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11.1 一阶隐方程 (可解出 y 或 x 的隐方程)

$$F(x,y,y')=0 \not\Rightarrow y'=f(y,x)$$

① 可以解出 y

$$F(x,y,y')=0 \longrightarrow y=f(x,y')$$

令 $y'=p(x)$ $\quad y'=f'(x,p)$, $p=f'(x,p) \longrightarrow$

关于 x 和 p 的方程

$$= f'_x(x,p)+f'_p(x,p)\frac{dp}{dx}$$

$$p=f'_x(x,p)+f_p(x,p)\frac{dp}{dx}$$ 全微分

$$\frac{dp}{dx}=\frac{p-f'_x(x,p)}{f'_p(x,p)} \longrightarrow \text{正规方程}$$

$$\frac{dy}{dx}=\frac{\partial f(x,p)}{\partial x}\cdot\frac{dx}{dx}+\frac{\partial f(x,p)}{\partial x}\cdot\frac{dp}{dx}$$

代入

$$p=\varphi(x) \Rightarrow y=f(x,\varphi(x))$$

e.g. $y=(y')^2-xy'+\frac{x^2}{2} \xrightarrow{\times} y'=-x$

① 令 $y'=p$

$$y=p^2-xp+\frac{x^2}{2}$$

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③ 两端求导：$p = 2p \cdot \frac{dp}{dx} - (p + x \cdot 2 \cdot \frac{dp}{dx}) + x$

关于 p 和 x 方程

$$2p - x = 2p - x\frac{dp}{dx}$$

$$(2p - x)(\frac{dp}{dx} - 1) = 0$$

$\Rightarrow$ ① $2p = x$，$p = \frac{x}{2}$，

$\Rightarrow y = \frac{x^2}{4} - \frac{x^2}{2} + \frac{x^2}{2} = \frac{x^2}{4}$

② $\frac{dp}{dx} = 2$，$p = x + c$

$\therefore y = (x+c)^2 - (x+c)x + \frac{(x+c)^2}{2}$

$= \frac{x^2}{2} + cx + c^2$

回代！

② 可以解出 x　　$x = f(y, y')$

令 $y' = p$，$x = f(y, p)$　　把 x 看成 y 的函数，同时对 y 求导

$\boxed{\frac{dx}{dy} = \frac{1}{p}}$，$\frac{1}{p} = f'_y(y, p) + f'_p(y, p)\frac{dp}{dy}$

$\Rightarrow p = \varphi(y)$ 回代

e.g. $(y')^3 - 4xyy' + 8y^2 = 0$

$$x = \frac{(y')^3 + 8y^2}{4yy'}, \quad x = f(y, y')$$

① 令 $y' = p$，$\frac{dx}{dy} = \frac{1}{p}$

$$x = \frac{p^3 + 8y^2}{4yp} = \frac{p^2}{4y} + \frac{2y}{p}$$

~~$= \frac{1}{4}p^2 \cdot y +$~~

$$\frac{dx}{dy} = \frac{1}{p} = -\frac{p^2}{4y^2} + \frac{2p}{4y}\frac{dp}{dy} + (-\frac{2y}{p^2}\cdot\frac{dp}{dy} + \frac{2}{p})$$

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$$\frac{1}{p} = \frac{1}{2}\frac{p}{y}\cdot\frac{dp}{dy} - \frac{p^2}{4y^2} + \frac{2}{p} - \frac{2y}{p^2}\frac{dp}{dy}$$

$$\frac{dp}{dy}\left(\frac{p}{2y} - \frac{2y}{p^2}\right) + \left(\frac{1}{p} - \frac{p^2}{4y^2}\right) = 0$$

$$\frac{dp}{dy}\left(\frac{p^3-4y^2}{2yp^2}\right) + \left(\frac{4y^2-p^3}{4y^2p}\right) = 0$$

$$(4y^2-p^3)\left(\frac{-1}{2yp^2}\cdot\frac{dp}{dy} + \frac{1}{4y^2p}\right) = 0$$

$$\cancel{(4y^2-p^3)\left(-p\right.}$$

$$(4y^2-p^3)\left(\frac{dp}{dy} - \frac{p}{2y}\right) = 0$$

① $p^3-4y^2=0$, $p=\sqrt[3]{4y^2}$，代回 $y=\frac{4x^2}{27}$

② $\frac{dp}{dy}-\frac{p}{2y}=0$ $\ln p=\frac{1}{2}\ln y$，$p=cy^{\frac{1}{2}}$

回代 $x=\frac{c^2}{4}+\frac{2}{c}\cdot y^{\frac{1}{2}}$

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11.2 一阶隐方程

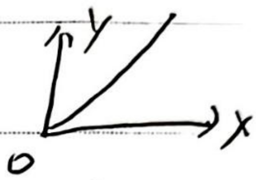
$F(x,y,y')=0$ — ① $y=f(x,y')$

before LEC — ② $x=f(y,y')$

$\Rightarrow$ 令 $y'=p$，两端求导

① 不显含 y $F(x,y')=0$

$F(x,y)=0$



曲线

令 $y'=p$ $F(x,p)=0$ 是 xop 上的曲线

求 dx $\Leftarrow$ $\begin{cases} x=\varphi(t) \\ p=\psi(t) \end{cases}$ $\boxed{\text{参数形式}}$

$dy=y'dx$，$y'=\frac{dy}{dx}$

$p=y'$，$\boxed{dy=p\,dx}$

$\boxed{dx=x'dt=\varphi'(t)dt}$

$\Rightarrow$ $\cancel{dx=x'dt=\varphi'(t)dt}$

$\cancel{x=\varphi(t)=}$ $dy=p\,dx=p\,x'dt=\psi(t)\varphi'(t)dt$

两边同时积分 $\boxed{y=\int\psi(t)\varphi'(t)dt+C}$

$\begin{cases} x=\varphi(t) \\ y=\int\psi(t)\varphi'(t)dt+C \end{cases}$ 参数方程形式的通解

Eg. $x^3+(y')^3-3xy'=0$

令 $y'=p$ $x^3+p^3-3xp=0$ $\begin{cases} x=\varphi(t) \\ p=\psi(t) \end{cases}$

$$\Rightarrow 1+\left(\frac{p}{x}\right)^3-\frac{3p}{x^2}=0$$

$$1+\left(\frac{p}{x}\right)^3-3\cdot\frac{p}{x}\cdot\frac{1}{x}=0$$

令 $\frac{p}{x}=t$, $1+t^3-3t\cdot\frac{1}{x}=0$

$$x=\frac{3t}{1+t^3}$$

$$\begin{cases} x=\frac{3t}{1+t^3}=\varphi(t) \\ p=xt=\frac{3t^2}{1+t^3}=\psi(t) \end{cases}$$

$$\Rightarrow y=\int\psi(t)\varphi'(t)\,dt+C$$

推导

$p=y'$

$dy=p\cdot dx$

$=p\cdot\varphi'(t)\cdot dt$

$y=\int p\cdot\varphi'(t)\cdot dt$

$=\int\psi(t)\varphi'(t)\,dt$

$$\varphi'(t)=\left(\frac{3t}{1+t^3}\right)'=\frac{3+3t^3-9t^3}{(1+t^3)^2}$$

$$\Rightarrow y=\int\frac{3t^2}{1+t^3}\cdot\frac{3-6t^3}{(1+t^3)^2}\,dt+C$$

$$=\int\frac{9t^2(1-2t^3)}{(1+t^3)^3}\,dt+C$$

令 $t^3=u$, $du=3t^2dt$, $dt=\frac{1}{3t^2}\cdot du$

$$=\int\frac{3(1-2u)}{(1+u)^3}\,du+C$$

$$=3\int\frac{(1-2u)}{(1+u)^3}\,du+C$$

$$=3\int\left(\frac{1}{(1+u)^3}-\frac{2u}{(1+u)^3}\right)du+C$$

↑ 裂项 (or $\boxed{\text{换元}}$

$1+u=t$, $u=t-1$, $du=$

$$=3\cdot\left(-\frac{1}{2}\frac{1}{(1+u)^2}-2\cdot\left(-\frac{1}{u+1}+\frac{1}{2(u+1)^2}\right)\right)+C$$

$$=\frac{3}{2}\frac{1+4u}{(1+u)^2}+C \Rightarrow \begin{cases} x=\dfrac{3t}{1+t^3}\\ y=\dfrac{3(1+4t^3)}{2(1+t^3)^2}+C\end{cases}$$

②不显含x，只有y和y'　　$F(y,y')=0$

令 $p=y'=\frac{dy}{dx}$　　$F(y,p)=0$　　yOp

求dy $\Leftarrow$ $\begin{cases} y=\varphi(t)\\ p=\psi(t)\end{cases}$　　$\frac{dx}{dt}=\frac{\frac{dy}{dt}}{\frac{dy}{dx}}=\frac{\varphi'(t)}{p}$　$\therefore dx=\frac{\varphi'(t)}{p}dt$

$$\Rightarrow\begin{cases} x=\int\dfrac{\varphi'(t)}{\psi(t)}dt\\ y=\varphi(t)\end{cases}$$

$$\begin{cases} p=y'=\dfrac{dy}{dx}\\ \dfrac{dx}{dt}=\dfrac{\frac{dy}{dt}}{\frac{dy}{dx}}=\dfrac{\varphi'(t)}{p}\\ dx=\dfrac{\varphi'(t)}{p}dt\\ x=\int\dfrac{\varphi'(t)}{\psi(t)}dt\end{cases}$$

例：$y^2(y'-1)-(2-y')^2=0$

令 $p=y'=\frac{dy}{dx}$

$\begin{cases} y=\varphi(t)\\ p=\psi(t)\end{cases}$　　$\frac{dx}{dt}=\frac{\frac{dy}{dt}}{\frac{dy}{dx}}=\frac{\varphi'(t)}{\psi(t)}$　　$x=\int\frac{\varphi'(t)}{\psi(t)}dt$

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$$y^2(p-1)-(2-p)^2=0$$

$$(p-1)-\left(\frac{2-p}{y}\right)^2=0$$

令 $\frac{2-p}{y}=t \Rightarrow \therefore p=t^2+1$

$$\therefore \begin{cases} p=t^2+1=\psi(t) \\ y=\frac{1-t^2}{t}=\varphi(t) \end{cases}$$

$$x=\int\frac{\varphi'(t)}{\psi(t)}dt=\int\frac{-\frac{1}{t^2}-1}{1+t^2}dt+c$$

$$=\int-\frac{1}{t^2}dt+c$$

$$=\frac{1}{t}+c$$

$$\begin{cases} x=\frac{1}{t}+c \\ y=\frac{1}{t}-t \end{cases} \Rightarrow \frac{1}{t}=x-c$$

$$y=x-c-\frac{1}{x-c}$$ (显式解)