Foundations of Computing

Lecture 18 – Exam Review

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March 25, 2025

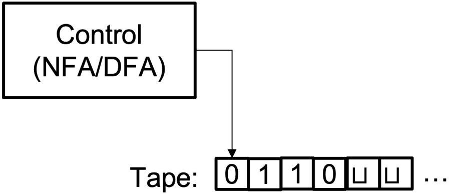
Outline

Lecture 17 Review

- Review of Reductions
- Types of Reductions – Mapping reductions, Turing reductions
- A brief intro into Kolmogorov complexity

Outline

The Turing Machine



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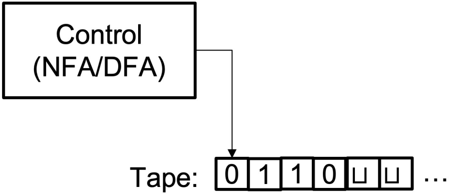

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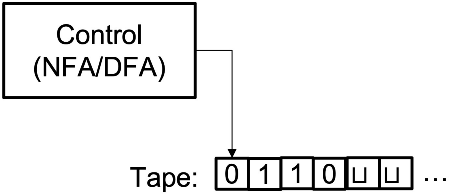

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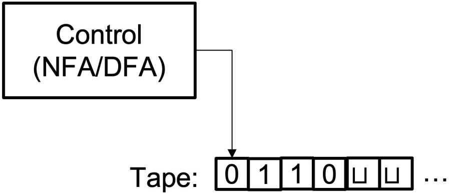

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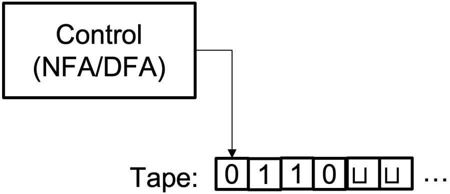

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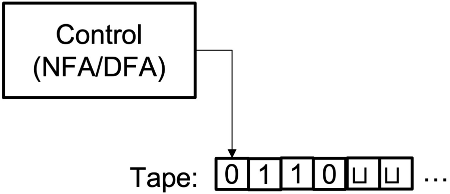

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- The read/write head can move to the right and to the left
- No separate input tape, input written onto memory tape at start
- The memory tape is infinite
- Control FA has accept and reject states. If entered, TM halts and outputs.

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3. When all symbols to the left of $\#$ have been crossed off, check that no uncrossed-off symbols remain to the right of $\#$. If any symbols remain, reject, otherwise accept.

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- To prove that some problem cannot be solved by an algorithm, enough to reason about Turing Machines
- This means that Turing Machines give an abstraction to capture "feasible computation"

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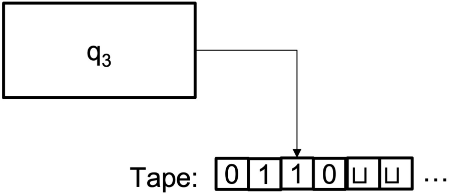
Computing on a Turing Machine



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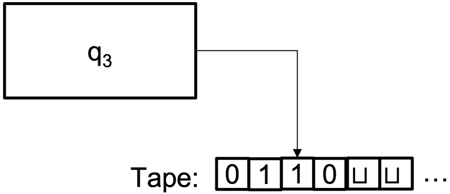

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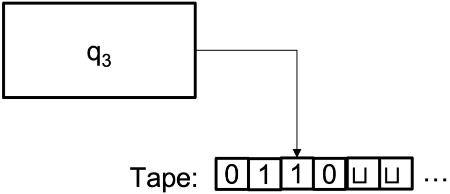

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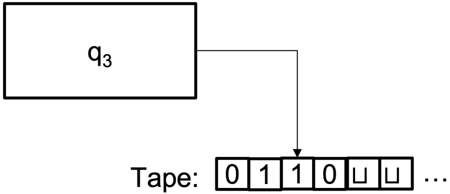

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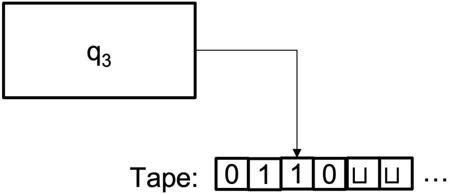

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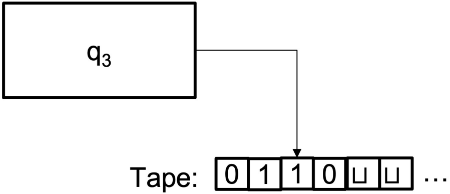

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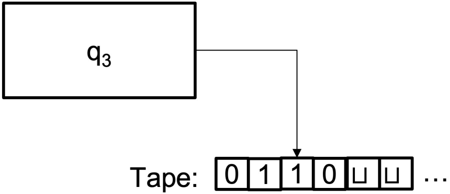

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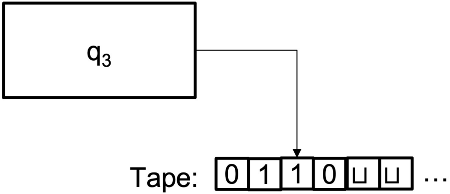

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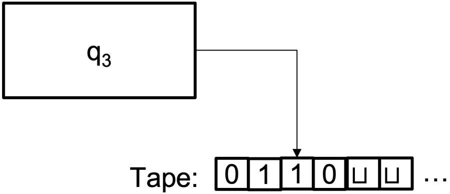

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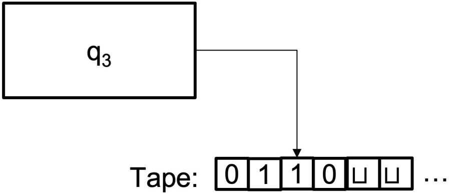

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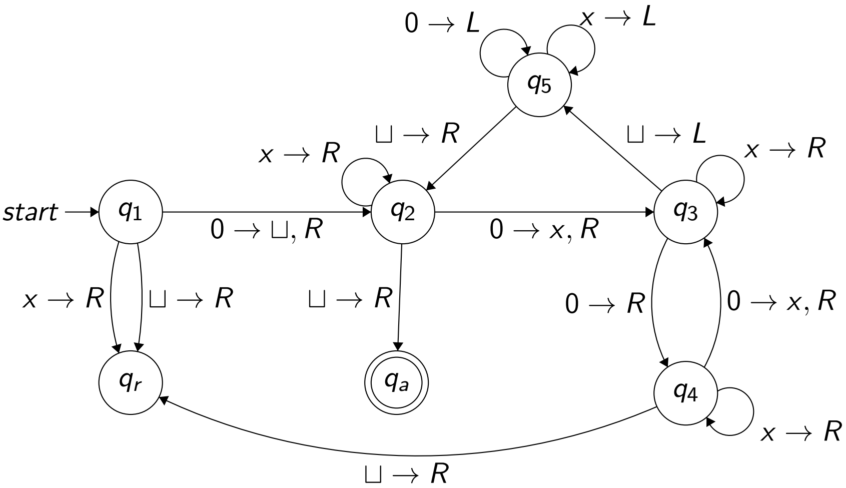
Full Specification: Running M on $w = 0000$



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A language L is *decidable* or *recursive* if some TM M decides it

- M halts on all inputs, accepting those in L and rejecting those not in L

Take Away

You should be able to show that a language is decidable or Turing-recognizable by designing a TM algorithm.

Important TM Notation / Observations

- TM always takes a string as input
  - Sometimes we want to talk about a TM taking another type of input (e.g., a graph, a FA, a TM)
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  - So, we can write this as a string and pass it to a TM
  - TM can then run the machine from this description
  - A TM that accepts any TM and runs it is called a *universal TM*

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3. Algorithm specification
   - Give algorithm in pseudocode
   - Don't explicitly spell out what happens on the tape

Turing Machine Variants

- Multi-tape Turing Machine
- Nondeterministic Turing Machine

What You Need to Know

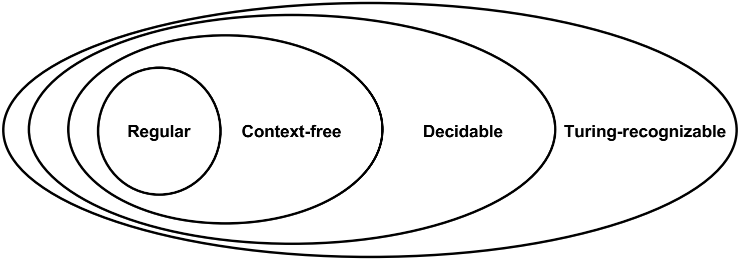
- Be able to explain what the variant is
- Know whether it is equivalent to standard TM
- Be able to explain why

Decidable Languages

We have seen many examples of decidable languages:

- Languages about strings
- Languages about DFAs/NFAs/CFGs – know which ones are decidable and which are not, why
- Be comfortable with TM's that take another machine as input

Relationships Among Language Classes



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- A set that is not countable is *uncountable*

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- Contradiction – f is not mapping between $\mathcal{R}$ and $\mathcal{N}$

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Proof: By construction of machine $M_{A_{TM}}$

$M_{A_{TM}}$: On input $\langle M, w \rangle$,

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- Note that $M_{A_{TM}}$ may not halt on all inputs – doesn't decide A_{TM}

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- Now consider what happens if we run D on $\langle D\rangle$

$$D(\langle D\rangle) = \begin{cases} \textit{accept} & \text{if } D \text{ does not accept } \langle D\rangle \\ \textit{reject} & \text{if } D \text{ accepts } \langle D\rangle \end{cases}$$

How Is This a Diagonalization?

	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\cdots$	$\langle D \rangle$	$\cdots$
M_1	$\underline{accept}$	$reject$	$accept$		$accept$	
M_2	$reject$	$\underline{reject}$	$reject$	$\dots$	$accept$	$\dots$
M_3	$accept$	$accept$	$\underline{accept}$		$reject$	
$\vdots$		$\vdots$		$\ddots$		
D	$reject$	$accept$	$reject$		$?$	

- We have defined D to do the opposite of what M_i does on input $\langle M_i \rangle$
- But what does D do on input $\langle D \rangle$??

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Reductions and Undecidability

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- But, this means that A is decidable by running the machine for B as needed by the reduction

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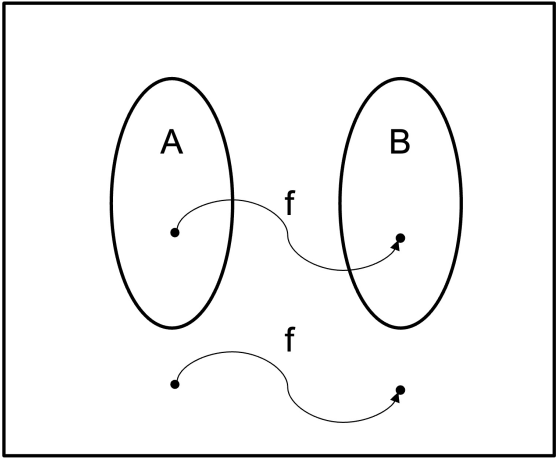
Reduction Types

Know the difference between:

- Mapping reductions
- Turing reductions

Know what each one implies

Mapping Reductions



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Language A is Turing reducible to language B ($A \leq_T B$) if can use a decider for B to decide A.

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- The reduction may make multiple calls to decider for B and may not directly use the result.

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   - If A is not Turing-recognizable, cannot say if B is Turing-recognizable

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$$K(x) = |d(x)|$$

$k(x) \leq \log n +$ $\quad$ $x = 0^n$

- $K(x)$ is the minimal description of x
- This captures the "amount of information" in x

What You Need to Know

- Basic definition of Kolmogorov complexity
- Be able to find rough bounds on Kolmogorov complexity
- Don't need to be able to prove anything