Foundations of Computing

Lecture 3

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January 21, 2025

Outline

Lecture 2 Review

- Language decided by DFA M
- Building DFAs
- Proving Correctness of DFAs

Outline

1. Lecture 2 Review
2. Regular Languages
3. Non-deterministic Finite Automata (NFA)
4. Example NFAs

From Machines to Languages

- Last lecture we saw how to build DFA M to decide a language L
- Learned to reason about machine M
- Recall that each machine M decides one language $L(M)$

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Let's switch our perspective

Instead of reasoning about machines, let's focus on languages decided by those machines.

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Definition

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Something to think about

Are all languages regular?

Properties of Regular Languages

Closure under Complement

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Intuition: Swap the accept and reject states

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Proof: Let $M = (Q, \Sigma, \delta, q, F)$ decide L

Construct $M' = (Q', \Sigma', \delta', q', F')$ that decides $\overline{L}$

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Intuition: Run both machines in parallel and accept if either of them stops in an accept state

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Intuition: Run both machines in parallel (same as for union) and accept if BOTH of them stop in an accept state

Closure Under Concatenation

If L_1 and L_2 are both regular languages then $L_1 \circ L_2$ is also regular

$L_1 \circ L_2 = \{xy | x \in L_1 \text{ and } y \in L_2\}$

Outline

Nondeterminism

Deterministic Finite Automaton

- For every state q and every symbol $x \in \Sigma$, exactly one value $\delta(q, x)$ is defined
- State transitions only on an input symbol
- Execution of DFA is fully determined

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- Allow multiple transitions for same state and symbol:
  e.g., $\delta(q1, 1) = \{q2, q3\}$
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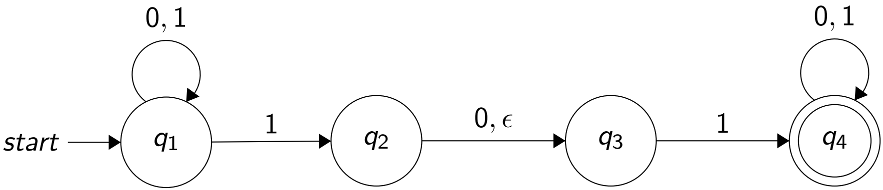
What is going on here?!?

What does non-determinism mean?

An Example NFA



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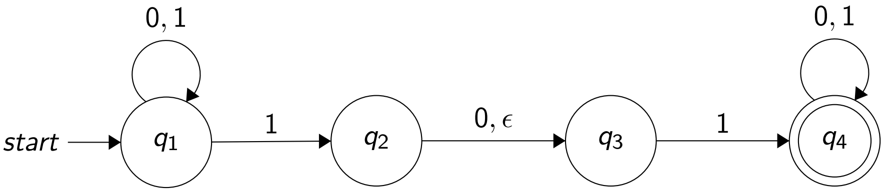

Input: 010

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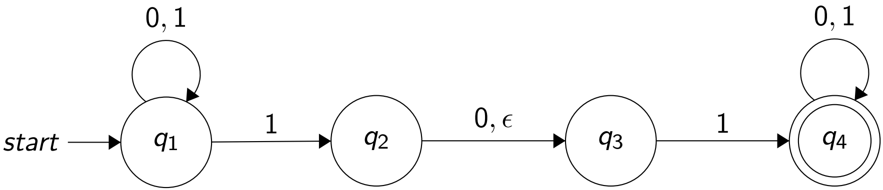

Input: 010
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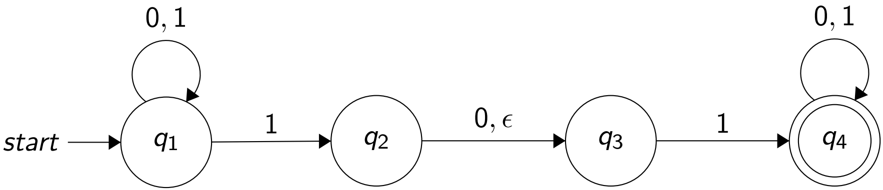
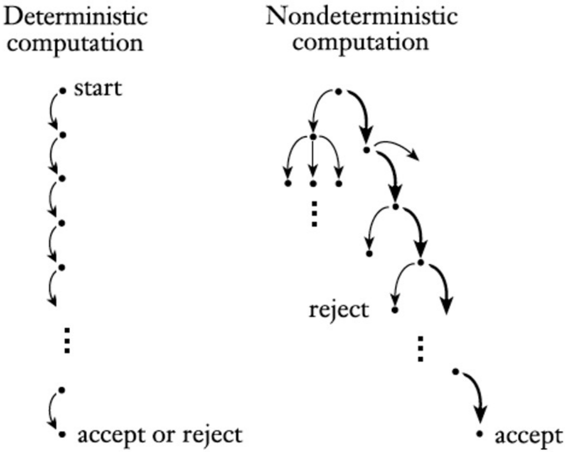


Input: 010
Input: 010110
Question: What language does this recognize?

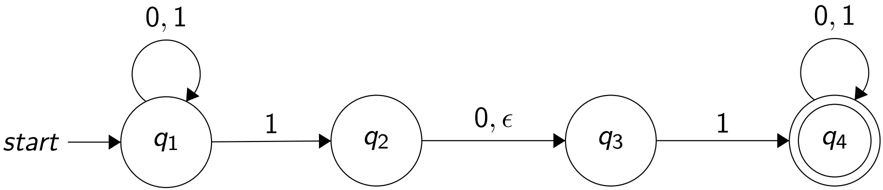
Understanding Nondeterminism

Interpretation 1: Try all paths in parallel



If any path leads to accept then accept

Interpretation 2: Guess and verify



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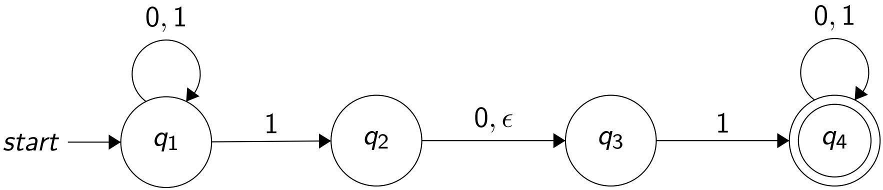

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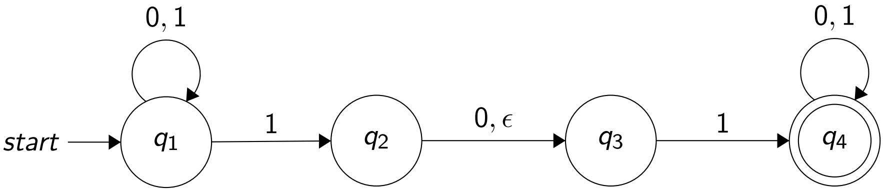

- M stays in q_1 until it "guesses" next input is 11 or 101
- Verifies that this guess was correct on path to q_4

Understanding Nondeterminism

Interpretation 3: Verifying a proof vs. finding a solution

Consider the execution of a finite automaton

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   - Input x alone does not necessarily take you to an accept state
   - Need to somehow choose which edge to take whenever there is a choice
   - Can view this sequence of nondeterministic choices as a "witness" w that allows you to verify that $x \in L(M)$

Important

For any $x \notin L$, there must be no path to an accepting state – no possible "witness" works

Nondeterministic Finite Automaton – Formal Definition

Nondeterministic Finite Automaton (NFA)

An NFA is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where:

- Q is a finite set of states
- Σ is a finite input alphabet
- $\delta : Q \times (\Sigma \cup \{\epsilon\}) \to P(Q)$ is the transition function
- $q_0 \in Q$ is the start state
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Recall:
$P(Q)$ is the power set of Q, i.e., the set of all subsets of Q

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Changes:

1. Transition function allows empty symbol (ϵ)
2. Output of transition function is a set of states $\in P(Q)$, not a single state in Q

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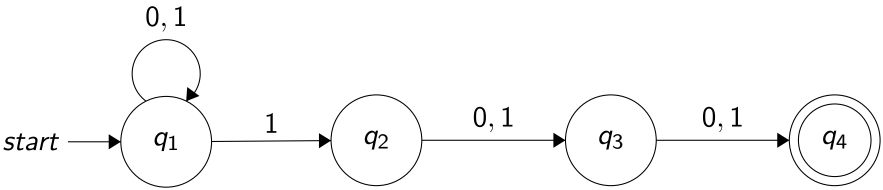
NFA Example 1



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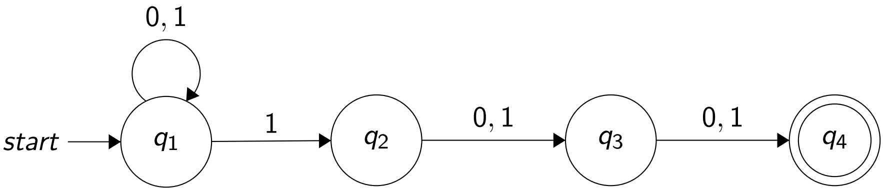

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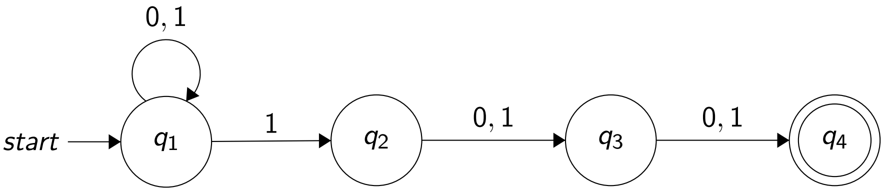

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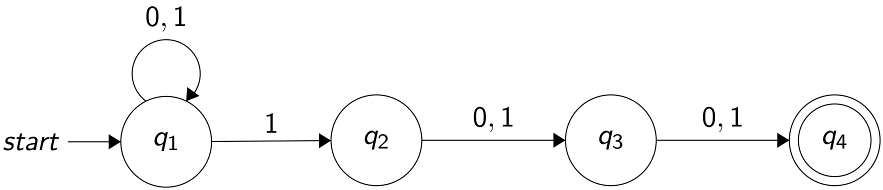

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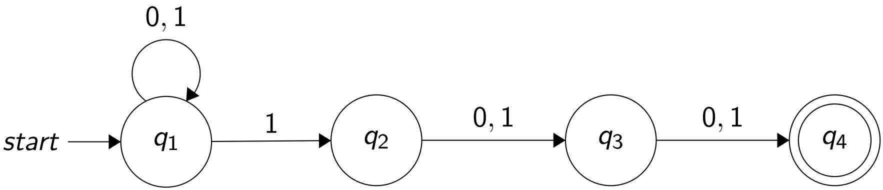

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- Uses the rest of the states to verify that 1 is third from the end
- DFA doing the same thing would have to track the last three bits seen – requires 8 states

Example 2 – OR statement

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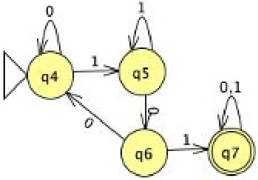

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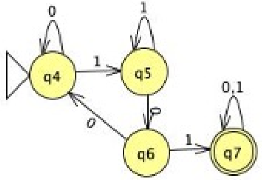

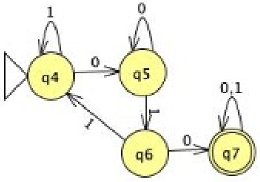

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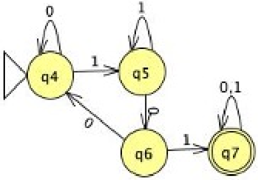
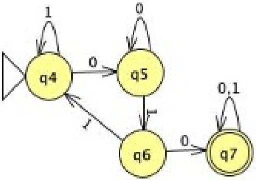
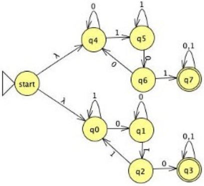


DFA for prop. (1)



DFA for prop. (2)



NFA for L