Foundations of Computing

Lecture 12

Arkady Yerukhimovich

February 25, 2025

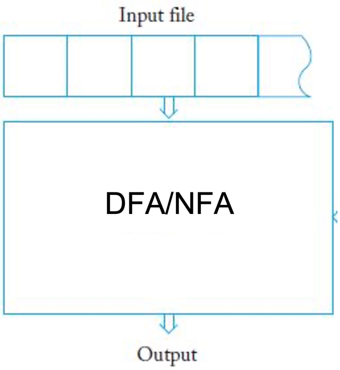
Outline

Lecture 10+11 Review

- Equivalence of CFGs and PDAs
- CFL Pumping Lemma
- Using the CFL Pumping Lemma

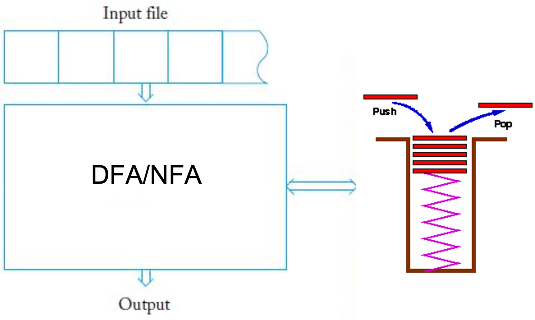
Outline

Finite Automata



Recall:

- An NFA/DFA has no external storage
- Only memory must be encoded in the finite number of states
- Can only decide regular languages



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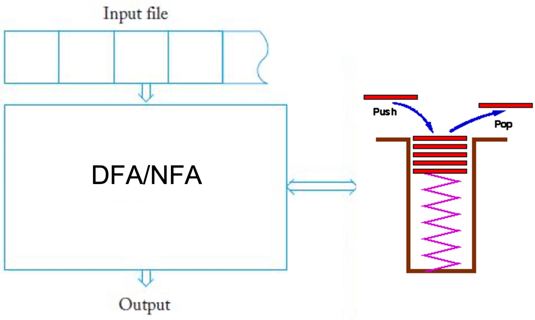
- An NFA for a control unit
- A Stack for storage

Pushdown Automata (PDA)



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Recall:

- Can only access memory in LIFO fashion
- Can only decide context-free languages

A Universal Computer

Question

All the prior machines couldn't decide some simple languages. Can we develop a machine that captures everything that can be computed?

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Our Goal

One model to rule them all!

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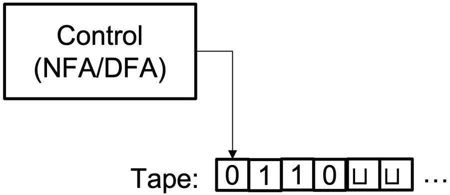
The Turing Machine



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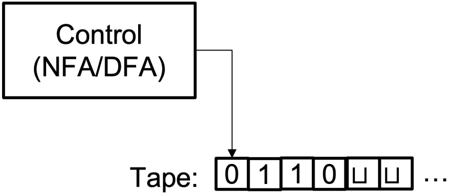

Key Differences:

- A TM can read and write to its tape

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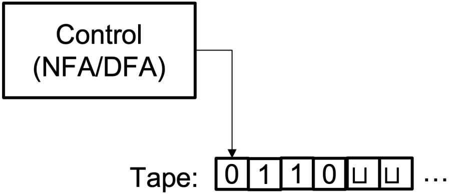

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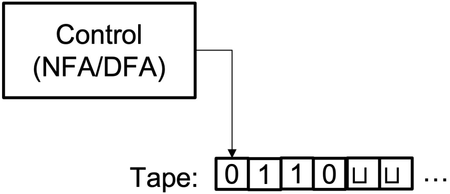

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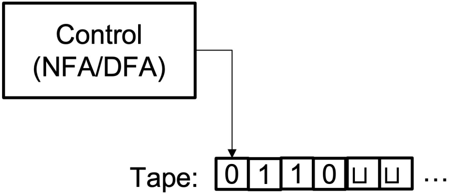

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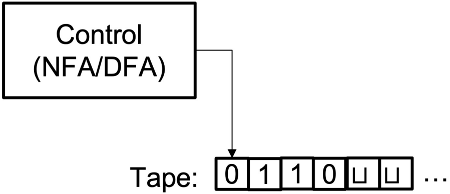

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- The memory tape is infinite
- Control FA has accept and reject states that are immediately output if entered

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3. When all symbols to the left of # have been crossed off, check that no uncrossed-off symbols remain to the right of #. If any symbols remain, reject, otherwise accept.

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- To reason about languages accepted by NFA/PDA, we designed algorithms
- How can we reason about the limits of what an algorithm can compute?

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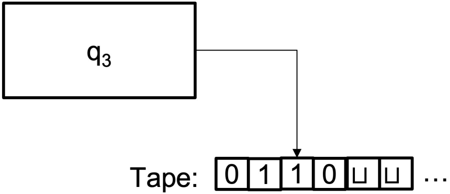
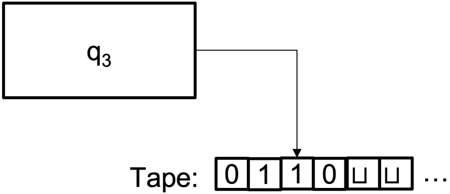
- and move the tape head one spot to either Left or Right

Computing on a Turing Machine



Configuration of a TM

Computing on a Turing Machine



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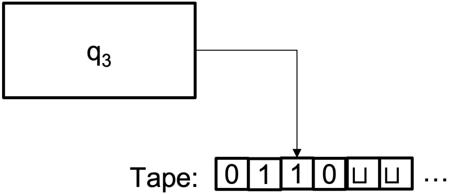

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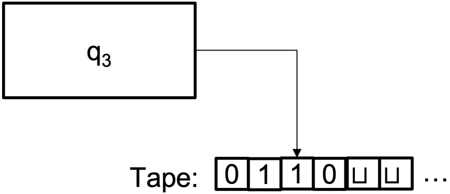

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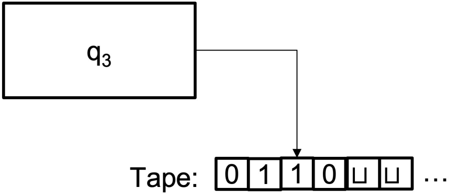

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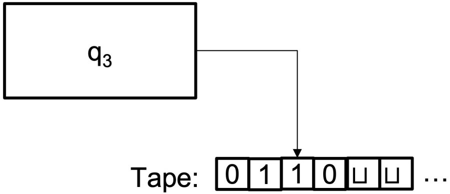

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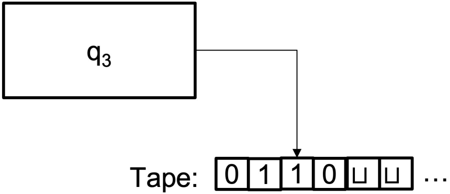

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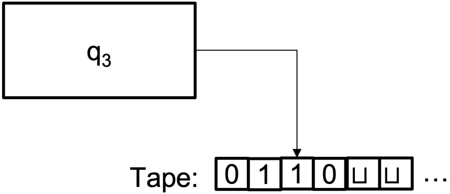

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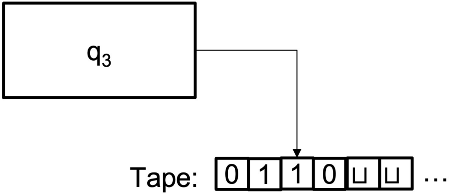

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- halting configuration – accepting or rejecting configuration

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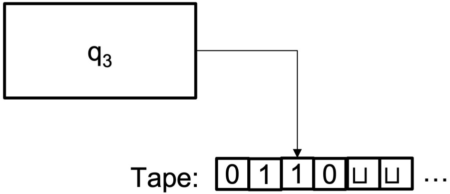

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Language $L(M)$

The collection of strings that M accepts

An Example

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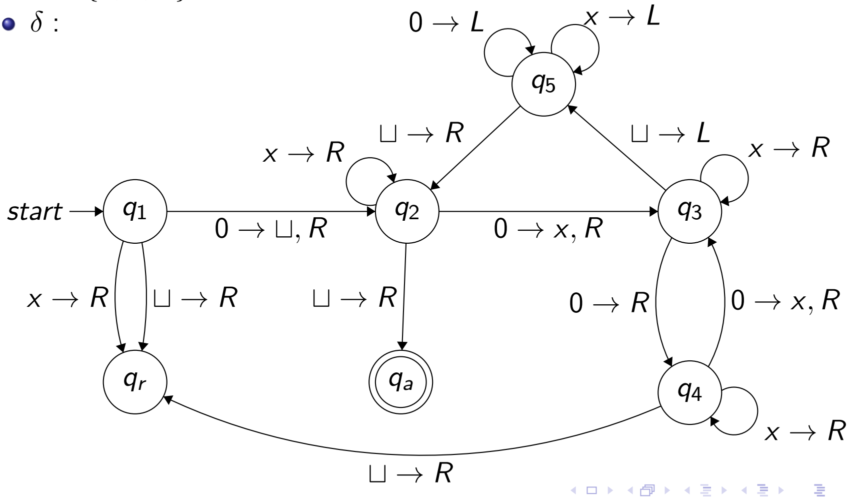
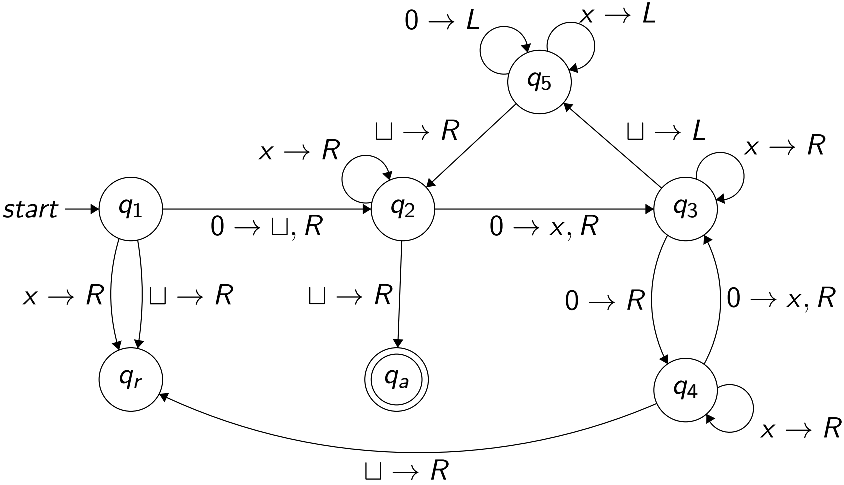


Running M on $w = 0000$



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A Universal Turing Machine

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Hint: Recall that $M = (Q, \Sigma, \Gamma, \delta, q_0, q_{accept}, q_{reject})$

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- While unproven, all modern computers satisfy Church-Turing thesis
- To prove that some problem cannot be solved by an algorithm, enough to reason about Turing Machines
- This means that Turing Machines give an abstraction to capture "feasible computation"

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