Foundations of Computing

Lecture 21

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April 11, 2023

Outline

Lecture 20 Review

- Verifying vs. Deciding
- The Complexity Class $\mathcal{NP}$

$$\mathcal{NP} = \bigcup_k NTIME(n^k)$$

Outline

$\mathcal{P}$ and $\mathcal{NP}$

- $L \in \mathcal{P}$ if there is a poly-time TM M

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Why Do We Study These?

Both $\mathcal{P}$ and $\mathcal{NP}$ contain many useful languages

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$\mathcal{NP}$-Completeness

There are problems in $\mathcal{NP}$ that are as hard as any other problem in $\mathcal{NP}$

Outline

Mapping Reductions

Mapping Reduction

Language A is mapping reducible to language B ($A \leq_m B$) if there is a computable function $f : \Sigma^* \to \Sigma^*$, where for every x,

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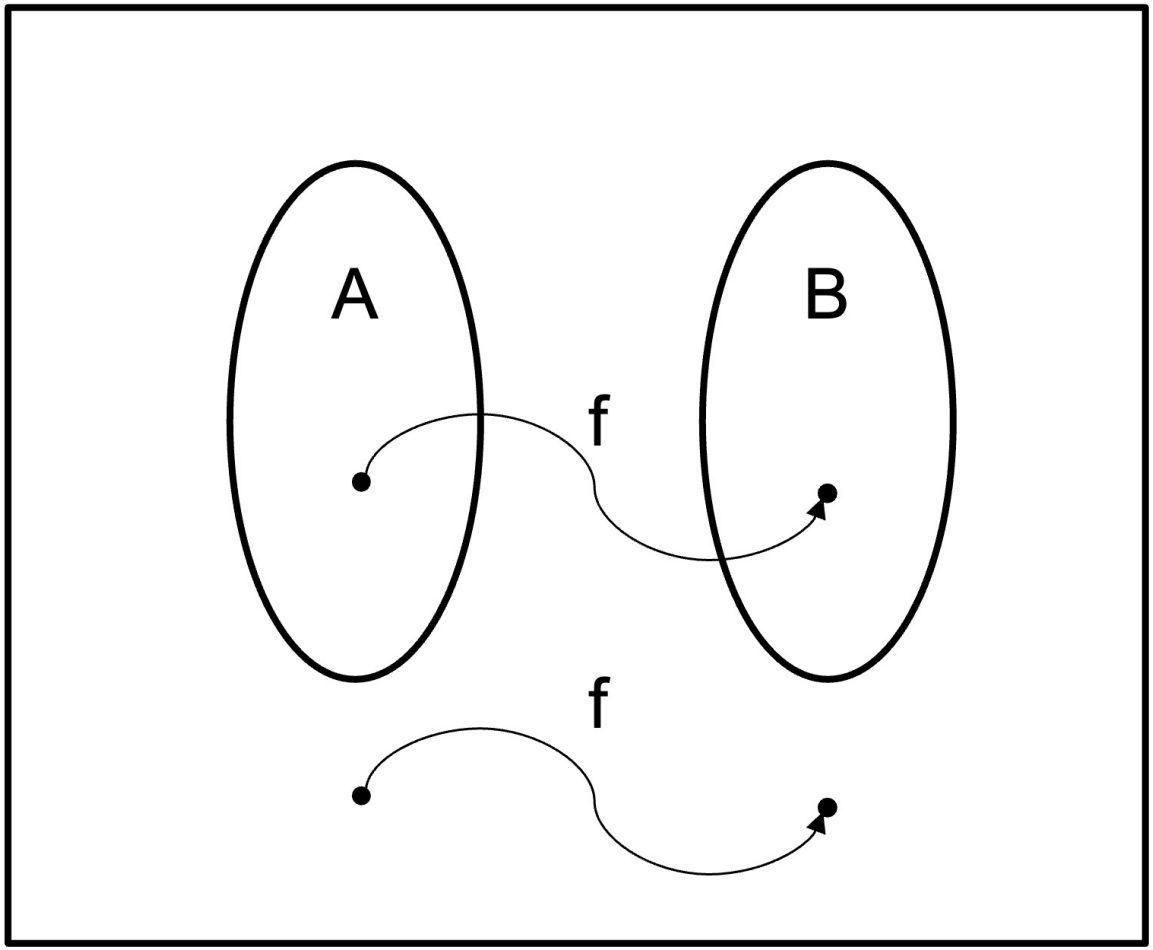
- Poly-time reductions give an efficient way to convert membership testing in A to membership testing in B
- If B has a poly-time solution so does A

Poly-time Mapping Reductions



f runs in time $poly(|x|)$ on all inputs x

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     - If $x \in A$, $f(x) \in B$ so M accepts
     - If $x \notin A$, $f(x) \notin B$, so M rejects
     - Since both f and M are poly-time, $M(f(x))$ is also poly-time

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3-SAT: ϕ in 3-CNF form is ϕ satisfiable

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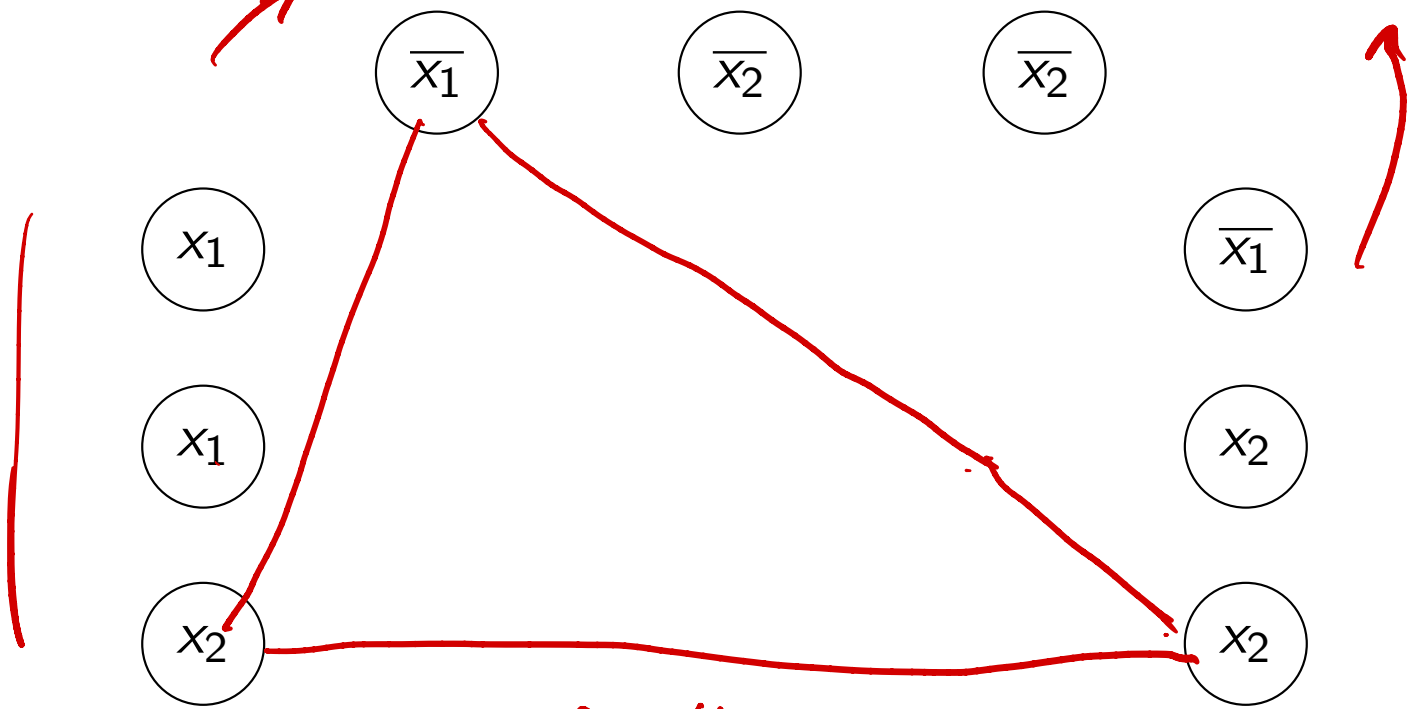

if $\phi \in$ 3SAT $\Rightarrow$ $f(\phi) \in$ CLIQUE$(a, 3)$

if $\phi \notin$ 3SAT $\Rightarrow$ $f(\phi) \notin$ CLIQUE

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     - Since any computation can be represented as a Boolean computation, this is always possible

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#	x_1	q_1	x_2	$\cdots$	x_n	$\sqcup$	$\cdots$	$\sqcup$	#
#									#
#									#

Table: Tableau of configurations of M

- configuration

$O(n^k)$

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M runs in time n^k

$$O((n^k)^2) = O(n^{2k}) = \text{poly}(n)$$

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- Every cell contains #, or a state $q \in Q$, or a tape symbol $\in \Gamma$

SAT is $\mathcal{NP}$-Complete

#	q_0	x_1	x_2	$\cdots$	x_n	$\sqcup$	$\cdots$	$\sqcup$	#
#									#
#									#
			q_{accept}						
#									#

Table: Tableau of configurations of M

- Every row is a configuration of M
- Two consecutive rows represent a valid transition if they follow rules of M
- Every cell contains #, or a state $q \in Q$, or a tape symbol $\in \Gamma$
- M accepts x if a row of this tableau is in q_{accept}

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We need to build a formula ϕ that checks the following four things:

1. Every cell contains a valid character in $C = Q \bigcup \Gamma \bigcup \{\#\}$
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4. Every pair of adjacent rows represents a valid transition of M

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   - The following equation ϕ_{cell} guarantees that a cell has a valid value

$$\phi_{cell} = \left(\bigvee_{s \in C} x_{i,j,s} \right) \wedge \left(\bigwedge_{s,t \in C, s \neq t} (\overline{x_{i,j,s}} \vee \overline{x_{i,j,t}}) \right)$$

cell[i,j] has at least 1 character $\in C$

exactly one character $\in C$

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   - Now, we just take the AND over all n^{2k} cells in the tableau

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3. Some row is in q_{accept}

SAT is $\mathcal{NP}$-Complete

3. Some row is in q_{accept}
   - Define a formula ϕ_{accept} that checks that some row contains q_{accept}

$$\phi_{accept} = \bigvee_{1 \leq i,j \leq n^k} x_{i,j,q_{accept}}$$

SAT is $\mathcal{NP}$-Complete

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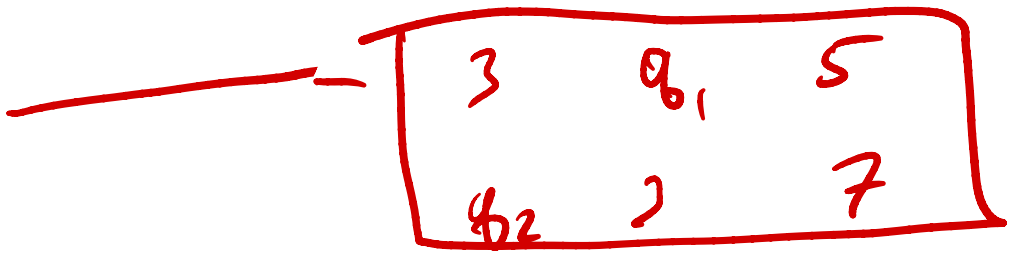

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- For cells after control head, can write to the cell and move left or right (depending on M)
- Every 2×3 window can be checked to follow these rules
- Now just take the $\wedge$ over all possible 6-cell windows

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- Summing up, we see $|\phi| = O(n^{2k})$
- Since $k = O(1)$, this is polynomial in n