

Foundations of Computing

Lecture 14

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March 4, 2025

Outline

- 1 Lecture 13 Review
- 2 Decidable and Turing-recognizable Languages
- 3 Languages With Machines as Input
- 4 Preliminaries – Countable and Uncountable Sets

Lecture 13 Review

- More Turing Machines
- Turing Machine Variants
 - Multi-tape Turing Machines
 - Non-deterministic Turing Machines

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- Are there some problems that inherently do not have any algorithmic solution?

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Observations:

- L is Turing-recognizable if can recognize yes instances, L is decidable if can also recognize no instances.
- Every Decidable language is also Turing-recognizable, but the reverse direction may not be true.

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A Second Question

What about Turing-unrecognizable languages?

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- This means that any such machine can be written down as a finite length string
- So, can give a description of a machine M to another machine M'
- We already talked about a universal TM which can run any other TM

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- 2 If simulation ends in an accept, then accept. If it ends in a non-accepting state, then reject

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- 1 Convert NFA B to equivalent DFA C
- 2 Run TM from previous slide on input $\langle C, w \rangle$
- 3 Output what this TM outputs

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$$A_{\text{REX}} = \{ \langle R, w \rangle \mid R \text{ is a reg. exp. that generates the string } w \}$$

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- If could determine that M will never halt (i.e, it has entered an infinite loop), could reject.

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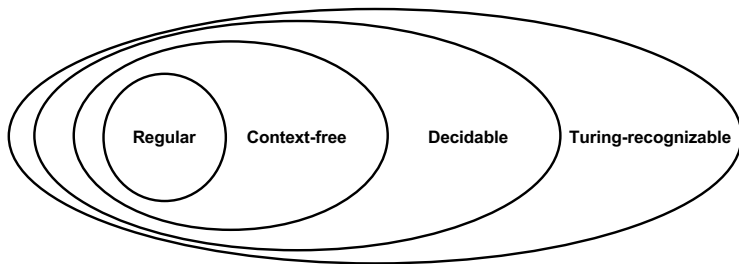
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- In this case, above algorithm will never output accept or reject
- If could determine that M will never halt (i.e, it has entered an infinite loop), could reject.
- But, how do we determine this?

Relationships Among Language Classes



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- 4 Preliminaries – Countable and Uncountable Sets**

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- $|S_1| = |S_2|$ if there's a one-to-one and onto mapping from S_1 to S_2
- Example:
 $A = \{0, 1, 2, 3\}$
 $B = \{a, b, c, d\}$
 $f(0) = a, f(1) = b, f(2) = c, f(3) = d$

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- An infinite set A is *countably infinite* if it has the same cardinality as the natural numbers: $\mathcal{N} = 1, 2, 3, \dots$
- A set A is countable if it is finite or countably infinite
- A set that is not countable is *uncountable*

Example 1: Evens

Evens

The set of even numbers is countable

Example 2: Rationals

Rationals

The set of rational numbers is

Example 2: Rationals

Rationals

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The set of rational numbers is countable

	1	2	3	...
1	1/1	1/2	1/3	
2	2/1	2/2	2/3	...
3	3/1	3/2	3/3	
4	4/1	4/2	⋮	

Example 3: Strings

Strings

The set of strings in $\{0, 1\}^*$ is countable

Diagonalization

Real Numbers

The set of real numbers (\mathcal{R}) is uncountable

Diagonalization

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Proof: By diagonalization

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- Then there is a one-to-one and onto mapping f from \mathcal{N} to \mathcal{R}

Diagonalization

Real Numbers

The set of real numbers (\mathcal{R}) is uncountable

Proof: By diagonalization

- Assume that \mathcal{R} is countable
- Then there is a one-to-one and onto mapping f from \mathcal{N} to \mathcal{R}

n	$f(n)$
1	1.234...
2	3.141...
3	5.556...
\vdots	\vdots

Diagonalization

Real Numbers

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- We construct a value $x \in \mathcal{R}$ s.t $x \neq f(n)$ for any n
Idea: For all $i \in \mathcal{N}$, make $x_i \neq f(i)_i$

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- We construct a value $x \in \mathcal{R}$ s.t $x \neq f(n)$ for any n
Idea: For all $i \in \mathcal{N}$, make $x_i \neq f(i)_i$
- Contradiction – f is not mapping between \mathcal{R} and \mathcal{N}