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Foundation Master 2020-2021
Mathematics
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Second order linear ordinary differential equations

with all solutions

Exercise 1 (Reduction of order for an homogeneous ODE)
For the following second order linear homogeneous ODE, check that the given function y_1 is a solution and find a function u such that $y_2 = uy_1$ is another solution. Deduce the general solution $y(x)$ of the ODE.

1. $(x^2 - x)y'' - xy' + y = 0$ on $]0, +\infty[$, with $y_1(x) = x$
2. $xy'' + 2y' + xy = 0$ on $]0, +\infty[$, with $y_1(x) = \frac{\cos(x)}{x}$
3. $x^2y'' - 5xy' + 9y = 0$ on $]0, +\infty[$, with $y_1(x) = x^3$
4. $x^2y'' - xy' + y = 0$ on $]0, +\infty[$, with $y_1(x) = x$

Solution Exo. 1:

1. Let's denote (1) the equation $(x^2-x)y''-xy'+y = 0$. First, we check that $y_1(x) = x$ is solution of (1) on $]0, +\infty[$. Then, we look for a second linear independent solution $y_2(x) = u(x)y_1(x)$ where $u(x)$ has to be found.

$$\begin{aligned}
y_2 \text{ solution } &\Leftrightarrow (x^2 - x)(u''y_1 + 2u'y_1' + uy_1'') - x(u'y_1 + uy_1') + uy_1 = 0\\
&\Leftrightarrow (x^2 - x)y_1u'' + (2(x^2 - x)y_1' - xy_1)u' = 0\\
&\Leftrightarrow (x^2 - x)xu'' + (x^2 - 2x)u' = 0\\
&\Leftrightarrow u'' - \frac{2 - x}{x^2 - x}u' = 0 \text{ (first order linear ODE in } u')
\end{aligned}$$

at $x \notin \{0, 1\}$.

$$\begin{aligned}
y_2 \text{ solution } &\Leftrightarrow u' = ce^{A(x)};\ c \in \mathbb{R},\ A(x) \text{ primitive of } \frac{2 - x}{x^2 - x} = \frac{1}{1 - x} - \frac{2}{x},\\
&\qquad A(x) = \ln|x - 1| - 2\ln|x| = \ln(\frac{|x - 1|}{x^2})\\
&\Leftrightarrow u' = c\frac{x - 1}{x^2};\ c \in \mathbb{R}
\end{aligned}$$

Thus $u'(x) = \frac{x-1}{x^2} = \frac{1}{x} - \frac{1}{x^2}$ is suitable. We deduce $u(x) = \ln|x| + \frac{1}{x}$ is suitable and derive the solution $y_2(x) = x\ln|x| + 1$. Thus the general solution of (1) on $]0, +\infty[$ without 1 is

$$y(x) = c_1 x + c_2(x\ln|x| + 1) \; ; \; c_1, c_2 \in \mathbb{R}$$

Finally we check that such a function is also solution on an interval containing 1, since it satisfies $-y'(1) + y(1) = 0$.

2. Let's denote (2) the equation $xy'' + 2y' + xy = 0$. First, we check that $y_1(x) = \frac{\cos(x)}{x}$ is solution of (2) on $]0, +\infty[$. Then, we look for a second linear independent solution $y_2(x) = u(x)y_1(x)$ where $u(x)$ has to be found.

$$\begin{aligned}
y_2 \text{ solution } &\Leftrightarrow x(u''y_1 + 2u'y_1' + uy_1'') + 2(u'y_1 + uy_1') + xuy_1 = 0 \\
&\Leftrightarrow xy_1u'' + (2xy_1' + 2y_1)u' = 0 \\
&\Leftrightarrow \cos(x)u'' - 2\sin(x)u' = 0 \\
&\Leftrightarrow u'' - 2\frac{\sin(x)}{\cos(x)}u' = 0 \text{ (first order linear ODE in } u') \\
&\Leftrightarrow u' = ce^{A(x)};\; c \in \mathbb{R},\; A(x) \text{ primitive of } 2\frac{\sin(x)}{\cos(x)}, \\
&\qquad\qquad A(x) = 2\ln|\cos(x)| = \ln(\cos^2(x)) \\
&\Leftrightarrow u' = \frac{c}{\cos^2(x)};\; c \in \mathbb{R}
\end{aligned}$$

Thus $u'(x) = \frac{1}{\cos^2(x)}$ is suitable. We deduce $u(x) = \tan(x)$ is suitable and derive the solution $y_2(x) = \tan(x)\frac{\cos(x)}{x} = \frac{\sin(x)}{x}$. Finally, the general solution of (2) on $]0, +\infty[$ is

$$y(x) = c_1\frac{\cos(x)}{x} + c_2\frac{\sin(x)}{x} \; ; \; c_1, c_2 \in \mathbb{R}$$

3. Let's denote (3) the equation $x^2y'' - 5xy' + 9y = 0$. First, we check that $y_1(x) = x^3$ is solution of (3) on $]0, +\infty[$. Then, we look for a second linear independent solution $y_2(x) = u(x)y_1(x)$ where $u(x)$ has to be found.

$$\begin{aligned}
y_2 \text{ solution } &\Leftrightarrow x^2(u''y_1 + 2u'y_1' + uy_1'') - 5x(u'y_1 + uy_1') + 9uy_1 = 0 \\
&\Leftrightarrow x^2y_1u'' + (2x^2y_1' - 5xy_1)u' = 0 \\
&\Leftrightarrow x^5u'' + x^4u' = 0 \\
&\Leftrightarrow xu'' + u' = 0 \text{ (first order linear ODE in } u') \\
&\Leftrightarrow u' = ce^{A(x)};\; c \in \mathbb{R},\; A(x) \text{ primitive of } -\frac{1}{x}, \\
&\qquad\qquad A(x) = -\ln|x| = \ln(\frac{1}{|x|}) \\
&\Leftrightarrow u' = \frac{c}{|x|};\; c \in \mathbb{R}
\end{aligned}$$

Thus $u'(x) = \frac{1}{x}$ is suitable on $]0, +\infty[$. We deduce $u(x) = \ln(x)$ is suitable and derive the solution $y_2(x) = x^3 \ln(x)$. Finally, the general solution of (3) on $]0, +\infty[$ is

$$y(x) = (c_1 + c_2 \ln(x))x^3 \; ; \; c_1, c_2 \in \mathbb{R}$$

4. Let's denote (4) the equation $x^2y'' - xy' + y = 0$. First, we check that $y_1(x) = x$ is solution of (4) on $]0, +\infty[$. Then, we look for a second linear independent solution $y_2(x) = u(x)y_1(x)$ where $u(x)$ has to be found.

$$\begin{aligned}
y_2 \text{ solution } &\Leftrightarrow x^2(u''y_1 + 2u'y_1' + uy_1'') - x(u'y_1 + uy_1') + uy_1 = 0 \\
&\Leftrightarrow x^2y_1u'' + (2x^2y_1' - xy_1)u' = 0 \\
&\Leftrightarrow xu'' + u' = 0 \text{ (first order linear ODE in } u') \\
&\Leftrightarrow u' = ce^{A(x)}; \; c \in \mathbb{R}, \; A(x) \text{ primitive of } -\frac{1}{x}, \\
&\qquad\qquad A(x) = -\ln|x| = \ln(\frac{1}{|x|}) \\
&\Leftrightarrow u' = \frac{c}{|x|}; \; c \in \mathbb{R}
\end{aligned}$$

Thus $u'(x) = \frac{1}{x}$ is suitable on $]0, +\infty[$. We deduce $u(x) = \ln(x)$ is suitable and derive the solution $y_2(x) = x\ln(x)$. Finally, the general solution of (4) on $]0, +\infty[$ is

$$y(x) = (c_1 + c_2 \ln(x))x \; ; \; c_1, c_2 \in \mathbb{R}$$

Exercise 2 (Homogeneous linear ODEs with constant coefficients)
Determine the general solution $y(x)$ of the following equations and derive the particular solution $y(x)$ with initial conditions if they are given.

1. $y'' + y' - 2y = 0$ with $y(0) = 4$ and $y'(0) = -5$
2. $y'' + 6y' + 9y = 0$
3. $y'' + 0.4y' + 9.04y = 0$ with $y(0) = 0$ and $y'(0) = 3$
4. $y'' + 9y' + 20y = 0$ with $y(0) = 2$ and $y'(0) = -1$
5. $9y'' - 30y' + 25y = 0$ with $y(0) = 1$ and $y'(0) = 0$
6. $y'' + 2y' + 5y = 0$ with $y(0) = -1$ and $y'(0) = 1$

Solution Exo. 2:

1. Let's denote (1) the ODE $y'' + y' - 2y = 0$. The characteristic equation is

$$r^2 + r - 2 = 0 \ ; \ \text{roots } \{-2,\ 1\}$$

We deduce that $y_1(x) = e^{-2x}$ and $y_2(x) = e^{x}$ form a basis of solutions of (1), that is the general solution of (1) is

$$y(x) = c_1 e^{-2x} + c_2 e^{x} \ ; \ c_1, c_2 \in \mathbb{R}$$

Then we look for c_1 and c_2 defining the particular solution satisfying the initial conditions $y(0) = 4$ and $y'(0) = -5$.

$$\begin{cases} y(0) = 4 \\ y'(0) = -5 \end{cases} \Leftrightarrow \begin{cases} c_1 + c_2 = 4 \\ -2c_1 + c_2 = -5 \end{cases} \Leftrightarrow \begin{cases} c_1 = 3 \\ c_2 = 1 \end{cases}$$

We conclude that the solution of (1) with the given initial condition is

$$y(x) = 3e^{-2x} + e^{x}$$

2. Let's denote (2) the ODE $y'' + 6y' + 9y = 0$. The characteristic equation is

$$r^2 + 6r + 9 = 0 \ ; \ \text{double root } \{3\}$$

We deduce that $y_1(x) = e^{3x}$ and $y_2(x) = xe^{3x}$ form a basis of solutions of (2), that is the general solution of (2) is

$$y(x) = (c_1 + c_2 x)e^{3x} \ ; \ c_1, c_2 \in \mathbb{R}$$

3. Let's denote (3) the ODE $y'' + 0.4y' + 9.04y = 0$. The characteristic equation is

$$r^2 + 0.4r + 9.04 = 0 \ ; \ \text{complex roots } \{-0.2 + 3i, -0.2 - 3i\}$$

We deduce that $y_1(x) = e^{-0.2x}e^{3ix}$ and $y_2(x) = e^{-0.2x}e^{-3ix}$ form a basis of solutions of (3). Since $\cos(3x) = \frac{e^{3ix}+e^{-3ix}}{2}$ and $\sin(3x) = \frac{e^{3ix}-e^{-3ix}}{2i}$, we can consider the basis $z_1(x) = e^{-0.2x}\cos(3x)$ and $z_2(x) = e^{-0.2x}\sin(3x)$. Thus the general solution of (3) is

$$y(x) = e^{-0.2x}(c_1 \cos(3x) + c_2 \sin(3x)) \ ; \ c_1, c_2 \in \mathbb{R}$$

Then we look for c_1 and c_2 defining the particular solution satisfying the initial conditions $y(0) = 0$ and $y'(0) = 3$. We calculate $y'(x) = e^{-0.2x}(-0.2c_1 \cos(3x) - 0.2c_2 \sin(3x) - 3c_1 \sin(3x) + 3c_2 \cos(3x))$.

$$\begin{cases} y(0) = 0 \\ y'(0) = 3 \end{cases} \Leftrightarrow \begin{cases} c_1 = 0 \\ -0.2c_1 + 3c_2 = 3 \end{cases} \Leftrightarrow \begin{cases} c_1 = 0 \\ c_2 = 1 \end{cases}$$

We conclude that the solution of (3) with the given initial condition is

$$y(x) = e^{-0.2x}\sin(3x)$$

4. Let's denote (4) the ODE $y'' + 9y' + 20y = 0$. The characteristic equation is

$$r^2 + 9r + 20 = 0 \ ; \ \text{roots } \{-4, \ -5\}$$

We deduce that $y_1(x) = e^{-4x}$ and $y_2(x) = e^{-5x}$ form a basis of solutions of (4), that is the general solution of (4) is

$$y(x) = c_1 e^{-4x} + c_2 e^{-5x} \ ; \ c_1, c_2 \in \mathbb{R}$$

Then we look for c_1 and c_2 defining the particular solution satisfying the initial conditions $y(0) = 2$ and $y'(0) = -1$.

$$\begin{cases} y(0) = 2 \\ y'(0) = -1 \end{cases} \Leftrightarrow \begin{cases} c_1 + c_2 = 2 \\ -4c_1 - 5c_2 = -1 \end{cases} \Leftrightarrow \begin{cases} c_1 = 9 \\ c_2 = -7 \end{cases}$$

We conclude that the solution of (4) with the given initial condition is

$$y(x) = 9e^{-4x} - 7e^{-5x}$$

5. Let's denote (5) the ODE $9y'' - 30y' + 25y = 0$. The characteristic equation is

$$9r^2 - 30r + 25 = 0 \ ; \ \text{double root } \{\frac{5}{3}\}$$

We deduce that $y_1(x) = e^{\frac{5}{3}x}$ and $y_2(x) = xe^{\frac{5}{3}x}$ form a basis of solutions of (5), that is the general solution of (5) is

$$y(x) = (c_1 + c_2 x)e^{\frac{5}{3}x} \ ; \ c_1, c_2 \in \mathbb{R}$$

Then we look for c_1 and c_2 defining the particular solution satisfying the initial conditions $y(0) = 1$ and $y'(0) = 0$.

$$\begin{cases} y(0) = 1 \\ y'(0) = 0 \end{cases} \Leftrightarrow \begin{cases} c_1 = 1 \\ \frac{5}{3}c_1 + c_2 = 0 \end{cases} \Leftrightarrow \begin{cases} c_1 = 1 \\ c_2 = -\frac{5}{3} \end{cases}$$

We conclude that the solution of (5) with the given initial condition is

$$y(x) = (1 - \frac{5}{3}x)e^{\frac{5}{3}x}$$

6. Let's denote (6) the ODE $y'' + 2y' + 5y = 0$. The characteristic equation is

$$r^2 + 2r + 5 = 0 \ ; \ \text{complex roots } \{-1 + 2i, -1 - 2i\}$$

We deduce that $y_1(x) = e^{-x}e^{2ix}$ and $y_2(x) = e^{-x}e^{-2ix}$ form a basis of solutions of (6). Since $\cos(2x) = \frac{e^{2ix} + e^{-2ix}}{2}$ and $\sin(2x) = \frac{e^{2ix} - e^{-2ix}}{2i}$, we can consider the basis $z_1(x) = e^{-x}\cos(2x)$ and $z_2(x) = e^{-x}\sin(2x)$. Thus the general solution of (6) is

$$y(x) = e^{-x}(c_1 \cos(2x) + c_2 \sin(2x)) \ ; \ c_1, c_2 \in \mathbb{R}$$

Then we look for c_1 and c_2 defining the particular solution satisfying the initial conditions $y(0) = -1$ and $y'(0) = 1$. We calculate $y'(x) = e^{-x}(-c_1\cos(2x) - c_2\sin(2x) - 2c_1\sin(2x) + 2c_2\cos(2x))$.

$$\begin{cases} y(0) = -1 \\ y'(0) = 1 \end{cases} \Leftrightarrow \begin{cases} c_1 = -1 \\ -c_1 + 2c_2 = 1 \end{cases} \Leftrightarrow \begin{cases} c_1 = -1 \\ c_2 = 0 \end{cases}$$

We conclude that the solution of (6) with the given initial condition is

$$y(x) = -e^{-x}\cos(2x)$$

Exercise 3 (Euler-Cauchy equations)
We consider the Euler-Cauchy equation

$$x^2y'' + axy' + by = 0 \text{ for } x > 0 \tag{1}$$

where a and b are constant real numbers.

1. We set $y(x) = x^m$.
   Prove that $y(x)$ is solution of (1) if and only if $m^2 + (a-1)m + b = 0$. We denote m_1 and m_2 the two roots

$$m_1 = \frac{1-a-\sqrt{\Delta}}{2}, \ m_2 = \frac{1-a+\sqrt{\Delta}}{2} \text{ where } \Delta = (a-1)^2 - 4b$$

2. Case $\Delta > 0$. Give the general solution of (1) depending on m_1 and m_2. Apply to

$$x^2y'' + 1.5xy' - 0.5y = 0 \text{ for } x > 0$$

3. Case $\Delta = 0$. In this case, $m_1 = m_2$. Give a first solution $y_1(x)$ of (1) depending on m_1. Then look for a second solution $y_2(x)$ by the method of reduction of order, setting $y_2 = uy_1$ where $u(x)$ is a suitable function to determine. Conclude by giving the general solution of (1) depending on m_1. Apply to

$$x^2y'' - 5xy' + 9y = 0 \text{ for } x > 0$$

4. Case $\Delta < 0$. In this case, m_1 and m_2 are complex conjugate numbers, $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$. Give a basis of solutions depending on α and β, using $\cos(\beta\ln(x))$ and $\sin(\beta\ln(x))$. Hint: $x^{i\beta} = (e^{\ln(x)})^{i\beta} = e^{i\beta\ln(x)}$. Apply to

$$x^2y'' + 0.6xy' + 16.04y = 0 \text{ for } x > 0$$

Solution Exo. 3:

1. Putting $y(x) = x^m$ into (1), we get

$$m(m-1)x^m + amx^m + bx^m = 0, \ \forall x > 0$$

which is equivalent to $m^2 + (a-1)m + b = 0$.

2. Case $\Delta > 0$. In this case, $m_1 \neq m_2$ so that $y_1(x) = x^{m_1}$ and $y_2(x) = x^{m_2}$ are linearly independent and form a basis of solutions of (1). The general solution of (1) is

$$y(x) = c_1 x^{m_1} + c_2 x^{m_2}, \text{ for } x > 0 \ \ c_1, c_2 \in \mathbb{R}$$

For the example $x^2y'' + 1.5xy' - 0.5y = 0$, we calculate $\Delta = \frac{9}{4}$, $m_1 = -1$, $m_2 = \frac{1}{2}$, and deduce the general solution

$$y(x) = c_1 \frac{1}{x} + c_2\sqrt{x}, \text{ for } x > 0 \ \ c_1, c_2 \in \mathbb{R}$$

3. Case $\Delta = 0$. In this case, $m_1 = m_2 = \frac{1-a}{2}$. We deduce a first solution $y_1(x) = x^{m_1}$. Then setting $y_2 = uy_1$, we get

$$\begin{aligned}
y_2 \text{ solution } &\Leftrightarrow x^2(u''y_1 + 2u'y_1' + uy_1'') + ax(u'y_1 + uy_1') + buy_1 = 0 \\
&\Leftrightarrow x^2y_1u'' + (2x^2y_1' + axy_1)u' = 0 \\
&\Leftrightarrow x^{2+m_1}u'' + (2m_1x^{1+m_1}y_1' + ax^{1+m_1})u' = 0 \\
&\Leftrightarrow xu'' + (2m_1 + a)u' = 0 \\
&\Leftrightarrow xu'' + u' = 0 \text{ (first order linear ODE in } u') \\
&\Leftrightarrow u' = ce^{A(x)}; \ c \in \mathbb{R}, \ A(x) \text{ primitive of } -\frac{1}{x}, \\
&\qquad\qquad A(x) = -\ln|x| = \ln(\frac{1}{|x|}) \\
&\Leftrightarrow u' = \frac{c}{|x|}; \ c \in \mathbb{R}
\end{aligned}$$

Thus $u'(x) = \frac{1}{x}$ is suitable on $]0, +\infty[$. We deduce $u(x) = \ln(x)$ is suitable and derive the solution $y_2(x) = x\ln(x)$. Finally, the general solution of (1) is in this case

$$y(x) = (c_1 + c_2\ln(x))x^{m_1}, \text{ for } x > 0 \ ; \ c_1, c_2 \in \mathbb{R}$$

For the example $x^2y'' - 5xy' + 9y = 0$ for $x > 0$, we calculate $\Delta = (-6)^4 - 4*9 = 0$, and $m_1 = m_2 = 3$, and deduce the general solution

$$y(x) = (c_1 + c_2\ln(x))x^3, \text{ for } x > 0 \ ; \ c_1, c_2 \in \mathbb{R}$$

4. Case $\Delta < 0$. In this case, m_1 and m_2 are complex conjugate numbers, $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$. Since $m_1 \neq m_2$, we derive that $y_1(x) = x^{m_1}$ and $y_2(x) = x^{m_2}$ form a basis of the solutions of (1). As

$$x^{m_1} = x^{\alpha+i\beta} = x^\alpha x^{i\beta} = x^\alpha e^{i\beta \ln(x)} = x^\alpha(\cos(\beta \ln(x)) + i \sin(\beta \ln(x))))$$

and the same for x^{m_2} with $-i$ instead of $+i$, we deduce a new basis $z_1(x) = \frac{y_1(x)+y_2(x)}{2} = x^\alpha \cos(\beta \ln(x))$ and $z_2(x) = \frac{y_1(x)-y_2(x)}{2i} = x^\alpha \sin(\beta \ln(x))$. Thus the general solution of (1) is

$$y(x) = (c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))x^\alpha, \text{ for } x > 0 \; ; \; c_1, c_2 \in \mathbb{R}$$

For the example $x^2y'' + 0.6xy' + 16.04y = 0$, we calculate $\Delta = -64$, $m_1 = 0.2 + 4i$, $m_2 = 0.2 - 4i$, and we deduce the general solution

$$y(x) = (c_1 \cos(4 \ln(x)) + c_2 \sin(4 \ln(x)))x^{0.2}, \text{ for } x > 0 \; ; \; c_1, c_2 \in \mathbb{R}$$

Exercise 4 (Method of undetermined coefficients)
For the following ODEs, solve the associated homogeneous equation, find a solution $y_p(x)$ of the non-homogeneous equation by the method of the undetermined coefficients, and conclude by giving the general solution.

1. $y'' - y' - 2y = e^{-x}$
2. $y'' + 2y' + y = e^{-x}$
3. $y'' + 2y' + y = e^{-x} + x$
4. $y'' + 3y' + 2y = 12x^2$
5. $y'' + 5y' + 4y = 10e^{-3x}$
6. $y'' + 4y' + 4y = e^{-x} \cos(x)$
7. $y'' - 3y' + 2y = e^x + \cos(x)$
8. $x^4y'' + 2x^3y' - y = \cosh(1/x)$. First transform this equation by using the new variable $t = 1/x$ and the new unknown function $z(t) = y(x)$.

Solution Exo. 4:

1. The characteristic equation associated to (1) $y'' - y' - 2y = e^{-x}$, is $r^2 - r - 2 = 0$, with roots $r_1 = -1$ and $r_2 = 2$. We deduce the general solution of the homogeneous ODE associated to (1) on $\mathbb{R}$

$$y_h(x) = c_1 e^{-x} + c_2 e^{2x} \; ; \; c_1, c_2 \in \mathbb{R}$$

Considering the right side of (1), putting $y_p(x) = cxe^{-x}$ in (1) (we multiply e^{-x} by x because e^{-x} is solution of the homogeneous ODE), we get

$$(-2ce^{-x} + cxe^{-x}) - (ce^{-x} - cxe^{-x}) - 2cxe^{-x} = e^{-x}$$

and deduce $c = -\frac{1}{3}$, so that $y_p(x) = -\frac{1}{3}xe^{-x}$ is a solution of (1). In conclusion, the general solution of (1) is defined on $\mathbb{R}$ by

$$y(x) = y_h(x) + y_p(x) = c_1 e^{-x} + c_2 e^{2x} - \frac{1}{3}xe^{-x} \; ; \; c_1, c_2 \in \mathbb{R}$$

2. The characteristic equation associated to (2) $y'' + 2y' + y = e^{-x}$, is $r^2 + 2r + 1 = 0$, with the double root $r_1 = -1$. We deduce the general solution of the homogeneous ODE associated to (2) on $\mathbb{R}$

$$y_h(x) = c_1 e^{-x} + c_2 xe^{-x} \; ; \; c_1, c_2 \in \mathbb{R}$$

Considering the right side e^{-x}, as e^{-x} and xe^{-x} are solutions of the homeneous ODE, we put $y_p(x) = cx^2e^{-x}$ in (2). We get after simplification $c = \frac{1}{2}$, so that $y_p(x) = \frac{1}{2}x^2e^{-x}$ is a solution of (2). In conclusion, the general solution of (2) is defined on $\mathbb{R}$ by

$$y(x) = y_h(x) + y_p(x) = c_1 e^{-x} + c_2 xe^{-x} + \frac{1}{2}x^2e^{-x} \; ; \; c_1, c_2 \in \mathbb{R}$$

3. As in the previous ODE, the solution of the associated homogeneous ODE of (3) $y'' + 2y' + y = e^{-x} + x$ on $\mathbb{R}$ is

$$y_h(x) = c_1 e^{-x} + c_2 xe^{-x} \; ; \; c_1, c_2 \in \mathbb{R}$$

First, considering the terme e^{-x} of the right side, putting $y_p(x) = cx^2e^{-x}$ in $y'' + 2y' + y = e^{-x}$, we get $y_p(x) = \frac{1}{2}x^2e^{-x}$ (already done for (2)).
Second, considering the term x of the right side of (3), putting $z_p(x) = c_0 + c_1 x$ in $y'' + 2y' + y = x$, we get

$$2c_1 + c_0 + c_1 x = x$$

and deduce $c_1 = 1$ and $c_0 = -2$, so that $z_p(x) = -2 + x$.
In conclusion, the general solution of (3) is defined on $\mathbb{R}$ by

$$y(x) = y_h(x) + y_p(x) + z_p(x) = (c_1 + c_2 x + \frac{1}{2}x^2)e^{-x} + x - 2 \; ; \; c_1, c_2 \in \mathbb{R}$$

4. The characteristic equation associated to (4) $y'' + 3y' + 2y = 12x^2$, is $r^2 + 3r + 2 = 0$, with roots $r_1 = -1$ and $r_2 = -2$. We deduce the general solution of the homogeneous ODE associated to (4) on $\mathbb{R}$

$$y_h(x) = c_1 e^{-x} + c_2 e^{-2x} \ ; \ c_1, c_2 \in \mathbb{R}$$

Putting $y_p(x) = a_0 + a_1 x + a_2 x^2$ in (4) we get

$$(2a_2 + 3a_1 + 2a_0) + (6a_2 + 2a_1)x + 2c_2 x^2 = 12x^2$$

and deduce $a_2 = 6$, $a_1 = -18$, $a_0 = 21$, so that $y_p(x) = 21 - 18x + 6x^2$ is a solution of (4). In conclusion, the general solution of (4) is defined on $\mathbb{R}$ by

$$y(x) = y_h(x) + y_p(x) = c_1 e^{-x} + c_2 e^{-2x} + 21 - 18x + 6x^2 \ ; \ c_1, c_2 \in \mathbb{R}$$

5. The characteristic equation associated to (5) $y'' + 5y' + 4y = 10e^{-3x}$, is $r^2 + 5r + 4 = 0$, with roots $r_1 = -1$ and $r_2 = -4$. We deduce the general solution of the homogeneous ODE associated to (5) on $\mathbb{R}$

$$y_h(x) = c_1 e^{-x} + c_2 e^{-4x} \ ; \ c_1, c_2 \in \mathbb{R}$$

Putting $y_p(x) = ce^{-3x}$ in (5) we get

$$9e^{-3x} - 15e^{-3x} + 4e^{-3x} = 10e^{-3x}$$

and deduce $c = -5$, so that $y_p(x) = -5e^{-3x}$ is a solution of (5). In conclusion, the general solution of (5) is defined on $\mathbb{R}$ by

$$y(x) = y_h(x) + y_p(x) = c_1 e^{-x} + c_2 e^{-4x} - 5e^{-3x} \ ; \ c_1, c_2 \in \mathbb{R}$$

6. The characteristic equation associated to (6) $y'' + 4y' + 4y = e^{-x}\cos(x)$, is $r^2 + 4r + 4 = 0$, with the double root $r_1 = -2$. We deduce the general solution of the homogeneous ODE associated to (6) on $\mathbb{R}$

$$y_h(x) = c_1 e^{-2x} + c_2 x e^{-2x} \ ; \ c_1, c_2 \in \mathbb{R}$$

Putting $y_p(x) = e^{-x}(a_1 \cos(x) + a_2 \sin(x))$ in (6), we get after simplification

$$e^{-x}(-2a_1 \sin(x) + 2a_2 \cos(x)) = e^{-x}\cos(x)$$

and we deduce $a_2 = \frac{1}{2}$, $a_1 = 0$, so that $y_p(x) = \frac{1}{2}e^{-x}\sin(x)$ is a solution of (6). In conclusion, the general solution of (6) is defined on $\mathbb{R}$ by

$$y(x) = y_h(x) + y_p(x) = c_1 e^{-2x} + c_2 x e^{-2x} + \frac{1}{2}e^{-x}\sin(x) \ ; \ c_1, c_2 \in \mathbb{R}$$

7. The characteristic equation associated to (7) $y'' - 3y' + 2y = e^x + \cos(x)$, is $r^2 - 3r + 2 = 0$, with roots $r_1 = 1$ and $r_2 = 2$. We deduce the general solution of the homogeneous ODE associated to (7) on $\mathbb{R}$

$$y_h(x) = c_1 e^x + c_2 e^{2x} \; ; \; c_1, c_2 \in \mathbb{R}$$

First, considering the term e^x of the right side of (7), putting $y_p(x) = cxe^x$ in $y'' - 3y' + 2y = e^x$ (in the expression of $y_p(x)$ we multiply e^x by x since e^x is solution of the homogeneous ODE), we get

$$ce^x(x + 2 - 3(x + 1) + 2x) = e^x$$

and deduce $c = -1$, so that $y_p(x) = -xe^x$.
Second, considering the term $\cos(x)$ of the right side of (7), putting $z_p(x) = a_1\cos(x) + a_2\sin(x)$ in $y'' - 3y' + 2y = \cos(x)$, we get

$$(-a_1 - 3a_2 + 2a_1)\cos(x) + (-a_2 + 3a_1 + 2a_2)\sin(x) = \cos(x)$$

and deduce $a_1 = 0.1$ and $a_2 = -0.3$, so that $z_p(x) = 0.1\cos(x) - 0.3\sin(x)$.
In conclusion, the general solution of (7) is defined on $\mathbb{R}$ by

$$y(x) = y_h(x) + y_p(x) + z_p(x) = c_1e^x + c_2e^{2x} - xe^x + 0.1\cos(x) - 0.3\sin(x) \; ; \; c_1, c_2 \in \mathbb{R}$$

8. Let's denote (8) $x^4y'' + 2x^3y' - y = \cosh(1/x)$. We put $t = 1/x$ and $z(t) = y(x)$. Then, $y(x) = z(t) = z(\frac{1}{x})$ so that

$$y'(x) = -\frac{1}{x^2}z'(\frac{1}{x}) = -t^2z'(t)$$

and

$$y''(x) = (-\frac{1}{x^2})^2 z''(\frac{1}{x}) + \frac{2}{x^3}z'(\frac{1}{x}) = t^4z''(t) + 2t^3z'(t)$$

Rewriting (8) with t and $z(t)$ we get

$$\frac{1}{t^4}(t^4z'' + 2t^3z') + \frac{2}{t^3}(-t^2z') - z = \cosh(t)$$

which can be reduced in

$$z'' - z = \cosh(t)$$

The general solution of $z'' - z = 0$ is well known as

$$z_h(t) = c_1\cosh(t) + c_2\sinh(t) \; ; \; c_1, c_2 \in \mathbb{R}$$

Then putting $z_p(t) = t(a_1\cosh(t) + a_2\sinh(t))$ in (8) (in the expression of $z_p(t)$ we multiply by t the linear combination of $\cosh(t)$ and $\sinh(t)$ since $\cosh(t)$ is solution of the homogeneous ODE), we get, after simplification

$$2a_1\cosh(t) + 2a_2\sinh(t) = \cosh(t)$$

and deduce $a_1 = 0$ and $a_2 = 0.5$, so that $z_p(t) = \frac{t}{2}\sinh(t)$.
In conclusion, the general solution of (8) is defined on $\mathbb{R}^*$ by

$$z(t) = z_h(t) + z_p(t) = c_1\cosh(t) + c_2\sinh(t) + \frac{t}{2}\sinh(t)\ ;\ c_1, c_2 \in \mathbb{R}$$

and the general solution of (8) is defined on $\mathbb{R}^*$ by

$$y(x) = z(\frac{1}{x}) = c_1\cosh(\frac{1}{x}) + c_2\sinh(\frac{1}{x}) + \frac{1}{2x}\sinh(\frac{1}{x})\ ;\ c_1, c_2 \in \mathbb{R}$$
