Foundations of Computing

Lecture 14

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Outline

Lecture 13 Review

- More Turing Machines
- Turing Machine Variants
  - Multi-tape Turing Machines
  - Non-deterministic Turing Machines

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Important TM Notation / Observations

- TM always takes a string as input
  - Sometimes we want to talk about a TM taking another type of input (e.g., a graph, a FA, a TM)
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- Can give a machine as an input to another machine
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  - A TM that accepts any TM and runs it is called a *universal TM*

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3. Algorithm specification
   - Give algorithm in pseudocode
   - Don't explicitly spell out what happens on the tape

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- Are there some problems that inherently do not have any algorithmic solution?

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Observation

Every Decidable language is also Turing-recognizable, but the reverse direction may not be true

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A Second Question

What about Turing-unrecognizable languages?

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- So, can give a description of a machine M to another machine M'
- Today, we will talk about TM's that run another machine M'

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2. If simulation ends in an accept, then accept. If it ends in a non-accepting state, then reject

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2. Run TM from previous slide on input $\langle C, w\rangle$
3. Output what this TM outputs

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Algorithm to decide L_{REX}:
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1. Convert R to DFA A

2. Run Algorithm to decide $L(A)$

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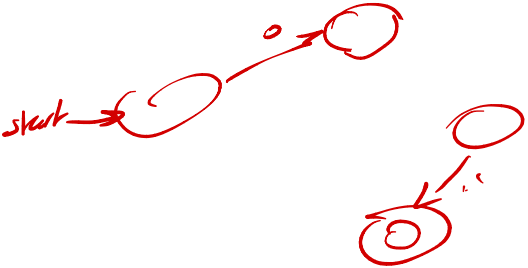
$$L_{\emptyset-DFA} = \{\langle A \rangle \mid A \text{ is a DFA and } L(A) = \emptyset\}$$

A :



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