Foundations of Computing

Lecture 17

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Outline

Lecture 16 Review

- Proving undecidability
- Reductions
- Examples of undecidable languages

Outline

Reductions

Reductions Between Problems

There is a reduction from a problem A to a problem B if we can use a solution to problem B to solve problem A

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We have actually seen two different types of reductions

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$\langle M, w \rangle \rightarrow \langle M_2 \rangle$ s.t. $\langle M, w \rangle \in L_{TM} \iff M_2 \in EQ_{TM}$

- If R decides $EMPTY - STRING_{TM}$, S decides L_{TM}, contradiction

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- If R decides $EMPTY - STRING_{TM}$, S decides L_{TM}, contradiction
- S maps input of L_{TM} to an input of $EMPTY - STRING_{TM}$

$$\langle M, w \rangle \in L_{TM} \iff \langle M_2 \rangle \in EMPTY - STRING_{TM}$$

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- If R decides $L_{E_{TM}}$, S decides L_{TM}, contradiction
- But, S does not map input in L_{TM} into an input in $L_{E_{TM}}$

$$\langle M, w \rangle \in L_{TM} \iff \langle M_2 \rangle \notin L_{E_{TM}}$$

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Mapping Reductions

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- Works by mapping input $\in A$ to input $\in B$ and vice-versa

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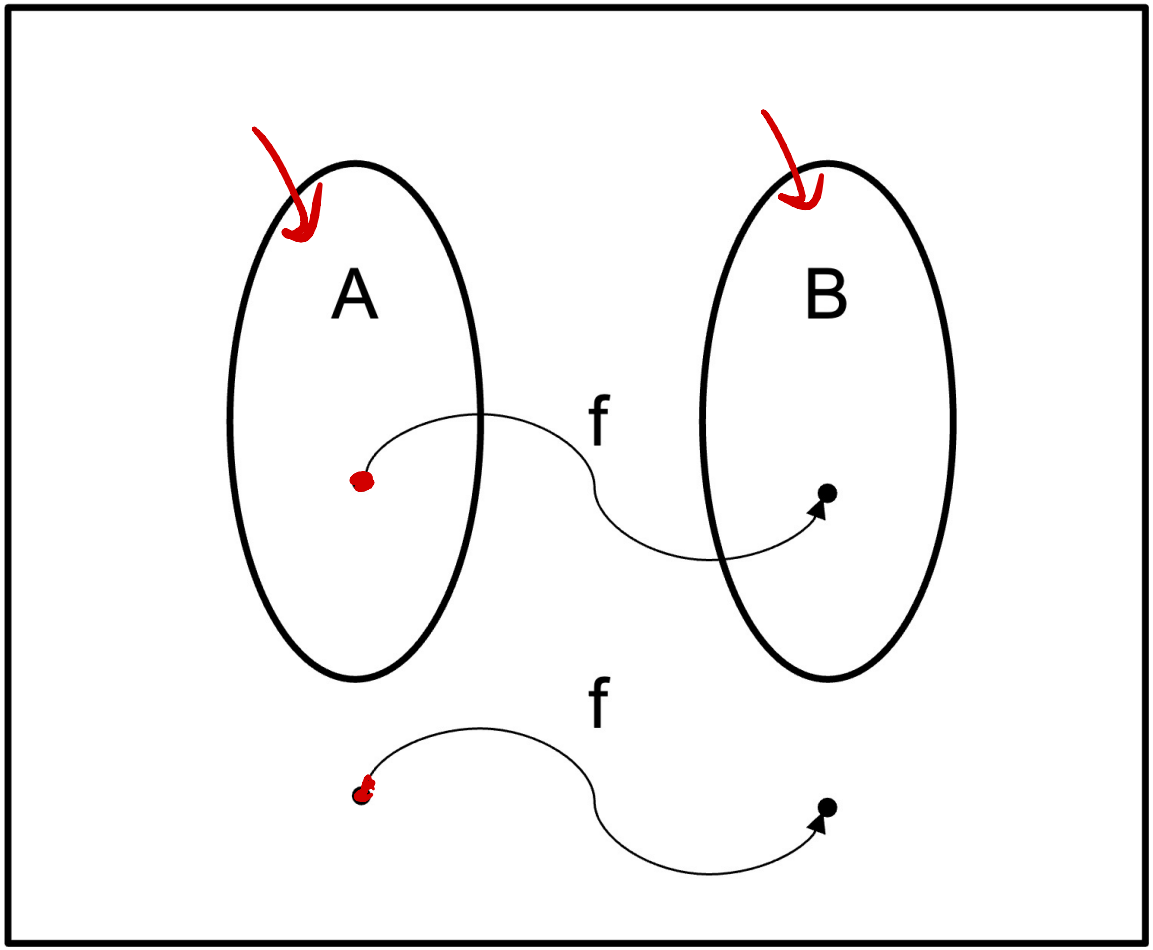

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- For example, in the proof that $L_{TM} \leq L_{E_{TM}}$, we flipped the result of R deciding $L_{E_{TM}}$

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4. If $A \leq_T B$
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   - If A is not Turing-recognizable, cannot say if B is Turing-recognizable

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Information in a String

$$A = 01010101010101010101010101$$

$$B = 11010010001110001011111 1$$

Question

Which of these strings contains more information?

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- $K(x)$ is the minimal description of x
- This captures the "amount of information" in x

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3. $\forall x, y, K(xy) \leq 2 \cdot K(x) + K(y) + c$

$x = 0\,0\,1$

$0\,0\,0\,0\,1\,1\,0\,1$ w y

$x \cdot 2$ w

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2. In fact, incompressible strings look like random strings

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   - There are 2^n strings of length n
   - The number of strings of length less than n is

$$\sum_{0 \leq i \leq n-1} 2^i = 1 + 2 + 4 + \cdots + 2^{n-1} = 2^n - 1$$

   - So, there exists at least one string that is incompressible
2. In fact, incompressible strings look like random strings
3. But, $K(x)$ is not computable, moreover cannot decide whether a string is incompressible