Foundations of Computing

Lecture 19

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Question

Suppose we want to solve a problem in real life, is knowing that it is decidable enough?

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Complexity

The study of decidability under bounded models of computation

Outline

- To measure runtime of an algorithm, we need to count the number of steps it takes

1. $f(n) = 5n^2 \log n + 10n + 6$

2. $f(n) = \begin{cases} 6n & \text{for } n < 1000 \\ 3n^3 & n \geq 1000 \end{cases}$

3. $f(n) = \begin{cases} 2^n & n < 10000000 \\ 5n & n \geq 100000000 \end{cases}$

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- We write $f = O(n^3)$

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Definition

Let $f, g : \mathbb{N} \to \mathbb{R}$, we say that $f(n) = O(g(n))$ if

- There exist positive integers c, n_0 s.t. for all $n \geq n_0$

$$f(n) \leq cg(n)$$

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- Note that $f(n) = O(n^4)$

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Time Complexity Classes

Let $t : \mathbb{N} \to \mathbb{N}$. Define time complexity class $TIME(t(n))$ as

$$TIME(t(n)) = \{L \mid L \text{ is a language decided by an } O(t(n)) \text{ time TM}\}$$

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$O(n)$

Important

Time complexity depends on the exact model of computation

Dependence on Model of Computation

Theorem

For any function $t(n) \geq n$, every multi-tape TM (with $O(1)$ tapes) running in time $t(n)$ has an equivalent 1-tape TM running in time $O(t^2(n))$.

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  - $f(n) = 2^n$: If $n = 1000$, $f(n) >$ number of atoms in the universe

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- All "reasonable" deterministic computation models are polynomially equivalent

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  - $f(n) = 2^n$: If $n = 1000$, $f(n) >$ number of atoms in the universe
- All "reasonable" deterministic computation models are polynomially equivalent
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Efficient Computation

We define computation to be efficient if it runs in time bounded by some polynomial of the input size n

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Outline

1. Polynomial Time
2. The Complexity Class $\mathcal{P}$

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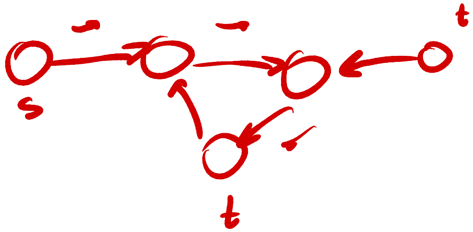
- $\mathcal{P}$ has nice closure properties

Problems in $\mathcal{P}$

PATH problem

$PATH = \{\langle G, s, t\rangle \mid G \text{ is a directed graph that has a path from } s \text{ to } t\}$



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$O(\log a)$ recursion

$O(|a|)$

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- We often more naturally think of computation as search problems (i.e., find a path from s to t)
- For some complexity classes, but not all, the two are equivalent – we will talk about this more later

Next Class

- Nondeterministic computation and the class $\mathcal{NP}$

$$P \stackrel{?}{=} NP$$