$$y = c_1 x + c_2 \ln x \, x + \frac{1}{6}(\ln x)^3 x$$

10.1 全微分方程

偏导

一元函数 $f(x) = x^2$　　$y' = 2x$

二元函数 $f(x,y) = xy + 3x^2y + 3xy^2$

$$\begin{cases} \dfrac{\partial z}{\partial x} = y + 6xy + 3y^2 \\ \dfrac{\partial z}{\partial y} = x + 6xy + 3x^2 \end{cases}$$

$\dfrac{dy}{dx} = f(x,y)$ $\longrightarrow$ 一阶正规形方程

$\Rightarrow f(x,y)dx - dy = 0$　　$\Rightarrow M(x,y)dx + N(x,y)dy = 0$

↑ 全微分方程

有一个二元函数 $du(x,y) = M(x,y)dx + N(x,y)dy$

判断方程是否为全微分方程：

若 $\boxed{\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}\text{，}\Rightarrow \text{全微分方程}}$

Memo No.

Date / /

$\Rightarrow$ ① $\int_{x_0}^{x} M(x,y)dx + \int_{y_0}^{y} M(x_0,y)dy = C$

任意 常量

② $\int_{y_0}^{y} N(x,y)dy + \int_{x_0}^{x} M(x,y_0)dx = C$

常量

$M(x,y)$ $N(x,y)$

例 $(4x^3y^3+1)dx + (3x^4y^2-2)dy = 0$

$\longrightarrow$ 判断是否为全微分方程

$$\frac{\partial M}{\partial y} = 12x^3y^2 \quad == \quad \frac{\partial N}{\partial x} = 12x^3y^2$$

$$\int_{x_0}^{x} M(x,y)dx + \int_{y_0}^{y} N(x_0,y)dy = C$$

$\Rightarrow$

$$\int_0^x (4x^3y^3+1)dx + \int_0^y -2\,dy = C$$

$$[x^4y^4+x]_0^x + [-2y]_0^y = C$$

$$x^4y^4+x-2y = C$$

Mo Tu We Th Fr Sa Su

Memo No. ____________
Date / /

10.2 全微分方程的积分因子.

$M(x,y)dx + N(x,y)dy = 0$

$\frac{\partial(M,y)}{\partial y} \neq \frac{\partial N(x,y)}{\partial x}$ ⇒ 不可以用公式

⇒ $\cancel{\mu(x,y)}\, M(x,y)dx + \mu(x,y)\, N(x,y)\, dy = 0$

⇒ $\boxed{\mu M}dx + \boxed{\mu N}dy = 0$

set $\frac{\partial \mu M}{\partial y} = \frac{\partial(\mu N)}{\partial x}$, μ: 积分因子

↓

让不是全微分方程的方程变为全微分方程

$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}$

$\cancel{\frac{\partial \mu}{\partial y} \cdot M + \frac{\partial M}{\partial y} \cdot \mu} =$

$\frac{\partial(\mu)}{\partial y} \cdot M + \frac{\partial M}{\partial y} \cdot \mu$

$= \frac{\partial \mu}{\partial x} \cdot N + \frac{\partial N}{\partial x} \cdot \mu$

$$\boxed{\mu\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = N\frac{\partial \mu}{\partial x} - M\frac{\partial \mu}{\partial y}}$$

Memo No.
Date / /

$\Rightarrow$ ① 想让 $\frac{\partial \mu}{\partial y}=0$ $\Rightarrow$ 说明 μ 只关于 x 的函数

$\Rightarrow \mu(x,y)=\mu(x)$, $\mu(x)$, $\frac{\partial \mu}{\partial x}=\frac{d\mu}{dx}$

$\mu\cdot\boxed{\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)}=N\cdot\frac{d\mu}{dx}$ $\Rightarrow$ 分离变量

$\int\frac{1}{\mu}\cdot d\mu=\int\frac{\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)}{N}\cdot dx$

$\therefore \ln\mu= \ldots$

$\therefore \mu=e^{\int\boxed{\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}}\cdot dx}$

以一定要和 y 无关

只有当与 x 有关才从写成这个形式

② 如果 $\frac{\partial \mu}{\partial x}=0$

同理, $\mu=e^{\int\boxed{\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{-M}}dy}$

只与 y 有关

Example: $(2xy+x^2y+\frac{y^3}{3})dx+(x^2+y^2)dy=0$

$$\begin{cases}\frac{\partial M}{\partial y}=2x+x^2+y^2\\ \frac{\partial N}{\partial x}=2x+0\end{cases}$$

$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=\frac{x^2+y^2}{N}=1$

积分因子

$\therefore \mu=e^{\int\frac{x^2+y^2}{N}dx}=e^x$,

Memo No.
Date / /

M N

$\therefore e^x(2xy + x^2y + \frac{y^3}{3})dx + e^x(x^2 + y^2)dy = 0$

⇒全微分方程

$\Rightarrow \int_0^y e^x N dy + \int_0^x e^x M(x,0) dx = C$

$\therefore \int_0^y e^x(x^2+y^2) dy + \underbrace{\int_0^x 0 dx}_{C} = C$

$\therefore [y \cdot e^x x^2 + \frac{1}{3} e^x y^3]_0^y = C$

$\therefore e^x y x^2 + \frac{1}{3} e^x y^3 = C$

(10.3) 单组积分因子

$M(x,y)dx + N(x,y)dy = 0$

$d\mu = M(x,y)dx + N(x,y)dy$

$\mu = e^{\int \frac{M_y - N_x}{N} dx}$ ①

$\mu = e^{\int \frac{M_y - N_x}{-M} dy}$ ②

if $M_y - N_x$ is complex ⇒