= \sum_{i=1}^{d} a_i f(n-i)$, for fixed a_i, d=order

~~Stoution~~ Solution

Try $f(n) = \alpha^n$ for constant α

$$f(n) = f(n-1) + f(n-2)$$

$$\Rightarrow \alpha^n = \alpha^{n-1} + \alpha^{n-2} \Rightarrow \alpha^2 = \alpha + 1$$

$$\Rightarrow \alpha^2 - \alpha - 1 = 0, \quad \alpha = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

$f(n) = \alpha_1^n$ or α_2^n where $\alpha_1 = \frac{1+\sqrt{5}}{2}$, $\alpha_2 = \frac{1-\sqrt{5}}{2}$

Fact: If $f(n) = \alpha_1^n$ & $f(n) = \alpha_2^n$ are solutions to a linear recurrence (w/o boundary condition) then $f(n) = c_1\alpha_1^n + c_2\alpha_2^n$ is also a solution for any constants c_1 & c_2

without

$$\Rightarrow f(n) = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n \text{ is a solution}$$

depend on $f(0)$ and $f(1)$

Determine the Constant Factors

$$f(0) = 0 = c_1 \cdot (\;)^0 + c_2(\;)^0 = c_1 + c_2 \Rightarrow c_2 = -c_1$$

$$f(1) = 1 = c_1 \cdot \left(\frac{1+\sqrt{5}}{2}\right)^1 + c_2\left(\frac{1-\sqrt{5}}{2}\right)^1$$

$$= c_1\left(\frac{1+\sqrt{5}}{2}\right) - c_1\left(\frac{1-\sqrt{5}}{2}\right)$$

$$= c_1 \cdot \sqrt{5}$$

$$\Rightarrow c_1 = 1/\sqrt{5}, \quad c_2 = -1/\sqrt{5}$$

Solution : $f(n) = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$

for $f(0)=0$, $f(1)=1$

$f(6) = 8 = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^6 - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^6$ amazing

$f(n) = \frac{1}{\sqrt{5}}\cdot\left(\frac{1+\sqrt{5}}{2}\right)^n + \delta(n)$, $|\delta(n)| \le \frac{1}{10}$ for $n \ge 4$

$= O(1)$

All m jobs filled when $f(n) \ge m$

$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n + \delta(n) \ge m$

$\Rightarrow \left(\frac{1+\sqrt{5}}{2}\right)^n \ge \sqrt{5}(m - \delta(n))$

$n \ge \frac{\log(\sqrt{5}(m-\delta(n)))}{\log\left(\frac{1+\sqrt{5}}{2}\right)} = \Theta(\log m)$

$\boxed{\text{Solve General Linear Recurrence}}$

$f(n) = \sum_{i=1}^{d} a_i f(n-i)$ $f(0) = b_0$, $f(1) = b_1, \cdots, f(d-1) = b_{d-1}$

Try $f(n) = \alpha^n$

$\alpha^n = a_1\alpha^{n-1} + a_2\alpha^{n-2} + \cdots + a_d\alpha^{n-d}$

$\Rightarrow \alpha^d = \cancel{a_1 d} \; a_1\alpha^{d-1} + a_2\alpha^{d-2} + \cdots + a_d$

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$\Rightarrow \alpha^d - a_1 \alpha^{d-1} - a_2 \alpha^{d-2} - \cdots - a_d = 0$

↑ the characteristic equation of the recurrence

① Simple Case: All d roots are different:

$\alpha_1, \alpha_2, \ldots \alpha_d$

Solution: $f(n) = c_1\alpha_1^n + c_2\alpha_2^n + \cdots + c_d\alpha_d^n$

Solve for $c_1, c_2, \ldots, c_d$ from $f(i) = b_i$ for $0 \le i < d$

Ex: $f(0) = c_1 + c_2 + \cdots + c_d = b_0$

$\vdots$

② Tricky case: repeated roots

Thm: If α is a root of characteristic Equation and it is repeated r times, then $\boxed{\alpha^n, n\alpha^n, n^2\alpha^n, \ldots, n^{r-1}\alpha^n}$ are ~~also~~ all solutions to the recurrence

Ex: Plant reproduces (one for one) during 1^{st} year of life, then never again. Plant lives forever.

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Solution: let $f(n)$ = # plants in year n.

$f(0)=0,\ f(1)=1$

$$f(n) = f(n-1) + (f(n-1) - f(n-2))$$

$$= 2f(n-1) - f(n-2), \quad f(n) = \alpha^n \Rightarrow$$

$$\Rightarrow \alpha^2 - 2\alpha + 1 = 0$$

$\Rightarrow \alpha = 1$ double root

$$f(n) = c_1(1)^n + c_2(n \cdot (1)^n)$$

$$= c_1 + c_2 n$$

$$f(0) = 0 = c_1, \quad f(1) = 1 = c_1 + c_2 = c_2$$

$$\Rightarrow f(n) = n$$

Linear Homogeneous

$$f(n) - a_1 f(n-1) - \cdots - a_d f(n-d) = 0$$

$$\left.\begin{array}{l} = 1 \\ = n^2 \\ = g(n) \end{array}\right\} \text{inhomogeneous}$$

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Inhomogeneous Recurrence

$f(n) - a_1 f(n-1) - \cdots - a_d f(n-d) = g(n)$

Step 1: Replace $g(n)$ by 0 & Solve the homogeneous recurrence (ignore bdry conds for now)

Step 2: Restore $g(n)$ & find any particular solution (ignore bdry conds)

Step 3: Add the homogeneous & particular solutions together & use bdry conds to determine constant factors.

Ex: $\underbrace{f(n) = 4f(n-1)}$ $+ 3^n$, $f(1) = 1$

Step 1: $\alpha - 4 = 0$ $\qquad \alpha = 4$

homo solution: $f(n) = c_1 4^n$

Step 2: Find a particular soln to $f(n) - 4f(n-1) = 3^n$

Guess $f(n) = c 3^n$ $\qquad c3^n - c3^{n-1} = 3^n$

$3c - c = 3$