Foundations of Computing

Lecture 19

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April 4, 2023

Outline

Lecture 17 Review

- Review of Reductions
- Types of Reductions – Mapping reductions, Turing reductions
- A brief intro into Kolmogorov complexity

Computability Theory

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Question

Suppose we want to solve a problem in real life, is knowing that it is decidable enough?

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Complexity

The study of decidability under bounded models of computation

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- We write $f = O(n^3)$

Asymptotic Notation – Big-O

Definition

Let $f, g : \mathbb{N} \to \mathbb{R}$, we say that $f(n) = O(g(n))$ if

- There exist positive integers c, n_0 s.t. for all $n \geq n_0$

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- Note that $f(n) = O(n^4)$

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$w = 000111$

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Time Complexity Classes

Let $t : \mathbb{N} \to \mathbb{N}$. Define time complexity class $TIME(t(n))$ as

$$TIME(t(n)) = \{L \mid L \text{ is a language decided by an } O(t(n)) \text{ time TM}\}$$

TIME (n) = {L decidable in O(n) time}

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Counting number of steps on $|w| = n$:

$2^{\log_2(n)} = n$

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$\log_2 n = \frac{\log_a n}{\log_a 2}$

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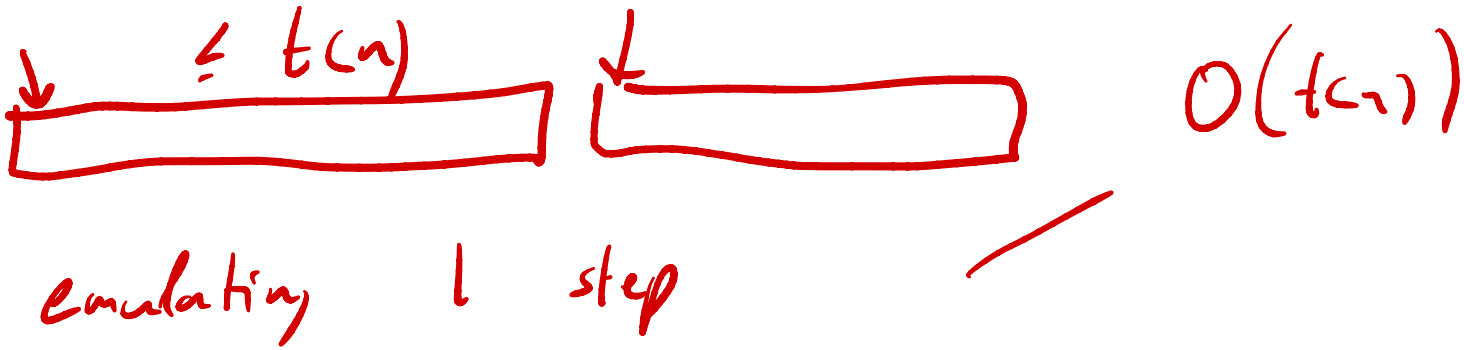
Important

Time complexity depends on the exact model of computation

Dependence on Model of Computation

Theorem

For any function $t(n) \geq n$, every multi-tape TM running in time $t(n)$ has an equivalent 1-tape TM running in time $O(t^2(n))$.



2-tape machine takes $O(t(n))$ steps

$O(t(n) \cdot t(n)) = O(t^2(n))$

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Why polynomial:

- Polynomials grow much slower than exponentials:
  - $f(n) = n^3$: If $n = 1000$, $f(n) = 1,000,000,000$ – large, but not unreasonable for today's PCs
  - $f(n) = 2^n$: If $n = 1000$, $f(n) >$ number of atoms in the universe

Polynomial Time

Efficient Computation

We define computation to be efficient if it runs in time bounded by some polynomial of n

Why polynomial:

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We define computation to be efficient if it runs in time bounded by some polynomial of n

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Outline

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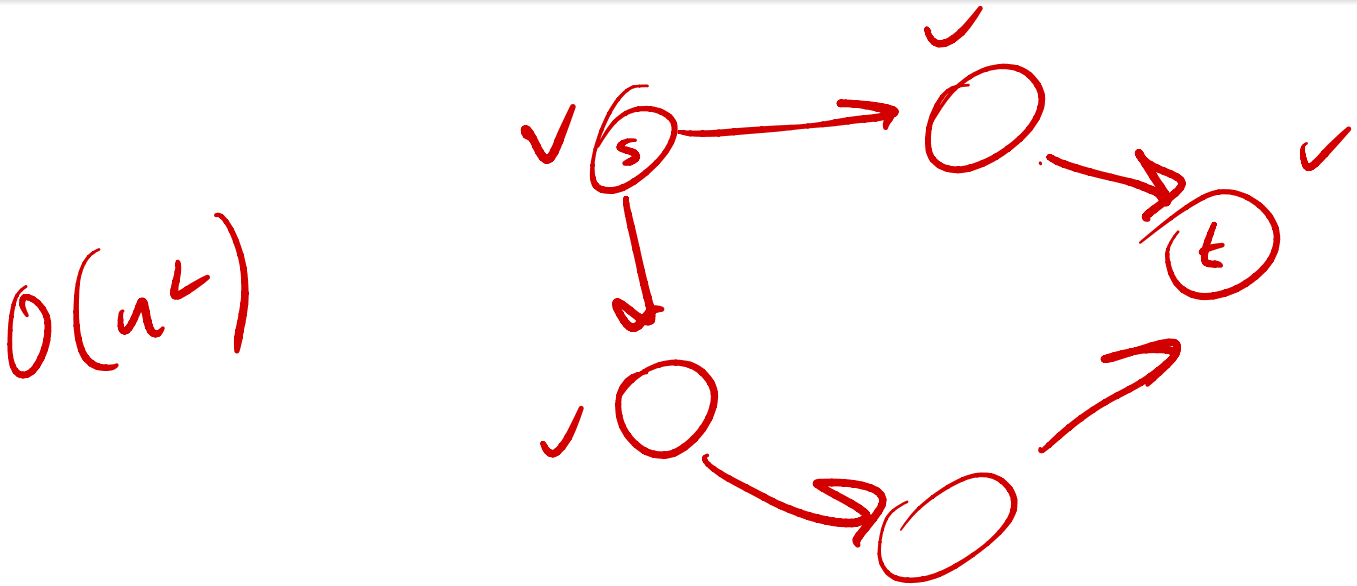
Problems in $\mathcal{P}$

PATH problem

$$PATH = \{\langle G, s, t\rangle \mid G \text{ is a directed graph that has a path from } s \text{ to } t\}$$



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- Important to remember that complexity classes are always defined wrt decision problems, not search problems
- For some classes, the two are equivalent – we will talk about this more later

Next Class

- Nondeterministic computation and the class $\mathcal{NP}$