Foundations of Computing

Lecture 10

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February 13, 2025

Outline

1. Lecture 9 Review
2. CFG == PDA
3. The CFL Pumping Lemma
4. Using the CFL Pumping Lemma

Lecture 9 Review

- Context-Free Grammars
  - Strings generated by grammars
  - Building CFGs
  - Parse Trees

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Today

Connect CFGs and PDAs and look at their limitations

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Proof:

We need to prove both directions:

1. If a language is context free, then some PDA decides it
2. If a language is decided by a PDA, then it is context free

Proof of CFG $G \to$ PDA M

Idea: Construct PDA M s.t. $M(w) = 1$ if there is derivation for w in G

- Recall: Derivation of w in G – sequence of substitutions resulting in w
- Each step gives intermediate string of variables and terminals
- M decides if $\exists$ sequence of substitutions in G leads from start to w

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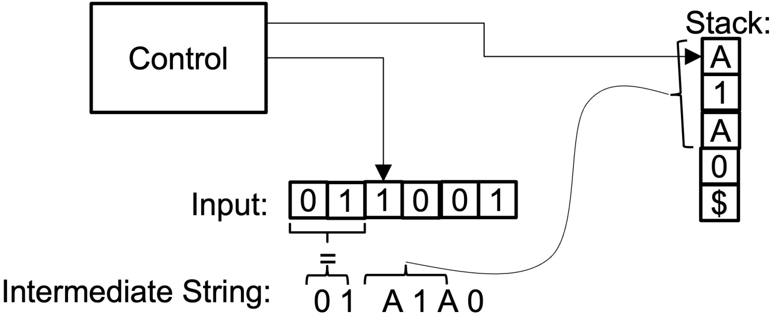

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Picture version of the resulting PDA is in the book

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- Thus, $A_{q_0 q_{accept}}$ generates all strings $w \in L(M)$

Proof of PDA $M \to$ CFG G: Building A_{pq}

Assume that M has the following properties:

1. Only one accept state: q_{accept}
2. M empties its stack before accepting
3. All transitions either have form $x, \epsilon \to a$ (push an item on the stack) or $x, a \to \epsilon$ (pop an item off the stack), but not both.

Easy to turn any PDA M into one satisfying these properties

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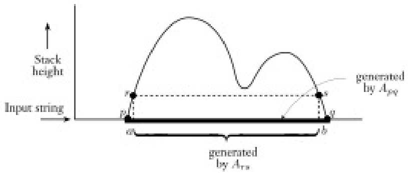
    - Add rule $A_{pq} \to aA_{rs}b$:
  - Case 2: Symbol popped in last step not same symbol pushed in first step
    - Symbol pushed in first step, must be popped before the end, so stack becomes empty at some middle state r
    - Add rule $A_{pq} \to A_{pr}A_{rq}$

The Same Thing in Pictures

Case 1: $A_{pq} \to aA_{rs}b$



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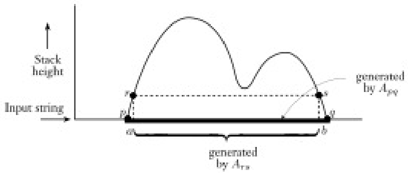

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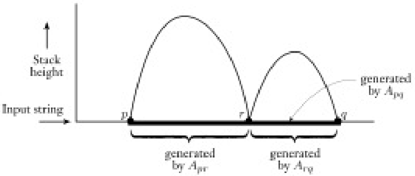

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We have shown conversions for:

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Question

Are all languages context-free?

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The CFL Pumping Lemma

Theorem

If L is a CFL, then there exists a pumping length p s.t. for any $s \in L$, with $|s| \geq p$, s can be divided into 5 pieces $s = uvxyz$ satisfying:

1. For each $i \geq 0$, $uv^ixy^iz \in L$
2. $|vy| > 0$
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Pumping lemma in math notation:
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Negation of pumping lemma:
$\forall p$, $\exists s \in L, |s| \geq p$ s.t. $\forall$ partitions $s = uvxyz$ $\exists$ i s.t. $uv^ixy^iz \notin L$

Proving the CFL Pumping Lemma (Intuition)

Consider the parse tree for some very long $s \in L$

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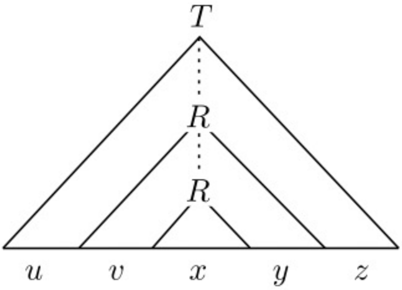

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- Consider the negation:

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- So, we need to find such an s and prove that for any way to partition it, it cannot be pumped

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To use the pumping lemma to prove that L is not CFL, we do the following:

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4. Demonstrate that s cannot be pumped
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5. Contradiction!!!

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Exam 1

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