Foundations of Computing

Lab 11 – $\mathcal{P}$, $\mathcal{NP}$, $co-\mathcal{NP}$

April 16, 2025

Outline

Boolean Formulas

- Boolean formula: A Boolean formula of size n is a logic equation with n letters (e.g., $x_1, \overline{x_1}$)

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- Formula ϕ is in CNF if it is an AND-of-ORs formula

$$(x_1 \vee x_2) \wedge (x_3 \vee x_4)$$

Why Boolean Formulas

- Boolean formulas give us many interesting problems to study
- Formulas are easy to reason about, and you've seen them before.
- SAT, 3-SAT are $\mathcal{NP}$-complete

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1. Satisfiability of Boolean Formulas

2. Complexity Classes We've Seen

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Exercises:

1. Show that 2-COLORING $\in \mathcal{P}$
   2-Coloring is the problem given a graph G can you color it with 2 colors such that no edge has the same color on both ends.

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Example Problems: SAT, 3-SAT, 3-Coloring, Vertex Cover, etc.

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Exercises:

3. Show that 2-COLORING $\in$ co-$\mathcal{NP}$.
4. Give some other examples of languages in co-$\mathcal{NP}$, and justify why they are in co-$\mathcal{NP}$.