Foundations of Computing

Lecture 24

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Outline

Lecture 23 Review

- Vertex Cover is $\mathcal{NP}$-complete
- Ladner's Theorem
- The class co-$\mathcal{NP}$

$\mathcal{NP}$ vs co-$\mathcal{NP}$

$\mathcal{NP}$ – Yes instances are efficiently verifiable

$L \in \mathcal{NP}$ if there exists poly-time DTM V s.t. for $x \in L$ there exists a witness w s.t. $V(x, w) = 1$

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Is $SAT \in$ co-$\mathcal{NP}$?

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- Can be an interactive procedure
- The verifier (and prover) can use randomness to decide whether to accept

An Example – Aladdin's Cave

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- Interactive proofs can be much more efficient (e.g., shorter) than non-interactive ones
- Can have additional properties that traditional proofs cannot satisfy.
  - Zero-knowledge

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- $\mathcal{NP} \subseteq \mathcal{IP}$

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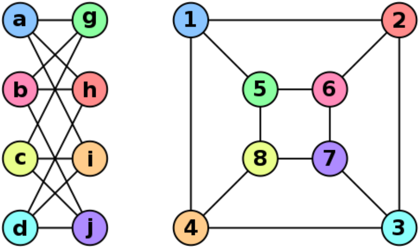

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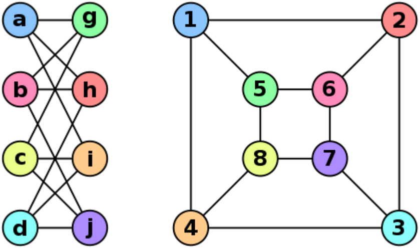

Claim

Graph Isomorphism $\in \mathcal{IP}$

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   - Thus, $\Pr[b' = b] = 1/2$

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- The power of interaction and randomness has allowed us to do what we couldn't do before

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3. P^* wins with probability $\leq 1/2$ in each run, so

$$\Pr[\langle P^*, V\rangle(x) = 1] \leq 1/2^n$$

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Another Example – Polynomial Identity Testing

Polynomial

A polynomial is an equation in one-variable

$$\begin{aligned} f(x) &= x^3 - 6x^2 + 11x - 7 = \\ &\quad (x-1)(x-2)(x-3) \end{aligned}$$

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- Suppose that V is deterministic.
- What if you allow V to be randomized?

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Next Week

We have seen the power of interactive proofs in convincing a verifier of the truth of some statement.

Question:

What does the verifier learn from seeing the proof?