Foundations of Computing

Lecture 2

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Outline

Homework Policies

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You may NOT do the following:

- Copy or provide answers to any hw problems to others
- Use ChatGPT or any other LLM to produce your answers
- Search the web for solutions or use services like chegg.com or StackExchange

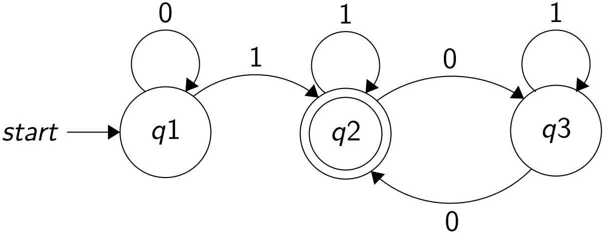
Outline

Lecture 1 Review

- Syllabus review and course details
- Strings, languages, and functions
- Finite automata

Outline

Language accepted by M



Accepting a string

- M accepts a string x (over Σ) if $M(x)$ stops in an accept state
- What strings does M accept?

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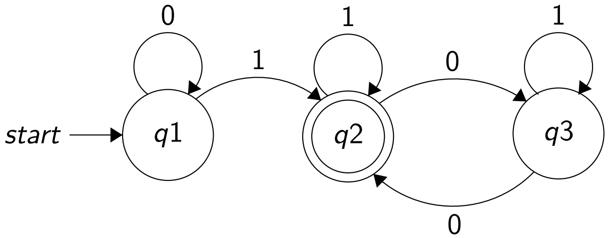

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- M <u>decides</u> a language L if it accepts:
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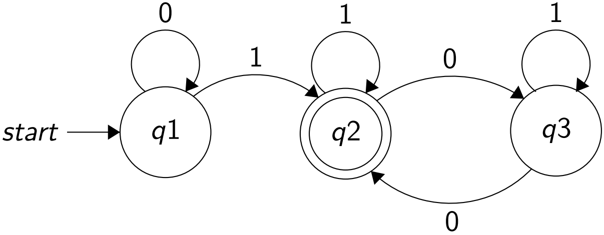

Accepting a string

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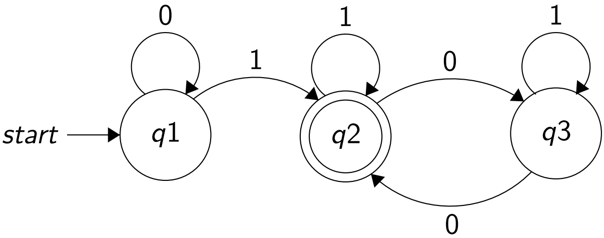
Deciding a language

- M decides a language L if it accepts:
  - ALL strings in L, and
  - NO strings not in L
- Every M accepts exactly one language $L(M)$

What language does M accept?

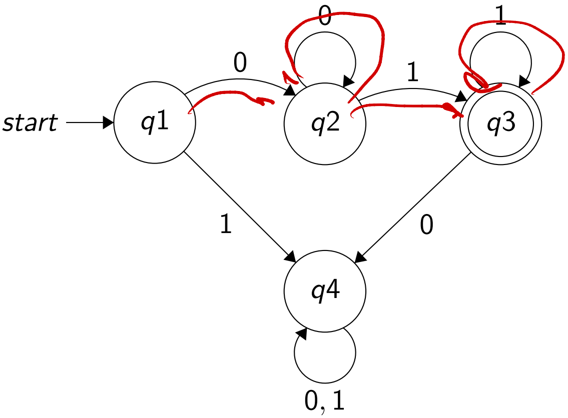


$L(M)$:

- String must contain at least one 1
- After the first string of 1's, there must be an even number of 0's or no 0's

Outline

Quiz Solutions



- Does M accept 00011?:

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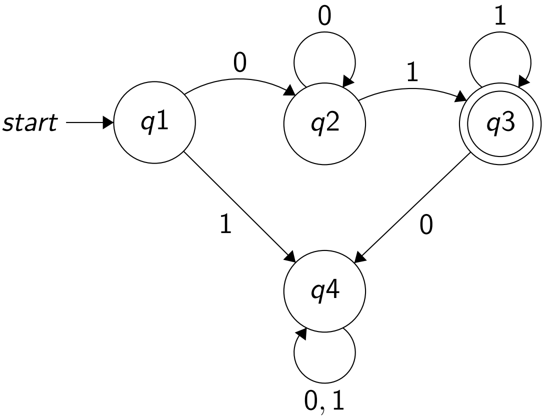

- Does M accept 00011?:
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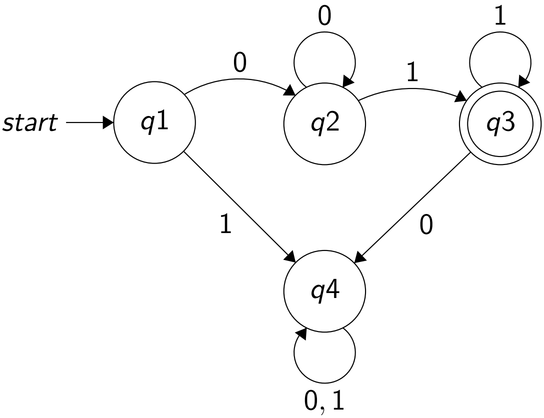

- Does M accept 00011?:
- Does M accept 01100?
- Describe the language $L(M)$:

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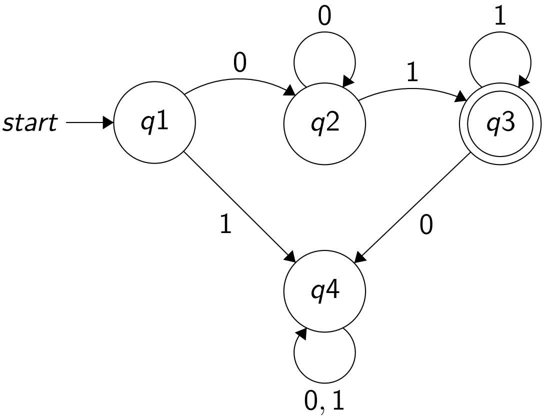

- Does M accept 00011?:
- Does M accept 01100?
- Describe the language $L(M)$: all strings with one or more 0s followed by one or more 1s

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Important Rules of Deterministic Finite Automata

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- Transition function must be fully defined:
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Important: Deterministic means that the execution of M on any input in Σ^* must be fully specified.

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DFA Execution

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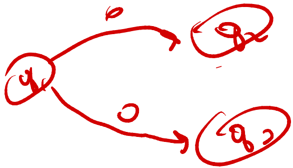
Memory in a DFA:

- Each state stores a summary of the input seen so far
- Next state depends on the current state and the next symbol
- Think of this as an "if" statement



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Important

Since $|Q|$ is finite, need to be able to take in inputs longer than the number of states

- Cannot just store the entire string!

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Problem

Build a DFA that decides

$$L = \{w | w \in \{0,1\}^* \text{ and } w \text{ contains the substring } 101\}$$

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- If see a 0:
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- If see a 1:
  - this can be the first character of 101
  - or, it can be the last character if we previously saw 10 – in this case, we should accept

Example 1 – The Algorithm

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Algorithm:

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3. Step 3:
   - If read a 0, goto step 1 – this is not 101, time to start over
   - If read a 1, goto step 4 – we have seen 101

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   - On any input, stay in step 4 and accept

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The DFA

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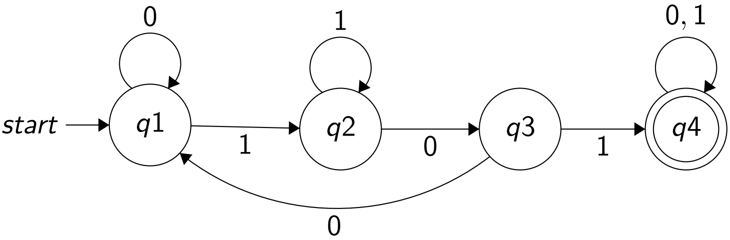

1. $q1$ – not yet read first 1 in 101
2. $q2$ – last input was a 1, could be start of 101
3. $q3$ – have read 10
4. $q4$ – have read 101

Trap States

A useful tool for designing DFAs:

- Trap states allow you to "reject" as soon as you know that $w \notin L$

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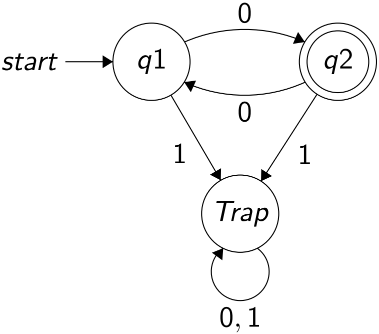

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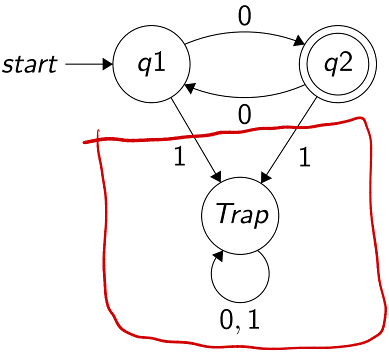

For convenience

You can omit edges from transition diagram that point to the trap state

Exercise

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Build a DFA that decides:
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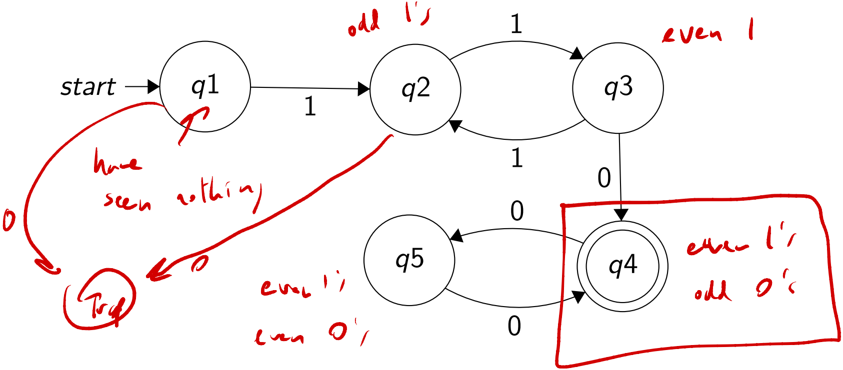

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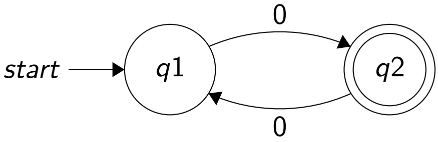
Another Example

Consider the following DFA M



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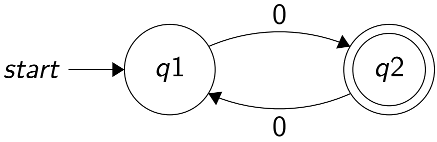

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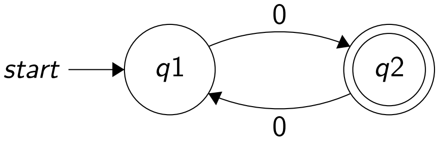

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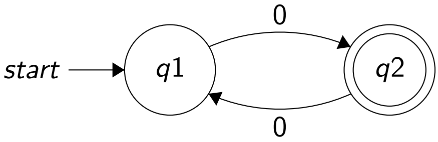
Proof:

- Need to prove that $L = L(M)$
- Instead we prove the $L \subseteq L(M)$ and $L(M) \subseteq L$

$L \subseteq L(M)$



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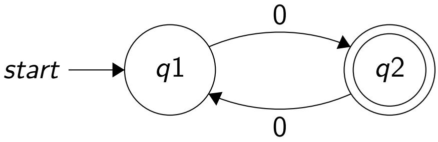

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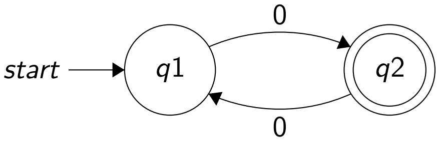

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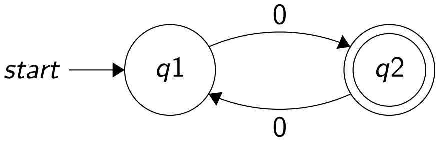

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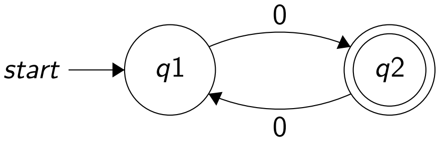

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Proof by Induction:
Consider $|w| = k + 2$ and let w' be the prefix of w of length k.
By hypothesis $\delta^*(q1, w') = q2$, and last two bits of w must be 0's
Hence $\delta^*(q1, w) = q2$

$L(M) \subseteq L$

Claim: Every w accepted by M is in L.

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Assume there exists a string w accepted by M that is not in L

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Proof:

1. w cannot have a 1, as any such input will not stop in $q2$
2. By similar proof to before, any w with even number of 0's must stop in $q1$
3. Contradiction!