Foundations of Computing

Lecture 7

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Outline

1. Lecture 6 Review
2. The Pumping Lemma for Regular Languages
3. Using the Pumping Lemma
4. Using Closure Properties

Lecture 6 Review

- Nonregular languages
- Proving The NFA pumping lemma

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Today

Using the pumping lemma to prove languages are not regular.

HW2 Problem 4

Let L be a regular language, prove that the following languages are regular.

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2. $NOEXTEND(L) = \{w \in L | w$ is not a proper prefix of any string in $L\}$

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Example:

- $L = \{00, 11, 001, 101\}$
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- We saw how to prove the pumping lemma last week
- Today we will learn how to use it

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3. Choose $w \in L$ with $|w| \geq p$
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5. Contradiction!!!

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3. Easy to see that $0^* 1^*$ is regular
4. Since regular languages are closed under $\cap$, if L is regular then L_1 must be regular
5. Since we know L_1 is nonregular, this means that L must be nonregular

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   Equivalently, we need $k = (n-m)/\beta + 1$ to be an integer
7. We only know $\beta \leq p$, how can we guarantee $(n-m)$ is divisible by β?
8. Set $n = 2p!$, $m = p!$, can guarantee $(n-m) = p!$ is divisible by β, so there is k s.t. $xy^k z \notin L$

Hints for Using the Pumping Lemma

To use the pumping lemma, need to do the following

- Identify what it means for $x \notin L$
- Choose w such that any valid split xyz can lead to a contradiction
- Prove that $w' = xy^k z \notin L$ form some k

Choosing w is often tricky, requires intuition and some trial and error.

Outline

Using Closure Properties of Regular Languages

We have seen a number of closure properties of REs

1. Closure under complement: $\overline{L}$ is regular if L is
2. Closure under union: $L_1 \cup L_2$ is regular if L_1, L_2 are
3. Closure under intersection: $L_1 \cap L_2$ is regular if L_1, L_2 are
4. Closure under reversal: L^R is regular if L is
5. NOPREFIX, NOEXTEND
6. There are many more (e.g., set difference, cross product, ...)

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Important

- It is much easier to prove non-regularity using closure properties
- Try this first before you turn to pumping lemma

Exercise

Prove that the following language is nonregular:

$$L = \{0^i 1^j 2^i 3^j | i, j > 0\}$$

What's Next?

- We will add (a little) memory to our machines to recognize a richer class of languages