Foundations of Computing

Lecture 22

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Outline

Lecture 21 Review

- $\mathcal{P}$ and $\mathcal{NP}$
- Polynomial-Time Reductions
- $\mathcal{NP}$-completeness of SAT

Outline

What We Already Know

1. SAT is $\mathcal{NP}$-complete
2. 3-SAT is $\mathcal{NP}$-complete
3. 3-SAT $\leq_P$ CLIQUE
4. CLIQUE $\leq_P$ Independent Set

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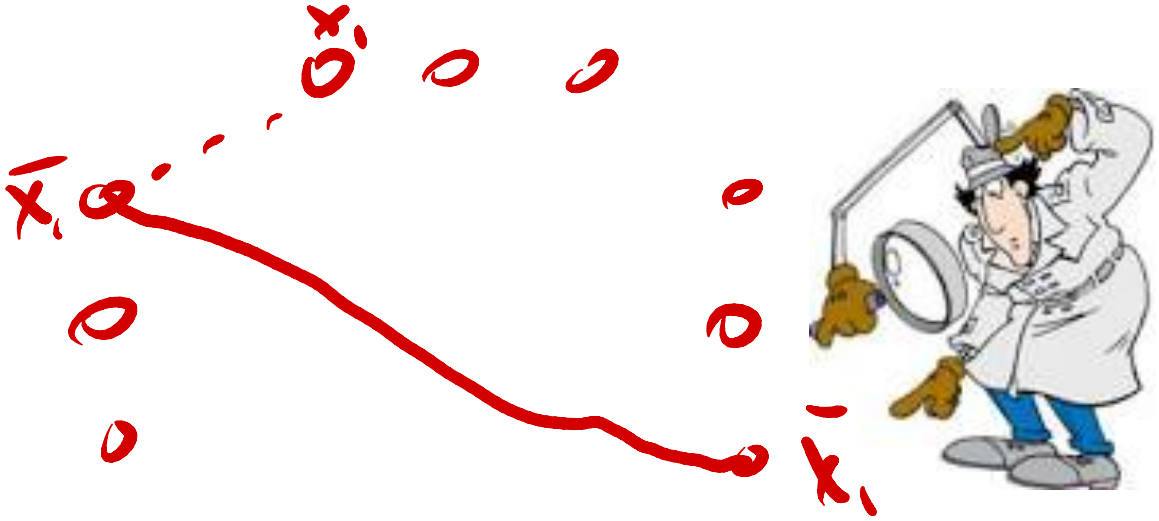
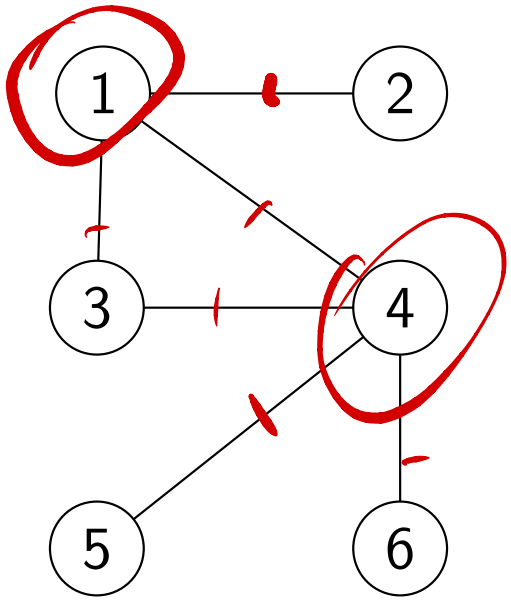


Gadgets

- Gadgets are structures in the target problem that can simulate structures in the source problem
- For example, in proof of 3SAT $\leq_P$ CLIQUE
  - We replaced each variable with a node
  - We replaced each clause with 3 nodes (1 for each variable)
  - Edges capture independent variables between clauses

Vertex Covers

Given a graph $G = (V, E)$, a vertex cover is a subset of the nodes $C \subseteq V$ s.t. each edge in E has an end-point in V.



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1. Show that VC $\in \mathcal{NP}$

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2. Show that 3-SAT $\leq_p$ VC

3-SAT $\leq_p$ VC

Goal: Show reduction f from 3-SAT to VC s.t.

- if ϕ is satisfiable, $f(\phi) = \langle G, k \rangle$ s.t. G has VC of size $\leq k$
- if ϕ is not satisfiable, $f(\phi) = \langle G, k \rangle$ s.t. G has no VC of size $\leq k$

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Variable gadget:

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Variable gadget: For every variable x_1, draw pair of nodes

Clause gadget:

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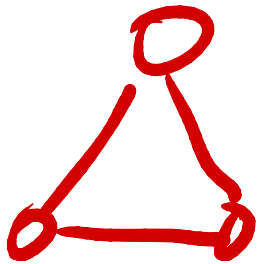
Variable gadget: For every variable x_1, draw pair of nodes

Clause gadget: For every (3-term) clause draw a triangle



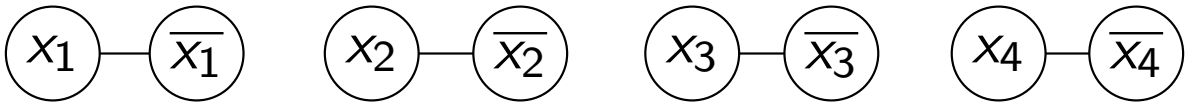
Observations:

- For each variable need 1 node in cover
- For each triangle need at least 2 nodes
- Need to connect variables to clauses

3-SAT $\leq_p$ VC Example

1, 0, 0, 1

$$\phi = (x_1 \vee \overline{x_2} \vee x_3) \wedge (\overline{x_2} \vee \overline{x_3} \vee x_4)$$



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2. No satisfying assignment implies smallest cover needs $|C| > 2c + v$

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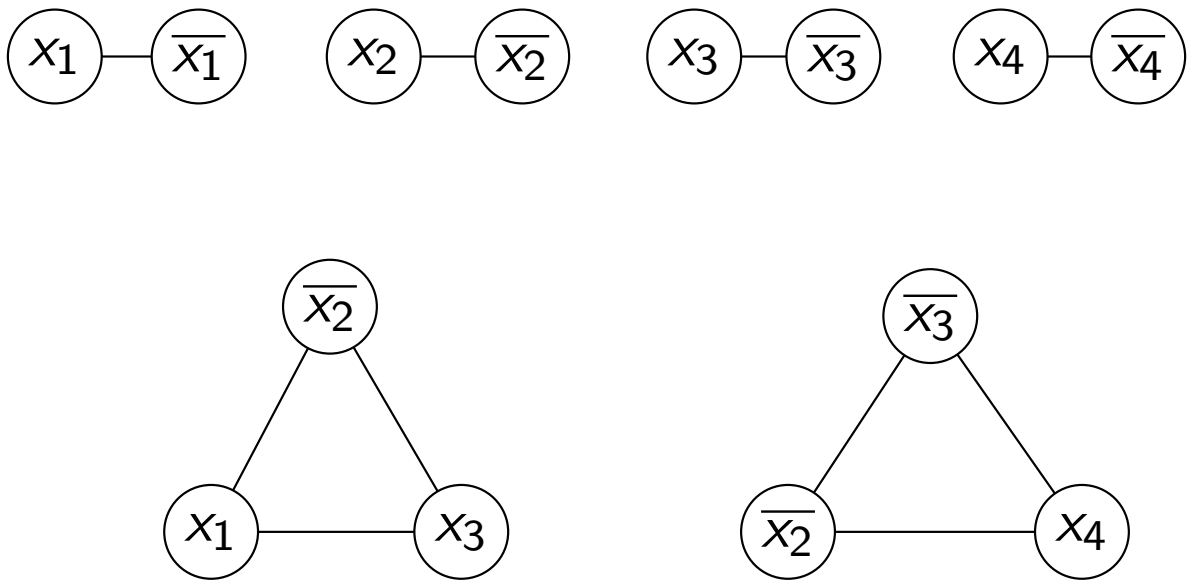

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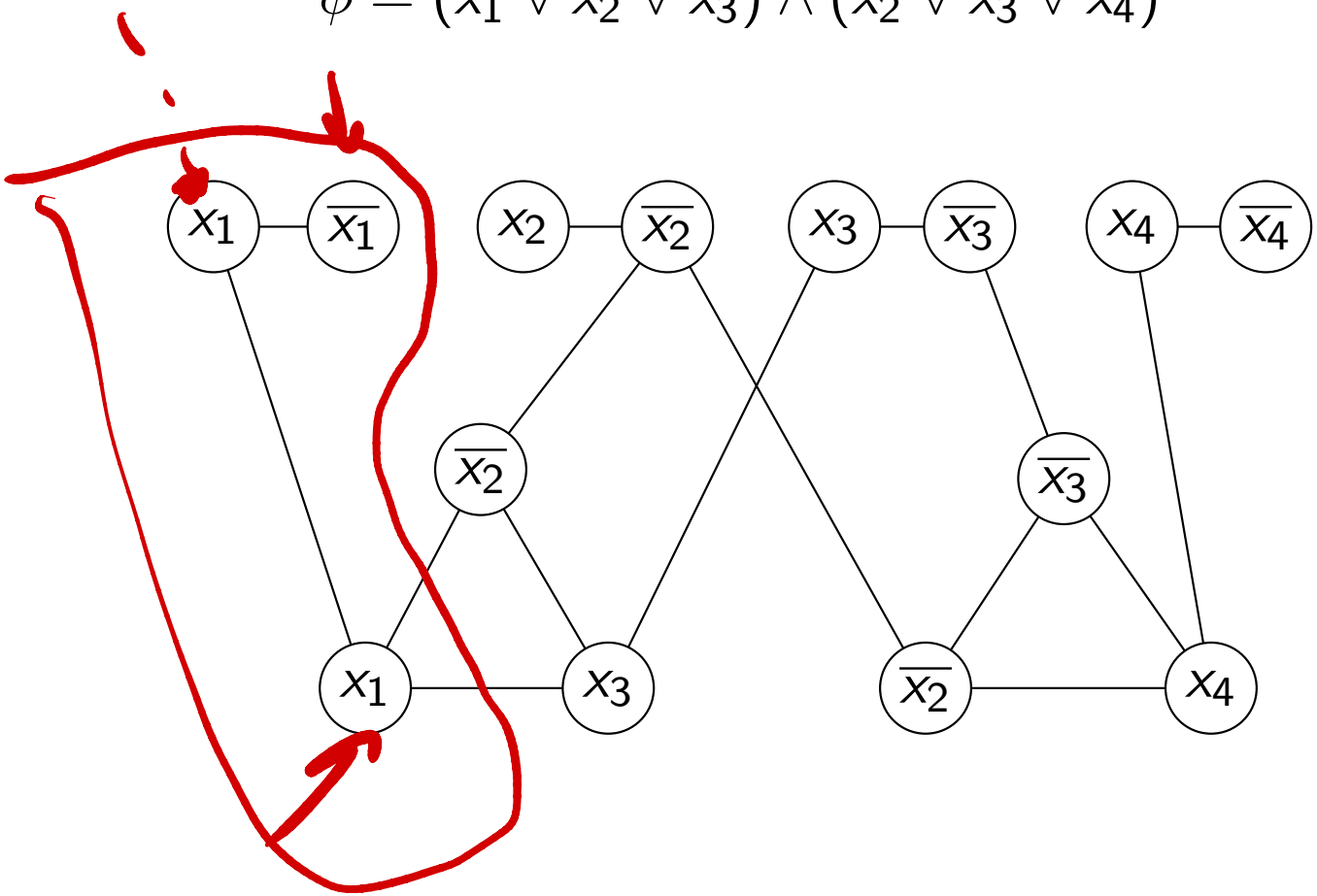

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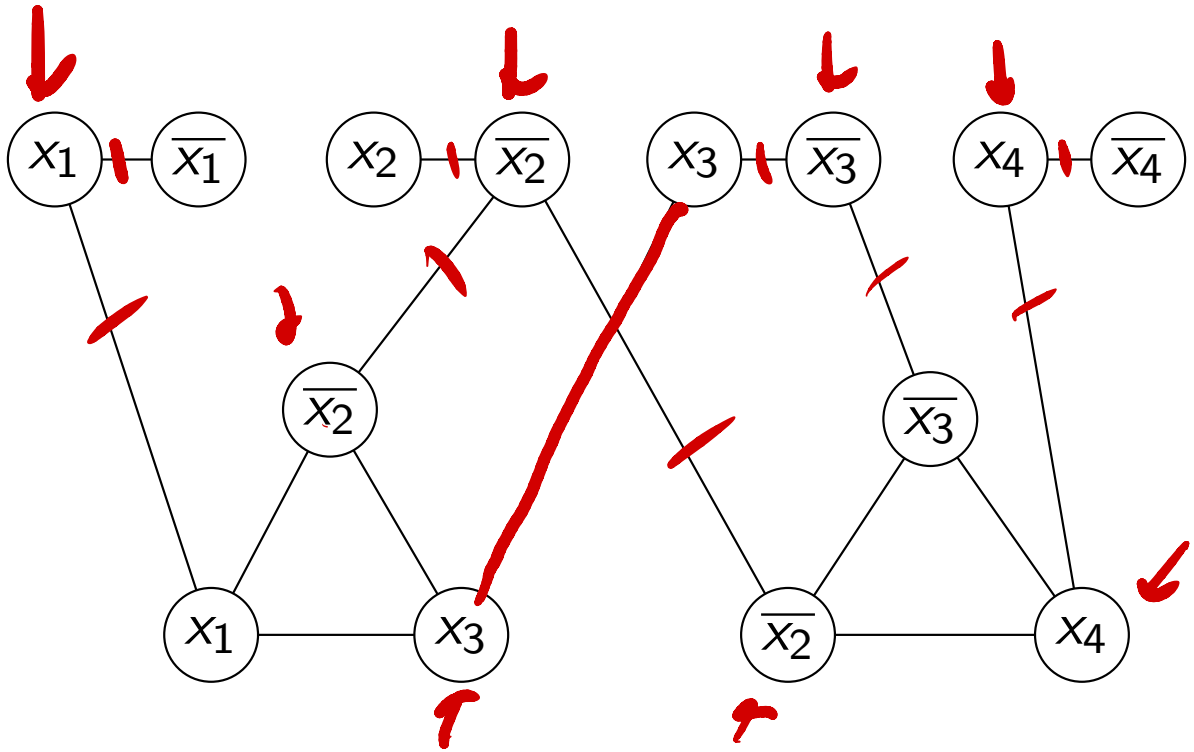

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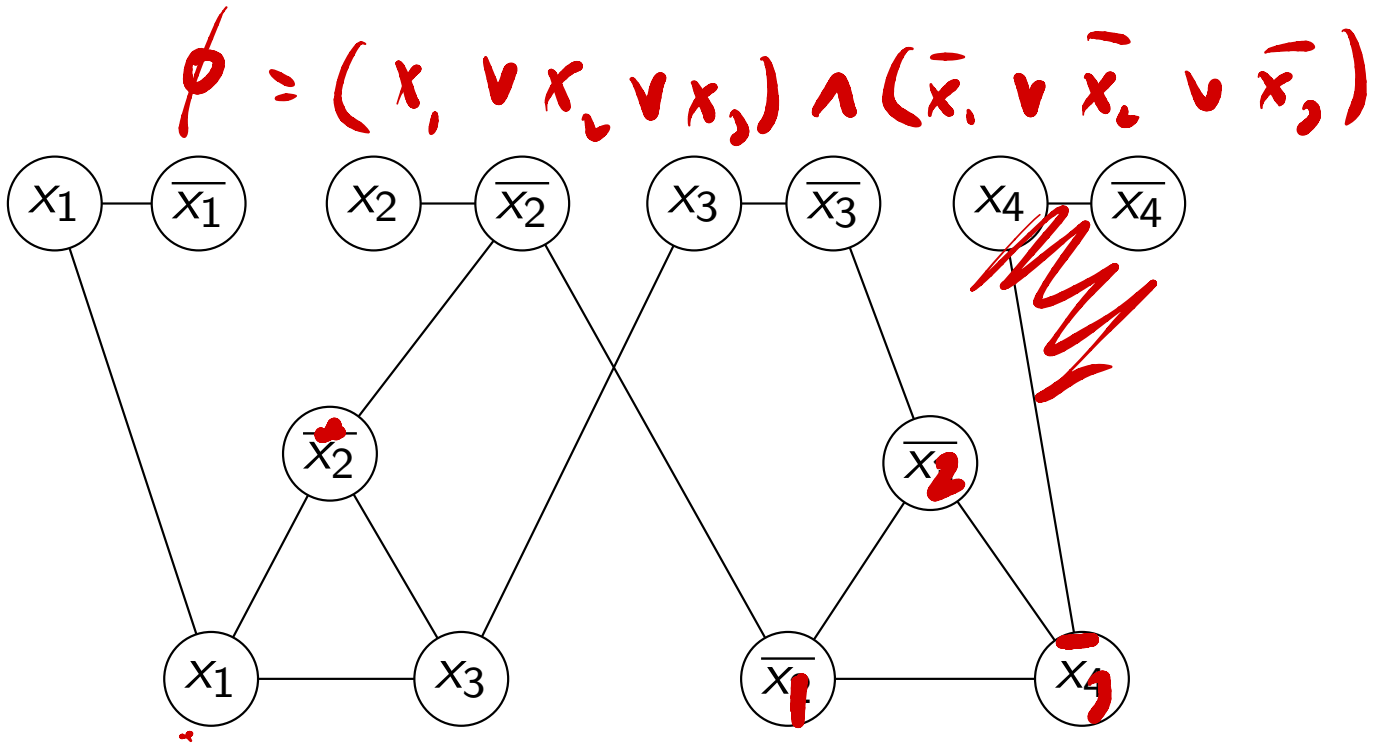

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2. No satisfying assignment implies smallest cover needs $|C| > 2c + v$

Why This Works

Satisfying assignment $\Rightarrow |C| = 2c + v$:

- Include the node corresponding to the 1 value in C (i.e., if $x_1 = 1$ then include x_1, otherwise include $\overline{x_1}$).
- Since $\phi(x) = 1$, for every triangle at least one edge between triangle and variable gadgets is already covered (i.e., at least one variable in each clause is satisfied).
- Can include the remaining two nodes in triangle in C to cover the remaining edges.
- This results in a cover of size $2c + v$.

$|C| = 2c + v \Rightarrow$ Satisfying assignment

- C must contain 2 nodes per triangle and 1 node per variable gadget
- One edge connecting each triangle to variable gadgets must be not covered by the triangle nodes (that would require all 3 nodes).
- This edge needs to be covered by the variable node.
- Variable nodes in C must cover at least one edge to each triangle implying a satisfying assignment.

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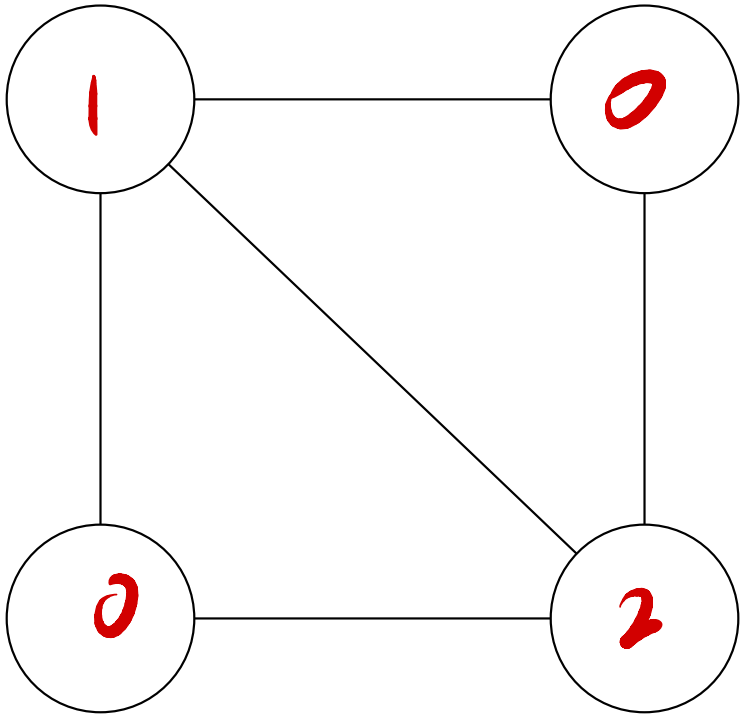

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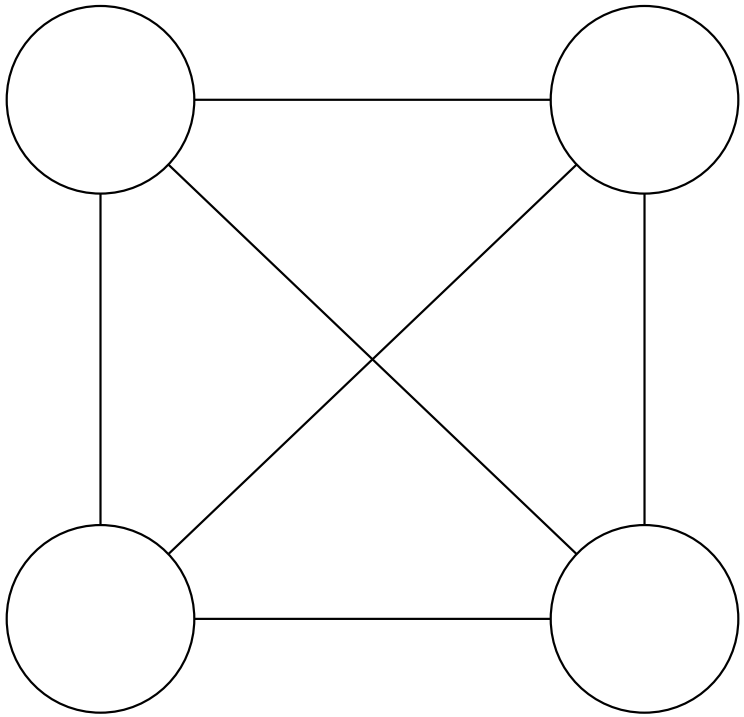

Goal: Prove than 3-Coloring is $\mathcal{NP}$-Complete

NAE-3SAT

NAE-kSAT Problem

$$\text{NAE-kSAT} = \{\langle \phi \rangle \mid \phi \text{ is in } k\text{-CNF and } \phi \text{ has a satisfying assignment s.t. each clause has at least one 0 and at least one 1}\}$$

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- x is an NAE-assignment of ϕ if $\phi(x) = 1$ and x does not assign all 1s or all 0s to any clause

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Lemma: If x is NAE-assignment of ϕ then $\overline{x}$ is NAE-assignment of ϕ

Proof:

- x must assign at least one 1 and at least one 0 to every clause
- $\overline{x}$ must also have at least one 1 and one 0 in every clause
- This means every clause is satisfied, and ϕ is satisfied since it's CNF

Goal

Prove that NAE-3SAT is $\mathcal{NP}$-complete: 3SAT $\leq_P$ NAE-3SAT

3SAT $\leq_P$ NAE-3SAT

- Turns out it's not easy to directly prove this
- Instead we take two steps:

$$\text{3SAT} \leq_P \text{NAE-4SAT} \leq_P \text{NAE-3SAT}$$

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    - If $S = 0$, then at least one $x_i = 1$, so $(x_1 \vee x_2 \vee x_3) = 1$
    - If $S = 1$, then $(\overline{x_1} \vee \overline{x_2} \vee \overline{x_3} \vee 0)$ is also NAE-assignment. So, $(\overline{x_1} \vee \overline{x_2} \vee \overline{x_3}) = 1$, so just flip all assignments in ϕ'

NAE-4SAT $\leq_P$ NAE-3SAT

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Theorem

$$3SAT \leq_P NAE\text{-}4SAT \leq_P NAE\text{-}3SAT$$

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$$\phi = (x_1 \vee \overline{x_2} \vee x_3) \wedge (\overline{x_2} \vee \overline{x_3} \vee x_4)$$

1 0 0 0 1 0

1, 1, 0, 0

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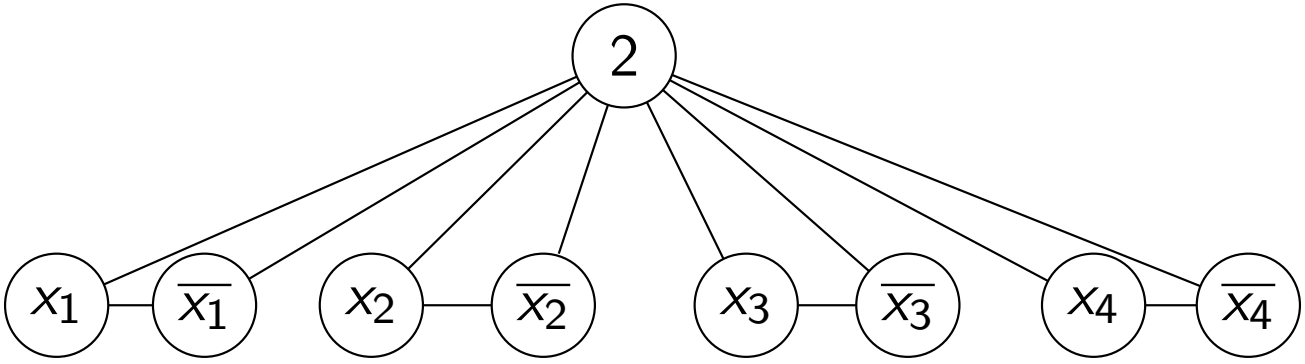

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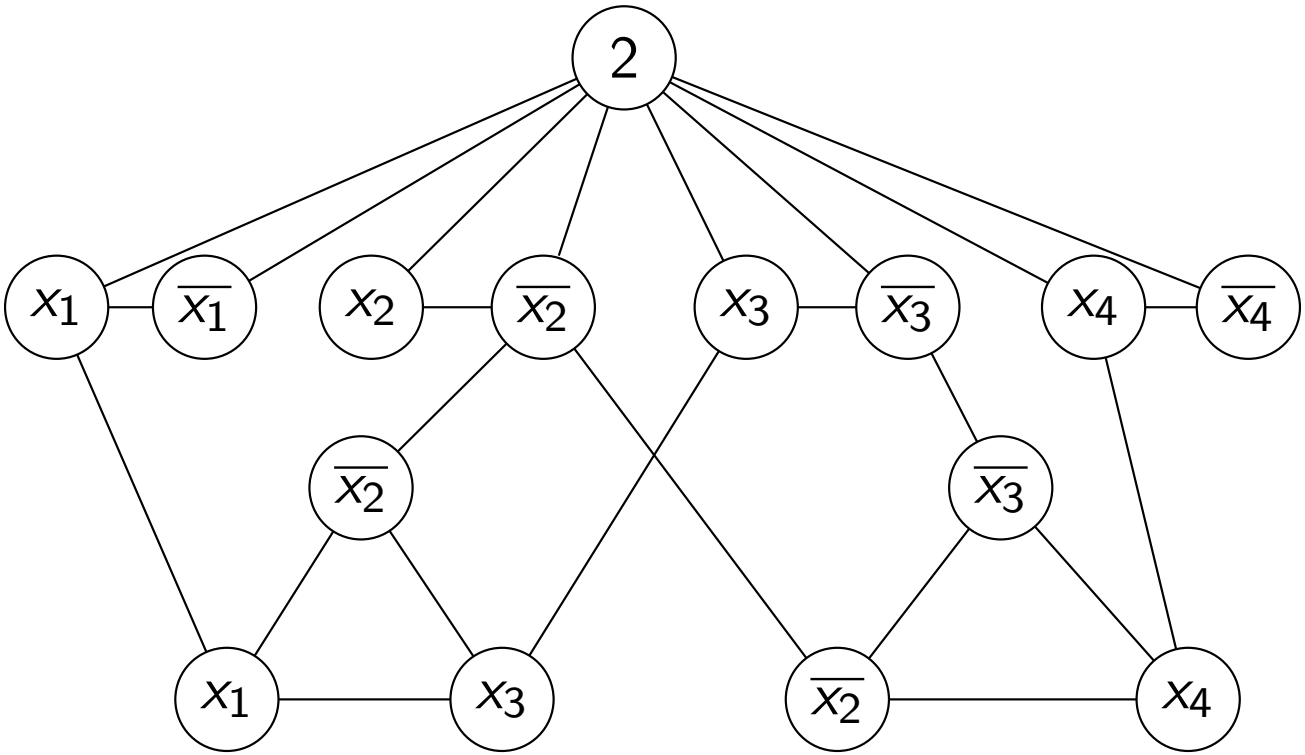

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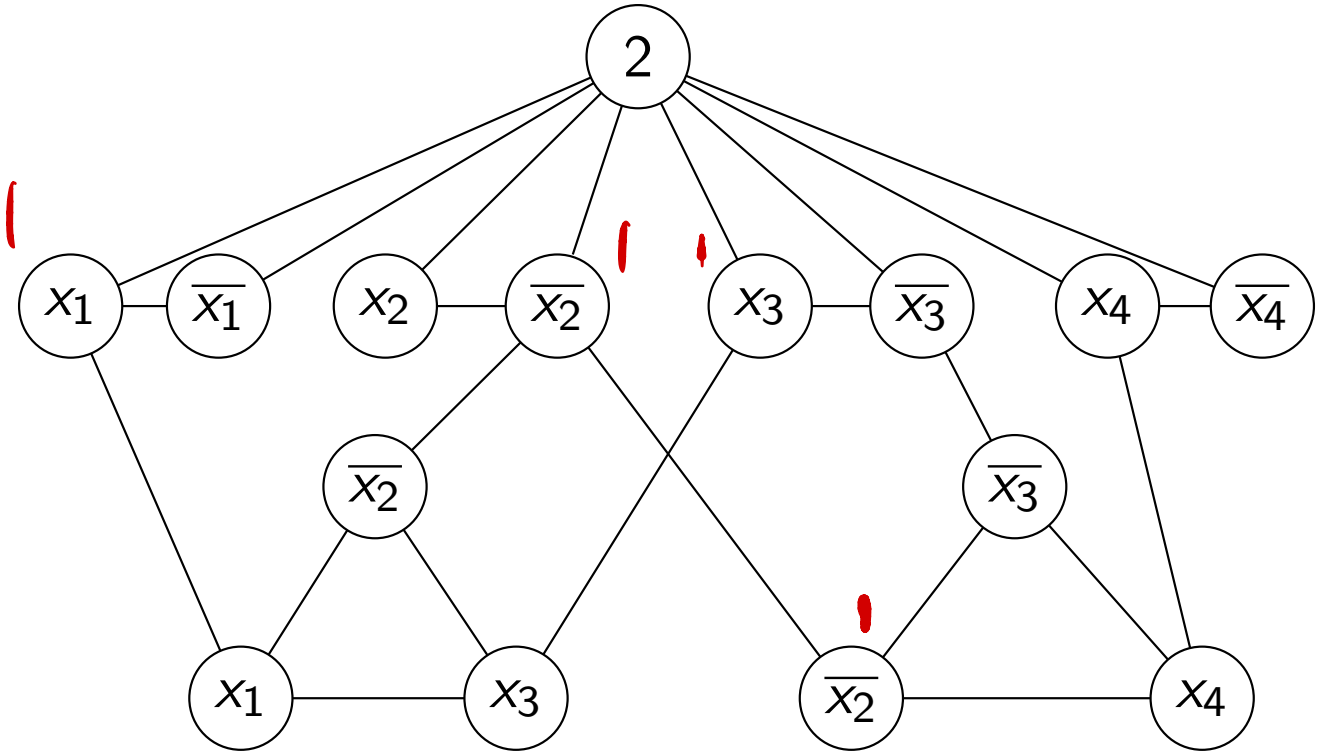

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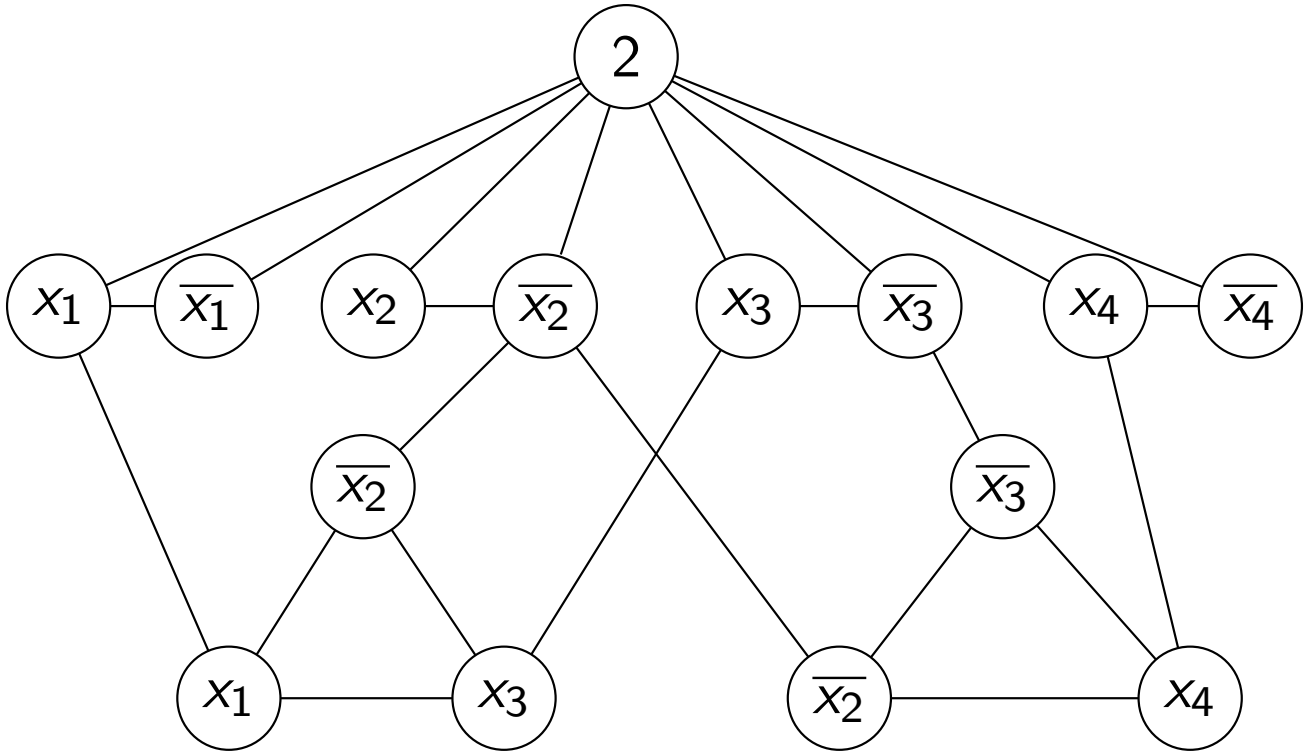

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  3-Coloring is $\mathcal{NP}$-complete, but 2-Coloring $\in \mathcal{P}$

Outline

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UNSAT

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Problems like UNSAT are in co-$\mathcal{NP}$

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$L \in \Sigma_2^p$ if there exists poly-time DTM V s.t. for $x \in L$, there exists a w_1 s.t. for all w_2, $V(x, w_1, w_2) = 1$

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We believe that there are infinitely many levels of the polynomial hierarchy and that $\Pi_i^p \neq \Sigma_i^p$ for $i > 0$, but can't prove it.

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Complexity Zoo

The complexity zoo (`https://complexityzoo.net/Complexity_Zoo`) now has 546 complexity classes.