Foundations of Computing

Lecture 17

Arkady Yerukhimovich

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Outline

1. Lecture 16 Review
2. Where Are We Now?
3. Reduction Types
4. A Computational Definition of Information – Kolmogorov Complexity

Lecture 16 Review

- Proofs by reduction
- Undecidable languages
  - $HALT_{TM}$
  - $REGULAR_{TM}$

Exercise

$$EMPTY-STRING_{TM} = \{\langle M \rangle \mid M \text{ is a TM and } M(\epsilon) = 1\}$$

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Algorithms

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1. To show that a problem is decidable: Give an algorithm that always terminates and outputs the answer
2. To show that a problem is undecidable: Give an algorithm (a reduction) that shows that this problem can be used to solve an undecidable problems

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Takeaway: General reductions do not work to prove Turing-unrecognizability

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Solution

We need to restrict what our reductions can do.

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$$w \in A \iff f(w) \in B$$

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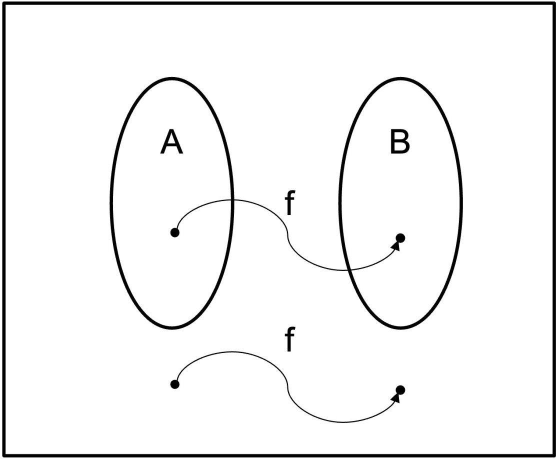
- Works by mapping input in A to input in B and vice-versa

Mapping Reductions



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- If A is not Turing-recognizable than B is not Turing-recognizable

Observation:

Mapping reductions work for both decidability and Turing-recognizability.

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- The reduction may make multiple calls to decider for B and may not directly use the result.
- For example, in the proof that checking whether $L(M) = \emptyset$ is undecidable (Exercise 1 from lab), we flipped the result of the decider.

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3. If $A \leq_T B$
   - If B is decidable then A is decidable
   - If A is not decidable, then B is not decidable

   - If B is Turing-recognizable, A is not necessarily Turing-recognizable
   - If A is not Turing-recognizable, cannot say if B is Turing-recognizable

Outline

Information in a String

$$A = 010101010101010101010101$$

$$B = 110100100011100010111111$$

Question

Which of these strings contains more information?

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- This captures the "amount of information" in x

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   - So, there exists at least one string that is incompressible
2. In fact, incompressible strings look like random strings
3. But, $K(x)$ is not computable, moreover it is undecidable whether a string is incompressible