Foundations of Computing

Lecture 17

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Outline

Lecture 16 Review

- Proofs by reduction
- Undecidable languages
  - $HALT_{TM}$
  - $REGULAR_{TM}$

Exercise

$(M, w) \overset{?}{\in} A_{TM} \longrightarrow M' \overset{?}{\in} ES_{TM}$ | $M' \in ES_{TM} \iff (M, w) \in A_{TM}$

$EMPTY-STRING_{TM} = \{\langle M \rangle \mid M \text{ is a TM and } M(\epsilon) = 1\}$

$A_{TM} = \{ (\langle M \rangle, w) \mid M(w) = 1 \}$

$A_{TM} \le ES_{TM}$

D_{ES} - decider ES_{TM}

$R(\langle M \rangle, w)$

1. M': on input ϵ, Run $M(w)$ output what it outputs

   on any other input, output 0

2. Run $D_{ES}(M')$ output what it outputs

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1. To show that a problem is decidable: Give an algorithm that always terminates and outputs the answer
2. To show that a problem is undecidable: Give an algorithm (a reduction) that shows that this problem can be used to solve an undecidable problems

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D

$R\left(\overline{M, w}\right)$

1. Run $D(M, w)$, output the opposite of what D outputs

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Takeaway: General reductions do not work to prove Turing-unrecognizability

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Takeaway: General reductions do not work to prove Turing-unrecognizability

Solution

We need to restrict what our reductions can do.

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Mapping Reductions

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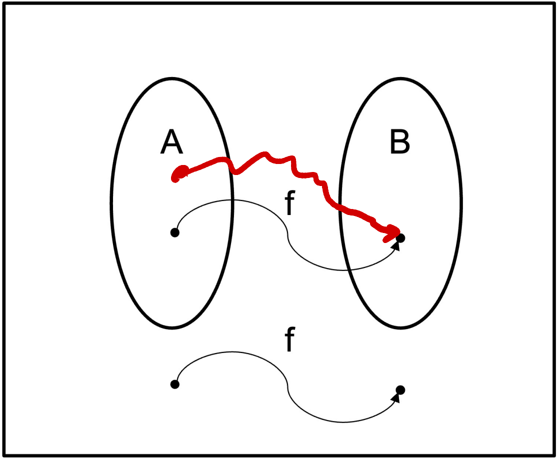
  - Karp reductions (when only considering poly-time reductions)
- Works by mapping input in A to input in B and vice-versa

Mapping Reductions



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- If A is not Turing-recognizable than B is not Turing-recognizable

Observation:

Mapping reductions work for both decidability and Turing-recognizability.

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- The reduction may make multiple calls to decider for B and may not directly use the result.
- For example, in the proof that checking whether $L(M) = \emptyset$ is undecidable (Exercise 1 from lab), we flipped the result of the decider.

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    - If A is not decidable, then B is not decidable

    - If B is Turing-recognizable, A is not necessarily Turing-recognizable
    - If A is not Turing-recognizable, cannot say if B is Turing-recognizable

Information in a String

$$A = 010101010101010101010101$$

$$B = 110100100011100010111111$$

Question

Which of these strings contains more information?

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- This captures the "amount of information" in x

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   - So, there exists at least one string that is incompressible
2. In fact, incompressible strings look like random strings
3. But, $K(x)$ is not computable, moreover it is undecidable whether a string is incompressible