Foundations of Computing

Lecture 21

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Outline

1. Lecture 20 Review
2. A Review of $\mathcal{P}$ and $\mathcal{NP}$
3. Polynomial-Time Reductions
4. $\mathcal{NP}$-Completeness
5. $\mathcal{NP}$-Completeness Using Reductions

Lecture 20 Review

- Verifying vs. Deciding
- The Complexity Class $\mathcal{NP}$

$$\mathcal{NP} = \bigcup_k NTIME(n^k)$$

Outline

$\mathcal{P}$ and $\mathcal{NP}$

- $L \in \mathcal{P}$ if there is a poly-time DTM M that decides L:

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Both $\mathcal{P}$ and $\mathcal{NP}$ contain many useful languages

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$\mathcal{NP}$-Completeness

There are problems in $\mathcal{NP}$ that are as hard as any other problem in $\mathcal{NP}$

Outline

Mapping Reductions

Mapping Reduction

Language A is mapping reducible to language B ($A \leq_m B$) if there is a computable function $f : \Sigma^* \to \Sigma^*$, where for every x,

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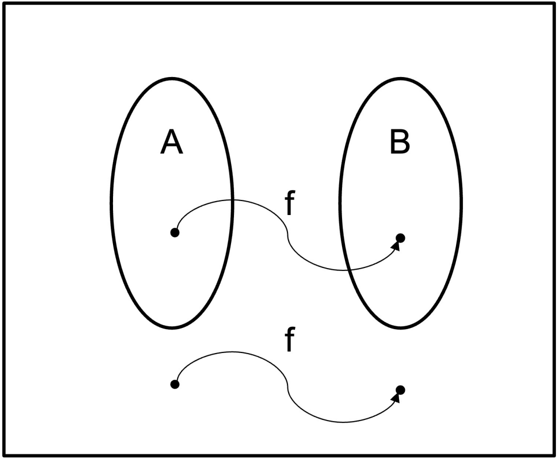
- Poly-time reductions give an efficient way to convert membership testing in A to membership testing in B
- If B has a poly-time solution so does A

Poly-time Mapping Reductions



f runs in time $poly(|x|)$ on all inputs x

Why Poly-Time Reductions

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     - If $x \in A$, $f(x) \in B$ so M accepts
     - If $x \notin A$, $f(x) \notin B$, so M rejects
     - Since both f and M are poly-time, $M(f(x))$ is also poly-time

Using Poly-Time Reductions to Prove Hardness

Theorem

If $A \leq_P B$ and $A \notin \mathcal{P}$, then $B \notin \mathcal{P}$

Outline

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If B is $\mathcal{NP}$-complete and $B \leq_P C$ for $C \in \mathcal{NP}$, then C is $\mathcal{NP}$-complete

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SAT Problem

$$SAT = \{\langle \phi \rangle \mid \phi \text{ is a satisfiable Boolean formula}\}$$

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     - Since any computation can be represented as a Boolean computation, this is always possible

Execution of Turing Machine M deciding A

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#									#
#									#
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Execution of Turing Machine M deciding A

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- M accepts x if a row of this tableau is in q_{accept}

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   - The following equation $\phi_{i,j}^{cell}$ guarantees that a cell has a valid value

$$\phi_{i,j}^{cell} = \underbrace{\left(\bigvee_{s \in C} x_{i,j,s}\right)}_{\text{cell } i,j \text{ has at least 1 value}} \wedge \underbrace{\left(\bigwedge_{s,t \in C, s \neq t} (\overline{x_{i,j,s}} \vee \overline{x_{i,j,t}})\right)}_{\text{cell } i,j \text{ has at most 1 value}}$$

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- Now, we just take the AND over all n^{2k} cells in the tableau

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- Define a formula ϕ_{start} that checks that all the cells in the top row are correct

$$\phi_{start} = x_{1,1,\#} \wedge x_{1,2,q_0} \cdots \wedge x_{1,n^k,\#}$$

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3. Some row is in q_{accept}

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- Define a formula ϕ_{accept} that checks that some row contains q_{accept}

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- For cells after control head, can write to the cell and move left or right (depending on M)
- Every 2×3 cell window can be checked to follow these rules
- Now just take the $\wedge$ over all possible 6-cell windows

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- ϕ_{cell} has fixed size for each cell, so $O(n^{2k})$ total
- ϕ_{start} has fixed size for each cell in top row, so $O(n^k)$ total

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Outline

1. Lecture 20 Review
2. A Review of $\mathcal{P}$ and $\mathcal{NP}$
3. Polynomial-Time Reductions
4. $\mathcal{NP}$-Completeness
5. $\mathcal{NP}$-Completeness Using Reductions

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Can show that 3SAT is $\mathcal{NP}$-complete using similar proof to SAT

Recall The Clique Problem

Clique

A clique in and undirected graph is a subset of nodes s.t. every two nodes are connected by an edge. A k-clique is a clique containing k nodes

$$CLIQUE = \{\langle G, k\rangle \mid G \text{ is an undirected graph with a } k\text{-clique}\}$$

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Prove that CLIQUE is $\mathcal{NP}$-complete

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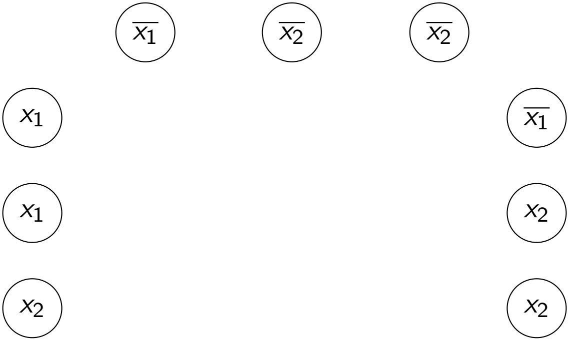

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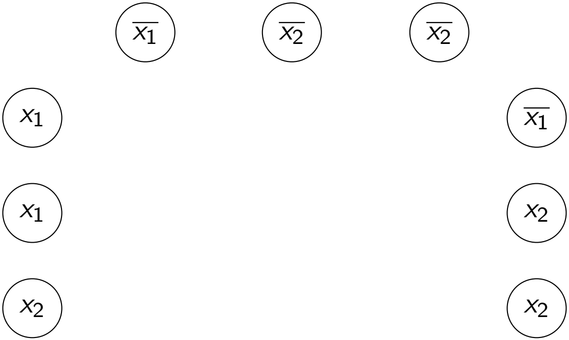

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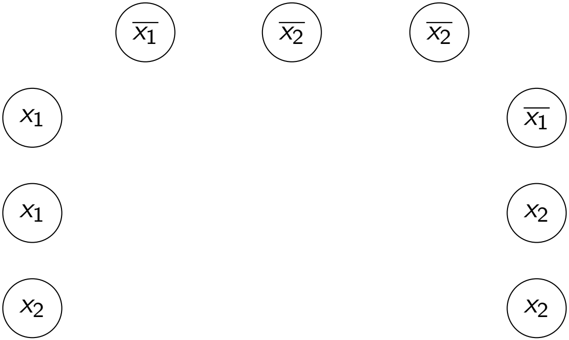

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