Foundations of Computing

Lecture 25

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April 22, 2025

Outline

1. Lecture 24 Review
2. A New Goal for Proofs
3. Defining Knowledge
4. Examples of Zero-Knowledge Proofs
5. Zero-Knowledge on the Blockchain

Lecture 24 Review

- Interactive Proofs
- Proof for Graph Non-Isomorphism
- Polynomial Identity Testing

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Reviewing the Definition of $\mathcal{IP}$

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$L \in \mathcal{IP}$ if there exist a pair of interactive algorithms (P, V) with V being poly-time (in $|x|$) s.t.

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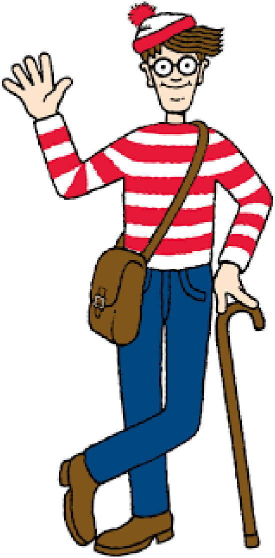
A New Property

We say that a proof is *zero-knowledge* if the verifier learns nothing (other than the truth of the statement) from seeing the proof.

An Example – Where's Waldo



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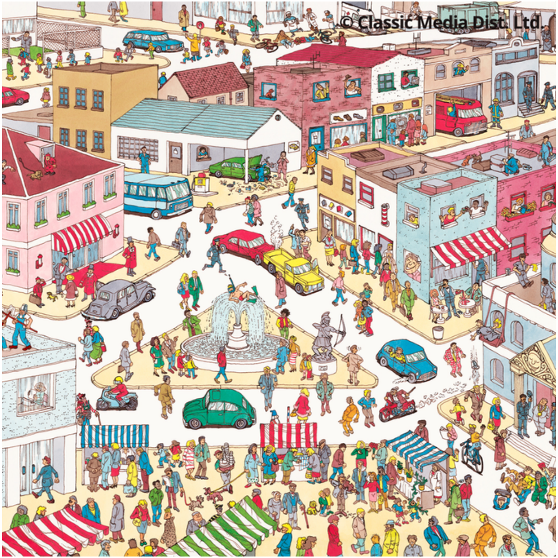
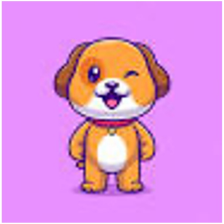


A Second Example – Puppy and Panda



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What does it mean for a machine to learn nothing from a proof?

Answer: Whatever it can (efficiently) compute after seeing the proof, it could have efficiently computed before seeing the proof.

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- Thus, the proof is zero-knowledge: V learns nothing more than that $x \in L$
- IMPORTANT: $VIEW_V^*$ and $S(x)$ are both distributions, not values. So, equality is of distributions

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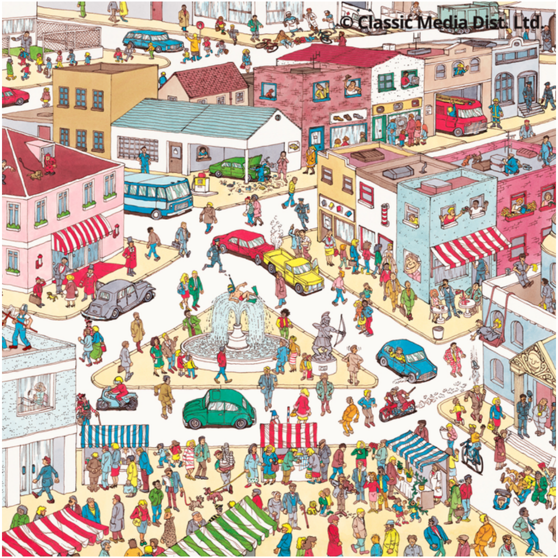

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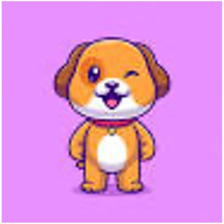

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2. Soundness: Suppose G_0 is not isomorphic to G_1, so there is no such π. Then, if $b \neq b'$, there is no permutation that P can give that V will accept

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- When S stops, he produces a perfect simulation

Graph 3-Coloring

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ZK Proofs enable privacy-preserving transactions on a public Blockchain!