Foundations of Computing

Lecture 13

Arkady Yerukhimovich

March 2, 2023

Outline

Lecture 12 Review

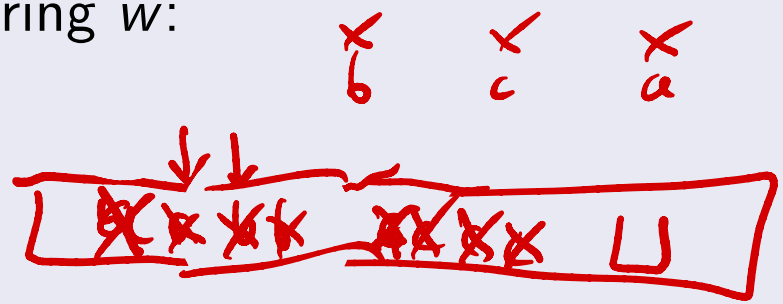
- Turing Machines
  - Definition
  - Examples
- Church-Turing Thesis
  Informally: Anything that can be computed can be computed by a Turing Machine.

Outline

Example 1: $L = \{a^i b^j c^k \mid i \times j = k \text{ and } i, j, k \geq 1\}$

Machine M deciding L

On input string w:



for i = 0 ; i < # a's

→ for j = 0 ; j < # b's

remove a c

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3. Cross off an a and scan to the right until you find a b. Zig zag between b's and c's crossing off one of each until all b's are gone.

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5. If all a's are crossed off, check if all c's are crossed off. Accept if yes, reject if no.

Example 2 – Build a TM deciding L Below

$$L = \{\#x_1\#x_2\#\cdots\#x_\ell \mid \text{ each } x_i \in \{0,1\}^* \text{ and } x_i \neq x_j \text{ for all } i \neq j\}$$

$L = \emptyset$

$\#x_1$

$\#x_1 \#x_2$ s.t. $x_1 \neq x_2$

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1. Look at first symbol, If $\sqcup$, accept. If $\#$ goto step 2. Else, reject

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2. Place mark on top of first # and scan to next # and mark it. If no second # found, accept.

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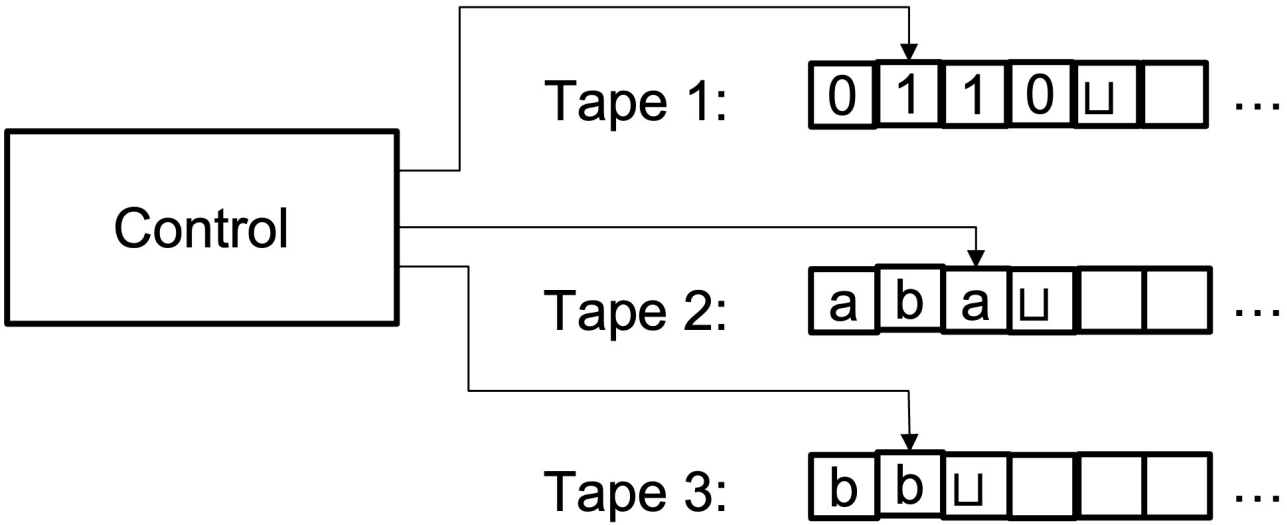
5. Goto step 3

Outline

Multi-Tape Turing Machines



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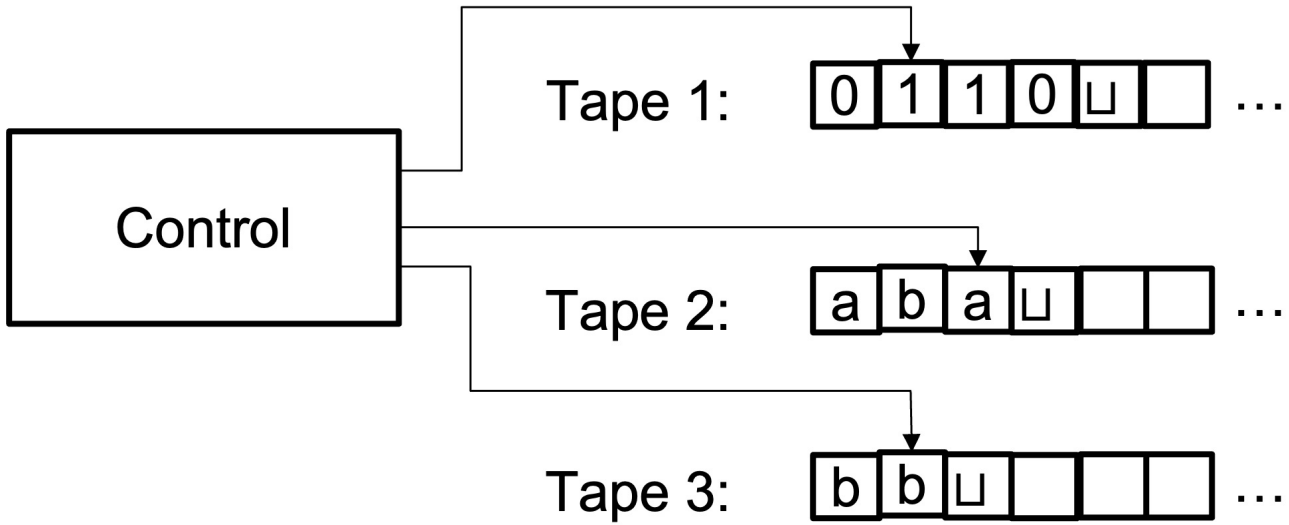

In each step:

- M can read each tape
- M can write to each tape
- M can move each tape head Left or Right

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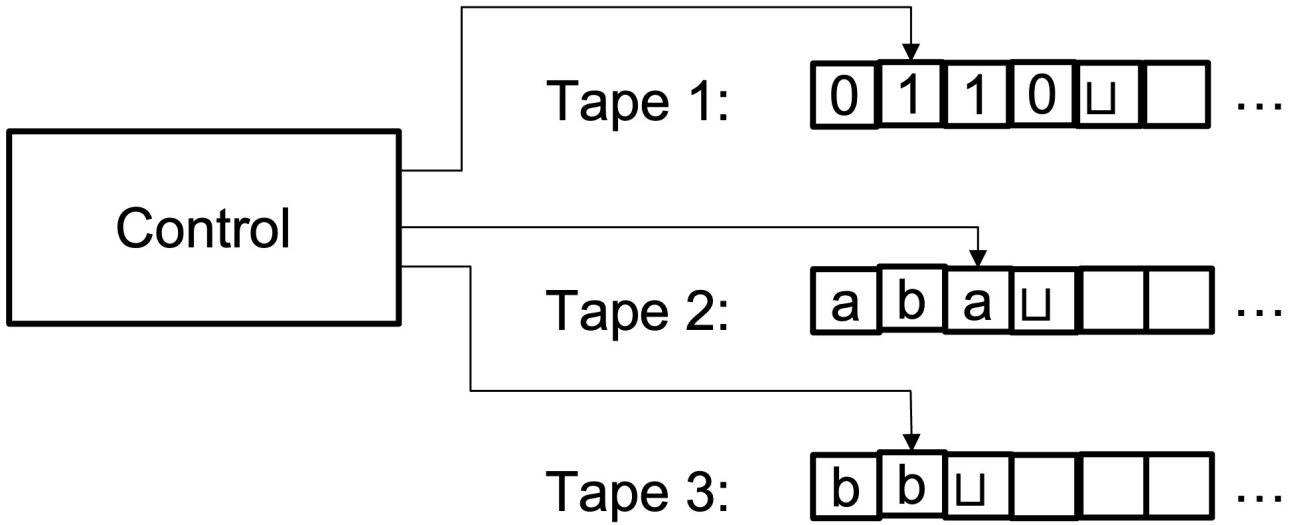

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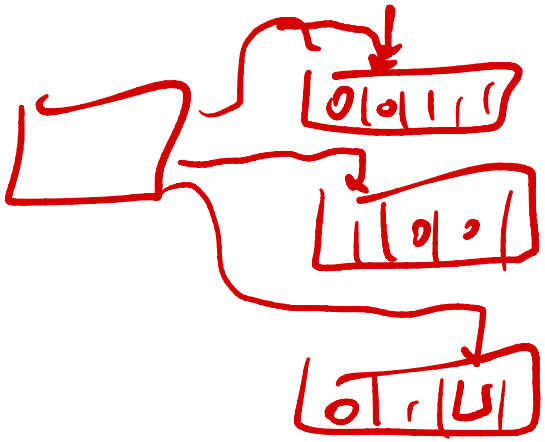
Formally, for k tapes

$$\delta : Q \times \Gamma^k \to Q \times \Gamma^k \times \{L, R\}^k$$

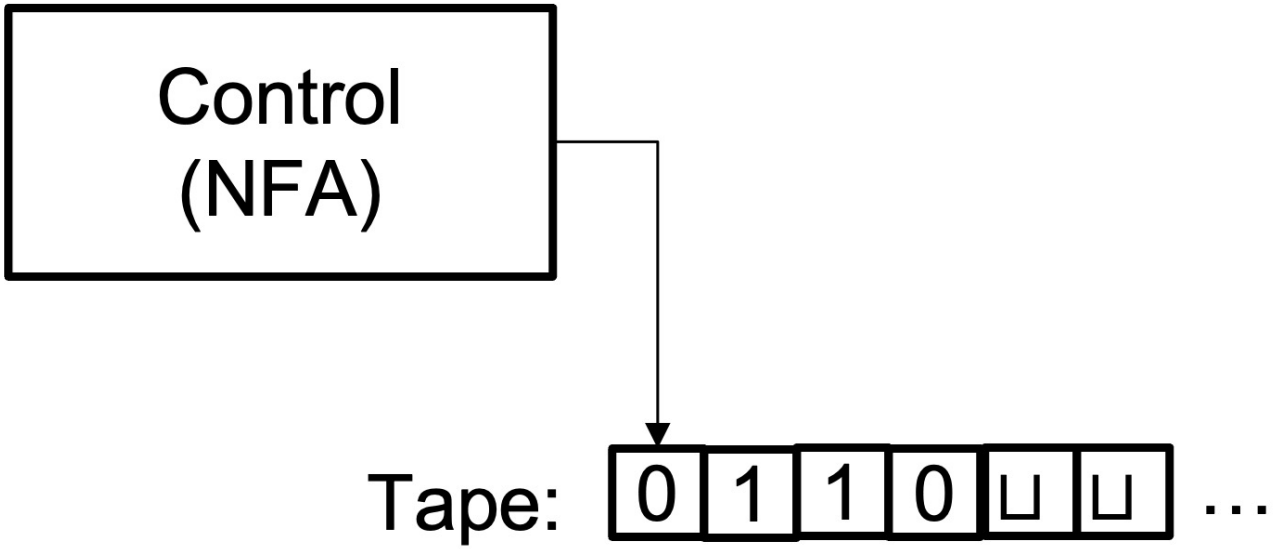
Multi-Tape Turing Machines

Theorem

Every multi-tape TM has an equivalent single-tape TM



Nondeterministic Turing Machines



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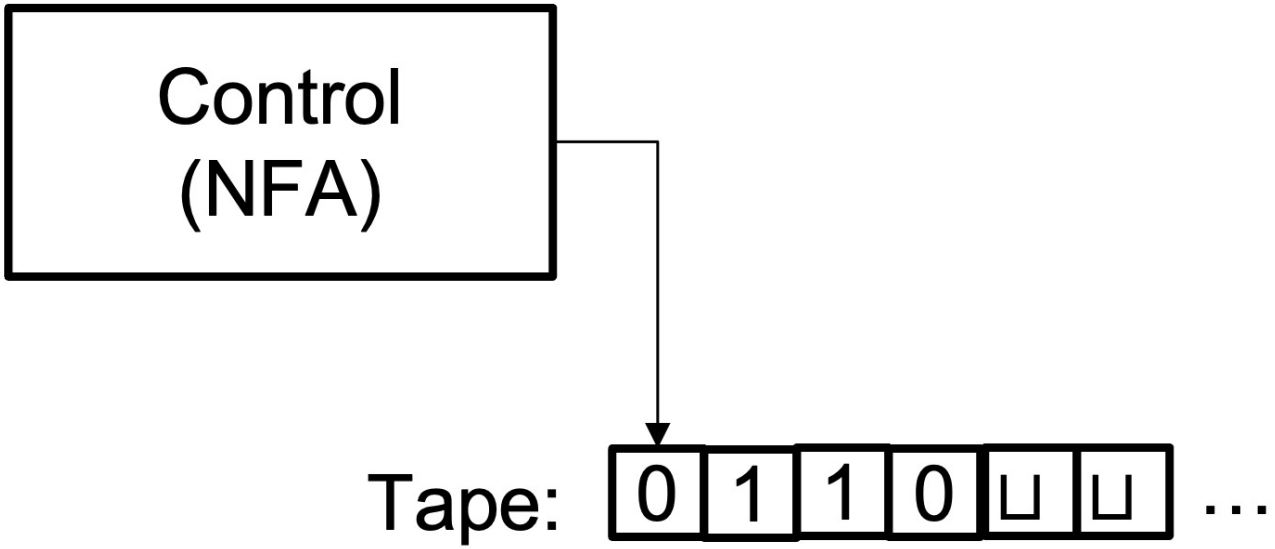
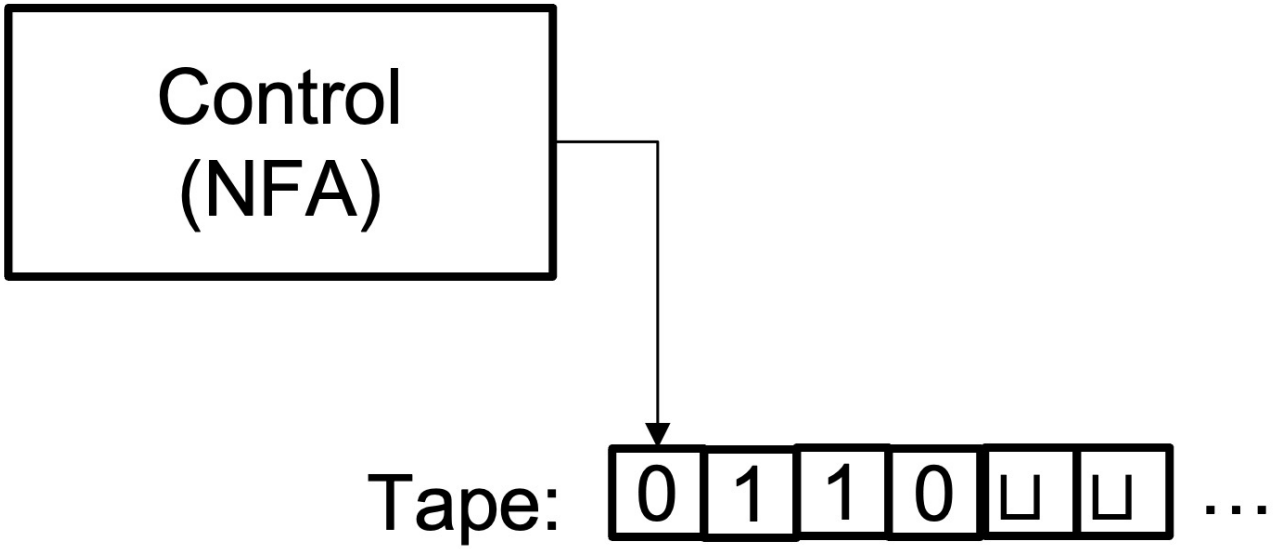


Formally,

$$\delta : Q \times \Gamma \to \mathcal{P}(Q \times \Gamma \times \{L, R\})$$

Nondeterministic Turing Machines



Formally,

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Intuition:

- The control unit is non-deterministic - many transitions possible on each input
- Execution corresponds to a tree of possible executions
- Accept if any of possible execution leads to accept

Nondeterministic Turing Machine

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Every nondeterministic TM has an equivalent deterministic TM.

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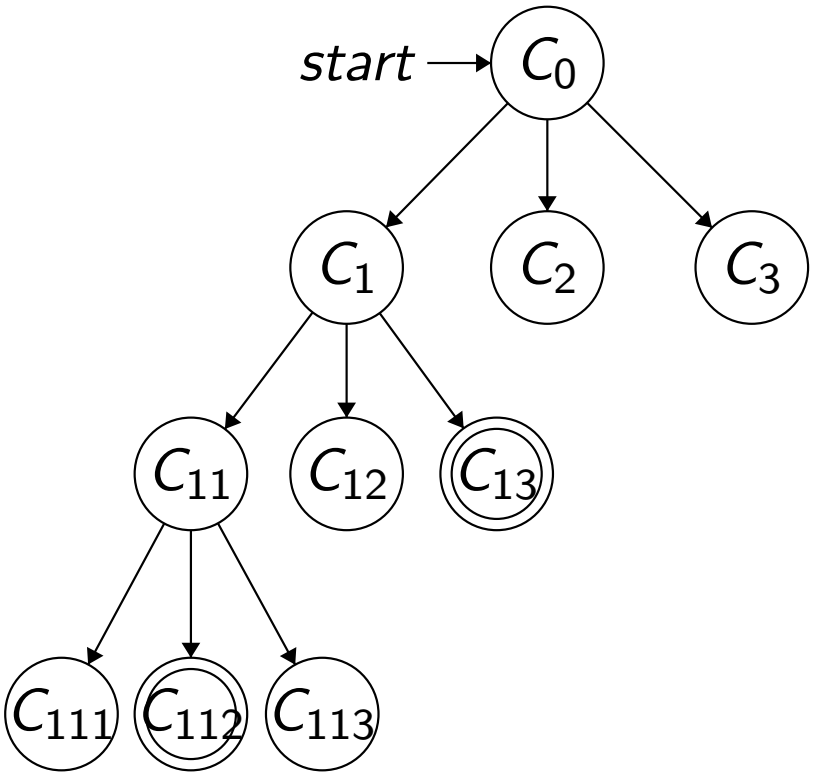

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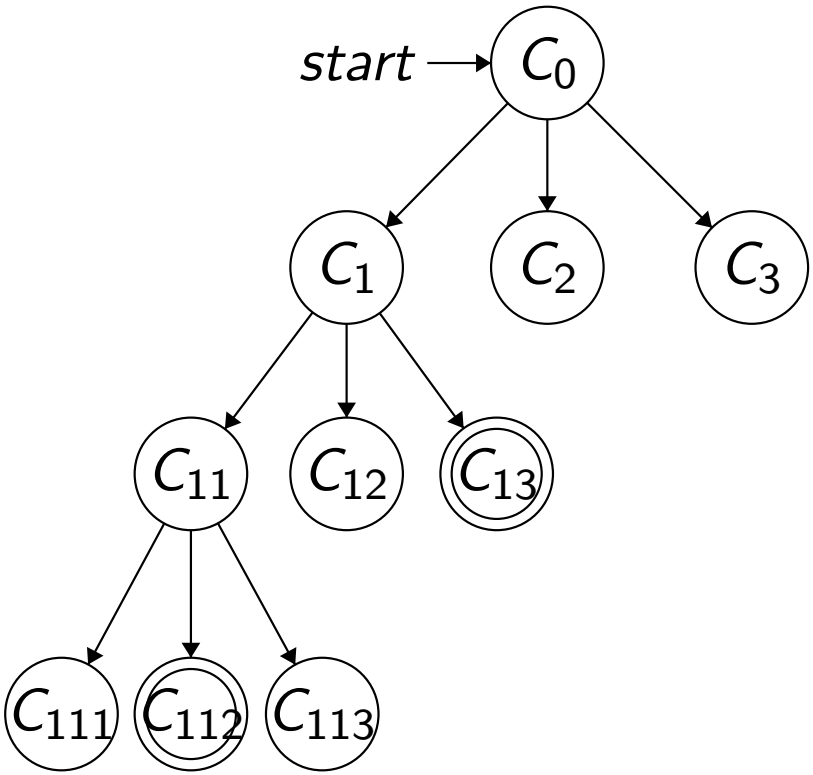

- Recall that an execution of a DTM is a sequence of configurations

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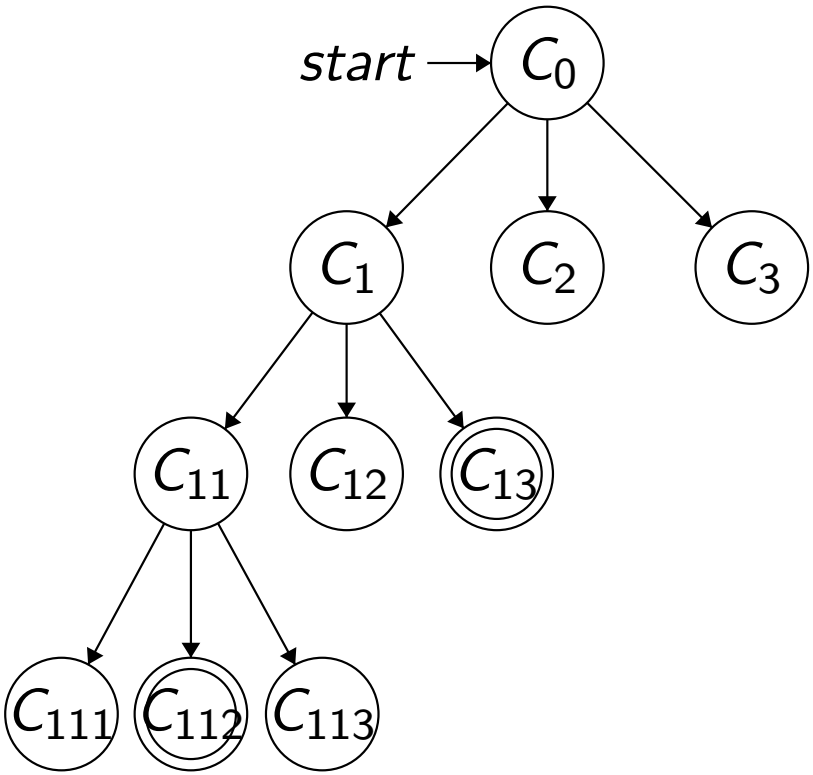

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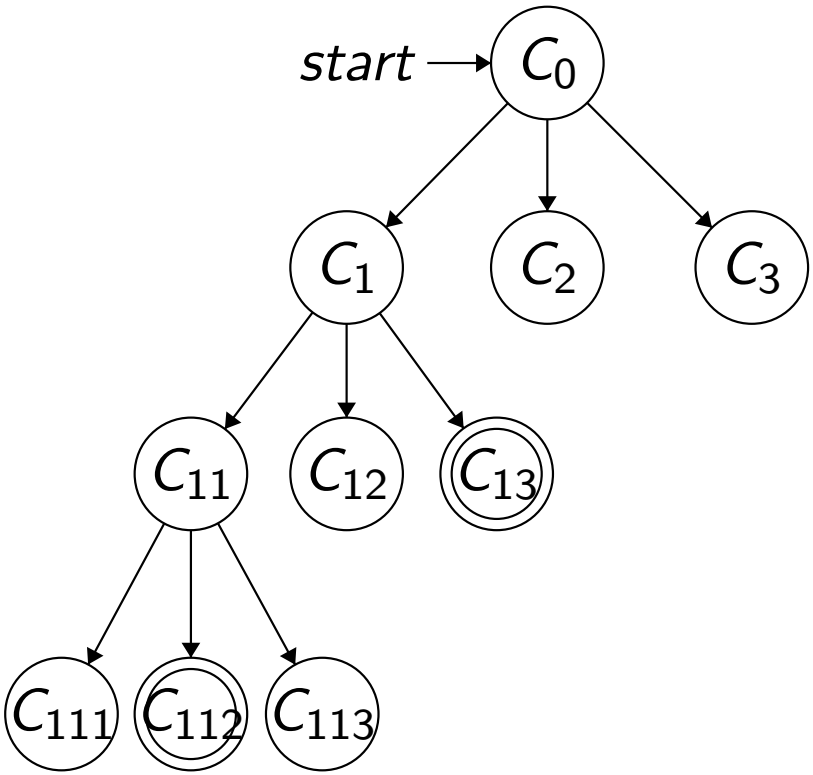

- Recall that an execution of a DTM is a sequence of configurations
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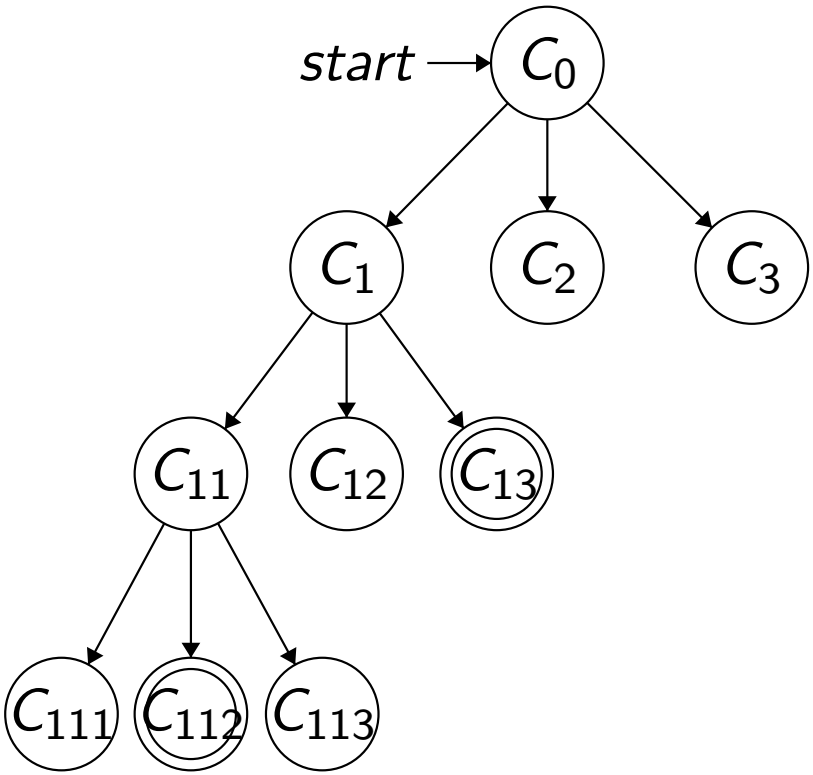
- If any node in the tree is an accept node, the NTM accepts

Nondeterministic Turing Machine

Theorem

Every nondeterministic TM has an equivalent deterministic TM.



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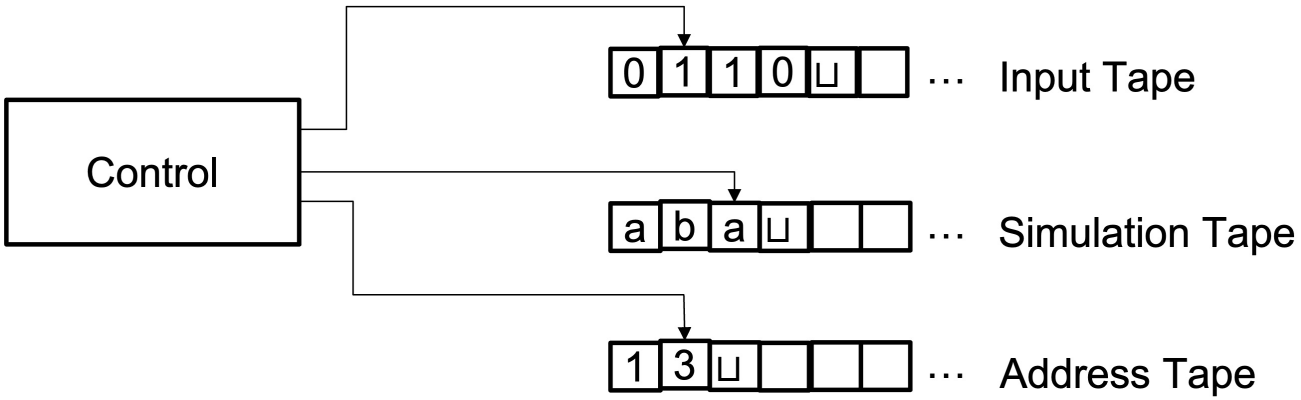
- Execution of an NTM is a tree of configurations (branches correspond to non-deterministic choices)
- If any node in the tree is an accept node, the NTM accepts
- To simulate an NTM by a DTM, need to try all configurations in the tree to see if we find an accepting one

Nondeterministic Turing Machines



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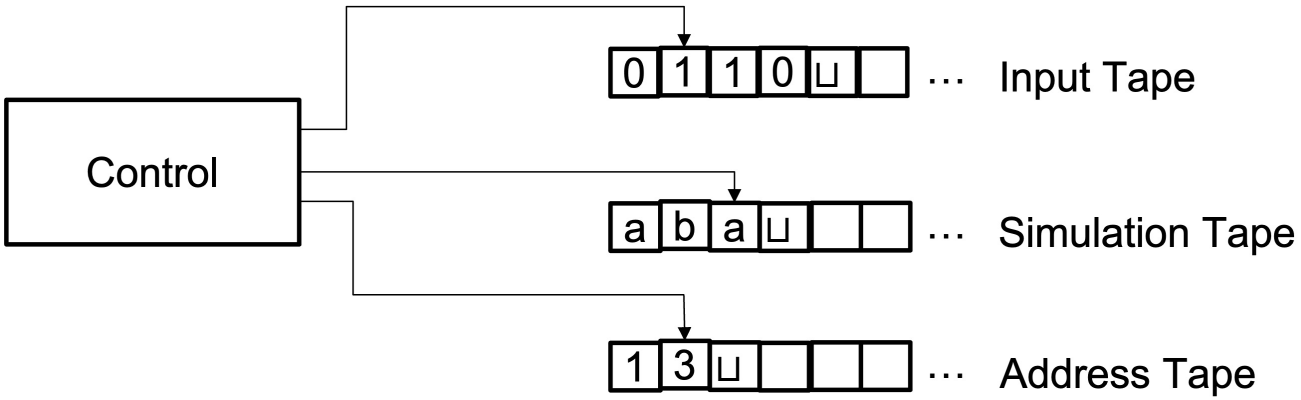

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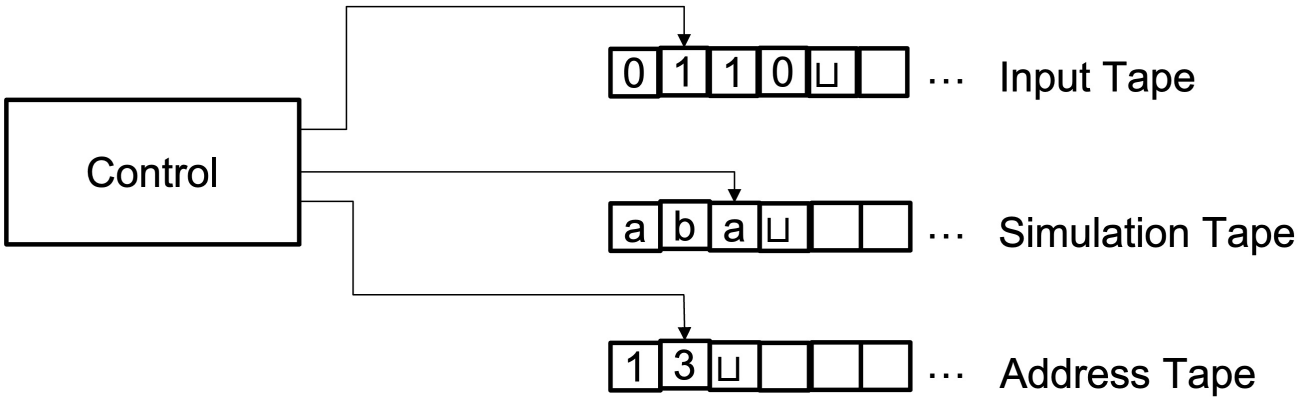
Nondeterministic Turing Machines



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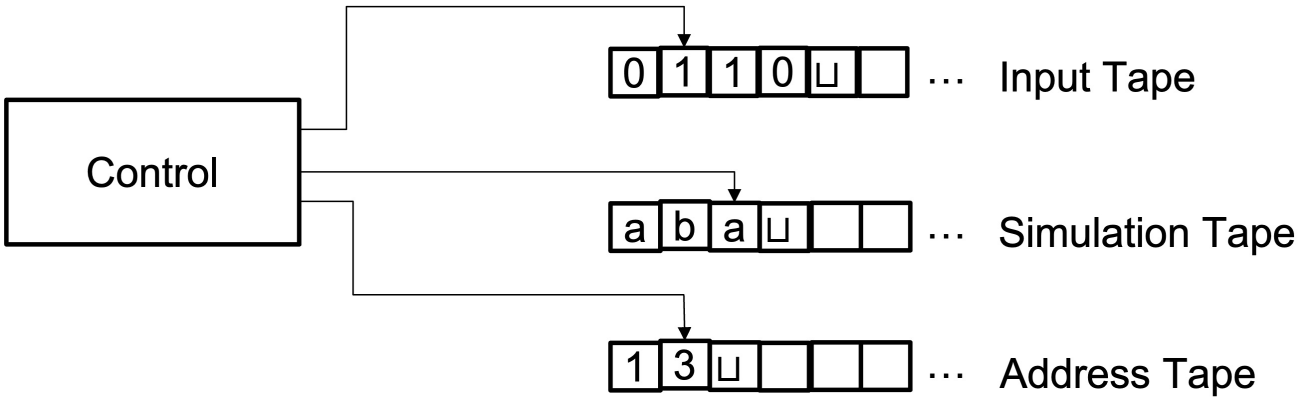
Nondeterministic Turing Machines



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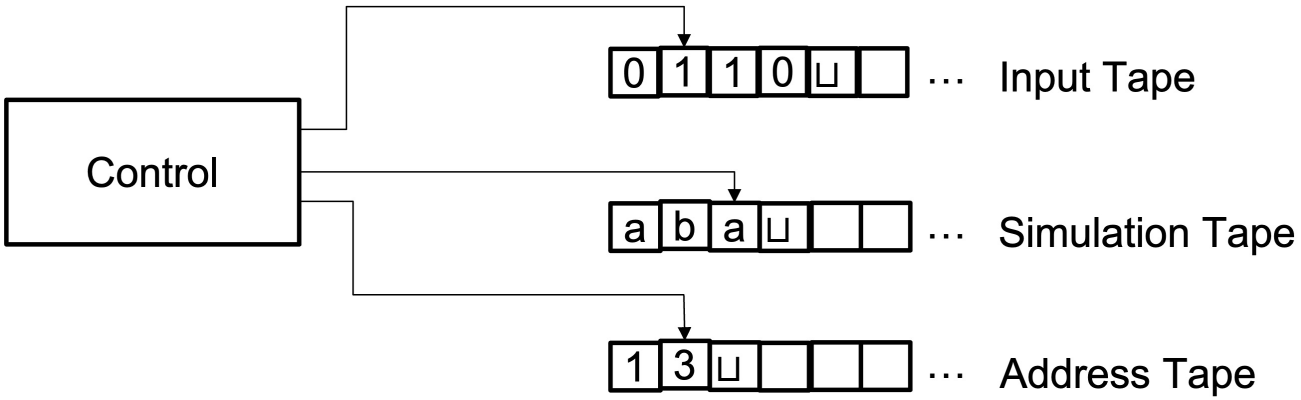

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1. Input tape – stores the input and doesn't change
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3. Address tape – use to store which nondeterministic branch you are on

Simulating an NTM N

Nondeterministic Turing Machines

Simulating an NTM N

1. Start with input w on tape 1, and tapes 2,3 empty

Nondeterministic Turing Machines

Simulating an NTM N

1. Start with input w on tape 1, and tapes 2,3 empty
2. Copy w to tape 2

Nondeterministic Turing Machines

Simulating an NTM N

1. Start with input w on tape 1, and tapes 2,3 empty
2. Copy w to tape 2
3. Use tape 2 to simulate a run of N. Whenever it needs to make a non-deterministic choice, see next symbol on tape 3 for which branch to take. If no symbols left, go to step 4

Nondeterministic Turing Machines

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Nondeterministic Turing Machines

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Important

Must traverse NTM tree in breadth-first, not depth-first order

Nondeterministic Turing Machines

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Important

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- Depth-first traversal may get stuck in an infinite loop, and not detect terminating branch