Foundations of Computing

Lecture 25

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Outline

1. Lecture 24 Review
2. A New Goal for Proofs
3. Defining Knowledge
4. Examples of Zero-Knowledge Proofs
5. Zero-Knowledge on the Blockchain

Lecture 24 Review

- Interactive Proofs
- Proof for Graph Non-Isomorphism
- Polynomial Identity Testing

Outline

Reviewing the Definition of $\mathcal{IP}$

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A New Property

We say that a proof is *zero-knowledge* if the verifier learns nothing (other than the truth of the statement) from seeing the proof.

An Example – Where's Waldo

An Example

A Second Example – Puppy and Panda

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Defining Knowledge

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Answer: A poly-time TM M "knows" x, if it can output x after an efficient computation.

Question

What does it mean for a machine to learn nothing from a proof?

Answer: Whatever it can (efficiently) compute after seeing the proof, it could have efficiently computed before seeing the proof.

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- IMPORTANT: $VIEW_V^*$ and $S(x)$ are both distributions, not values. So, equality is of distributions

Outline

Where's Waldo

Puppy and Panda

accept if $b = b'$

Graph Isomorphism

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Prover's goal: Prove that he knows permutation π s.t. $\pi(G_0) = G_1$

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1. Completeness: If $\pi(G_0) = G_1$, then π' correctly maps $G_{b'}$ to H
2. Soundness: Suppose G_0 is not isomorphic to G_1, so there is no such π. Then, if $b \neq b'$, there is no permutation that P can give that V will accept

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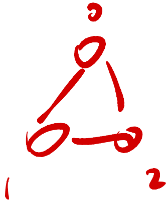
- So, $b = b'$ with probability $1/2$. Thus, S expected to stop after two rounds
- When S stops, he produces a perfect simulation

Graph 3-Coloring

X = G



V

P(w)

1. Permute the colors

2. Commit to the colors

commited colors

pick a random edge e

e

Open this edge

Sim

1. Color all nodes red except 1 node is yellow

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ZK Proofs enable privacy-preserving transactions on a public Blockchain!