Foundations of Computing

Lecture 4

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January 26, 2023

Outline

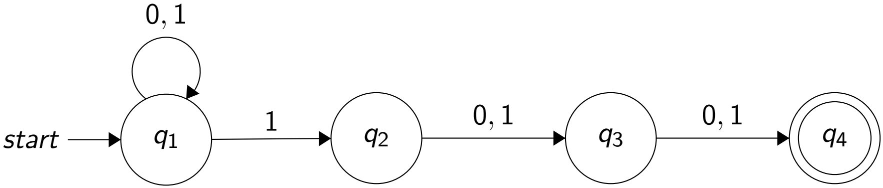
1. Lecture 3 Review
2. Example NFAs
3. Equivalence of NFAs and DFAs
4. Properties of Regular Languages Using NFAs
5. Regular Expressions

Lecture 3 Review

- Regular Languages
- Nondeterministic Finite Automata
- Understanding Nondeterminism

Outline

NFA Example 1



Question: What is $L(M)$?

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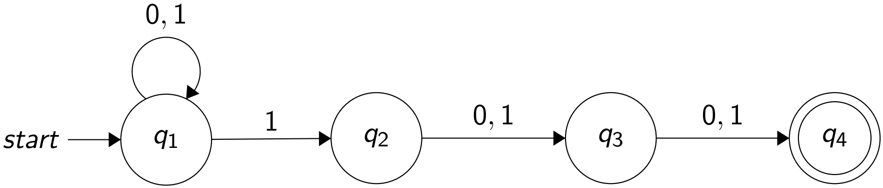

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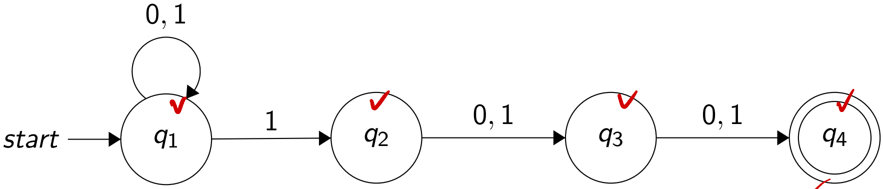

0, 1

$\boxed{\text{Trap}}$

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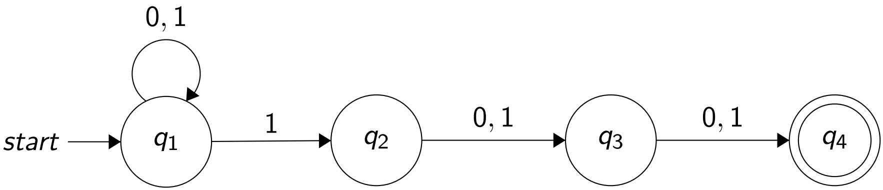

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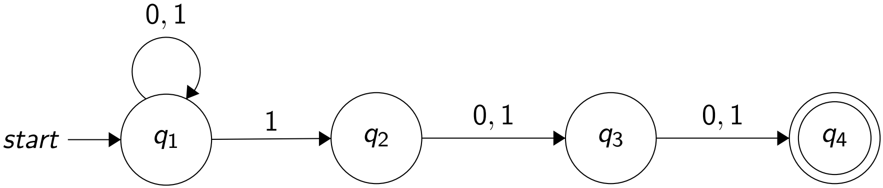

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- Uses the rest of the states to verify that 1 is third from the end
- DFA doing the same thing would have to track the last three bits seen – requires 8 states

Example 2 – OR statement

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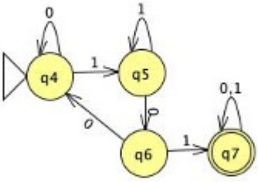

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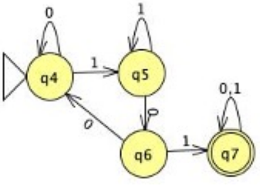
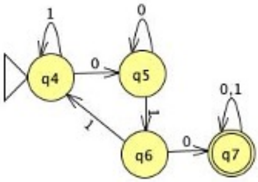


DFA for prop. (1)



DFA for prop. (2)

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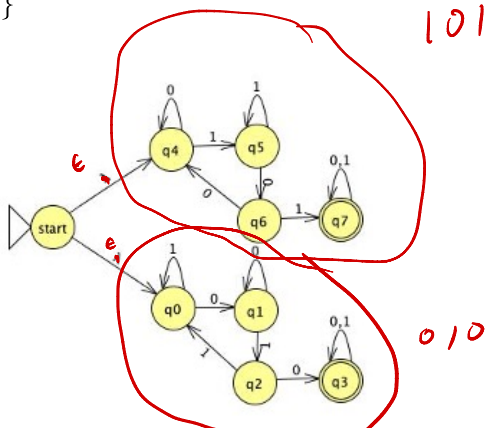

NFA for L

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Are NFAs more powerful than DFAs?

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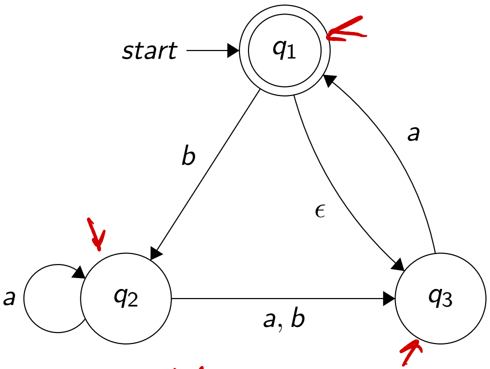

1. Does N accept $w = \epsilon$? Yes
2. Does N accept $w = aaa$? Yes
3. Does N accept $w = babba$? No
4. Does N accept $w = abaaba$? Yes

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DFAs == NFAs

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- Define δ to move to new set of highlighted nodes
- Accept states are ones in which at least one node is an accept node
- Can deal with ϵ edges by "placing more fingers" on resulting nodes

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Let N be an NFA recognizing L. Contruct DFA M recognizing L

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Look at transitions from all states in set R and map to set that gives results of all these transitions

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3. $q'_0 = \{q_0\}$
4. $F' = \{R \in Q' | R \text{ contains an accept state of } N\}$
   Accept if any state in R is an accept state

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Formally:

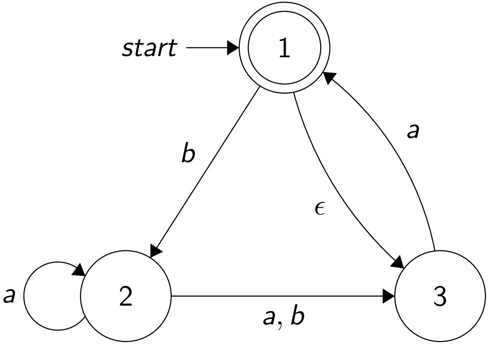
1. Let $E(R) = \{q | q \text{ can be reached from } R \text{ along } \epsilon \text{ arrows}\}$
2. Define extended transition function

$$\delta'(R, a) = \cup_{r \in R} E(\delta(r, a))$$

Map to set of states that can be reached on input a or $a\epsilon$

An Example: NFA $\rightarrow$ DFA

NFA $N = (Q, \Sigma, \delta, q_0, F)$



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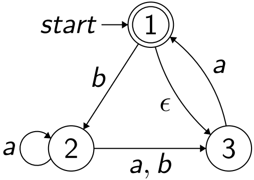

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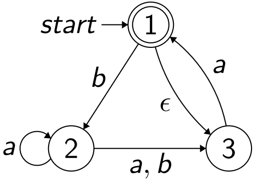

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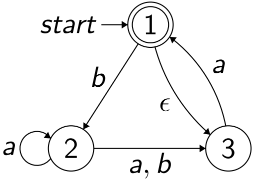

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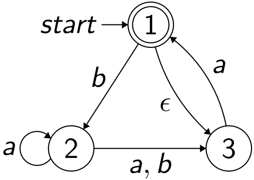

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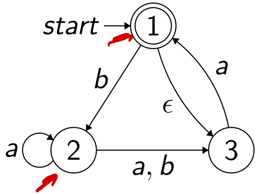

4. Transition function δ':

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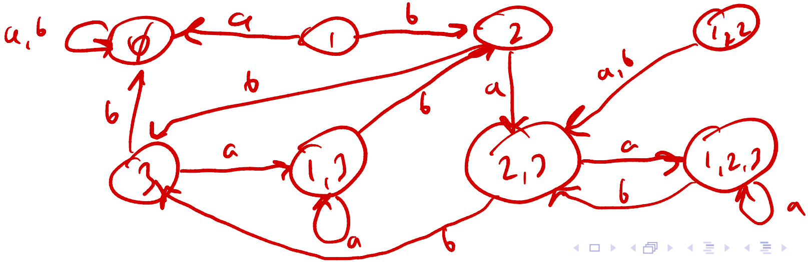

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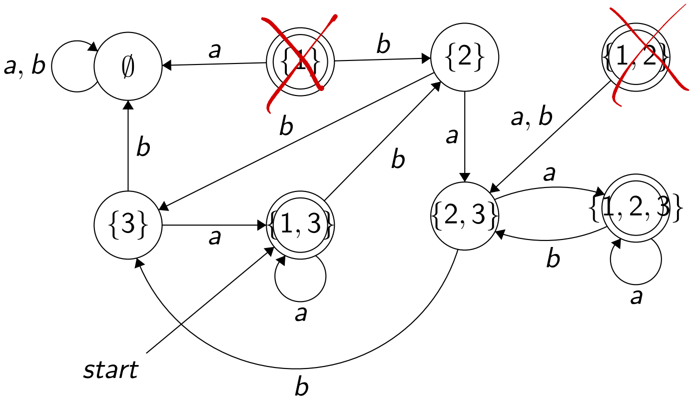

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A language L is regular if and only if there is a DFA that recognizes it

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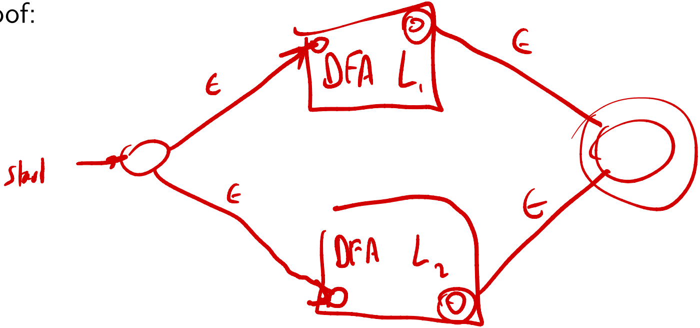
We can now use NFAs to argue the properties of regular languages

Closure Under Union

If L_1 and L_2 are both regular languages then $L_1 \cup L_2$ is also regular

$L_1 \cup L_2$ is the language consisting of all strings either in L_1 or L_2

Proof:



Closure Under Concatenation

If L_1 and L_2 are both regular languages then $L_1 \circ L_2$ is also regular

$L_1 \circ L_2 = \{xy | x \in L_1 \text{ and } y \in L_2\}$

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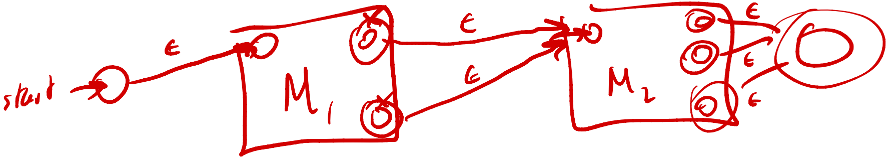

Closure Under the Star Operation

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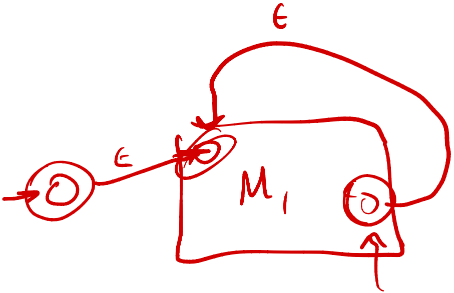
If L is a regular languages then L^* is also regular

$L^* = \{0 \text{ or more strings from } L\}$

$L = \{1\}$

Proof:



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You've seen this before

Regular expressions very useful in compilers, and string search (e.g., grep)

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5. $(R_1 \circ R_2)$ – R_1 concatenated with R_2 where R_1 and R_2 are regular expressions
6. (R_1^*) – 0 or more repetitions of R_1 where R_1 is a regular expression