Foundations of Computing

Lecture 4

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January 23, 2025

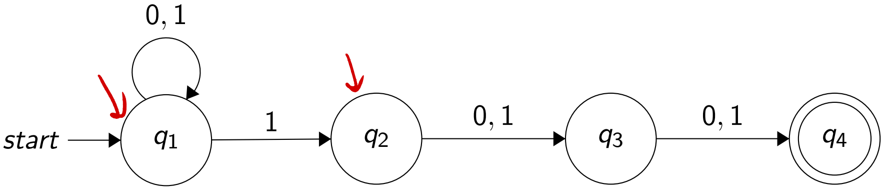
Outline

Lecture 3 Review

- Regular Languages
- Nondeterministic Finite Automata
- Understanding Nondeterminism

Outline

NFA Example 1



Question: Does M accept 01101? 01001?

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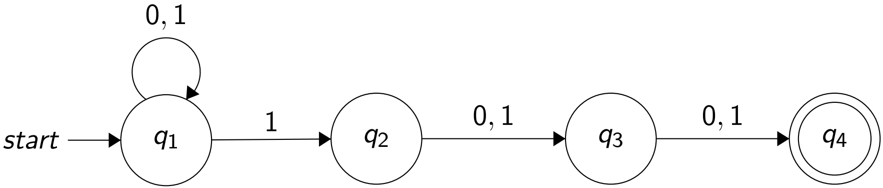

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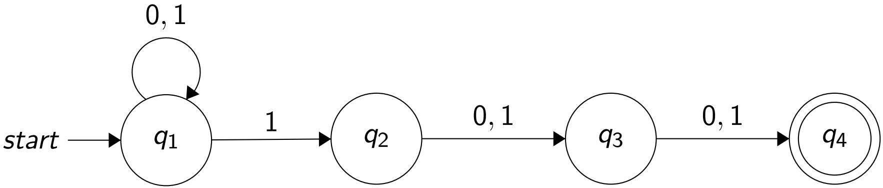

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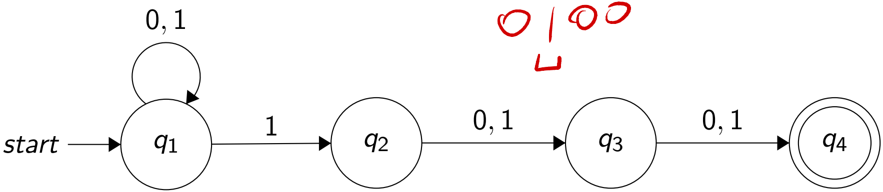

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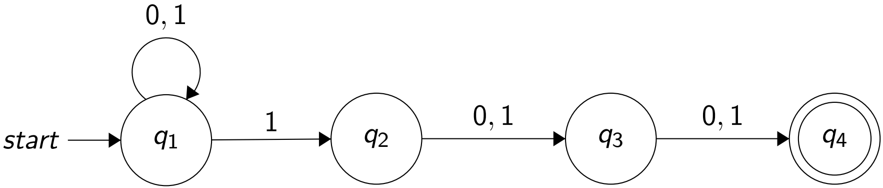

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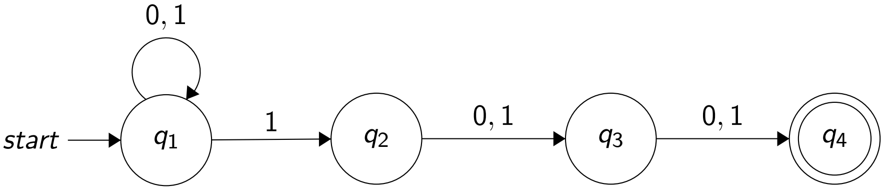

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- Uses the rest of the states to verify that 1 is third from the end
- DFA doing the same thing would have to track the last three bits seen – requires 8 states

Example 2 – OR statement

$L = \{x | x \in \{0, 1\}^*$ and x contains

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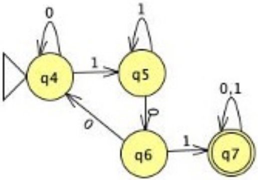

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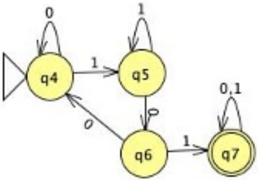
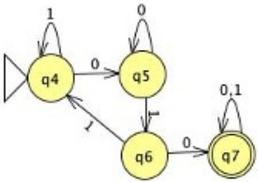


DFA for prop. (1)



DFA for prop. (2)

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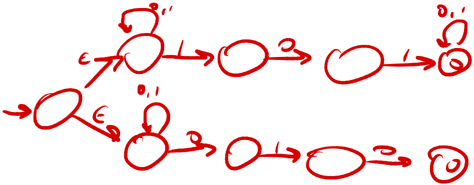

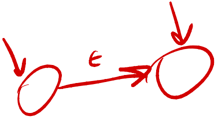

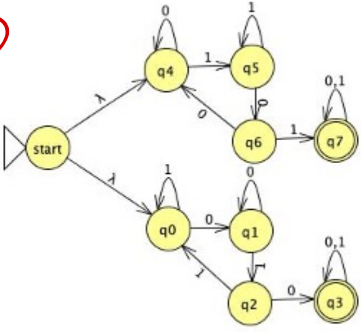

NFA for L

Quiz

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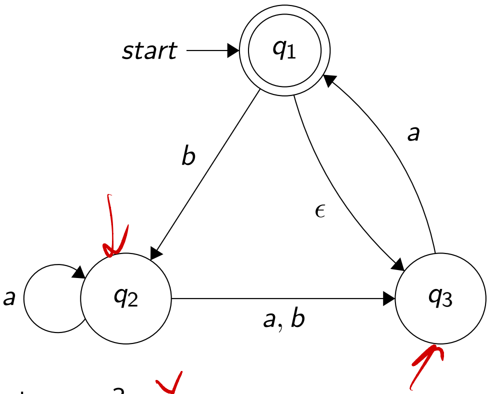

1. Does N accept $w = \epsilon$? Y
2. Does N accept $w = aaa$? Y
3. Does N accept $w = babba$? N
4. Does N accept $w = abaaba$? Y

NFA Summary

- NFAs are much simpler to design
- Only need to verify that inputs have correct form
- Ability to "guess" when some checkable property occurs is very useful

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Question

Are NFAs more powerful than DFAs?

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Nondeterministic Finite Automaton – Formal Definition

Nondeterministic Finite Automaton (NFA)

An NFA is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where:

- Q is a finite set of states
- Σ is a finite input alphabet
- $\delta : Q \times (\Sigma \cup \{\epsilon\}) \to P(Q)$ is the transition function
- $q_0 \in Q$ is the start state
- $F \subseteq Q$ is the set of accept states

Recall:
$P(Q)$ is the power set of Q, i.e., the set of all subsets of Q

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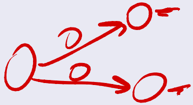

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Changes:

1. Transition function allows empty symbol (ϵ)
2. Output of transition function is a set of states $\in P(Q)$, not a single state in Q

DFAs == NFAs

Theorem

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- Let node of DFA M represent set of "highlighted" nodes
- Define δ to move to new set of highlighted nodes
- Accept states are ones in which at least one node is an accept node
- Can deal with ϵ edges by "placing more fingers" on resulting nodes

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Let N be an NFA recognizing L. Contruct DFA M recognizing L

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2. For $R \in Q'$ and $a \in \Sigma$, let

$R = \{r_1, r_2\}$

$$\delta'(R, a) = \cup_{r \in R} \delta(r, a)$$

Look at transitions from all states in set R and map to set that gives results of all these transitions

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3. $q_0' = \{q_0\}$
4. $F' = \{R \in Q' | R \text{ contains an accept state of } N\}$
   Accept if any state in R is an accept state

Handling ϵ transitions

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Formally:

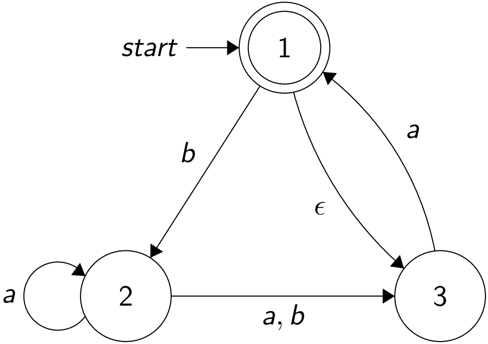
1. Let $E(R) = \{q | q \text{ can be reached from } R \text{ along } \epsilon \text{ arrows}\}$
2. Define extended transition function

$$\delta'(R, a) = \cup_{r \in R} E(\delta(r, a))$$

Map to set of states that can be reached on input a or $a\epsilon$

An Example: NFA $\rightarrow$ DFA

NFA $N = (Q, \Sigma, \delta, q_0, F)$



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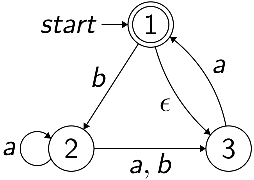

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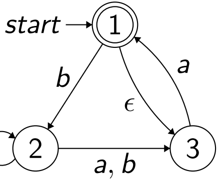

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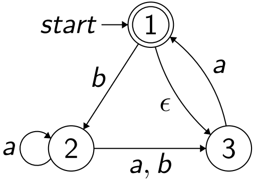

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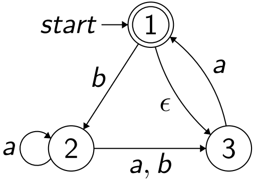

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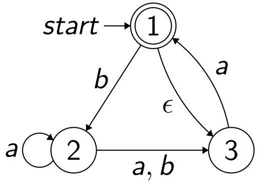

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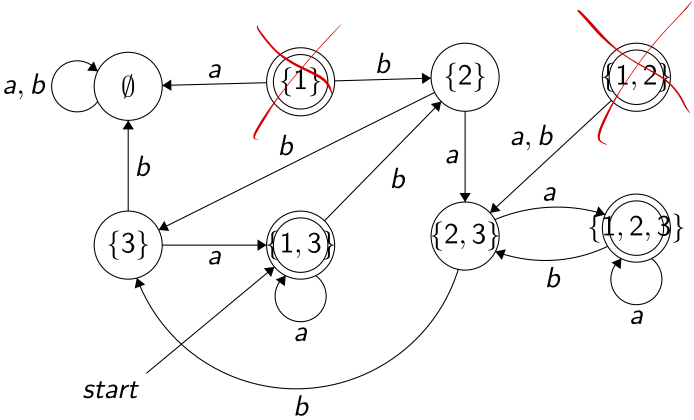

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A Useful Corollary

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We can now use NFAs to argue the properties of regular languages

Closure Under Union

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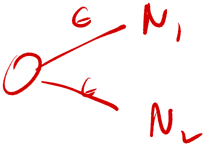
If L_1 and L_2 are both regular languages then $L_1 \cup L_2$ is also regular

$L_1 \cup L_2$ is the language consisting of all strings either in L_1 or L_2

Proof:

N_1 N_2



Closure Under Concatenation

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$L_1 \circ L_2 = \{xy | x \in L_1 \text{ and } y \in L_2\}$

Proof:

Closure Under the Star Operation

Closure Under Star Operation

If L is a regular languages then L^* is also regular

$L^* = \{0 \text{ or more strings from } L\}$

Proof: