# Foundations of Computing

## Lecture 6

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February 2, 2023

# Outline

1. Lecture 5 Review
2. A Nonregular Language
3. The Pumping Lemma for Regular Languages
4. Using the Pumping Lemma

# Lecture 5 Review

- Regular expressions
- Equivalence of regular expressions and NFAs/DFAs

# Quiz Solutions

For each of the following languages over $\Sigma = \{a, b\}$, give two strings that are in the language and two strings not in the language.

1. $a^* \cup b^*$

   $\in L$: a , b

   $\notin L$: ab

2. $(aa \cup bb)^*$

   $\in L$: aa , bb , aabb

   $\notin L$: aba , abb

3. $\Sigma^* a \Sigma^* b \Sigma^* a \Sigma^*$

   $\in L$: aba , aa ab aa

   $\notin L$: aaa , bbb

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# What We Know So Far

The following four things are equivalent:

1. Regular languages
2. Languages recognized by a DFA
3. Languages recognized by an NFA
4. Languages described by a regular expression

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$L = \{ w \mid w = \text{all strings on } 0,1 \}$

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**Are all languages regular?**

Today we will see that there are languages that are not regular

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## Important

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This means that:

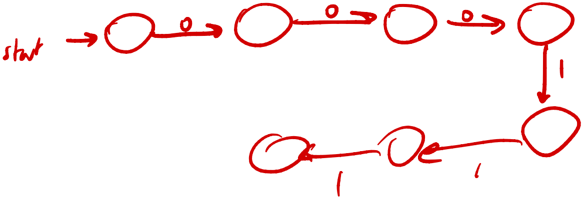
- The number of states is fixed independently of the input size
- An automaton must be able to process strings $w$ s.t. $|w| > |Q|$
- Thus, a finite automaton cannot store its whole input

# A Nonregular Language

Consider the following language:

$$L = \{0^n 1^n | n \geq 0\}$$

Let's try to build a DFA for $L$:



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**The Problem**

We need to count the number of 0s, but this is unbounded so can't have a state for each value

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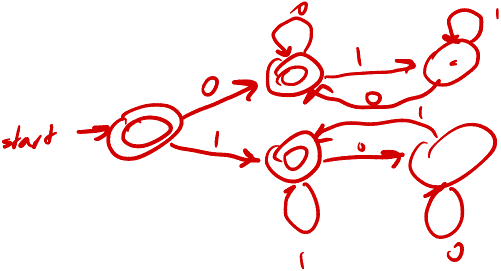
Consider the following language:

$L = \{w | w \text{ has an equal number of occurrences of 01 and 10 as substrings}\}$



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2. Pick a string $w \in L$
3. Show that if $M(w) = 1$ then there exists a string $w' \notin L$ s.t. $M(w') = 1$
4. Conclude that $L$ is not regular since any $M$ that accepts all strings in $L$ must also accept strings not in $L$

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If $L$ is a regular language, then there exists an integer $p$ (the pumping length) where any string $w \in L$ such that $|w| \geq p$ can be divided into three pieces $w = xyz$ satisfying:

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Next steps:

1. Prove the pumping lemma
2. Show how to use the pumping lemma to prove languages nonregular

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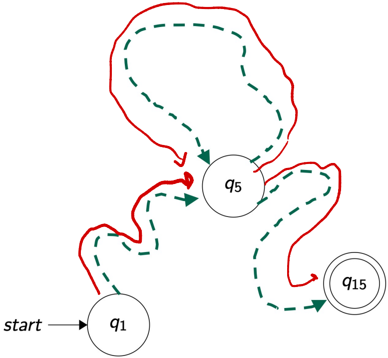

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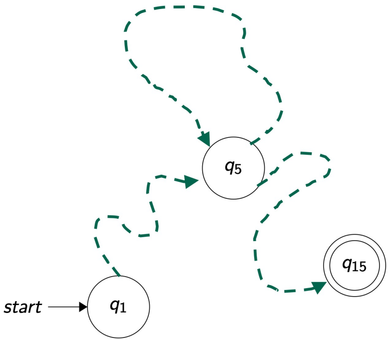


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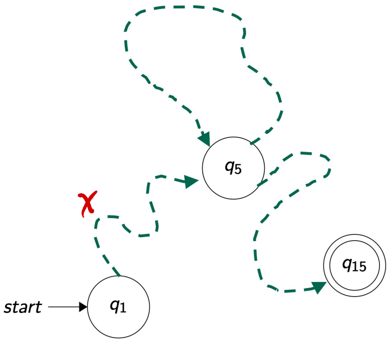


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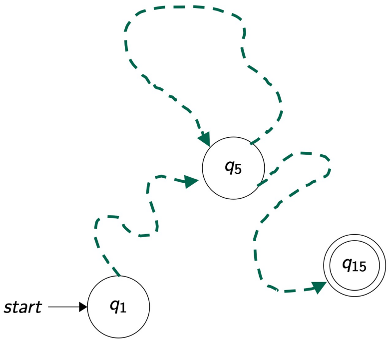


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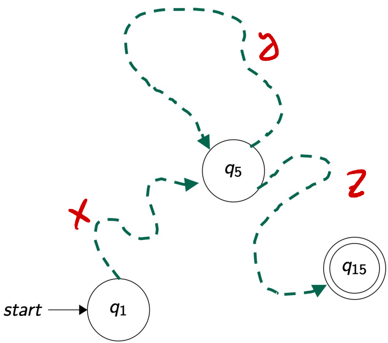


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3. $|xy| \leq p$
   Proof: if $q_5$ is the first repetition in $M(w)$, then this repetition must occur in the first $p + 1$ states, hence $|xy| \leq p$

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5. Contradiction!!!

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2. Observe that $L_1 = L \cap 0^* 1^*$
3. Since regular languages are closed under $\cap$, if $L$ is regular then $L_1$ must be regular
4. Since we know $L_1$ is nonregular, this means that $L$ must be nonregular

# Exercise

Prove that the following language is nonregular:

$$L = \{0^i 1^j 2^i 3^j | i, j > 0\}$$

# What's Next?

- We will get plenty of practice with proving languages nonregular
- We will add (a small amount of) memory to our machines to recognize a richer class of languages