Mo Tu We Th Fr Sa Su

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齐次方程

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

$P(x,y)dy = Q(x,y)dy$

项的次数和相同

$(x^2y + x^3 + xy^2)$

$$\Rightarrow \frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)} = \frac{2xy^2+y^3}{x^2y+x^3+xy^2}$$

$$\Rightarrow = \frac{2\frac{y^2}{x^2} + \frac{y^3}{x^3}}{\frac{y}{x} + 1 + \frac{y^2}{x^2}}$$

$f\left(\frac{y}{x}\right)$

$\Rightarrow$ 整体换元，$\frac{y}{x} = u$

$$f(u) = \frac{2u^2+u^3}{u+1+u^2}$$

$\Rightarrow y = u \cdot x$ $\quad \to \varphi(x)$

同时对 x 求导 $\Rightarrow \frac{dy}{dx} = \frac{du}{dx} \cdot x + u \cdot 1$

$$\Rightarrow \frac{du}{dx} \cdot x + u = f(u) = \frac{2u^2+u^3}{u+1+u^2}$$

$$\Rightarrow \frac{1}{x}dx = \frac{1}{f(u)-u} \cdot du$$ $\Rightarrow$ 分离变量法求解

$\frac{u}{u}$

例3.

$y' = \frac{y}{x} + \tan\frac{y}{x}$

(1) set $\frac{y}{x} = u$ $du/dx \cdot x + u = u + \tan u$

$\Rightarrow \frac{1}{\tan u}\cdot du = \frac{1}{x}\cdot dx$

$\int \frac{\cos u}{\sin u} du = \int \frac{1}{x} dx$

$\Rightarrow \ln(|\sin u|) = \ln|x| + c = \ln e^{c}\cdot x$

$|\sin u| = e^{c}\cdot|x|$

$\sin\frac{y}{x} = C_0\cdot x$ $C_0 = \pm e^{c}$

$\frac{y}{x} = \arcsin C_0 x$

$y = x\cdot\arcsin C_0\cdot x$

例2: $(x^2+y^2)\cdot dy - 2xy\,dx = 0$ $P(x,y)dy + Q(x,y)dx$

$\frac{dy}{dx} = \frac{2xy}{x^2+y^2} = \frac{2\frac{y}{x}}{1+\frac{y^2}{x^2}}$ $P(x,y), Q(x,y)$ 项次表一样

set $\frac{y}{x} = u$ $\frac{dy}{dx} = \frac{du}{dx}\cdot x + u$

$y = ux$

$\frac{dy}{dx} = \frac{du}{dx}\cdot x + u$

$\Rightarrow \frac{du}{dx}\cdot x + u = \frac{2u}{1+u^2}$

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$$\Rightarrow \quad \frac{u-u^3}{1+u^2} = x \cdot \frac{du}{dx}$$

$$\frac{y}{x} = u$$

$$\frac{dy}{dx} = \frac{du}{dx} \cdot x + u$$

$$\int \frac{1}{x}\, dx = \int \frac{1+u^2}{u-u^3} \cdot du$$

$$\ln x + c = \int \frac{1+u^2}{u(1-u^2)} \cdot du$$

need u^2

$$\frac{1+u^2}{u(1-u^2)} = \cancel{\frac{A}{u} + \frac{Bu+C}{1-u^2}}$$

$$A = \frac{1}{1} = 1, \quad B$$

$$1+u^2 = A\cdot(1-u^2) + Bu^2 + Cu$$

$$1+u^2 = (B-A)\cdot u^2 + Cu + A$$

$$\Rightarrow C = 0, \quad B-A = 1, \quad B = 2$$

$$= \int \frac{1}{u} + \frac{2u}{1-u^2}$$

$$= \ln u + (-\ln(1-u^2)) = \ln \frac{u}{1-u^2}$$

$\Rightarrow$

$$C_0 \cdot x = \frac{u}{1-u^2}, \quad C_0 x = \frac{\frac{y}{x}}{1-\frac{y^2}{x^2}}$$

$$\frac{xy}{x^2-y^2} = C_0 x, \Rightarrow \boxed{y = C_0 \cdot (x^2-y^2)}$$

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③ 可化为齐次方程的方程

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

$$\frac{dy}{dx} = f\left(\frac{ax+by+c}{a_1x+b_1y+c}\right)$$

主要是为了消除常数项

① $\begin{vmatrix} a & b \\ a_1 & b_1 \end{vmatrix}$ 不成比例

$\begin{matrix} 1 & 2 \\ 2 & 4 \end{matrix} \Rightarrow$ 成比例

$$\begin{cases} x = \xi + \alpha \\ y = \eta + \beta \end{cases}$$

$\beta \rightarrow$ constant

$\eta \downarrow$ variable

$$\begin{cases} \alpha = x_1 \\ \beta = x_2 \end{cases}$$

$\Uparrow$

set $\begin{cases} a\alpha + b\beta + c = 0 \\ a_1\alpha + b_1\beta + c_1 = 0 \end{cases}$

eg. $\begin{cases} 3x+2y+1=0 \\ 2x+3y+3=0 \end{cases}$

$\uparrow$

所以成比例的解不了

$$\frac{a\xi + a\alpha + b\eta + b\beta + c}{a_1\xi + a_1\alpha + b_1\eta + b_1\beta + c_1}$$

$$= \frac{a\xi + b\eta}{a_1\xi + b_1\eta} = \frac{a + b\frac{\eta}{\xi}}{a_1 + b_1\frac{\eta}{\xi}}$$

$$\frac{d\eta}{d\xi} = \downarrow$$

ex1. $\dfrac{dy}{dx} = \dfrac{x-y+1}{x+y-3}$

$\Rightarrow$ 令 $\begin{cases} x = \xi + \alpha \\ y = \eta + \beta \end{cases}$ or $\left(\begin{matrix} x = X + a \\ y = Y + b \end{matrix}\right)$

$\Rightarrow$ 变 $\dfrac{d\eta}{d\xi} = \dfrac{\xi+\alpha-\eta-\beta+1}{\xi+\alpha+\eta+\beta-3}$

$\Rightarrow \begin{cases} \alpha - \beta + 1 = 0 \\ \alpha + \beta - 3 = 0 \end{cases} \Rightarrow \begin{cases} \alpha = 1 \\ \beta = 2 \end{cases}$

$\dfrac{d\eta}{d\xi} = \dfrac{\xi-\eta}{\xi+\eta} = \dfrac{1-\frac{\eta}{\xi}}{1+\frac{\eta}{\xi}}$ $\left\{ \begin{matrix} \text{令 } u = \frac{\eta}{\xi} \\ \cancel{\frac{du}{d\xi}\,\xi + u = \frac{1-u}{1+u}} \end{matrix} \right.$

$u = \dfrac{\eta}{\xi}$

$\eta = u \cdot \xi$

$\cancel{\text{由}}$ $\dfrac{d\eta}{d\xi} = \dfrac{du}{d\xi}\xi + u. \Rightarrow \dfrac{1-u}{1+u} = \dfrac{du}{d\xi}\cdot\xi + u$

$\dfrac{1+u}{u^2+2u-1}$ $\quad \xi \cdot \dfrac{du}{d\xi} = \dfrac{1-2u-u^2}{1+u}$

$\Rightarrow \int \cancel{\dfrac{u^2+2u-1}{1+u}} \cdot du = \int -\dfrac{1}{\xi} \cdot d\xi$

$\ln(u^2+2u-1) = -2\ln\xi + 2C$

$= \ln \xi^{-2} \cdot e^{2C}$

$$u^2+2u-2=\xi^{-2}\cdot C_0$$

$$u=\frac{\eta}{\xi} \Rightarrow \frac{\eta^2}{\xi^2}+2\left(\frac{\eta}{\xi}\right)-1=C_0\,\xi^{-2}$$

$$\Rightarrow \eta^2+2\eta\xi-\xi^2=C_0$$

$$\begin{cases} x=\xi+1 \\ y=\eta+2 \end{cases} \Rightarrow \begin{cases} \xi=x-1 \\ \eta=y-2 \end{cases}$$

$$\Rightarrow (y-2)^2+2(y-2)(x-1)-(x-1)^2=C_1$$

$$y^2-4y+4+(2y-4)(x-1)-x^2+2x-1=C_1$$

$$y^2-4y+2xy-2y-4x+\cancel{0}-x^2+2x=C_1$$

$$\underline{y^2+2xy-x^2-2x-6y=C_1}$$

隐式通解

② $\begin{matrix} a & b \\ a_1 & b_1 \end{matrix}$ <u>成比例</u>

set $\frac{a}{a_1}=\frac{b}{b_1}=\lambda \Rightarrow \quad a=\lambda a_1,\ b=\lambda b_1$

$$\frac{dy}{dx}=f\left(\frac{\overset{\lambda a_1}{a}x+\overset{\lambda b_1}{b}y+c}{a_1x+b_1y+c_1}\right)=f\left(\frac{\lambda(a_1x+b_1y)+c}{a_1x+b_1y+c_1}\right)$$

set $z=a_1x+b_1y$, $\frac{dz}{dx}=a_1+b_1\cdot\frac{dy}{dx}$

$$\Rightarrow \frac{dz}{dx} = a_1 + b_1 \cdot f\left(\frac{\lambda z + c}{z + c_1}\right)$$ ⇒ 分离变量

Ex2. $$\frac{dy}{dx} = \frac{x-y+4}{2x-2y+5}$$

$$\Rightarrow \cancel{\frac{dy}{dx} = \frac{x-y+\frac{5}{2}+\frac{3}{2}}{2x-2y+5} = \frac{1}{2} + \frac{\frac{3}{2}}{2x-2y+5}}$$

set $$\frac{dy}{dx} = \frac{\lambda(2x-2y)+4}{2x-2y+5}$$ $\lambda = \frac{1}{2}$

$z = 2x - 2y$, $\frac{dz}{dx} = 2 - 2\frac{dy}{dx}$

$z+8$

$2z+10-z-8$

$$\frac{dz}{dx} = 2 - 2\cdot\frac{\frac{1}{2}z+4}{z+5} = \frac{z+2}{z+5}$$

$$\int dx = \frac{z+5}{z+2}\cdot dz = \int\left(\frac{3}{z+2}\right)+1\cdot dz$$

$$x = z + 3\ln(z+2) + C$$

$$x = 2x - 2y + 3\ln(2x-2y+2) + C$$

$$\Rightarrow x - 2y + 3\ln|2x-2y+2| + c = 0$$

$$x - 2y + 3\ln 2\cdot(x-y+1) = -C$$

$$x - 2y + 3\ln(x-y+1) = C$$

$$e^{3\ln(x-y+1)} = 2y - x + c$$

$$(x-y+1)^3 = e^{2y-x}\cdot C_0$$