Foundations of Computing

Lecture 26 – Final Exam Review

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Outline

Lecture 25 Review

- Zero-Knowledge Proofs
- Where's Waldo
- Puppy and Panda
- Graph Isomorphism

We Are Done!

Welcome to the last lecture of CS 3313!!!

- Complete course evaluation form for 5 points on final exam

- Get a cookie

Outline

Complexity Classes

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- $\mathcal{P}$

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- $\mathcal{P}$
- $\mathcal{NP}$

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- $\mathcal{P}$
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Important

Make sure you know the definitions and relationships between these complexity classes.

Asymptotic Notation – Big-O

Definition

Let $f, g : \mathbb{N} \to \mathbb{R}$, we say that $f(n) = O(g(n))$ if

- There exist positive integers c, n_0 s.t. for all $n \geq n_0$

$$f(n) \leq cg(n)$$

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- For every $n \geq 6$, $f(n) \leq 6n^3$
- I.e., $n_0 = 6, c = 6$

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- I.e., $n_0 = 6, c = 6$
- Note that $f(n) = O(n^4)$

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- $\mathcal{P}$ corresponds to the class of "efficiently-solvable" problems
- $\mathcal{P}$ is invariant for all models of computation polynomially-equivalent to 1-tape TM
- $\mathcal{P}$ has nice closure properties

Problems in $\mathcal{P}$

- PATH
- RELPRIME
- Anything you saw in algorithms class

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Intuition

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- $\mathcal{NP}$ is the class of problems where you can verify a solution in poly-time
- Question: $\mathcal{P} \stackrel{?}{=} \mathcal{NP}$

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Proof Idea:

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- An NTM simulates the verifier by guessing the witness w
- A verifier simulates the NTM by using the accepting branch as the witness

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Question:

$$\text{Is } \mathcal{NP} = \text{co-}\mathcal{NP}?$$

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- Independent set
- and many more

Important

Make sure you know how to prove $L \in \mathcal{NP}$

Outline

Mapping Reductions

Mapping Reduction

Language A is mapping reducible to language B ($A \leq_m B$) if there is a computable function $f : \Sigma^* \to \Sigma^*$, where for every x,

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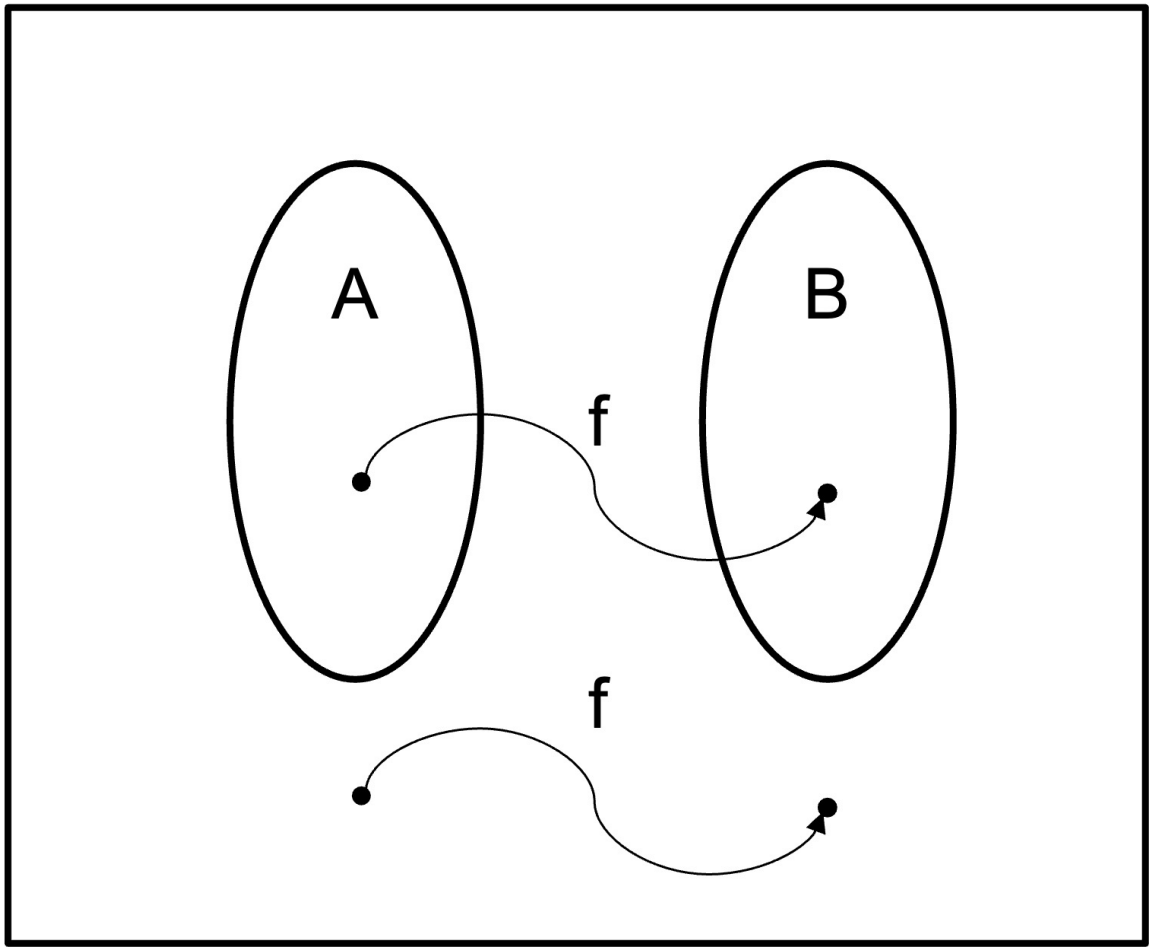
- Poly-time reductions give an efficient way to convert membership testing in A to membership testing in B
- If B has a poly-time solution so does A

Poly-time Mapping Reductions



f runs in time $poly(|x|)$ on all inputs x

Why Poly-Time Reductions

Theorem

If $A \leq_P B$ and $B \in \mathcal{P}$, then $A \in \mathcal{P}$

Proof:

- Let M be the poly-time TM deciding B
- Let f be the poly-time reduction from A to B
- Can construct M' deciding A:
  $M' =$ On input x:
  1. Compute $f(x)$
  2. Run $M(f(x))$ and output whatever M outputs

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     - If $x \in A$, $f(x) \in B$ so M accepts
     - If $x \notin A$, $f(x) \notin B$, so M rejects
     - Since both f and M are poly-time, $M(f(x))$ is also poly-time

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Theorem

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If B is $\mathcal{NP}$-complete and $B \leq_P C$ for $C \in \mathcal{NP}$, then C is $\mathcal{NP}$-complete

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Important

Make sure you remember what direction the reduction should go.

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2. (Soundness) Suppose that G_0 and G_1 are isomorphic

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How can we prove that two graphs G_0 and G_1 are NOT isomorphic?

The Protocol:

1. V chooses $b \leftarrow \{0,1\}$, and applies a random permutation π to the vertices of G_b and sends this graph G^* to P
2. P determines if G^* is isomorphic to G_0 and sends $b' = 0$ if so, or $b' = 1$ otherwise back to V
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Why This Works:

1. (Completeness) Suppose that G_0 and G_1 are not isomorphic.
   - Then G^* can only be isomorphic to one of the two graphs
   - P can perfectly determine which one this is
   - So $\Pr[b' = b] = 1$
2. (Soundness) Suppose that G_0 and G_1 are isomorphic
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   - Thus, $\Pr[b' = b] = 1/2$

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Question: What should V do?

- Suppose that V is deterministic:
- What if you allow V to be randomized:

Arithmetization

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Idea

For n inputs $x_1, x_2, \ldots, x_n$, construct a polynomial $\Phi(x_1, \ldots, x_n)$ s.t.

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$$\phi(x) = (x_1 \vee x_2 \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee x_3)$$

Remember

Arithmetization is key component of proof that co-$\mathcal{NP} \subseteq \mathcal{IP}$

Languages in $\mathcal{IP}$

- $\mathcal{P} \subseteq \mathcal{IP}$
- $\mathcal{NP} \subseteq \mathcal{IP}$
- co-$\mathcal{NP} \subseteq \mathcal{IP}$
- PSPACE$=\mathcal{IP}$

$\neq$

$P \subsetneq NP \subsetneq PSPACE$

Outline

Zero-Knowledge Proofs

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Zero-Knowledge Proof

A proof $\langle P, V \rangle(x)$ for a language L is *zero-knowledge* if

- For any (possibly malicious) poly-time verifier V^*
- There exists a poly-time *Simulator* S s.t.

$$\forall x \in L, \qquad VIEW_{V^*}(\langle P, V^* \rangle(x)) = S(x)$$

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4. V accepts iff $H = \pi'(G_{b'})$

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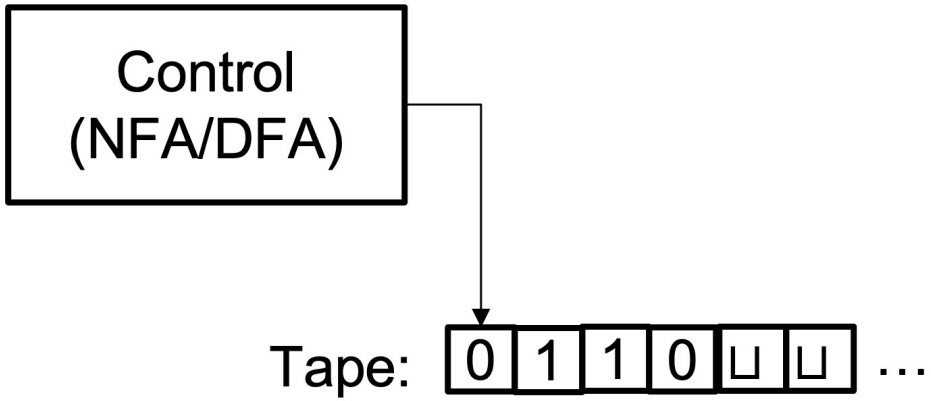
The Turing Machine



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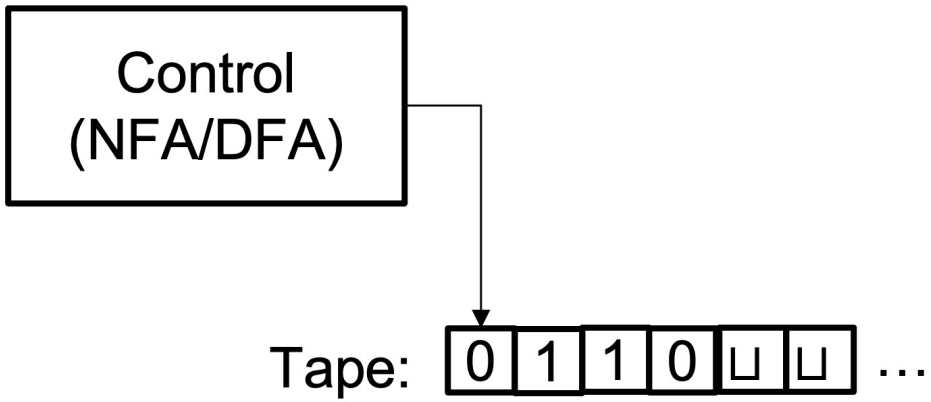

Key Differences:

- A TM can read and write to its tape
- The read/write head can move to the right and to the left
- No separate input tape, input written onto memory tape at start
- The memory tape is infinite
- Control FA has accept and reject states that are immediately output if entered

Turing Machines and Algorithms

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Observations:

- While unproven, all modern computers satisfy Church-Turing thesis
- To prove that some problem cannot be solved by an algorithm, enough to reason about Turing Machines
- This means that Turing Machines give an abstraction to capture "feasible computation"

Outline

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A language L is *decidable* or *recursive* if some TM M decides it

- M halts on all inputs, accepting those in L and rejecting those not in L

Take Away

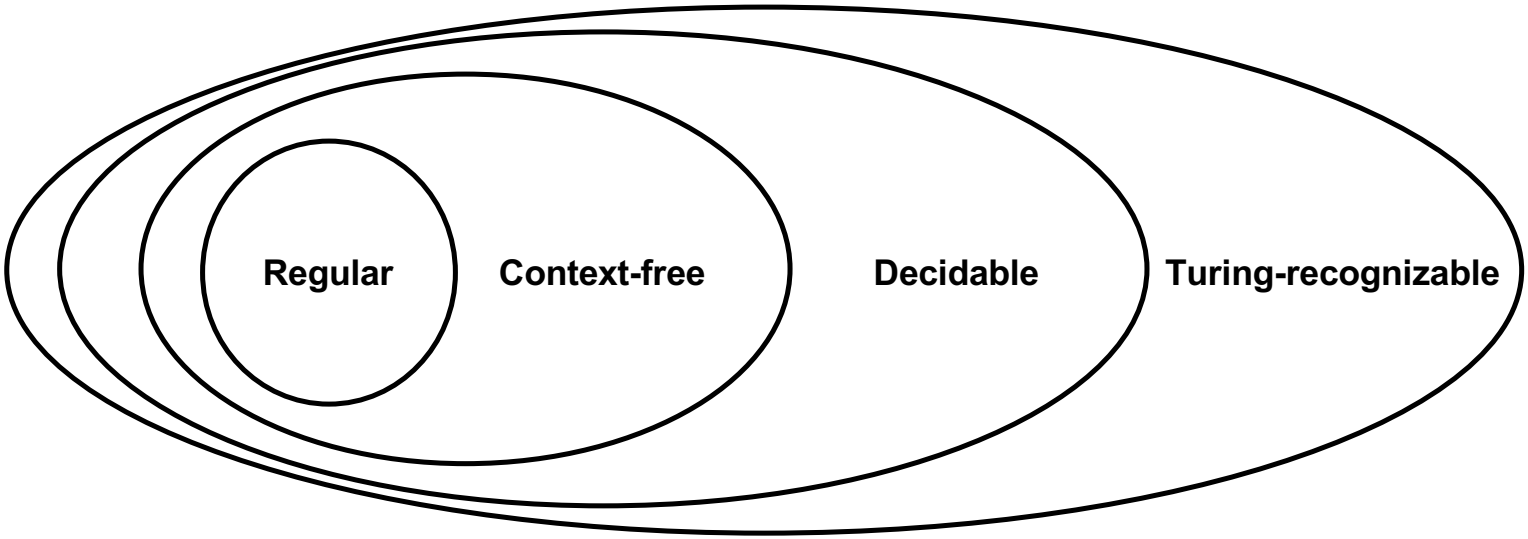
You should be able to show that a language is decidable or Turing-recognizable by designing a TM algorithm.

Decidable Languages

We have seen many examples of decidable languages:

- Languages about strings
- Languages about DFAs/NFAs/PDAs/CFGs – know which ones are decidable and which are not, why
- Be comfortable with TM's that take another machine as input

Relationships Among Language Classes



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Preliminaries – Countable and Uncountable Sets

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- A set that is not countable is *uncountable*

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- Contradiction – f is not mapping between $\mathcal{R}$ and $\mathcal{N}$

Take Aways

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- Note that $M_{L_{TM}}$ may not halt on all inputs – doesn't decide L_{TM}

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- Now consider what happens if we run D on $\langle D\rangle$

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- But, this means that A is decidable by running the machine for B as needed by the reduction

Outline

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Exam

Exam Details:

- Tuesday, May 9, 10:20-12:20
- In the classroom
- 2 sheets (back-and-front) of notes are allowed

See you all there!