Foundations of Computing

Lecture 23

Arkady Yerukhimovich

April 15, 2025

Remaining Class Schedule

- We have only 2 weeks left of lectures!
- HW7 is due tomorrow
- HW8 is out, due next Wednesday
- Thursday, April 24 will be a review lecture

Final Exam

Final exam will be on Tuesday, May 6, 10:20-12:20.

Outline

1. Lecture 22 Review
2. $\mathcal{NP}$-Intermediate Languages
3. co-$\mathcal{NP}$

Lecture 22 Review

- More $\mathcal{NP}$-complete problems
  - SAT
  - 3SAT
  - CLIQUE
  - VERTEX-COVER
  - NAE-SAT
  - 3-coloring

Vertex Cover Problem

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$$\text{VERTEX-COVER} = \{\langle G, k\rangle \mid G \text{ has a vertex cover of size } \leq k\}$$

Goal: Prove that VC is $\mathcal{NP}$-Complete

1. Show that VC $\in \mathcal{NP}$
2. Show that 3-SAT $\leq_p$ VC

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Goal: Show reduction f from 3-SAT to VC s.t.

- if ϕ is satisfiable, $f(\phi) = \langle G, k \rangle$ s.t. G has VC of size $\leq k$
- if ϕ is not satisfiable, $f(\phi) = \langle G, k \rangle$ s.t. G has no VC of size $\leq k$

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Variable gadget: For every variable x_1, draw pair of nodes

Clause gadget: For every (3-term) clause draw a triangle

Observations:

- For each variable need 1 node in cover
- For each triangle need at least 2 nodes
- Need to connect variables to clauses

3-SAT $\leq_p$ VC Example

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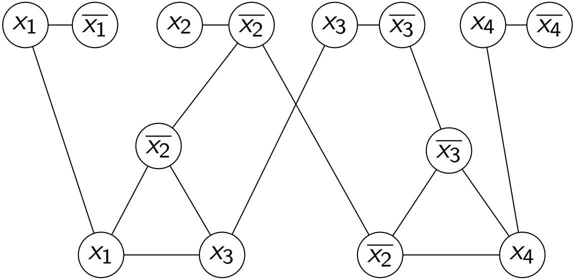

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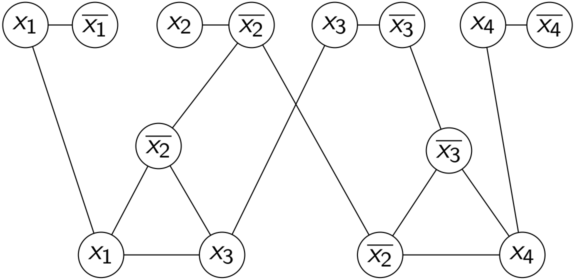

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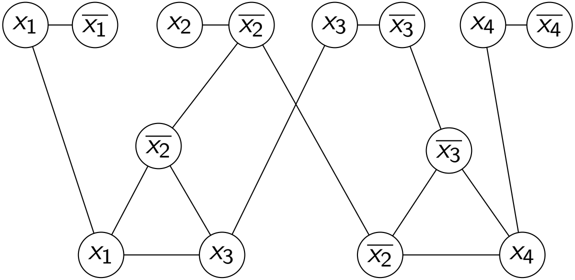

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2. No satisfying assignment implies smallest cover needs $|C| > 2c + v$

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- This edge needs to be covered by the variable node.
- Variable nodes in C must cover at least one edge to each triangle implying a satisfying assignment.

Outline

Ladner's Theorem

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Comment: All languages useful for crypto are such $\mathcal{NP}$-intermediate languages

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1. If $H(n) = n$, then $SAT_H \in \mathcal{P}$
2. If $H(n) \leq c$, then SAT_H is $\mathcal{NP}$-Complete
3. We will define H to be in between these two cases

Defining $H(n)$

Let $M_1, M_2, \ldots$ be an enumeration of all TM's (can do this since TM's are countable)

$H(n)$

- Smallest $i \leq \log\log n$ s.t. for all $x \in \{0,1\}^*$, with $|x| \leq \log n$

$$M_i(x) \text{ halts in } i|x|^i \text{ steps and accepts iff } x \in SAT_H$$

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2. Claim: $SAT_H \in \mathcal{P}$ iff $H(n) < c$ for all n
   ($\Rightarrow$) By definition of $\mathcal{P}$, there is machine M_k that decides SAT_H in kn^k steps so $H(n) = k$

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  - So, $H(n) \neq i$ for all $n > 2^{|x|}$. Contradiction!

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   - But $H(n) \to \infty$ so the padding is super-poly in size of ϕ
   - Hence $|\phi| << n$, so have reduced solving long formula to solving a much shorter one.
   - Repeat this enough times to make $|\phi| = O(1)$ and solve.

Takeaway

If $\mathcal{P} \neq \mathcal{NP}$, then $\mathcal{NP}$-intermediate languages exist!

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Is UNSAT in $\mathcal{NP}$?

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Co-NP is the class of languages that:

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2. There exists poly-time DTM V s.t. for $x \in L$ <u>for all</u> w, $V(x, w) = 0$

Observations:

- co-$\mathcal{NP}$ is not the complement of $\mathcal{NP}$ (i.e., it does not consist of all languages not in $\mathcal{NP}$)

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- In fact, $\mathcal{P} \subseteq (\mathcal{NP} \cap$ co-$\mathcal{NP})$

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$L \in$ co-$\mathcal{NP}$ if there exists poly-time DTM V s.t. for $x \in L$ for all w, $V(x, w) = 0$

Question:

Can you prove that $x \in L$, when $L \in$co-$\mathcal{NP}$?

Proving that $x \in L$ for $L \in$co-$\mathcal{NP}$

The Problem

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- I.e., $\mathcal{NP} \neq$ co-$\mathcal{NP}$

Proving that $x \in L$ for $L \in$co-$\mathcal{NP}$

The Problem

Suppose, I am given an input formula ϕ and I want to prove that ϕ is not satisfiable.

- It is widely believed that there is no poly-size, efficiently verifiable proof w that you could give for UNSAT
- I.e., $\mathcal{NP} \neq$ co-$\mathcal{NP}$
- But, we don't know how to prove this

The Complexity Zoo

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Complexity Zoo

The complexity zoo (`https://complexityzoo.net/Complexity_Zoo`) now has 550 complexity classes.