Foundations of Computing

Lecture 26 – Final Exam Review

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April 24, 2025

Outline

1. Lecture 25 Review
2. Complexity Theory
   - $\mathcal{P}$
   - $\mathcal{NP}$
   - Poly-time Reductions and $\mathcal{NP}$-Completeness
   - Interactive Proofs
   - Zero-Knowledge Proofs
3. Exam Details

Lecture 25 Review

- Zero-Knowledge Proofs
- Where's Waldo
- Puppy and Panda
- Graph Isomorphism
- 3-Coloring

We Are Done!

Welcome to the last lecture of CS 3313!!!

- Complete course evaluation form for 5 points on final exam

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Complexity Classes

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Important

Make sure you know the definitions and relationships between these complexity classes.

Asymptotic Notation – Big-O

Definition

Let $f, g : \mathbb{N} \to \mathbb{R}$, we say that $f(n) = O(g(n))$ if

- There exist positive integers c, n_0 s.t. for all $n \geq n_0$

$$f(n) \leq cg(n)$$

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- I.e., $n_0 = 6, c = 6$

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- I.e., $n_0 = 6, c = 6$
- Note that $f(n) = O(n^4)$

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- $\mathcal{P}$ corresponds to the class of “efficiently-solvable” problems
- $\mathcal{P}$ is invariant for all models of computation polynomially-equivalent to 1-tape TM
- $\mathcal{P}$ has nice closure properties

Problems in $\mathcal{P}$

- PATH
- RELPRIME
- Anything you saw in algorithms class

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Intuition

- $\mathcal{P}$ is the class of problems where you can find a solution in poly-time
- $\mathcal{NP}$ is the class of problems where you can verify a solution in poly-time
- Question: $\mathcal{P} \stackrel{?}{=} \mathcal{NP}$

The Class $\mathcal{NP}$ – Another Formulation

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- An NTM simulates the verifier by guessing the witness w
- A verifier simulates the NTM by using the accepting branch as the witness

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Question:

Is $\mathcal{P} = \mathcal{NP} =$ co-$\mathcal{NP}$?

Problems in $\mathcal{NP}$

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Problems in $\mathcal{NP}$

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- Satisfiability
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- Independent set
- and many more

Important

Make sure you know how to prove $L \in \mathcal{NP}$

Outline

Mapping Reductions

Mapping Reduction

Language A is mapping reducible to language B ($A \leq_m B$) if there is a computable function $f : \Sigma^* \to \Sigma^*$, where for every x,

$$x \in A \iff f(x) \in B$$

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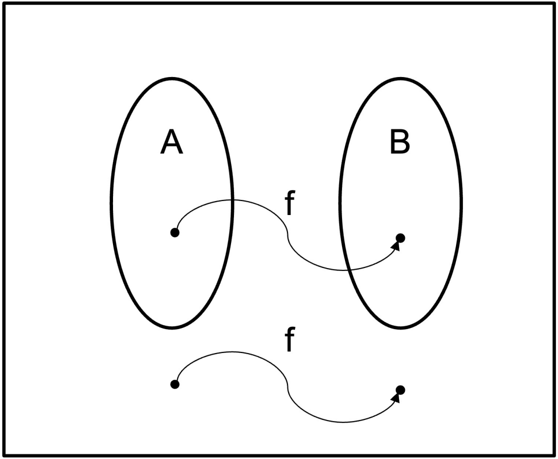
- Poly-time reductions give an efficient way to convert membership testing in A to membership testing in B
- If B has a poly-time solution so does A

Poly-time Mapping Reductions



f runs in time $poly(|x|)$ on all inputs x

Why Poly-Time Reductions

Theorem

If $A \leq_P B$ and $B \in \mathcal{P}$, then $A \in \mathcal{P}$

Proof:

- Let M be the poly-time TM deciding B
- Let f be the poly-time reduction from A to B
- Can construct M' deciding A:
  $M' =$ On input x:
  1. Compute $f(x)$
  2. Run $M(f(x))$ and output whatever M outputs

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  $M' =$ On input x:
  1. Compute $f(x)$
  2. Run $M(f(x))$ and output whatever M outputs
  - If $x \in A$, $f(x) \in B$ so M accepts
  - If $x \notin A$, $f(x) \notin B$, so M rejects
  - Since both f and M are poly-time, $M(f(x))$ is also poly-time

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- Consider $\phi = (x_1 \vee x_1 \vee x_2) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_2}) \wedge (\overline{x_1} \vee x_2 \vee x_2)$

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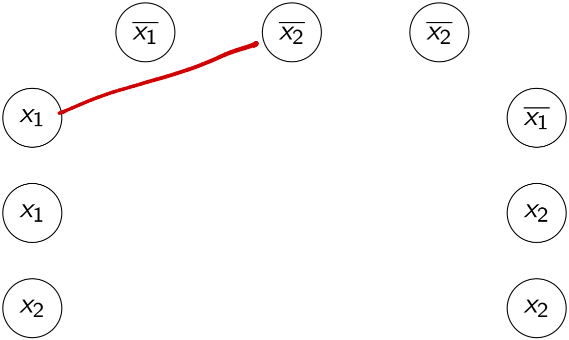

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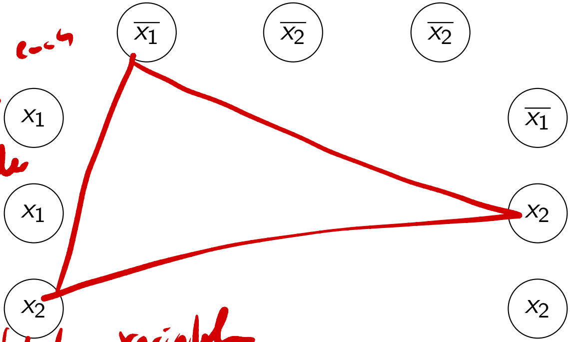
1. Create node for each variable
2. Group nodes by clause
3. Connect non-conflicting variables



- If ϕ is satisfiable then G has a k-clique
- If G has a k-clique then ϕ is satisfiable

$\mathcal{NP}$-Completeness

Definition

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Theorem

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If B is $\mathcal{NP}$-complete and $B \leq_P C$ for $C \in \mathcal{NP}$, then C is $\mathcal{NP}$-complete

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Important

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1. (Completeness) If $x \in L$, then $\Pr[\langle P, V\rangle(x) = 1] = 1$
2. (Soundness) If $x \notin L$, then for any (possibly unbounded) P^*, we have $\Pr[\langle P^*, V\rangle(x) = 1] \leq 1/2$

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1. V chooses $b \leftarrow \{0, 1\}$, and applies a random permutation π to the vertices of G_b and sends this graph G^* to P

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Why This Works:

1. (Completeness) Suppose that G_0 and G_1 are not isomorphic.
    - Then G^* can only be isomorphic to one of the two graphs
    - P can perfectly determine which one this is

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The Protocol:

1. V chooses $b \leftarrow \{0, 1\}$, and applies a random permutation π to the vertices of G_b and sends this graph G^* to P
2. P determines if G^* is isomorphic to G_0 and sends $b' = 0$ if so, or $b' = 1$ otherwise back to V
3. V accepts if $b' = b$

Why This Works:

1. (Completeness) Suppose that G_0 and G_1 are not isomorphic.
   - Then G^* can only be isomorphic to one of the two graphs
   - P can perfectly determine which one this is
   - So $\Pr[b' = b] = 1$

Graph Non-Isomorphism

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   - Thus, $\Pr[b' = b] = 1/2$

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Question: What should V do?

- Suppose that V is deterministic:
- What if you allow V to be randomized:

Languages in $\mathcal{IP}$

- $\mathcal{P} \subseteq \mathcal{IP}$
- $\mathcal{NP} \subseteq \mathcal{IP}$
- Graph Non-Isomorphism $\in \mathcal{IP}$

Outline

1. Lecture 25 Review

2. Complexity Theory
   - $\mathcal{P}$
   - $\mathcal{NP}$
   - Poly-time Reductions and $\mathcal{NP}$-Completeness
   - Interactive Proofs
   - Zero-Knowledge Proofs

3. Exam Details

Zero-Knowledge Proofs

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$$\langle P, V \rangle (x)$$

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Zero-Knowledge Proof

A proof $\langle P, V \rangle(x)$ for a language L is *zero-knowledge* if

- For any (possibly malicious) poly-time verifier V^*
- There exists a poly-time *Simulator* S s.t.

$$\forall x \in L, \qquad VIEW_{V^*}(\langle P, V^* \rangle(x)) = S(x)$$

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Input: $x = (G_0, G_1)$
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4. V accepts iff $H = \pi'(G_{b'})$

Outline

Exam

Exam Details:

- Tuesday, May 6, 10:20-12:20
- In the classroom
- 2 sheets (back-and-front) of notes are allowed

See you all there!