Foundations of Computing

Lecture 7

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February 4, 2025

Outline

ACM Hackathon

First Midterm

- First midterm exam will be in class on Thursday, February 20
- Tuesday, February 18 will be a review lecture
- 5 points bonus for anyone who attends ACM Hackathon

Important

If you need to miss lecture on February 20, let me know ASAP.

Outline

1. Announcements
2. Lecture 6 Review
3. Proving Languages Not Regular
4. Proving L Not Regular Using the Pumping Lemma
5. Proving L Not Regular Using Closure Properties

Lecture 6 Review

- Nonregular languages
- Proving The NFA pumping lemma
- Using the pumping lemma

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- Nonregular languages
- Proving The NFA pumping lemma
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Today

- Some more examples proving languages are not regular
- Going beyond regular languages

HW2 Problem 4

Let L be a regular language, prove that the following languages are regular.

1. $NOPREFIX(L) = \{w \in L |$ no proper prefix of w is a member of $L\}$
2. $NOEXTEND(L) = \{w \in L | w$ is not a proper prefix of any string in $L\}$

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Example:

- $L = \{00, 11, 001, 101\}$
- $NOPREFIX(L) = \{00, 11, 101\}$
- $NOEXTEND(L) = \{11, 001, 101\}$

$001 \notin NP(L)$

$001 \in L$ $\quad 00 \in L$

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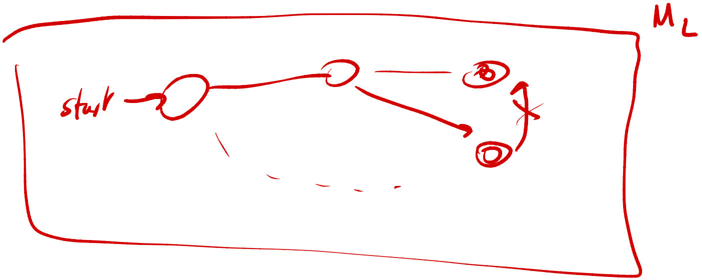

$\bar{w} \in L$ and a prefix of $\bar{w} \in L$

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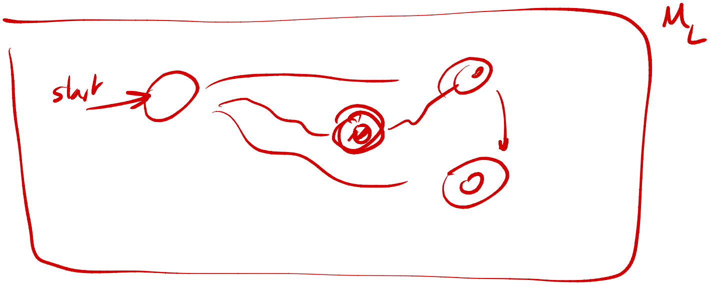

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The Regular Language Pumping Lemma

Pumping Lemma

If L is a regular language, then there exists an integer p (the pumping length) where any string $w \in L$ such that $|w| \geq p$ can be divided into three pieces $w = xyz$ satisfying:

1. For each $i \geq 0$, $xy^iz \in L$
2. $|y| > 0$, and
3. $|xy| \leq p$

The Proof Procedure

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3. Choose $w \in L$ with $|w| \geq p$
4. Demonstrate that w cannot be pumped
   - For each possible division $w = xyz$ (with $|y| > 0$ and $|xy| \leq p$), find an integer i such that $xy^iz \notin L$

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5. Contradiction!!!

Prior Examples

We've already seen how to prove:

- $L = \{0^n 1^n | n \geq 0\}$ is not regular

Let's try something a little harder

Another Example

Consider $L = \{w | w \text{ has an equal number of 0s and 1s}\}$, prove L is not regular

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2. Choose $w = 0^p 1^p$

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Proof:

1. Assume L is regular, and let p be the pumping length this implies
2. Choose $w = 0^p 1^p$
3. By pumping lemma, $w = xyz$ s.t. $xy^i z \in L$

$y = 0^\alpha$ $\quad$ $y = 01$

$z = 1^p$

0001011

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4. Problem: If $y = 0^m 1^m$, then w can be pumped – no contradiction

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6. Complete proof by considering all possible values for y
   - y consists of only 0s – then $xyyz$ has more 0s than 1s, so $w \notin L$

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6. Complete proof by considering all possible values for y
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7. Contradiction – hence, L is not regular

A More Challenging Example

Consider $L = \{0^m 1^n | m \neq n\}$, prove L is not regular

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   That is, $xy^i z = 0^{m'} 1^{n'}$ with $m' = n'$.

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Question

What w should we choose?

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$\beta \geq 1$

2. Consider what happens when we pump k times:

$$xy^k z = 0^{\alpha + k\beta + p - (\alpha+\beta)} 1^{p+1}.$$

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For this to give a contradiction we need

$$m' = n', \text{i.e. } \cancel{\alpha +} k\beta + p - (\cancel{\alpha} + \beta) = \underline{p + (k-1)}\beta = p + 1$$

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Equivalently, we need

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Equivalently, we need
$$(k-1)\beta = 1$$
3. But, we can't control β, so this w does not work

Let's try again!!!

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2. Consider what happens when we pump k times:

$$xy^kz = 0^{\overset{\text{from } x}{\alpha}+\overset{\text{from } y}{k\beta}+\underbrace{m-(\alpha+\beta)}_{\text{from } z}}1^n.$$

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We need a k s.t. $m + (k-1)\beta = n$ for a contradiction

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$p!$

$1 + \frac{n-m}{\beta}$ $1 \leq \beta \leq p$

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   Equivalently, we need $k = 1 + (n-m)/\beta$ to be an integer
3. We only know $\beta \leq p$, how can we guarantee $(n-m)$ is divisible by β?

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Hint: What number is divisible by all integers $\leq p$?

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3. We only know $\beta \leq p$, how can we guarantee $(n-m)$ is divisible by β?

   Hint: What number is divisible by all integers $\leq p$?
4. Set $n = 2(p!)$, $m = p!$, then $(n-m) = p!$ is divisible by β, so there is k s.t. $xy^k z \notin L$

Hints for Using the Pumping Lemma

To use the pumping lemma, need to do the following

- Identify what it means for $x \notin L$
- Choose w such that any valid split xyz can lead to a contradiction
- Prove that $w' = xy^k z \notin L$ form some k

Choosing w is often tricky, requires intuition and some trial and error.

Outline

Proving Non-Regularity Using Closure Properties

Consider $L = \{w | w$ has an equal number of 0s and 1s$\}$, prove L is not regular

A simpler proof:

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1. We already proved that $L_1 = \{0^n 1^n | n \geq 0\}$ is nonregular

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2. Observe that $L_1 = L \cap 0^* 1^*$

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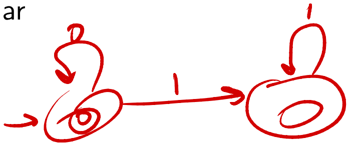
1. We already proved that $L_1 = \{0^n 1^n | n \geq 0\}$ is nonregular
2. Observe that $L_1 = L \cap 0^* 1^*$
3. Easy to see that $0^* 1^*$ is regular
4. Since regular languages are closed under $\cap$, if L is regular then L_1 must be regular



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2. Observe that $L_1 = L \cap 0^* 1^*$
3. Easy to see that $0^* 1^*$ is regular
4. Since regular languages are closed under $\cap$, if L is regular then L_1 must be regular
5. Since we know L_1 is nonregular, this means that L must be nonregular

1. Assume L is regular
2. $\Rightarrow L_1$ is regular

contradiction

Using Closure Properties of Regular Languages

We have seen a number of closure properties of REs

1. Closure under complement: $\overline{L}$ is regular if L is
2. Closure under union: $L_1 \cup L_2$ is regular if L_1, L_2 are
3. Closure under intersection: $L_1 \cap L_2$ is regular if L_1, L_2 are
4. Closure under reversal: L^R is regular if L is
5. NOPREFIX, NOEXTEND
6. There are many more (e.g., set difference, cross product, ...)

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6. There are many more (e.g., set difference, cross product, ...)

Important

- It is often much easier to prove non-regularity using closure properties
- Try this first before you turn to pumping lemma

Exercise

Prove that the following language is nonregular:

$$L = \{0^i 1^j 2^i 3^j | i, j > 0\}$$