Foundations of Computing

Lecture 16

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Outline

Lecture 15 Review

- Countable and Uncountable Sets
  - Diagonalization
- Proving A_{TM} is Undecidable

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Another Way to Prove Undecidability

Reductions Between Problems

There is a reduction from a problem A to a problem B if we can use a solution to problem B to solve problem A

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- Equivalently, problem B is no easier than problem A

Reductions and Undecidability

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Suppose that $A \leq B$, then:

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- Suppose that B is decidable
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- But, this means that A is decidable by running the reduction using the decider machine for B.

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On input $\langle M, w\rangle$, R does the following: R is trying to decide if

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- Run $D(\langle M, w \rangle)$
- If D rejects – $M(w)$ doesn't halt – halt and reject
- if D accepts – $M(w)$ halts – Simulate $M(w)$ until it halts, and output whatever M outputs

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- If we can decide whether M' recognizes a regular language or not, can use that to decide whether M accepts w or not

1. On input $\langle M, w \rangle$ build M'
2. Run $D(M')$
3. if $D(M') = 1 \Rightarrow M(w) = 1$, if $D(M') = 0 \Rightarrow M$ does not accept w

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2. Run D on input $\langle M' \rangle$
3. Output what D outputs

$M(w) = 1$: M' accepts all strings $L(M') = \Sigma^* -$ Reg

M does not accept w : M' accept $0^n 1^n$ $L(M')$ - Non Reg.

Other Undecidable Languages – Exercise

$$EMPTY-STRING_{TM} = \{\langle M\rangle \mid M \text{ is a TM and } M(\epsilon) = 1\}$$

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1. To show that a problem is decidable – give an algorithm that always terminates and outputs the answer
2. To show that a problem is undecidable – give an algorithm (a reduction) that shows that this problem can be used to solve one of the undecidable problems

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Can reductions help us determine if a language is Turing-unrecognizable?

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Solution

We need to restrict what our reductions can do.

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Language A is mapping reducible to language B ($A \leq_m B$) if there is a computable function $f : \Sigma^* \to \Sigma^*$, where for every w,

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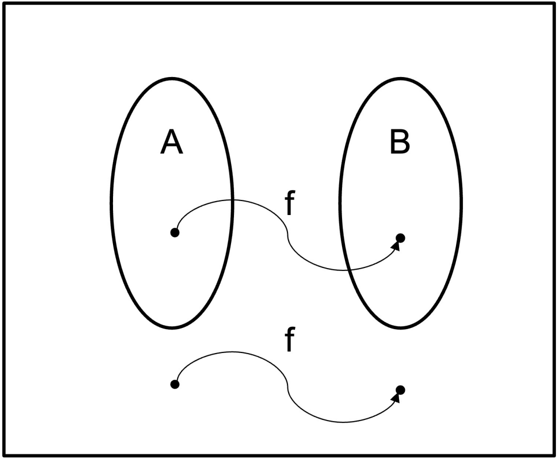
- Works by mapping input $\in A$ to input $\in B$ and vice-versa

Mapping Reductions



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- For example, in the proof that $L_{TM} \leq L_{E_{TM}}$, we flipped the result of R deciding $L_{E_{TM}}$

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4. If $A \leq_T B$
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    - If A is not Turing-recognizable, cannot say if B is Turing-recognizable