INSTITUTO POLITÉCNICO NACIONAL

ESCUELA SUPERIOR DE COMPUTO

LISTA DE EJERCICIOS 1-12

SEMANA 4

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GRUPO: 4CV3

MATERIA: MATEMATICAS AVANZADAS PARA LA INGENIERIA

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FECHA: 21/03/2023

3.47 Libro: Schaum / Ejercicio 1

Verifique que la parte real y la imaginaria de las siguientes funciones satisfacen las ecuaciones de Cauchy-Riemmann y concluya si estas funciones son analíticas:

a) $f(z) = z^2 + 5iz + 3 - i$

$z = x + iy$

$z^2 = x^2 - y^2 + 2ixy$

$x^2 - y^2 + 2ixy + 5i(x+iy) + 3 - i$

$x^2 - y^2 + 3 - i + 2ixy + 5ix - 5y$

$(x^2 - y^2 + 3 - 5y) + i(-1 + 2xy + 5x)$

Condiciones Cauchy-Riemmann

$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}$ $\rightarrow$ $\boxed{\begin{array}{l} U_x = V_y \\ U_y = -V_x \end{array}}$

$\frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}$

$f(z) = U(x,y) + iV(x,y)$

$U(x,y) = x^2 - y^2 + 3 - 5y$

$V(x,y) = -1 + 2xy + 5x$

$2x \neq 2y$ $\boxed{U_x \neq V_y}$ No es analítica

b) $f(z) = ze^{-z}$

$(x+iy)e^{-x-iy} = (x+iy)e^{-x}e^{-yi}$

$(x+iy)e^{-x}(\cos(-y) + i\,\mathrm{sen}(-y))$

$(x+iy)e^{-x}\cos(-y) + ix\,\mathrm{sen}(-y))$

$xe^{-x}\cos(-y) - y\,\mathrm{sen}(-y) + i(ye^{-x}\cos(-y) + x\,\mathrm{sen}(-y))$

$U_x = V_y$ $\quad e^{-x}\cos(-y) - ye^{-x}\mathrm{sen}(-y) - x\cos(-y)$

$\cos(-y)(-xe^{-x} + e^{-x}) \neq$

No es analítica

c) $f(z) = \mathrm{sen}(2z)$

$\mathrm{sen}(2x + 2iy) = \mathrm{sen}(2x)\cosh(2y) + i\,\mathrm{senh}(2y)\cos(2x)$

$\boxed{U_x = V_y}$

$2\cos 2x \cosh 2y = 2\cosh 2y \cos 2x$

$\boxed{U_y = -V_x}$

$2\,\mathrm{sen}\,2x\,\mathrm{senh}\,2y = 2\,\mathrm{sen}\,2x\,\mathrm{senh}\,2y$

Es analítica

1.56 Suponga $\operatorname{Im}\{f'(z)\} = 6x2y - 1$ y
$f(0) = 3 - 2i$, $f(1) = 6 - 5i$

Encuentre $f(1+i)$

Ejercicio 2

3.50 a) Compruebe que la función es armonica
$v = 2x(1-y)$

b) Encuentre una función v tal que $f(z) = v + iz$ sea analítica (es decir encuentre la conjugada de v)

c) Exprese $f(z)$ en terminos de z.

a) $v = 2x(1-y)$ Condición de Laplace

$$\boxed{\nabla^2 v = 0}$$
$$v_{xx} + v_{yy} = 0$$

$v_x = 2(1-y)$ $\boxed{v_{xx} = 0}$

$v_y = -2x$ $\boxed{v_{yy} = 0}$

Es armonica

b) $v = 2x - 2xy$ condición Cauchy-Riemmann

$u_x = v_y$
$u_y = -v_x$

$u_x = v_y$ $v = \int (2-2y)\,dy = 2y - \frac{2y^2}{2} + g(x)$

$$\boxed{v = 2y - y^2 + g(x)}$$

$-v_x = -g'(x)\,dx = -2x\,dx$

$u_y = -v_x$
$u_y = -2x$

$g(x) = \int 2x\,dx = x^2 + C$

$$\boxed{v = 2y - y^2 + x^2}$$

c) $f(z) = u(x,y) + i\,v(x,y) = (2x - 2xy) + i(2y - y^2 + x^2)$

$\boxed{x = \frac{z + \bar{z}}{2}}$ $\boxed{y = \frac{z - \bar{z}}{2i}}$

$2\left(\frac{z+\bar{z}}{2}\right) - 2\left(\frac{z+\bar{z}}{2}\right)\left(\frac{z-\bar{z}}{2i}\right) + i\left[2\left(\frac{z-\bar{z}}{2i}\right) - \left(\frac{z-\bar{z}}{2i}\right)^2 + \left(\frac{z+\bar{z}}{2}\right)^2\right]$

$z + \bar{z} - \frac{(z+\bar{z})(z-\bar{z})}{2i} + z - \bar{z} - i\left(\frac{z-\bar{z}}{2i}\right)^2 + i\left(\frac{z+\bar{z}}{2}\right)^2$

$2z + i\,\frac{z^2 - \cancel{z\bar{z}} + z\bar{z} - \bar{z}^2}{2} + i\,\frac{(z-\bar{z})^2}{4} + i\,\frac{(z+\bar{z})^2}{4}$

$2z + i\,\frac{z^2 - \bar{z}^2}{2} + i\,\frac{z^2 + \bar{z}^2 - 2z\bar{z} + z^2 + \bar{z}^2 + 2z\bar{z}}{4}$

$2z + i\,\frac{z^2 - \bar{z}^2}{2} + i\,\frac{2(z^2 + \bar{z}^2)}{4} = \boxed{iz^2 + 2z}$

3.51 Respondo el problema 3.50 con $U = x^2 - y^2 (+2xy) - 2x + 3y$
Ejercicio 3

a) Función armónica

$\nabla^2 U = 0 \rightarrow U_{xx} + U_{yy} = 0 \rightarrow 2 - 2 = 0$

$U_x = 2x + 2y - 2$

$U_{xx} = 2$

$U_y = -2y (+2x) + 3$

$U_{yy} = -2$

b) Encontrar $v(x, y)$

$U_x = V_y$

$V = \int V_y dy = \int (2x (+2y) - 2)\, dy = 2xy (+\frac{2y^2}{2}) - 2y + g^+(x)$

$U_y = -V_x$

$-V_x = -(2y + g'(x)) \partial_x = -V_x$

$-2y - g'(x) = -2y - (+2x) + 3$

$\int dx\, g'(x) = \int (+2x - 3)\, dx$

$g(x) = +x^2 - 3x$

$\boxed{V = 2xy (+\frac{2y^2}{2}) - 2y + x^2 - 3x}$ $V = 2xy - 2y - 3x - y^2 + x^2$

c) $F(z) = U(x, y) + i V(x, y) = x^2 - y^2 - 2xy - 2x + 3y + i[x^2 - y^2 - 2y - 3y + 2xy]$

3.52 Verifique que las ecuaciones de Cauchy Riemann se satisfacen para

a) $e^{z^2} = e^{x^2-y^2} e^{i2xy} = e^{x^2-y^2}[\cos(2xy) + i\,\text{sen}(2xy)]$

$U_x = V_y$

$-e^{x^2-y^2} 4\sin 2xy + (+2x)e^{x^2-y^2}\cos(2xy) = U_x$

$-2y e^{x^2-y^2}\,\text{sen}\,2xy + 2x\cos 2xy\, e^{x^2-y^2} = V_y$

$\boxed{U_x = V_y}$ ó $\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}$

$U_y = V_x$

$-2x e^{x^2-y^2}\,\text{sen}(2xy) + e^{x^2-y^2}(-2y)\cos(2xy) = U_y$

$-[2x e^{x^2-y^2}\,\text{sen}(2xy) + 2y e^{x^2-y^2}\cos(2xy)] = -V_x$

$\boxed{U_y = -V_x}$ ó $\frac{\partial V}{\partial y} = \frac{-\partial V}{\partial x}$

Son analiticas

b) $\cos(2z) = \cos(2x+2iy) = \cos(2x)\cosh(2y) - i\,\text{sen}(2x)\,\text{senh}(2y)$

$\cos(z) = \cos(a+ib) = \cos(a)\cosh(b) - i\,\text{sen}(a)\,\text{senh}(b)$

$\boxed{U_x = V_y}$

$U_x = -2\cosh(2y)\,\text{sen}(2x)$

$U_y = -2\,\text{sen}\,2x\,\cosh(2y)$

$\boxed{U_y = -V_x}$

$U_y = -2\,\text{senh}(2y)\cos(2x)$

$-V_x = \cos(2x)\,\text{senh}(2y)$ Son analiticas

c) $\text{senh}(4z) = -i\,\text{sen}(i4z) = i\,\text{sen}\,4y\,\cosh 4x + \text{senh}(4x)\cos(4y)$

$\text{sen}(i4z) = \text{sen}(4xi - 4y)$

$\text{sen}(x+iy) = \text{sen}(x)\cosh(y) + \text{senh}(x)\cos(x)$

$\boxed{U_x = V_y}$

$U_x = V_y$

$4\,\text{sen}\,4y\,\text{senh}\,4x$

$U_x \neq \text{sen}\,4y$

$U_x = 4\cosh(4x)\cos(-4y)$

$V_y = 4\cos(-4y)\cosh(4x)$

$\boxed{U_y = -V_x}$

$U_y = 4\,\text{sen}(-4y)\,\text{senh}(4x)$

$-V_x = 4\,\text{senh}(4x)\,\text{sen}(-4y)$

Es analitica

Libro 'Makarenko   Ejercicio 5

106 Mostrar que en el dominio $Re\, z > 0$ $w = \ln z$ es la función analítica

$$\ln|x+iy| = \ln\left|\sqrt{x^2+(y^2)}\, e^{\tan^{-1}\left(\frac{y}{x}\right)i}\right| = \frac{1}{2}\ln(x^2+y^2) + \ln\left|e^{\tan^{-1}\left(\frac{y}{x}\right)i}\right|$$

$$\frac{1}{2}\ln|x^2+y^2| + i\tan^{-1}\left(\frac{y}{x}\right) = \frac{1}{2}\ln|x^2+y^2| + \tan^{-1}\left(\frac{y}{x}\right)i$$

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y} \rightarrow \boxed{U_x = V_y}$$

$$\frac{1}{2}\,\frac{2x}{x^2+y^2} + 0 = U_x \qquad V_y = \frac{\frac{1}{x}}{1+\left(\frac{y}{x}\right)^2} = \frac{\frac{1}{x}}{\frac{x^2+y^2}{x^2}}$$

$$\boxed{\frac{x}{x^2+y^2} = \frac{x}{x^2+y^2}}$$

$$\boxed{U_y = -V_x}$$

$$\left(\frac{1}{2}\right)\left(\frac{2y}{x^2+y^2}\right) = \boxed{\frac{y}{x^2+y^2}} \qquad \left[-\frac{\frac{y}{x^2}}{\frac{x^2+y^2}{x^2}}\right](-1) = -V_x = \boxed{\frac{y}{x^2+y^2}}$$

Es analítica

114. a) $v = \dfrac{x}{x^2+y^2}$ $\qquad f(\pi) = \dfrac{1}{\pi} \qquad z_0 = \pi \qquad c_0 = \dfrac{1}{\pi}$

$$f(z) = 2v\left(\frac{z+\bar{z}_0}{2}, \frac{z-\bar{z}_0}{2i}\right) - \bar{c}_0$$

$$f(z) = 2\left(\frac{\frac{z+\pi}{2}}{\left(\frac{z+\pi}{2}\right)^2 + \left(\frac{z-\pi}{2i}\right)^2}\right) - \frac{1}{\pi}$$

$$2\left(\frac{\frac{z+\pi}{2}}{\frac{z^2+\pi^2+2\pi}{4} + \frac{z^2+\pi^2-2\pi z}{4i^2}}\right) - \frac{1}{\pi}$$

$$2\left(\frac{\frac{z+\pi}{2}}{\frac{z^2+\pi+2\pi_z - z^2 - \pi^2 + 2\pi z}{4}}\right) - \frac{1}{\pi}$$

$$\frac{z+\pi}{\pi z} - \frac{1}{\pi} = \left(\frac{z+\pi}{\frac{4\pi z}{4}}\right) - \frac{1}{\pi}$$

$$\boxed{\frac{1}{z}} = \frac{z-z+\pi}{\pi z}$$

119. ¿Para que condiciones el trinomio $v = ax^2 + 2bxy + cy^2$ es la función armonica? Ejercicio 7

$v_x = 2ax + 2by$

$\boxed{v_{xx} = 2a} + 0$

$v_y = 2bx + 2cy$

$\boxed{v_{yy} = 2c}$

$\nabla^2 v = v_{xx} + v_{yy} = 0$

$2a + 2c = 0$

$2(a+c) = 0$

$\boxed{a + c = 0}$

Esta es la condición

Ejercicio 8

123. Sea analítica la función $w = f(z)$ en el dominio D ¿Que funciones de las siguientes serán armonicas en el dominio D?

a) $z = |w| = |x+iy| = \sqrt{x^2+y^2}$

$z_{xx} + z_{yy} = 0$

$z_x = \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}} 2x$ $\quad z_y = \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}} 2y$

$z_{xx} = -\frac{1}{4}(x^2+y^2)^{-\frac{3}{2}} 4x^2$ $\quad z_{yy} = -\frac{1}{4}(x^2+y^2)^{-\frac{3}{2}} 4y^2$

$-x^2(x^2+y^2)^{-\frac{3}{2}} - y^2(x^2+y^2)^{-\frac{3}{2}} = 0$

No es armonica.

b) $z = \arg w = \tan^{-1}\frac{y}{x}$

$z_x = \dfrac{-\frac{y}{x^2}}{1+\left(\frac{y}{x}\right)^2} = \dfrac{-y}{x^2+y^2}$ $\quad z_{xx} = +y(x^2+y^2)^{-2} 2x$

$z_y = \dfrac{+\frac{1}{x}}{1+\left(\frac{y}{x}\right)^2} = \dfrac{+x}{x^2+y^2}$ $\quad z_{yy} = -x(x^2+y^2)^{-2} 2y$

$(x^2+y^2)^{-2} 2xy - 2xy(x^2+y^2)^{-2} = 0 = 0$

Si es armonica

c) $z = \ln|w| = \ln|x+iy| = \frac{1}{2}\ln(x^2+y^2)$

$z_x = \frac{1}{2}\left(\frac{2x}{x^2+y^2}\right)$ $\quad z_{xx} = x\left((x^2+y^2)^{-2}(-2x)\right) + \frac{1}{x^2+y^2}$

$z_y = \frac{1}{2}\left(\frac{2y}{x^2+y^2}\right)$ $\quad z_{yy} = y\left((x^2+y^2)^{-2}(-2y)\right) + \frac{1}{x^2+y^2}$

$\dfrac{-2x^2-2y^2}{(x^2+y^2)^2} + \dfrac{2}{x^2+y^2} = 0 = 0$

Si es armonica

Ejercicio 9

118. ¿Pueden ser la parte real e imaginaria de una función analítica $f(z) = u(x,y) + v(x,y)$ las funciones siguientes?

a) $u = x^2 - y^2 + 2xy$

$v_y = u_x = 2x + 2y$

$v = \int (2x+2y)\,dy = 2xy + y^2 + g(x)$

$-v_x = -(2y + g'(x))\,dx$

$u_y = -2y + 2x$

$-2y + 2x = -2y - g'(x)$

$\int dx\, g'(x) = g(x) = \int -2x\,dx = -x^2$

$\boxed{v = 2xy + y^2 - x^2}$ Si puede ser analítica

B $u = x^2$

$\nabla^2 u = 0$

$u_{xx} + u_{yy} = 0$

$\boxed{2 = 0}$ No es analítica

$u_x = 2x$ $\quad u_{xx} = 2$

$u_y = 0$ $\quad u_{yy} = 0$

c) $u = \ln(x^2 + y^2)$

$\nabla^2 u = 0$

$u_x = \dfrac{2x}{x^2 + y^2}$ $\quad \boxed{u_{xx} = 2x\left(\dfrac{-2x}{(x^2+y^2)^2}\right) + 2\left(\dfrac{1}{x^2+y^2}\right)}$

$u_y = \dfrac{2y}{x^2+y^2}$ $\quad \boxed{u_{yy} = 2y\left(\dfrac{-2y}{(x^2+y^2)^2}\right) + 2\left(\dfrac{1}{x^2+y^2}\right)}$

$\dfrac{-4x^2 - 4y^2}{(x^2+y^2)^2} + \dfrac{2+2}{x^2+y^2} = 0 = \dfrac{4}{x^2+y^2} + (-4)\left(\dfrac{x^2+y^2}{(x^2+y^2)^2}\right)$

$0 = \dfrac{4}{x^2+y^2} - \dfrac{4}{x^2+y^2} = 0$

Si puede ser analítica

d) $u = \dfrac{x^2+1}{2}\,y^2 = \dfrac{x^2y^2}{2} + \dfrac{y^2}{2}$

$\nabla^2 u = u_{xx} + u_{yy} = 0$

$u_x = \dfrac{2xy^2}{2} = xy^2$ $\quad u_{xx} = y^2$

$u_y = 2y\left(\dfrac{x^2+1}{2}\right)$ $\quad u_{yy} = 2\left(\dfrac{x^2+1}{2}\right)$

$\boxed{y^2 + x^2 + 1 = 0}$

No puede ser analítica

Excersise 3.2 Libro: Zill

Ejercicio 10

6. $\bar{z}^2 = (x-iy)^2 = x^2 - y^2 - 2ixy$

Cauchy-Riemann

$U_x = V_y$ $\qquad$ $U_y = -V_x$

$\boxed{\begin{array}{l} U_x = 2x \\ V_y = -2x \end{array}}$ $\qquad$ $\boxed{\begin{array}{l} U_y = -2y \\ -V_x = +2y \end{array}}$

No es analitica

Ejercicio 11

7. $x^2 + y^2$

Cauchy-Riemann

$\boxed{\begin{array}{l} U_x = 2x \\ V_y = 0 \end{array}}$ $\qquad$ $\boxed{\begin{array}{l} U_y = 2y \\ -V_x = 0 \end{array}}$

$U_x \neq V_y$ $\qquad$ $U_y = -V_x$

No es analitica

Ejercicio 12

4. $f(z) = y + ix$

Cauchy-Riemann

$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}$ $\qquad$ $\frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}$

$U_x = 0$

$U_y = 0$

$\boxed{\begin{array}{l} U_y 1 \\ -U_x = -1 \end{array}}$

No es analitica

5. $f(z) = 4z - 6\bar{z} + 3$

$4(x+iy) - 6(x-iy) + 3 = 4x - 6x + 4iy + 6iy + 3$

$8x - 10x + 10iy + 3$

$f(z) = -2x + 10iy + 3$

$U_x = V_y$ $\qquad$ $U_y = -V_x$

$\boxed{\begin{array}{l} U_x = 10-12 \\ V_y = 10 \end{array}}$ $\qquad$ $\begin{array}{l} U_y = 0 \\ -V_x = 0 \end{array}$

No es analitica