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④ 一阶线性微分方程

y' $\qquad \frac{dy}{dx} = p(x)\cdot y + q(x)$

① $q(x) \equiv 0$ $\qquad \frac{dy}{dx} = p(x)\cdot y$ —— 一阶齐次线性

方程

$\int \frac{1}{y}\, dy = \int p(x)\, dx$

$\ln y = \int p(x)\, dx + C$ $\rightarrow$ $y = C\cdot e^{\int p(x)\, dx}$

求 $\int p(x)\, dx$

② $q(x) \neq 0$ $\qquad \frac{dy}{dx} = \cancel{q(x)}\ p(x)\cdot y + q(x)$ —— 一阶非齐次线性

$\boxed{y = C(x)\cdot e^{\int p(x)dx}\ ?}$ $\qquad$ (变动参数法)

$f = uv \qquad f' = u'v + uv'$

$y' = C'(x)\cdot e^{\int p(x)dx} + \boxed{C(x)\cdot e^{\int p(x)dx}\cdot p(x)} = \boxed{p(x)\cdot C(x)\cdot e^{\int p(x)dx}} + q(x)$

$q(x) = C'(x)\cdot e^{\int p(x)dx}$

$C'(x) = \frac{dC(x)}{dx}$

$\int dC(x) = \int q(x)\, e^{-\int p(x)dx} dx + C$

$C(x) = \int q(x)\, e^{-\int p(x)dx} dx + C$

$y = \underbrace{\left(\int q(x)\cdot e^{-\int p(x)dx} dx + C\right)}_{C(x)} \cdot e^{\int p(x)dx}$

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$$\boxed{y=\left(\int g(x)\cdot e^{-\int p(x)dx}dx+C\right)\cdot e^{\int p(x)\cdot dx}}$$

例1: $xy'=y+x^2$

$$\Rightarrow y'=y/x+x \quad =p(x)\cdot y+g(x)$$

$$p(x)=\frac{1}{x} \quad g(x)=x$$

$$y=e^{\int\frac{1}{x}dx}\left(C+\int x\cdot e^{-\int\frac{1}{x}dx}\cdot dx\right)$$

$$=x\cdot\left(C+\int x\cdot\frac{1}{x}\cdot dx\right)=x^2+C\cdot x$$

例2: $y'=\frac{y\ln y}{\ln y-x}$ $\quad (y'=g(x)+p(x)y$

$$\Rightarrow \frac{dx}{dy}=\frac{\ln y-x}{y\ln y}=\frac{1}{y}-\frac{x}{y\ln y}$$

$$=\cancel{p(y)}\ g(y)+p(y)\cdot x$$

$(uv)'=u'v+uv'$

$$\boxed{y=e^{\int p(x)\cdot dx}\left(\int g(x)\cdot e^{-\int p(x)dx}dx+C\right)}$$

$\int\frac{1}{y\ln y}dy = \ln\ln y$

$$g(y)=\frac{1}{y},\quad p(y)=-\frac{1}{y\ln y}$$

$$x=\left(\int\frac{1}{y}\cdot e^{-\int-\frac{1}{y\ln y}dy}\cdot dy+C\right)\cdot e^{\int\frac{-1}{y\ln y}\cdot dy}$$

$$=\left(\int\frac{1}{y}\cdot e^{\ln(\ln y)}\cdot dy+C\right)\cdot -\ln y$$

$$=-\ln y\left(\int\frac{\ln y}{y}dy+C\right)=-\ln y\left(\frac{1}{2}\ln^2 y+C\right)$$

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⑤ 伯努利方程

Bernoulli

$$\frac{dy}{dx} = p(x)y + q(x)\cdot \underline{y^{\alpha}}$$

① $\alpha = 0 \Rightarrow$ 非齐次线性微分方程

$\alpha = 2$ 齐次

$$\Rightarrow \boxed{y^{-\alpha}\frac{dy}{dx}} = p(x)y^{1-\alpha} + q(x)$$

$$[y^{(1-\alpha)}]' = \frac{d y^{1-\alpha}}{dx} = (1-\alpha)\boxed{y^{-\alpha}\cdot\frac{dy}{dx}}$$

$$\Rightarrow \frac{1}{1-\alpha}\cdot\frac{dy^{1-\alpha}}{dx} = p(x)\cdot y^{1-\alpha} + q(x), \quad z = y^{1-\alpha}$$

$$\frac{1}{1-\alpha}\cdot\frac{dz}{dx} = p(x)\cdot z + q(x)$$

$$\Rightarrow \frac{dz}{dx} = \boxed{(1-\alpha)p(x)}\cdot z + \boxed{(1-\alpha)\cdot q(x)}$$

$P(x)$ $Q(x)$

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Example:

$$x \cdot y' = 4y + x^2 y^{\frac{1}{2}} \to \alpha$$

$$y' = 4\,y/x + x y^{\frac{1}{2}}$$

$$= 4\frac{1}{x} \cdot y + x \cdot y^{\frac{1}{2}}$$

$$\downarrow p(x) y + q(x) \cdot y^{\alpha}, \quad \alpha = \frac{1}{2}$$

令 $z = y^{1-\alpha} = y^{\frac{1}{2}}$

$$\frac{dz}{dx} = \cancel{(1-\frac{1}{2})} \cdot \frac{1}{2} \cdot 4 \cdot \frac{1}{x} z + \frac{1}{2} \cdot x$$

$$= \underbrace{2\frac{1}{x}}_{p(x)} \cdot z + \underbrace{\frac{1}{2}x}_{Q(x)}$$

$$\Rightarrow z = e^{\int 2\frac{1}{x} dx} \cdot \left(C + \int \frac{1}{2} x \cdot e^{-\int 2\frac{1}{x} dx} dx \right)$$

$$= e^{2\ln x} \cdot \left(C + \int \frac{1}{2} x \cdot e^{-2\ln x} dx \right)$$

$$= x^2 \cdot \left(C + \int \frac{1}{2} x \cdot x^{-2} dx \right)$$

$$= x^2 \left(C + \frac{1}{2} \cdot \ln|x| \right)$$

$$y^{\frac{1}{2}} = x^2 \cdot \left(C + \frac{1}{2} \ln|x| \right) \checkmark$$

$$y = x^4 \left(C + \frac{1}{2} \ln|x| \right)^2$$