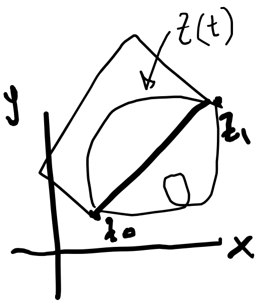
RESUMEN DE INTEGRACIÓN DE FUNCIONES DE VARIABLE COMPLEJA

Integral de una función compleja por una trayectoria C de un punto z_0 un punto z_1

Se parametriza la trayectoria C en función de un parámetro, por ejemplo, t, de manera que la integral se hace con la variable t



$$z = z(t);\ z(t_0) = z_0;\ z(t_1) = z_1$$

$$\int_{z_0}^{z_1} f(z)\,dz = \int_{t_0}^{t_1} f(z(t))\frac{dz(t)}{dt}\,dt$$

Ejemplo

$$\int_0^i z\,dz$$

$$\int_1^i z\,dz = \frac{i^2 - 1^2}{2} = -1$$

$$\int_1^i z\,dz = \int_1^i (x+iy)(dx+idy) = \int_0^1 ((1-t)+it)(-dt+idt)$$

$$= (-1+i)\int_0^1 (1-t+it)\,dt$$

$$= (-1+i)\left(t - \frac{t^2}{2} + \frac{it^2}{2}\right)_0^1$$

$$= \frac{1}{2}(-1+i)(1+i) = \frac{1}{2}(-1-1) = -1$$

$$(x,y) = (1,0) + (-1,1)t$$
$$(dx,dy) = (-1,1)dt$$

$$\int_1^i z\,dz = \int_0^{\frac{\pi}{2}} (cos(\theta) + isen(\theta))i(cos(\theta) + isen(\theta))d\theta$$

$$= \int_0^{\frac{\pi}{2}} (e^{i\theta})ie^{i\theta}d\theta = \int_0^{\frac{\pi}{2}} (e^{i\theta})ie^{i\theta}d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} (e^{2i\theta})2id\theta$$

$$= \frac{1}{2}e^{2i\theta} = \frac{1}{2}\left(e^{\frac{2i\pi}{2}} - 1\right) = \frac{1}{2}(-1-1) = -1$$

Ejemplo, sea $f(z) = z^2 = x^2 - y^2 + i\, 2xy$, integrar de a=0 a b=1+i

a) En línea recta

Parametrizando la trayectoria tenemos

$$(x, y) = (0,0) + (1,1)t$$
$$x = t; \qquad dx = dt$$
$$y = t; \qquad dy = dt$$

O también

$$z = (1 + i)t; \qquad dz = (1 + i)dt$$

$$\int_0^{1+i} z^2\, dz = \int_0^{1+i} (x^2 - y^2 + i\, 2xy)(dx + idy) = i2(1+i)\int_0^1 t^2 dt = \frac{(-2+2i)1^3}{3} = \frac{2}{3}(-1+i)$$

b) Por la curva $z(t) = t + it^2$

Parametrizando la trayectoria tenemos

$$x = t; \qquad dx = dt$$
$$y = t^2; \qquad dy = 2tdt$$

O también

$$z(t) = t + it^2\,; \quad dz = (1 + 2ti)dt$$

$$\int_0^{1+i} z^2\, dz = \int_0^1 (t^2 - t^4 + i\, 2t^3)(1 + 2ti)dt =$$

$$= \int_0^1 (t^2 - t^4 + i\, 2t^3 + 2it^3 - 2it^5 - 4t^4)dt =$$

$$\int_0^1 (-2it^5 - 5t^4 + 4it^3 + t^2)dt = \left[-\frac{2it^6}{6} - \frac{5t^5}{5} + \frac{4it^4}{4} + \frac{t^3}{3}\right]_0^1 = -\frac{2i}{6} - \frac{5}{5} + \frac{4i}{4} + \frac{1}{3} = -\frac{2}{3} + \frac{i2}{3}$$

$$= \frac{2}{3}(-1+i)$$

En este caso se puede hacer directamente por ser analítica la función

$$\int_0^{1+i} z^2\, dz = \frac{(1+i)^3 - (0)^3}{3} = \frac{1 + 3i - 3 - i}{3} = \frac{2}{3}(-1+i)$$

Ejemplo, integrar por una trayectoria semicircular C: $|z| = 2$ de 2 a -2

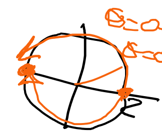
$x = r\cos\theta$

$y = r\,sen\,\theta$

$z = r\cos\theta + i\,r\,sen\,\theta$

$\theta = 0 \to \pi$

$\theta = 0 \to -\pi$



La parametrización será

$$z = 2e^{i\theta}; \qquad dz = 2e^{i\theta} i d\theta$$

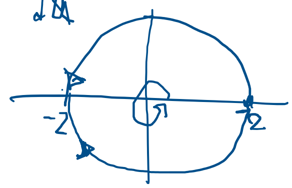
u du

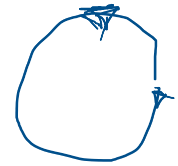
z^2 dz

$$\int_2^{-2} z^2\,dz = \frac{-8-8}{3} = \int_0^{\pi} 4e^{2i\theta}2e^{i\theta}i\,d\theta = \frac{8}{3}\int_0^{\pi} e^{i3\theta}i3\,d\theta = 8\frac{e^{i3\pi}-1}{3} = 8\frac{-1-1}{3} = -\frac{16}{3}$$

$\int e^u\,du$

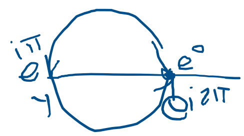
$\sqrt{x^2+y^2} = 2$ $\quad$ $\boxed{x^2+y^2 = 2^2}$





$z = x+iy = r\cos\theta + i\,r\,\mathrm{sen}\,\theta = re^{i\theta} = 2e^{i\theta}$ $\quad$ $z^2 = 4e^{i2\theta}$

$dz = 2e^{i\theta}\,i\,d\theta$



Ejercicio, integrar

$$\int_0^{1+i}(1+i-2\bar{z})dz = (1+i)\int_0^{1+i}dz - 2\int_0^{1+i}\bar{z}dz = (1+i)^2 - 2\int_0^{1+i}(x-iy)(dx+idy)$$

a) por una trayectoria recta, y=x

$$\int_0^{1+i}(1+i-2\bar{z})dz = (1+i)^2 - 2\int_0^1(x-ix)(dx+idx) = (1+i)^2 - 2(1-i)(1+i)\int_0^1 xdx$$

$$= (1+i)^2 - \cancel{2}(1-i)(1+i)\frac{1}{2} = (1+i)\big((1+i)-(1-i)\big) = (1+i)(2i) = 2(-1+i)$$

b) por una parabola, $y=x^2$

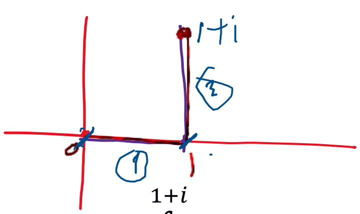
$$\int_0^{1+i}(1+i-2\bar{z})dz = (1+i)^2 - 2\int_0^1(x-ix^2)(dx+i2xdx) = (1+i)^2 - 2\int_0^1 x(1-ix)(1+i2x)dx$$

$$= (1+i)^2 - 2\int_0^1 x(1+2x^2+2xi-xi)dx = (1+i)^2 - 2\int_0^1(x+2x^3+x^2i)dx$$

$$= (1+i)^2 - 2\left(\frac{x^2}{2}+\frac{2x^4}{4}+\frac{x^3}{3}i\right)\Bigg|_0^1 = (1+i)^2 - 2\left(\frac{1}{2}+\frac{1}{2}+\frac{1}{3}i\right) = 1+2i-1-\left(2+\frac{2}{3}i\right)$$

$$= 2\left(-1+\frac{2}{3}i\right)$$

c) De 0 a 1 y de 1 a 1+i



$y=0$ $\quad dy=0$ $\qquad x=1$ $\quad dx=0$

$0<x\le 1$ $\qquad 0<y<1$

$$I = \int_0^{1+i}(1+i-2\bar{z})dz = (1+i)\int_0^{1+i}dz - 2\int_0^{1+i}\bar{z}dz = (1+i)^2 - 2\int_0^{1+i}(x-iy)(dx+idy)$$

i) De 0 a 1; $x = x,\ y = 0,\ dx = dx,\ dy = 0$

$$I_1 = 2\int_0^1 x dx = 1$$

ii) De 1 a 1+i

$$x = 1, \qquad y = y, \qquad dx = 0, \qquad dy = dy$$

$$I_2 = 2\int_0^1 (1 - iy) i dy = 2i\left(1 - \frac{i1}{2}\right) = 1 + 2i$$

Por lo tanto

$$I = (1+i)^2 - (I_1 + I_2) = 1 + 2i - 1 - (1 + 1 + 2i) = -2$$

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140. $\int_C z \operatorname{Im} z^2\, dz,\ C\colon\ |z| = 1\ (-\pi \leqslant \arg z \leqslant 0).$

141. $\int_C e^{|z|^2} \operatorname{Re} z\, dz$, C es la recta que conecta los puntos $z_1 = 0,\ z_2 = 1 + i$.

142. $\int_C \ln z\, dz$ (ln z es el valor principal del logaritmo), C: $|z| = 1$, a) el punto inicial del proceso de integración $z_0 = 1$; b) $z_0 = -1$. El recorrido se efectúa contra el sentido horario.

143. $\int_C z \operatorname{Re} z\, dz,\ C\colon\ |z| = 1$. El recorrido se realiza contra el sentido horario.

144. $\int_C z\bar{z}\, dz,\ C\colon\ |z| = 1$. El recorrido se efectúa contra el sentido horario.

Ejercicio **140.** $\int_C z \operatorname{Im} z^2\, dz,\ C\colon\ |z| = 1\ (-\pi \leqslant \arg z \leqslant 0).$

$$z = 1e^{i\theta}; z^2 = e^{i2\theta} = \cos(2\theta) + i\ sen(2\theta)$$

$$\int z\ Im\ (z^2) dz = \int_{-\pi}^{0} e^{\theta i} sen(2\theta) e^{\theta i} i d\theta$$

$$= i\int_{-\pi}^{0} e^{2\theta i} sen(2\theta) d\theta = i\int_{-\pi}^{0} \left(\cos(2\theta) + isen(2\theta)\right) sen(2\theta) d\theta$$

$$= i\int_{-\pi}^{0} (sen(2\theta)\cos(2\theta)) d\theta - \int_{-\pi}^{0} \left(sen^2(2\theta)\right) d\theta$$

$$= \left[\frac{isen^2(2\theta)}{2}\right]_{-\pi}^{0} - \frac{1}{2}\int_{-\pi}^{0} (1 - \cos(4\theta))\, d\theta$$

$$= 0 - \frac{1}{2}\left(\theta - \frac{sen(4\theta)}{4}\right)_{-\pi}^{0} = -\frac{1}{2}\left(0 + \pi - \frac{sen(4\theta)}{4}\right)_{-\pi}^{0} = -\frac{\pi}{2}$$

$$= \int_{-\pi}^{0} e^{2\theta i}\frac{\left(e^{2\theta i} - e^{-2\theta i}\right)}{2i} id\theta$$

$$\frac{1}{2}\int_{-\pi}^{0}\left(e^{4\theta i} - 1\right)d\theta = \frac{1}{2}\int_{-\pi}^{0}\left(\frac{e^{4\theta i}4i}{4i} - 1\right)d\theta = \left[\frac{e^{4\theta i}}{8i} - \frac{1}{2}\theta\right]_{-\pi}^{0} = \frac{e^{0} - e^{-4\pi i}}{8i} - \frac{0 - (-\pi)}{2} = -\frac{\pi}{2}$$

$$e^{-4\pi i} = \cos(4\pi) - isen(4\pi)$$

141

$$\int_C e^{|z|^2}\, x\, dz\,; C\colon 0 \to 1 + i$$

$$\int_C e^{x^2+y^2}\, x\, dz = \int_C e^{x^2+x^2}\, x\,(dx + idx)$$

$$(1 + i)\int_0^1 e^{2x^2}\, xdx$$

$$\frac{(1 + i)}{4}\int_C e^{2x^2}\, 4x\, dx = \left[\frac{(1 + i)}{4}e^{2x^2}\right]_0^1 = \frac{(1 + i)}{4}(e^2 - 1)$$

$$b)\ si\ y = x^2;\ \int_C e^{x^2+x^4}\, x\, dz = \int_C e^{x^2+x^4}\, x\,(dx + i2xdx)$$

$$= \int_0^1 e^{x^2(1+x^2)}\, x\, dx + i2\int_0^1 e^{x^2+x^4}\, x^2\, dx$$

142

$$\oint_{|z|=1} \ln(z)\, dz$$

$$z = e^{i\theta};\ \ln(z) = i\theta;\ dz = e^{i\theta}id\theta$$

$$a)\ \int_0^{2\pi} i\theta e^{i\theta}id\theta = \int_0^{2\pi} u(e^u du) = \left[e^{i\theta}(i\theta - 1)\right]_0^{2\pi} = e^{i2\pi}(2\pi i - 1) - (0 - 1) = 2\pi i$$

$$b)\ \int_{-\pi}^{\pi} i\theta e^{i\theta}id\theta = \int_{-\pi}^{\pi} u(e^u du) = \left[e^{i\theta}(i\theta - 1)\right]_{-\pi}^{\pi} = e^{i\pi}(\pi i - 1) - e^{-i\pi}(-\pi i - 1) = -2\pi i$$

$$b)\ \int_{\pi}^{3\pi} i\theta e^{i\theta}id\theta = \int_{\pi}^{3\pi} u(e^u du) = \left[e^{i\theta}(i\theta - 1)\right]_{\pi}^{3\pi} = e^{i3\pi}(3\pi i - 1) - e^{i\pi}(\pi i - 1) = -2\pi i$$

143

$$\int_{|z|=1} z\, x\, dz$$

$$= \int_0^{2\pi} e^{i\theta}\cos(\theta)\, e^{i\theta}id\theta$$

$$= \frac{i}{2}\int_0^{2\pi} e^{i2\theta}(e^{i\theta} + e^{-i\theta})\, d\theta =$$

$$= \frac{i}{2}\left(\frac{1}{3i}\int_0^{2\pi} e^{i3\theta}3id\theta + \frac{1}{i}\int_0^{2\pi} e^{i\theta}id\theta\right)$$

$$= \frac{i}{2}\left(\frac{1}{3i}\left(e^{i6\pi}-1\right)+\frac{1}{i}\left(e^{i2\pi}-1\right)\right)$$
$$= \left(\frac{1}{6}(1-1)+\frac{1}{2}(1-1)\right)=0$$

144

$$z = 1e^{i\theta}; dz = e^{i\theta} i d\theta; \oint |z|^2 dz = 1\int_0^{2\pi} e^{i\theta} i d\theta = \left[e^{i\theta}\right]_0^{2\pi} = e^{i2\pi} - e^0 = 0$$

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a)

$$z = (2t+it); dz = (2+i)dt \; 0<t<1$$
$$\oint Re(z)dz = \int_C x\,(dx+idy) = \int_0^1 2t(2+i)dt = 2(2+i)\int_0^1 tdt = \frac{(4+2i)t^2}{2} = (2+i)$$

b)

$$z_1 = 0; z_2 = 2; z_3 = 2+i$$
$$\oint Re(z)dz = \int_0^1 2t2dt + \int_0^1 2\,i\,dt = 4\int_0^1 tdt + 2i\int_0^1 dt = \left[\frac{4t^2}{2}+2it\right]_0^1 = 2+2i$$

Respuesta: 6+2i correcta 2+2i

152. $\int_C \frac{dz}{\sqrt{z}}$, C: a) es la mitad superior de la circunferencia $|z|=1$; se elige aquella rama de la función $\sqrt{z}$ para la cual $\sqrt{1}=1$;

b) $|z|=1$, $\operatorname{Re} z \geqslant 0$, $\sqrt{-i} = \frac{\sqrt{2}}{2}(1-i)$.

152

$$\int_C \frac{dz}{\sqrt{z}} = \int_0^{\pi} \frac{e^{i\theta} i d\theta}{e^{\frac{i\theta}{2}}} = 2\int_0^{\pi} e^{\frac{i\theta}{2}}\left(\frac{id\theta}{2}\right) = 2\left(e^{\frac{i\pi}{2}}-1\right) = 2(i-1)$$

a)

b)

$$z_k = e^{\frac{i\left(-\frac{\pi}{2}+2k\pi\right)}{2}} = e^{\frac{i(-\pi+4k\pi)}{4}}$$
$$z_0 = e^{\frac{i(-\pi)}{4}}$$
$$z_1 = e^{\frac{i(3\pi)}{4}}$$

$$\int_C \frac{dz}{\sqrt{z}} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{e^{i\theta} i d\theta}{e^{\frac{i\theta}{2}}} = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{\frac{i\theta}{2}}\left(\frac{id\theta}{2}\right) = 2\left(e^{\frac{i\pi}{4}} - e^{-\frac{i\pi}{4}}\right) = 2\sqrt{2}i$$

154. $\int_{1+i}^{2i} (z^3 - z) e^{\frac{z^2}{2}} dz.$ **155.** $\int_0^i z \cos z \, dz.$

156. $\int_1^i z \operatorname{sen} z \, dz.$ **157.** $\int_0^i (z-i) e^{-z} dz.$

158. $\int_1^i \frac{\ln(z+1)}{z+1} dz$ por el arco de la circunferencia $|z|=1$, $\operatorname{Im} z \geqslant 0$, $\operatorname{Re} z \geqslant 0$.

154

$$\int_{1+i}^{2i} (z^3 - z) e^{\frac{z^2}{2}} dz = \int_{1+i}^{2i} z^2 e^{\frac{z^2}{2}} z dz - \int_{1+i}^{2i} e^{\frac{z^2}{2}} z dz$$

$$= \int_{1+i}^{2i} z^2 e^{\frac{z^2}{2}} z dz - \left(e^{\frac{(2i)^2}{2}} - e^{\frac{(1+i)^2}{2}} \right)$$

Ejercicio 156, integrar:

$$\int_1^i z \, sen(z) \, dz = -z \, cos(z) + \text{sen(z)} =$$

$$= \left(-i \, cos(i) + \text{sen(i)}\right) - \left(-1 \, cos(1) + \text{sen}(1)\right)$$

$$= -i \, cosh(1) - i\text{senh}(1) - (-1 \, cos(1) + \text{sen}(1))$$

$$= cos(1) - \text{sen}(1) - i \, (cosh(1) + \text{senh}(1))$$

$$= cos(1) - \text{sen}(1) - ie^1$$

Si $f(z)$ es analítica dentro de C

$$\oint_c f(z) \, dz = 0$$

Esto se puede verificar por el teorema de Green y las condiciones de Cauchy Riemann

$$\oint_c (M \, dx + N \, dy) = \iint_S \left(\partial_x N - \partial_y M\right) dx \, dy$$

$$\oint_c f(z) \, dz = \oint_c (u + iv)(dx + idy) = \oint_c (u \, dx - v \, dy) + i \oint_c (v \, dx + u \, dy)$$

$$= \iint_S \left(-\partial_x v - \partial_y u\right) dx \, dy + i \iint_S \left(\partial_x u - \partial_y v\right) dx \, dy$$

$$= \iint_S \left(\partial_y u - \partial_y u\right) dx \, dy + i \iint_S (\partial_x u - \partial_x u) dx \, dy = 0$$

Ejemplo:

$$\oint_{|z|=10} (z^{20} + sen(z) - e^{5z})\, dz = 0$$

Si $f(z)$ es analítica en la región limitada entre dos curvas simples cerradas C y C_1, con C_1 dentro de C, entonces

$$\oint_C f(z)\, dz = \oint_{C_1} f(z)\, dz$$

Si $f(z)$ es analítica en la región limitada entre las curvas simples cerradas C y C_1, C_2, … C_n, con C_1, C_2, … C_n dentro de C, entonces

$$\oint_C f(z)\, dz = \sum_k^n \oint_{C_k} f(z)\, dz$$

Ejemplo:
a) calcular la integral

$$\oint_c f(z)\, dz = \oint_{|z|=r} \frac{1}{z} dz = \int_0^{2\pi} \frac{re^{i\theta} id\theta}{re^{i\theta}} = 2\pi i$$

$$z = re^{i\theta} \; dz = re^{i\theta} id\theta$$

b)

$$\oint_c f(z)\, dz = \oint_{C:|z-z_0|=r} \frac{1}{z - z_0} dz = \int_0^{2\pi} \frac{re^{i\theta} id\theta}{(re^{i\theta} + z_0) - z_0} = \int_0^{2\pi} \frac{re^{i\theta} id\theta}{re^{i\theta}} = 2\pi i$$

$$z = re^{i\theta} + z_0; \; dz = re^{i\theta} id\theta$$

Si $f(z) = \frac{\varphi(z)}{z - z_0}$ donde $\varphi(z)$ es analítica dentro de la curva C y z_0 está dentro de C

$$\oint_c f(z)\, dz = \oint_c \frac{\varphi(z)}{z - z_0} dz = 2\pi i\, \varphi(z_0)$$

Esto porque si cambiamos la curva de integración por una circunferencia de radio r centrada en z_0 y parametrizamos la curva

$$z = r\, e^{i\theta} + z_0$$
$$dz = r\, e^{i\theta} id\theta$$

Podemos tomar r muy pequeña, en el límite cuando r tiende a cero

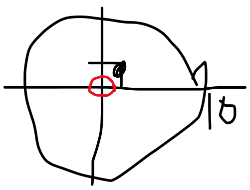
$$\oint_{|z-z_0|=\epsilon} \frac{\varphi(z)}{z - z_0} dz = \int_0^{2\pi} \frac{\varphi\left(\epsilon\, e^{i\theta} + z_0\right)\epsilon e^{i\theta} id\theta}{\epsilon e^{i\theta}} = i \int_0^{2\pi} \varphi\left(\epsilon\, e^{i\theta} + z_0\right) d\theta$$

$$= i2\pi\varphi(z_0)$$

Ejemplo, calcular

$$\oint_{c:|z|=10} \frac{sen(z)}{z-(2+i4)} dz = 2\pi i \operatorname{sen}(2+4i)$$

Usando $\oint_c \frac{\varphi(z)}{z-z_0} dz = 2\pi i\, \varphi(z_0)$



$$\oint_{c:|z|=1} \frac{sin(z)}{z-(2+i4)} dz = 0$$

Ejemplo, calcular

$$\oint_{c:|z|=2} \frac{z^3+4z^2-3}{z-i} dz = 2\pi i(i^3+4i^2-3) = 2\pi(1-7i)$$

Ejemplo, calcular

$$\oint_{c:|z-5|=6} \frac{z^3+4z^2-3}{z-i} dz = 2\pi i(i^3+4i^2-3) = 2\pi(-7i+1)$$

Ejemplo, calcular

$$\oint_{c:|z-5|=6} \frac{z^3+4z^2-3}{z-(3+i)} dz = 2\pi i((3+i)^3+4(3+i)^2-3) = \pi(94i-100)$$

Ejemplo, calcular

$$\oint_{c:|z|=1} \frac{z^3+4z^2-3}{z-(3+i)} dz = 0$$

Ejemplo, calcular

a)

$$\oint_{c:|z-1|=\frac{1}{2}} \frac{e^{3z}}{(z-1)(z-2)}dz = \oint_{c:|z-1|=\frac{1}{2}} \frac{\left[\frac{e^{3z}}{(z-2)}\right]}{(z-1)}dz = 2\pi i\left(\frac{e^{3}}{(1-2)}\right) = -2\pi i e^{3}$$

b)

$$\oint_{c:|z-2|=\frac{1}{2}} \frac{e^{3z}}{(z-1)(z-2)}dz = \oint_{c:|z-2|=\frac{1}{2}} \frac{\left[\frac{e^{3z}}{(z-1)}\right]}{(z-2)}dz = 2\pi i\left(\frac{e^{3(2)}}{(2-1)}\right) = 2\pi i e^{6}$$

c)

$$\oint_{c:|z|=3} \frac{e^{3z}}{(z-1)(z-2)}dz = \oint_{c:|z-1|=\frac{1}{2}} \frac{\left[\frac{e^{3z}}{(z-2)}\right]}{(z-1)}dz + \oint_{c:|z-2|=\frac{1}{2}} \frac{\left[\frac{e^{3z}}{(z-1)}\right]}{(z-2)}dz$$

$$= 2\pi i\left(\frac{e^{3}}{(1-2)}\right) + 2\pi i\left(\frac{e^{3(2)}}{(2-1)}\right) = 2\pi i(-e^{3}+e^{6})$$

Ejemplo, calcular

Separando en dos curvas cerradas

$$\oint_{c:|z|=3} \frac{e^{3z}}{(z-1)(z-2)}dz$$

$$= \oint_{c:|z-1|=.1} \frac{e^{3z}}{(z-1)(z-2)}dz + \oint_{c:|z-2|=.1} \frac{e^{3z}}{(z-1)(z-2)}dz$$

$$= 2\pi i(-e^{3}+e^{6})$$

$$\frac{e^{3z}}{(z-1)(z-2)} = e^{3z}\frac{1}{(z-1)(z-2)}$$

$$\frac{1}{(z-1)(z-2)} = \frac{a}{z-1} + \frac{b}{z-2}$$

$$1 = a(z-2) + b(z-1)$$

$$1 = z(a+b) - 2a - b$$

$$(a+b) = 0$$

$$-2a - b = 1$$

$$a = -1; b = 1$$

$$1 = a(z-2) + b(z-1)$$

Si z=2

$$1 = a(2-2) + b(2-1)$$

$$b = 1$$

Si z=1

$$1 = a(1-2) + b(1-1)$$

$$a = -1$$

$$\oint_{c1:|z-1|=\frac{1}{2}} \frac{\left(\frac{e^{3z}}{z-2}\right)}{z-1} dz + \oint_{c2:|z-2|=\frac{1}{2}} \frac{\left(\frac{e^{3z}}{z-1}\right)}{z-2} dz = 2\pi i \left(\frac{e^{3(1)}}{(1)-2}\right) + 2\pi i \left(\frac{e^{3(2)}}{(2)-1}\right)$$

$$= -2\pi i e^{3} + 2\pi i e^{6} = 2\pi i(e^{6} - e^{3})$$

Por fracciones parciales

$$\oint_{c:|z|=3} \frac{e^{3z}}{(z-1)(z-2)} dz = \oint_{c:|z|=3} \left(\frac{e^{3z}}{(z-2)}\right) dz - \oint_{c:|z|=3} \left(\frac{e^{3z}}{(z-1)}\right) dz = 2\pi i(e^{6} - e^{3})$$

$$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2} = \frac{(z-2)A + (z-1)B}{(z-1)(z-2)}$$

$$1 = A(z-2) + B(z-1) = z(A+B) + (-2A - B)$$

Se puede resolver el sistema

$$A + B = 0$$

$$-2A - B = 1$$

$$A = -1; B = 1$$

Pero también se puede resolver así:

$$(z-2)A + (z-1)B = 1$$

Supongamos que z=1, A=-1
Supongamos que z=2, B=1

Ejemplos:

$$z^{2} + 1 = (z+1)(z-1) = (z+i)(z-i)$$

$$z = \frac{0 \pm \sqrt{-4}}{2} = \pm i$$

167. $\displaystyle\oint_{|z|=1} \frac{e^z}{z^2+2z}\,dz.$

168. $\displaystyle\oint_{|z-i|=1} \frac{e^{iz}}{z^2+1}\,dz.$ $= 2\pi i$

169. $\displaystyle\oint_{|z-1|=2} \frac{\operatorname{sen}\frac{\pi z}{2}}{z^2+2z-3}\,dz.$

170. $\displaystyle\oint_{|z|=2} \frac{\operatorname{sen} iz}{z^2-4z+3}\,dz.$ $= 2\pi i$

171. $\displaystyle\oint_{|z|=1} \frac{\operatorname{tg} z}{z e^{1/(z+2)}}\,dz.$

172. $\displaystyle\oint_{|z|=3} \frac{\cos(z+\pi i)}{z(e^z+2)}\,dz.$ $= 2\pi i$

173. $\displaystyle\oint_{|z|=5} \frac{dz}{z^2+16}.$

174. $\displaystyle\oint_{|z|=4} \frac{dz}{(z^2+9)(z+9)}.$

175. $\displaystyle\oint_{|z|=1} \frac{\operatorname{sh}\frac{\pi}{2}(z+i)}{z^2-2z}.$

176. $\displaystyle\oint_{|z|=2} \frac{\operatorname{sen} z \operatorname{sen}(z-1)}{z^3-z}\,dz.$

174

$$\oint_{|z|=4} \frac{dz}{(z^2+9)(z+9)} = \oint_{|z|=4} \frac{dz}{(z-(-3i))(z-3i)(z+9)}$$

$$= \oint_{|z-3i|=1} \frac{\left(\frac{1}{(z+3i)(z+9)}\right)dz}{(z-3i)} + \oint_{|z+3i|=1} \frac{\left(\frac{1}{(z-3i)(z+9)}\right)dz}{(z-(-3i))} =$$

$$= 2\pi i\left(\frac{1}{(3i+3i)(3i+9)} + \frac{1}{(-3i-3i)(-3i+9)}\right)$$

$$= \frac{2\pi i}{9}\left(\frac{1}{(2i)(i+3)} + \frac{1}{(-2i)(-i+3)}\right)$$

$$= \frac{2\pi i}{9}\left(\frac{1}{(-2+6i)} + \frac{1}{-(2+6i)}\right)$$

$$= \frac{2\pi i}{9}\left(\frac{-2+6i-(2+6i)}{-(-36-4)}\right) = -\frac{\pi i}{9}\left(\frac{1}{5}\right)$$
$$= -\frac{\pi i}{45}$$

Si $f(z) = \frac{\varphi(z)}{(z-z_0)^n}$ donde $\varphi(z)$ es analítica dentro de la curva C y z_0 está dentro de C

$$\oint_c f(z)\,dz = \oint_C \frac{\varphi(z)}{(z-z_0)^n}\,dz = 2\pi i\,\frac{\varphi^{(n-1)}(z_0)}{(n-1)!}$$

Justificación:

$$\oint_c \frac{\varphi(z)}{z-z_0}\,dz = 2\pi i\,\phi(z_0)$$

$$\frac{d}{dz_0}\oint_c \frac{\varphi(z)}{z-z_0}\,dz = 2\pi i\,\varphi'(z_0)$$

Primera derivada

$$\oint_c \frac{\partial}{\partial z_0}\left(\frac{\varphi(z)}{z-z_0}\,dz\right) = 2\pi i\,\varphi'(z_0)$$

$$1\oint_c \varphi(z)(z-z_0)^{-2}\,dz = 2\pi i\,\varphi'(z_0)$$

Segunda derivada

$$1\oint_c \frac{\partial}{\partial z_0}\left(\varphi(z)(z-z_0)^{-2}\right)dz = 2\pi i\,\varphi''(z_0)$$

$$2\times 1\oint_c \left(\varphi(z)(z-z_0)^{-3}\right)dz = 2\pi i\,\varphi''(z_0)$$

Tercera derivada

$$2\times 1\oint_c \frac{\partial}{\partial z_0}\left(\varphi(z)(z-z_0)^{-3}\right)dz = 2\pi i\,\varphi'''(z_0)$$

$$3\times 2\times 1\oint_c \left(\varphi(z)(z-z_0)^{-4}\right)dz = 2\pi i\,\varphi'''(z_0)$$

…

$$(n-1)\times \ldots \times 3\times 2\times 1\oint_c \left(\varphi(z)(z-z_0)^{-(n)}\right)dz = 2\pi i\,\varphi^{(n-1)}(z_0)$$

$$\oint_c \frac{\varphi(z)}{(z-z_0)^n}\,dz = 2\pi i\,\frac{\varphi^{(n-1)}(z_0)}{(n-1)!}$$

Ejemplo, calcular

$$\oint_{|z|=4} \frac{e^{2z}}{(z-3i)^3} dz = 2\pi i \left[\frac{4e^{2z}}{2!}\right]_{3i} = 4\pi i e^{6i}$$

Ejemplo, calcular

$$\oint_{c:|z-3i|=1} \frac{sen(3z^2-1)}{(z^2-6iz-9)^2} dz = \oint_c \frac{sen(3z^2-1)}{(z-3i)^4} dz = 2\pi i \ \frac{\varphi'''(3i)}{(3)!} =$$

$$= 2\pi i \ \frac{-108\ (3i)\ sen(3(3i)^2-1) - 216\ (3i)^3\ cos(3(3i)^2-1)}{6}$$

$$= \pi\ (-108\ sen(28) - 1944\ cos(28))$$

$$\phi(z) = sen(3z^2-1)$$

$$\phi'^{(z)} = 6z\ cos(3z^2-1)$$

$$\phi''(z) = 6cos(3z^2-1) - 36z^2 sen(3z^2-1)$$

$$\phi'''(z) = -36zsen(3z^2-1) - \left(72z\ sen(3z^2-1) + 216z^3 cos(3z^2-1)\right)$$

$$\phi'''(z) = -108\ z\ sen(3z^2-1) - 216\ z^3\ cos(3z^2-1)$$

---

182. $\displaystyle\oint_{|z-2|=3} \frac{\text{ch}\, e^{i\pi z}}{z^3-4z^2} dz.$

$$\oint_c \frac{\cosh\left(e^{i\pi z}\right)}{z^3-4z^2} dz = \oint_{|z-2|=3} \frac{\cosh\left(e^{i\pi z}\right)}{z^2(z-4)} dz$$

$$= \oint_{|z|=1} \frac{\left(\dfrac{\cosh\left(e^{i\pi z}\right)}{z-4}\right)}{(z-0)^2} dz + \oint_{|z-4|=1} \frac{\dfrac{\cosh\left(e^{i\pi z}\right)}{z^2}}{(z-4)} dz$$

$$= 2\pi i\left(senh\left(e^{i\pi 0}\right)e^{i\pi 0}\pi i(0-4)^{-1} - \cosh\left(e^{i\pi 0}\right)(0-4)^{-2}\right) + 2\pi i \frac{\cosh\left(e^{i\pi 4}\right)}{4^2}$$

$$= 2\pi i(-\frac{senh(1)\pi i}{4} - \frac{\cosh(1)}{16}) + \pi i \frac{\cosh(1)}{8}$$

$$= \pi i\left(-\frac{senh(1)\pi i}{2} - \frac{\cosh(1)}{8}\right) + \pi i \frac{\cosh(1)}{8}$$

$$= \frac{senh(1)\pi^2}{2}$$

$$\phi(z) = \cosh\left(e^{i\pi z}\right)(z-4)^{-1}$$

$$\phi'(z) = senh\left(e^{i\pi z}\right)e^{i\pi z}\pi i(z-4)^{-1} - \cosh\left(e^{i\pi z}\right)(z-4)^{-2}$$

$$\cosh(\text{z}) = \frac{\text{e}^{\text{z}} + e^{-z}}{2} \rightarrow senh(z) = \frac{\text{e}^{\text{z}} - e^{-z}}{2}$$

Ejercicios:

167. $\displaystyle\int_{|z|=1} \frac{e^z}{z^2+2z}\,dz.$ **168.** $\displaystyle\int_{|z-i|=1} \frac{e^{iz}}{z^2+1}\,dz.$

169. $\displaystyle\int_{|z-1|=2} \frac{\operatorname{sen}\frac{\pi z}{2}}{z^2+2z-3}\,dz.$ **170.** $\displaystyle\int_{|z|=2} \frac{\operatorname{sen} iz}{v^2-4z+3}\,dz.$

171. $\displaystyle\int_{|z|=1} \frac{\operatorname{tg} z}{ze^{1/(z+2)}}\,dz.$ **172.** $\displaystyle\int_{|z|=3} \frac{\cos(z+\pi i)}{z(e^z+2)}\,dz.$

173. $\displaystyle\int_{|z|=5} \frac{dz}{z^2+16}.$ **174.** $\displaystyle\int_{|z|=4} \frac{dz}{(z^2+9)(z+9)}.$

175. $\displaystyle\int_{|z|=1} \frac{\operatorname{sh}\frac{\pi}{2}(z+i)}{z^2-2z}.$ **176.** $\displaystyle\int_{|z|=2} \frac{\operatorname{sen} z \operatorname{sen}(z-1)}{z^3-z}\,dz.$

172

$$\int_{|z|=3} \frac{\cos(z+\pi i)}{z(e^z+2)}dz = 2\pi i\left(\frac{\cosh(\pi)}{3}\right)$$

167

$$\int_{|z|=1} \frac{e^z}{z^2+2z}dz = \int_{|z|=1} \frac{e^z}{z(z+2)}dz = \int_{|z|=1} \frac{\left(\frac{e^z}{z+2}\right)}{z}dz = 2\pi i\frac{e^0}{2} = \pi i$$

168

$$\int_{|z-i|=1} \frac{e^{iz}}{z^2+1}dz = \int_{|z-i|=1} \frac{e^{iz}}{z^2-(i^2)}dz = \int_{|z-i|=1} \frac{e^{iz}}{(z-(-i))(z-i)}dz$$

$$= \int_{|z-i|=1} \frac{\frac{e^{iz}}{z+i}}{(z-i)}dz = 2\pi i\frac{e^{ii}}{i+i} = \frac{\pi}{e}$$

170

$$\oint_{|z|=2} \frac{sen(iz)}{z^2-4z+3}dz = \oint_{|z|=2} \frac{sen(iz)}{(z-3)(z-1)}dz$$

$$= 2\pi i\frac{sen(i1)}{(1-3)} = -\pi i sen(i) = -\pi i\frac{\left(e^{ii}-e^{-ii}\right)}{2i} = \pi\frac{\left(e^{1}-e^{-1}\right)}{2} = \pi\ senh(1)$$

171

$$\int_{|z|=1} \frac{tg(z)}{ze^{\frac{1}{z+2}}}dz = \int_{|z|=1} \frac{\left(\frac{tg(z)}{e^{\frac{1}{z+2}}}\right)}{z}dz = 2\pi i\left(\frac{tg(0)}{e^{\frac{1}{0+2}}}\right) = 0$$

177. $\displaystyle\int_{|z|=1} \frac{\cos z}{z^3}\, dx.$ **178.** $\displaystyle\int_{|z|=1} \frac{\mathrm{sh}^2 z}{z^3}\, dz.$

179. $\displaystyle\int_{|z-1|=1} \frac{\sin \frac{\pi}{4} z}{(z-1)^3 (z-3)}\, dz.$ **180.** $\displaystyle\int_{|z|=2} \frac{z \,\mathrm{sh}\, z}{(z^2-1)^2}\, dz.$

181. $\displaystyle\int_{|z-3|=6} \frac{z\, dz}{(z-2)^3 (z+4)}.$ **182.** $\displaystyle\int_{|z-2|=3} \frac{\mathrm{ch}\, e^{i\pi z}}{z^3-4z^2}\, dz.$

183. $\displaystyle\int_{|z|=1/2} \frac{1}{z^3} \cos \frac{\pi}{z+1}\, dz.$ **184.** $\displaystyle\int_{|z-2|=1} \frac{e^{\frac{1}{z}}}{(z^2+4)^2}\, dz.$

185. $\displaystyle\int_{|z|=1/2} \frac{1-\mathrm{sen}\, z}{z^2}\, dz.$ **186.** $\displaystyle\int_{|z-1|=1/2} \frac{e^{iz}}{(z^2-1)^2}\, dz.$

$$\oint_C \frac{\varphi(z)}{(z-z_0)^n} dz = 2\pi i \, \frac{\varphi^{(n-1)}(z_0)}{(n-1)!}$$

177

$$\oint_{|z|=1} \frac{\cos(z)}{(z-0)^3} dz = \frac{2\pi i \phi''(0)}{2!} = -\pi i cos(0) = -\pi i$$

177

$$\int_{|z|=1} \frac{\cos(z)}{z^3} dz = \frac{2\pi i}{2!} \left[\frac{d^2[\cos(z)]}{dz^2}\right]_{z=0} = -cos(0)\pi i = -\pi i$$

178

$$\oint_{|z|=1} \frac{\mathrm{senh}^2(z)}{(z-0)^3} dz = \frac{2\pi i}{2!} \phi''(0) = 2\pi i (\cosh^2(0) + senh^2(0)) = 2\pi i$$

$$\phi(z) = senh^2(z)$$

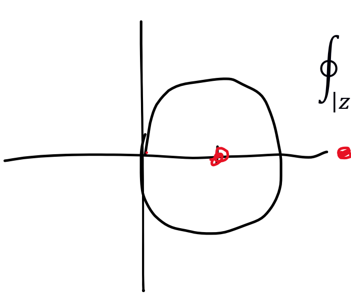
$$\phi'(z) = 2 senh(z) \cosh(z)$$

$$\phi'(z) = 2\big(cosh(z)\cosh(z) + senh(z) senh(z)\big)$$

$$\cosh^2(0) + senh^2(0) = \left(\frac{e^z + e^{-z}}{2}\right)^2 + \left(\frac{e^z - e^{-z}}{2}\right)^2$$

$$= \frac{1}{4}(2e^{2z} + 2e^{-2z}) = \frac{(e^{2z} + e^{-2z})}{2} = \cosh(2z)$$

179



$$\oint_{|z-1|=1} \frac{\left(\dfrac{\mathrm{sen}\left(\frac{\pi}{4} z\right)}{z-3}\right)}{(z-1)^3} dz = \frac{2}{2!} \pi i \phi''(1) = \frac{\pi i}{8}\left(-\frac{2\sqrt{2}}{2} \quad - \frac{\sqrt{2}}{2}\frac{\pi}{}\right)$$

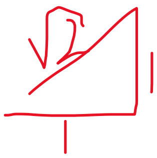
$$\phi(z) = (z-3)^{-1} \mathrm{sen}\left(\frac{\pi}{4} z\right)$$

$$\phi'(z) = -(z-3)^{-2}\text{sen}\left(\frac{\pi}{4}z\right) + (z-3)^{-1}\cos\left(\frac{\pi}{4}z\right)\frac{\pi}{4}$$

$$\phi''(z) = 2(z-3)^{-3}\text{sen}\left(\frac{\pi}{4}z\right) - (z-3)^{-2}\cos\left(\frac{\pi}{4}z\right)\frac{\pi}{4} - (z-3)^{-2}\cos\left(\frac{\pi}{4}z\right)\frac{\pi}{4} - (z-3)^{-1}\text{sen}\left(\frac{\pi}{4}z\right)\left(\frac{\pi}{4}\right)^2$$

$$\phi''(1) = 2(1-3)^{-3}\text{sen}\left(\frac{\pi}{4}\right) - (1-3)^{-2}\cos\left(\frac{\pi}{4}\right)\frac{\pi}{4} - (1-3)^{-2}\cos\left(\frac{\pi}{4}\right)\frac{\pi}{4} - (1-3)^{-1}\text{sen}\left(\frac{\pi}{4}\right)\left(\frac{\pi}{4}\right)^2$$

$$\phi''(1) = \frac{2(-2)^{-3}}{\sqrt{2}} - \frac{(-2)^{-2}}{\sqrt{2}}\frac{\pi}{4} - \frac{(-2)^{-2}}{\sqrt{2}}\frac{\pi}{4} - \frac{(-2)^{-1}}{\sqrt{2}}\left(\frac{\pi}{4}\right)^2$$



$$= -\frac{1}{4\sqrt{2}} - \frac{\pi}{8\sqrt{2}} + \frac{\pi^2}{32\sqrt{2}} = -\frac{\sqrt{2}}{4(\sqrt{2}\sqrt{2})} - \frac{\pi 2\sqrt{2}}{16\sqrt{2}\sqrt{2}} + \frac{\pi^2\sqrt{2}}{32\sqrt{2}\sqrt{2}}$$

$$= \frac{\pi i}{4\sqrt{2}}\left(-1 - \frac{\pi}{2} + \frac{\pi^2}{8}\right) =$$

$$\phi'(1) = -(1-3)^{-2}\text{sen}\left(\frac{\pi}{4}\right) + (1-3)^{-1}\cos\left(\frac{\pi}{4}\right)\frac{\pi}{4}$$

$$= -2\pi\mathrm{i}\left(\frac{1}{4\sqrt{2}} + \frac{1}{2\sqrt{2}}\frac{\pi}{4}\right) = -\frac{\pi\mathrm{i}}{4\sqrt{2}}(2+\pi)$$

180

$$\int_{|z|=2} \frac{z\,senh(z)}{(z^2-1)^2}dz = \int_{|z|=2} \frac{z\,senh(z)}{(z+1)^2(z-1)^2}dz$$

$$= \int_{|z+1|=1} \frac{z\,senh(z)(z-1)^{-2}}{(z+1)^2}dz + \int_{|z-1|=1} \frac{z\,senh(z)(z+1)^{-2}}{(z-1)^2}dz$$

$$= 2\pi i\left[\frac{d[z\,senh(z)(z-1)^{-2}]}{dz}\right]_{z=-1} + 2\pi i\left[\frac{d[z\,senh(z)(z+1)^{-2}]}{dz}\right]_{z=1} =$$

$$= 2\pi i[senh(z)(z-1)^{-2} + z\cosh(z)(z-1)^{-2} - 2z\,senh(z)(z-1)^{-3}]_{z=-1}$$

$$+2\pi i[senh(z)(z+1)^{-2} + z\cosh(z)(z+1)^{-2} - 2z\,senh(z)(z+1)^{-3}]_{z=1}$$

$$= 2\pi i[senh(-1)(-1-1)^{-2} - 1\cosh(-1)(-1-1)^{-2} + 2\,senh(-1)(-1-1)^{-3}]$$

$$+2\pi i[senh(1)(1+1)^{-2} + 1\cosh(1)(1+1)^{-2} - 2\,senh(1)(1+1)^{-3}]$$

$$= 2\pi i\left[\frac{senh(-1)}{4} - \frac{\cosh(-1)}{4} - 2\frac{senh(-1)}{8}\right]$$

$$+2\pi i\left[\frac{senh(1)}{4} + \frac{\cosh(1)}{4} - 2\frac{senh(1)}{8}\right]$$

$$= \frac{\pi i}{2}[senh(-1) - \cosh(-1) - senh(-1) + senh(1) + \cosh(1) - senh(1)]$$

$$= \frac{\pi i}{2}[-senh(1) - \cosh(1) + senh(1) + senh(1) + \cosh(1) - senh(1)]$$

$$= \frac{\pi i}{2}[+senh(1) - senh(1)] = 0$$

182

$$\oint_{c_1} \frac{ch\left(e^{i\pi z}\right)}{(z-4)z^2}dz = \oint_{c_1} \frac{\left(\frac{ch\left(e^{i\pi z}\right)}{z-4}\right)}{z^2}dz + \oint_{c_2} \frac{\left(\frac{ch\left(e^{i\pi z}\right)}{z^2}\right)}{z-4}dz = 2\pi i \left(\frac{d}{dz}\left(\frac{ch\left(e^{i\pi z}\right)}{z-4}\right)_{z=0} + \frac{ch\left(e^{i\pi 4}\right)}{4^2}\right)$$

$$= 2\pi i \left(\frac{sh\left(e^{i\pi 0}\right)e^{i\pi 0}(i\pi)(0-4) - ch\left(e^{i\pi 0}\right)}{(0-4)^2} + \frac{ch\left(e^{i\pi 4}\right)}{4^2}\right)$$

$$= 2\pi i \left(-\frac{ch(1)}{16} - \frac{sh(1)\pi i}{4} + \frac{ch(1)}{16}\right)$$

$$= -\frac{2\pi i}{4}\left(sh(1)(i\pi)\right)$$

$$= \frac{\pi^2 sh(1)}{2}$$

183

$$\oint_{|z|=\frac{1}{2}} \frac{\cos\left(\frac{\pi}{z+1}\right)}{z^3}dz = \frac{2\pi i\phi''(0)}{2!} = \pi i\pi^2$$

$$\varphi(z) = \cos(\pi(z+1)^{-1})$$

$$\varphi'(z) =$$

$$\varphi''(z) =$$

186

$$= \int_{|z-1|=\frac{1}{2}} \frac{e^{iz}}{(z^2-1)^2}dz$$

$$= \int_{|z-1|=\frac{1}{2}} \frac{e^{iz}}{((z+1)(z-1))^2}dz$$

$$= \int_c \frac{\left(\frac{e^{iz}}{(z+1)^2}\right)}{(z-1)^2}dz = 2\pi i\phi'(1)$$

$$\phi(z) = e^{iz}(z+1)^{-2}$$

$$\phi'(z) = e^{iz}(z+1)^{-2}$$

$$2\pi i \frac{e^{i1}i(1+1)^2 - e^{i1}2(1+1)}{(1+1)^4}$$

$$= 2\pi i \frac{e^{i}i2^2 - e^{i1}2^2}{2^4} = \pi i e^{i}\frac{i-1}{2} = -\frac{\pi e^{i}}{2}(1+i)$$

$$\left(\frac{e^{iz}}{(z+1)^2}\right)' = \frac{e^{iz}i(z+1)^2 - e^{iz}2(z+1)}{(z+1)^4}$$

Si $\boldsymbol{\varphi(z_0) \neq 0,\ \psi(z_0) = 0, \psi'(z_0) \neq 0\ ,\ \phi\ y\ \psi\ analíticas\ dentro\ de\ C, y\ z_0}$ está dentro de la curva C

$$\oint_c f(z)\,dz = \oint_c \frac{\varphi(z)}{\psi(z)}dz = 2\pi i\ \frac{\varphi(z_0)}{\psi'(z_0)}$$

Demostración:
Si $\psi(z)$ tiene una raíz simple en $\boldsymbol{z_0}$, $\psi(z) = (z - z_0)g(z)$

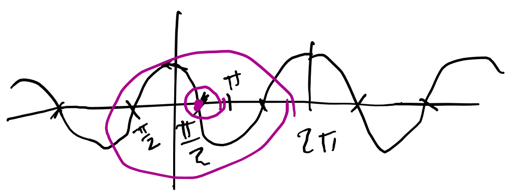
$$\psi'(z) = g(z) + (z - z_0)g'(z)$$

$$\psi'(z_0) = g(z_0) + (z_0 - z_0)g'(z_0) = g(z_0)$$

$$\oint_c \frac{\varphi(z)}{\psi(z)}dz = \oint_c \frac{\varphi(z)}{(z - z_0)g(z)}dz = 2\pi i \frac{\varphi(z_0)}{g(z_0)} = 2\pi i \frac{\varphi(z_0)}{\psi'(z_0)}$$

Ejemplo, calcular

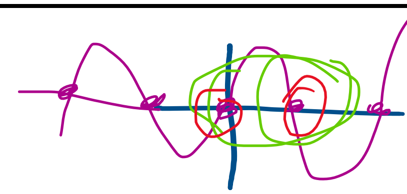
$$\oint_{\left|z-\frac{\pi}{2}\right|=1} \tan(z)\,dz = \oint_{\left|z-\frac{\pi}{2}\right|=1} \frac{sen(z)}{cos(z)}dz = 2\pi i\ \frac{sen\left(\frac{\pi}{2}\right)}{-sen\left(\frac{\pi}{2}\right)} = -2\pi i$$



Ejemplo, calcular

$$\oint_{\left|z-\frac{\pi}{2}\right|=1} \frac{z^2}{cos(z)}dz = 2\pi i\ \frac{\left(\frac{\pi}{2}\right)^2}{-sen\left(\frac{\pi}{2}\right)} = -\frac{\pi^3 i}{2}$$

Ejemplo, calcular

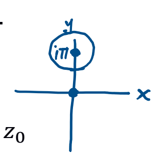


$$\oint_{|z-1.5|=3} \cot(z)\,dz$$

$$= \oint_{|z|=.1} \frac{\cos(z)}{\mathrm{sen}(z)}dz + \oint_{|z-\pi|=.1} \frac{\cos(z)}{\mathrm{sen}(z)}dz = 2\pi i\left(\frac{cos(0)}{\cos(0)} + \frac{\cos(\pi)}{\cos(\pi)}\right) = 4\pi i$$

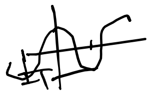
$$\oint_C \frac{\varphi(z)}{\psi(z)}dz = 2\pi i\ \frac{\varphi(z_0)}{\psi'(z_0)}$$

Ejemplo, calcular

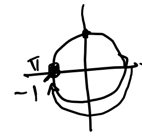


$$\oint_{|z-i\pi|=1} \frac{e^z}{\sinh(z)}dz = 2\pi i\ \frac{e^{z_0}}{\cosh(z_0)}$$

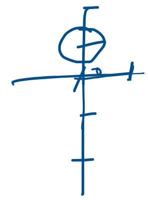
$$= 2\pi i \frac{e^{i\pi}}{\cosh(i\pi)} = 2\pi i \frac{\overset{-1}{2e^{i\pi}}}{\underset{-1}{e^{i\pi}} + \underset{-1}{e^{-i\pi}}} = 2\pi i$$



$\overset{-1}{\cos\pi} + i\,\overset{0}{\sin\pi}$



$$\sinh(z) = \frac{e^{z} - e^{-z}}{2} = 0$$
$$e^{z} = e^{-z} = \frac{1}{e^{z}}$$
$$e^{2z} = 1$$



$$e^{2z} = 1e^{i(0+2k\pi)}$$
$$2z = +i2k\pi$$
$$z = ik\pi$$

$$\oint_{|z|=10} \frac{e^{z}}{\sinh(z)} dz =$$

$$\frac{\left(e^{i\pi k} - e^{-i\pi k}\right)}{2} = 0$$

$$\frac{\left(e^{z} - e^{-z}\right)}{2} = 0$$
$$\left(e^{2z} - 1\right) = 0$$
$$e^{2z} = 1e^{i(2k\pi)}$$
$$z = \frac{1}{2}\ln\left(1e^{i(2k\pi)}\right) = \frac{1}{2}\left(\ln(|1|) + i(0 + 2k\pi)\right) = ik\pi$$

$$\oint_{|z|=10} \frac{e^{z}}{\sinh(z)} dz = 2\pi i \left(\frac{e^{0}}{\cosh(0)} + \frac{e^{\pi i}}{\cosh(\pi i)} + \frac{e^{-\pi i}}{\cosh(-\pi i)} + \frac{e^{2\pi i}}{\cosh(2\pi i)} + \frac{e^{-2\pi i}}{\cosh(-2\pi i)} + \frac{e^{3\pi i}}{\cosh(3\pi i)} + \frac{e^{-3\pi i}}{\cosh(-3\pi i)} \right)$$
$$= 2\pi i \left(1 + \frac{-1}{-1} + \frac{-1}{-1} + \frac{1}{1} + \frac{1}{1} + \frac{-1}{-1} + \frac{-1}{-1} \right) = 14\pi i$$

Ejercicio

$$\int_{|z|=1} z\tan(\pi z)dz = \int_{|z|=1} z\frac{sen(\pi z)}{\cos(\pi z)}dz$$

$$= \int_{\left|z-\frac{1}{2}\right|=.1} \frac{z\,sen(\pi z)}{\cos(\pi z)}dz + \int_{\left|z+\frac{1}{2}\right|=.1} \frac{z\,sen(\pi z)}{\cos(\pi z)}dz = 2\pi i\left(\frac{\frac{1}{2}\,sen\left(\frac{\pi}{2}\right)}{-\mathrm{sen}\left(\frac{\pi}{2}\right)} + \frac{\left(-\frac{1}{2}\right)\,sen\left(-\frac{\pi}{2}\right)}{-\mathrm{sen}\left(-\frac{\pi}{2}\right)}\right) = 0$$

Serie de Taylor de f(z) alrededor de z_0:

$$f(z) = \sum_{k=0}^{\infty} a_k\,(z-z_0)^k = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

$$f(z_0) = \sum_{k=0}^{\infty} a_k\,(z_0-z_0)^k = a_0 + a_1(z_0-z_0) + a_2(z_0-z_0)^2 + \cdots$$

$$a_0 = \frac{f(z_0)}{0!}$$

$$f'(z_0) = \sum_{k=0}^{\infty} a_k\,k(z_0-z_0)^{k-1} = a_1 + a_2 2(z_0-z_0)^1 + a_3 3(z_0-z_0)^2 + \cdots$$

$$a_1 = \frac{f'(z_0)}{1!}$$

$$f''(z_0) = \sum_{k=0}^{\infty} a_k\,k(k-1)(z_0-z_0)^{k-2} = a_2 2\times 1 + a_3 3\times 2(z_0-z_0)^1 + a_4 4\times 3(z_0-z_0)^2 + \cdots$$

$$a_2 = \frac{f''(z_0)}{2!}$$

$$a_3 = \frac{f'''(z_0)}{3!}$$

$$a_4 = \frac{f^{(IV)}(z_0)}{4!}$$

La serie de Taylor es analítica, tiene derivada en z_0, los coeficientes son $a_k = \frac{f^{(k)}(z_0)}{k!}$

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k = \frac{f(z_0)(z-z_0)^0}{(0)!} + \frac{f'(z_0)(z-z_0)^1}{(1)!} + \frac{f''(z_0)(z-z_0)^2}{(2)!} + \cdots + \frac{f^n(z_0)(z-z_0)^n}{(n)!} + \cdots$$

Si $z_0 = 0$, la serie se llama serie de McLaurin:

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}(z)^k = \frac{f(0)(z)^0}{(0)!} + \frac{f'(0)(z)^1}{(1)!} + \frac{f''(0)(z)^2}{(2)!} + \cdots + \frac{f^n(0)(z)^n}{(n)!} + \cdots$$

Ejemplos, calcular las series de:

---

a)

$$f(z) = e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} \ldots + \frac{(z)^n}{(n)!} + \cdots = \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

b)

$$f(z) = \cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \frac{z^8}{8!} - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!}$$

c)

$$f(z) = sen(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \frac{z^9}{9!} - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{z^{(2k+1)}}{(2k+1)!}$$

$$e^{(i\theta)} = 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} \dots + \frac{(i\theta)^n}{(n)!}$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right)$$

$$e^{i\theta} = \cos(\theta) + i\ sen(\theta)$$

$$e^{i\pi} = \cos(\pi) + i\ sen(\pi)$$

$$e^{i\pi} + 1 = 0$$

Ejercicio, verificar con series de Taylor que $e^{i\theta} = \cos(\theta) + i\ sen(\theta)$

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

$$e^{i\theta} = 1 + \frac{i\theta}{1!} + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \cdots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + i\left(\frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right)$$

$$cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots$$

$$sen(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots$$

$$e^{i\theta} = \cos(\theta) + isen(\theta)$$

$$e^{i\pi} + 1 = 0$$

d)

Serie de Laurent de f(z) alrededor de z_0:

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$$

$$= \cdots + a_{-3}(z - z_0)^{-3} + a_{-2}(z - z_0)^{-2} + a_{-1}(z - z_0)^{-1} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$$

$$= \cdots + \frac{a_{-3}}{(z - z_0)^3} + \frac{a_{-2}}{(z - z_0)^2} + \frac{a_{-1}}{(z - z_0)^1} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$$

$$\oint_C f(z)\,dz = \sum_{k=-\infty}^{\infty} a_k \oint_C (z-z_0)^k\,dz$$

La serie de Laurent tiene también potencias negativas, es una generalización de la serie de Taylor para funciones no analíticas en z_0

Ejemplos de series de potencias:

1. Para la función exponencial natural
   a) Usando la fórmula $a_k = \frac{f^{(k)}(z_0)}{k!}$

$$f(z) = e^z = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k = \frac{f(z_0)(z-z_0)^0}{(0)!} + \frac{f'(z_0)(z-z_0)^1}{(1)!} + \frac{f''(z_0)(z-z_0)^2}{(2)!} + \cdots$$

$$= \frac{e^{z_0}(z-z_0)^0}{(0)!} + \frac{e^{z_0}(z-z_0)^1}{(1)!} + \frac{e^{z_0}(z-z_0)^2}{(2)!} + \frac{e^{z_0}(z-z_0)^3}{(3)!} + \cdots$$

Si $z_0 = 0$, tenemos:

$$f(z) = e^z = \frac{e^0(z-0)^0}{(0)!} + \frac{e^0(z-0)^1}{(1)!} + \frac{e^0(z-0)^2}{(2)!} + \frac{e^0(z-0)^3}{(3)!} + \cdots$$

$$e^z = \frac{z^0}{0!} + \frac{z^1}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots = \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

Ejercicio, verificar con series de Taylor que $e^{i\theta} = \cos(\theta) + i\,sen(\theta)$

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

$$e^{i\theta} = 1 + \frac{i\theta}{1!} + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \cdots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + i\left(\frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right)$$

$$cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots$$

$$sen(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots$$

$$e^{i\theta} = \cos(\theta) + i sen(\theta)$$

$$e^{i\pi} + 1 = 0$$

b) Es común aprovechar series conocidas para obtener otras series, sin usar la fórmula $a_k = \frac{f^{(k)}(z_0)}{k!}$

$$f(z) = z^2 e^z = z^2 \sum_{k=0}^{\infty} \frac{z^k}{k!} = \sum_{k=0}^{\infty} \frac{z^{k+2}}{k!} = z^2\left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right) = z^2 + z^3 + \frac{z^4}{2!} + \frac{z^5}{3!} + \cdots$$

c) En el caso siguiente a partir de una serie de Taylor se obtiene una serie de Laurent (si hay potencias negativas es una serie de Laurent)

$$f(z) = e^{\left(\frac{1}{z}\right)} = e^{(z^{-1})} = 1 + (z^{-1}) + \frac{(z^{-1})^2}{2!} + \frac{(z^{-1})^3}{3!} + \cdots = 1 + z^{-1} + \frac{z^{-2}}{2!} + \frac{z^{-3}}{3!} + \frac{z^{-4}}{4!} + \cdots$$

$$= 1 + 1\frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \frac{1}{3!}\frac{1}{z^3} + \frac{1}{4!}\frac{1}{z^4} + \cdots = \sum_{k=0}^{\infty} \frac{z^{-k}}{k!} = \sum_{k=0}^{\infty} \frac{1}{k!\,z^k}$$

d) Otro ejemplo

$$f(z) = z^3 e^{\left(\frac{1}{z}\right)} = z^3\left(1 + 1z^{-1} + \frac{z^{-2}}{2!} + \frac{z^{-3}}{3!} + \frac{z^{-4}}{4!} + \cdots\right) = z^3 + z^2 + \frac{z^1}{2!} + \frac{1}{3!} + \frac{z^{-1}}{4!} + \frac{z^{-2}}{5!} \ldots$$

$$= z^3 \sum_{k=0}^{\infty} \frac{z^{-k}}{k!} = \sum_{k=0}^{\infty} \frac{z^{3-k}}{k!}$$

e) Podemos generalizar el ejemplo anterior

$$f(z) = z^n e^{\frac{1}{z}} = z^n \sum_{k=0}^{\infty} \frac{z^{-k}}{k!} = \sum_{k=0}^{\infty} \frac{z^{n-k}}{k!}$$

---

2. La serie de Taylor para la función seno

$$f(z) = sen(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k$$

$$= \frac{f(z_0)(z - z_0)^0}{(0)!} + \frac{f'(z_0)(z - z_0)^1}{(1)!} + \frac{f''(z_0)(z - z_0)^2}{(2)!} + \frac{f'''(z_0)(z - z_0)^3}{(3)!} + \cdots$$

$$= \frac{sen(z_0)(z - z_0)^0}{(0)!} + \frac{cos(z_0)(z - z_0)^1}{(1)!} + \frac{-sen(z_0)(z - z_0)^2}{(2)!} + \frac{-cos(z_0)(z - z_0)^3}{(3)!} + \cdots$$

a) Si $z_0 = 0$, tenemos:

$$f(z) = sen(z) = \frac{sen(0)(z)^0}{(0)!} + \frac{\cos(0)\,(z)^1}{(1)!} + \frac{-sen(0)(z)^2}{(2)!} + \frac{-\cos(0)\,(z)^3}{(3)!} + \cdots$$

$$= z^1 - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k z^{(2k+1)}}{(2k+1)!}$$

b) Podemos aprovechar la serie anterior para calcular la serie de

$$f(z) = z^3 sen(z) = z^3\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right) = z^4 - \frac{z^6}{3!} + \frac{z^8}{5!} - \frac{z^{10}}{7!} + \cdots$$

c) También se puede calcular la serie de Laurent siguiente (recuerden que si hay potencias negativas es serie de Laurent)

$$f(z) = \frac{sen(z)}{z^2} = z^{-2}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right) = z^{-1} - \frac{z^1}{3!} + \frac{z^3}{5!} - \frac{z^5}{7!} + \cdots$$

d) Otra serie de Laurent muy usual es

$$f(z) = sen(z^{-1}) = \sum_{k=0}^{\infty} \frac{(-1)^k \,(z^{-1})^{\,(2k+1)}}{(2k+1)!} = z^{-1} - \frac{z^{-3}}{3!} + \frac{z^{-5}}{5!} - \frac{z^{-7}}{7!} + \cdots$$

e) O también la serie de Laurent de la función

$$f(z) = z^2 sen(z^{-1}) = z^2\left(z^{-1} - \frac{z^{-3}}{3!} + \frac{z^{-5}}{5!} - \frac{z^{-7}}{7!} + \cdots\right) = z - \frac{z^{-1}}{3!} + \frac{z^{-3}}{5!} - \frac{z^{-5}}{7!} + \cdots$$

f) Podemos generalizar la anterior:

$$f(z) = z^n \sin\left(\frac{1}{z}\right) = z^n \sum_{k=0}^{\infty} \frac{(-1)^k \, z^{-(2k+1)}}{(2k+1)!} = \sum_{k=0}^{\infty} \frac{(-1)^k \, z^{n-(2k+1)}}{(2k+1)!}$$

---

3. Análogamente, la serie de Taylor para la función coseno

$$f(z) = cos(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k$$

$$= \frac{f(z_0)(z-z_0)^0}{(0)!} + \frac{f'(z_0)(z-z_0)^1}{(1)!} + \frac{f''(z_0)(z-z_0)^2}{(2)!} + \frac{f'''(z_0)(z-z_0)^3}{(3)!} + \cdots$$

$$= \frac{cos(z_0)(z-z_0)^0}{(0)!} + \frac{-sen(z_0)(z-z_0)^1}{(1)!} + \frac{-cos(z_0)(z-z_0)^2}{(2)!} + \frac{sen(z_0)(z-z_0)^3}{(3)!} + \cdots$$

a) Si $z_0 = \frac{\pi}{4}$, tenemos:

$$f(z) = cos(z) = \frac{cos\left(\frac{\pi}{4}\right)\left(z-\frac{\pi}{4}\right)^0}{(0)!} + \frac{-\mathrm{sen}\left(\frac{\pi}{4}\right)\left(z-\frac{\pi}{4}\right)^1}{(1)!} + \frac{-cos\left(\frac{\pi}{4}\right)\left(z-\frac{\pi}{4}\right)^2}{(2)!} + \frac{\mathrm{sen}\left(\frac{\pi}{4}\right)\left(z-\frac{\pi}{4}\right)^3}{(3)!} + \cdots$$

b) Si $z_0 = -\frac{\pi}{4}$, tenemos:

$$f(z) = cos(z)$$

$$= \frac{cos\left(-\frac{\pi}{4}\right)\left(z+\frac{\pi}{4}\right)^0}{(0)!} + \frac{-\mathrm{sen}\left(-\frac{\pi}{4}\right)\left(z+\frac{\pi}{4}\right)^1}{(1)!} + \frac{-cos\left(-\frac{\pi}{4}\right)\left(z+\frac{\pi}{4}\right)^2}{(2)!}$$

$$+ \frac{\mathrm{sen}\left(-\frac{\pi}{4}\right)\left(z+\frac{\pi}{4}\right)^3}{(3)!} + \cdots$$

$$= \frac{1}{\sqrt{2}}\left(1 + \left(z+\frac{\pi}{4}\right)^1 - \frac{\left(z+\frac{\pi}{4}\right)^2}{2!} - \frac{\left(z+\frac{\pi}{4}\right)^3}{3!} + \frac{\left(z+\frac{\pi}{4}\right)^4}{4!} + \frac{\left(z+\frac{\pi}{4}\right)^5}{5!} - \cdots\right)$$

c) Si $z_0 = 0$, tenemos:

$$f(z) = cos(z) = \frac{cos(0)(z)^0}{(0)!} + \frac{-\mathrm{sen}(0)\,(z)^1}{(1)!} + \frac{-cos(0)(z)^2}{(2)!} + \frac{\mathrm{sen}(0)\,(z)^3}{(3)!} + \cdots$$

$$cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k z^{(2k)}}{(2k)!}$$

d) Podemos aprovechar la serie anterior para calcular la serie de

$$f(z) = z^3 cos(z) = z^3\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots\right) = z^3 - \frac{z^5}{2!} + \frac{z^7}{4!} - \frac{z^9}{6!} + \cdots$$

b) También se puede calcular la serie de Laurent siguiente

$$f(z) = cos(z^{-1}) = \sum_{k=0}^{\infty} \frac{(-1)^k\, z^{-(2k)}}{(2k)!} = 1 - \frac{z^{-2}}{2!} + \frac{z^{-4}}{4!} - \frac{z^{-6}}{6!} + \cdots$$

c) También la serie de Laurent de la función

$$f(z) = z^2 cos(z^{-1}) = z^2 \left(1 - \frac{z^{-2}}{2!} + \frac{z^{-4}}{4!} - \frac{z^{-6}}{6!} + \cdots\right) = z^2 - \frac{1}{2!} + \frac{z^{-2}}{4!} - \frac{z^{-4}}{6!} + \cdots$$

d) generalizando

$$f(z) = z^n \cos\left(\frac{1}{z}\right) = z^n \sum_{k=0}^{\infty} \frac{(-1)^k\, z^{-(2k)}}{(2k)!} = \sum_{k=0}^{\infty} \frac{(-1)^k\, z^{n-(2k)}}{(2k)!}$$

Si a_{-1} es el coeficiente de $(z - z_0)^{-1}$ en la serie de Laurent de

$$f(z) = \sum_{k=-\infty}^{\infty} a_k\, (z - z_0)^k$$

$$\oint_c f(z)\, dz = 2\pi i\, a_{-1}$$

Se puede obtener integrando la serie de Laurent en un contorno cerrado alrededor de z_0

$$\oint_c f(z)\, dz = \cdots + a_{-3} \oint_c dz \frac{1}{(z - z_0)^3} + a_{-2} \oint_c dz \frac{1}{(z - z_0)^2} + a_{-1} \oint_c dz \frac{1}{(z - z_0)^1}$$

$$+ a_0 \oint_c dz + a_1 \oint_c dz\, (z - z_0) + a_2 \oint_c dz\, (z - z_0)^2 + \cdots = 2\pi i\, a_{-1}$$

Serie aritmética

$$1 + 2 + 3 + 4 + 5 + \cdots + n = \frac{n(n+1)}{2}$$

Otros ejemplos de series de Laurent se obtienen de la fórmula:

$$\frac{1}{1 - z} = 1 + z + z^2 + z^3 + \cdots$$

La cual solo es válida solo cuando $|z| < 1$, lo cual define su radio de convergencia
Veamos la deducción para entender

De la serie geométrica:

$$S = z^0 + z^1 + z^2 + z^3 + \cdots + z^n =$$
$$zS = z^1 + z^2 + z^3 + z^4 + \cdots + z^{n+1}$$
$$S(1-z) = z^0 - z^{n+1}$$
$$S = \frac{1 - z^{n+1}}{1-z}$$
$$S = \frac{1 - \frac{1}{2}^{n+1}}{1-\frac{1}{2}}$$

Si z=1/2

Consideremos la suma parcial

$$S_n = 1 + z + z^2 + z^3 + \cdots + z^n$$

Multiplicando ambos miembros por z

$$zS_n = z + z^2 + z^3 + \cdots + z^n + z^{n+1}$$

De las dos anteriores se obtiene

$$S_n - zS_n = S_n(1-z) = 1 - z^{n+1}$$

Despejando

$$S_n = \frac{1 - z^{n+1}}{1-z}$$

Si $|z| < 1$ entonces $\lim_{n\to\infty} z^{n+1} = 0$ y se obtiene

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots$$

Ejemplos de series de Laurent que se obtienen usando la fórmula anterior

a) $f(z) = \frac{1}{1-az} = 1 + az + (az)^2 + (az)^3 + \cdots$ (solo potencias positivas de z)
   siempre que $|az| < 1$ o bien $|z| < \frac{1}{|a|}$, la serie converge para todo z dentro del disco de radio $\frac{1}{|a|}$
b) $f(z) = \frac{1}{1-\left(\frac{a}{z}\right)} = 1 + \frac{a}{z} + \left(\frac{a}{z}\right)^2 + \left(\frac{a}{z}\right)^3 + \cdots$ (potencias negativas de z)

   siempre que $\left|\frac{a}{z}\right| < 1$ o bien $|a| < |z|$, la serie converge para todo z fuera del disco de radio $|a|$
c) $f(z) = \frac{1}{1-a(z-z_0)} = 1 + a(z-z_0) + (a(z-z_0))^2 + (a(z-z_0))^3 + \cdots$

siempre que $|a(z-z_0)|<1$, o bien $|z-z_0|<\frac{1}{|a|}$, la serie converge para todo z dentro del disco centrado en z_0 de radio $\frac{1}{|a|}$

d) $f(z)=\frac{1}{1-\frac{a}{(z-z_0)}}=1+\frac{a}{z-z_0}+\left(\frac{a}{z-z_0}\right)^2+\left(\frac{a}{z-z_0}\right)^3+\cdots$

siempre que $\left|\frac{a}{z-z_0}\right|<1$, o bien $|a|<|z-z_0|$, la serie converge para todo z fuera del disco centrado en z_0 de radio $|a|$

Observar que para las potencias positivas se habla de la convergencia dentro de un disco y para las potencias negativas se habla de la convergencia fuera de un disco. Para una serie de Laurent con potencias positivas y negativas se hablará de un anillo de convergencia $r_1<|z-z_0|<r_2$, el cual corresponde a la intersección de las dos regiones de convergencia.

$$f(z)=\sum_{k=-\infty}^{\infty} a_k\,(z-z_0)^k$$

$$=\cdots+a_{-3}(z-z_0)^{-3}+a_{-2}(z-z_0)^{-2}+a_{-1}(z-z_0)^{-1}+a_0+a_1(z-z_0)+a_2(z-z_0)^2+\cdots$$

Más ejemplos, ahora una función con desarrollos distintos de serie de Laurent:

a) $f(z)=\frac{1}{az-1}=\frac{-1}{1-az}=-(1+az+(az)^2+(az)^3+\cdots)\qquad |z|<\frac{1}{|a|}$

b) $f(z)=\frac{1}{az-1}=\frac{1}{az}\left(\frac{1}{1-\frac{1}{az}}\right)=\frac{1}{az}\left(1+\frac{1}{az}+\left(\frac{1}{az}\right)^2+\left(\frac{1}{az}\right)^3+\cdots\right)\qquad \frac{1}{|a|}<|z|$

c) $g(z)=\frac{1}{1-z}=(1+z+(z)^2+(z)^3+\cdots)\qquad |z|<1$ disco de radio 1

d) $g(z)=\frac{1}{1-z}=\frac{1}{1-(z-z_0)-z_0}=\frac{1}{(1-z_0)-(z-z_0)}=\frac{1}{(1-z_0)}\left(\frac{1}{1-\frac{z-z_0}{1-z_0}}\right)$

$$=\frac{1}{(1-z_0)}\left(1+\left(\frac{z-z_0}{1-z_0}\right)+\left(\frac{z-z_0}{1-z_0}\right)^2+\left(\frac{z-z_0}{1-z_0}\right)^3+\cdots\right)$$

$\left|\frac{z-z_0}{1-z_0}\right|<1=,\ |z-z_0|<|1-z_0|$ Disco de radio $|1-z_0|$ centrado en z_0

Ejercicio

$$\oint_{|z|=\frac{1}{2}} z^2 sen\left(\frac{1}{z}\right)dz=2\pi i a_{-1}$$

$$= 2\pi i\left(-\frac{1}{3!}\right) = -\frac{\pi i}{3}$$

$$f(z) = z^2 sen(z^{-1}) = z^2\left(z^{-1} - \frac{z^{-3}}{3!} + \frac{z^{-5}}{5!} - \frac{z^{-7}}{7!} + \cdots\right) = 1z - \frac{1}{3!}z^{-1} + \frac{1}{5!}z^{-3} - \frac{z^{-5}}{7!} + \cdots$$

Ejercicio

$$\oint_{|z|=\frac{1}{2}} z^3 sen\left(\frac{1}{z}\right) dz = 2\pi i a_{-1} = 0$$

$$z^3 sen\left(\frac{1}{z}\right) = z^3\left(z^{-1} - \frac{z^{-3}}{3!} + \frac{z^{-5}}{5!} - \frac{z^{-7}}{7!} + \cdots\right) = \left(z^2 - \frac{1}{3!}z^0 + \frac{z^{-2}}{5!} - \frac{z^{-4}}{7!} + \cdots\right)$$

Ejemplo de integración usando series de Laurent

1. Ejemplo: Calcular

$$I = \oint_{|z|=1} z^7 e^{\frac{1}{z}} dz = 2\pi i a_{-1} = \frac{2\pi i}{8!}$$

Solución: Expandir en serie de Laurent el integrando y encontrar el residuo (coeficiente de $\frac{1}{z}$)

$$f(z) = z^7 e^{\frac{1}{z}} = z^7 \sum_{k=0}^{\infty} \frac{z^{-k}}{k!} = \sum_{k=0}^{\infty} \frac{z^{7-k}}{k!} = \cdots + \frac{z^{7-(7+1)}}{(7+1)!} + \cdots$$

$$z^7 e^{\frac{1}{z}} = z^7 + z^6 + \frac{z^5}{2!} + \frac{z^4}{3!} + \frac{z^3}{4!} + \frac{z^2}{5!} + \frac{z^1}{6!} + \frac{z^0}{7!} + \frac{1}{8!}z^{-1} + \frac{z^{-2}}{9!} \ldots$$

Por tanto, $I = \frac{2\pi i}{8!}$

2. Calcular $I = \oint_{|z|=1} z^n\, e^{\frac{1}{z}} dz$ donde $n \in \mathbb{N}$

Solución:

$$f(z) = z^n e^{\frac{1}{z}} = z^n \sum_{k=0}^{\infty} \frac{z^{-k}}{k!} = \sum_{k=0}^{\infty} \frac{z^{n-k}}{k!} = \cdots + \frac{1}{(n+1)!} z^{n-(n+1)} + \cdots$$

Por tanto, $I = \frac{2\pi i}{(n+1)!}$

$$I = \oint_{|z|=1} (2z^2 - 3z + 1) e^{\frac{1}{z}} dz = 2\pi i(a_{-1} + a_{-1} + a_{-1})$$

$$= 2\oint_{|z|=1} z^2 e^{\frac{1}{z}} dz - 3\oint_{|z|=1} z e^{\frac{1}{z}} dz + \oint_{|z|=1} e^{\frac{1}{z}} dz = 2\pi i\left(\frac{2}{3!} - \frac{3}{2!} + 1\right) = 2\pi i\left(\frac{1}{3} - \frac{3}{2} + 1\right) = \frac{4\pi i}{3!} - \frac{6\pi i}{2!} + 2\pi i$$

$$e^{z^{-1}} = 1 + 1\frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \frac{1}{3!}\frac{1}{z^3} + \frac{1}{4!}\frac{1}{z^4} + \cdots$$

3. Calcular $I = \oint_{|z|=1} P_N(z)\, e^{\frac{1}{z}} dz$ donde

$$P_N(z) = \sum_{n=0}^{N} a_n z^n$$

Solución:

$$I = \oint_{|z|=1} \left(\sum_{n=0}^{N} a_n z^n\right) e^{\frac{1}{z}} dz$$

$$f(z) = \left(\sum_{n=0}^{N} a_n z^n\right) e^{\frac{1}{z}} = \left(\sum_{n=0}^{N} a_n z^n\right) \sum_{k=0}^{\infty} \frac{z^{-k}}{k!} = \sum_{n=0}^{N} \sum_{k=0}^{\infty} \frac{a_n z^{n-k}}{k!} = \cdots + \sum_{n=0}^{N} \frac{a_n z^{n-(n+1)}}{(n+1)!} + \cdots$$

$$= z^{-1} \sum_{n=0}^{N} \frac{a_n}{(n+1)!} + \cdots$$

Por tanto, $\oint_{|z|=1} z^n e^{\frac{1}{z}} dz = 2\pi i \sum_{n=0}^{N} \frac{a_n}{(n+1)!}$

Ejemplo:

$$\oint_{C:|z|=1} (z^3 + 2z^2 - 3z + 5) e^{\frac{1}{z}}$$
$$= \oint_{C:|z|=1} \left(z^3 e^{\frac{1}{z}}\right) dz$$
$$+ \oint_{C:|z|=1} \left(2z^2 e^{\frac{1}{z}}\right) dz$$
$$+ \oint_{C:|z|=1} \left(-3z e^{\frac{1}{z}}\right) dz$$
$$+ \oint_{C:|z|=1} \left(5 e^{\frac{1}{z}}\right) dz$$
$$= \oint_{C:|z|=1} \left(z^3 + z^2 + \frac{z^1}{2!} + \frac{z^0}{3!} + \frac{z^{-1}}{4!} + \frac{z^{-2}}{5!} + \cdots\right) dz$$
$$+ \oint_{C:|z|=1} 2\left(z^2 + z^1 + \frac{z^0}{2!} + \frac{z^{-1}}{3!} + \frac{z^{-2}}{4!} \ldots\right) dz$$
$$+ \oint_{C:|z|=1} -3\left(z + z^0 + \frac{z^{-1}}{2!} + \frac{z^{-2}}{3!} \ldots\right) dz$$
$$+ \oint_{C:|z|=1} 5\left(1 + z^{-1} + \frac{z^{-2}}{2!} + \frac{z^{-3}}{3!} \ldots\right) dz$$

$$= 2\pi i \left(\frac{1}{4!} + \frac{2}{3!} - \frac{3}{2!} + 5\right)$$

1.

2. Ejemplo: Calcular $I = \oint_{|z|=1} z^7 \sin\left(\frac{1}{z}\right) dz$

Solución: Expandir en serie de Laurent el integrando y encontrar el residuo (coeficiente de $\frac{1}{z}$)

$$f(z) = z^7 \sin\left(\frac{1}{z}\right) = z^7 \sum_{k=0}^{\infty} \frac{(-1)^k\, z^{-(2k+1)}}{(2k+1)!} = \sum_{k=0}^{\infty} \frac{(-1)^k\, z^{7-(2k+1)}}{(2k+1)!} = \cdots + 0\, z^{-1} + \cdots$$

Por tanto, $I = 2\pi i\; 0 = 0$

3. Calcular $I = \oint_{|z|=1} z^n \sin\left(\frac{1}{z}\right) dz$ donde $n \in \mathbb{N}$

   Solución:

$$f(z) = z^n \sin\left(\frac{1}{z}\right) = z^n \sum_{k=0}^{\infty} \frac{(-1)^k\, z^{-(2k+1)}}{(2k+1)!} = \sum_{k=0}^{\infty} \frac{(-1)^k\, z^{n-(2k+1)}}{(2k+1)!}$$

   El coeficiente de $z^{-1}\; será\; \frac{(-1)^k\, z^{2k-(2k+1)}}{(2k+1)!}\;\; si\; n = 2k$ ó será cero $si\; n\; es\; impar$

   Por tanto, $I = 0\; si\; n\; es\; impar$ ó $I = 2\pi i\; \frac{(-1)^k}{(2k+1)!}, si\; n = 2k$

4. Calcular $I = \oint_{|z|=1} P_N(z) \sin\left(\frac{1}{z}\right) dz$ donde $P_N(z) = \sum_{n=0}^{N} a_n z^n$

   Solución:

$$f(z) = \left(\sum_{n=0}^{N} a_n z^n\right) \sin\left(\frac{1}{z}\right) = \sum_{n=0}^{N} a_n z^n \sum_{k=0}^{\infty} \frac{(-1)^k\, z^{-(2k+1)}}{(2k+1)!} = \sum_{n=0}^{N} \sum_{k=0}^{\infty} \frac{(-1)^k\, a_n z^{n-(2k+1)}}{(2k+1)!}$$

$$= z^{-1}\left(\frac{a_0}{1!} - \frac{a_2}{3!} + \frac{a_4}{5!} - \frac{a_6}{7!} + \cdots\right) + \cdots$$

   Por tanto, I=$2\pi i\; \left(\frac{a_0}{1!} - \frac{a_2}{3!} + \frac{a_4}{5!} - \frac{a_6}{7!} + \cdots\right)$

1. Calcular $I = \oint_{|z|=1} z^7 \cos\left(\frac{1}{z}\right) dz$

Solución: Expandir en serie de Laurent el integrando y encontrar el residuo (coeficiente de $\frac{1}{z}$)

$$f(z) = z^7 \cos\left(\frac{1}{z}\right) = z^7 \sum_{k=0}^{\infty} \frac{(-1)^k\, z^{-(2k)}}{(2k)!} = \sum_{k=0}^{\infty} \frac{(-1)^k\, z^{7-(2k)}}{(2k)!} = \cdots + \frac{z^{7-(8)}}{(8)!} + \cdots$$

Por tanto, $I = 2\pi i \frac{1}{8!}$

2. Calcular $I = \oint_{|z|=1} z^n \cos\left(\frac{1}{z}\right) dz$ para $n \in \mathbb{N}$

   Solución:

   $$f(z) = z^n \cos\left(\frac{1}{z}\right) = z^n \sum_{k=0}^{\infty} \frac{(-1)^k\, z^{-(2k)}}{(2k)!} = \sum_{k=0}^{\infty} \frac{(-1)^k\, z^{n-(2k)}}{(2k)!}$$

   El coeficiente de z^{-1} será igual a 0 si n es par y será igual a $\frac{(-1)^k\, z^{n-(2k)}}{(2k)!}$ Si n=2k-1 y k>0

   Por tanto, $I = 0$ si n es par, ó $I = 2\pi i\ \frac{(-1)^k}{(n+1)!}$ si n es impar y k=$\frac{n+1}{2}$

3. Calcular $I = \oint_{|z|=1} P_N(z) \cos\left(\frac{1}{z}\right) dz$ donde $P_N(z) = \sum_{n=0}^{N} a_n z^n$

   Solución:

   $$f(z) = \left(\sum_{n=0}^{N} a_n z^n\right) \cos\left(\frac{1}{z}\right) = \sum_{n=0}^{N} a_n z^n \sum_{k=0}^{\infty} \frac{(-1)^k\, z^{-(2k)}}{(2k)!} = \sum_{n=0}^{N} \sum_{k=0}^{\infty} \frac{(-1)^k\, a_n z^{n-(2k)}}{(2k)!}$$

   $$= z^{-1}\left(\frac{a_0}{1!} - \frac{a_2}{3!} + \frac{a_4}{5!} - \frac{a_6}{7!} + \cdots\right) + \cdots$$

   Por tanto, $I = 2\pi i\ \left(-\frac{a_1}{2!} + \frac{a_3}{4!} - \frac{a_5}{6!} + \frac{a_7}{8!} + \cdots\right)$

Formulario:

1.

$$\int_{a}^{b} f(z)\,dz = \int_{t_0}^{t_1} f(z(t))\,\frac{dz(t)}{dt}\,dt$$

2.

$$\oint_{c} f(z)\,dz = 2\pi i \sum_{k=1}^{n} res\big(f(z_k)\big)$$

$$res\big(f(z_k)\big) = a_{-1}\ donde\ f(z) = \sum_{k=-\infty}^{\infty} a_k\,(z-z_0)^k$$

3. **Si** $\boldsymbol{f(z) = \frac{\varphi(z)}{\psi(z)}, donde\ \varphi(z_0) \neq 0,\ \psi(z_0) = 0, \psi'(z_0) \neq 0,\ \phi\ y\ \psi\ analíticas\ dentro\ de\ C, y\ z_0}$ **está dentro de la curva** $res\big(f(z_0)\big) = \frac{\varphi(z_0)}{\psi'(z_0)}$

4. **Si** $\boldsymbol{f(z) = \frac{\varphi(z)}{(z-z_0)^n}}$ **donde** $\boldsymbol{\varphi(z)}$ **es analítica dentro de la curva C y** $\boldsymbol{z_0}$ **está dentro de C, n es natural**

$$res\big(f(z_0)\big) = \frac{\varphi^{(n-1)}(z_0)}{(n-1)!} = \frac{1}{(n-1)!}\lim_{z\to z_0}\left\{\frac{d^{n-1}}{dz^{n-1}}\big(f(z)(z-z_0)^n\big)\right\}$$

337. $\displaystyle\int\limits_{|z|=1} z \operatorname{tg} \pi z \, dz.$

338. $\displaystyle\int\limits_{C} \frac{z\,dz}{(z-1)^2(z+2)}$, donde C: $x^{2/3}+y^{2/3}=3^{2/3}$.

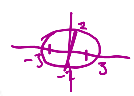
339. $\displaystyle\int\limits_{|z|=2} \frac{e^z\,dz}{z^3(z+1)}$. 340. $\displaystyle\int\limits_{|z-i|=3} \frac{e^{z^2}-1}{z^3-iz^2}\,dz.$

341. $\displaystyle\int\limits_{|z|=1/2} z^2 \operatorname{sen} \frac{1}{z}\,dz.$ 342. $\displaystyle\int\limits_{|z|=\sqrt{3}} \frac{\operatorname{sen} \pi z}{z^2-z}\,dz.$

343. $\displaystyle\int\limits_{|z+1|=4} \frac{z\,dz}{e^z+3}$. 344. $\displaystyle\int\limits_{|z|=1} \frac{z^2\,az}{\operatorname{sen}^3 z \cos z}$.

345. $\displaystyle\int\limits_{|z-i|=1} \frac{e^z\,dz}{z^4+2z^2+1}$. 346. $\displaystyle\int\limits_{|z|=4} \frac{e^{iz}}{(z-\pi)^3}$.

347. $\displaystyle\int\limits_{C} \frac{\cos\frac{z}{2}}{z^2-4}\,dz$, C: $\dfrac{x^2}{\underset{3^2}{9}}+\dfrac{y^2}{\underset{2^2}{4}}=1.$



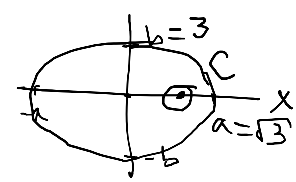
348. $\displaystyle\int\limits_{C} \frac{e^{2z}}{z^3-1}\,dz$, C: $x^2+y^2-2x=0.$ $z=\sqrt[3]{1}=e^{i}$

349. $\displaystyle\int\limits_{C} \frac{\operatorname{sen} \pi z}{(z^2-1)^2}\,dz$, C: $\dfrac{x^2}{4}+y^2=1.$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

350. $\displaystyle\int\limits_{C} \frac{z+1}{z^2+2z-3}\,dz$; C: $x^2+y^2=16.$

351. $\displaystyle\int\limits_{C} \frac{z \operatorname{sen} z}{(z-1)^5}\,dz$, C: $\dfrac{x^2}{3}+\dfrac{y^2}{9}=1.$ $\longrightarrow$



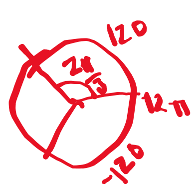
352. $\displaystyle\int\limits_{C} \frac{dz}{z^4+1}$, C: $x^2+y^2=2x.$

$$\oint_C \frac{\varphi(z)}{(z-z_0)^5}\,dz = 2\pi i\,\frac{\varphi^{(IV)}(z_0)}{4!}$$

$$= \frac{2\pi i}{24}\left(-4\cos(1)+1\operatorname{sen}(1)\right)$$

353. $\displaystyle\int\limits_{|z|=1} z^3 \operatorname{sen}\frac{1}{z}\,dz.$

$$\varphi(z) = z \operatorname{sen} z$$



$$z = e^{i\left(\frac{2\pi k}{3}\right)}$$

354. $\displaystyle\int_{|z|=1/3} (z+1)\, e^{1/z}\, dz.$

355. $\displaystyle\int_{|z|=2/3} \left(\operatorname{sen}\frac{1}{z^2} + e^{z^2}\cos z\right) dz.$

$\varphi'(z) = 1\,\mathrm{Sen}\,z + z\,\mathrm{Cos}\,z$

$\varphi''(z) = \mathrm{Cos}\,z + 1\,\mathrm{Cos}\,z + z(-\mathrm{Sen}\,z)$

$\varphi'''(z) = -\mathrm{Sen}\,z - \mathrm{Sen}\,z - 1\,\mathrm{Sen}\,z - z\,\mathrm{Cos}\,z$

$= -3\,\mathrm{Sen}\,z - z\,\mathrm{Cos}\,z$

$\varphi^{(IV)}(z) = -3\,\mathrm{Cos}\,z - 1\,\mathrm{Cos}\,z + z\,\mathrm{Sen}\,z = -4\,\mathrm{Cos}\,z + z\,\mathrm{Sen}\,z$

337

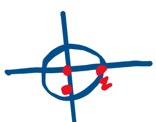
$$\oint_{|z|=1} \frac{z\sin(\pi z)}{\cos(\pi z)}dz = 2\pi i\left(-\frac{\frac{1}{2}\sin\left(\frac{\pi}{2}\right)}{\operatorname{sen}\left(\frac{\pi}{2}\right)\pi} + \left(-\frac{-\frac{1}{2}\sin\left(-\frac{\pi}{2}\right)}{\operatorname{sen}\left(-\frac{\pi}{2}\right)\pi}\right)\right)$$
$$= 2\pi i\left(-\frac{1}{2\pi} + \frac{1}{2\pi}\right) = 0$$

338

$$\oint_C \frac{z}{(z-1)^2(z+2)}dz = \oint_{|z-1|=.1} \frac{\left(\frac{z}{(z+2)}\right)}{(z-1)^2}dz + \oint_{|z+2|=.1} \frac{\left(\frac{z}{(z-1)^2}\right)}{(z+2)}dz$$
$$= 2\pi i\left(((1+2)^{-1} - 1(1+2)^{-2}) + \frac{(-2)}{(-2-1)^2}\right) = 2\pi i\left(((3)^{-1} - 3^{-2}) + \frac{(-2)}{(-3)^2}\right)$$
$$= 2\pi i\left(\frac{1}{3} - \frac{1}{9} - \frac{2}{9}\right) = 0$$

$$\int_{-\infty}^{\infty} \frac{1-\cos(w)}{w^2} e^{iwt} dw$$

175



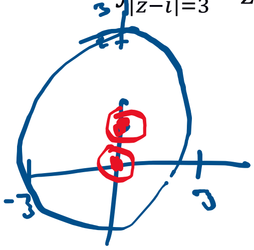
$$\oint_{|z|=1} \frac{senh\left(\frac{\pi}{2}(z+i)\right)dz}{z(z-2)} = 2\pi i\left(\frac{senh\left(\frac{\pi}{2}(0+i)\right)}{(0-2)}\right) = -\pi i\,\frac{e^{\frac{\pi i}{2}} - e^{-\frac{\pi i}{2}}}{2} = \pi\,\frac{e^{i\frac{\pi}{2}} - e^{-i\frac{\pi}{2}}}{2i} = \pi\, sen\left(\frac{\pi}{2}\right) = \pi$$

5.5 Zill

$$\oint_{|z-i|=1} \frac{e^{z^2}dz}{(z-i)^3} = \frac{2\pi i}{2}\lim_{z\to i}\left\{\frac{d^2}{dz^2}\left(e^{z^2}\right)\right\} = \pi i\lim_{z\to i}\left(e^{z^2}(2z)^2 + e^{z^2}2\right) = \pi i(-4e^{-1} + 2e^{-1})$$
$$= -\frac{2\pi i}{e}$$

340

$$\oint_{|z-i|=3} \frac{(e^{z^2}-1)dz}{z^3 - iz^2} = \oint_{|z-i|=3} \frac{(e^{z^2}-1)dz}{z^2(z-i)} = 2\pi i\left(\lim_{z\to 0}\left\{\frac{d}{dz}\frac{(e^{z^2}-1)]}{(z-i)}\right\} + \frac{(e^{i^2}-1)}{i^2}\right)$$
$$= 2\pi i\left(0 + \frac{(e^{i^2}-1)}{i^2}\right) = -2\pi i(e^{-1}-1)$$



$-2, 1$

$$\frac{d}{dz}\left[\left(e^{z^2}-1\right)(z-i)^{-1}\right]=e^{z^2}2z(z-i)^{-1}-\left(e^{z^2}-1\right)(z-i)^{-2}$$

341

$$\oint_{|z|=\frac{1}{2}} z^2 sen\left(\frac{1}{z}\right)dz = 2\pi i a_{-1} = 2\pi i\left(-\frac{1}{3!}\right) = -\frac{\pi i}{3}$$

$$z^2 sen\left(\frac{1}{z}\right) = z^2\left(z^{-1}-\frac{z^{-3}}{3!}+\frac{z^{-5}}{5!}-\cdots\right)$$

$$=\left(z^1-\frac{1}{3!}z^{-1}+\frac{z^{-3}}{5!}-\cdots\right)$$

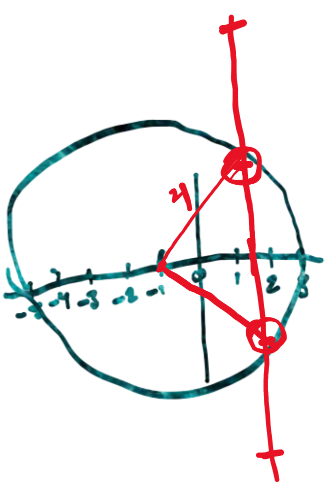
$$\oint_{|z|=\frac{1}{2}} \frac{sen\left(\frac{1}{z}\right)}{z^2}dz = 2\pi i = 0$$

I

$$\left(z^{-3}-\frac{z^{-5}}{3!}+\frac{z^{-7}}{5!}-\cdots\right)$$

343

$$\oint_{|z+1|=4}\frac{z}{(e^z+3)}dz = 2\pi i\left(\frac{\ln(3)+i(\pi)}{-3}+\frac{\ln(3)-i(\pi)}{-3}\right)=2\pi i\frac{2\ln(3)}{-3}=\frac{4}{3}\pi i\, ln(3)$$



$$e^z = -3$$

$+\pi$
$-\pi$

$$z = \ln(-3) = \underbrace{\ln(3)} + \underbrace{i(\pi+2k\pi)}$$

$$\sqrt{\ln(3)^2+\pi^2} < 4$$

343

$$\oint_{|z+1|=4}\frac{z}{e^z+3}dz = \oint_{c1}\frac{z}{e^z+3}dz+\oint_{c2}\frac{z}{e^z+3}dz = 2\pi i\left(\frac{\phi(z_0)}{\psi'^{(z_0)}}+\frac{\phi(z_1)}{\psi'^{(z_1)}}\right)$$

$$=2\pi i\left(\frac{z_0}{e^{z_0}}+\frac{z_1}{e^{z_1}}\right)=2\pi i\left(\frac{\ln(3)+i(\pi)}{e^{\ln(3)+i(\pi)}}+\frac{\ln(3)-i\pi}{e^{\ln(3)-i\pi}}\right)$$

$$=2\pi i\left(\frac{\ln(3)+i(\pi)}{e^{\ln(3)}e^{i\pi}}+\frac{\ln(3)-i\pi}{e^{\ln(3)}e^{-i\pi}}\right)$$

$$= 2\pi i\left(\frac{\ln(3)+i(\pi)}{-3}+\frac{\ln(3)-i(\pi)}{-3}\right)=-\frac{4\pi i\ ln(3)}{3}$$

Raíces del denominador

$$\mathrm{z}=\ln(-3)=\ln\left(3e^{i(\pi+2k\pi)}\right)=\ln(3)+i(\pi+2k\pi)$$

$$\mathrm{e}^{\ln(3)\pm i(\pi)}=\mathrm{e}^{\ln(3)}\mathrm{e}^{\pm i(\pi)}=3\left(\cos(\pi)\pm isen(\pi)\right)=-3$$

$$z=\ln(-3)=\ln(3)+i(\pi+2k\pi), con\ k=0\ y\ k=-1$$

344

$$\oint_{|z|=1}\frac{z^2}{sen^3(z)\cos(z)}dz$$

$$=\oint_{|z|=1}\frac{z^2}{\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\cdots\right)^3\cos\,(z)}dz$$

$$=\oint_{|z|=1}\frac{z^2}{z^3\left(1-\frac{z^2}{3!}+\frac{z^4}{5!}-\cdots\right)^3\cos\,(z)}dz=2\pi i$$

$$\oint_{|z|=1}\frac{1}{z\left(1-\frac{z^2}{3!}+\frac{z^4}{5!}-\cdots\right)^3\cos\,(z)}dz=2\pi i$$

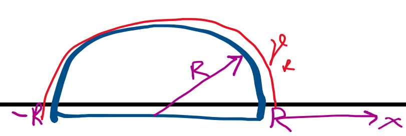
345

$$\oint_{|z-i|=1}\frac{e^z}{z^4+2z^2+1}dz=\oint_{|z-i|=1}\frac{e^z}{(z^2+1)^2}dz=\oint_{|z-i|=1}\frac{e^z(z+i)^{-2}}{(z-i)^2}dz$$

$$=2\pi i\left(e^i(i+i)^{-2}-2e^i(i+i)^{-3}\right)=2\pi ie^i((2i)^{-2}-2(2i)^{-3})=\frac{\pi e^i}{2}(1-i)$$

[

$$=\frac{\pi}{2}(sen(1)+\cos(1)+i(sen(1)-\cos(1))\,)$$



Algunas integrales de variable real que se resuelven con técnicas de integrales de variable compleja

En las siguientes 3 fórmulas $p_n(x)\,,q_m(x)$ son polinomios, $n\leq m+2$, si $q_m(z_k)=0$ entonces $z_k\notin\mathbb{R}$, C es una curva simple cerrada que contiene todas las raíces del semiplano superior de $q_m(x)$.

a) $\displaystyle\int_{-\infty}^{\infty}\frac{p_n(x)}{q_m(x)}dx=\oint_C\frac{p_n(z)}{q_m(z)}dz$

b) $\displaystyle\int_{-\infty}^{\infty}\frac{p_n(x)}{q_m(x)}sen(kx)\,dx=Im\left(\oint_C\frac{p_n(z)}{q_m(z)}e^{ikz}\,dz\right)=Im\left(\oint_C\frac{p_n(z)}{q_m(z)}\cos(kz)\,dz+i\oint_c\frac{p_n(z)}{q_m(z)}sen(kz)\ dz\right)$

c) $\displaystyle\int_{-\infty}^{\infty}\frac{p_n(x)}{q_m(x)}cos(kx)\,dx=Re\left(\oint_C\frac{p_n(z)}{q_m(z)}e^{ikz}\,dz\right)=Re\left(\oint_C\frac{p_n(z)}{q_m(z)}\left(\cos(kz)+i\,sen(kz)\right)dz\right)$

d) $\int_0^{2\pi} f(sen(\theta), \cos(\theta))d\theta = \oint_{|z|=1} f\left(\frac{z-z^{-1}}{2i}, \frac{z+z^{-1}}{2}\right)\frac{dz}{iz}$ f puede ser cualquier función racional

369. $\int_0^{\infty} \frac{x^2+1}{x^4+1}\,dx.$

370. $\int_{-\infty}^{+\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}$ $(a>0,\ b>0).$

371. $\int_{-\infty}^{+\infty} \frac{dx}{(x^2+1)^3}.$

372. $\int_{-\infty}^{+\infty} \frac{dx}{(1+x^2)^{n+1}}.$

373. $\int_{-\infty}^{+\infty} \frac{x\,dx}{(x^2+4x+13)^2}.$

374. $\int_{-\infty}^{+\infty} \frac{dx}{(x^2+a^2)^2(x^2+b^2)^2}.$

375. $\int_0^{+\infty} \frac{x^4+1}{x^6+1}\,dx.$

376. $\int_{-\infty}^{+\infty} \frac{x^{2m}}{1+x^{2n}}\,dx.$

377. $\int_{-\infty}^{+\infty} \frac{dx}{1+x^6}.$

378. $\int_{-\infty}^{+\infty} \frac{dx}{(x^2+2x+2)^2}.$

379. $\int_{-\infty}^{+\infty} \frac{x^4\,dx}{(a+bx^2)^4}$ $(a>0,\ b>0).$

380. Demostrar la fórmula $\int_{-\infty}^{\infty} \frac{dz}{(1+x^2)^{n+1}} = \frac{1\cdot 3\ \ldots\ (2n-1)}{2\cdot 4\cdot 6\ \ldots\ 2n}\pi.$

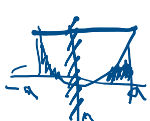
367. $-\frac{\pi}{3}i$. **368.** $2\pi i$. **369.** $\frac{\pi}{\sqrt{2}}$. **370.** $\frac{\pi}{ab(a+b)}$.

371. $\frac{3}{8}\pi$. **372.** $\frac{(2n)!}{(n!)^2}2^{-2n}\pi$. **373.** $-\frac{\pi}{27}$. **374.** $\frac{\pi}{2(b^2-a^2)^3}\times$ $\left(\frac{5b^2-a^2}{b^3}+\frac{b^2-5a^2}{a^3}\right)$. **375.** $\frac{2}{3}\pi$.

376. $\frac{\pi}{n\ \mathrm{sen}\ \frac{2m+1}{n}}$. **377.** $\frac{2}{3}\pi$. **378.** $\frac{\pi}{2}$.

379. $\frac{\pi}{16a^{3/2}b^{5/2}}$. **381.** $\frac{\pi}{3}e^{-3}(\cos 1 - 3\ \mathrm{sen}\ 1)$.

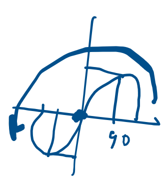
F es par si cumple que f(x)=f(-x)



$\int_{-a}^{a} = 2\int_0^a$

x^2

$5^2 = (-5)^2$



Ejemplos:

369

$$\int_0^\infty \frac{x^2+1}{x^4+1}dx = \frac{\pi}{\sqrt{2}}$$

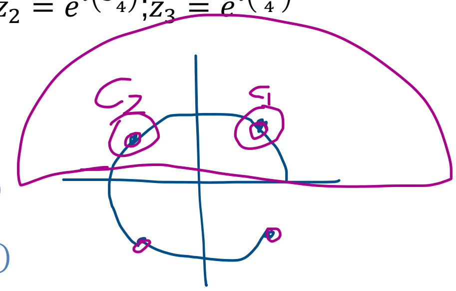
$$\int_0^\infty \frac{x^2+1}{x^4+1}dx = \frac{1}{2}\int_{-\infty}^\infty \frac{x^2+1}{x^4+1}dx = \frac{1}{2}\oint_{C_1}\frac{1+z^2}{1+z^4}dz + \frac{1}{2}\oint_{C_2}\frac{1+z^2}{1+z^4}dz = \frac{2\pi i}{2}\left(\frac{1+z_0^2}{4z_0^3}+\frac{1+z_1^2}{4z_1^3}\right)$$

$$= \frac{\pi i}{4}\left(z_0^{-3}(1+z_0^2)+z_1^{-3}(1+z_1^2)\right) = \frac{\pi i}{4}\left(e^{\frac{-i3\pi}{4}}\left(1+e^{\frac{i2(\pi)}{4}}\right)+e^{\frac{-9i\pi}{4}}\left(1+e^{\frac{i(6\pi)}{4}}\right)\right)$$

$$= \frac{\pi i}{4}\left(\left(e^{\frac{-i3\pi}{4}}+e^{-\frac{i(\pi)}{4}}\right)+\left(e^{\frac{-9i\pi}{4}}+e^{-\frac{i(3\pi)}{4}}\right)\right)$$

Raíces:

$$z = \left(1e^{i(\pi+2k\pi)}\right)^{1/4};\ z_0 = e^{i\left(\frac{\pi}{4}\right)};\ z_1 = e^{i\left(3\frac{\pi}{4}\right)};\ z_2 = e^{i\left(5\frac{\pi}{4}\right)};\ z_3 = e^{i\left(\frac{7\pi}{4}\right)}$$



Calcular las raíces de $1+z^4=0$

$$z^4 = -1 = 1e^{i(\pi+2k\pi)}$$

$$z_0 = e^{\frac{i(\pi)}{4}} = \frac{1}{\sqrt{2}}(1+i)$$

$$z_1 = e^{\frac{i(3\pi)}{4}} = \frac{1}{\sqrt{2}}(-1+i)$$

$$z_2 = e^{\frac{i(5\pi)}{4}} = \frac{1}{\sqrt{2}}(-1-i)$$

$$z_3 = e^{\frac{i(7\pi)}{4}} = \frac{1}{\sqrt{2}}(1-i)$$

$$I = \frac{\pi i}{4}\left(z_0^{-3}(1+z_0^2)+z_1^{-3}(1+z_1^2)\right) = \frac{\pi i}{4}\left(e^{\frac{-i3\pi}{4}}\left(1+e^{\frac{i2(\pi)}{4}}\right)+e^{\frac{-9i\pi}{4}}\left(1+e^{\frac{i(6\pi)}{4}}\right)\right)$$

$$\frac{\pi i}{4}\left(\left(e^{\frac{-i3\pi}{4}}+e^{-\frac{i(\pi)}{4}}\right)+\left(e^{\frac{-9i\pi}{4}}+e^{-\frac{i(3\pi)}{4}}\right)\right)$$

$$\frac{\pi i}{4}\left(\left(2e^{\frac{-i3\pi}{4}}+e^{-\frac{i(\pi)}{4}}\right)+\left(e^{-2i\pi}e^{\frac{-i\pi}{4}}\right)\right)$$

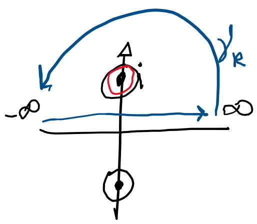
$$\frac{\pi i}{4}\left(\left(2e^{\frac{-i3\pi}{4}}+2e^{-\frac{i(\pi)}{4}}\right)\right)$$

$$= \frac{\pi i 2}{4}\left(\left(\frac{1}{\sqrt{2}}(-1-i)+\frac{1}{\sqrt{2}}(1-i)\right)\right)$$

$$= \frac{\pi}{\sqrt{2}}$$

371

$$\int_{-\infty}^\infty \frac{1dx}{(x^2+1)^3} = \int_{-\infty}^\infty \frac{dx}{(x+i)^3(x-i)^3} = \oint_\Gamma \frac{(z+i)^{-3}dz}{(z-i)^3} =$$



$\cdot - i$

1 obtener las raíces del denominador

2 observar que las raíces no son reales

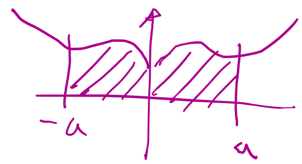
3 integrar sobre una curva cerrada que contenga todas las raíces con la parte imaginaria positiva

$$\oint_c \frac{(z+i)^{-3}dz}{(z-i)^3} = \oint_c \frac{\phi(z)dz}{(z-i)^3} = 2\pi i\left(\frac{\phi''(i)}{2!}\right) = \pi i 12(i+i)^{-5} = \frac{12\pi}{32} = \frac{3}{8}\pi$$

$$\phi = (z+i)^{-3}$$

$$\phi' = -3(z+i)^{-4}$$

$$\phi'' = 12(z+i)^{-5}$$



$$\int_{-a}^{a} f_{par}(x)\,dx = 2\int_0^a f(x)\,dx$$

In each of Problems 1 through 10, evaluate the integral. Wherever they appear, α and β are positive numbers.

1. $\int_0^{2\pi} \frac{1}{2-\cos(\theta)}\,d\theta$
2. $\int_{-\infty}^{\infty} \frac{1}{x^4+1}\,dx$
3. $\int_{-\infty}^{\infty} \frac{1}{x^6+1}\,dx$
4. $\int_0^{2\pi} \frac{1}{6+\sin(\theta)}\,d\theta$
5. $\int_{-\infty}^{\infty} \frac{x\sin(2x)}{x^4+16}\,dx$
6. $\int_{-\infty}^{\infty} \frac{1}{x^2-2x+6}\,dx$
7. $\int_{-\infty}^{\infty} \frac{\cos^2(x)}{(x^2+4)^2}\,dx$
8. $\int_0^{2\pi} \frac{2\sin(\theta)}{2+\sin^2(\theta)}\,d\theta$
9. $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+4)^2}\,dx$
10. $\int_{-\infty}^{\infty} \frac{\cos(\beta x)}{(x^2+\alpha^2)^2}\,dx$

In Problems 11 through 18, α and β are positive numbers wherever they occur.

11. Show that $\int_{-\infty}^{\infty} \frac{\cos(\alpha x)}{x^2+1}\,dx = \pi e^{-\alpha}$.
12. Show that $\int_{-\infty}^{\infty} \frac{x^2\cos(\alpha x)}{(x^2+\beta^2)^2}\,dx = \frac{\pi}{2\beta}e^{-\alpha\beta}(1-\alpha\beta)$.
13. Let $\alpha \neq \beta$. Show that $\int_0^{2\pi} \frac{1}{\alpha^2\cos^2(\theta)+\beta^2\sin^2(\theta)}\,d\theta = \frac{2\pi}{\alpha\beta}$.
14. Show that $\int_0^{\pi/2} \frac{1}{\alpha+\sin^2(\theta)}\,d\theta = \frac{\pi}{2\sqrt{\alpha(1+\alpha)}}$.
15. Show that $\int_0^{\infty} e^{-x^2}\cos(2\beta x)\,dx = \frac{\sqrt{\pi}}{2}e^{-\beta^2}$.
    Hint: Integrate e^{-z^2} about the rectangular path having corners at $\pm R$ and $\pm R+\beta i$. Use Cauchy's theorem

1.

$$I = \int_0^{2\pi} \frac{1}{2-\cos(\theta)}\,d\theta$$

Haciendo el cambio de variable:

$$z = e^{i\theta};\, dz = e^{i\theta} i\,d\theta \rightarrow d\theta = \frac{dz}{iz};\, \cos(\theta) = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+z^{-1}}{2}$$

$$I = \oint_{|z|=1} \left(\frac{1}{2-\frac{z+z^{-1}}{2}}\right)\frac{dz}{iz} = \frac{1}{i}\oint_{|z|=1} \frac{1}{2z-\frac{z^2+1}{2}}\frac{dz}{1} = -i\oint_{|z|=1} \frac{2}{4z-(z^2+1)}dz$$

$$= i\oint_{|z|=1} \frac{2}{z^2 - 4z + 1} dz$$

$$= i\oint_{|z|=1} \frac{2}{\left(z - (2+\sqrt{3})\right)\left(z - (2-\sqrt{3})\right)} dz$$

$$= i\oint_{|z|=1} \frac{\left[\frac{2}{\left(z-(2+\sqrt{3})\right)}\right]}{\left(z-(2-\sqrt{3})\right)} dz$$

$$= i2\pi i\left[\frac{2}{\left((2-\sqrt{3})-(2+\sqrt{3})\right)}\right]$$

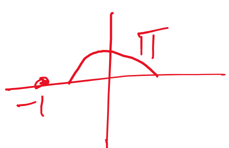
$$= -4\pi\left(\frac{1}{-2\sqrt{3}}\right)$$

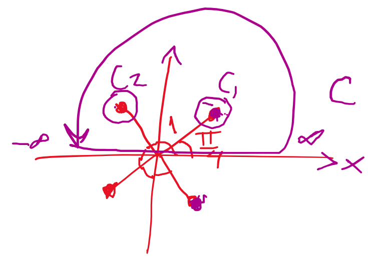
$$= \frac{2\pi}{\sqrt{3}}$$

2

$$I = \int_{-\infty}^{\infty} \frac{1}{1+x^4} dx = \oint_{c_1} \frac{1dz}{1+z^4} + \oint_{c_2} \frac{1dz}{1+z^4} = 2\pi i\left(\frac{1}{4z_0^3} + \frac{1}{4z_1^3}\right)$$

Calcular las raíces de $1 + z^4 = 0$





$$z^4 = -1 = 1e^{i(\pi+2k\pi)}$$

$$z_0 = e^{\frac{i(\pi)}{4}} = \frac{1}{\sqrt{2}}(1+i)$$

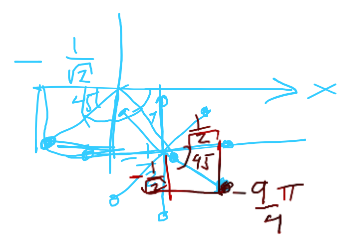
$$z_1 = e^{\frac{i(3\pi)}{4}} = \frac{1}{\sqrt{2}}(-1+i)$$

$$z_2 = e^{\frac{i(5\pi)}{4}} = \frac{1}{\sqrt{2}}(-1-i)$$

$$z_3 = e^{\frac{i(7\pi)}{4}} = \frac{1}{\sqrt{2}}(1-i)$$

$$\int \frac{\varphi}{\psi} dz = \frac{\varphi(z_0)}{\psi'(z_0)} 2\pi i$$

$$\int \frac{\Phi}{(z-z_0)(z-z_1)(z-z_3)(z-z_4)}$$



$$I = \frac{\pi i}{2}(z_0^{-3} + z_1^{-3}) = \frac{\pi i}{2}\left(e^{\frac{-i3\pi}{4}} + e^{\frac{-9i\pi}{4}}\right)$$

$$= \frac{\pi i}{2}\left(\frac{1}{\sqrt{2}}(\cancel{-1} - i) + \frac{1}{\sqrt{2}}(\cancel{1} - i)\right) = \frac{\pi}{\sqrt{2}}$$

Observar que se puede calcular los residuos con la fórmulas:

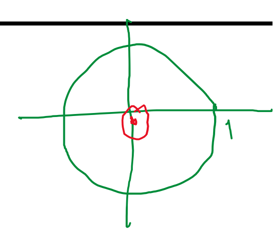
$$res(f(z_0)) = \frac{\varphi(z_0)}{\psi'(z_0)} = \frac{1}{4z_0^3}$$

$$res(f(z_0)) = \frac{\varphi^{(n-1)}(z_0)}{(n-1)!} = \frac{1}{(n-1)!}\lim_{z\to z_0}\left\{\frac{d^{n-1}}{dz}(f(z)(z-z_0)^n)\right\} = \lim_{z\to z_0}\frac{(z-z_0)}{1+z^4} = \frac{1}{4z_0^3}$$

$$res(f(z_0)) = \varphi(z_0) = \frac{1}{(z_0-z_1)(z_0-z_2)(z_0-z_3)} = \frac{1}{4z_0^3}$$

4

$$I =$$



$$\int_0^{2\pi} \frac{1}{6+\operatorname{sen}(\theta)} d\theta$$

$$z = e^{\theta i}$$

$$dz = e^{\theta i} i d\theta$$

$$\frac{dz}{iz} = d\theta$$

$$sen(\theta) = \frac{\left(e^{\theta i} - e^{-\theta i}\right)}{2i} = \frac{z - z^{-1}}{2i}$$

$$= \oint_{|z|=1} \left(\frac{1}{6 + \frac{1}{2i}\left(z - \frac{1}{z}\right)}\right)\frac{dz}{iz}$$

$$= \oint_{|z|=1} \frac{dz}{6iz + \frac{1}{2}(z^2 - 1)}$$

$$= \oint_{|z|=1} \frac{2dz}{12iz + (z^2 - 1)}$$

$$= 2\oint_{|z|=1} \frac{1}{(z^2 + 12iz - 1)} dz = 4\pi i \frac{1}{\left(2\left(i\left(-6+\sqrt{35}\right)\right) + 12i\right)}$$

$$= 4\pi i \frac{1}{\left(\left(i\left(-12 + 2\sqrt{35}\right)\right) + 12i\right)} = \frac{2\pi}{\sqrt{35}}$$

$$z = \frac{-12i \pm \sqrt{-144+4}}{2} = i\frac{\left(-12 \pm 2\sqrt{35}\right)}{2} = i\left(-6+\sqrt{35}\right) \quad , \qquad i\left(-6-\sqrt{35}\right)$$

$$= 2\oint_{|z|=1} \frac{dz}{\left(z - i\left(-6+\sqrt{35}\right)\right)\left(z + i\left(6+\sqrt{35}\right)\right)}$$

Las raíces del denominador son:

$$= 2\oint_{|z|=1} \frac{\left[\frac{1}{\left(z + i\left(6+\sqrt{35}\right)\right)}\right]}{\left(z - i\left(-6+\sqrt{35}\right)\right)} dz$$

$$= 4\pi i \frac{1}{\left(i\left(-6+\sqrt{35}\right) + i\left(6+\sqrt{35}\right)\right)} = \frac{2\pi}{\sqrt{35}}$$

5

$$I = \int_{-\infty}^{\infty} \frac{x}{16+x^4} \operatorname{sen}(2x)\, dx = Im\left(\oint_c \frac{z e^{i2z} dz}{16+z^4}\right)$$

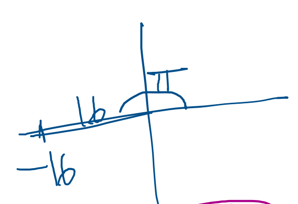
Primero resolvemos la integral:

$$I_2 = \oint_c \frac{z e^{i2z} dz}{16+z^4}$$

Calculamos los ceros del denominador y ubicamos los de la parte del semiplano superior

$$16 + z^4 = 0 \implies z^4 = -16 = 16 e^{i(\pi + 2k\pi)}$$

$$z_k = 2e^{\frac{i(\pi+2k\pi)}{4}}$$



$$z_0 = 2e^{\frac{i(\pi)}{4}} = 2\frac{1}{\sqrt{2}}(1+i) = \sqrt{2}(1+i)$$
$$z_1 = 2e^{\frac{i(3\pi)}{4}} = 2\frac{1}{\sqrt{2}}(-1+i) = \sqrt{2}(-1+i)$$
$$z_2 = 2e^{\frac{i(\pi+4\pi)}{4}}$$
$$z_3 = 2e^{\frac{i(\pi+6\pi)}{4}}$$

las dos integrales que resultan se calculan con la fórmula $\oint_C \frac{\varphi(z)}{\psi(z)}dz = 2\pi i\, \frac{\varphi(z_0)}{\psi'(z_0)}$

o también con

$$I_2 = \oint_c \frac{ze^{i2z}dz}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)} = \oint_{c_1} \frac{\frac{ze^{i2z}}{(z-z_2)(z-z_3)(z-z_4)}dz}{(z-z_1)} + \oint_{c_1} \frac{\frac{ze^{i2z}}{(z-z_1)(z-z_3)(z-z_4)}dz}{(z-z_2)}$$

$$= 2\pi i\left(\frac{z_1e^{i2z_1}}{(z_1-z_2)(z_1-z_3)(z_1-z_4)} + \frac{z_2e^{i2z_2}}{(z_2-z_1)(z_2-z_3)(z_2-z_4)}\right)$$

$$I_2 = \oint_c \frac{ze^{2z}dz}{16+z^4} = 2\pi i\left(\frac{z_0e^{i2z_0}}{4z_0^3} + \frac{z_1e^{i2z_1}}{4z_1^3}\right) = \frac{\pi i}{2}\left(z_0^{-2}e^{i2z_0} + z_1^{-2}e^{i2z_1}\right)$$

Calculamos:

$$z_0^{-2} = \frac{1}{4}e^{\frac{-2i(\pi)}{4}} = \frac{1}{4}e^{\frac{-i(\pi)}{2}} = \frac{1}{4}(-i)$$
$$z_1^{-2} = \frac{1}{4}e^{\frac{-2i(3\pi)}{4}} = \frac{1}{4}e^{\frac{-i(3\pi)}{2}} = \frac{1}{4}(i)$$
$$e^{i2z_0} = e^{2\sqrt{2}(1+i)}$$
$$e^{i2z_1} = e^{2\sqrt{2}(-1+i)}$$

Al sustituir obtenemos:

$$I_2 = \frac{\pi i}{8}\left(-i\,e^{2\sqrt{2}(1+i)} + i\,e^{2\sqrt{2}(-1+i)}\right)$$
$$= \frac{\pi}{8}\left(e^{2\sqrt{2}(1+i)} - e^{2\sqrt{2}(-1+i)}\right) = \frac{\pi}{8}\left(e^{2\sqrt{2}}e^{2\sqrt{2}i} - e^{-2\sqrt{2}}e^{2\sqrt{2}i}\right)$$
$$= \frac{\pi}{8}\left(e^{2\sqrt{2}}\left(\cos(2\sqrt{2}) + i\,sen(2\sqrt{2})\right) - e^{-2\sqrt{2}}\left(\cos(2\sqrt{2}) + isen(2\sqrt{2})\right)\right)$$
$$= \frac{\pi}{8}\cos(2\sqrt{2})\left(e^{2\sqrt{2}} - e^{-2\sqrt{2}}\right) + i\frac{\pi}{8}sen(2\sqrt{2})\left(e^{2\sqrt{2}} - e^{-2\sqrt{2}}\right)$$
$$= \frac{\pi}{4}\cos(2\sqrt{2})\,senh(2\sqrt{2}) + i\frac{\pi}{4}sen(2\sqrt{2})senh(2\sqrt{2})$$

Tomando la parte imaginaria obtenemos finalmente:

$$\int_{-\infty}^{\infty} \frac{x\,sen(x)\,dx}{16+x^4} = \frac{\pi}{4}sen(2\sqrt{2})senh(2\sqrt{2})$$

7

$$\cos(2\theta) = \cos^2\theta - (1-\cos^2\theta)$$

$$\cos^2\theta = \frac{1}{2}(1+\cos(2\theta))$$

$$\int_{-\infty}^{\infty}\frac{\cos^2(\mathrm{x})\,\mathrm{dx}}{(\mathrm{x}^2+4)^2} = \frac{1}{2}\int_{-\infty}^{\infty}\frac{1+\cos(2\mathrm{x})\,\mathrm{dx}}{(\mathrm{x}^2+4)^2} = \frac{1}{2}\left(\int_{-\infty}^{\infty}\frac{1\mathrm{dx}}{(\mathrm{x}^2+4)^2}+\int_{-\infty}^{\infty}\frac{\cos(2\mathrm{x})\,\mathrm{dx}}{(\mathrm{x}^2+4)^2}\right)$$

$$= \frac{1}{2}\left(\oint_\Gamma \frac{\mathrm{dz}}{(\mathrm{z}^2+4)^2} + Re\oint_\Gamma \frac{\mathrm{e}^{i2z}\mathrm{dz}}{(\mathrm{z}^2+4)^2}\right)$$

$$= \frac{1}{2}\left(\oint_\Gamma \frac{\mathrm{dz}}{(\mathrm{z}^2-(i2)^2)^2} + Re\oint_\Gamma \frac{\mathrm{e}^{i2z}\mathrm{dz}}{(\mathrm{z}^2-(i2)^2)^2}\right)$$

$$= \frac{1}{2}\left(\oint_\Gamma \frac{\mathrm{dz}}{(z-2i)^2(z+2i)^2} + Re\oint_\Gamma \frac{\mathrm{e}^{i2z}\mathrm{dz}}{(z-2i)^2(z+2i)^2}\right)$$

$$= \frac{1}{2}\left(-2\frac{2\pi i}{(2i+2i)^3} + Re\left(2\pi i\left(-2\frac{\mathrm{e}^{i2\cdot 2i}}{(2i+2i)^3}+\frac{2\mathrm{ie}^{i2\cdot i2}}{(2i+2i)^2}\right)\right)\right)$$

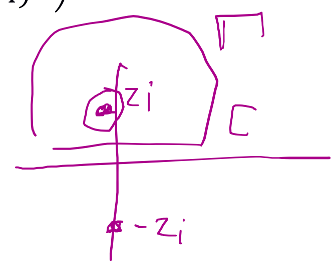
$$= \frac{1}{2}\left(-2\frac{2\pi i}{(4i)^3} + Re\left(2\pi i\left(-2i\frac{\mathrm{e}^{-4}}{64}+\frac{2\mathrm{ie}^{-4}}{(4i)^2}\right)\right)\right)$$

$$= \frac{1}{2}\left(-i\frac{2\pi i2}{64} + Re\left(2\pi i\;2i\left(-\frac{\mathrm{e}^{-4}}{64}-\frac{4\mathrm{e}^{-4}}{64}\right)\right)\right) =$$

$$= \frac{1}{2}\left(\frac{\pi 4}{64} + \left(4\pi\left(\frac{5\mathrm{e}^{-4}}{64}\right)\right)\right) = \frac{\pi(1+5e^{-4})}{32}$$

$$\phi_1(z) = \frac{1}{(z+2i)^2};\ \phi_1'(z) = -2\frac{1}{(z+2i)^3}$$

$$\phi_2(z) = \frac{\mathrm{e}^{i2z}}{(z+2i)^2};\ \phi_2'(z) = -2\frac{\mathrm{e}^{i2z}}{(z+2i)^3}+\frac{2\mathrm{ie}^{i2z}}{(z+2i)^2}$$



8

$$\int_0^{2\pi}\frac{2\,sen(\theta)}{2+sen^2(\theta)}d\theta = \oint_{|z|=1}\frac{2\left(\frac{z-z^{-1}}{2i}\right)}{\left(2+\left(\frac{z-z^{-1}}{2i}\right)^2\right)}\frac{dz}{iz}$$

$$= -\oint_{|z|=1}\frac{z-z^{-1}}{2-\frac{1}{4}(z-z^{-1})^2}\frac{dz}{z}$$

$$= -\frac{4}{4}\oint_{|z|=1}\frac{z-z^{-1}}{2z-\frac{1}{4}(z^3-2z+z^{-1})}dz$$

$$= -\frac{z}{z}\oint_{|z|=1}\left(\frac{4(z-z^{-1})}{8z-(z^3-2z+z^{-1})}\right)dz$$

$$= \oint_{|z|=1} \left(\frac{4(z^2-1)}{z^4-10z^2+1}\right) dz = 2\pi i\left(\left(\frac{4(z_0^2-1)}{4z_0^3-20z_0}\right)+\left(\frac{4(z_1^2-1)}{4z_1^3-20z_1}\right)\right)$$

$$= 2\pi i\left(\left(\frac{4\left(5-\sqrt{24}-1\right)-4\left(5-\sqrt{24}-1\right)}{4\left(5-\sqrt{24}\right)^{\frac{3}{2}}-20\left(5-\sqrt{24}\right)^{\frac{1}{2}}}\right)\right)$$

$$= 2\pi i\left(\frac{4\left(5-\sqrt{24}-1\right)-4\left(5-\sqrt{24}-1\right)}{4\left(5-\sqrt{24}\right)^{\frac{3}{2}}-20\left(5-\sqrt{24}\right)^{\frac{1}{2}}}\right) = 0$$

$$= (z^2)^2 - 10(z^2) + 1$$

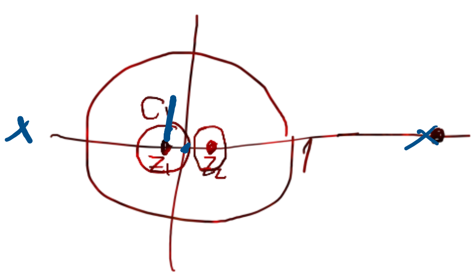
$$z^2 = \frac{\left(10 \pm \sqrt{100-4}\right)}{2} = \frac{\left(10 \pm \sqrt{96}\right)}{2} = 5 \pm \sqrt{24}$$

$$z_0 = +\sqrt{5-\sqrt{24}}$$

$$z_1 = -\sqrt{5-\sqrt{24}}$$

$$z = +\sqrt{5+\sqrt{24}}$$

$$z = -\sqrt{5+\sqrt{24}}$$



$2\pi i \;\; \phi(z_1)$

$2\pi i \;\; \dfrac{\phi(z_1)}{\psi'(z_1)}$

8

$$\int_0^{2\pi} \frac{2\,\mathrm{sen}(\theta)}{2+\mathrm{sen}^2(\theta)}\,d\theta = \oint_{|z|=1} \frac{2\,\frac{z-z^{-1}}{2i}}{2+\left(\frac{z-z^{-1}}{2i}\right)^2}\,\frac{dz}{iz}$$

$$= \frac{1}{i}\oint_{|z|=1}\left(\frac{z-z^{-1}}{2-\frac{z^2-2+z^{-2}}{4}}\right)\frac{dz}{iz} = \frac{4}{i}\oint_{|z|=1}\left(\frac{z-z^{-1}}{8-(z^2-2+z^{-2})}\right)\frac{dz}{iz}$$

$$= 4\oint_{|z|=1}\left(\frac{z-z^{-1}}{-8+(z^2-2+z^{-2})}\right)\frac{dz}{z} = 4\oint_{|z|=1}\left(\frac{z-z^{-1}}{z^3-10z+z^{-1}}\right)dz$$

$$= 4\oint_{|z|=1}\frac{z^2-1}{z^4-10z^2+1}\,dz = 8\pi i\left(\frac{z_1^2-1}{4z_1^3-20z_1}+\frac{z_2^2-1}{4z_2^3-20z_2}\right)$$

$$= 8\pi i\left(\frac{z_1^2-1}{4(z_1^2-5)z_1}+\frac{z_1^2-1}{4(z_1^2-5)(-z_1)}\right) = 8\pi i\left(\frac{5-2\sqrt{6}-1}{4\left(5-2\sqrt{6}-5\right)\sqrt{5-2\sqrt{6}}}-\frac{5-2\sqrt{6}-1}{4\left(5-2\sqrt{6}-5\right)\sqrt{5-2\sqrt{6}}}\right)$$

$$= 0$$

$$z^2 = \frac{10 \pm \sqrt{100-4}}{2} = 5 \pm 2\sqrt{6}$$
$$z = \pm\sqrt{5 \pm 2\sqrt{6}}$$
$$z_{1,2} = \pm\sqrt{5 - 2\sqrt{6}}$$
$$z_{3,4} = \pm\sqrt{5 + 2\sqrt{6}}$$

9

$$I = \int_{-\infty}^{\infty} \frac{x^2}{(x^2+4)^2}\,dx = \oint_\Gamma \frac{z^2}{(z^2-(2i)^2)^2}\,dz = \oint_\Gamma \frac{z^2}{(z+2i)^2(z-2i)^2}\,dz$$
$$= \oint_\Gamma \frac{z^2(z+2i)^{-2}}{(z-2i)^2}\,dz = 2\pi i(z^2(z+2i)^{-2})'$$
$$= 2\pi i(2(2i)(2i+2i)^{-2} - 2(2i)^2(2i+2i)^{-3})$$
$$= 2^3\pi i\left(-\frac{i}{2^4} + i\frac{2}{2^6}\right) = \pi i\left(-\frac{2^3 i}{2^4} + i\frac{2^4}{2^6}\right) = -\pi\left(-\frac{2}{4} + \frac{1}{4}\right) = \frac{\pi}{4}$$

$$(z^2(z+2i)^{-2})' = 2z(z+2i)^{-2} - 2z^2(z+2i)^{-3}$$

11

$$\int_{-\infty}^{\infty} \frac{\cos(\alpha x)\,dx}{x^2+1} = Re\left(\oint_C \frac{e^{i\alpha z}}{(z-i)(z+i)}\,dz\right) = Re\left(\oint_C \frac{\left(\frac{e^{i\alpha z}}{z+i}\right)}{(z-i)}\,dz\right)$$
$$= Re\left(2\pi i\left(\frac{e^{i\alpha i}}{i+i}\right)\right)$$
$$= Re\left(2\pi i\left(\frac{e^{i\alpha i}}{2i}\right)\right) = \pi e^{-\alpha}$$

13

$$\int_0^{2\pi} \frac{1}{\alpha^2\cos^2(\theta) + \beta^2 \operatorname{sen}^2(\theta)}\,d\theta = \oint_{|z|=1} \frac{1}{\alpha^2\left(\frac{z+z^{-1}}{2}\right)^2 + \beta^2\left(\frac{z-z^{-1}}{2i}\right)^2}\frac{dz}{iz}$$
$$= \oint_{|z|=1}\left(\frac{1}{\alpha^2\frac{z^2+2+z^{-2}}{4} - \beta^2\frac{z^2-2+z^{-2}}{4}}\right)\frac{dz}{iz} = \oint_{|z|=1} \frac{4}{\alpha^2(z^2+2+z^{-2}) - \beta^2(z^2-2+z^{-2})}\frac{dz}{iz}$$
$$= \frac{1}{i}\oint_{|z|=1} \frac{4}{\alpha^2(z^3+2z+z^{-1}) - \beta^2(z^3-2z+z^{-1})}\,dz = \frac{1}{i}\oint_{|z|=1} \frac{4z}{\alpha^2(z^4+2z^2+1) - \beta^2(z^4-2z^2+1)}\,dz$$
$$= \frac{1}{i}\oint_{|z|=1} \frac{4z}{(\alpha^2-\beta^2)z^4 + (\alpha^2+\beta^2)2z^2 + (\alpha^2-\beta^2)}\,dz = 2\pi\left(\frac{4z}{4(\alpha^2-\beta^2)z^3 + 4(\alpha^2+\beta^2)z}\right)$$
$$= 2\pi\left(\frac{1}{(\alpha^2-\beta^2)z^2 + (\alpha^2+\beta^2)}\right) = 2\pi\left(\frac{1}{(\alpha+\beta)(\alpha-\beta)z_0^2 + (\alpha^2+\beta^2)} + \frac{1}{(\alpha+\beta)(\alpha-\beta)z_1^2 + (\alpha^2+\beta^2)}\right)$$
$$= 2\pi\left(\frac{1}{(\alpha+\beta)(\alpha-\beta)z_0^2 + (\alpha^2+\beta^2)} + \frac{1}{(\alpha+\beta)(\alpha-\beta)z_1^2 + (\alpha^2+\beta^2)}\right)$$

$$= 2\pi\left(\frac{2}{(\alpha+\beta)(\alpha-\beta)\left(-\frac{\alpha-\beta}{(\alpha+\beta)}\right)+(\alpha^2+\beta^2)}\right)$$

$$= 2\pi\left(\frac{2}{-(\alpha-\beta)^2+(\alpha^2+\beta^2)}\right)$$

$$= 2\pi\left(\frac{2}{-(\alpha^2-2\alpha\beta+\beta^2)+(\alpha^2+\beta^2)}\right) = \frac{2\pi}{\alpha\beta}$$

$$z^2 = \frac{-(\alpha^2+\beta^2)2 \pm \sqrt{((\alpha^2+\beta^2)2)^{\,2} - 4(\alpha^2-\beta^2)^2}}{2(\alpha^2-\beta^2)}$$

$$= \frac{-(\alpha^2+\beta^2) \pm \sqrt{(\alpha^4+2\alpha^2\beta^2+\beta^4)-(\alpha^4-2\alpha^2\beta^2+\beta^4)}}{(\alpha^2-\beta^2)} = \frac{-(\alpha^2+\beta^2) \pm 2\alpha\beta}{(\alpha^2-\beta^2)} =$$

$$z^2 = \frac{-(\alpha^2+\beta^2)-2\alpha\beta}{(\alpha^2-\beta^2)} = -\frac{(\alpha+\beta)^2}{(\alpha+\beta)(\alpha-\beta)} = -\frac{\alpha+\beta}{(\alpha-\beta)} fuera$$

$$z^2 = \frac{-(\alpha^2+\beta^2)+2\alpha\beta}{(\alpha^2-\beta^2)} = -\frac{(\alpha-\beta)^2}{(\alpha+\beta)(\alpha-\beta)} = -\frac{\alpha-\beta}{(\alpha+\beta)} dentro$$

$$-\frac{3-1}{(3+1)} = -\frac{2}{4}$$

381. $\int_{-\infty}^{+\infty} \frac{x\cos x\,dx}{x^2-2x+10}$. **382.** $\int_{-\infty}^{+\infty} \frac{x \operatorname{sen} x}{x^2+4x+20}\,dx$.

383. $\int_{0}^{\infty} \frac{\cos x\,dx}{(x^2+1)(x^2+4)}$. **384.** $\int_{-\infty}^{+\infty} \frac{\cos x\,dx}{x^2+9}$.

385. $\int_{0}^{\infty} \frac{\cos ax}{1+x^4}\,dx\ (a>0)$. **386.** $\int_{0}^{\infty} \frac{\cos x}{x^2+a^2}\,dx\ (a>0)$.

387. $\int_{0}^{\infty} \frac{\cos mx}{a^2+x^2}\,dx\ (m>0,\ a>0)$.

388. $\int_{0}^{\infty} \frac{x \operatorname{sen} x}{1+x^2+x^4}\,dx$. **389.** $\int_{-\infty}^{+\infty} \frac{\cos \lambda x}{(x^2+1)(x^2+9)}\ (\lambda>0)$.

390. $\int_{0}^{\infty} \frac{x^3 \operatorname{sen} ax}{(1+x^2)^2}\,dx\ (a>0)$. **391.** $\int_{0}^{\infty} \frac{x^2\cos x\,dx}{(x^2+1)^2}$

392. $\int_{0}^{\infty} \frac{3x^2-a^2}{(x^2+b^2)^2}\cos mx\,dx$.

$$\int_{-\infty}^{\infty} \frac{x\cos(x)\,dx}{x^2 - 2x + 10}$$

$$x = \frac{2 \pm \sqrt{(4-40}}{2} = 1 \pm 3i$$

$$= Re\left(\oint_c \frac{ze^{iz}dz}{z^2 - 2z + 10}\right)$$

$$= Re\left(\frac{2\pi i(1+3i)e^{i(1+3i)}}{2(1+3i) - 2}\right)$$

$$I = \frac{2\pi i(1+3i)e^{i(1+3i)}}{6i} = \frac{\pi}{3}(1+3i)e^{i}e^{-3}$$

$$= \frac{\pi}{3}(1+3i)\big(\cos(1) + isen(1)\big)e^{-3}$$

$$= \frac{\pi}{3}\big(\cos(1) - 3sen(1) + i(sen(1) + 3\cos(1))\big)e^{-3}$$

$$= \frac{\pi}{3}e^{-3}\big(\cos(1) - 3\,sen(1)\big)$$

382

$$\int_{-\infty}^{\infty} \frac{x\cos(x)\,dx}{x^2 - 2x + 10}$$

384

$$\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2 + 9}dx$$

$$= \int_{-\infty}^{\infty} \frac{\cos(x)}{x^2 - i^2 3^2}dx = \int_{-\infty}^{\infty} \frac{\cos(x)}{(x-3i)(x+3i)}dx = Re\oint_c \frac{\mathrm{e}^{iz}}{(z-3i)(z+3i)}dz = \frac{\pi e^{-3}}{3}$$

$$\oint_c \frac{\frac{\mathrm{e}^{iz}}{z+3i}}{(z-3i)}\ dz = 2\pi i\frac{\mathrm{e}^{i3i}}{3i+3i} = \frac{\pi e^{-3}}{3}$$

$$\int_{-\infty}^{\infty} \frac{\mathrm{sen}(x)}{x^2 + 9}dx = 0$$

$$\oint_c \frac{\frac{\cos\,(\mathrm{z})}{z+3i}}{(z-3i)}\ dz = 2\pi i\frac{\cos\,(3\mathrm{i})}{3i+3i} = \frac{\pi}{3}\frac{(e^{-3}+e^{3})}{2}$$

Ejemplo

$$I=\int_{-\infty}^{\infty}\frac{\mathrm{x}^2\mathrm{dx}}{1+\mathrm{x}^4}=\oint_c\frac{\mathrm{z}^2\mathrm{dz}}{1+z^4}$$

Posteriormente se calculan las 4 raíces y se tienen dos raíces simples en el semiplano superior,

$$1+z^4=0\implies z^4=-1\implies z_k=e^{\frac{i(\pi+2k\pi)}{4}}$$

$$z_0=e^{\frac{i(\pi)}{4}}=\boxed{\frac{1}{\sqrt{2}}(1+i)}$$

$$z_1=e^{\frac{i(\pi+2\pi)}{4}}=\frac{1}{\sqrt{2}}(-1+i)$$

$$z_2=e^{\frac{i(\pi+4\pi)}{4}}=\frac{1}{\sqrt{2}}(-1-i)$$

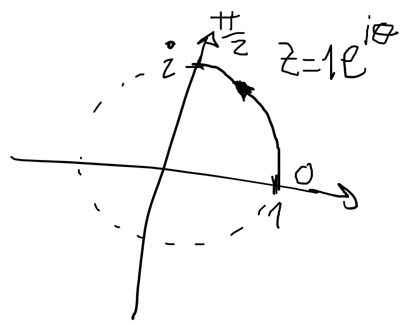
$$z_3=e^{\frac{i(\pi+6\pi)}{4}}=\frac{1}{\sqrt{2}}(1-i)$$

las dos integrales que resultan se calculan con la fórmula $\oint_C\frac{\varphi(z)}{\psi(z)}dz=2\pi i\,\frac{\varphi(z_0)}{\psi'(z_0)}$

$$I=\int_{-\infty}^{\infty}\frac{\mathrm{x}^2\mathrm{dx}}{1+\mathrm{x}^4}=\oint_{c_1}\frac{\mathrm{z}^2\mathrm{dz}}{1+z^4}+\oint_{c_2}\frac{\mathrm{z}^2\mathrm{dz}}{1+z^4}=2\pi\mathrm{i}\left(\frac{\mathrm{z}_0^2}{4\mathrm{z}_0^3}+\frac{\mathrm{z}_1^2}{4\mathrm{z}_1^3}\right)$$

$$=\frac{\pi\mathrm{i}}{2}\left(\mathrm{z}_0^{-1}+\mathrm{z}_1^{-1}\right)=\frac{\pi\mathrm{i}}{2}\left(e^{\frac{-i\pi}{4}}+e^{\frac{-3i\pi}{4}}\right)$$

$$=\frac{\pi\mathrm{i}}{2}\left(\frac{1}{\sqrt{2}}(1-i)+\frac{1}{\sqrt{2}}(-1-i)\right)=\frac{\pi}{\sqrt{2}}$$



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$u = z + 1 \quad du = dz$ $\qquad \int v\,dv = \frac{v^2}{2}$ $\qquad \int \frac{\ln u}{u}\,du$ $\qquad v = \ln u \quad dv = \frac{du}{u}$

$$\int_1^i \frac{\ln(z+1)}{z+1}\,dz = \frac{1}{2}\left(\ln^2(i+1) - \ln^2(1+1)\right)$$

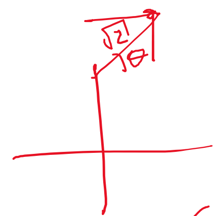
$$= \frac{1}{2}\left(\ln^2(i+1) - \ln^2(1+1)\right)$$

$$= \frac{1}{2}\left(\left(\frac{1}{2}\ln(2) + i\left(\frac{\pi}{4}\right)\right)^2 - \left(\ln(2) + i(0)\right)^2\right)$$

$$= \frac{1}{2}\left(\frac{1}{4}\ln^2(2) + \ln(2)\, i\left(\frac{\pi}{4}\right) - \left(\frac{\pi}{4}\right)^2 - \frac{4}{4}\ln^2(2)\right)$$

$$= \frac{1}{2}\left(\ln(2)\, i\left(\frac{\pi}{4}\right) - \frac{\pi^2}{4^2} - \frac{3}{4}\ln^2(2)\right)$$

$$= -\frac{1}{8}\left(\frac{\pi^2}{4} + 3\ln^2 2\right) + \frac{i\pi}{8}\ln(2)$$



$$(\ln(i+1))^2 = \left(\ln(\sqrt{2}) + i\left(\frac{\pi}{4} + 2k\pi\right)\right)^2$$

$\ln(\sqrt{2}\, e^{i\theta})$

$$(\ln(1+1))^2 = \left(\ln(2) + i(0 + 2k\pi)\right)^2$$

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$C: |\operatorname{Im} z| \leq 1$

$$\int_C Re\big(sen(z)\big)\, cos(z)dz =$$

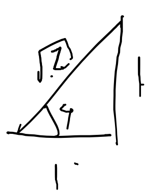
$\frac{e^{i(iy)} + e^{-i(iy)}}{2i} \quad -i\,\frac{e^{-y} - e^{y}}{2}$

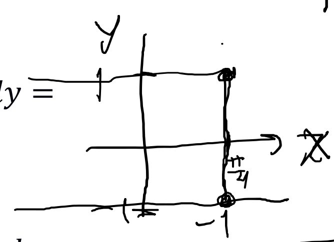
$\frac{e^{-y} + e^{y}}{2} \quad i\,\frac{e^{y} - e^{-y}}{2}$

$$\int_C Re\left(sen\left(\frac{\pi}{4} + iy\right)\right) cos\left(\frac{\pi}{4} + iy\right) idy =$$

$x = Re(z) = \frac{\pi}{4}$

$$\int_C Re\left(sen\left(\frac{\pi}{4}\right)\cos(iy) + sen(iy)\cos\left(\frac{\pi}{4}\right)\right) cos\left(\frac{\pi}{4} + iy\right) idy =$$





$$i\int_C sen\left(\frac{\pi}{4}\right)\cos h(y)\left(\cos\left(\frac{\pi}{4}\right)\cosh(y) - i sen\left(\frac{\pi}{4}\right) senh(y)\right) dy =$$

$z = \frac{\pi}{4} + iy \quad \boxed{-1 \leq y \leq 1}$

$dz = i\,dy$

$\cosh^2(y) = \left(\frac{e^{y} + e^{-y}}{2}\right)^2$

$$\frac{i}{2}\int_C \cos h(y)\left(\cosh(y) - i senh(y)\right) dy =$$

$= \frac{e^{2y} + 2 + e^{-2y}}{4}$

$$\frac{i}{2}\int_C \cosh^2(y)dy + \frac{1}{2}\int_C \left(\cosh(y)\, senh(y)\right) dy =$$

$$\frac{i}{2}\int_C \frac{1}{2}(\cosh(2y) + 1)\, dy + \frac{1}{2}\int_C \left(\cosh(y)\, senh(y)\right) dy =$$

$= \left(\frac{e^{2y} + e^{-2y}}{4} + \frac{1}{2}\right)$

$\int u\,du = \frac{u^2}{2}$

$\frac{1}{4}\left(u^2\Big|_{-1}^{1}\right)$

$$\frac{i}{4}\int_C (\cosh(2\mathrm{y})+1)\,dy = \left(\frac{senh(2)}{4}+\frac{1}{2}\right)i$$

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$$\int_C tg(z)dz = \int \frac{sen(z)}{\cos(z)}dz = -(\ln(cos(1+i)) - \ln(\cos(0)))$$
$$= (-\ln(cos(1+i)) + \ln(1))$$
$$= \left(-\ln\sqrt{\left(\cos^2(1) + senh^2(1)\right)} + i\arctan(\tan(1)\tanh(1))\right)$$

$$w = cos(1+i) = \cos(1)\cos(i) - sen(1)sen(i)$$
$$w = \cos(1)\cos h(1) - isen(1)senh(1)$$

$$|w| = \sqrt{\left(\cos^2(1)\cosh^2(1) + sen^2(1)senh^2(1)\right)}$$
$$= \sqrt{\left(\cos^2(1)\cosh^2(1) + (1-\cos^2(1))senh^2(1)\right)}$$
$$= \sqrt{\left(\cos^2(1)\left(\cosh^2(1) - senh^2(1)\right) + senh^2(1)\right)}$$
$$= \sqrt{\left(\cos^2(1) + senh^2(1)\right)}$$

$$\theta = \arctan\left(\frac{sen(1)senh(1)}{cos(1)\,cos\,h(1)}\right) = \arctan(\tan(1)\tanh(1))$$

Calcular las integrales siguientes:

399. $\int_0^{2\pi} \frac{dx}{1-2p\cos x+p^2} \quad (0<p<1)$.

400. $\int_0^{2\pi} \frac{\cos^2 3x\, dx}{1-2p\cos 2x+p^2} \quad (0<p<1)$.

401. $\int_0^{2\pi} \frac{\cos 2x\, dx}{1-2p\cos x+p^2} \quad (p>1)$.

402. $\int_0^{2\pi} \frac{\cos x\, dx}{1-2p\,\mathrm{sen}\, x+p^2} \quad (0<p<1)$.

403. $\int_0^{2\pi} \frac{dx}{a+\cos x} \quad (a>1)$.

404. $\int_0^{\pi} \mathrm{ctg}\,(x-a)\, dx \quad (\mathrm{Im}\, a>0)$.

405. $\int_0^{2\pi} \frac{\mathrm{sen}^2 x}{a+b\cos x}\, dx \quad (a>b>0)$.

399

$$\int_0^{2\pi} \frac{dx}{1-2p\cos(x)+p^2} = \int_0^{2\pi} \frac{d\theta}{1-2p\cos(\theta)+p^2} = \oint_{|z|=1} \frac{1}{1-2p\frac{(z+z^{-1})}{2}+p^2}\frac{dz}{iz}$$

Cambio de variable

$$z=e^{i\theta}; dz = ie^{i\theta}d\theta \rightarrow; d\theta = \frac{dz}{ie^{i\theta}} = \frac{dz}{iz}; \cos(\theta) = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+z^{-1}}{2}$$

$$= \frac{1}{i}\oint_{|z|=1} \frac{1}{z-p(z^2+1)+zp^2}dz = -\frac{1}{i}\oint_{|z|=1} \frac{1}{pz^2-z(p^2+1)+p}dz$$

$$= -\frac{1}{i}2\pi i\frac{1}{2p(p)-(p^2+1)} = \frac{1}{2p^2-(p^2+1)} = 2\pi\left(\frac{1}{1-p^2}\right)$$

$$z = \frac{(p^2+1) \pm \sqrt{(p^2+1)^2 - 4p^2}}{2p} =$$

$$= \frac{(p^2+1) \pm \sqrt{p^4 - 2p^2 + 1}}{2p} = \frac{(p^2+1) \pm \sqrt{(p^2-1)^2}}{2p}$$

$$= \frac{(p^2+1) \pm (1-p^2)}{2p}$$

$$= \frac{(p^2+1) + (1-p^2)}{2p} = \frac{2}{2p} = \frac{1}{p}$$

$$= \frac{(p^2+1) - (1-p^2)}{2p} = \frac{2p^2}{2p} = p$$

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2} = \frac{z + z^{-1}}{2}$$

$$\cos(2x) = \frac{e^{ix2} + e^{-ix2}}{2} = \frac{z^2 + z^{-2}}{2}$$

$$z = e^{ix}$$

Resolver

$$\int_0^{2\pi} e^{\cos(\theta)} \cos(sen(\theta))\, d\theta = Re\left(\oint_{|z|=1} e^z \frac{dz}{iz}\right) = \frac{2\pi i e^0}{i} = 2\pi$$

$$\oint_{|z|=1} e^z \frac{dz}{iz} = \int_0^{2\pi} e^{\cos(\theta)} e^{isen(\theta)}\, d\theta = \int_0^{2\pi} e^{\cos(\theta)} (\cos(sen(\theta)) + i\, \mathrm{sen}(sen(\theta)))\, d\theta$$

Resolver

$$\int_0^{2\pi} (1-\cos(\theta))^n \cos(n\theta)\, d\theta = \frac{(-1)^n \pi}{2^{n-1}}$$

$$\int_0^{2\pi} (1-\cos(\theta))^n \cos(n\theta)\, d\theta = Re\left(\int_0^{2\pi} (1-\cos(\theta))^n \left(\cos(n\theta) + i\, sen(n\theta)\right) d\theta\right)$$

$$\int_0^{2\pi} (1-\cos(\theta))^n \left(\cos(n\theta) + i\, sen(n\theta)\right) d\theta = \oint_{|z|=1} \left(1 - \frac{1}{2}\left(z + \frac{1}{z}\right)\right)^n z^n \frac{dz}{iz}$$

$$= \oint_{|z|=1} \left(1 - \left(\frac{z^2+1}{2z}\right)\right)^n z^n \frac{dz}{iz} = \oint_{|z|=1} \left(-\left(\frac{z^2 - 2z + 1}{2z}\right) z\right)^n \frac{dz}{iz}$$

$$= \frac{(-1)^n}{i2^n} \oint_{|z|=1} \frac{(z^2 - 2z + 1)^n}{z}\, dz = 2\pi i \frac{(-1)^n}{i2^n} = \frac{(-1)^n \pi}{2^{n-1}}$$