Foundations of Computing

Lecture 9

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February 11, 2025

Outline

Midterm 1 – February 20

- Exam 1 will be in class on February 20 (next Thursday)
- It will cover NFA/DFA/regular languages, and PDAs/Context-free grammars

Exam Policies

- The exam will be closed book and closed notes
- You will be allowed two 8.5×11 pieces of paper with notes – anything you choose
- No computers, calculators, or other digital devices – bring a pencil or pen

Important

If you have a conflict with this exam, let me know ASAP!

Next Week

- Lecture and lab next week will be largely for review
- This is your chance to clear things up before the midterm

Bring your questions!

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Lecture 8 Review

- Pushdown Automata
  - Using a stack to recognize non-regular languages
  - Examples of building PDAs

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Today

An alternative formulation for languages accepted by PDAs

Exercise – Work in Groups

Show a PDA that recognizes the language

$$L = \{w \mid w \text{ has an equal number of 0s and 1s}\}$$

1. Describe a PDA algorithm for this language
2. Write the states and transition function
3. Draw the PDA graph

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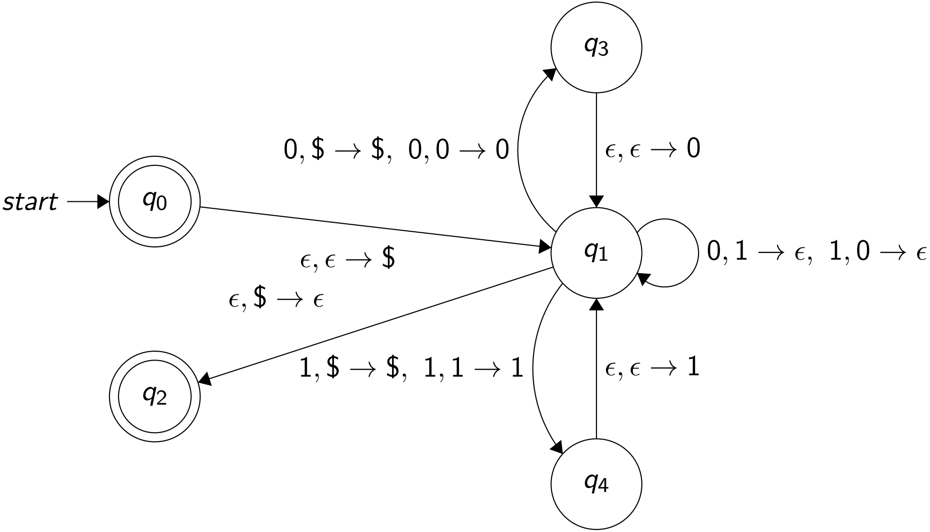
Solution:

1. Push \$ on the stack
2. If input is 0, pop value from the stack
   - If it's a 0 or \$ push it back on the stack and push another 0 on top
   - If it's a 1 pop it off the stack
3. If input is 1, pop value from the stack
   - If it's a 1 or \$ push it back and push another 1 on top
   - If it's a 0 pop it off the stack
4. When the input is done, if \$ is top of the stack, accept

Exercise – Work in Groups



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Representing Languages

Recall that a language L is a set of strings
We have seen several ways for describing a language L:

- DFA/NFA – the language of strings accepted by M
- Regular expressions
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- A grammar is a set of rules by which strings in L are constructed/derived

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Grammars

- A grammar is a set of rules by which strings in L are constructed/derived
- Today, we will focus on context-free grammars and the languages they represent

Grammar

A grammar G consists of:

- V – finite set of variables (usually Capital Letters)
- Σ – a finite set of symbols called the terminals (usually lower case letters)
- R – finite set of rules how strings in L can be produced
- $S \in V$ – start variable

If no S is specified, can assume it is the variable in the first rule.

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Definition

For a grammar G, the language L_G generated by G is the set of all terminal strings that can be produced by G starting with the start symbol by using a sequence of the production rules.

Example 1

Consider the following grammar G_1:

- $V = \{A, B\}$
- $\Sigma = \{0, 1, \#\}$
- $R =$

$$\begin{aligned} A &\rightarrow 0A1 \\ A &\rightarrow B \\ B &\rightarrow \# \end{aligned}$$

- $S = A$

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Rules Notation

- Rules can have multiple options separated by $|$ to indicate OR
- ϵ - empty string

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- This is called context-free since a variable (on left side of rule) always produces same output, regardless of "context"
- Context-free grammars originated in the study of human languages
- They capture recursive structures common in language (e.g., noun phrases can be made of verb-phrases and vice-versa)
- Also, very useful for describing programming languages:
  - Can capture matching, nested brackets:

    if $x > 3$ {
        if $y < 5$ {
            Do something
        }
    }

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This is Tricky

Designing CFGs is not natural, takes lots of practice

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Question

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3. Combine the two to give the grammar for the union

$$\begin{aligned} S &\rightarrow S_1 \mid S_2 \\ S_1 &\rightarrow 0S_11 \mid \epsilon \\ S_2 &\rightarrow 1S_20 \mid \epsilon \end{aligned}$$

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3. Concatenate the two to give the grammar for L

$$\begin{aligned} S &\to AC \\ C &\to aCb \mid \epsilon \\ A &\to aA \mid a \end{aligned}$$

Exercise

Give a CFG for $L = \{a^m b^n \mid m \neq n, m, n \geq 0\}$
Hint: Think of this as the union of two languages

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Why study parse trees?

- Parse trees help us understand the "meaning" of a string
- Also, how parsers can parse a string according to a grammar (e.g., of a programming language)

Parse Trees – An Example

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Parse tree for $000\#111$ in grammar G_1

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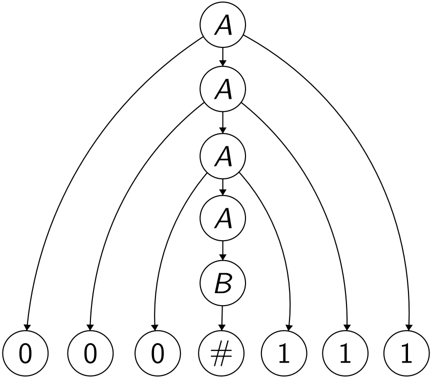

Another Example

A Grammar G_2 for Arithmetic Statements

- $V = \{\langle EXPR \rangle, \langle TERM \rangle, \langle FACTOR \rangle\}$
- $\Sigma = \{a, +, \times, (,)\}$
- $R =$

$$\begin{aligned} \langle EXPR \rangle &\rightarrow \langle EXPR \rangle + \langle TERM \rangle \mid \langle TERM \rangle \\ \langle TERM \rangle &\rightarrow \langle TERM \rangle \times \langle FACTOR \rangle \mid \langle FACTOR \rangle \\ \langle FACTOR \rangle &\rightarrow (\langle EXPR \rangle) \mid a \end{aligned}$$

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- What is $L(G_2)$?
- Parse tree for $a + a \times a$

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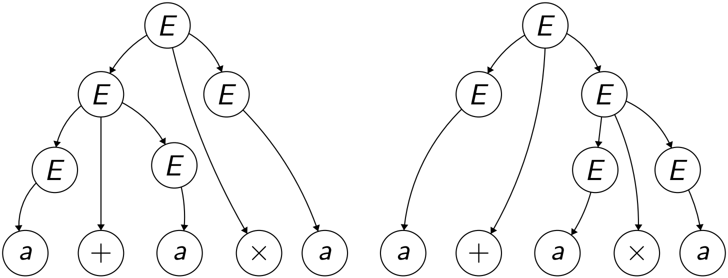
- Ambiguous derivation may lead to different meanings for the string
  Example: The girl touches the boy with the flower
- Unfortunately, ambiguous languages cannot be made unambiguous

An Example

Consider the following grammar G_3

$$E \to E + E \mid E \times E \mid (E) \mid a$$



Two parse trees for the string $a + a \times a$

On Thursday

- Equivalence between CFGs and PDAs
- A pumping lemma for CFGs