Foundations of Computing

Lecture 6

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Outline

Lecture 5 Review

- Regular expressions
- Equivalence of regular expressions and NFAs/DFAs

Quiz Solutions

For each of the following languages over $\Sigma = \{a, b\}$, give two strings that are in the language and two strings not in the language.

1. $a^* \cup b^*$

   in: aaa, bb

   out: ab, ba

2. $(aa \cup bb)^*$

   in: aa, bbbb

   out: abba, ab

3. $\Sigma^* a \Sigma^* b \Sigma^* a \Sigma^*$

   in: aba, aababab

   out: bbbbb, aaaaaa

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What We Know So Far

The following four things are equivalent:

1. Regular languages
2. Languages decided by DFAs
3. Languages decided by NFAs
4. Languages described by regular expressions

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Are all languages regular?

Today we will see that there are languages that are not regular

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- An automaton must be able to process strings w s.t. $|w| > |Q|$

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Important

In a *Finite* Automaton, the number of states is **finite**

This means that:

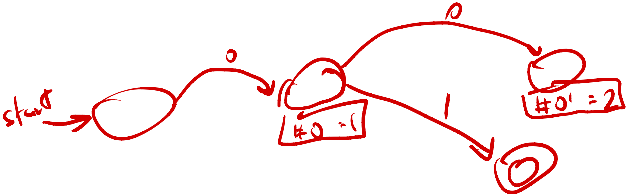
- The number of states is fixed independently of the input size
- An automaton must be able to process strings w s.t. $|w| > |Q|$
- Thus, a finite automaton cannot store its whole input

A Nonregular Language

Consider the following language:

$$L = \{0^n 1^n | n \geq 0\}$$

Let's try to build a DFA for L:



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The Problem

We need to count the number of 0s, but this is unbounded so can't have a state for each value

The Need for a Proof

What we just saw

Intuition: An NFA/DFA cannot count unbounded inputs

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What we just saw

Intuition: An NFA/DFA cannot count unbounded inputs

Why isn't this a proof?

Consider the following language:

0101010

$L' = \{w | w$ has an equal number of occurrences of 01 and 10 as substrings$\}$

General Proof Structure

We will prove that a language L is not regular by contradiction

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1. Assume L is regular – there is a NFA/DFA M accepting it
2. Pick a particular string $w \in L$
3. Show that if $M(w) = 1$ then there exists a string $w' \notin L$ s.t. $M(w') = 1$
4. Conclude that L is not regular since any M that accepts all strings in L must also accept strings not in L

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Let's write this using math:

$\exists p$ s.t. $\forall w \in L, |w| \geq p$ $\exists x, y, z$ s.t. $w = xyz$ $\forall i$ $xy^iz \in L$

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Next steps:

1. Prove the pumping lemma
2. Show how to use the pumping lemma to prove languages nonregular

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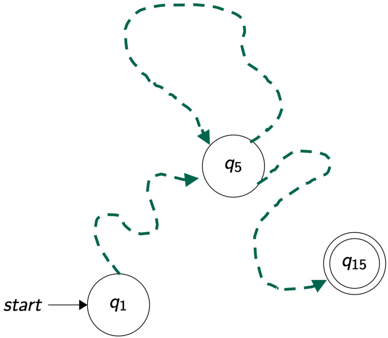
  - Since $n + 1 > p$, there must be some state that is visited twice



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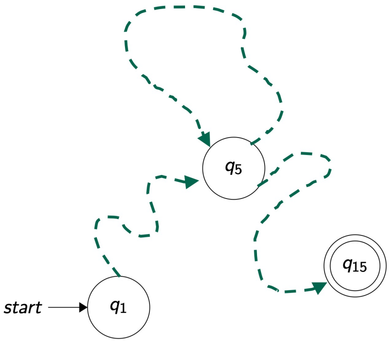

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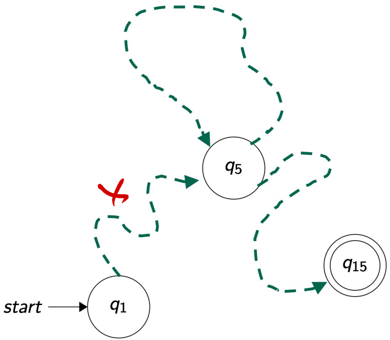

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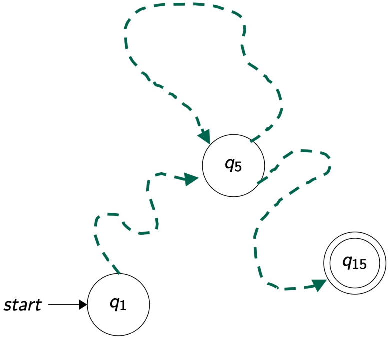

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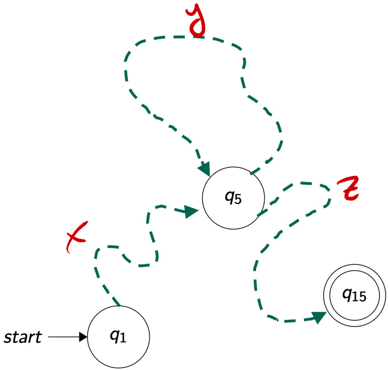

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3. $|xy| \leq p$
   Proof: if q_5 is the first repetition in $M(w)$, then this repetition must occur in the first $p + 1$ states, hence $|xy| \leq p$

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Observe that:

- x takes M from $r_1 = q_1$ to r_j, y takes M from r_j to r_k, and z takes M from r_k to r_{n+1}, which is an accept state. So, M must accept $xy^i z$ for $i \geq 0$

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- $j \neq k$ so, $|y| > 0$
- $k \leq p + 1$, so $|xy| \leq p$

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If L is a regular language, then there exists an integer p (the pumping length) where any string $w \in L$ such that $|w| \geq p$ can be divided into three substrings $w = xyz$ satisfying:

1. For each $i \geq 0$, $xy^iz \in L$
2. $|y| > 0$, and
3. $|xy| \leq p$

Mathematically:

$$\forall w \in L, |w| \geq p \; \exists \text{ substrings } w = xyz \text{ s.t. } \forall i \geq 0, xy^iz \in L$$

Let's negate this:

$$\exists w \in L, |w| \geq p \; \forall \text{ substrings } w = xyz \; \exists i \geq 0, \text{ s.t. } xy^iz \notin L$$

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The Recipe

Negated Pumping Lemma

$$\exists w \in L, |w| \geq p \; \forall \text{ substrings } w = xyz \; \exists i \geq 0, \text{ s.t. } xy^iz \notin L$$

To use the pumping lemma to prove that L is not regular, we do the following:

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5. Contradiction – Pumping lemma must hold for any regular L

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5. Contradiction – hence, L is not regular

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7. Contradiction – hence, L is not regular

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3. Easy to see that $0^* 1^*$ is regular
4. Since regular languages are closed under $\cap$, if L is regular then L_1 must be regular
5. Since we know L_1 is nonregular, this means that L must be nonregular

Exercise

Prove that the following language is nonregular:

$$L = \{0^i 1^j 2^i 3^j | i, j > 0\}$$

What's Next?

- We will get plenty of practice with proving languages nonregular
- We will add (a small amount of) memory to our machines to recognize a richer class of languages