Foundations of Computing

Lecture 13

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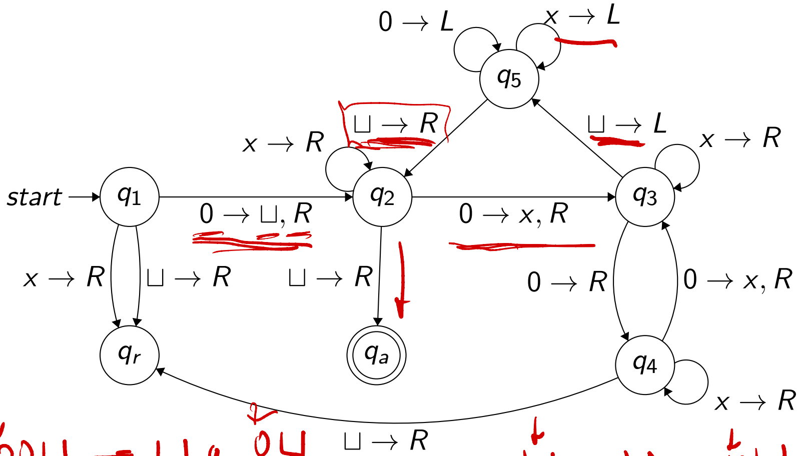
February 27, 2025

Outline

Lecture 12 Review

- Turing Machines
  - Definition
  - Examples

Running M on $w = 00$



$q_1 00\sqcup \to \sqcup q_2 0\sqcup \to \sqcup x q_3 \sqcup \to \sqcup q_5 x \sqcup$

$\to q_5 \sqcup x \sqcup \to \sqcup q_2 x \sqcup \to \sqcup x q_2 \sqcup$

$\to \sqcup x \sqcup q_a$

Outline

Specification of a Turing Machine

There are several levels of detail for specifying a TM

1. Full specification
    - Give full detail of transition function δ
    - This is very tedious

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   - Explain algorithmically what happens on the tape
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   - Explain algorithmically what happens on the tape
   - For example, scan the tape until you find a $\#$, zig-zag on the tape, etc.
   - Don't bother specifying a DFA for the control state
3. Algorithm specification
   - Give algorithm in pseudocode
   - Don't explicitly spell out what happens on the tape

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Machine M deciding L

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5. If all a's are crossed off, check if all c's are crossed off. Accept if yes, reject if no.

Example 2 – Build a TM deciding L Below

$$L = \{\#x_1\#x_2\#\cdots\#x_\ell \mid \text{ each } x_i \in \{0,1\}^* \text{ and } x_i \neq x_j \text{ for all } i \neq j\}$$

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5. Goto step 3

Outline

Turing Machines and Algorithms

Church-Turing Thesis (1936)

Anything that can be computed by an algorithm can be computed by a Turing Machine

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Observations:

- While unproven, all modern computers satisfy Church-Turing thesis
- To prove that some problem cannot be solved by an algorithm, enough to reason about Turing Machines
- This means that Turing Machines give an abstraction to capture "feasible computation"

Outline

A Universal Turing Machine

Question

A TM that can run any other TM

$$\hat{M}(M, x) = M(x)$$

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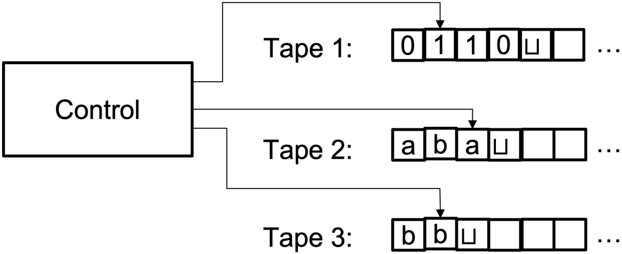
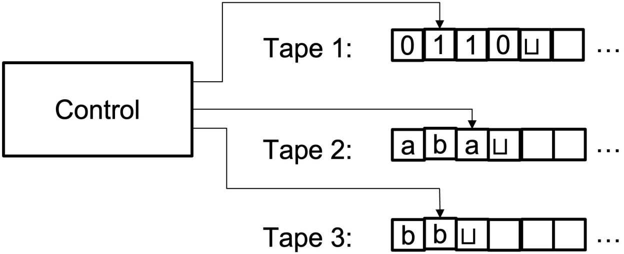
Observations:

- $\hat{M}$ can take any TM M as an input
- In particular $|\hat{M}|$ can be much less than $|M|$

Multi-Tape Turing Machines



Multi-Tape Turing Machines



In each step:

- M can read each tape
- M can write to each tape
- M can move each tape head Left or Right

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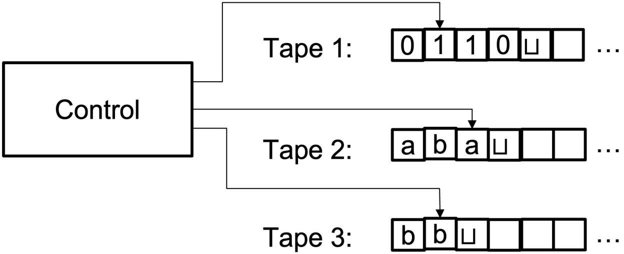

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Formally, for k tapes

$$\delta : Q \times \Gamma^k \to Q \times \Gamma^k \times \{L, R\}^k$$

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Theorem

Every multi-tape TM has an equivalent single-tape TM

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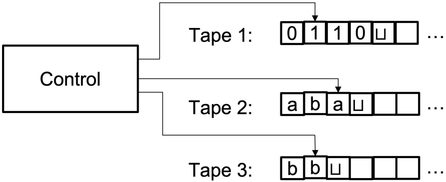
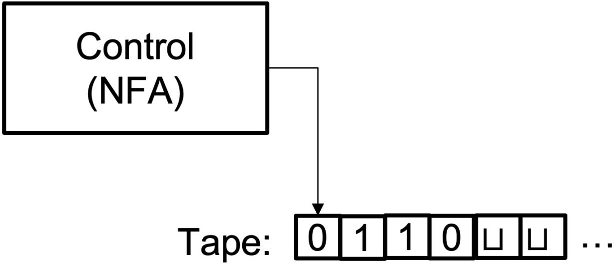


0 1 1 0 ⊔ # a b a ⊔ # b b ⊔

Nondeterministic Turing Machines



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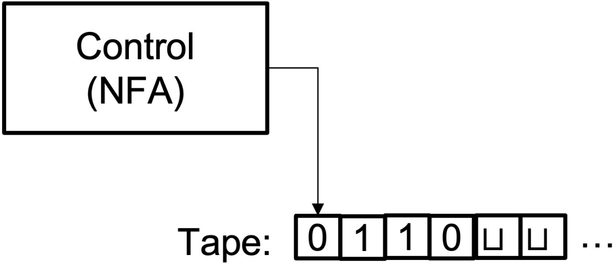

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$$\delta : Q \times \Gamma \to \mathcal{P}(Q \times \Gamma \times \{L, R\})$$

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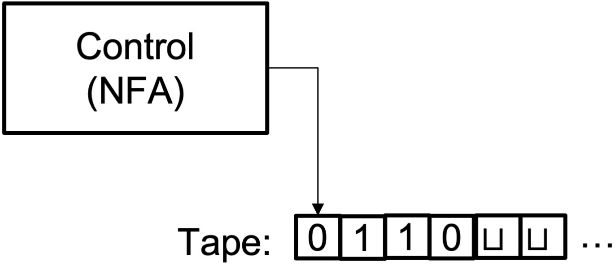

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Intuition:

- The control unit is non-deterministic - many transitions possible on each input
- Execution corresponds to a tree of possible executions
- Accept if any of possible execution leads to accept

Nondeterministic Turing Machine

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Every nondeterministic TM has an equivalent deterministic TM.

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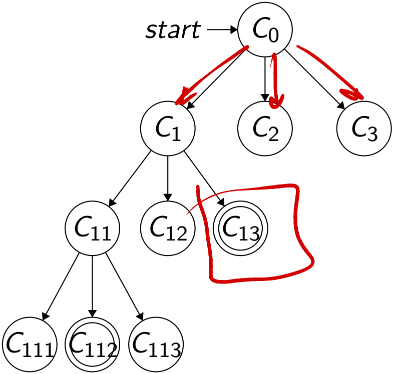

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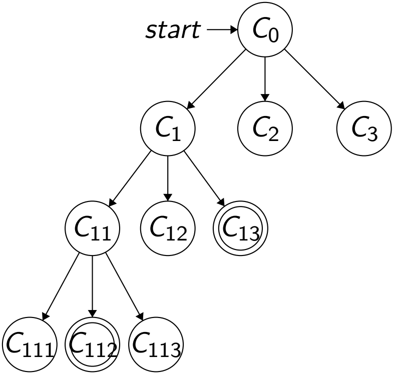

- Recall that an execution of a DTM is a sequence of configurations

$C_0 \to C_1 \to C_2 \cdots \to C_a$

Nondeterministic Turing Machine

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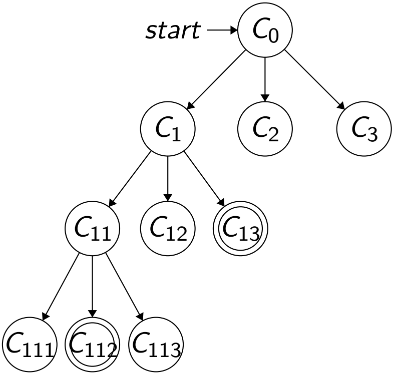

- Recall that an execution of a DTM is a sequence of configurations
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Nondeterministic Turing Machine

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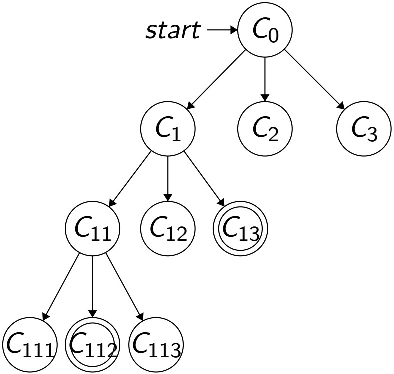

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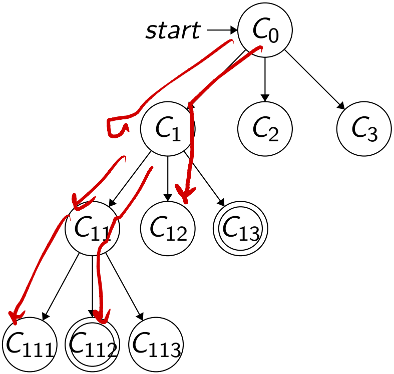
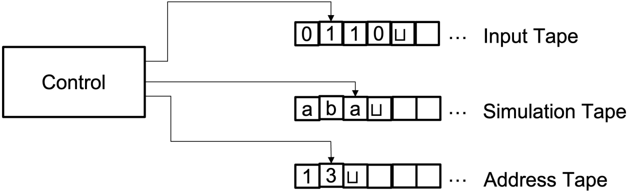


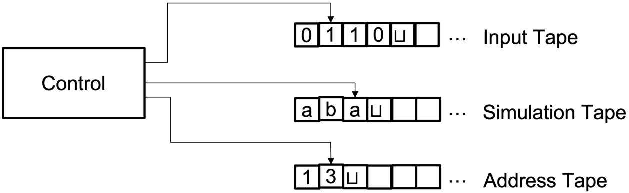
- Recall that an execution of a DTM is a sequence of configurations
- Execution of an NTM is a tree of configurations (branches correspond to non-deterministic choices)
- If any node in the tree is an accept node, the NTM accepts
- To simulate an NTM by a DTM, need to try all configurations in the tree to see if we find an accepting one

Nondeterministic Turing Machines



To simulate an NTM N by a DTM D, we use three tapes:

Nondeterministic Turing Machines



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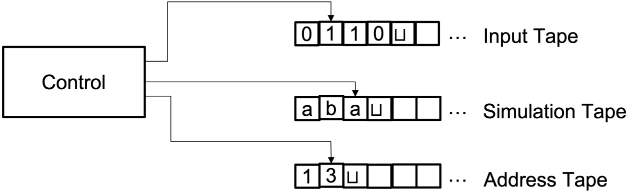
1. Input tape – stores the input and doesn't change

Nondeterministic Turing Machines



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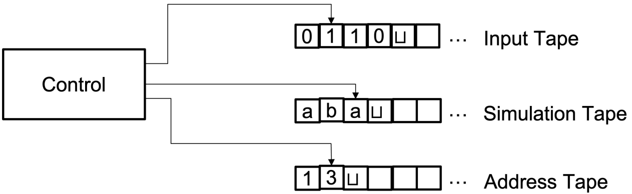

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3. Address tape – use to store which nondeterministic branch you are on

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Simulating an NTM N

1. Start with input w on tape 1, and tapes 2,3 empty

Nondeterministic Turing Machines

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Nondeterministic Turing Machines

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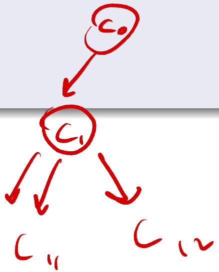
3. Use tape 2 to simulate a run of N. Whenever it needs to make a non-deterministic choice, see next symbol on tape 3 for which branch to take. If no symbols left, go to step 4



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Important

Must traverse NTM tree in breadth-first, not depth-first order

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Important

Must traverse NTM tree in breadth-first, not depth-first order

- Depth-first traversal may get stuck in an infinite loop, and not detect terminating branch

Next Week

- Decidable and undecidable languages
- I.e., are there things that no TM can compute?