Foundations of Computing

Lecture 15

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March 9, 2023

Outline

1. Lecture 14 Review
2. Languages on Machines
3. Preliminaries – Countable and Uncountable Sets
4. Proving L_{TM} Undecidable

Lecture 14 Review

- Decidable and Turing-recognizable languages
- Decidability of regular languages

Outline

Characterizing Computability of Languages

Definition: Decidable languages

A language L is *decidable* or *recursive* if some TM M decides it

- M halts on ALL inputs, accepts those in L and rejects those not in L
- Seems to match informal definition we wanted before

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Observation

Every Decidable language is also Turing-recognizable, but the reverse direction is not true.

Problems About Regular Languages

Last time, we showed the following languages are decidable:

1. $L_{DFA} = \{\langle B, w\rangle \mid B \text{ is a DFA that accepts input string } w\}$
2. $L_{NFA} = \{\langle B, w\rangle \mid B \text{ is a NFA that accepts input string } w\}$
3. $L_{REX} = \{\langle R, w\rangle \mid R \text{ is a reg. exp. that generates the string } w\}$
4. $L_{\emptyset-DFA} = \{\langle A\rangle \mid A \text{ is a DFA and } L(A) = \emptyset\}$
5. $L_{EQ-DFA} = \{\langle A, B\rangle \mid A, B \text{ are DFAs and } L(A) = L(B)\}$

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Try 1:

- Every CFG has an equivalent PDA
- Use a TM to run the PDA (easy to simulate stack using TM's tape)

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- A PDA may have some branches that go on forever – keep pushing and popping things on the stack
- This would mean that on such an input the resulting TM would not halt – i.e., not be a decider

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Corollary

Every CFL is decidable

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   - Mark any variable A where G has a rule $A \to U_1 U_2 \cdots U_k$ and each symbol $U_1, U_2, \ldots, U_k$ has already been marked

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3. If starts symbol is not marked, accept. Otherwise, reject

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We will prove after spring break that this is undecidable.

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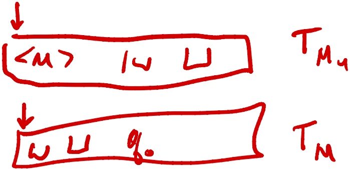
$M_u \;(\langle M \rangle)$



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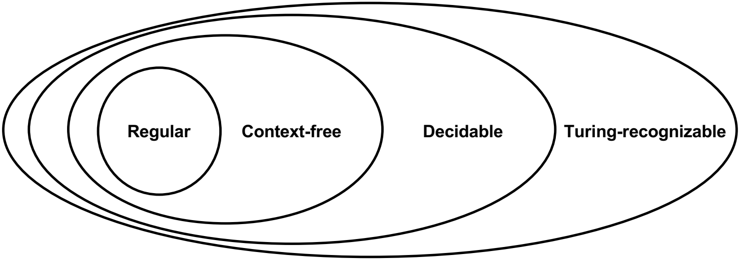
An Undecidable Problem

- We will prove that L_{TM} is undecidable

Relationships Among Language Classes



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- Example:
  $A = \{0, 1, 2, 3\}$
  $B = \{a, b, c, d\}$
  $f(0) = a, f(1) = b, f(2) = c, f(3) = d$

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- A set A is countable if it is finite or countably infinite
- A set that is not countable is *uncountable*

Example 1: Evens

Evens

The set of even numbers is countable

$\mathbb{N} \to$ Evens

$f(x) = 2x$

$f(1) = 2$

$f(2) = 4$

Example 2: Rationals

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	1	2	3	$\cdots$
1	1/1	1/2	1/3	
2	2/1	2/2	2/3	$\cdots$
3	3/1	3/2	3/3	
4	4/1	4/2	$\vdots$	

Example 3: Strings

Strings

The set of strings in $\{0,1\}^*$ is countable

$\epsilon, 0, 1, 00, 01, 10, 11, \ldots$

Diagonalization

Real Numbers

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- Assume that $\mathcal{R}$ is countable
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n	f(n)
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- We construct a value $x \in \mathcal{R}$ s.t $x \neq f(n)$ for any n
  Idea: For all $i \in \mathcal{N}$, make $x_i \neq f(i)_i$ $x = 0.314...$

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- Contradiction – f is not mapping between $\mathcal{R}$ and $\mathcal{N}$

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- Can similarly show that for any finite alphabet Σ, Σ^* is countable
- But, a TM M can be written as a string $\langle M \rangle \in \Sigma^*$
- Hence, by omitting all strings that are not encodings of valid TMs we get a mapping of TMs to $\mathcal{N}$

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Languages over alphabet Σ

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Proof:

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$B = \{ 0101100\ldots,\ 1100\ldots,\ \}$

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   - Define the characteristic sequence χ_A of language $A \in \mathcal{L}$

$$\begin{array}{ccccccccccc}
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3. Therefore, $\mathcal{L}$ is uncountable

Some Languages are not Turing-recognizable

We have proven:

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2. The set of languages is uncountable

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- We have proven that some languages are not Turing-recognizable
- But, we have not given any examples of such a language

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- Use H to build a TM D that checks whether a TM M accepts its own description, and then does the opposite:
  On Input $\langle M\rangle$, where M is a TM
  1. Run H on input $\langle M, \langle M\rangle\rangle$
  2. Output the opposite of what H outputs

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$$H(\langle M, w\rangle) = \begin{cases} \textit{accept} & \text{if } M \text{ accepts } w \\ \textit{reject} & \text{if } M \text{ does not accept } w \end{cases}$$

- Use H to build a TM D that checks whether a TM M accepts its own description, and then does the opposite:
  On Input $\langle M\rangle$, where M is a TM
  1. Run H on input $\langle M, \langle M\rangle\rangle$
  2. Output the opposite of what H outputs

$$D(\langle M\rangle) = \begin{cases} \textit{accept} & \text{if } M \text{ does not accept } \langle M\rangle \\ \textit{reject} & \text{if } M \text{ accepts } \langle M\rangle \end{cases}$$

L_{TM} is Undecidable

$$L_{TM} = \{\langle M, w\rangle \mid M \text{ is a TM and } M(w) = 1\}$$

Proof: By contradiction

- Assume that L_{TM} is decided by TM H

$$H(\langle M, w\rangle) = \begin{cases} \textit{accept} & \text{if } M \text{ accepts } w \\ \textit{reject} & \text{if } M \text{ does not accept } w \end{cases}$$

- Use H to build a TM D that checks whether a TM M accepts its own description, and then does the opposite:
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- Now consider what happens if we run D on $\langle D\rangle$

$$D(\langle D\rangle) = \begin{cases} \textit{accept} & \text{if } D \text{ does not accept } \langle D\rangle \\ \textit{reject} & \text{if } D \text{ accepts } \langle D\rangle \end{cases}$$

How Is This a Diagonalization?

	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\cdots$	$\langle D \rangle$	$\cdots$
M_1	$\underline{accept}$	$reject$	$accept$		$accept$	
M_2	$reject$	$\underline{reject}$	$reject$	$\ldots$	$accept$	$\ldots$
M_3	$accept$	$accept$	$\underline{accept}$		$reject$	
$\vdots$		$\vdots$		$\ddots$		
D	$reject$	$accept$	$reject$		?	

- We have defined D to do the opposite of what M_i does on input $\langle M_i \rangle$
- But what does D do on input $\langle D \rangle$??

A Turing-unrecognizable Language

$\overline{L_{TM}}$

The language

$$\overline{L_{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM and } M(w) \neq 1\}$$

is not Turing-recognizable