Foundations of Computing

Lecture 5

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Outline

Lecture 4 Review

- More NFAs
- Equivalence of NFAs and DFAs
- NFAs for union, composition, and star – closure of regular languages

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A Useful Corollary

Recall that:

Definition

A language L is regular if and only if there is a DFA that recognizes it

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We can now use NFAs to argue the properties of regular languages

Closure Under Union

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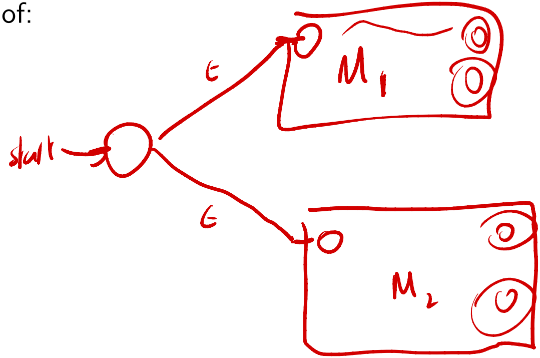
If L_1 and L_2 are both regular languages then $L_1 \cup L_2$ is also regular

$L_1 \cup L_2$ is the language consisting of all strings either in L_1 or L_2

Proof:



Closure Under Concatenation

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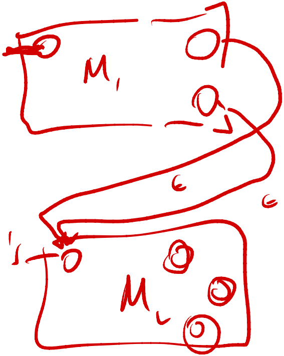
If L_1 and L_2 are both regular languages then $L_1 \circ L_2$ is also regular

$L_1 \circ L_2 = \{xy | x \in L_1 \text{ and } y \in L_2\}$

Proof:

00101



$x = y \in L_1 \,||\, z \in L_2$

L_1 = strings of all 0's

L_2 = strings 01 01 ..

Closure Under the Star Operation

Closure Under Star Operation

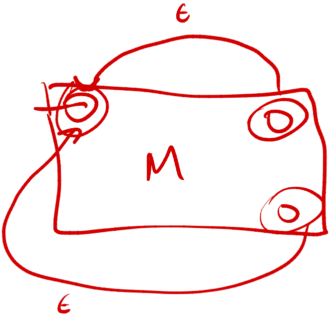
If L is a regular languages then L^* is also regular

$L^* = \{0 \text{ or more strings from } L\}$

0^*

Proof:

$\{0, 1\}^*$



$L = \{01, 10\}$

$L^* = \{\epsilon, 01, 10, 0101,$

0110

1001

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Regular Expressions

- Strings that describe a language

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You've seen this before

Regular expressions very useful in compilers, and string search (e.g., grep)

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5. $(R_1 \circ R_2)$ – R_1 concatenated with R_2 where R_1 and R_2 are regular expressions
6. (R_1^*) – 0 or more repetitions of R_1 where R_1 is a regular expression

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- $1^*\emptyset = \emptyset$ $(\{\epsilon, 1, 11, 111, \ldots\})\ \emptyset$
- $\emptyset^* =$

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$1^* \epsilon = 1^*$

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- $(0 \cup \epsilon)(1 \cup \epsilon) = \{\epsilon, 0, 1, 01\}$
- $1^*\emptyset = \emptyset$
- $\emptyset^* = \{\epsilon\}$
- $0\Sigma^*0 \cup 1\Sigma^*1 \cup 0 \cup 1 = \{w | w \text{ starts and ends with the same symbol}\}$

$\phi = \{\}$ $\{\epsilon\}$

Languages to Regular Expressions Examples

Consider languages over the alphabet $\{0, 1, 2\}$

1. $L_1 = \{w | w \text{ has 2 consecutive 0's}\}$

$$\Sigma^* 00 \Sigma^*$$

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$\Sigma^* 101 \Sigma^* 22$

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$R_1 \cup R_2$

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Question:

What does this have to do with FAs and regular languages?

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Regular Expressions == Regular Languages == NFA

Theorem

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1. $R = a$ for some $a \in \Sigma$

2. $R = \epsilon$

3. $R = \emptyset$

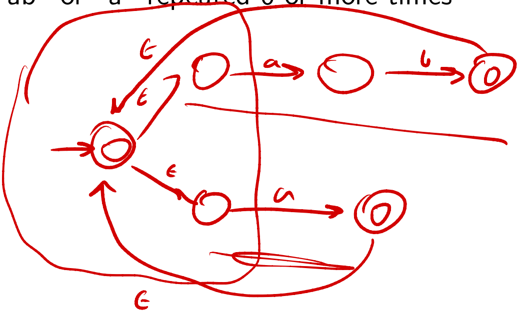
4. $R = R_1 \cup R_2$
5. $R = R_1 \circ R_2$
6. $R = R_1^*$

An Example

Problem: Convert $(ab \cup a)^*$ to an NFA

In English: Either "ab" or "a" repeated 0 or more times

- a:
- b:
- ab:
- $ab \cup a$:
- $(ab \cup a)^*$:



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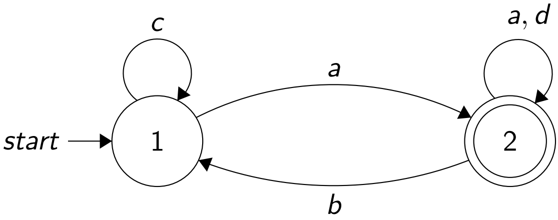

Question

How do we represent L by a regular expression?

Step 1: NFA $\rightarrow$ generalized NFA

A generalized NFA has 3 important properties:

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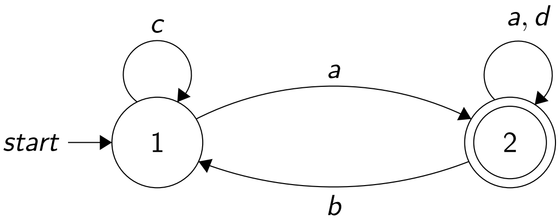
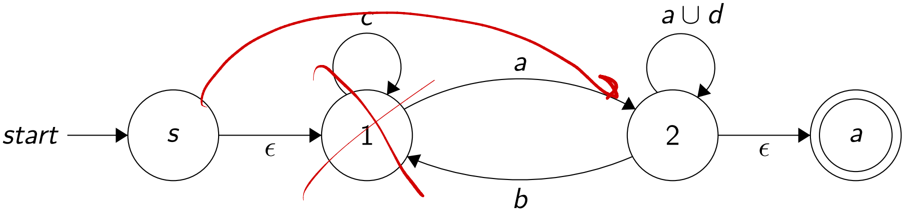


Step 2: Node Elimination – Remove Node 1

Remove nodes one-by-one (in any order) until only start and accept states left:

- Need to update reg. exp.'s for all paths through removed nodes



Step 2: Node Elimination – Remove Node 2

Remove nodes one-by-one (in any order) until only start and accept states left:

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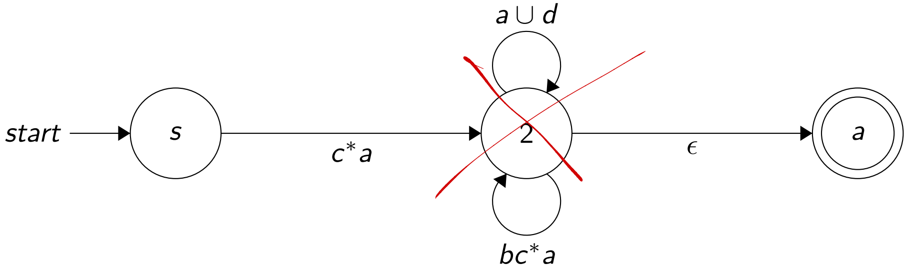
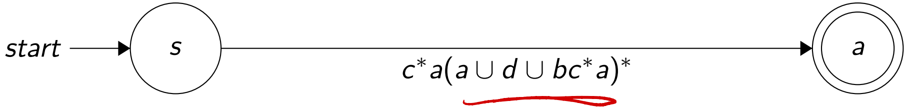
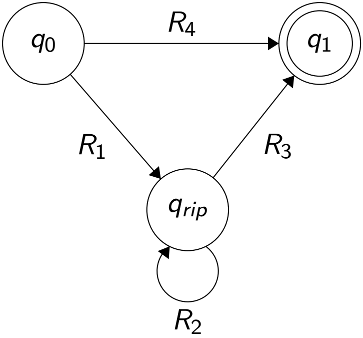


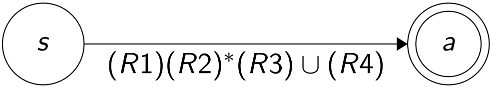
We are Done

Output label of final edge from start to accept state.



Generalized Node Elimination





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Base Case: For $|G| = 2$, G consists of start and accept states and arrow between them. The label on this arrow exactly describes the language of strings accepted by G.

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For any GNFA G, G'=NODE-ELIMINATE(G) is equivalent to G

Inductive step: Assume true for $|G| = k - 1$, prove true for $|G| = k$. (i.e., prove that $G' = G$)

- Assume some w s.t. $G(w) = 1$, then on input w, G goes through

$$q_{start}, q_1, q_2, \ldots, q_{accept}$$

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  - If it would have gone through q_{rip} then the modified edge accepts w, so there is a path through q_{rip} in G that accepts w.

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  - If the accepting path would have not gone through q_{rip}, then G must also have the same path to accept w

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By equivalence between regular expressions and NFAs, we have:

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Proof:

- Build NFA M corresponding to each clause
- Since we already showed how to build NFA to show closure, can convert that to regular expression to prove the claim.