RESUMEN DE SERIES DE FOURIER

Usaremos adelante las siguientes dos propiedades de las integrales

a)

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

b)

$$\int_a^b f(x)\,dx = \int_{-a}^{-b} f(-x)\,d(-x)$$

Definición de función par: $f(x)\ es\ par\ si\ f(x) = f(-x)$

Ejemplos de funciones pares

$$f(x) = x^2, \qquad f(x) = \cos(x), \qquad f(x) = 5$$

La integral de una función par en un intervalo simétrico es:

$$\int_{-L}^{L} f(x)\,dx = 2\int_0^L f(x)\,dx$$

Esto porque

$$\int_{-L}^{L} f(x)\,dx = \int_{-L}^{0} f(x)\,dx + \int_0^L f(x)\,dx = -\int_0^{-L} f(x)\,dx + \int_0^L f(x)\,dx$$

$$= -\int_0^L f(-x)\,d(-x) + \int_0^L f(x)\,dx = \int_0^L f(x)\,dx + \int_0^L f(x)\,dx = 2\int_0^L f(x)\,dx$$

Definición de función impar: $f(x)\ es\ impar\ si\ f(x) = -f(-x)$

Ejemplos de funciones impares:

$$f(x) = x, \qquad \mathrm{f(x)} = \sin(x), \qquad f(x) = x^3, \qquad f(x) = \tan(x) = \frac{\sin(x)}{\cos(x)}$$

La integral de una función impar en un intervalo simétrico es:

$$\int_{-L}^{L} f(x)\,dx = 0$$

Esto porque

$$\int_{-L}^{L} f(x)\,dx = \int_{-L}^{0} f(x)\,dx + \int_0^L f(x)\,dx = -\int_0^{-L} f(x)\,dx + \int_0^L f(x)\,dx$$

$$= -\int_0^L f(-x)\,d(-x) + \int_0^L f(x)\,dx = -\int_0^L f(x)\,dx + \int_0^L f(x)\,dx = 0$$

El producto de dos funciones pares es una función par

Esto porque si f y g son funciones pares, entonces

$$h(x) = f(x)g(x) = f(-x)g(-x) = h(-x)$$

El producto de dos funciones impares es una función par

Esto porque si f y g son funciones impares, entonces

$$h(x) = f(x)g(x) = \big(-f(-x)\big)\big(-g(-x)\big) = f(-x)g(-x) = h(-x)$$

El producto de una función par por una función impar es una función impar

Esto porque si f es función par y g es función impar, entonces

$$h(x) = f(x)g(x) = f(-x)\left(-g(-x)\right) = -f(-x)g(-x) = -h(-x)$$

Definición de función periódica

$$f(x)\ es\ periodica\ de\ periodo\ T\ si\ f(x+T) = f(x)$$

El producto de dos funciones (supongamos f y g) de periodo T es una función de periodo T

Esto porque, si

$h(x) = f(x)g(x)$ entonces $h(x+T) = f(x+T)g(x+T) = f(x)g(x) = h(x)$

La integral de una función periódica en un periodo no depende de donde inicie el periodo de integración

$$\int_0^T f(x)dx = \int_{0+C}^{T+C} f(x)\, dx$$

puede integrarse a partir del límite inferior que represente menor dificultad, por ejmplo, si

$$C = -\frac{T}{2} = -L$$

$$\int_o^T f(x)dx = \int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\, dx = \int_{-L}^{L} f(x)\, dx$$

Es útil integrar en un intervalo simétrico cuando se integran funciones pares o impares.

Una función f(t) de periodo T puede expresarse como combinacion lineal de la base de Fourier, la cual es una base en un espacio de Hilbert (Espacio vectorial de funciones)

La base de Fourier es el conjunto:

$$\{1, \cos(\omega_0 x), \sin(\omega_0 x), \cos(2\omega_0 x), \sin(2\omega_0 x), \dots, \cos(n\omega_0 x), \sin(n\omega_0 x), \dots\}$$

Donde la frecuencia angular es

$$\omega_0 = \frac{2\pi}{T} = 2\pi\nu \quad \left[\frac{rad}{s}\right]$$

$$\nu = \frac{1}{T}\left[\frac{1}{s}\right] Hz, ciclos\ por\ segundo$$

El producto interno entre dos funciones de esta base se define como:

$$\langle \phi_n(x), \phi_m(x) \rangle = \int_0^T \phi_n(x)\, \phi_m(x)\, dx$$

Ejercicio: verificar que las funciones de la base de Fourier son ortogonales:

$$\langle \phi_n(x), \phi_m(x) \rangle = \delta_{nm} L$$

$$\delta_{nm} = \begin{cases} 1\ si\ n = m \\ 0\ si\ n \neq m \end{cases}$$

Donde δ_{nm} es la delta de Kronecker y $L = \frac{T}{2}$

Solución:

a)

$$\langle 1,1 \rangle = \int_0^T 1 dx = T$$

b)

$$\langle 1, \cos(n\omega_0 x)\rangle = \int_0^T 1 \cos(n\omega_0 x)\, dx = \frac{sen\left(\frac{n2\pi}{T}T\right) - 0}{n\omega_0} = 0$$

c)

$$\langle 1, \text{sen}(n\omega_0 x)\rangle = \int_0^T 1 \,\text{sen}(n\omega_0 x)\, dx = \frac{-cos\left(\frac{n2\pi}{T}T\right) + 1}{n\omega_0} = 0$$

d)

$$\langle \cos(n\omega_0 x), \text{sen}(m\omega_0 x)\rangle = \int_0^T \cos(n\omega_0 x)\,\text{sen}(m\omega_0 x)\, dx$$

$$= \frac{1}{2}\left(\int_0^T \text{sen}\big((n+m)\omega_0 x\big)\, dx + \int_0^T \text{sen}\big((n-m)\omega_0 x\big)\, dx\right)$$

Si $n \neq m$

$$= \frac{1}{2}\left[\frac{-\cos\big((n+m)\omega_0 x\big)}{\omega_0(n+m)} + \frac{-\cos\big((n-m)\omega_0 x\big)}{\omega_0(n-m)}\right]_0^T = 0$$

Si $n = m$

$$\langle \cos(n\omega_0 x), \text{sen}(n\omega_0 x)\rangle = \int_0^T \cos(n\omega_0 x)\,\text{sen}(n\omega_0 x)\, dx$$

$$= \frac{1}{2}\left[\frac{\text{sen}^2(n\omega_0 x)}{n\omega_0}\right]_0^T = 0$$

e)

$$\langle \cos(n\omega_0 x), \cos(m\omega_0 x)\rangle = \int_0^T \cos(n\omega_0 x)\cos(m\omega_0 x)\, dx = \delta_{nm} L$$

$$= \frac{1}{2}\left(\int_0^T \cos(\omega_0(n+m)x)\, dx + \int_0^T \cos(\omega_0(n-m)x)\, dx\right)$$

Si $n \neq m$

$$= \frac{1}{2}\left[\frac{\text{sen}(\omega_0(n+m)x)}{\omega_0(n+m)} + \frac{\text{sen}(\omega_0(n-m)x)}{\omega_0(n-m)}\right]_0^T = 0$$

Si $n = m$

$$\int_0^T \cos^2(n\omega_0 x)\, dx = \frac{1}{2}\left[\frac{\text{sen}\big(\omega_0 x(2n)\big)}{n\omega_0 2} + x\right]_0^T = \frac{T}{2} = L$$

f)

$$\langle \text{sen}(n\omega_0 x), \text{sen}(m\omega_0 x)\rangle = \int_0^T \text{sen}(n\omega_0 x)\,\text{sen}(m\omega_0 x)\, dx = \delta_{nm} L$$

$$= \frac{1}{2}\left(\int_0^T -\cos\big(\omega_0 x(n+m)\big)\, dx + \int_0^T \cos\big(\omega_0 x(n-m)\big)\, dx\right)$$

Si $n \neq m$

$$= \frac{1}{2}\left[\frac{-\operatorname{sen}\big(\omega_0 x(n+m)\big)}{\omega_0(n+m)} + \frac{\operatorname{sen}\big(\omega_0 x(n-m)\big)}{\omega_0(n-m)}\right]_0^T = 0$$

Si $n = m$

$$= \frac{1}{2}\left[\frac{-\operatorname{sen}\big(\omega_0 x(2n)\big)}{\omega_0(2n)} + T\right]_0^T = \frac{T}{2} = L$$

Una función periódica tiene expansión en serie de Fourier bajo las condiciones de Dirichlet (investigarlas).

SERIE DE FOURIER TRIGONOMÉTRICA

La serie de Fourier trigonométrica para una función periódica $f(x+T) = f(x)$

$$con\, T = 2L, \qquad frecuencia\ angular\colon \omega_0 = \frac{2\pi}{T} = 2\pi\nu$$

es:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos(n\omega_0 x) + b_n \operatorname{sen}(n\omega_0 x)\right)$$

Los coeficientes se calculan proyectando la función en cada elemento de la base de Fourier mediante el producto interno

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$$\langle f(x),1\rangle = \frac{a_0}{2}\langle 1,1\rangle + \sum_{n=1}^{\infty}\left(a_n\langle\cos(n\omega_0 x),1\rangle + b_n\langle\operatorname{sen}(n\omega_0 x)\,,1\rangle\right)$$

$$\langle f(x),1\rangle = \frac{a_0}{2}T$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$$

$$\langle f(x),\cos(k\omega_0 x)\rangle = \frac{a_0}{2}\langle 1,\cos\,(k\omega_0 x)\rangle + \sum_{n=1}^{\infty}\left(a_n\langle\cos(n\omega_0 x),\cos(k\omega_0 x)\rangle + b_n\langle\operatorname{sen}(n\omega_0 x),\cos(k\omega_0 x)\rangle\right)$$

$$\langle f(x),\cos(n\omega_0 x)\rangle = \sum a_n\delta_{nk}L = a_k L$$

$$a_k = \frac{1}{L}\int_{-L}^{L} f(x)\cos(k\omega_0 x)\,dx\ \ para\ k = 1,2,3,\ldots$$

$$\langle f(x),\operatorname{sen}(k\omega_0 x)\rangle = \frac{a_0}{2}\langle 1,\operatorname{sen}(k\omega_0 x)\rangle + \sum_{n=1}^{\infty}\left(a_n\langle\cos(n\omega_0 x),\operatorname{sen}(k\omega_0 x)\rangle + b_n\langle\operatorname{sen}(n\omega_0 x),\operatorname{sen}(k\omega_0 x)\rangle\right)$$

$$\langle f(x),\operatorname{sen}(k\omega_0 x)\rangle = \sum b_n\delta_{nk}L = b_k L$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\operatorname{sen}(n\omega_0 x)\,dx \qquad para\ n = 1,2,3,\ldots$$

Si la función es par $f(x) = f(-x)$ la expansión en serie de Fourier solo contiene el termino constante y combinación de funciones pares coseno.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\big(a_n\cos(n\omega_0 x)\big)$$

Los coeficientes se pueden calcular así.

$$a_0 = \frac{2}{L}\int_0^L f(x)\,dx\,; \quad a_n = \frac{2}{L}\int_0^L f(x)\cos(n\omega_0 x)\,dx\,; \quad b_n = 0$$

Si la función es impar $f(x) = -f(-x)$ la expansión en serie de Fourier solo contiene funciones seno que son impares.

$$f(x) = \sum_{n=1}^{\infty} \left(b_n \text{sen}(n\omega_0 x)\right)$$

Los coeficientes se pueden calcular así.

$$a_n = 0; \quad b_n = \frac{2}{L}\int_0^L f(x)\text{sen}(n\omega_0 x)\, dx$$

También se puede llevar la serie de Fouerier a la expresión

$$f(x) \cong \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(n\omega_0 x) + b_n \text{sen}(n\omega_0 x)\right)$$

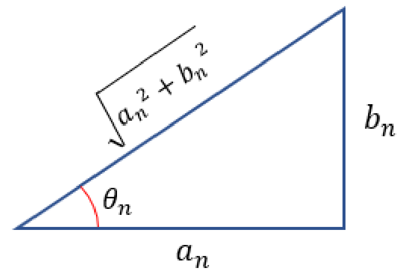
$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2}\left(\frac{a_n}{\sqrt{a_n^2 + b_n^2}} \cos(n\omega_0 x) + \frac{b_n}{\sqrt{a_n^2 + b_n^2}} \text{sen}(n\omega_0 x)\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2}\left(\cos(\phi_n) \cos(n\omega_0 x) + sen(\phi_n)\text{sen}(n\omega_0 x)\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} K_n(\cos(n\omega_0 x - \phi_n))$$

Donde $K_n = \sqrt{a_n^2 + b_n^2}\ y\ \phi_n = \tan^{-1}\left(\frac{b_n}{a_n}\right)$

$Sen\ \phi_n = \frac{b_n}{\sqrt{{a_n}^2 + {b_n}^2}} \quad Cos\ \phi_n = \frac{a_n}{\sqrt{{a_n}^2 + {b_n}^2}}$



Ejemplos de serie de Fourier:

Ejemplo 1: Cálculo de la serie de Fourier de una función periódica par de periodo $T = 2\pi,\ L = \pi$

$$f(x) = |x| \quad si -\pi \leq x \leq \pi$$

Los coeficientes:

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\, dx = \pi$$

$$a_n = \frac{2}{\pi n^2}\int_0^{\pi} nx\ \cos(nx)\ ndx = \frac{2}{\pi n^2}(nx\ sen(nx) + \cos(nx))_0^{\pi} = \frac{2}{n^2\,\pi}((-1)^n - 1)$$

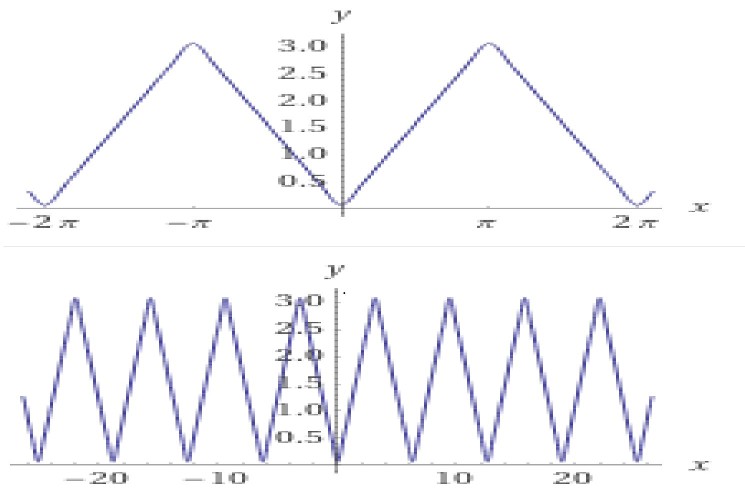
$$b_n = 0$$

La serie queda

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{10} \frac{2}{n^2\,\pi}\left((-1)^n - 1\right)\cos\,(n\,x)$$

$$= \frac{\pi}{2} - 4\sum_{k=0}^{10} \frac{1}{(2k+1)^2\,\pi}\cos\big((2k+1)x\big)$$

Al graficar usando solo 10 términos de la suma tenemos:



Ejemplo 2: Cálculo de la serie de Fourier de una función periódica impar de periodo $T = 2\pi,\ L = \pi$

$$f(x) = x \; si - \pi \le x \le \pi$$

Los coeficientes:

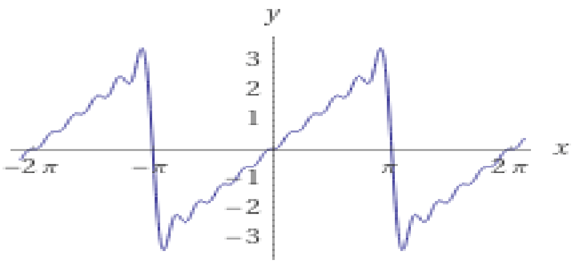
$$a_n = 0 \; para \; n = 0,1,2,3, \ldots$$

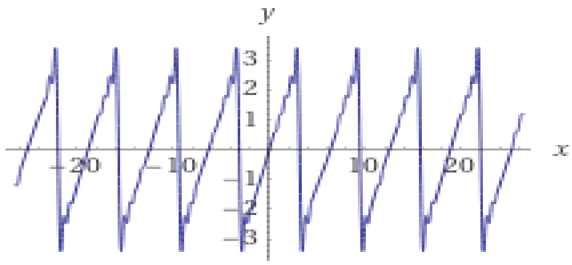
$$b_n = \frac{2}{\pi}\int_0^{\pi} x\ \operatorname{sen}(nx)\ dx = \frac{2(-1)^{n+1}}{n}$$

La serie queda

$$f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \operatorname{sen}(n\,x)$$

Al graficar usando solo 10 términos de la suma tenemos:





Ejemplo 3 de cálculo de la serie de Fourier de una función periódica que no es par ni impar, periodo $T = 2\pi,\ L = \pi$

Definida en un periodo como $f(x) = \begin{cases} 0 & si\ -\pi < x \le 0 \\ x & si\ 0 < x \le \pi \end{cases}$

Los coeficientes se calculan con las fórmulas:

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} 0\, dx + \int_{0}^{\pi} x\, dx \right) = \frac{\pi}{2}$$

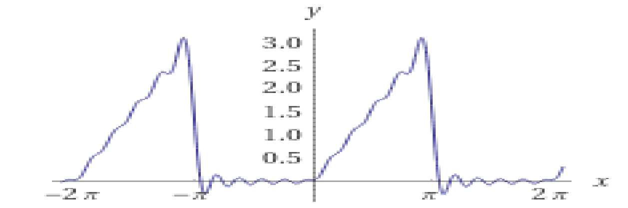
$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos\frac{n\pi x}{\pi}\, dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} 0\ \cos\frac{n\pi x}{\pi} dx + \int_{0}^{\pi} x\ \cos\frac{n\pi x}{\pi}\, dx \right) = \frac{((-1)^n - 1)}{n^2\,\pi}$$

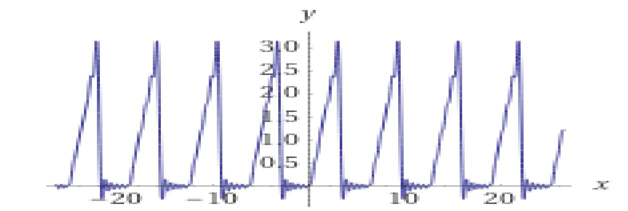
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{sen}\frac{n\pi x}{\pi}\,dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} 0\ \mathrm{sen}\frac{n\pi x}{\pi}\,dx + \int_{0}^{\pi} x\ \mathrm{sen}\frac{n\pi x}{\pi}\,dx\right) = \frac{((-1)^{n+1})}{n}$$

La serie queda

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right) = \frac{\pi}{4} + \sum_{n=1}^{10}\left(\frac{(-1)^n - 1}{n^2\,\pi}\cos(n\,x) + \frac{(-1)^{n+1}}{n}\mathrm{sen}(n\,x)\right)$$

Al graficar usando solo 10 términos de la suma tenemos:





Ejercicio, obtener la serie de Fourier de la función definida por

$$f(t) = \begin{cases} -1 & si\ -\pi < t < 0 \\ 1 & si\ 0 \le t < \pi \end{cases}$$

$$f(t) = f(t + 2\pi)$$

Solución:

$$a_0 = -\frac{1}{\pi}\int_{-\pi}^{0} dx + \frac{1}{\pi}\int_{0}^{\pi} dx = -\frac{1}{\pi}\big(0 - (-\pi)\big) + \frac{1}{\pi}(\pi - 0) = 0$$

$$a_n = -\frac{1}{\pi}\int_{-\pi}^{0}\cos(nx)\,dx + \frac{1}{\pi}\int_{0}^{\pi}\cos(nx)\,dx = 0$$

$$b_n = -\frac{1}{\pi}\int_{-\pi}^{0}\mathrm{sen}(nx)\,dx + \frac{1}{\pi}\int_{0}^{\pi}\mathrm{sen}(nx)\,dx$$

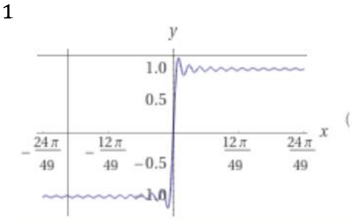
$$= \frac{1}{\pi}\left(\frac{1-\cos(-n\pi)}{n}\right) - \frac{1}{\pi}\left(\frac{\cos(n\pi)-1}{n}\right)$$

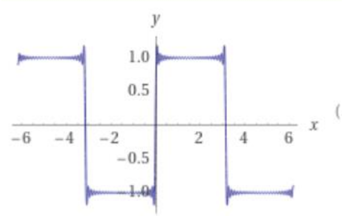
$$= \frac{2}{\pi}\left(\frac{1-\cos(n\pi)}{n}\right)$$

$$= \frac{2}{\pi}\left(\frac{1-(-1)^{\mathrm{n}}}{n}\right)$$

$$f(x) \cong \frac{2}{\pi}\sum_{n=1}^{10}\left(\left(\frac{1-(-1)^{\mathrm{n}}}{n}\right)\mathrm{sen}(\mathrm{nx})\right)$$

$$\frac{4}{\pi}\sum_{k=1}^{10}\left(\left(\frac{1}{(2\mathrm{k}-1)}\right)\mathrm{sen}((2\mathrm{k}-1)\mathrm{x})\right)$$





Ejercicio: Aplicar la identidad de Parseval a la serie anterior

$$f(x) = \frac{4}{\pi}\sum_{k=1}^{10}\left(\left(\frac{1}{(2\mathrm{k}-1)}\right)\mathrm{sen}((2\mathrm{k}-1)\mathrm{x})\right)$$

$$\frac{4}{\pi}\sum_{k=1}^{10}\left(\left(1-\frac{2\mathrm{k}-1}{10}\right)\left(\frac{1}{(2\mathrm{k}-1)}\right)\mathrm{sen}((2\mathrm{k}-1)\mathrm{x})\right)$$

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$$f(x) = \cos(x)\, si -3 \leq x \leq 3$$

$$a_0 = \frac{2}{3}\int_0^3 \cos(x)\,dx = \frac{2}{3}\big(sen(3) - sen(0)\big) = \frac{2}{3}sen(3)$$

$$a_n = \frac{2}{3}\int_0^3 \cos(x)\cos\left(\frac{\mathrm{n\pi x}}{3}\right)\,dx = \frac{1}{3}\int_0^3 \cos\left(x\left(1+\frac{n\pi}{3}\right)\right) + \cos\left(x\left(1-\frac{n\pi}{3}\right)\right)\,dx = 0$$

$$= \frac{1}{3}\frac{\left(\operatorname{sen}\left(3\left(1+\frac{n\pi}{3}\right)\right)\right)}{1+\frac{n\pi}{3}} + \frac{1}{3}\frac{\left(\operatorname{sen}\left(3\left(1-\frac{n\pi}{3}\right)\right)\right)}{1-\frac{n\pi}{3}}$$

$$= \frac{\operatorname{sen}(3+n\pi)}{3+n\pi} + \frac{\operatorname{sen}(3-n\pi)}{3-n\pi}$$

$$= (-1)^n \operatorname{sen}(3)\left(\frac{1}{3+n\pi} + \frac{1}{3-n\pi}\right)$$

$$= \left(\frac{6\,(-1)^n \operatorname{sen}(3)}{9-n^2\pi^2}\right)$$

$$b_n = \frac{1}{3}\int_{-3}^{3} \cos(x)\operatorname{sen}\left(\frac{\mathrm{n\pi x}}{3}\right) dx = 0$$

$$f(x) \cong \frac{1}{3} sen(3) + 6sen(3)\sum_{n=1}^{10}\left(\frac{(-1)^n}{9-n^2\pi^2}\right)\cos\left(\frac{n\pi x}{3}\right)$$

Otro ejemplo (1.29):

$$f(t) = \begin{cases} 1 \; si -\pi < t < 0 \\ 0 \quad si \; 0 < t < \pi \end{cases}$$

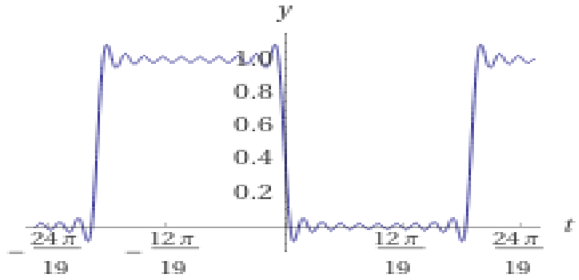
$$a_0 = \frac{1}{\pi}\int_{-\pi}^{0} 1\, dx = 1$$

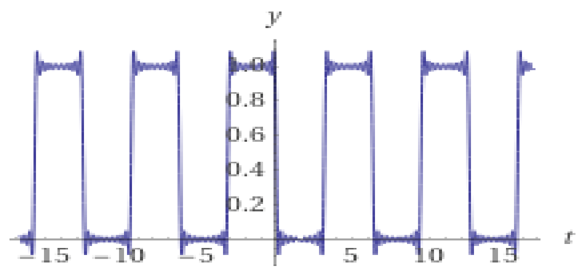
$$a_n = \frac{1}{\pi}\int_{-\pi}^{0} 1 \cos \mathrm{nx}\; dx = \frac{1}{\pi}\left[\frac{sen(nx)}{n}\right]_{-\pi}^{0} = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{0} 1 \operatorname{sen} \mathrm{nx}\, dx = \frac{-1}{\pi}\left[\frac{cos(nx)}{n}\right]_{-\pi}^{0} = \frac{-1}{n\pi}\left(1-(-1)^n\right)$$

$$f(t) \cong \frac{1}{2} + \sum_{n=1}^{\infty}\frac{1}{n\pi}\left((-1)^n - 1\right)\; \operatorname{sen}(\mathrm{nt})$$

$$= \frac{1}{2} + \sum_{k=1}^{\infty}\frac{1}{(2k-1)\pi}(-2)\operatorname{sen}\big((2\mathrm{k}-1)\mathrm{t}\big) = \frac{1}{2} - \frac{2}{\pi}\sum_{k=1}^{10}\frac{\operatorname{sen}(2\mathrm{k}-1)t}{2k-1}$$





Otro ejemplo

$$T = 2\pi, \qquad L = \pi$$

Solo es necesario definir la función en un intervalo correspondiente a un periodo

$$f(x) = \begin{cases} 0 & si \ -\pi < x \leq 0 \\ 1 & si \ 0 < x \leq \pi \end{cases}$$

Los coeficientes se calculan con las fórmulas:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi} 1\,dx\right) = 1$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos\frac{n\pi x}{\pi}\,dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} 0\ \cos\frac{n\pi x}{\pi}dx + \int_{0}^{\pi} 1\ \cos\frac{n\pi x}{\pi}\,dx\right) = \frac{sen(n\pi)}{n\,\pi} = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{sen}\frac{n\pi x}{\pi}\,dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} 0\ \mathrm{sen}\frac{n\pi x}{\pi}dx + \int_{0}^{\pi} 1\ \mathrm{sen}\frac{n\pi x}{\pi}\,dx\right)$$

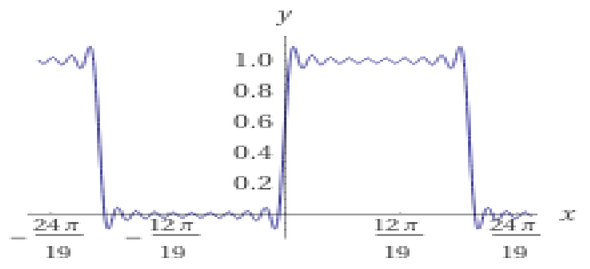
$$= \frac{(-\cos(n\pi) + 1)}{n\,\pi} = \frac{(1 - (-1)^{\mathrm{n}})}{n\,\pi}$$

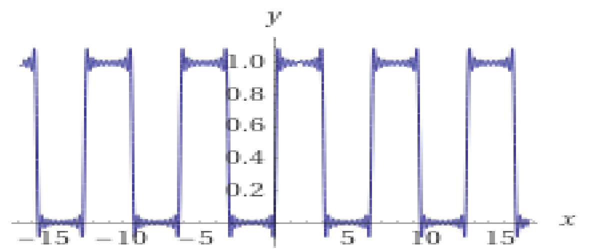
La serie queda como

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right) = \frac{1}{2} + \sum_{n=1}^{\infty}\left(0\cos(n\,x) + \frac{(1-(-1)^{\mathrm{n}})}{n\,\pi}\mathrm{sen}(n\,x)\right)$$

$$= \frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{(1-(-1)^{\mathrm{n}})}{n\,\pi}\mathrm{sen}(n\,x)\right) = \frac{1}{2} + \frac{2}{\pi}\sum_{k=1}^{10}\left(\frac{\mathrm{sen}\big((2k-1)x\big)}{2k-1}\right)$$

Al graficar usando solo 10 términos de la suma tenemos:





Otro ejemplo (1.31)

$$f(t) = t^2 \; si -\pi < t < \pi$$
$$f(t) = f(t + 2\pi)$$
$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} t^2 dt = \frac{1}{\pi}\left[\frac{t^3}{3}\right]_{-\pi}^{\pi} = \frac{2}{3}\pi^2$$
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t^2 \cos(\mathrm{nt})\ dt = \frac{2}{\pi}\int_{0}^{\pi} t^2 \cos(\mathrm{nt})\ dt = \frac{2\left(2\,n\,\pi\,cos(n\,\pi) + \ (-2\ +\ n^2\pi^2) sin(n\,\pi)\right)}{n^3\pi}$$
$$= \frac{2\ (2\,n\,\pi\,(-1)^n)}{n^3\pi} = \frac{4(-1)^n}{n^2}$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t^2 \,\mathrm{sen}(\mathrm{nt})\, dt = 0$$
$$f(t) \cong \frac{1}{3}\pi^2 + \sum_{n=1}^{10} \frac{4(-1)^n}{n^2}\cos(nt)$$
$$f'(t) \cong 4\sum_{n=1}^{10} \frac{(-1)^{n+1}}{n}\mathrm{sen}(nt)$$

(1.33)

$$a_0 = \frac{2A}{\frac{\pi}{2w_0}}\int_{0}^{\frac{\pi}{2w_0}} sen(w_0 t) dt = -\frac{2A}{\pi/2w_0}\left[\frac{\cos(w_0 t)}{w_0}\right]_0^{\frac{\pi}{2w_0}} = \frac{4A}{\pi}$$
$$a_n = \frac{2A}{\pi/2w_0}\int_{0}^{\pi/2w_0} sen(w_0 t)\cos(2w_0 nt)\ dt$$

$$= \frac{A}{\pi/2w_0} \int_0^{\pi/2w_0} [sen(w_0t + 2w_0nt) + sen(w_0t - 2w_0nt)]dt$$

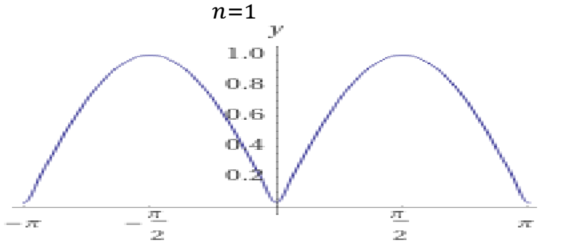
$$= -\frac{A}{\frac{\pi}{2w_0}} \left[\frac{cos(w_0t + 2w_0nt)}{(w_0 + 2w_0n)} + \frac{cos(w_0t - 2w_0nt)}{(w_0 - 2w_0n)}\right]_0^{\frac{\pi}{2w_0}}$$

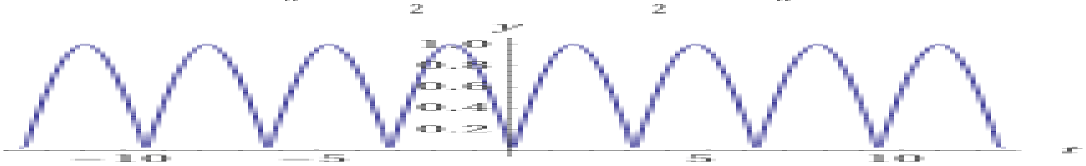
$$= -\frac{A}{\frac{\pi}{2w_0}} \left[\frac{cos\left(\frac{\pi}{2} + n\pi\right)}{(w_0 + 2w_0n)} + \frac{cos\left(\frac{\pi}{2} - n\pi\right)}{(w_0 - 2w_0n)} - \frac{1}{(w_0 + 2w_0n)} - \frac{1}{(w_0 - 2w_0n)}\right]$$

$$= \frac{4A}{\pi}\left[\frac{1}{1 - 4n^2}\right]$$

$$b_n = 0$$

$$f(t) \cong \frac{2A}{\pi} + \frac{4A}{\pi} \sum_{n=1}^{10} \left(\frac{1}{1 - 4n^2}\right) \cos(2w_0\, nt)$$





(1.38)

Haciendo t=π

$$\pi^2 \cong \frac{1}{3}\pi^2 + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos(n\pi)$$

$$\pi^2 \cong \frac{1}{3}\pi^2 + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} (-1)^n$$

$$\frac{3}{3}\pi^2 - \frac{1}{3}\pi^2 \cong +4 \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\frac{2}{3 \times 4}\pi^2 \cong \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\frac{\pi^2}{6} \cong \sum_{n=1}^{\infty} \frac{1}{n^2}$$

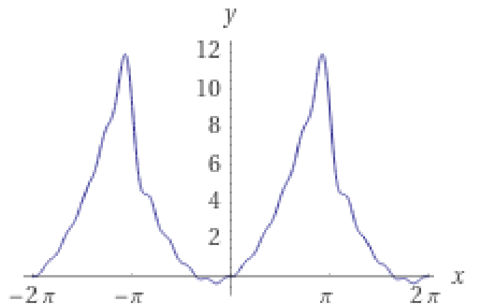
Otro ejemplo

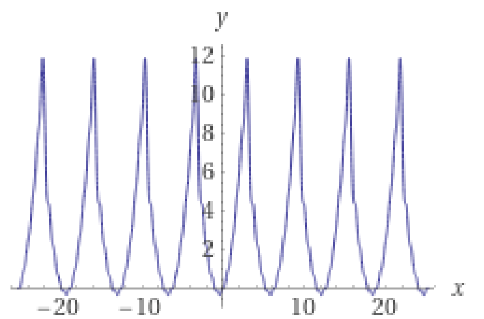
$$f(x) = x^2 + x \quad -\pi < x < \pi$$

$$f(x) \cong \frac{\frac{2\pi^2}{3}}{2} + \sum_{n=1}^{\infty}\left(\frac{4\cos(\pi n)}{n^2}\cos\frac{n\pi x}{\pi} + \left(-\frac{2\cos(\pi n)}{n}\right)\operatorname{sen}\frac{n\pi x}{\pi}\right)$$

$$\cong \frac{\pi^2}{3} + \sum_{n=1}^{10}\left(\frac{4(-1)^{\mathrm{n}}}{n^2}\cos \mathrm{nx} - \left(\frac{2(-1)^{\mathrm{n}}}{n}\right)\operatorname{sen} \mathrm{nx}\right)$$

$$\frac{\pi^2}{3} + \sum_{n=1}^{10}\left(\frac{4(-1)^{\mathrm{n}}}{n^2}\cos \mathrm{nx} - \left(\frac{2(-1)^{\mathrm{n}}}{n}\right)\operatorname{sen} \mathrm{nx}\right)$$





$$f(x) = x - \pi \quad \pi < x < 2\pi$$

$$a_0 = \frac{2}{\pi}\int_{\pi}^{2\pi}(x-\pi)dx = \pi$$

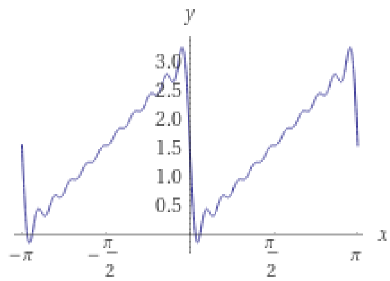
$$a_n = \frac{2}{\pi}\int_{\pi}^{2\pi}(x-\pi)\cos(2nx)\ dx = \frac{-cos(2\,n\,\pi) + \ cos(4\,n\,\pi) + \ 2\,n\,\pi\ sin(4\,n\,\pi)}{2\,n^2\pi}$$

$$= \frac{-1 + \ 1 + \ 0}{2\,n^2\pi} = 0$$

$$b_n = \frac{2}{\pi}\int_{\pi}^{2\pi}(x-\pi)\operatorname{sen}(2nx)\ dx = -\frac{2\,n\,\pi\ cos(4\,n\,\pi) + \ sin(2\,n\,\pi) - \ sin(4\,n\,\pi)}{2\,n^2\pi}$$

$$= -\frac{2\,n\,\pi}{2\,n^2\pi} = -\frac{1}{n}$$

$$f(x) \cong \frac{\pi}{2} - \sum_{n=1}^{10}\left(\frac{1}{\mathrm{n}}\operatorname{sen}(2nx)\right)$$



IDENTIDAD DE PARSEVAL

Partiendo de la serie de Fourier

$$f(x) \cong \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{L} + b_n \operatorname{sen}\frac{n\pi x}{L} \right)$$

Multiplicando por f(x) e integrando en un periodo T

$$\int_0^T f^2(x)dx \;\cong \frac{a_0}{2}\int_0^T f(x)dx + \sum_{n=1}^{\infty}\left(a_n \int_0^T f(x)\cos\left(\frac{n\pi x}{L}\right)dx + b_n \int_0^T f(x)\, sen\left(\frac{n\pi x}{L}\right)\, dx \right)$$

Sabiendo que

$$a_0 = \frac{1}{L}\int_0^T f(x)dx, \quad a_n = \frac{1}{L}\int_0^T f(x)\cos\frac{n\pi x}{L}\, dx, \quad b_n = \frac{1}{L}\int_0^T f(x)\operatorname{sen}\frac{n\pi x}{L}\, dx$$

$$\int_0^T f^2(x)dx \;\cong \frac{a_0}{2} a_0 L + \sum_{n=1}^{\infty}(a_n a_n L + b_n b_n L)$$

Logramos la Identidad de Parseval

$$\frac{1}{T}\int_0^T f^2(x)dx = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

Ejercicio, Aplicar la identidad anterior a la serie siguiente

$$f(x) = \frac{4}{\pi}\sum_{k=1}^{10}\left(\left(\frac{1}{(2\mathrm{k}-1)}\right)\operatorname{sen}((2\mathrm{k}-1)\mathrm{x})\right)$$

$$\frac{2}{2\pi}\int_0^{\pi} 1dx = \frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{4}{\pi(2\mathrm{k}-1)}\right)^2$$

$$\frac{2}{2\pi}\pi = \frac{1}{2}\sum_{k=1}^{\infty}\frac{16}{\pi^2(2\mathrm{k}-1)^2}$$

$$\pi^2 = 8\sum_{k=1}^{\infty}\frac{1}{(2\mathrm{k}-1)^2} = 8\left(1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \cdots\right)$$

$$\pi^2 = 8\sum_{k=1}^{\infty}\frac{1}{(2\mathrm{k}-1)^2} = 8\left(1+\frac{1}{9}+\frac{1}{25}+\frac{1}{49}+\cdots\right)$$

$$\pi = \sqrt{8\left(1+\frac{1}{9}+\frac{1}{25}+\frac{1}{49}+\cdots\right)}$$

Ejemplo

$$f(x) = |x| \qquad -\pi < x < \pi$$

$f(x) = \frac{\pi}{2} + \sum_{n=1}^{10}\frac{2}{n^2\,\pi}((-1)^n - 1)\cos(n\,x)$Aplicaremos la identidad de Parseval

$$\frac{1}{T}\int_0^T f^2(x)dx = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

$$\frac{2}{2\pi}\int_0^{\pi} x^2 dx = \frac{\pi^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{2}{n^2\,\pi}((-1)^n - 1)\right)^2$$

$$\frac{2}{2\pi}\frac{\pi^3}{3} = \frac{\pi^2}{4} + \frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{-4}{(2k+1)^2\,\pi}\right)^2$$

$$\frac{\pi^2}{3} - \frac{\pi^2}{4} = \frac{\pi^2}{12} = \frac{16}{2}\sum_{k=0}^{\infty}\left(\frac{1}{(2k+1)^2\,\pi}\right)^2$$

Obtenemos la siguiente serie

$$\pi^4 = 96\sum_{k=0}^{\infty}\frac{1}{(2k+1)^4} = 96\left(1+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\cdots\right)$$

$$\pi = \left(96\left(1+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\cdots\right)\right)^{\frac{1}{4}}$$

$$Si\ f(x) = x \qquad\qquad -\pi < x < \pi$$

$$La\ serie\ f(x) = \sum_{n=1}^{\infty}\left(b_n \operatorname{sen}\frac{n\pi x}{L}\right) = \sum_{n=1}^{10}\frac{2(-1)^{n+1}}{n}\operatorname{sen}\,(n\,x)$$

$$Aplicando\ Parseval\ \frac{1}{T}\int_0^T f^2(x)dx = \frac{a_0^2}{4}+\frac{1}{2}\sum_{n=1}^{\infty}(a_n^2+b_n^2)$$

$$\frac{2}{2\pi}\int_0^{\pi}x^2dx = \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{2(-1)^{n+1}}{n}\right)^2$$

$$\frac{\pi^2}{3} = \frac{4}{2}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$\frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2}$ Problema de Basilea

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2} = 1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\cdots$$

$$\pi = \sqrt{\left(6\left(1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\cdots\right)\right)}$$

Problema 1.31

$$f(x) = f(x+2\pi) = x^2 = \frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\left(\frac{(-1)^n}{n^2}\cos nx\right)$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi}x^2dx = 2\frac{\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi}x^2\cos(nx)\,dx = \frac{2\,n\,\pi\,cos(n\,\pi)+\ (-2\ +\ n^2\pi^2)sin(n\,\pi)}{n^3}$$

$$= \frac{2\,n\,\pi\,(-1)^n}{n^3}$$

Problema 1.38

$$\pi^2 = \frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\left(\frac{(-1)^n}{n^2}\cos n\pi\right)$$

$$\pi^2 = \frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\left(\frac{(-1)^n}{n^2}(-1)^{\mathrm{n}}\right)$$

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty}\left(\frac{1}{n^2}\right)$$

$$f(x) = |x|^m \qquad -\pi < x < \pi$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx) =$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^m dx = \frac{\pi^m}{m+1}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^m \cos(\mathrm{nx})\ dx =$$

SERIE DE FOURIER EXPONENCIAL

La serie de Fourier exponencial se puede obtener a partir de la serie de Fourier trigonométrica, reemplazando la función coseno y seno

$$f(x) \cong \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 x + b_n \operatorname{sen} n\omega_0 x)$$

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad sen(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$f(x) \cong \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \frac{\left(e^{in\omega_0 x} + e^{-in\omega_0 x}\right)}{2} + b_n \frac{\left(e^{in\omega_0 x} - e^{-in\omega_0 x}\right)}{2i} \right)$$

$$f(x) \cong \frac{a_0}{2} + \sum_{n=1}^{\infty} e^{in\omega_0 x} \frac{1}{2}(a_n - i\, b_n) + \sum_{n=1}^{\infty} e^{-in\omega_0 x} \frac{1}{2}(a_n + i\, b_n)$$

Definiendo

$$c_n = \frac{1}{2}(a_n - i\, b_n) = \frac{1}{2}\left(\frac{1}{L}\int_0^T f(x)(cos(n\omega_0 x) - i \operatorname{sen}(n\omega_0 x))\ dx \right) = \left(\frac{1}{T}\int_0^T f(x) e^{-in\omega_0 x}\, dx \right)$$

Observar que

$$c_0 = \left(\frac{1}{T}\int_0^T f(x)1\,dx\right) = \frac{1}{2}a_0$$

y

$$c_{-n} = \left(\frac{1}{T}\int_0^T f(x)e^{-i\frac{-n\pi x}{L}}\,dx\right) = \left(\frac{1}{T}\int_0^T f(x)e^{i\frac{n\pi x}{L}}\,dx\right) = c_n^* = \frac{1}{2}(a_n + i\,b_n)$$

$$f(x) \cong c_0 1 + \sum_{n=1}^{\infty} c_n e^{i\frac{n\pi x}{L}} + \sum_{n=1}^{\infty} c_{-n} e^{-i\frac{n\pi x}{L}}$$

$$\sum_{n=1}^{3} c_n = c_1 + c_2 + c_3 = \sum_{n=-1}^{-3} c_{-n}$$

$$f(x) \cong c_0 e^{i\frac{(0)\pi x}{L}} + \sum_{n=1}^{\infty} c_n e^{i\frac{n\pi x}{L}} + \sum_{n=-1}^{-\infty} c_n e^{i\frac{n\pi x}{L}} = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi x}{L}}$$

Obtenemos finalmente:

$$f(x) = \sum_{n=-\infty}^{\infty} \left(c_n \mathrm{e}^{in\omega_0 x}\right)$$

$$c_n = \frac{1}{T}\int_o^T f(x)\mathrm{e}^{-in\omega_0 x} dx$$

Ortogonalidad de las funciones $\mathrm{e}^{in\frac{2\pi}{T}t} = ` = \mathrm{e}^{i\omega_n t}$

El producto interno se define para funciones complejas como

$$\langle \phi_n(t), \phi_m(t)\rangle = \int_0^T \phi_n(t)\,\phi_m^*(t)\,dt = \delta_{mn}T$$

$$= \int_0^T \mathrm{e}^{in\frac{2\pi}{T}t}\,\mathrm{e}^{-im\frac{2\pi}{T}t}\,dt$$

Si n distinto de m

$$= \frac{1}{i\,\frac{2\pi}{T}(n-m)}\int_o^T \mathrm{e}^{i\frac{2\pi}{T}(n-m)t}\,i\,\frac{2\pi}{T}(n-m)dt = \frac{\mathrm{e}^{i\frac{2\pi}{T}(n-m)T} - \mathrm{e}^0}{i\,\frac{2\pi}{T}(n-m)} = \frac{\mathrm{e}^{i2\pi(n-m)} - \mathrm{e}^0}{i\,\frac{2\pi}{T}(n-m)} = 0$$

Si n=m

$$= \int_o^T \mathrm{e}^0\,dt = T$$

Ejercicio, calcular la serie de Fourier de la función $f(x) = e^x \ si \ 0 \leq x \leq 2\pi \ y \ f(x+2\pi) = f(x)$

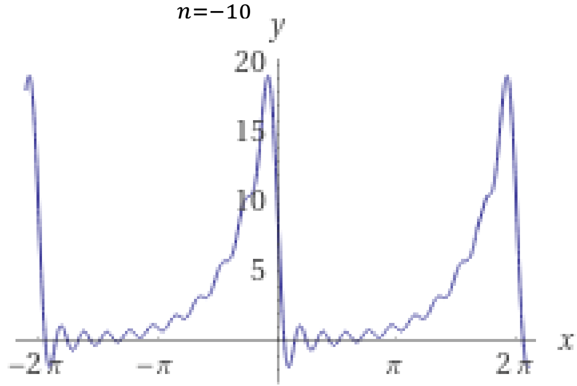
$$f(x) = \sum_{n=-\infty}^{\infty} \left(c_n \mathrm{e}^{inx}\right)$$

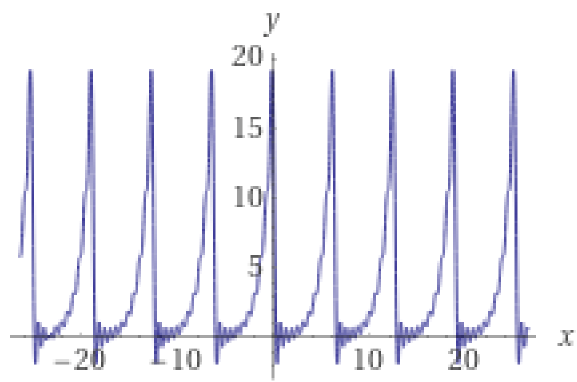
$$c_n = \frac{1}{T}\int_o^T f(x)\mathrm{e}^{-inx}dx = \frac{1}{2\pi}\int_o^{2\pi} e^x\mathrm{e}^{-inx}dx = \frac{1}{2\pi}\int_o^{2\pi} \mathrm{e}^{x-inx}dx$$

$$= \frac{1}{2\pi}\int_o^{2\pi} \mathrm{e}^{x(1-in)}dx = \frac{1}{2\pi(1-in)}\int_o^{2\pi} \mathrm{e}^{x(1-in)}(1-in)dx$$

$$c_n = \frac{\mathrm{e}^{2\pi(1-in)}-1}{2\pi(1-in)} = \frac{\mathrm{e}^{2\pi}\mathrm{e}^{-2\pi in}-1}{2\pi(1-in)} = \frac{\mathrm{e}^{2\pi}-1}{2\pi(1-in)}$$

$$f(x) = \sum_{n=-10}^{10}\left(\frac{\mathrm{e}^{2\pi}-1}{2\pi(1-in)}\ \mathrm{e}^{i\mathrm{nx}}\right)$$





(1.32) Ejercicio, calcular la serie de Fourier de la función $f(x) = e^x \ si -\pi \leq x \leq \pi \ y \ f(x+2\pi) = f(x)$

$$f(x) = \sum_{n=-\infty}^{\infty}\left(c_n \mathrm{e}^{i\frac{n\pi x}{\pi}}\right)$$

Solución:

$$= \frac{2senh(\pi)}{\pi}\left(\frac{1}{2} + \frac{(-1)^{\mathrm{n}}}{1+n^2}\sum_{n=1}^{\infty}(\cos(\mathrm{nx}) - \mathrm{n}\ \mathrm{sen}(\mathrm{nx}))\right)$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x \mathrm{e}^{-i\frac{n\pi x}{\pi}} dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{x(1-\mathrm{in})} dx = \frac{\mathrm{e}^{\pi(1-in)} - \mathrm{e}^{-\pi(1-in)}}{2\pi(1-in)}$$

$$f(x) = \sum_{n=-10}^{10}\left(\frac{\mathrm{e}^{\pi(1-in)} - \mathrm{e}^{-\pi(1-in)}}{2\pi(1-in)}\ \mathrm{e}^{i\frac{n\pi x}{\pi}}\right)$$

$$= \sum_{n=-10}^{10}\left(\frac{\mathrm{e}^{\pi} e^{-\pi in} - \mathrm{e}^{-\pi} e^{\pi in}}{2\pi(1-in)}\ \left(\cos(\mathrm{nx}) + \mathrm{isen}(\mathrm{nx})\right)\right)$$

$$= \sum_{n=-\infty}^{\infty}\left(\frac{\mathrm{e}^{\pi}\left(\cos(n\pi) - isen(n\pi)\right) - \mathrm{e}^{-\pi}\left(\cos(n\pi) + isen(n\pi)\right)}{2\pi(1-in)}\ \left(\cos(\mathrm{nx}) + \mathrm{isen}(\mathrm{nx})\right)\right)$$

$$= \sum_{n=-\infty}^{\infty}\frac{(-1)^{\mathrm{n}}}{\pi(1-in)}\left(\left(\frac{\mathrm{e}^{\pi} - \mathrm{e}^{-\pi}}{2}\right)\ \left(\cos(\mathrm{nx}) + \mathrm{isen}(\mathrm{nx})\right)\right)$$

$$= \frac{senh(\pi)}{\pi}\sum_{n=-\infty}^{\infty}\frac{(-1)^{\mathrm{n}}}{(1-in)}\frac{(1+in)}{(1+in)}\ \left(\ \left(\cos(\mathrm{nx}) + \mathrm{isen}(\mathrm{nx})\right)\right)$$

$$= \frac{senh(\pi)}{\pi}\sum_{n=-\infty}^{\infty}\frac{(-1)^{\mathrm{n}}(1+in)}{1+n^2}\ \left(\ \left(\cos(\mathrm{nx}) + \mathrm{isen}(\mathrm{nx})\right)\right)$$

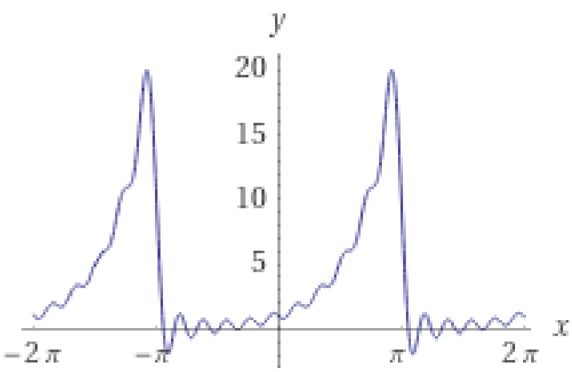
$$= \frac{senh(\pi)}{\pi}\left(\sum\nolimits_{n=-\infty}^{-1}\frac{(-1)^{\mathrm{n}}(1+in)}{1+n^2}\ \left(\ \left(\cos(\mathrm{nx}) + \mathrm{isen}(\mathrm{nx})\right)\right) + 1 + \sum\nolimits_{n=1}^{\infty}\frac{(-1)^{\mathrm{n}}(1+in)}{1+n^2}\ \left(\ \left(\cos(\mathrm{nx}) + \mathrm{isen}(\mathrm{nx})\right)\right)\right)$$

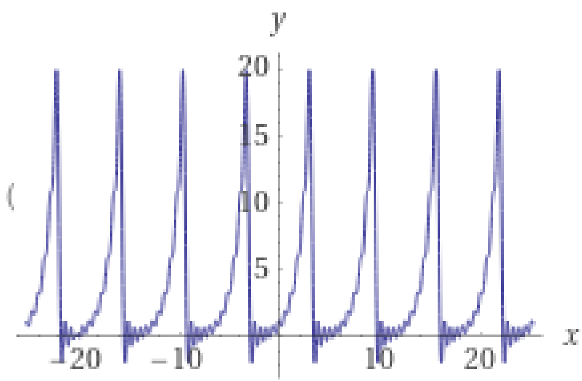
$$= \frac{senh(\pi)}{\pi}\left(\sum_{n=1}^{\infty}\frac{(-1)^{\mathrm{n}}(1-in)}{1+n^2}\left(\left(\cos(\mathrm{nx}) - \mathrm{isen}(\mathrm{nx})\right)\right) + 1\right.$$

$$\left. + \sum_{n=1}^{\infty}\frac{(-1)^{\mathrm{n}}(1+in)}{1+n^2}\ \left(\ \left(\cos(\mathrm{nx}) + \mathrm{isen}(\mathrm{nx})\right)\right)\right)$$

$$= \frac{senh(\pi)}{\pi}\left(\sum_{n=1}^{\infty}\frac{(-1)^{\mathrm{n}}}{1+n^2}\left((2\cos(\mathrm{nx}))\right) + 1 + \sum_{n=1}^{\infty}\frac{(-1)^{\mathrm{n}}(n)}{1+n^2}\ \left(\ \left(-2\mathrm{sen}(\mathrm{nx})\right)\right)\right)$$

$$= \frac{2senh(\pi)}{\pi}\left(\frac{1}{2} + \frac{(-1)^{\mathrm{n}}}{1+n^2}\sum_{n=1}^{\infty}(\cos(\mathrm{nx}) - \mathrm{n}\ \mathrm{sen}(\mathrm{nx}))\right)$$





Identidad de Parseval para la serie exponencial

$$f(x) = \sum_{n=-\infty}^{\infty} \left(c_n e^{i\frac{n\pi x}{L}}\right)$$

$$c_n = \frac{1}{T}\int_0^T f(x) e^{-i\frac{n\pi x}{L}} dx$$

$$\int_0^T f^*(x) f(x) dx = \int_0^T |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} c_n \int_0^T f^*(x) e^{i\frac{n\pi x}{L}} dx = \sum_{n=-\infty}^{\infty} c_n \left(\int_0^T f(x) e^{-i\frac{n\pi x}{L}} dx\right)^*$$

$$= \sum_{n=-\infty}^{\infty} c_n T c_n^* = T \sum_{n=-\infty}^{\infty} |c_n|^2$$

$$\frac{1}{T}\int_0^T |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |c_n|^2$$

Ejemplo, aplicar la identidad de parseval al problema anterior:

$$\frac{1}{2\pi}\int_o^{2\pi} e^{2x} dx = \sum_{n=-\infty}^{\infty} \left(\frac{e^{2\pi}-1}{2\pi(1-in)}\right)\left(\frac{e^{2\pi}-1}{2\pi(1-in)}\right)^*$$

$$\frac{1}{4\pi}(e^{4\pi}-1) = \frac{(e^{2\pi}-1)^2}{(2\pi)^2} \sum_{n=-\infty}^{\infty} \left(\frac{1}{(1+n^2)}\right)$$

$$\pi \frac{(e^{2\pi}-1)(e^{2\pi}+1)}{(e^{2\pi}-1)^2} = \sum_{n=-\infty}^{\infty} \left(\frac{1}{(1+n^2)}\right) = \cdots + \frac{1}{10} + \frac{1}{5} + \frac{1}{2} + 1 + \frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \cdots$$

$$\pi \frac{(e^{2\pi}+1)}{(e^{2\pi}-1)} = 1 + 2\left(\frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \frac{1}{17} + \cdots\right)$$

PROBLEMA 1.35 Desarrollar $f(t) = e^{r\cos t}\cos(r \text{ sen } t)$ en serie de Fourier. [Sugerencia: usar la serie de potencias para e^z cuando $z = re^{jt}$.]

Respuesta: $1 + \sum_{n=1}^{\infty} \frac{r^n}{n!} \cos nt$.

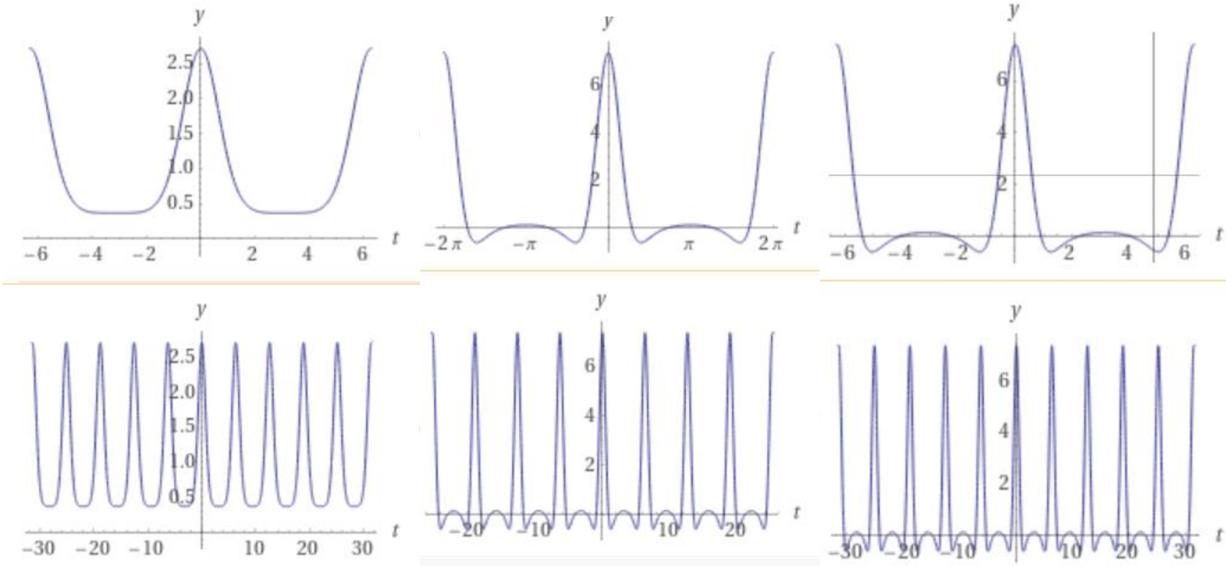
$$f(t) = e^{rcos(t)}\cos(rsen(t))$$
$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + \cdots = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$
$$z = re^{it} = r\,cos(t) + i\,r\,sen(t)$$
$$e^{r\,cos(t) + i\,r\,sen(t)} = e^{r\,cos(t)} e^{i\,r\,sen(t)}$$
$$= e^{r\,cos(t)}\left(\cos(r\,sen(t)) + isen(r\,sen(t))\right)$$
$$= e^{r\,cos(t)}\cos(r\,sen(t)) + i\,e^{r\,cos(t)} sen(r\,sen(t))$$

$$= \sum_{n=0}^{\infty} \frac{r^n\left(\,cos(t) + i\;\,sen(t)\right)^n}{n!}$$
$$= \sum_{n=0}^{\infty} \frac{r^n}{n!}\left(\,cos(nt) + i\,sen(nt)\right)$$
$$= \sum_{n=0}^{\infty} \frac{r^n}{n!}\,cos(nt) + i\sum_{n=0}^{\infty} \frac{r^n}{n!}\,sen(nt)$$

Igualar parte real con parte real

$$f(t) = e^{r\,cos(t)}\cos(r\,sen(t)) = \sum_{n=0}^{\infty} \frac{r^n}{n!}\,cos(nt)$$

$$\sum_{n=0}^{10} \frac{2^n}{n!}\,cos(nt)$$



a)r=1 b)r=$2e^{r\,cos(t)}\cos\big(r\,sen(t)\big)$ c)r=2

$$\sum_{n=0}^{10} \frac{2^n}{n!}\,cos(nt)$$

PROBLEMA 1.44 Utilizar el teorema de Parseval (1.72) para probar que $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$.

[Sugerencia: utilizar el resultado del problema 1.10.]

$$\frac{1}{T}\int_0^T f^2(x)dx = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

$$1 = +\frac{1}{2}\sum_{n=1}^{\infty}\left(2\frac{1-(-1)^n}{n\pi}\right)^2$$

$$\frac{\pi^2}{8} = +\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}$$

$$\frac{1}{2} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}sen(nt) = \frac{1}{2} - \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{2k-1}sen\big((2k-1)t\big)$$

$$\sum_{n=0}^{\infty}\frac{r^n}{n!}\left(cos(nt) + i\ sen(nt)\right)$$

RESUMEN DE TRANSFORMADA DE FOURIER

Recordemos la serie de Fourier exponencial

$$f(t) = \sum_{n=-\infty}^{\infty}\left(C_n \mathrm{e}^{i\frac{n2\pi t}{T}}\right)$$

$$C_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(u)\mathrm{e}^{-i\frac{n2\pi u}{T}}du$$

La serie de Fourier se aplica solo a funciones periódicas, la transformada de Fourier es la generalización para funciones no necesariamente periódicas (puede decirse que su periodo es infinito). Para pasar de la serie a la transformada podemos proceder así: Se juntan ambas expresiones anteriores y se obtiene:

$$f(t) = \sum_{n=-\infty}^{\infty}\left(\left(\frac{1}{2\pi}\ \frac{2\pi}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(u)\mathrm{e}^{-i\frac{n2\pi u}{T}}du\right)\mathrm{e}^{i\frac{n2\pi t}{T}}\right)$$

Haciendo tender a T a infinito y definiendo $\Delta w = \frac{2\pi}{T}$

$$f(t) = \sum_{n=-\infty}^{\infty}\left(\left(\frac{1}{2\pi}\ \Delta w\int_{-\infty}^{\infty} f(u)\mathrm{e}^{-i(n\Delta w)u}du\right)\mathrm{e}^{i(n\Delta w)t}\right)$$

ordenando

$$f(t) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\left(\left(\int_{-\infty}^{\infty} f(u)\mathrm{e}^{-i(n\Delta w)u}du\right)\mathrm{e}^{i(n\Delta w)t}\right)\Delta w$$

Usando la integral de Riemann $\int_{-\infty}^{\infty} G(w)dw = \sum_{n=-\infty}^{\infty} G(n\,\Delta w)\,\Delta w$,se obtiene la integral de Fourier siguiente

$$f(t) = \frac{1}{2\pi}\int_{w=-\infty}^{\infty}\left(\int_{u=-\infty}^{\infty} f(u)e^{-iwu}du\right)e^{iwt}dw = \frac{1}{2\pi}\int_{w=-\infty}^{\infty} F(w)e^{iwt}dw$$

De donde se define la transformada y transformada inversa de Fourier

TRANSFORMADA DE FOURIER

$$\mathcal{F}\{f(t)\} = F(w) = \int_{-\infty}^{\infty} f(t)e^{-iwt}dt$$

$$F(\nu) = \int_{-\infty}^{\infty} f(t)e^{-i2\pi\nu t}dt$$

TRANSFORMADA INVERSA DE FOURIER

$$\mathcal{F}^{-1}\{F(w)\} = f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(w)e^{iwt}dw$$

$$f(t) = \int_{-\infty}^{\infty} F(\nu)e^{i2\pi\nu t}d\nu$$

Esto porque $w = \frac{2\pi}{T} = 2\pi\nu$

Ejemplo: Calcular la transformada de Fourier de la función rectangular de área unidad, el ancho de la función es 2a, el alto es $(2a)^{-1}$, el área es A= (2a) $(2a)^{-1}$ =1.

Matemáticamente la función es:

$$f(t) = \begin{Bmatrix} \frac{1}{2a} & si\ |t| \leq a \\ 0 & si\ a < |t| \end{Bmatrix}$$

En este caso, aplicando la definición de la transformada de Fourier se obtiene una función conocida como seno cardinal sinc(wa)

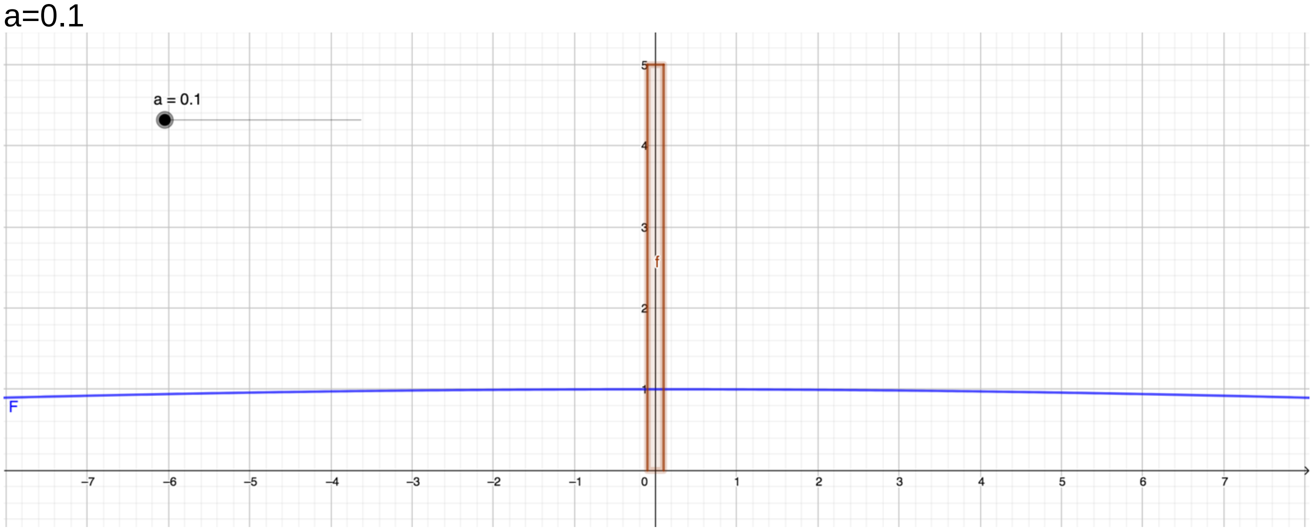
$$F(w) = \int_{-a}^{a} \frac{1}{2a}e^{-iwt}dt = \frac{1}{2a}\frac{\left(e^{-iwa} - e^{iwa}\right)}{(-iw)} = \frac{1}{wa}\frac{\left(e^{iwa} - e^{-iwa}\right)}{(2i)} = \frac{sen(wa)}{wa} = sinc(wa)$$

1

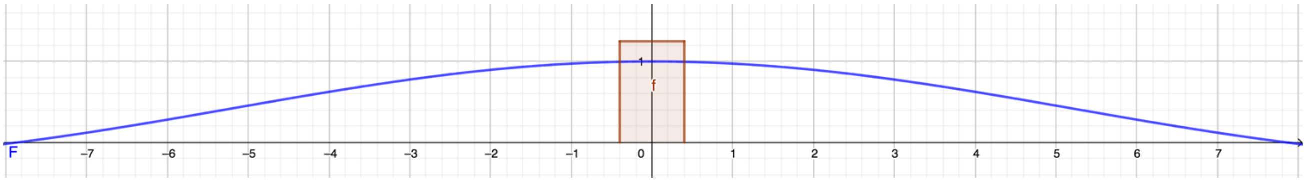
En el límite cuando $a \to 0,\ f(t)$ se transforma en una función conocida como delta de Dirac $\delta(t)$ y la transformada de la delta es $F(w) = \lim_{a \to 0}\frac{sen(wa)}{wa} = 1$

En las imágenes siguientes se muestra la función rectangular para diferentes valores de a y su transformada de Fourier en color azul

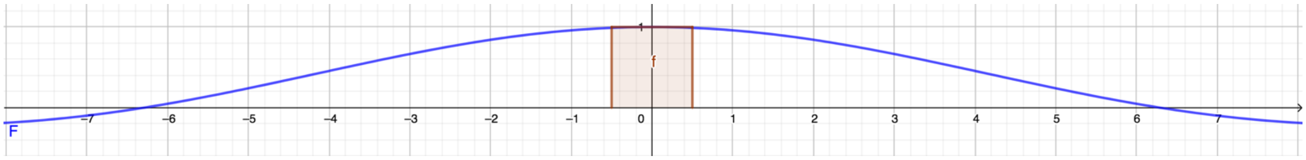
a=0.1



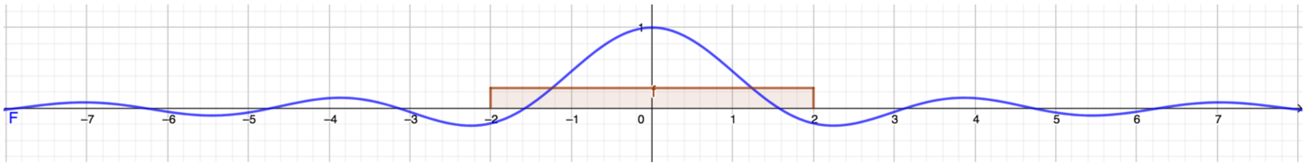
a=0.4



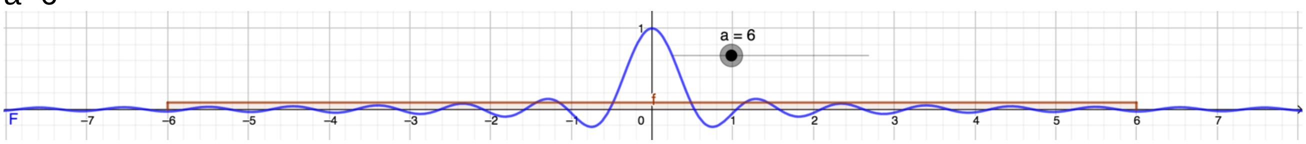
a=0.5



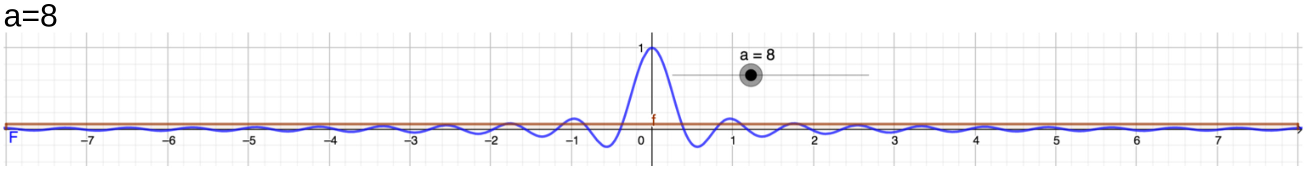
a=2



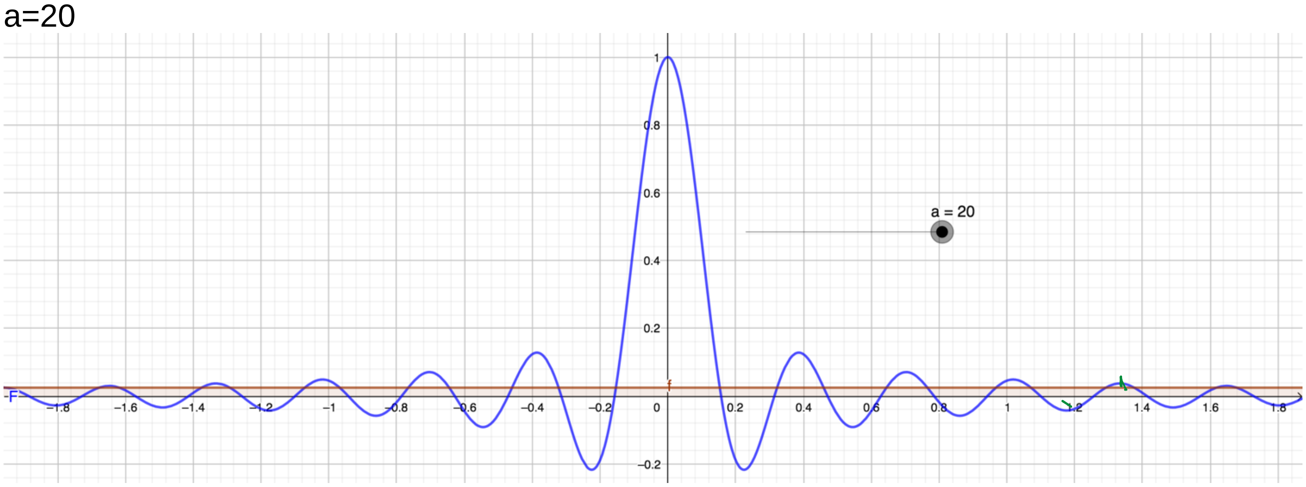
a=6



a=8



a=20



Ejercicio

Calcular la transformada de Fourier de un pulso de ancho 2a

$$f(t) = \begin{Bmatrix} 1 \; si \; |t| \leq a \\ 0 \quad si \; a < |t| \end{Bmatrix}$$

$$F(w) = \int_{-a}^{a} 1e^{-iwt}dt = \left[\frac{e^{-iwt}}{-iw}\right]_{-a}^{a} = 2a\frac{-e^{-iwa} + e^{iwa}}{2aiw} = 2a\frac{sen(wa)}{wa}$$

Aunque siempre se pueden usar la definición para calcular la transformada y la transformada inversa, es más fácil usar las propiedades de la transformada de Fourier, en la tabla siguiente se resumen algunas de las más usuales, se pueden demostrar mediante la definición.

	$f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(w)e^{iwt}dw$	$F(w)=\int_{-\infty}^{\infty} f(t)e^{-iwt}dt$	
1	$a\,f(t)+b\,g(t)$	$a\,F(w)+b\,G(w)$	Linealidad
2	$f(a\,t)$	$\frac{1}{\|a\|}F\left(\frac{w}{a}\right)$	Escalamiento
3	$f(t-t_0)$	$F(w)\,e^{-iwt_0}$	Corrimiento en el tiempo
4	$f(t)\,e^{iw_0t}$	$F(w-w_0)$	Corrimiento en la frecuencia
5	$f(t)\cos(w_0t)$	$\frac{1}{2}\left(F(w-w_0)+F(w+w_0)\right)$	Modulación
6	$f(t)\,\mathrm{sen}\,(w_0t)$	$\frac{1}{2i}\left(F(w-w_0)-F(w+w_0)\right)$	Modulación
7	$f(t)=f(-t)$	$R(w)=R(-w)$	La transformada de una función par es real y par en w
8	$f(t)=-f(-t)$	$iI(w)=-iI(-w)$	La transformada de una función impar es imaginaria e impar en w
9	$F(t)$	$2\pi f(-w)$	Simetría
10	$f^{(n)}(t)$	$(iw)^nF(w)$	Derivada n-ésima
11	$f(t)$ g(t)	$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)G(w-u)du$	Convolución en la frecuencia
12	$\int_{-\infty}^{\infty} f(u)g(t-u)du$	$F(w)G(w)$	Convolución en el tiempo
13	$P_{2a}(t)=\begin{Bmatrix}1 & si\ \|t\|\le a\\ 0 & si\ a\le\|t\|\end{Bmatrix}$	$2a\frac{sen(wa)}{wa}$	Función rectangular de ancho 2a
14	$\delta(t)$	1	Delta de Dirac
15	$H(t)$=$\begin{Bmatrix}1 & si & 0\le t\\ 0 & si & t<0\end{Bmatrix}$	$\pi\,\delta(w)+\frac{1}{iw}$	Función de Heaviside
16	$H(t)\,e^{-at}$	$\frac{1}{iw+a}$	
17	$e^{-a\|t\|}$	$\frac{2a}{w^2+a^2}$	
18	e^{-at^2}	$\sqrt{\frac{\pi}{a}}e^{-\frac{w^2}{4a}}$	
19	$\frac{1}{a^2+t^2}$	$\frac{\pi}{a}e^{-a\|w\|}$	

Explicación de algunas propiedades:

1 Linealidad

$$\mathcal{F}\{a\,f(t)+b\,g(t)\}=\int_{-\infty}^{\infty}[a\,f(t)+b\,g(t)]e^{-iwt}dt=\int_{-\infty}^{\infty}\left[a\,f(t)e^{-iwt}+b\,g(t)e^{-iwt}\right]dt$$

$$=a\int_{-\infty}^{\infty}f(t)e^{-iwt}dt+b\int_{-\infty}^{\infty}g(t)e^{-iwt}dt=a\,F(w)+b\,G(w)$$

2 Escalamiento

$$Si\ a>0$$

$$\mathcal{F}\{f(at)\}=\int_{-\infty}^{\infty}f(at)e^{-iwt}dt=\frac{1}{a}\int_{-\infty}^{\infty}f(at)e^{-i\left(\frac{w}{a}\right)(at)}(adt)=\frac{1}{a}\int_{-\infty}^{\infty}f(\xi)e^{-i\left(\frac{w}{a}\right)(\xi)}(d\xi)=\frac{1}{a}F\left(\frac{w}{a}\right)$$

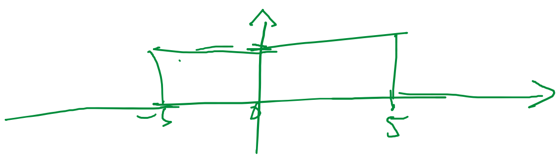
$$Si\ a<0$$

$$\mathcal{F}\{f(at)\}=\int_{-\infty}^{\infty}f(at)e^{-iwt}dt=\frac{1}{a}\int_{-\infty}^{\infty}f(at)e^{-i\left(\frac{w}{a}\right)(at)}(adt)=\frac{1}{a}\int_{\infty}^{-\infty}f(\xi)e^{-i\left(\frac{w}{a}\right)(\xi)}(d\xi)$$

$$=-\frac{1}{a}\int_{-\infty}^{\infty}f(\xi)e^{-i\left(\frac{w}{a}\right)(\xi)}(d\xi)=-\frac{1}{a}F\left(\frac{w}{a}\right)=\frac{1}{|a|}F\left(\frac{w}{a}\right)$$

3 Corrimiento en el tiempo

$$\mathcal{F}\{f(t-t_0)\}=\int_{-\infty}^{\infty}f(t-t_0)e^{-iwt}dt=\int_{-\infty}^{\infty}f(\xi)e^{-iw(\xi+t_0)}d\xi=e^{-iw(t_0)}\int_{-\infty}^{\infty}f(\xi)e^{-iw(\xi)}d\xi=e^{-iw(t_0)}F(w)$$



Ejemplo:Calcular la TF de el pulso de altura 1 con ancho 10 centrado en 5

$$\int_{0}^{10}1e^{-iwt}dt=\frac{e^{-iw10}-1}{-iw}=2\frac{sen(w5)}{w}e^{-iw(5)}$$

$$2\frac{sen(w5)}{w}e^{-iw(5)}=2\frac{(e^{i5w}-e^{-i5w})}{2iw}e^{-iw(5)}=2\frac{(1-e^{-i10w})}{2iw}=\frac{(1-e^{-i10w})}{iw}$$

$$\frac{e^{-iw10}-1}{-iw}=\frac{2(e^{-iw5}-e^{iw5})e^{-iw5}}{-2iw}=2\frac{sen(w5)}{w}e^{-iw5}$$

4 Corrimiento en la frecuencia

a)

$$\mathcal{F}\{f(t)\, e^{iw_0t}\} = \int_{-\infty}^{\infty} f(t)\, e^{iw_0t} e^{-iwt} dt \;=\; \int_{-\infty}^{\infty} f(t) e^{-i(w-w_0)t} dt \;= F(w-w_0)$$

b)

$$\mathcal{F}^{-1}\{F(w-w_0)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(w-w_0) e^{iwt} dw = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\chi) e^{i(\chi+w_0)t} d(\chi+w_0)$$

$$= e^{iw_0t}\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\chi) e^{i\chi t} d\chi = f(t)\, e^{iw_0t}$$

Haciendo $\chi = w - w_0; w = \chi + w_0; d\chi = dw$

5 Modulación

$$\mathcal{F}\{f(t)\cos(w_0t)\} = \int_{-\infty}^{\infty} f(t)\cos(w_0t)\, e^{-iwt} dt = \int_{-\infty}^{\infty} f(t)\frac{\mathrm{e}^{\mathrm{i}\mathrm{w}_0\mathrm{t}} + \mathrm{e}^{-\mathrm{i}\mathrm{w}_0\mathrm{t}}}{2} e^{-iwt} dt$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} f(t)\left(\mathrm{e}^{-\mathrm{i}(\mathrm{w}-\mathrm{w}_0)\,\mathrm{t}} + \mathrm{e}^{-\mathrm{i}(\mathrm{w}+\mathrm{w}_0)\mathrm{t}}\right) dt$$

$$= \frac{1}{2}\left(\int_{-\infty}^{\infty} f(t)\mathrm{e}^{-\mathrm{i}(\mathrm{w}-\mathrm{w}_0)\mathrm{t}} dt + \int_{-\infty}^{\infty} f(t)\mathrm{e}^{-\mathrm{i}(\mathrm{w}+\mathrm{w}_0)\mathrm{t}} dt\right)$$

$$= \frac{1}{2}\left(F(w-w_0) + F(w+w_0)\right)$$

$$\mathcal{F}\{f(t)\,\mathrm{sen}\,(w_0t)\} = \int_{-\infty}^{\infty} f(t)\,\mathrm{sen}(w_0t)\, e^{-iwt} dt = \int_{-\infty}^{\infty} f(t)\frac{\mathrm{e}^{\mathrm{i}\mathrm{w}_0\mathrm{t}} - \mathrm{e}^{-\mathrm{i}\mathrm{w}_0\mathrm{t}}}{2i} e^{-iwt} dt$$

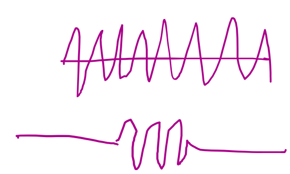
$$= \frac{1}{2i}\int_{-\infty}^{\infty} f(t)\left(\mathrm{e}^{-\mathrm{i}(\mathrm{w}-\mathrm{w}_0)\mathrm{t}} - \mathrm{e}^{-\mathrm{i}(\mathrm{w}+\mathrm{w}_0)\mathrm{t}}\right) dt$$

$$= \frac{1}{2i}\left(\int_{-\infty}^{\infty} f(t)\mathrm{e}^{-\mathrm{i}(\mathrm{w}-\mathrm{w}_0)\mathrm{t}} dt - \int_{-\infty}^{\infty} f(t)\mathrm{e}^{-\mathrm{i}(\mathrm{w}+\mathrm{w}_0)\mathrm{t}} dt\right)$$

$$= \frac{1}{2i}\left(F(w-w_0) - F(w+w_0)\right)$$

Ejemplo: Calcular la TF de $\mathrm{f(t)}\cos(w_0t)$, donde

$$f(t) = \begin{Bmatrix} 1\; si\; |t| \le \frac{1}{2} \\ 0 \quad si\; a < |t| \end{Bmatrix}$$



i) integrando

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \cos(w_0t)\, e^{-iwt} dt = \int_{-\frac{1}{2}}^{\frac{1}{2}} \cos(w_0t)\cos(wt)\, dt - i\int_{-\frac{1}{2}}^{\frac{1}{2}} \cos(w_0t) sen(wt)\, dt$$

$$= \frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}}\left[\cos\big((w_0-w)t\big)+\cos\big((w_0+w)t\big)\right]dt - i\int_{-\frac{1}{2}}^{\frac{1}{2}}\cos(w_0t)sen(wt)\,dt$$

$$= \frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}}\left[\cos\big((w_0-w)t\big)+\cos\big((w_0+w)t\big)\right]dt - \frac{i1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} sen\big((w_0-w)t\big)+sen\big((w_0+w)t\big)dt$$

$$= \frac{\operatorname{sen}\left(\frac{(w_0-w)}{2}\right)}{(w_0-w)} + \frac{\operatorname{sen}\left(\frac{(w_0+w)}{2}\right)}{(w_0+w)} - 0$$

ii) integrando

$$\int_{-\frac{1}{2}}^{\frac{1}{2}}\cos(w_0t)\,e^{-iwt}dt = \frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}}\left[e^{-i(w-w_0)t}+e^{-i(w+w_0)t}\right]dt = \frac{1}{2}\left[\frac{e^{-i(w-w_0)t}}{-i(w-w_0)}+\frac{e^{-i(w+w_0)t}}{-i(w+w_0)}\right]_{-\frac{1}{2}}^{\frac{1}{2}}$$

$$= \frac{e^{-\frac{i(w-w_0)}{2}}-e^{\frac{i(w-w_0)}{2}}}{-2i(w-w_0)} + \frac{e^{-\frac{i(w+w_0)}{2}}-e^{\frac{i(w+w_0)}{2}}}{-2i(w+w_0)}$$

$$= \frac{e^{\frac{i(w-w_0)}{2}}-e^{-\frac{i(w-w_0)}{2}}}{2i(w-w_0)} + \frac{e^{\frac{i(w+w_0)}{2}}-e^{-\frac{i(w+w_0)}{2}}}{2i(w+w_0)}$$

$$= \left(\frac{sen\left(\frac{(w-w_0)}{2}\right)}{(w-w_0)} + \frac{sen\left(\frac{(w+w_0)}{2}\right)}{(w+w_0)}\right)$$

Usando el teorema de modulación

$$\mathcal{F}\{f(t)\cos(w_0t)\} = \frac{1}{2}\big(F(w-w_0)+F(w+w_0)\big)$$

$$= \left(\frac{sen\left(\frac{w-w_0}{2}\right)}{w-w_0} + \frac{sen\left(\frac{w+w_0}{2}\right)}{w+w_0}\right)$$

$$F(w) = 2\frac{sen\left(\frac{w}{2}\right)}{w}$$

7 La transformada de una función par es real y par en w

$$\mathcal{F}\{f_{par}(t)\} = \int_{-\infty}^{\infty} f_{par}(t)e^{-iwt}dt = \int_{-\infty}^{\infty} f_{par}(t)\cos(wt)\,dt - i\int_{-\infty}^{\infty} f_{par}(t)sen(wt)dt = R(w)$$

$$R(w) = R(-w)$$

8 La transformada de una función impar es imaginaria e impar en w

$$\mathcal{F}\{f_{impar}(t)\} = \int_{-\infty}^{\infty} f_{impar}(t)e^{-iwt}dt = \int_{-\infty}^{\infty} f_{impar}(t)\cos(wt)\,dt - i\int_{-\infty}^{\infty} f_{impar}(t)\,sen(wt)dt = iI(w)$$

$$I(w) = -I(-w)$$

Observación: Cualquier función real se puede separar en parte par y parte impar

$$f(t) = f_{par}(t) + f_{impar}(t)$$

$$f(-t) = f_{par}(-t) + f_{impar}(-t) = f_{par}(t) - f_{impar}(t)$$

$$f(t) + f(-t) = 2f_{par}(t)$$

$$f_{par}(t) = \frac{f(t) + f(-t)}{2}$$

$$f(t) - f(-t) = 2f_{impar}(t)$$

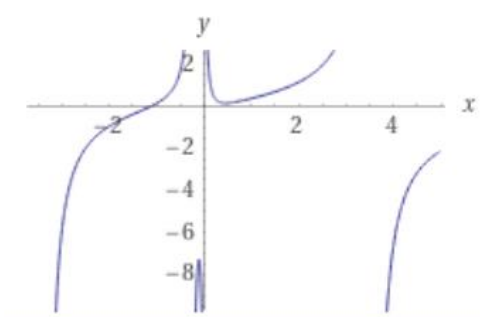
$$f_{impar}(t) = \frac{f(t) - f(-t)}{2}$$

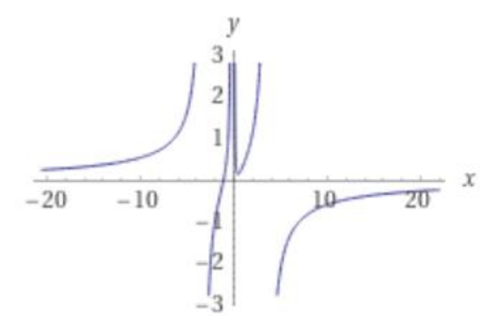
Ejemplo, calcular la parte par e impar de:

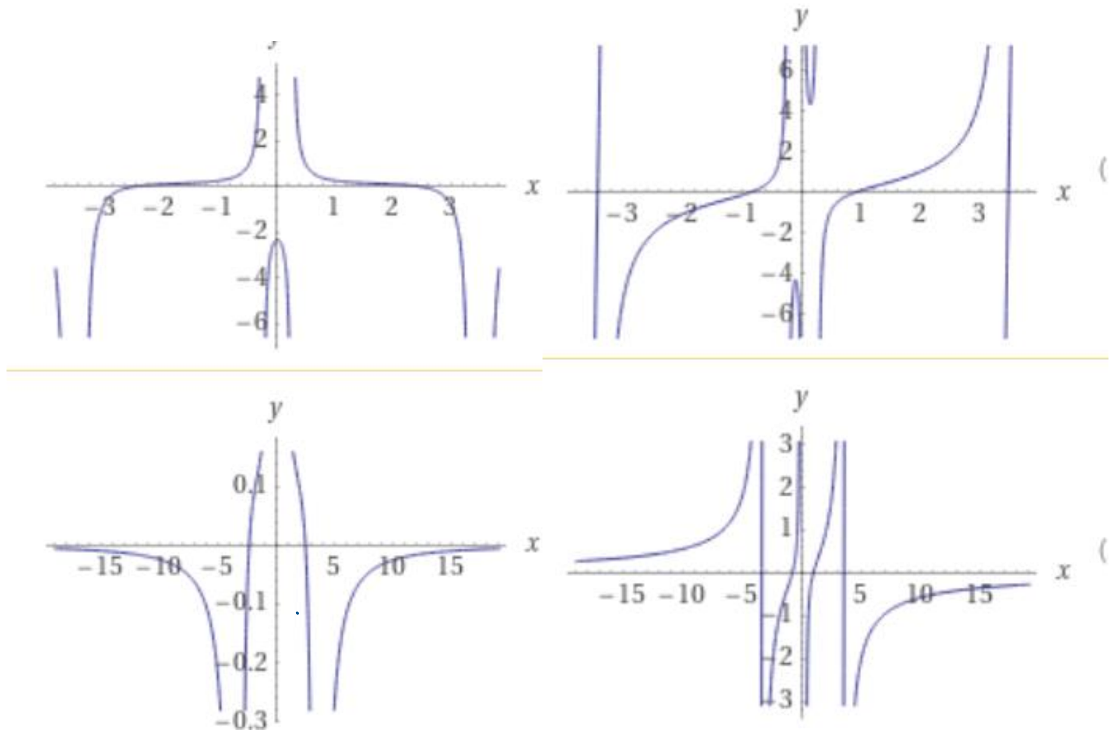
$$f(t) = \frac{5x^3 + 2x^2 - 3x + 1}{-x^4 + 12x^2 + 3x}$$

$$f_{par}(t) = \frac{\left(\frac{5x^3 + 2x^2 - 3x + 1}{-x^4 + 12x^2 + 3x}\right) + \left(\frac{-5x^3 + 2x^2 + 3x + 1}{-x^4 + 12x^2 - 3x}\right)}{2}$$

$$f_{impar}(t) = \frac{\left(\frac{5x^3 + 2x^2 - 3x + 1}{-x^4 + 12x^2 + 3x}\right) - \left(\frac{-5x^3 + 2x^2 + 3x + 1}{-x^4 + 12x^2 - 3x}\right)}{2}$$

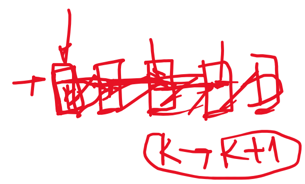






10 Transformada de la derivada n-ésima de $f(t)$ contínua

Demostración por inducción:



P(n)

i) P(1) verificar

ii) P(k) -> P(k+1)

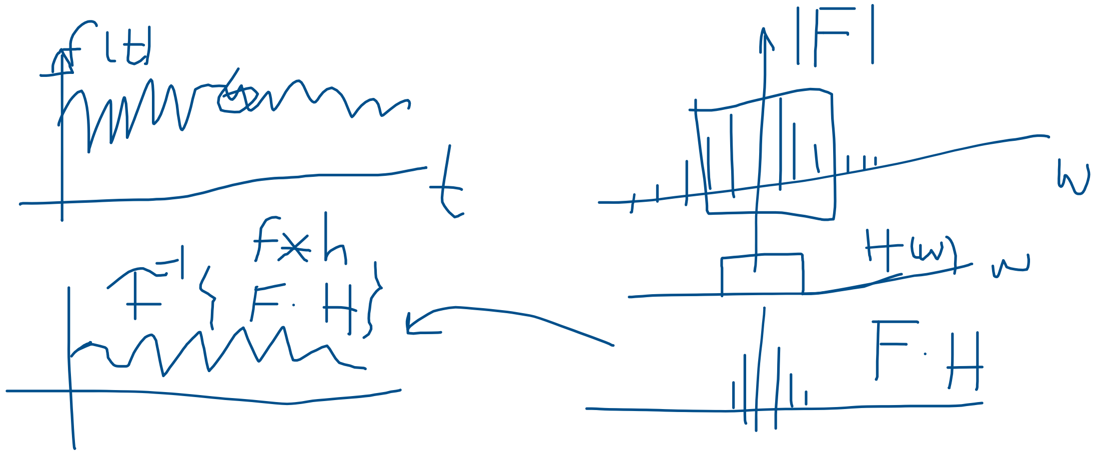
1) Verificar para n=1 (se aplican las condiciones de Dirichlet)

$$\mathcal{F}\{f'(t)\} = \int_{-\infty}^{\infty} f'(t)e^{-iwt}dt = uv - \int vdu = \left[e^{-iwt}f(t)\right]_{-\infty}^{\infty} + iw\int_{-\infty}^{\infty} f(t)e^{-iwt}dt = iwF(w)$$

2) Paso inductivo $\mathcal{F}\{f^{(k)}(t)\} = (iw)^k F(w) => \mathcal{F}\{f^{(k+1)}(t)\} = (iw)^{k+1}F(w)$

$$\mathcal{F}\{f^{(k+1)}(t)\} = \int_{-\infty}^{\infty} f^{(k+1)}(t)e^{-iwt}dt = uv - \int vdu = \left[e^{-iwt}f^{(k)}(t)\right]_{-\infty}^{\infty} + iw\int_{-\infty}^{\infty} f^{(k)}(t)e^{-iwt}dt$$

$$= iw\,\mathcal{F}\{f^{(k)}(t)\} = iw\,(iw)^k F(w) = (iw)^{k+1}F(w)$$

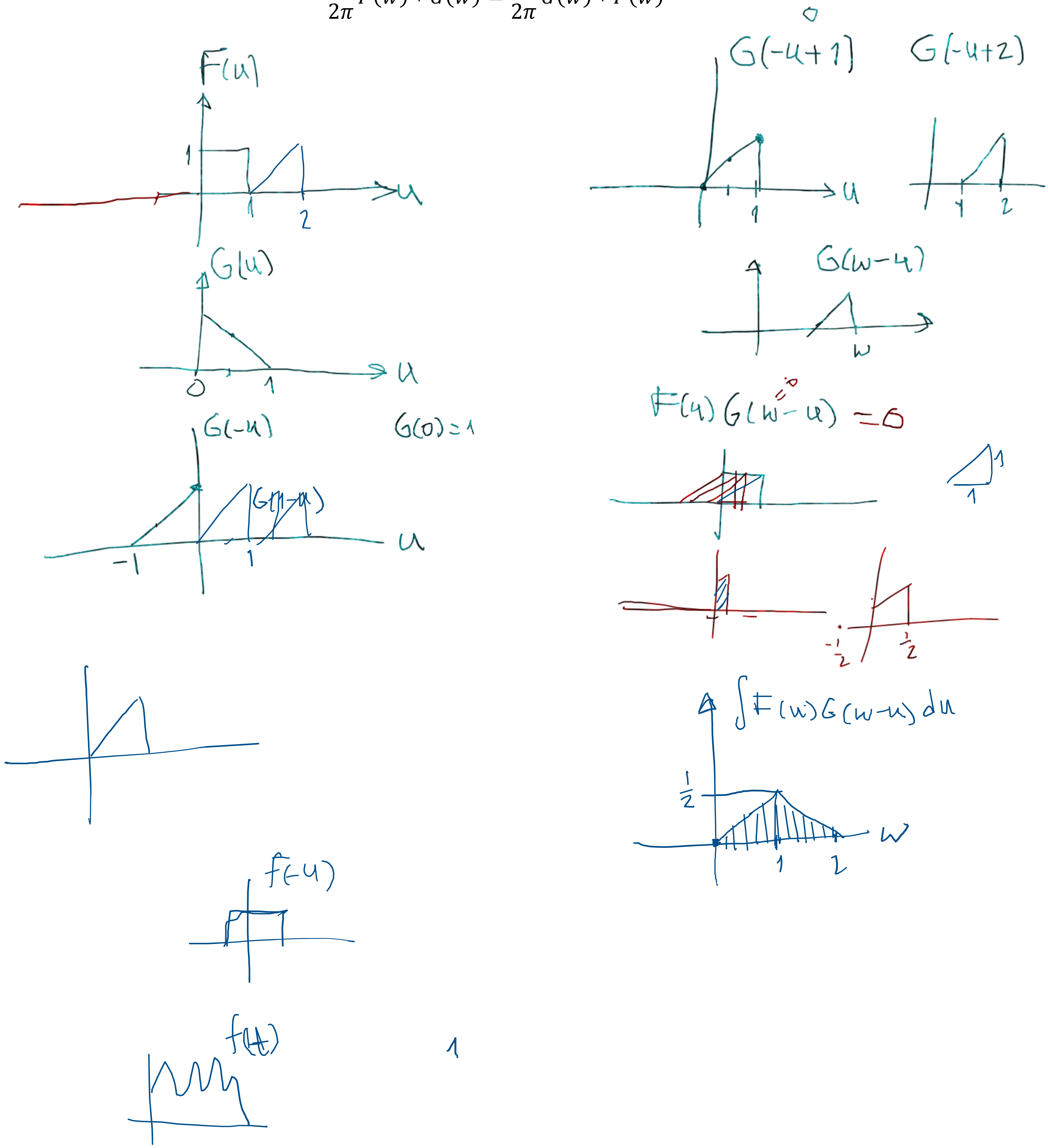


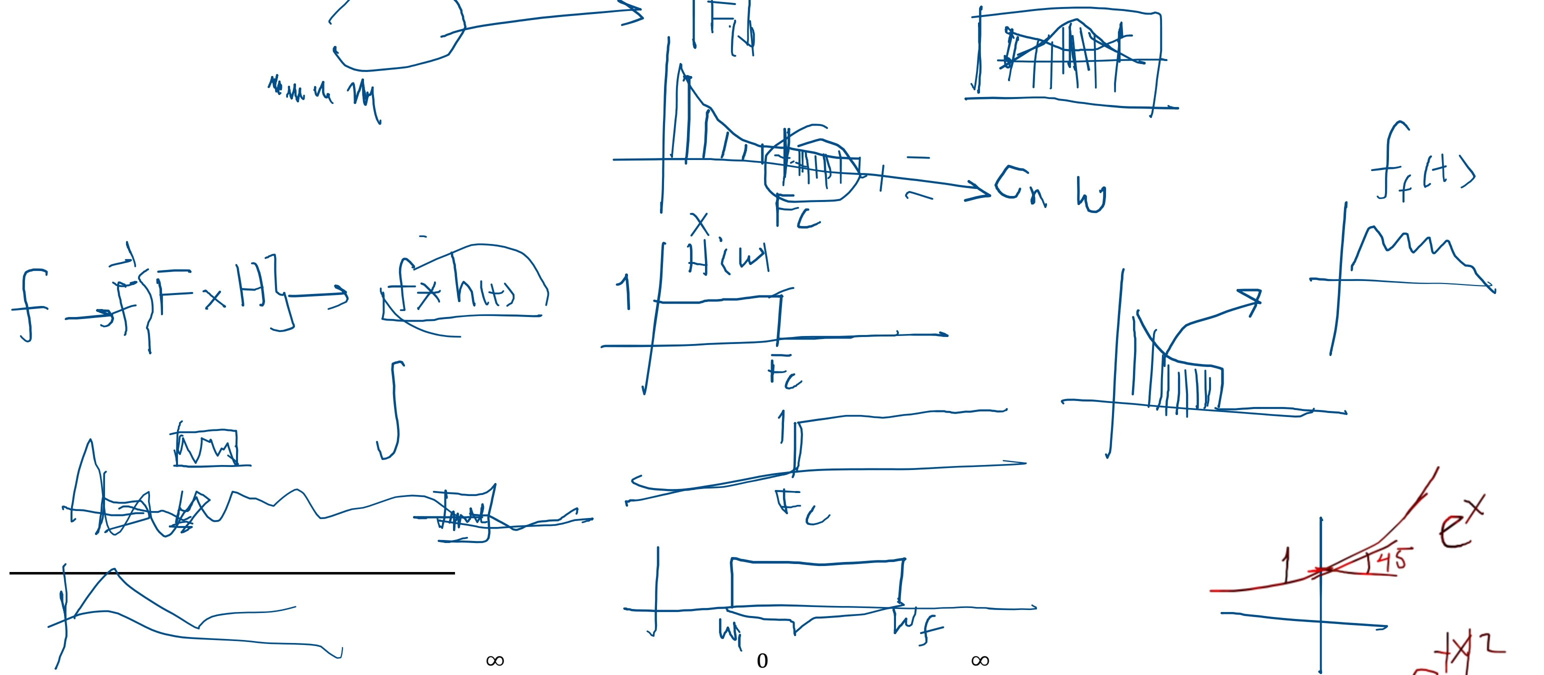
11 Convolución en la frecuencia

$$H(w) = \mathcal{F}\{f(t)g(t)\} = \int_{-\infty}^{\infty} f(t)g(t)e^{-iwt}dt = \int_{t=-\infty}^{\infty} \left(\frac{1}{2\pi} \int_{u=-\infty}^{\infty} F(u)e^{iut}du \right) g(t)e^{-iwt}dt$$

$$= \frac{1}{2\pi} \int_{u=-\infty}^{\infty} F(u) \left(\int_{t=-\infty}^{\infty} g(t)e^{-i(w-u)t}dt \right) du = \frac{1}{2\pi} \int_{u=-\infty}^{\infty} F(u)G(w-u)du$$

$$\frac{1}{2\pi}F(w) * G(w) = \frac{1}{2\pi}G(w) * F(w)$$

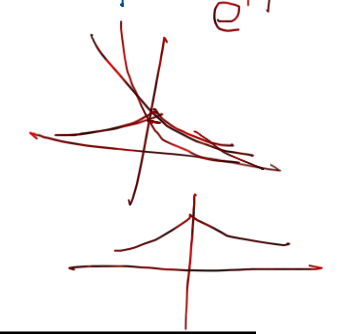




$$\mathcal{F}\{e^{-a|t|}\} = \int_{-\infty}^{\infty} e^{-a|t|} e^{-iwt} dt = \int_{-\infty}^{0} e^{at} e^{-iwt} dt + \int_{0}^{\infty} e^{-at} e^{-iwt} dt$$

$$= \int_{-\infty}^{0} e^{(a-iw)t} dt + \int_{0}^{\infty} e^{-(a+iw)t} dt = \frac{e^{(a-iw)0}}{(a-iw)} - \frac{e^{-(a+iw)0}}{-(a+iw)}$$

$$= \frac{1}{(a-iw)} + \frac{1}{(a+iw)} = \frac{(a+iw)+(a-iw)}{a^2+w^2} = \frac{2a}{a^2+w^2}$$



Ejemplo: Cálculo de Transformadas de Fourier usando las propiedades:

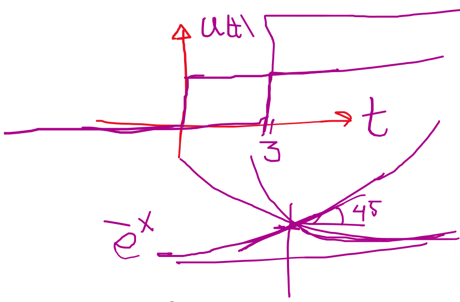
$$f(t) = 2H(t-3)e^{-4t}$$

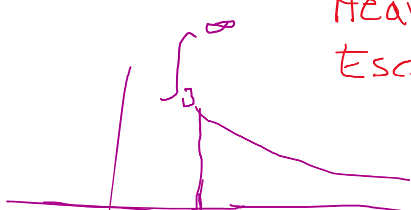
Heaviside
Escalon

$f(t-3)$

$$H(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$U(t)$





Solución: Completaremos la función para poder aplicar la propiedad de corrimiento en el tiempo:

$$f(t) = 2H(t-3)e^{-4(t-3+3)}$$

$$f(t) = 2H(t-3)e^{-4(t-3)-12}$$

$$f(t) = 2\,H(t-3)\,e^{-4(t-3)}\,e^{-12} = 2\,e^{-12}\,H(t_1)\,e^{-4t_1}$$ donde $t_1 = t-3 = t-t_0$

$\mathcal{F}\{f(t-t_0)\} = F(...)e^{...}$

De la fórmula 16 tenemos: $\mathcal{F}\{H(t_1)e^{-4t_1}\} = \frac{1}{iw+4}$

De la fórmula 3 (corrimiento en el tiempo) tenemos finalmente la transformada:

$$F(w) = 2\,e^{-12}\frac{1}{iw+4}\,e^{-iw3}$$

Ejemplo: Cálculo de la Transformada Inversa de Fourier usando las propiedades:

$$F(w) = \frac{1+iw}{6-w^2+5iw}$$

Solución: Factorizaremos el denominador y separamos en fracciones parciales

$$F(w) = \frac{1+iw}{-w^2+5iw+6} = \frac{1+iw}{(iw+3)(iw+2)} = \frac{A}{(iw+3)} + \frac{B}{(iw+2)}$$

Multiplicando por el denominador $(iw+3)(iw+2)$ obtenemos una ecuación válida para todo w:

$$1 + iw = A(iw+2) + B(iw+3)$$

$$1 + iw = iw(A + B) + 2A + 3B$$

Implica 2A+3B=1 y A+B=1 y se resuelven las ecuaciones simultaneas, o bien

Si hacemos $iw = -3$, $1 - 3 = A(-3 + 2) + B(-3 + 3)$
podemos despejar fácilmente $A = 2$
Si hacemos $iw = -2, 1 - 2 = A(-2 + 2) + B(-2 + 3)$
podemos despejar fácilmente $B = -1$

$$F(w) = \frac{2}{(iw + 3)} - \frac{1}{(iw + 2)}$$

Aplicando la fórmula 16 a los dos términos se obtiene finalmente la transformada inversa:

$$f(t) = 2\, H(t)\, e^{-3t} - H(t)\, e^{-2t}$$

Ejemplo: Aplicación de la Transformada de Fourier para resolver una ecuación diferencial

$$y'' + 6y' + 5y = \delta(t - 3)$$

Solución: Aplicaremos la transformada de Fourier a la ecuación diferencial, usaremos las fórmulas 10 para las derivadas, 3 para la delta y 14 para el corrimiento en el tiempo de la delta, obtenemos:

$$(iw)^2\, Y(w) + 6(iw)Y(w) + 5\, Y(w) = 1\, e^{-iw3}$$

Despejemos Y(w) para aplicar la inversa y obtener y(t)

$$Y(w) = \frac{1}{(iw)^2 + 6(iw) + 5}\, e^{-iw3}$$

Podemos factorizar el denominador y aplicar fracciones parciales, el factor e^{-iw3} lo dejaremos pendiente para aplicar el corrimiento en el tiempo (fórmula 3).

$$\frac{1}{(iw)^2 + 6(iw) + 5} = \frac{1}{(iw + 5)(iw + 1)} = \frac{A}{(iw + 5)} + \frac{B}{(iw + 1)}$$

Multiplicando por el denominador obtenemos una ecuación válida para todo w:

$$1 = A(iw + 1) + B(iw + 5)$$

Si hacemos $iw = -5$ podemos despejar fácilmente $A = -\frac{1}{4}$

Si hacemos $iw = -1$ podemos despejar fácilmente $B = \frac{1}{4}$ y obtenemos

$$\frac{1}{(iw)^2 + 6(iw) + 5} = \frac{1}{4}\left(\frac{1}{(iw + 1)} - \frac{1}{(iw + 5)}\right)$$

Aplicando la fórmula 16 a los dos términos se obtiene la transformada inversa de estos términos:

$$\mathcal{F}^{-1}\left\{\frac{1}{4}\left(\frac{1}{(iw + 1)} - \frac{1}{(iw + 5)}\right)\right\} = \frac{1}{4}\left(H(t)\, e^{-t} - H(t)\, e^{-5t}\right)$$

Considerando ahora el factor e^{-iw3} que quedó pendiente, usamos la propiedad de corrimiento en el tiempo (fórmula 3) y tenemos finalmente la solución de la ecuación diferencial:

$$y(t) = \frac{1}{4}\left(H(t - 3)\, e^{-(t-3)} - H(t - 3)\, e^{-5(t-3)}\right)$$

1

In each of Problems 1 through 15, find the Fourier transform of the function and graph the amplitude spectrum. Wherever k appears it is a positive constant. Use can be made of the following transforms:

$$\mathcal{F}[e^{-kt^2}](\omega) = \sqrt{\frac{\pi}{k}}e^{-\omega^2/4k}$$

and

$$\mathcal{F}\left[\frac{1}{k^2+t^2}\right](\omega) = \frac{\pi}{k}e^{-k|\omega|}$$

1. $f(t) = \begin{cases} 1 & \text{for } 0 \le t \le 1 \\ -1 & \text{for } -1 \le t < 0 \\ 0 & \text{for } |t| > 1 \end{cases}$
2. $f(t) = \begin{cases} \sin(t) & \text{for } -k \le t \le k \\ 0 & \text{for } |t| > k \end{cases}$
3. $f(t) = 5[H(t-3) - H(t-11)]$
4. $f(t) = 5e^{-3(t-5)^2}$
5. $f(t) = H(t-k)e^{-t/4}$
6. $f(t) = H(t-k)t^2$
7. $f(t) = 1/(1+t^2)$
8. $f(t) = 3H(t-2)e^{-3t}$
9. $f(t) = 3e^{-4|t+2|}$
10. $f(t) = H(t-3)e^{-2t}$

In each of Problems 11 through 15, find the inverse Fourier transform of the function.

11. $9e^{-(\omega+4)^2/32}$
12. $e^{(20-4\omega)i}/(3-(5-\omega)i)$
13. $e^{(2\omega-6)i}/(5-(3-\omega)i)$
14. $10\sin(3\omega)/(\omega+\pi)$
15. $(1+i\omega)/(6-\omega^2+5i\omega)$ *Hint*: Factor the denominator and use partial fractions.

In each of Problems 16, 17, and 18, use convolution to find the inverse Fourier transform of the function.

16. $1/((1+i\omega)(2+i\omega))$
17. $1/(1+i\omega)^2$
18. $\sin(3\omega)/\omega(2+i\omega)$
19. Prove the following version of Parseval's theorem:

$$\int_{-\infty}^{\infty} |f(t)|^2\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2\, d\omega.$$

20. Compute the total energy of the signal $f(t) = H(t)e^{-2t}$.
21. Compute the total energy of the signal $f(t) = (1/t)\sin(3t)$. *Hint*: Use Parseval's theorem, Problem 19.
22. Use the Fourier transform to solve

$$y'' + 6y' + 5y = \delta(t-3).$$

In each of Problems 23 through 28, compute the windowed Fourier transform of f for the given window function w. Also compute the center and RMS bandwidth of the window function.

23. $f(t) = t^2, \quad w(t) = \begin{cases} 1 & \text{for } -5 \le t \le 5, \\ 0 & \text{for } |t| > 5. \end{cases}$

Resolver el problema 3

Tiempo estimado: 5 minutos…

$$f(t) = 5(H(t-3) - H(t-11))\,;$$

$$F(w) = \frac{1}{-iw}5\int_3^{11} e^{-iwt}(-iwdt) = \frac{5}{-iw}\left(e^{-iw11} - e^{-iw3}\right)$$

$$= \frac{5}{-iw}\left(e^{-iw11} - e^{-iw3}\right)$$

$$= \frac{5}{-iw}\left(e^{-iw4} - e^{iw4}\right)e^{-iw7}$$

$$= \frac{10}{w}\frac{\left(e^{iw4} - e^{-iw4}\right)}{2i}e^{-iw7}$$

$$= \frac{10}{\omega} sen(4\omega) e^{-i\omega 7}$$

c) Usando las propiedades

$$f(t) = 5\big(H(t-3) - H(t-11)\big)$$

$$F(\omega) = 5\left(\left(\pi\,\delta(w) + \frac{1}{iw}\right) e^{-iw3} - \left(\pi\,\delta(w) + \frac{1}{iw}\right) e^{-iw11}\right)$$

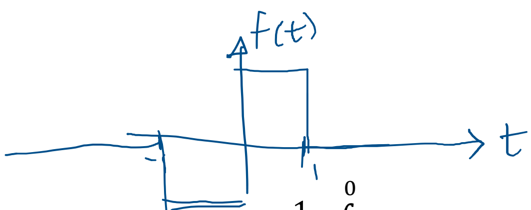
$$= 5\left(\left(\pi\,\delta(w) + \frac{1}{iw}\right) e^{-iw3} - \left(\pi\,\delta(w) + \frac{1}{iw}\right) e^{-iw11}\right)$$

$$= 5\left(\left(+\frac{1}{iw}\right) e^{-iw3} - \left(+\frac{1}{iw}\right) e^{-iw11}\right)$$

$$= \frac{5}{iw}\left(e^{iw4} - e^{-iw4}\right) e^{-iw7}$$

$$= \frac{10}{w} \frac{\left(e^{iw4} - e^{-iw4}\right)}{2i} e^{-iw7}$$

$$= \frac{10}{\omega} sen(4\omega) e^{-i\omega 7}$$



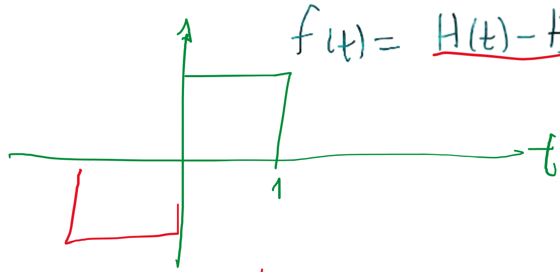
$$F(w) = \frac{1}{iw}\int_{-1}^{0} e^{-iwt}(-iw)dt + \frac{1}{-iw}\int_{0}^{1} e^{-iwt}(-iw)dt = \frac{1-e^{iw}}{iw} + \frac{e^{-iw}-1}{-iw}$$

$$= \frac{1-e^{iw}}{iw} + \frac{-e^{-iw}+1}{iw} = \frac{2-e^{iw}-e^{-iw}}{iw} = \frac{2}{iw} - \frac{2}{iw}\left(\frac{e^{iw}+e^{-iw}}{2}\right)$$

$$= \frac{2}{iw}(1-\cos(w)) = i\frac{2}{w}(\cos(w)-1)$$

$U(t)$

$f(t) = H(t) - H(t-1) + H(-t) - H(-t-1)$



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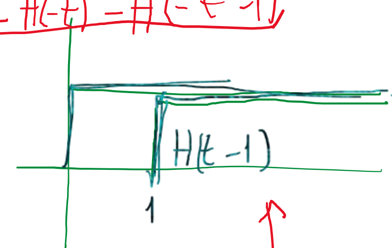
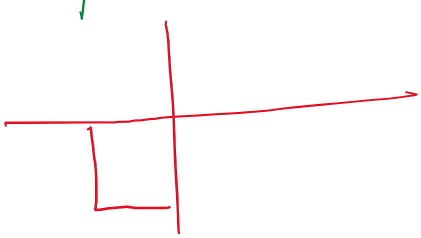
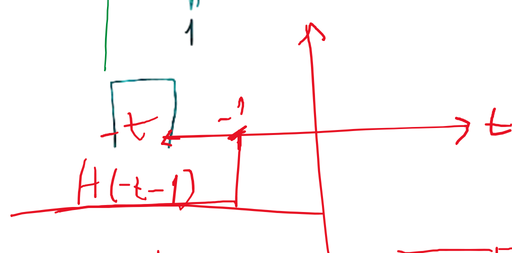
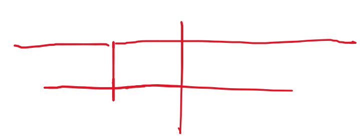


$H(t) - H(t-1)$







$H(-t) - H(-t-1)$

$H(-(t+1))$

$H(t)$

1

$-(-1)$

$H(t+1)$

4 $f(t) = 5e^{-3(t-5)^2}$

$$\mathcal{F}\left\{5e^{-3(t-5)^2}\right\} = 5\mathcal{F}\left\{e^{-3(t-5)^2}\right\} = 5\mathcal{F}\left\{e^{-3t_1^2}\right\} e^{-iw5} = 5\sqrt{\frac{\pi}{3}}e^{-\frac{w^2}{12}} e^{-iw5} = 5\sqrt{\frac{\pi}{3}}e^{-\left(\frac{w^2}{12}+i5w\right)}$$

Section 14.3 The Fourier Transform

1. $2i[\cos(\omega) - 1]/\omega$ **3.** $10e^{-7i\omega}\sin(4\omega)/\omega$

5. $\frac{4}{1+4i\omega}e^{-(1+4i\omega)k/4}$ **7.** $\pi e^{-|\omega|}$

9. $\frac{24}{16+\omega^2}e^{2i\omega}$

11. $18\sqrt{\frac{2}{\pi}}e^{-8t^2}e^{-4it}$ **13.** $H(t+2)e^{-10-(5-3i)t}$

15. $H(t)[2e^{-3t} - e^{-2t}]$ **17.** $H(t)te^{-t}$

19. $\int_{-\infty}^{\infty}|f(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\omega)\overline{\hat{f}(\omega)}\,d\omega = \frac{1}{2\pi}|\hat{f}(\omega)|^2\,d\omega$

21. 3π

23. $(2/\omega^3)[25\omega^2\sin(5\omega) + 10\omega\cos(5\omega) - 2\sin(5\omega)]$

25. $\frac{1}{1+\omega^2}(1 - e^{-4}\cos(4\omega) + e^{-4}\sin(4\omega)) + \frac{i}{1+\omega^2}(e^{-4}\sin(4\omega) + (e^{-4}\cos(4\omega) - 1)\omega)$

27. $\frac{4}{\omega^3}(\sin(2\omega)(4\omega^2 - 1) + 2\omega\cos(2\omega)) + \frac{8i}{\omega^2}(2\omega\cos(2\omega) - \sin(2\omega))$

1

$$f(t) = \begin{cases} 1 & 0 \le t \le 1 \\ -1 & -1 \le t < 0 \\ 0 & |t| < 1 \end{cases}$$

$$f(t) = P_1\left(t - \frac{1}{2}\right) - P_1\left(t + \frac{1}{2}\right)$$

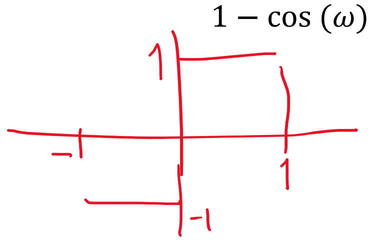
$$F(\omega) = 1\frac{sen\left(\frac{\omega}{2}\right)}{\frac{\omega}{2}}\ e^{-\frac{i\omega}{2}} - \frac{sen\left(\frac{\omega}{2}\right)}{\frac{\omega}{2}}\ e^{+\frac{i\omega}{2}}$$

$$= 1\frac{sen\left(\frac{\omega}{2}\right)}{\frac{\omega}{2}}\frac{\left(e^{-\frac{i\omega}{2}} - e^{+\frac{i\omega}{2}}\right)}{2i}2i \ = -\frac{sen\left(\frac{\omega}{2}\right)}{\frac{\omega}{2}}sen\left(\frac{\omega}{2}\right)\ 2i$$

$$= 2i\frac{\cos(\omega) - 1}{\omega}$$

$$cos\left(\frac{\omega}{2} - \frac{\omega}{2}\right) - cos\left(\frac{\omega}{2} + \frac{\omega}{2}\right) = 2sen\left(\frac{\omega}{2}\right)sen\left(\frac{\omega}{2}\right)$$

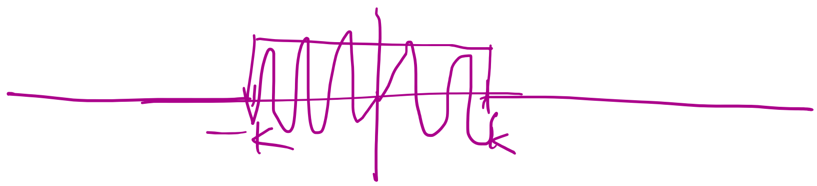
$$1 - \cos{(\omega)} = 2sen\left(\frac{\omega}{2}\right)sen\left(\frac{\omega}{2}\right)$$



$$\int_{-\infty}^{\infty} f(t)e^{-iwt}dt = -\int_{-1}^{0} 1\, e^{-iwt}dt + \int_{0}^{1} 1\, e^{-iwt}dt = -\frac{1-e^{iw}}{-iw} + \frac{e^{-iw}-1}{-iw}$$

$$= \frac{\frac{e^{iw}+e^{-iw}}{2}2-2}{-iw} = 2i\frac{\cos(\omega)-1}{w}$$

2



a) Usando propiedades

$$f(t) = sen(t)\, P_{2k}(t)$$

$$P_{2k}(t) \to 2k\frac{sen(wk)}{wk}$$

$$\frac{1}{2i}\big(F(w-w_0) - F(w+w_0)\big) = \frac{1}{i}\left(\frac{sen((w-1)k)}{(w-1)} - \frac{sen((w+1)k)}{(w+1)}\right)$$

$$\left(\frac{sen\big(k(w-1)\big)}{i(w-1)} - \frac{sen\big(k(1+w)\big)}{i(1+w)}\right)$$

b) Integrando

$$F(w) = \int_{-k}^{k} sen(t)e^{-iwt}dt = \int_{-k}^{k} sen(t)\big(\cos(wt) - isen(wt)\big)dt$$

$$= \int_{-k}^{k} \frac{e^{ti}-e^{-it}}{2i}e^{-iwt}dt = \int_{-k}^{k} \frac{e^{it(1-w)} - e^{-it(1+w)}}{2i}dt = \frac{1}{2i}\left(\frac{e^{it(1-w)}}{i(1-w)} - \frac{e^{-it(1+w)}}{-i(1+w)}\right)_{-k}^{k}$$

$$= \frac{1}{2i}\left(\frac{e^{ik(1-w)}}{i(1-w)} - \frac{e^{-ik(1-w)}}{i(1-w)} - \frac{e^{-ik(1+w)}}{-i(1+w)} + \frac{e^{ik(1+w)}}{-i(1+w)}\right) = \left(\frac{sen\big(k(1-w)\big)}{i(1-w)} + \frac{sen\big(k(1+w)\big)}{-i(1+w)}\right)$$

7

$$\mathcal{F}\left\{\frac{1}{1+t^2}\right\} = \pi e^{-|\omega|}$$

11

$$\mathcal{F}^{-1}\left\{9e^{-\frac{(\omega+4)^2}{32}}\right\} = 9\mathcal{F}^{-1}\left\{e^{-\frac{(\omega-(-4))^2}{32}}\right\}$$

$$= \sqrt{\frac{8}{\pi}}9\mathcal{F}^{-1}\left\{\sqrt{\frac{\pi}{8}}e^{-\frac{(w_1)^2}{4(8)}}\right\} e^{i(-4)t}$$

$$= \sqrt{\frac{8}{\pi}}9(e^{-8t^2})\, e^{i(-4)t} = 18\sqrt{\frac{2}{\pi}}e^{-8t^2}e^{-i4t}$$

13

$$\mathcal{F}^{-1}\left\{\frac{e^{(2\omega-6)i}}{5-(3-\omega)i}\right\}=\mathcal{F}^{-1}\left\{\frac{e^{2(\omega-3)i}}{5+(\omega-3)i}\right\}$$
$$=\mathcal{F}^{-1}\left\{\frac{1}{5+(\omega_1)i}e^{-i(\omega_1)(-2)}\right\}\ e^{i3t} por\ propiedad\ de\ corrimiento\ en\ la\ frecuencia$$
$$=\mathcal{F}^{-1}\left\{\frac{1}{5+\omega i}\right\}_{t\to t-(-2)}\ e^{i3t} por\ Corrimiento\ en\ el\ tiempo$$
$$=\{H(t)e^{-5t}\}_{t\to t-(-2)}\ e^{i3t}=H(t+2)e^{-5(t+2)}\ e^{i3t}$$
$$=H(t+2)e^{3it-5(t+2)}=H(t+2)e^{-10-(5-3i)t}$$

---

15 $\mathcal{F}^{-1}\left\{\frac{1+i\omega}{(i\omega)^2+5(i\omega)+6}\right\}=\mathcal{F}^{-1}\left\{\frac{1+i\omega}{-(w-2i)(w-3i)}\right\}=\mathcal{F}^{-1}\left\{\frac{1+i\omega}{(iw+2)(iw+3)}\right\}=\mathcal{F}^{-1}\left\{\frac{A}{(iw+2)}+\frac{B}{iw+3}\right\}$

$$=+2\,\mathcal{F}^{-1}\left\{\frac{1}{iw+3}\right\}-\mathcal{F}^{-1}\left\{\frac{1}{(iw+2)}\right\}=H(t)(+2\,e^{-3t}-1\,e^{-2t})$$
$$\omega=\frac{-5i\pm\sqrt{-25+24}}{-2}=\frac{-5i\pm i}{-2}=2i,3i$$

$$\frac{1+i\omega}{(iw+2)(iw+3)}=\frac{A}{(iw+2)}+\frac{B}{iw+3}$$
$$1+i\omega=A(iw+3)+B(iw+2)$$

Haciendo iw=-2, A=-1
Haciendo iw=-3, B=2

16

a) Por convolución

$$H(\omega)=\frac{1}{(1+i\omega)(2+i\omega)}=\frac{1}{(1+i\omega)}\frac{1}{(2+i\omega)}=F(w)G(w)$$
$$f(u)=H(u)\,e^{-u}$$
$$g(t-u)=H(t-u)\,e^{-2(t-u)}$$
$$h(t)=\int_{-\infty}^{\infty}f(u)g(t-u)du=\int_{-\infty}^{\infty}H(u)\,e^{-u}\,H(t-u)\,e^{-2(t-u)}du$$
$$=\int_0^{\infty}e^{-u}\,H(t-u)\,e^{-2(t-u)}du=H(t)\int_0^{t}e^{-u}\ e^{-2(t-u)}du$$
$$=H(t)\int_0^{t}e^{-u-2t+2u}\,du=H(t)e^{-2t}\int_0^{t}e^{u}du=H(t)e^{-2t}(e^{t}-1)=H(t)(e^{-t}-e^{-2t})$$

b) Por fracciones parciales

$$\frac{1}{(1+i\omega)(2+i\omega)}=\frac{A}{(1+i\omega)}+\frac{B}{(2+i\omega)}$$

$$1=A(2+i\omega)+B(1+i\omega)$$
iw=-2 =>B=-1
iw=-1 => A=1
$$\frac{1}{(1+i\omega)(2+i\omega)}=\frac{1}{(1+i\omega)}-\frac{1}{(2+i\omega)}$$
$$H(t)\,e^{-1t}-H(t)\,e^{-2t}=H(t)(e^{-t}-e^{-2t})$$

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17 Por convolución

$$\mathcal{F}^{-1}\left\{\frac{1}{(1+i\omega)^2}\right\} = \mathcal{F}^{-1}\left\{\frac{1}{(1+i\omega)}\frac{1}{(1+i\omega)}\right\} = \int_{-\infty}^{\infty} f(u)\, g(t-u)du$$

$$f(t) = H(t)\, e^{-t}$$
$$g(t) = H(t)\, e^{-t}$$
$$f(u) = H(u)\, e^{-u}$$
$$g(t-u) = H(t-u)\, e^{-(t-u)}$$

$$\mathcal{F}^{-1}\left\{\frac{1}{(1+i\omega)^2}\right\} = \int_{-\infty}^{\infty} H(u)\, e^{-u}\; H(t-u)\, e^{-(t-u)}du =$$

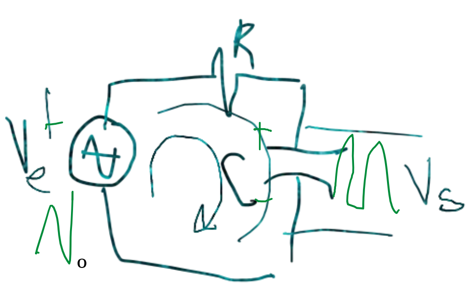
$$= \int_{0}^{\infty} e^{-u}\; H(t-u)\, e^{-(t-u)}du$$

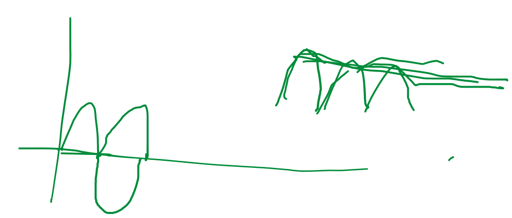
$$= H(t)\int_{0}^{t} e^{-u}\, e^{-(t-u)}du = H(t)e^{-t}\int_{0}^{t} 1\, du = H(t)e^{-t}t$$

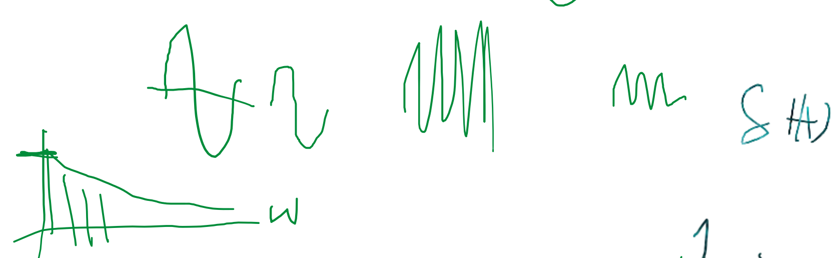
$$E_5 = \frac{1}{2\pi}\int_{-\pi}^{\pi} t^2\, dt - \frac{1}{2}\left[\left(\frac{2}{1}\right)^2 + \left(\frac{2}{2}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{4}\right)^2 + \left(\frac{2}{5}\right)^2\right] = \frac{\pi^2}{3} - \frac{1}{2}\left[\left(\frac{2}{1}\right)^2 + \left(\frac{2}{2}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{4}\right)^2 + \left(\frac{2}{5}\right)^2\right]$$
$$= \frac{\pi^2}{3} - \frac{1}{2}\left[\left(\frac{2}{1}\right)^2 + \left(\frac{2}{2}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{4}\right)^2 + \left(\frac{2}{5}\right)^2\right] = 0.3626459$$

Ejemplo de función de transferencia en un sistema continuo
Sistema RC

$$\boxed{R\, i + v_s(t) = v_e(t)}$$

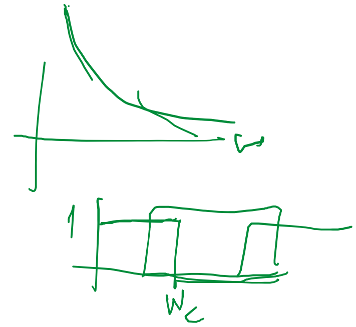






$$R\, i + \frac{1}{C}q(t) = v_e(t)$$
$$R\frac{dq(t)}{dt} + \frac{1}{C}q(t) = v_e(t)$$
$$(iw)RQ(w) + \frac{1}{C}Q(w) = V_e(w)$$

H



$$\left((iw)RC+1\right)\frac{Q(w)}{C}=V_e(w)$$

$$V_s(w)=\frac{Q(w)}{C}=\left[\frac{1}{(iw)RC+1}\right]V_e(w)$$

$$H(w)=\frac{V_s(w)}{V_e(w)}=\frac{1}{RC}\left[\frac{1}{(iw)+\frac{1}{RC}}\right]\ \textit{Función de transferencia}$$

$$h(t)=\frac{1}{RC}u(t)e^{-\frac{t}{RC}}\ \textit{Respuesta al impulso unitario}$$

$$V_s(w)=H(w)V_e(w)$$

$$v_s(t)=h(t)*v_e(t)$$

Si $v_e=\delta(t)$

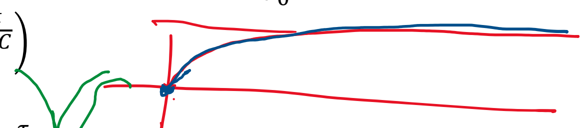
$$v_s(t)=h(t)*\delta(t)=h(t)$$

$$|H(w)|^2=\left(\frac{1}{RC}\right)^2\left[\frac{1}{\left(\frac{1}{RC}\right)^2+w^2}\right]$$

Si $v_e=u(t)$

$$v_s(t)=h(t)*u(t)=\int_{-\infty}^{\infty}\frac{1}{RC}\cancel{u}(\tau)e^{-\frac{\tau}{RC}}u(t-\tau)d\tau=\frac{1}{RC}u(t)\left(-\frac{1}{RC}\right)^{-1}\int_0^t e^{-\frac{\tau}{RC}}\left(-\frac{1}{RC}\right)d\tau$$

$$=u(t)\left(1-e^{-\frac{t}{RC}}\right)$$

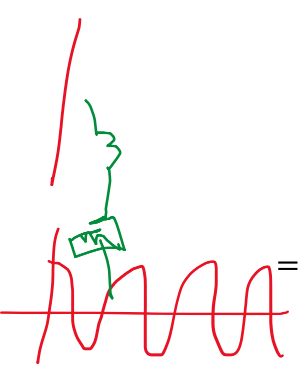


Si $v_e=sen(w_0t)$

$$v_s(t)=h(t)*sen(w_0t)=\frac{1}{2i}\int_{-\infty}^{\infty}\frac{1}{RC}u(\tau)e^{-\frac{\tau}{RC}}\left(e^{iw_0(t-\tau)}-e^{-iw_0(t-\tau)}\right)d\tau$$

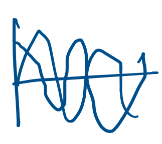
$$=\frac{1}{2i}\frac{1}{RC}\left(e^{iw_0t}\int_0^{\infty}e^{\left(-\frac{1}{RC}-iw_0\right)\tau}-e^{-iw_0t}\int_0^{\infty}e^{\left(-\frac{1}{RC}+iw_0\right)\tau}\right)d\tau$$

$$=\frac{1}{2i}\frac{1}{RC}\left(\frac{e^{iw_0t}}{\left(\frac{1}{RC}+iw_0\right)}-\frac{e^{-iw_0t}}{\left(\frac{1}{RC}-iw_0\right)}\right)=\frac{1}{2i}\left(\frac{e^{iw_0t}}{(1+iw_0RC)}-\frac{e^{-iw_0t}}{(1-iw_0RC)}\right)$$



$$=\frac{1}{2i}\left(\frac{(1-iw_0RC)e^{iw_0t}-(1+iw_0RC)e^{-iw_0t}}{(1+(w_0RC)^2)}\right)=\frac{1}{2i}\left(\frac{-iw_0RC\left(e^{iw_0t}+e^{-iw_0t}\right)}{(1+(w_0RC)^2)}\right)$$

$$=-\left(\frac{w_0RC\cos(w_0t)}{(1+(w_0RC)^2)}\right)$$



Si $v_e = cos(w_0 t)$

$$v_s(t) = h(t) * cos(w_0 t) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{RC} u(\tau) e^{-\frac{\tau}{RC}} \left(e^{iw_0(t-\tau)} + e^{-iw_0(t-\tau)}\right) d\tau$$

$$= \frac{1}{2}\frac{1}{RC}\left(e^{iw_0 t}\int_0^{\infty} e^{\left(-\frac{1}{RC}-iw_0\right)\tau} + e^{-iw_0 t}\int_0^{\infty} e^{\left(-\frac{1}{RC}+iw_0\right)\tau}\right) d\tau$$

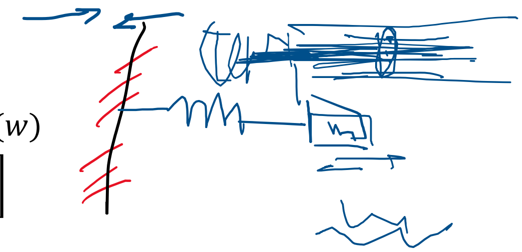
$$= \frac{1}{2}\frac{1}{RC}\left(\frac{e^{iw_0 t}}{\left(\frac{1}{RC}+iw_0\right)} + \frac{e^{-iw_0 t}}{\left(\frac{1}{RC}-iw_0\right)}\right) = \frac{1}{2}\left(\frac{e^{iw_0 t}}{(1+iw_0 RC)} + \frac{e^{-iw_0 t}}{(1-iw_0 RC)}\right)$$

$$= \frac{1}{2}\left(\frac{(1-iw_0 RC)e^{iw_0 t} + (1+iw_0 RC)e^{-iw_0 t}}{(1+(w_0 RC)^2)}\right) = \frac{1}{2}\left(\frac{\left(e^{iw_0 t}+e^{-iw_0 t}\right) - iw_0 RC\left(e^{iw_0 t}-e^{-iw_0 t}\right)}{(1+(w_0 RC)^2)}\right)$$

$$= \left(\frac{\cos(w_0 t) + w_0 RC sen(w_0 t)}{(1+(w_0 RC)^2)}\right)$$

Sistema Masa-Resorte-Amortiguado

$$F = \boxed{mx'' = -kx - cx' + f_e(t)}$$



$$m(iw)^2 X(w) + c(iw)X(w) + kX(w) = F_e(w)$$

$$H(w) = \frac{X_s(w)}{F_e(w)} = \left[\frac{1}{m(iw)^2 + c(iw) + k}\right]$$

$$= \frac{1}{m}\left[\frac{1}{\left(iw + i\sqrt{\frac{k}{m}}\right)\left(iw - i\sqrt{\frac{k}{m}}\right)}\right] \text{Función de transferencia para } c = 0$$

KX

$$\frac{1}{2im}\sqrt{\frac{m}{k}}\left[\frac{1}{\left(iw - i\sqrt{\frac{k}{m}}\right)} - \frac{1}{\left(iw + i\sqrt{\frac{k}{m}}\right)}\right]$$

$$h(t) = \frac{1}{2i\sqrt{km}} u(t)\left(e^{-i\sqrt{\frac{k}{m}}t} - e^{i\sqrt{\frac{k}{m}}t}\right) = -\frac{sen\left(\sqrt{\frac{k}{m}}t\right)}{\sqrt{km}} u(t) \text{ Respuesta al impulso unitario}$$

Sistema RLC

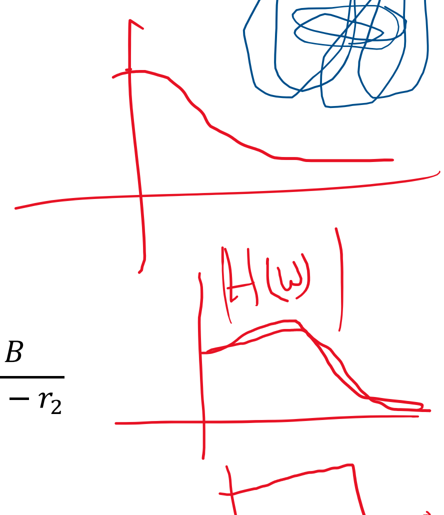
$$L\frac{di}{dt} + R\, i + \frac{1}{C} q(t) = v(t)$$

$$L\frac{d^2 q}{dt} + R\frac{dq}{dt} + \frac{1}{C} q(t) = v(t)$$

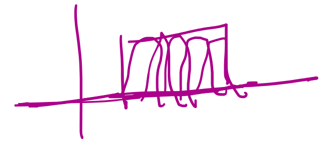
$$(iw)^2 LQ(w) + (iw)RQ(w) + \frac{1}{C} Q(w) = V(w)$$

$$V_s(w) = \frac{Q(w)}{C} = \left[\frac{1}{(iw)^2 LC + (iw)RC + 1}\right] V(w)$$

$$\frac{V_s(w)}{V_e(w)} = H(w) = \left[\frac{1}{(iw)^2 LC + (iw)RC + 1}\right] = \frac{A}{iw - r_1} + \frac{B}{iw - r_2}$$



$$iw = \frac{-RC \pm \sqrt{(RC)^2 - 4LC}}{2LC}$$
$$h(t) = u(t)[A\, e^{r_1 t} + B^{r_2 t}]$$

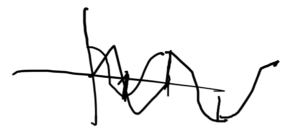


Tarea terminar

SECTION 13.6 PROBLEMS

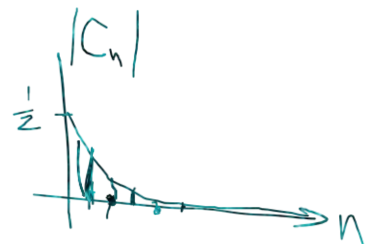
In each of Problems 1 through 7, write the complex Fourier series of f, determine the sum of the series, and plot some points of the frequency spectrum.

1. $f(x) = 2x$ for $0 \le x < 3$, period 3
2. $f(x) = x^2$ for $0 \le x < 2$, period 2
3. $f(x) = \begin{cases} 0 & \text{for } 0 \le x < 1 \\ 1 & \text{for } 1 \le x < 4 \end{cases}$, f has period 4
4. $f(x) = 1 - x$ for $0 \le x < 6$, period 6
5. $f(x) = \begin{cases} -1 & \text{for } 0 \le x < 2 \\ 2 & \text{for } 2 \le x < 4 \end{cases}$, f has period 4
6. $f(x) = e^{-x}$ for $0 \le x < 5$, period 5
7. $f(x) = \begin{cases} x & \text{for } 0 \le x < 1 \\ 2 - x & \text{for } 1 \le x < 2 \end{cases}$, f has period 2



7

$$c_n = \left|\frac{1}{2}(a_n - i\cancel{b_n})\right|$$



$$a_n = 2\int_0^1 x\cos(n\pi x)\,dx = \frac{2((-1)^n - 1)}{\pi^2 n^2}$$

$$f(x) = \frac{1}{2} + \sum_{n=1}^{10} \frac{2((-1)^n - 1)}{\pi^2 n^2}\cos(n\pi x)$$

$$= \frac{1}{2} + \sum_{n=1}^{10} \frac{2((-1)^n - 1)}{\pi^2 n^2}\,\frac{\left(e^{in\pi x} + e^{-in\pi x}\right)}{2}$$

$$f(x) = \sum_{n=-\infty}^{\infty}\left(c_n e^{i\frac{n\pi x}{L}}\right)$$

$$c_n = \frac{1}{2}\int_0^1 x e^{-in\pi x}\,dx + \frac{1}{2}\int_1^2 (2-x)e^{-in\pi x}\,dx = \frac{\cos(n\pi)\,(in\pi - \cos(n\pi) + 1) - 1\cos(\pi n)(1 - i\pi n)}{2\pi^2 n^2}$$
$$= \frac{\cos(n\pi)\,(in\pi - \cos(n\pi) + 1) - 1\cos(\pi n)(1 - i\pi n)}{2\pi^2 n^2}$$
$$= \frac{(in\pi(-1)^n) - (-1)^n 1 - (-1)^n i\pi n)}{2\pi^2 n^2}$$
$$= \frac{(-1)^{n+1}}{2\pi^2 n^2}$$

$$\frac{e^{-i\pi n}\left(i\pi n - e^{i\pi n} + 1\right)}{2\pi^2 n^2} + \frac{e^{-2i\pi n}\left(-1 + e^{i\pi n}(1 - i\pi n)\right)}{2\pi^2 n^2}$$

$$c_n = \frac{1}{2}\int_{-1}^{0} -x\mathrm{e}^{-i\frac{n\pi x}{L}}dx + \frac{1}{2}\int_{0}^{1} x\mathrm{e}^{-i\frac{n\pi x}{L}}dx = \int_{0}^{1} x\cos(n\pi x)dx$$

$$= \frac{1}{\pi^2 n^2}(nx\pi sen(nx\pi) + \cos(nx\pi)) = \frac{(-1)^n}{\pi^2 n^2}$$

(-1 + cos(n π) + n π sin(n π))/(n^2 π^2)

$$f(x) = \frac{1}{2} + \sum_{n=1}^{10}\left(\frac{2((-1)^n}{\pi^2 n^2}\frac{\left(\mathrm{e}^{in\pi x} + \mathrm{e}^{-in\pi x}\right)}{2}\right)$$

$$= \frac{1}{2} + \sum_{n=1}^{2}\left(\frac{2(-1)^n - 1}{\pi^2 n^2}\cos(n\pi x)\right)$$

$$c_0 = \frac{1}{2}\int_{0}^{1} xdx + \frac{1}{2}\int_{1}^{2}(2 - x)dx =$$

$$c_0 = \int_{0}^{1} xdx = \frac{1}{2}$$