Foundations of Computing

Lecture 23

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Outline

Lecture 22 Review

- More $\mathcal{NP}$-complete problems
- The class co-$\mathcal{NP}$

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Are All Problems in $\mathcal{NP}$?

Question

Do all languages have poly-size proofs?

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- We define co-$\mathcal{NP}$ to contain all such languages that are complements of languages in $\mathcal{NP}$

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Question:

Can you prove that $x \in L$, when $L \in$co-$\mathcal{NP}$?

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- $\mathcal{NP} \neq$ co-$\mathcal{NP}$

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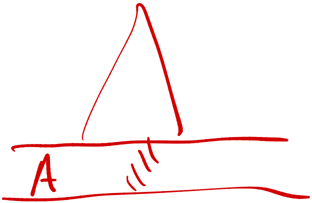
- Can be an interactive procedure
- The verifier (and prover) can use randomness to decide whether to accept

An Example – Aladdin's Cave



S: flip a coin
if heads, come out on L
if tails, come out on R

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The Class $\mathcal{IP}$

Definition

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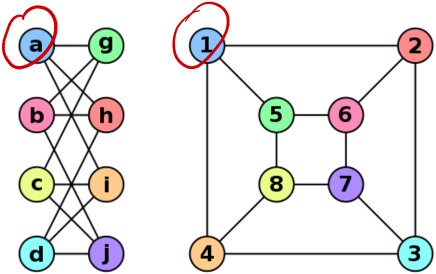
- $\mathcal{NP} \subseteq \mathcal{IP}$

Another Example – Graph Isomorphism



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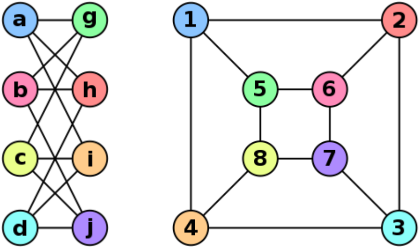

Claim

Graph Isomorphism $\in \mathcal{IP}$

Graph Non-Isomorphism

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   - Thus, $\Pr[b' = b] = 1/2$

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  I.e., GNI $\notin \mathcal{NP}$
- The power of interaction and randomness has allowed us to do what we couldn't do before

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3. P^* wins with probability $\leq 1/2$ in each run, so

$$\Pr[\langle P^*, V\rangle(x) = 1] \leq 1/2^n$$