CS535 Assignment 2

1. **Let G = (U; W; E) be a bipartite graph with maximal vertex degree k. Show that E can be decomposed into k matchings.**

Solution: It seems the requirement is to show that bipartite graph G can be decomposed into k matchings which means the graph should have perfect matching. We can prove this using the Hall's Theorem.

Hall's Theorem: A bipartite graph G consisting of sets u and w, |u| <= |w| has a matching of size |u|(smaller side) if and only if it satisfies Hall's condition which is |N(x)| >= |x| for every non empty set x subset of u. N(x) is neighborhood of the set x.

Step -1: Assuming G = (U; W; E) is a bipartite graph with maximal vertex degree of k. Now remove the vertex which have degree 0. After this find the perfect matching using Hall's theorem considering the smaller side.

Proof to find whether graph has perfect matching or not using Hall's theorem:

If G is k-regular bipartite graph then by the definition G is (u, v) and

|E| = k| u |=k| v | since k > 0 we can conclude that |u| = |v|. let x be the subset of u,

For the x_2 set of edges touching the x_1 set of N(x) by the definition x2 is subset or equal to x1 which is k|N(x)|= $|x_1|$ >= $|x_2|$ =|x|

Which shows |N(x)| >= |x| and $|x_1|$ = $|x_2|$ supporting result we can conclude that G has perfect matching.

Step-2: After finding the perfect matching of a smaller side using the above theorem remove this and repeat this step by considering the next smaller side.

We will run this k times and after running for k times there will be some edges remaining in the graph. These edges which are not part of the previous k matchings might have its end point in all matchings which shows that the degree of the vertex is greater than k (which will be the contradiction to our theorem). So this process must continue to atmost k times.

Therefore we can conclude that E can be decomposed into k matchings.

2. **Let G = (U; W; E) be a bipartite graph and M be a maximum matching of G. Suppose that M does not cover U. Show that the set U∩R_M is a Hall set, i.e.** $|N(U \cap Rm)| < |U \cap Rm||$

Solution: U is set of unmatched vertices and Rm is set of reachable vertices of matching M which contains both matched vertices and unmatched vertices of U and W.

$U \cap R_M$ is a set of unmatched vertices on U side. We have to prove |N(unmatched vertices on U side)| < |unmatched vertices on U side|. we can also say this N(matched vertices on U side) > matched vertices on U side. Which can be proved using Hall's theorem.

Hall's theorem:

If G is bipartite graph then by the definition G is (U, W, E) and

|E| = k| U |=k| W | since k > 0 we can conclude that |U| = |W|. let x be the subset of U matched vertices.

Number of edges incident on x be n.

Therefore |x| < |N(x)| we can conclude that G has maximum matching.

3. **Let G = (U;W;E) be a bipartite graph. Let τ denote the collection of subsets of U which can be covered by some matching of G:Suppose that $S, T \in \tau$ and $|S| < |T|$. Show that there exists $v \in T \setminus S$ such that $S \cup \{v\} \in \tau$.**

Solution:

Bipartite graph has a matching M which is subset of E if and only if each vertex appears atmost edge in M.

Given that U is collection of subsets which can be covered by matching M.

Let us consider U = {x,y,z,m,n,o,p}

T={m,p,x}

S ={x,y,}

|S| < |T|

T\S = {m,p}

S union T\S = {x,y,m,p} which belongs to set U.

Using this example we can state that there exists a vertex of T\S when S unions that vertex is a part of the U.

4. **Let G = (U, W, E) be a bipartite graph. Show that α'(G) = min(|U| - |S| + |N(S)|) for S be a subset of U.**

Solution:

From konig's theorem we know that minimum vertex cover is equal to maximum matching.

For G let H be the set of vertices which are reachable from unmatched vertices along the m-alternating path

Then α'(G) or maximum matching = (U-H) union (W intersection H)

If H is some subset of U = S then

W intersection H will be $N(S)$

Then $\alpha'(G) = (U-S) \text{ union } (N(S))$

$$\alpha'(G) = (U - S + N(S)).$$

In this way minimum vertex cover using konig’s theorem statements we can prove that $\alpha'(G) = (U - S + N(S))$ where S is subset of U.