Foundations of Computing

Lecture 8

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February 9, 2023

Outline

1. Lecture 7 Review

2. Pushdown Automata

Lecture 7 Review

- Proving languages not regular
  - Using the pumping lemma
  - Using closure properties

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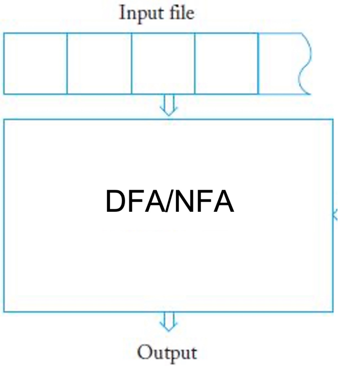
Today

Going beyond regular languages.

Outline

1. Lecture 7 Review
2. Pushdown Automata

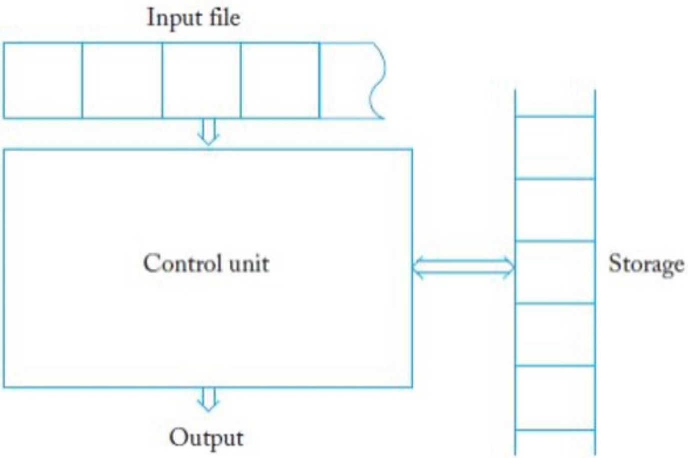
Let's Add Some Storage



Recall:

- An NFA/DFA has no external storage
- Only memory must be encoded in the finite number of states
- Can only recognize regular languages

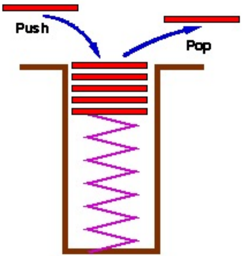
Let's Add Some Storage



Question

What kind of storage should we add?

A Stack



Let's add a Stack for storage

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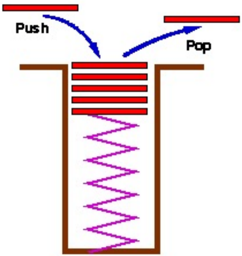

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A stack has the following operations:

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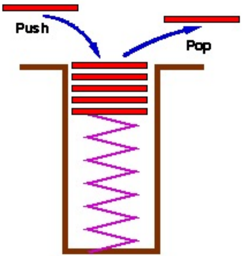

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A stack has the following operations:

- push value - push a value onto the top of the stack
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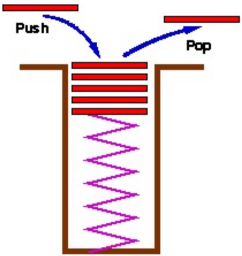

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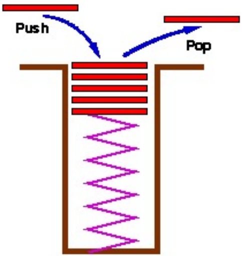

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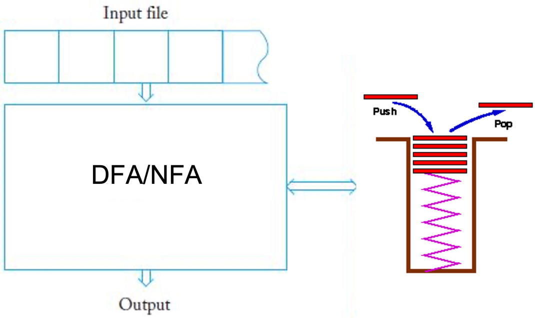
A stack is a Last-In First-Out (LIFO) data structure, that can hold an infinite amount of information

Pushdown Automata (PDA)



A PDA consists of:

- An NFA for a control unit
- A Stack for storage

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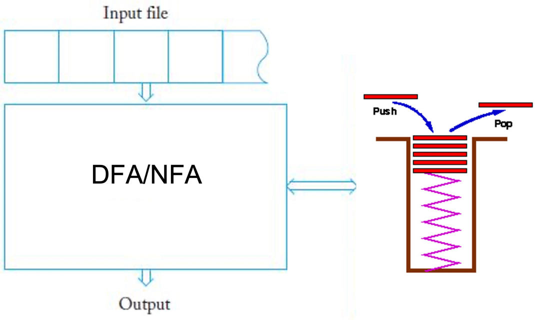

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Is this any more powerful than an NFA?

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Observations:

- Since the control is an NFA, ϵ transitions are allowed
- A PDA may choose not to touch the stack in a particular step
- Unlike the case for DFA/NFA, deterministic PDA's are not equal to non-deterministic ones. We will only study non-deterministic PDAs.

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Question

What language does this recognize?

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This recognizes

$$L = \{0^n 1^n \mid n \geq 0\}$$

Formal Definition of PDAs

A PDA M is a 6-tuple $(Q, \Sigma, \Gamma, \delta, q_0, F)$ where

- Q – set of state of the NFA
- Σ – input alphabet
- Γ – Stack alphabet
- $\delta : Q \times \Sigma_\epsilon \times \Gamma_\epsilon \rightarrow P(Q \times \Gamma_\epsilon)$ – transition function
- $q_0 \in Q$ – start state
- $F \subseteq Q$ – accept states

Recall that $P(Q \times \Gamma_\epsilon)$ is the power set of the set of pairs $\{(q \in Q, a \in \Gamma_\epsilon)\}$

Computing with a PDA – Formal Notation

A PDA M accepts a string $w = w_1 w_2 \cdots w_m$ with $w_i \in \Sigma_\epsilon$ if there exist

- A sequence of states $q_0, q_1, \ldots q_m \in Q$, and
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3. $r_m \in F$ – stop in an accept state

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Recall the PDA we described before:

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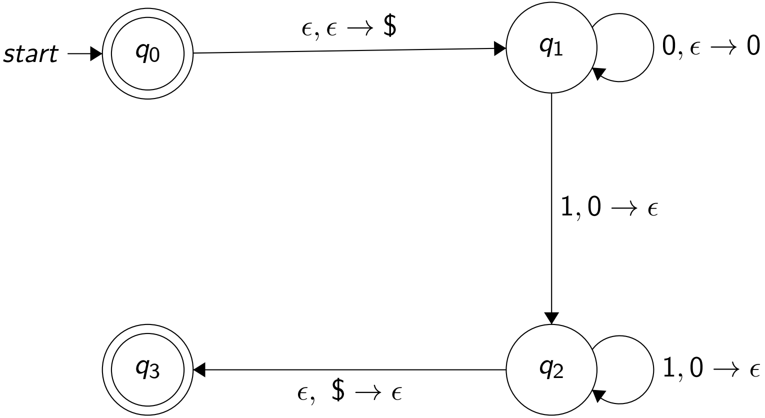
Transition Function

Input:	0			1			ϵ		
Stack:	0	\$	ϵ	0	\$	ϵ	0	\$	ϵ
q_0									$\{(q_1, \$)\}$
q_1			$\{(q_1, 0)\}$	$\{(q_2, \epsilon)\}$					
q_2				$\{(q_2, \epsilon)\}$				$\{(q_3, \epsilon)\}$	
q_3									

Table: Transition Function δ

Empty cells correspond to output of $\emptyset$

Example PDA as a Graph



Exercise – Work in Groups

Show a PDA that recognizes the language

$$L = \{w \mid w \text{ has an equal number of 0s and 1s}\}$$

1. Describe a PDA algorithm for this language
2. Write the states and transition function
3. Draw the PDA graph

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Solution:

1. Push \$ on the stack
2. If input is 0, pop value from the stack
   - If it's a 0 or \$ push it back on the stack and push another 0 on top
   - If it's a 1 pop it off the stack
3. If input is 1, pop value from the stack
   - If it's a 1 or \$ push it back and push another 1 on top
   - If it's a 0 pop it off the stack
4. When the input is done, if \$ is top of the stack, accept

Exercise – Work in Groups