CS 3313
Foundations of Computing:

Lab 7

Decidable vs Undecidable problems

- Algorithm = Turing machine that halts on all inputs (always halts)
  - Decision problem: the answer is “Yes” or “No”
- A problem is undecidable if there is no algorithm (Turing machine that always halts) that solves the problem
  - Problem = language
  - How do we show a problem is undecidable – need to prove the problem is undecidable
- A problem is decidable if there is an algorithm (Turing machine that always halt) to solve the problem
  - How do we show a problem is solvable – provide an algorithm that solves the problem
  - *Key observation: the algorithm can be deterministic or non-deterministic when we are trying to prove it is solvable/decidable*

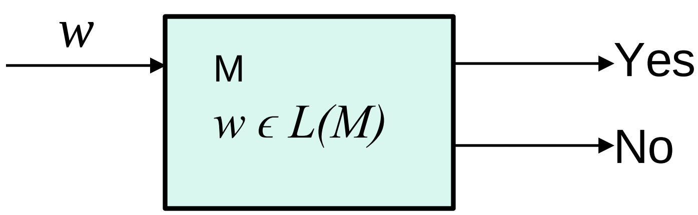
Decidable Problems

- A problem is *decidable* if there is an algorithm to answer it
  - Recall: An "algorithm," formally, is a TM that halts on all inputs, accepted or not
- Otherwise, the problem is *undecidable*.

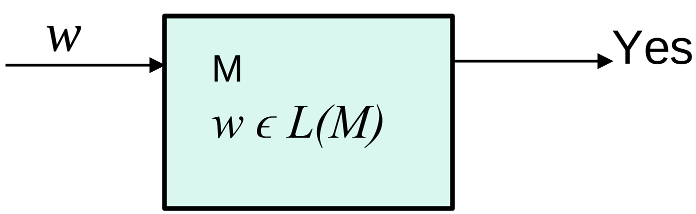
- Language is *Turing-recognizable* if it is accepted by a TM
  - TM halts and accepts if the string is in the language
  - However, TM may not halt if the string is not in the language

Recall Definitions

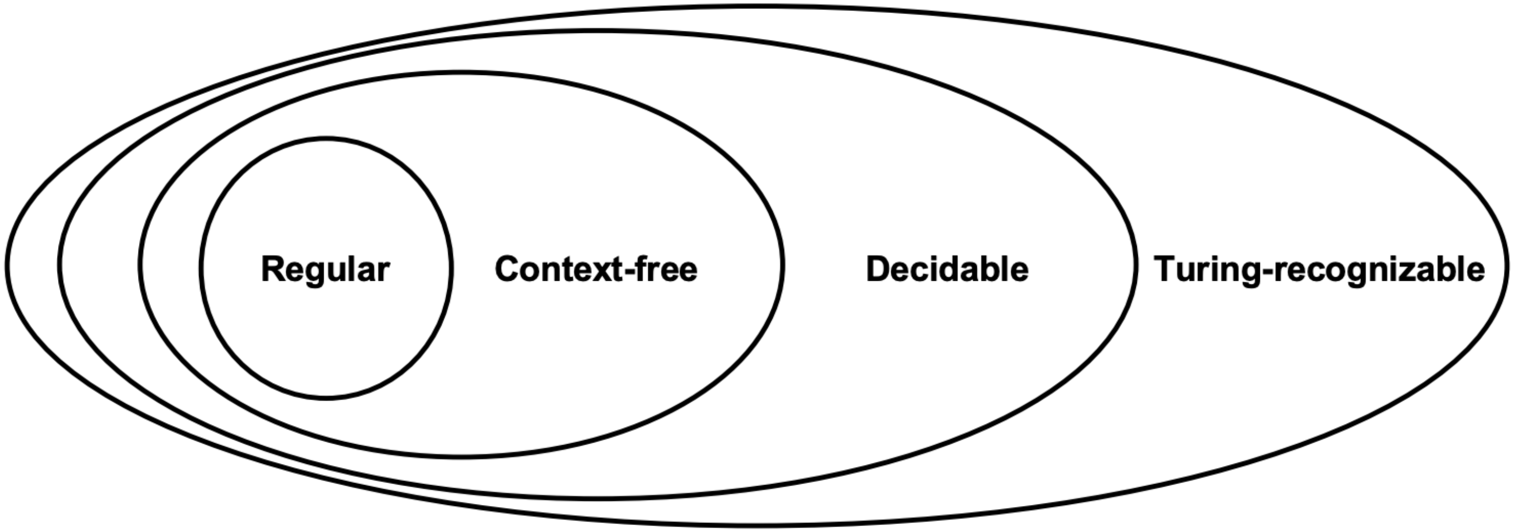
- Decidable Language: A language L is decidable if there is a Turing machine that accepts the language and *halts on all inputs*



- Turing-recognizable Language: if there is a Turing machine that accepts the language by *halting when the input string is in the language*
  - The machine may or may not halt if the string is not in the language



Recall the Relationships Among Language Classes



Recall Proof that A_{TM} is Undecidable

$$A_{TM} = \{\langle M, w\rangle \mid M \text{ is a TM and } M(w) = 1\}$$

- Assume that A_{TM} is decided by TM H

$$H(\langle M, w\rangle) = \begin{cases} \text{accept} & \text{if } M \text{ accepts w} \\ \text{reject} & \text{if } M \text{ does not accept w} \end{cases}$$

- Use H to build a TM D that checks whether a TM M accepts its own description, and then does the opposite:

On Input $\langle M\rangle$, where M is a TM

1. Run H on input $\langle M, \langle M\rangle\rangle$
2. Output the opposite of what H outputs

$$D(\langle M\rangle) = \begin{cases} \text{accept} & \text{if } M \text{ does not accept } \langle M\rangle \\ \text{reject} & \text{if } M \text{ accepts } \langle M\rangle \end{cases}$$

- Now consider what happens if we run D on $\langle D\rangle$

$$D(\langle D\rangle) = \begin{cases} \text{accept} & \text{if } D \text{ does not accept } \langle D\rangle \\ \text{reject} & \text{if } D \text{ accepts } \langle D\rangle \end{cases}$$

Decidability…and Reducibility proof technique

- Reducibility of a problem A to problem B
- Given two problems A and B, problem A is reducible to problem B if an algorithm for solving B can be used to solve problem A
  - Therefore, solving A cannot be harder than solving B
  - *If A is undecidable and A is reducible to B, then B is undecidable*
- Idea: If you had a black box that can solve instances of B, can you solve instances of A using calls to this Black box?
  - The black box is the assumed Algorithm for B
- Crucial step in the proof is the reduction "algorithm"
  - This process should be an "algorithm" – i.e., a TM that always halts

Example: Proof that the halting problem is undecidable

- $HALT_{TM} = \{ <M,w> \mid M \text{ halts on } w \}$
- Given any input and any machine, will the machine terminate or run forever?
- Assume algorithm B for HALT
- Reducibility algorithm R ($HALT_{TM}$ reducible to A_{TM}):
  - Run B(<M,w>), if it rejects then reject – M does not halt on w
  - *Otherwise Run M(w) and output what it outputs*
  - *This algorithm R decides* A_{TM}

Exercise 1: L = { $\langle M \rangle$ | M is a TM and L(M)=Ø }

$$L = \{ \langle M \rangle \mid M \text{ is a TM and } L(M)=\emptyset \}$$

- Given a Turing machine M, does M accept any input?
  - (i.e., does M accept the empty set)

Exercise 2: $L = \{\langle M_1, M_2\rangle \mid L(M_1) \subseteq L(M_2)\}$ is Undecidable

- Given any two Turing machines $M_{1,}$ M_2 is the language accepted by M_1 a subset of language accepted by M_2?
  - Hint: Reduce to Exercise 1