Memo No.

Date / /

⑥ 二阶微分方程 (special)

y'' no special: $(\ )y''+(\ )y'+(\ )y+(\ )x=0$

① $y''=f(x)$

$y'=\int f(x)dx+C_1$

$y=\int(\int f(x)dx)dx+C_1x+C_2$

eg. $y''=x\cdot e^x$

$y'=xe^x-e^x+C_1$

$y=xe^x-2e^x+C_1x+C_2$

② $y''=f(x,y')$

$\Rightarrow y'=z,\quad y''=z'$

$z'=f(x,z)$ 一阶

eg. $x\underbrace{y''}_{z'}=\underbrace{y'}_{z}+x^2e^x$

$z'=\underbrace{\frac{1}{x}}_{p(x)}z+\underbrace{xe^x}_{Q(x)}$

$e^{\int p(x)dx}(C+\int q(x)e^{-\int p(x)dx}$

$=x(C+\int e^x dx)$

$z=Cx+e^x\cdot x$

$y' = e^{x} \cdot x + cx$, $y = \int ((ce^{x} \cdot x) + (cx)) dx$

$= \frac{c_1}{2} x^2 + xe^{x} - e^{x} + c_2$

③ $y'' = f(y, y')$

$\downarrow z \qquad y'' = z' = \boxed{\frac{dz}{dx}}$

$\Rightarrow$ 化成关于 y 的方程 $\qquad = \frac{dz}{dy} \cdot \frac{dy}{dx} \to y' = z$

$\to = \frac{dz}{dy} \cdot \frac{dy}{dx} = z \cdot \frac{dz}{dy} = (z)'$

$\Rightarrow z \cdot \frac{dz}{dy} = f(y, z)$

$\Rightarrow z = \cdot \; y$

$\downarrow$
$y' \Rightarrow \int \cdots = \int \cdots$

e.g. $yy'' - y'^2 = y'y^2$

$\Rightarrow y' = z, \quad y'' = z \cdot \frac{dz}{dy}$

$z \cdot \frac{dz}{dy} \cdot y - z^2 = zy^2$ (一阶)

$\frac{dz}{dy} = y + \frac{z}{y}$

$Q(y) \quad P(y)$

$dx)$

$\frac{dz}{dy} = \frac{1}{y} z + y \Rightarrow z = e^{\int \frac{1}{y} dy} (C + \int y \cdot e^{-\int \frac{1}{y} dy} dy)$

$\frac{1}{y} \to P(y) \qquad y \to Q(y)$

$= y(c + \int -y^2 dy)$

$= y(c_1 - \frac{1}{2} y^3) = c_1 y - \frac{1}{3} y^4 = z = y' \to \int \int$

② 二阶常系数齐次线性方程.

$y''+py'+qy=0$ (*) p, q 常数

定理: 若 $y_1(x)$ 与 $y_2(x)$ 是方程(*)的解, 那么

$y=C_1y_1(x)+C_2y_2(x)$ 也是(*)的解

$y_1''+py_1'+qy_1=0$ $\square''+p\square'+q\square=0$

$y_2''+py_2'+qy_2=0$

$\Rightarrow y''+py'+qy=0$ $\frac{y_1}{y_2}\neq k$

$y=C_1y_1+C_2y_2$

Euler 待定 指数函数 法

e^x — ke^x, $e^{\lambda x}$

令 $y=e^{\lambda x}$

$(e^{\lambda x})''+p(e^{\lambda x})'+q\cdot e^{\lambda x}=0$

$\Rightarrow \lambda^2e^{\lambda x}+p\lambda e^{\lambda x}+qe^{\lambda x}=0$

$e^{\lambda x}[\lambda^2+p\lambda+q]=0$

$\Rightarrow \lambda^2 + p\lambda + q = 0$ 特征多项式

↳ 一元二次方程

① $\lambda_1 \neq \lambda_2$ 为实数

$\Rightarrow y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$

② $\lambda_1 = \lambda_2$ 为实数 $e^{\lambda_1 x} = e^{\lambda_2 x}$ ⇒ 线性相关

⇒ 找 y_2，$\frac{y_2}{e^{\lambda_1 x}} \neq k$

令 $\frac{y_2}{e^{\lambda x}} = z(x)$，$\boxed{y_2 = e^{\lambda x} z(x)}$ → 代入

$\cancel{e^{\lambda x}(z'' + (2\lambda_1 + p) z' + (\lambda^2 + p\lambda + q) z) = 0}$

$\cancel{(e^{\lambda x} z(x))' = \lambda e^{\lambda x} z(x) + e^{\lambda(x)} z'(x)}$

$y'' + py' + qy = 0$ 代入 y_2

$(\lambda e^{\lambda x} \cdot z(x) + e^{\lambda x} z'(x))' + p(\lambda e^{\lambda x} z(x) + e^{\lambda x} z'(x))$

$+ q \cdot e^{\lambda x} \cdot z(x) = 0$

$(\lambda^2 e^{\lambda x} z(x) + \lambda e^{\lambda x} z'(x) + \lambda e^{\lambda x} \cdot z'(x) + e^{\lambda x} z''(x))$

$+ p\lambda e^{\lambda x} z(x) + p e^{\lambda x} z'(x) + q e^{\lambda x} z(x) = 0$

$\Rightarrow e^{\lambda(x)}(\lambda^2 z(x) + \lambda z'(x) + \lambda z'(x) + z''(x) + p\lambda z(x) +$

$p z'(x) + q z(x)) = 0$

$e^{\lambda x}[z''(x) + (2\lambda_1 + p) z'(x) + (\lambda_1^2 + p\lambda_1 + q) z(x)] = 0$

λ_1 为方程 $\lambda^2 + p\lambda + q = 0$ 的根

又∵ λ_1 是 ↓ 的二重根 $\lambda_1 + \lambda_2 = -p$

$\lambda_1 \cdot \lambda_2 = q$

Mo Tu We Th Fr Sa Su　Date　/　/

$\Rightarrow \lambda_1^2 = q$　　$2\lambda_1 + p = 0$

$2\lambda_1 = p$

$\therefore e^{\lambda x}[z''] = 0$

$\Rightarrow z'' = 0$　$\cancel{\Rightarrow z = C(x)}$

$z' = C$　$\Rightarrow y_2 = Cx \cdot e^{\lambda_1 x}$

$z = Cx$　　$= Cx \cdot e^{\lambda_1 x}$

$\therefore y = C_1 e^{\lambda_1 x} + C_2 x \cdot e^{\lambda_1 x}$

特征方程是同根

③ λ_1, λ_2 为复数

$\lambda_1 = \alpha + i\beta$

$\lambda_2 = \bar{\lambda_1} = \alpha - i\beta$　　$\boxed{e^{ix} = \cos x + i\sin x}$

$e^{\lambda_1 x} = e^{(\alpha + i\beta)x} = e^{\alpha x} \cdot e^{i\beta x}$　　Euler公式

$= e^{\alpha x}[\cos\beta x + i\sin\beta x]$

$e^{\lambda_2 x} = e^{\alpha x}[\cos\beta x - i\sin\beta x]$

$$\begin{cases} e^{\lambda_1 x} = e^{\alpha x}[\cos\beta x + i\sin\beta x] & ① \\ e^{\lambda_2 x} = e^{\alpha x}[\cos\beta x - i\sin\beta x] & ② \end{cases}$$

①+② $= 2e^{\alpha x} \cdot \cos\beta x = e^{\lambda_1 x} + e^{\lambda_2 x}$

①−② $= 2i e^{\alpha x} \sin\beta x = e^{\lambda_1 x} - e^{\lambda_2 x}$

$\Rightarrow e^{\alpha x}\sin\beta x = \frac{e^{\lambda_1 x}-e^{\lambda_2 x}}{2i}$

$e^{\alpha x}\cos\beta x = \frac{e^{\lambda_1 x}+e^{\lambda_2 x}}{2}$

实数且非线性相关

$\Rightarrow y = c_1 \cdot e^{\alpha x}\cos\beta x + c_2 \cdot e^{\alpha x}\sin\beta x$

$\boxed{\text{总结}}$ $y''+py'+qy=0$

$\lambda^2+p\lambda+q=0$

$\boxed{\text{天知道}}$

① $\lambda_1 \neq \lambda_2$ 为实数

$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$

② $\lambda_1 = \lambda_2 = \lambda$

$y = c_1 e^{\lambda_1 x} + c_2 x \cdot e^{\lambda x}$

③ $\lambda_1 = \alpha + i\beta \qquad \lambda_2 = \alpha - i\beta$

$y = \boxed{c_1 e^{(\alpha x)}\cos(\beta x) + c_2 e^{(\alpha x)}\sin(\beta x)}$

Examples:

e.g. 1. $y''-4y'+3y=0 \qquad y|_{x=0}=6,\ y'|_{x=0}=10$

$\lambda^2-4\lambda+3=0$

$\frac{4\pm\sqrt{16-12}}{2}$

$\begin{cases}\lambda_1 = 1\\ \lambda_2 = 3\end{cases}$

$(\lambda-1)(\lambda-3)=0$

$\Rightarrow y = c_1 e^{x} + c_2 e^{3x}$

$x=0$

$\begin{cases} y = c_1\cdot 1 + c_2 = 6 \\ y' = c_1 + 3c_2 = 10 \end{cases}$

$\Rightarrow c_1 = 4, c_2 = 2$

$\Rightarrow y = 4e^{x} + 2e^{3x}$

Mo | Tu | We | Th | Fr | Sa | Su

Memo No.

Date / /

e.g. 2 $y'' + 4y' + 4y = 0$

$$\lambda^2 + 4\lambda + 4 = 0$$

$$\Rightarrow \begin{cases} \lambda_1 = -2 \\ \lambda_2 = -2 \end{cases}$$

$$y = c_1 e^{-2x} + c_2 x e^{-2x}$$

e.g. 3 $y'' + 2y' + 4y = 0$

$\lambda^2 + 2\lambda + 4 = 0$ $(\lambda + 1)^2 = -3$

$$\Rightarrow \begin{cases} \lambda_1 = -1 + \sqrt{3}i \\ \lambda_2 = -1 - \sqrt{3}i \end{cases}$$ $\alpha = -1,\ \beta = \sqrt{3}$

$$\Rightarrow y = c_1 e^{\alpha x} \cancel{\sin} \cos \beta x + c_2 e^{\alpha x} \sin \beta x$$

$$= c_1 e^{-x} \cos \sqrt{3} x + c_2 e^{-x} \sin \sqrt{3} x$$