MATHS

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IT-3

Assignment-2

① Show that $\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$

Sol: We know that $J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$

multiplying on both sides with x^n

$$x^n. J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} . x^n \left(\frac{x}{2}\right)^{n+2r}$$

$$x^n. J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} . \frac{x^{2n+2r}}{2^{n+2r}}$$

Differentiating the above equation wrt x, we have

$$\frac{d}{dx}\left[x^n. J_n(x)\right] = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} (2n+2r) . \frac{x^{2n+2r-1}}{2^{n+2r}}$$

$$\left[\because \Gamma(n+r+1) = (n+r)\Gamma(n+r)\right]$$

$$\frac{d}{dx}\left[x^n. J_n(x)\right] = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r)} . x^n . \frac{x^{n+2r-1}}{2^{n+2r-1}}$$

$$= x^n \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma((n-1)+r+1)} . \left(\frac{x}{2}\right)^{(n-1)+2r}$$

$$\frac{d}{dx}\left[x^n. J_n(x)\right] = x^n J_{n-1}(x)$$

Hence proved.

② Prove that $\int_{-1}^{1} P_n(x)\, P_m(x)\, .dx = 0$

sol: Legendre's equation $(1-x^2)y'' - 2xy' + n(n+1)y = 0 \longrightarrow ①$

$\because m \neq n$

We know that $P_m(x)$ and $P_n(x)$ are solutions of legendre's equation.

$$\Rightarrow (1-x^2)\,P_m''(x) - 2x\,P_m'(x) + m(m+1)\,P_m(x) = 0 \longrightarrow (2)$$

$$\Rightarrow (1-x^2)\,P_n''(x) - 2x\,P_n'(x) + n(n+1)\,P_n(x) = 0 \longrightarrow (3)$$

$(2) \times P_n(x) - (3) \times P_m(x)$

$$\Rightarrow (1-x^2)\,P_m''P_n - 2x\,P_m'P_n + m(m+1)\,P_mP_n = 0$$

$$\underset{(-)}{(1-x^2)\,P_n''P_m} \underset{(+)}{- 2x\,P_n'P_m} \underset{(-)}{+ n(n+1)\,P_nP_m} = 0$$

$$\Rightarrow (1-x^2)\left[P_m''P_n - P_n''P_m\right] - 2x\left(P_m'P_n - P_n'P_m\right] + \left\{m^2+m-n^2-n\right\}P_mP_n = 0$$

$$\therefore \frac{d}{dx}(1-x^2)\,(P_m'P_n - P_mP_n') = (1-x^2)\left[P_m''P_n + P_m'P_n' - P_m'P_n' - P_mP_n''\right] - 2x\left[P_m'P_n - P_mP_n'\right]$$

$$\therefore m^2+m-n^2-n = (m^2-n^2) + (m-n)$$

$$m^2+m-n^2-n = (m-n)\,(m+n+1)$$

$$\frac{d}{dx}\left[(1-x^2)\left[P_m''P_n - P_n''P_m\right]\right] = -(m-n)\,(m+n+1)\,P_mP_n$$

Integrating above equation w.r.t. "x" between the limits -1 and 1 we have

$$\Rightarrow \left[\frac{(1-x^2)\,(P_m'P_n - P_mP_n')}{-(m-n)\,(m+n+1)}\right]_{-1}^{1} = \int_{-1}^{1} P_m(x).P_n(x).dx$$

$\Downarrow$

$m \neq n$

$$\Rightarrow \int_{-1}^{1} P_m(x).P_n(x).dx = 0, \quad \text{where } m \neq n$$

(3) Determine the value of $J_{-\frac{3}{2}}(x)$.

Sol: We know that, $J_n(x) = \sum_{n=0}^{\infty} \frac{(-1)^r}{r! \, \Gamma(n+r+1)} \left(\frac{x}{2}\right)^{2r+n}$

Put $n = \frac{3}{2}$

$\Rightarrow J_{\frac{3}{2}}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r! \, \Gamma(-\frac{3}{2}+r+1)} \left(\frac{x}{2}\right)^{2r-\frac{3}{2}}$, is not defined

So multiplying and dividing the numerotor by $\Rightarrow \left(-\frac{3}{2}+1\right) = -\frac{1}{2}$

$$J_{-\frac{3}{2}}(x) = \sum_{r=0}^{\infty} \frac{(-\frac{1}{2})(-1)^r}{r!(-\frac{1}{2})\Gamma(-\frac{3}{2}+r+1)} \cdot \left(\frac{x}{2}\right)^{2r-\frac{3}{2}}$$

$$= \frac{(-\frac{1}{2})}{(-\frac{1}{2})\Gamma(-\frac{1}{2})} \cdot \left(\frac{x}{2}\right)^{-\frac{3}{2}} - \frac{(-\frac{1}{2})\left(\frac{x}{2}\right)^{\frac{1}{2}}}{1!\,(-\frac{1}{2})\Gamma(-\frac{1}{2}+1)} + \frac{(-\frac{1}{2}).\left(\frac{x}{2}\right)^{5/2}}{2!\,(-\frac{1}{2})\Gamma(\frac{3}{2})}$$

$$= \frac{-1}{2\,\Gamma(-\frac{1}{2}+1)} \cdot \left(\frac{x}{2}\right)^{-\frac{3}{2}} - \frac{1}{1!\,\Gamma(\frac{1}{2})} \cdot \left(\frac{x}{2}\right)^{\frac{1}{2}} + \left(\frac{1}{2!\,\Gamma(\frac{3}{2})}\left(\frac{x}{2}\right)^{\frac{5}{2}}\right.$$

$\left[\because \Gamma\frac{1}{2} = \sqrt{\pi},\ n\Gamma n = \Gamma(n+1),\ \Gamma\frac{3}{2} = \Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2}\Gamma\frac{1}{2} = \frac{1}{2}\sqrt{\pi}\right]$

$$= \frac{-1}{2\sqrt{\pi}} \cdot \left(\frac{x}{2}\right)^{-\frac{3}{2}} - \frac{1}{\sqrt{\pi}} \cdot \left(\frac{x}{2}\right)^{\frac{1}{2}} + \frac{1}{2(\frac{1}{2})\sqrt{\pi}} \cdot \left(\frac{x}{2}\right)^{\frac{5}{2}} \ldots$$

$$= -\sqrt{\frac{2}{\pi x}}\left(\frac{x}{2} + \frac{x}{4} - \frac{1}{(\frac{x}{2})^2} - \left(\frac{x}{2}\right)\left(\frac{x}{2}\right)^2 \ldots\right)$$

$$= -\sqrt{\frac{2}{\pi x}}\left[\frac{x}{2} + \frac{x}{4} \times \frac{4}{x^2} - \frac{x^3}{8} \ldots\right]$$

$$= -\sqrt{\frac{2}{\pi x}}\left(\frac{1}{x}\left(x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} \ldots\right) + \frac{1}{x}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\right)\right]$$

$$= -\sqrt{\frac{2}{\pi x}}\left(\sin x + \frac{\cos x}{x}\right]$$

$$\therefore J_{-\frac{3}{2}}(x) = -\sqrt{\frac{2}{\pi x}}\left[\sin x + \frac{\cos x}{x}\right]$$

(4) Stat Rodrigue's formula

Sol: Statement : $P_n(x) = \frac{1}{2^n n!} \cdot \frac{d^n}{dx^n}\left[(x^2-1)^n\right]$

Note: $(1-x^2)y'' - 2xy' + n(n+1)y = 0$ then

$P_n(x)$ is the solution $\ni (1-x^2) P_n''(x) - 2x P_n'(x) + n(n+1) P_n(x) = 0$

Proof: Let $V = (x^2-1)^n \longrightarrow (1)$

Differentiat on both sides with respect to x

$$\frac{dv}{dx} = n(x^2-1)^{n-1}(2x) = \frac{2xn(x^2-1)^n}{(x^2-1)}$$

$$\Rightarrow (x^2-1)\frac{dv}{dx} = 2nx(x^2-1)^n$$

$$\Rightarrow \underset{\mathrm{II}}{(1-x^2)}\underset{\mathrm{I}}{v_1} + \underset{\mathrm{II}}{2nx}\underset{\mathrm{I}}{v} = 0 \longrightarrow (2)$$

[$\because$ Leibnitz rule from differentiation]

$$\frac{d^n}{dx^n}(uv) = n[0 u_n v + n[-u_{n-1} v_1 + nC_2 u_{n-2} v_2 + \cdots + nC_n u v_n$$

$\Rightarrow$ suffix stands for derivative

Differentiating eq.(2) successively $(n+1)$ times w.r.t. x, we have

By applying Leibnitz rule

$(n+1)C_0 v_{n+2}(1-x^2) + (n+1)C_1 v_{n+1}(-2x) + (n+1)C_2 v_n(-2) +$

$2n[(n+1)C_0 . v_{n+1}(x) + (n+1)C_1 . v_n(1)] = 0.$

$$\Rightarrow (1-x^2)v_{n+2} - 2x(n+1)v_{n+1} - \frac{2(n+1)n}{2!}v_n + 2nx v_{n+1} + 2n(n+1) = 0$$

$(1-x^2)(v_{n+2}) - 2x v_{n+1} + n(n+1) v_n = 0 \longrightarrow (3)$

Let $v_n = 4$

$\Rightarrow (1-x^2) u_2 - 2x u_1 + n(n+1) u = 0$ [which is legendre's eq. in 4]

Let its solution be $c_1 = c P_n(x)$ [where c is arbitrary constant]

$\Rightarrow v_n = c P_n(x)$ [$\because$ from (1)] $[u = v_n]$

$$\frac{d^n}{dx^n}(x^2-1)^n = c P_n(x) \longrightarrow (4)$$

To find 'c'

Put $x=1$, $c P_n(1) = \left[\frac{d^n}{dx^n}(x^2-1)^n\right]_{x=1}$

Here $P_n(1) = 1$

$$\Rightarrow c \cdot 1 = \left[\frac{d^n}{dx^n} \underset{\text{II}}{(x+1)^n} \underset{\text{I}}{(x-1)^n}\right]_{x=1}$$

$$c = \left[{}^nC_0 \left[\frac{d^n}{dx^n}(x-1)^n\right] + {}^nC_1 \left[\frac{d^{n-1}}{dx^{n-1}}(x-1)^n\right] \frac{d}{dx}(x+1)^n + \dots + {}^nC_n (x-1)^n \frac{d^n}{dx^n}(x+1)^n\right]_{x=1}$$

$\therefore$ In the above equation except first term all terms will have $(x-1)$ as factor, when we take $x=1$, all will be zero except first term.

$\Rightarrow c = n! 2^n$

Putting 'c' value in eq. (4)

$$\Rightarrow \frac{d^n}{dx^n}(x^2-1)^n = n!\, 2^n P_n(x)$$

$$P_n(x) = \frac{1}{n! 2^n} \cdot \frac{d^n}{dx^n}(x^2-1)^n$$

5. Express the polynomial $f(x)=x^3+3x^2+1$ in Legendre's polynomials.

Sol: $P_0(x)=1, P_1(x)=x, \quad P_2(x)=\frac{1}{2}(3x^2-1) \Rightarrow x^2=\frac{2P_2(x)+P_0(x)}{3}$

$\Rightarrow x^3+3x^2+1=0, \quad P_3(x)=\frac{1}{2}(5x^3-3x) \Rightarrow x^3=\frac{2P_3(x)+3P_1(x)}{5}$

$\Rightarrow \frac{2P_3(x)+3P_1(x)}{5}+3\left[\frac{2P_2(x)+P_0(x)}{3}\right]+P_0(x)$

$\Rightarrow \frac{2}{5}P_3(x)+2P_2(x)+\frac{3}{5}P_1(x)+2P_0(x)$

6. Find the fourier transform of

$$f(x)=\begin{cases}1-x\ ; & |x|\leq a\\ 0\ ; & |x|>a\end{cases}$$

Sol: $|x|\leq a \Rightarrow -a\leq x\leq a$

$|x|>a \Rightarrow x<-a;\ x>a$

$x=(-\infty,-a)\cup(a,\infty)$

$$F\{f(x)\}=\int_{-\infty}^{\infty} f(x).e^{isx}.dx$$

$$=\int_{-\infty}^{-a} f(x).e^{isx}.dx+\int_{-a}^{a} f(x).e^{isx}.dx+\int_{a}^{\infty} f(x).e^{isx}.dx$$

$$=0+\int_{-a}^{a}(1-x).e^{isx}.dx+0$$

$$=\left[(1-x).\frac{e^{isx}}{is}-\int (-1)\frac{e^{isx}}{(is)}dx\right]_{-a}^{a}$$

$$= \left[(1-x).\frac{e^{isx}}{(is)} + \frac{e^{isx}}{(-s^2)}\right]_{-a}^{a}$$

$$=(1-a).\frac{e^{isa}}{is} - (1+a).\frac{e^{-isa}}{is} - \frac{e^{isa}}{s^2} - \frac{e^{-isa}}{s^2}$$

$$= \frac{1}{is}\left(e^{isa} - e^{-isa}\right) - \left(\frac{a}{is} + \frac{1}{s^2}\right)\left(e^{isa} + e^{-isa}\right)$$

$$F\{f(x)\} = \frac{1}{is}\sin as - \left(\frac{a}{is} + \frac{1}{s^2}\right)\cos as$$

(7) Find the fourier transform of $f(x) = \begin{cases} 1-x^2, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$. Hence evaluate $\int_0^\infty \frac{x\cos x - \sin x}{x^3}\frac{\cos x}{2}$

Sol: $|x| \le 1 \Rightarrow -1 \le x \le 1$

$|x| \ge 1 \Rightarrow x < -1$ & $x > 1 \Rightarrow x \in (-\infty, -1) \cup (1, \infty)$

$$f(x) = \int_{-\infty}^{\infty} f(x).e^{isx}dx$$

$$= \int_{-\infty}^{-1} (0).e^{isx}.dx - \int \left([-2x]\int e^{isx}.dx\right]_{-1}^{1}$$

$$= \left((1-x^2).\frac{e^{isx}}{is} + 2\left[\frac{x.e^{isx}}{(is)^2} - \frac{1}{(is)}\int\frac{e^{isx}}{(is)}.dx\right]\right]_{-1}^{1}$$

$$= \left((1-x^2).\frac{e^{isx}}{is} + 2\frac{x.e^{isx}}{(is)^2} - \frac{2}{(is)^3}.e^{isx}\right]_{-1}^{1}$$

$$= \frac{2.e^{is}}{s^2} - \frac{2e^{is}}{is^3} + \frac{(-2.e^{-is})}{s^2} + \frac{2e^{-is}}{is^3}$$

$$= \frac{-4}{s^2}\cos s + \frac{4i\sin s}{s^3}$$

$$f\{f(x)\} = \frac{-4}{s^3}(s\cos s - \sin s) = F(s)\ \text{(say)}$$

To evaluate integral taking inverse ~~functio~~ fourier transform on above result we have

$$F^{-1}\{F(s)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s).e^{-isx}.ds = f(x)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{-4}{s^3}(s\cos s - \sin s)(\cos sx - i\sin sx).ds = f(x)$$

$$= \frac{-4}{\pi}\int_0^{\infty}\left(\frac{s\cos s - \sin s}{s^3}\right)\cos x . ds = f(x)$$

Put $x = \frac{1}{2}$

$$= \frac{-4}{\pi}\int_0^{\infty}\left(\frac{s\cos s - \sin s}{s^3}\right)\cos\frac{s}{2}\, ds = f\left(\frac{1}{2}\right) \cdot (1-x^2)\Big|_{x=\frac{1}{2}}$$

Replacing s by 'x' we have

$$\int_0^{\infty}\left(\frac{x\cos x - \sin x}{x^3}\right)\cos x/2 . dx = \frac{3\pi}{16}$$

(8) Find $z(\cos n\theta)$

Sol: $e^{-in\theta} = \cos n\theta - i\sin n\theta$

$$z(e^{-in\theta}) = z((e^{-i\theta})^n) = \frac{z}{z-e^{-i\theta}}$$

$$z(e^{-in\theta}) = \frac{z}{(z-e^{-i\theta})} \times \left(\frac{z-e^{i\theta}}{z-e^{i\theta}}\right)$$

$$= \frac{z(z-e^{i\theta})}{(z^2 - z(e^{-i\theta}+e^{i\theta})+1)} = \frac{z(z-e^{i\theta})}{(z^2 - 2z\cos\theta + 1)}$$

$$z(\cos n\theta - i\sin n\theta) = \frac{z(z-\cos\theta) - iz\sin\theta}{z^2 - 2z\cos\theta + 1}$$

$$z(\cos n\theta) - iz(\sin n\theta) = \frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1} - \frac{iz\sin\theta}{z^2-2z\cos\theta+1}$$

Equating the real terms

$$z(\cos n\theta) = \frac{z(z-\cos\theta)}{z^2+2z\cos\theta+1}.$$

(9) State and prove initial value thm. Hence calculate u_0, u_1, u_2 for

$$U(z) = \frac{z^2+z}{(z-1)(z+3)}.$$

Sol: Statement: If $z(u_n) = U(z)$ then $U_z = \underset{z\to\infty}{Lt}\, U(z)$

Proof: We know that $U(z) = z(u_n) = U_0 + \frac{U_1}{z} + \frac{U_2}{z^2} + \cdots$

taking Limit $z \to \infty$

$$\underset{z\to\infty}{Lt}\, U(z) = \underset{z\to\infty}{Lt}\left[U_0 + \frac{U_1}{z} + \frac{U_2}{z^2} + \cdots\right]$$

$$\underset{z\to\infty}{Lt}\, U(z) = U_0$$

i) $U_0 = \underset{z\to\infty}{Lt}\, U(z) = \underset{z\to\infty}{Lt}\, \frac{z^2+z}{(z-1)(z+3)}$

$$= \underset{z\to\infty}{Lt}\, \frac{z^2\left(1+\frac{1}{z}\right)}{z^2\left(1+\frac{2}{z}-\frac{3}{z^2}\right)}$$

$$= \underset{z\to\infty}{Lt}\, \frac{1+\frac{1}{z}}{1+\frac{2}{z}-\frac{3}{z^2}}$$

$$= 1$$

$U_0 = 1$

(ii) $U_1 = \underset{z\to\infty}{Lt}\left(z\left[U(z) - U_0\right]\right) = \underset{z\to\infty}{Lt}\; z\left[\frac{z^2+z}{(z-1)(z+3)} - 1\right]$

$$= \underset{z\to\infty}{Lt}\; \frac{-z^2+3z}{z^2+2z-3}$$

$$= \underset{z\to\infty}{Lt}\; \frac{z^2\left(-1+3/z\right)}{z^2\left(1+\frac{2}{z}-3/z^2\right)} \Rightarrow -1$$

$U_1 = -1$

(iii) $U_2 = \underset{z\to\infty}{Lt}\; z^2\left[U(z) - U_0 - \frac{U_1}{z}\right]$

$$= \underset{z\to\infty}{Lt}\; z^2\left[\frac{z^2+z}{(z-1)(z+3)} - 1 + \frac{1}{z}\right]$$

$$= \underset{z\to\infty}{Lt}\left[\frac{z^4+z^3}{(z-1)(z+3)} - z^2 + z\right]$$

$$= \underset{z\to\infty}{Lt}\left[\frac{z^4+z^3 - z^2(z^2+2z-3) + z(z^2+2z-3)}{z^2+2z-3}\right]$$

$$= \underset{z\to\infty}{Lt}\; \frac{5z^2-3z}{z^2+2z-3}$$

$$= \underset{z\to\infty}{Lt}\; \frac{z^2\left(5-3/z\right)}{z^2\left(1+2/z-3/z^2\right)}$$

$U_2 = 5$

$U_0 = 1,\; U_1 = -1,\; U_2 = 5$

(10) Solve the difference equation

$u_{n+1} - 2u_{n+1} + u_n = 2^n$, where $u_0 = 1, u_1 = 2$

Sol: Applying z-transform on both sides,

$$z(u_{n+1} - 2u_{n+1} + u_n) = z(2^n)$$

$$z(u_{n+1}) - 2z(u_n+1) + z(u_n) = \frac{z}{z-2}$$

$$z^2\left[u(z) - u_0 - \frac{u_1}{z}\right] - 2z[u(z) - u_0] + u(z) = \frac{z}{z-2}$$

$$u(z)\left[z^2 - 2z + 1\right] = \frac{z}{z-2} + z^2u_0 + zu_1 - 2zu_0$$

$$= \frac{z}{z-2} + z^2 + 2z - 2z$$

$$= \frac{z^3 - 2z^2 + z}{z-2}$$

$$u(z) = \frac{z^3 - 2z^2 + z}{(z-1)(z^2-2z+1)} = \frac{z}{z-2}$$

Res: $u(z) = \underset{z\to 2}{Lt}\ (z-2)\left[\frac{z}{(z-2)}\right]z^{n-1}$

$z = 2$

$$= \underset{z\to 2}{Lt}\ z^n \Rightarrow 2^n$$

$\therefore$ The solution of difference equation $u_{n+1} - 2u_{n+1} + u_n = 2^n$

where $u_0 = 1, u_1 = 2$, is 2^n.