Foundations of Computing

Lecture 23

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April 15, 2025

Remaining Class Schedule

- We have only 2 weeks left of lectures!
- HW7 is due tomorrow
- HW8 is out, due next Wednesday
- Thursday, April 24 will be a review lecture

Final Exam

Final exam will be on Tuesday, May 6, 10:20-12:20.

Outline

Lecture 22 Review

- More $\mathcal{NP}$-complete problems
  - SAT
  - 3SAT
  - CLIQUE
  - VERTEX-COVER
  - NAE-SAT
  - 3-coloring

Vertex Cover Problem

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$$\text{VERTEX-COVER} = \{\langle G, k\rangle \mid G \text{ has a vertex cover of size } \leq k\}$$

Goal: Prove that VC is $\mathcal{NP}$-Complete

1. Show that VC $\in \mathcal{NP}$
2. Show that 3-SAT $\leq_p$ VC

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Goal: Show reduction f from 3-SAT to VC s.t.

- if ϕ is satisfiable, $f(\phi) = \langle G, k \rangle$ s.t. G has VC of size $\leq k$
- if ϕ is not satisfiable, $f(\phi) = \langle G, k \rangle$ s.t. G has no VC of size $\leq k$

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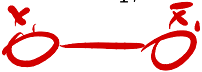
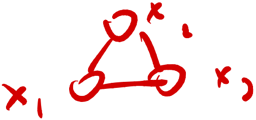
Variable gadget: For every variable x_1, draw pair of nodes



Clause gadget: For every (3-term) clause draw a triangle



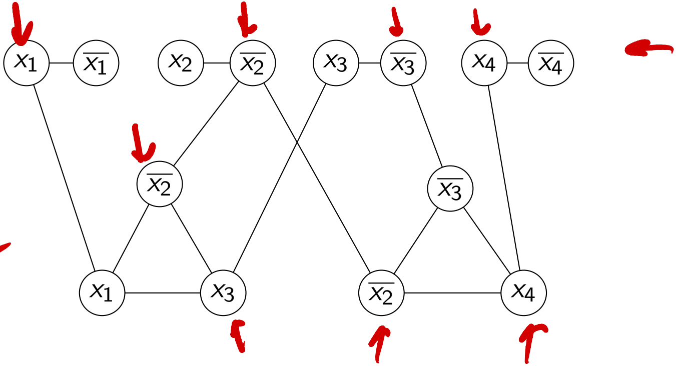
Observations:

- For each variable need 1 node in cover
- For each triangle need at least 2 nodes
- Need to connect variables to clauses

3-SAT $\leq_p$ VC Example

$$\phi = (x_1 \vee \overline{x_2} \vee x_3) \wedge (\overline{x_2} \vee \overline{x_3} \vee x_4)$$

1 0 0 1



V+2c

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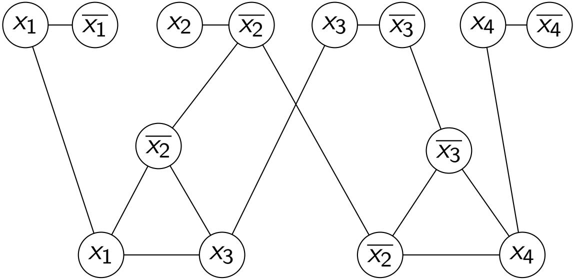

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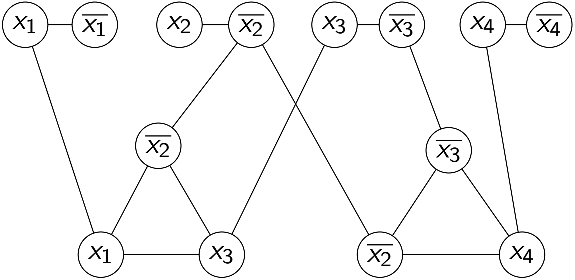

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2. No satisfying assignment implies smallest cover needs $|C| > 2c + v$

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$\Rightarrow G, k \in VC$

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- This edge needs to be covered by the variable node.
- Variable nodes in C must cover at least one edge to each triangle implying a satisfying assignment.

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2. $\mathcal{NP}$-Intermediate Languages
3. co-$\mathcal{NP}$

Ladner's Theorem

- Recall that we know that $\mathcal{P} \subseteq \mathcal{NP}$

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Comment: All languages useful for crypto are such $\mathcal{NP}$-intermediate languages

Proof

A Useful Language

$$SAT_H = \{\phi 0 1^{n^{H(n)}} \mid \phi \in SAT, n = |\phi|\}$$

Proof

$n+1+n^n$

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$SAT \leq_p SAT_H$

$\phi \rightarrow \phi 0 1^{n^c}$

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A Useful Language

$$SAT_H = \{\phi 01^{n^{H(n)}} \mid \phi \in SAT, n = |\phi|\}$$

1. If $H(n) = n$, then $SAT_H \in \mathcal{P}$
2. If $H(n) \leq c$, then SAT_H is $\mathcal{NP}$-Complete
3. We will define H to be in between these two cases

Defining $H(n)$

Let $M_1, M_2, \ldots$ be an enumeration of all TM's (can do this since TM's are countable)

$H(n)$

- Smallest $i \leq \log \log n$ s.t. for all $x \in \{0,1\}^*$, with $|x| \leq \log n$

$$M_i(x) \text{ halts in } i|x|^i \text{ steps and accepts iff } x \in SAT_H$$

- If no such M_i exists, $H(n) = \log \log n$

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2. Claim: $SAT_H \in \mathcal{P}$ iff $H(n) < c$ for all n
   ($\Rightarrow$) By definition of $\mathcal{P}$, there is machine M_k that decides SAT_H in kn^k steps so $H(n) = k$

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0 0 1 1 1 1 2 2 2

0, 1, 2, 2, ...

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  - So, $H(n) \neq i$ for all $n > 2^{|x|}$. Contradiction!

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   - Hence $|\phi| << n$, so have reduced solving long formula to solving a much shorter one.
   - Repeat this enough times to make $|\phi| = O(1)$ and solve.

Takeaway

If $\mathcal{P} \neq \mathcal{NP}$, then $\mathcal{NP}$-intermediate languages exist!

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Observations:

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- In fact, $\mathcal{P} \subseteq (\mathcal{NP} \cap$ co-$\mathcal{NP})$

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co-$\mathcal{NP}$

$L \in$ co-$\mathcal{NP}$ if there exists poly-time DTM V s.t. for $x \in L$ for all w, $V(x, w) = 0$

Question:

Can you prove that $x \in L$, when $L \in$co-$\mathcal{NP}$?

Proving that $x \in L$ for $L \in$co-$\mathcal{NP}$

The Problem

Suppose, I am given an input formula ϕ and I want to prove that ϕ is not satisfiable.

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Complexity Zoo

The complexity zoo (`https://complexityzoo.net/Complexity_Zoo`) now has 550 complexity classes.