Foundations of Computing

Lecture 12

Arkady Yerukhimovich

February 25, 2025

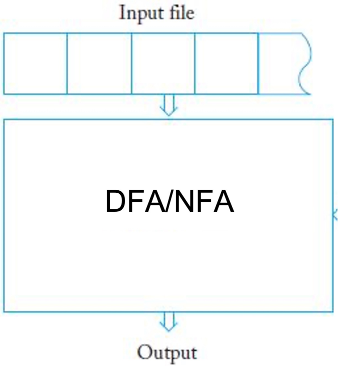
Outline

Lecture 10+11 Review

- Equivalence of CFGs and PDAs
- CFL Pumping Lemma
- Using the CFL Pumping Lemma

Outline

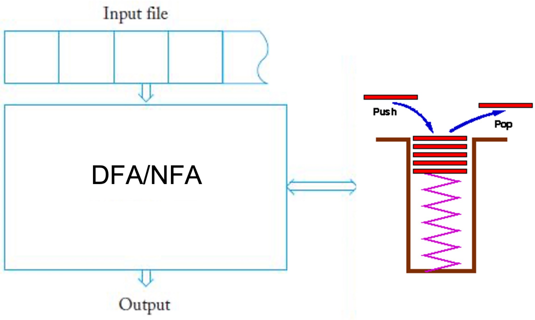
Finite Automata



Recall:

- An NFA/DFA has no external storage
- Only memory must be encoded in the finite number of states
- Can only decide regular languages

Pushdown Automata (PDA)



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- A Stack for storage

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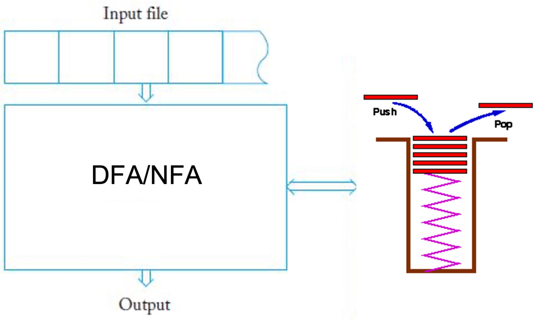

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Recall:

- Can only access memory in LIFO fashion
- Can only decide context-free languages

A Universal Computer

Question

All the prior machines couldn't decide some simple languages. Can we develop a machine that captures everything that can be computed?

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Our Goal

One model to rule them all!

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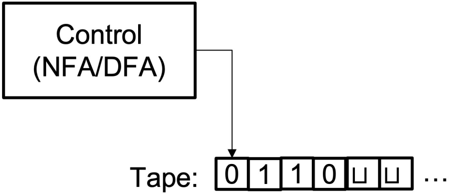
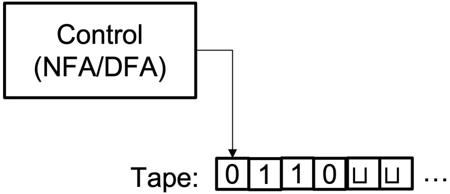
The Turing Machine



The Turing Machine



Key Differences:

- A TM can read and write to its tape

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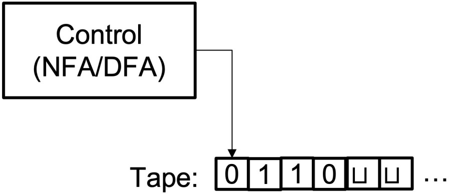

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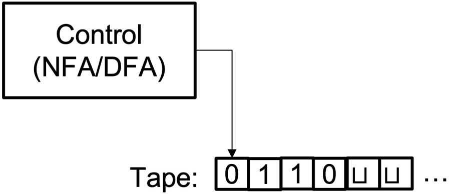

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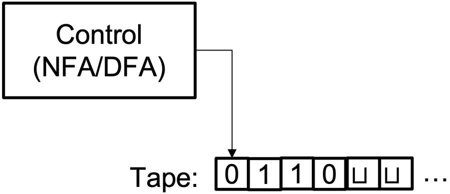

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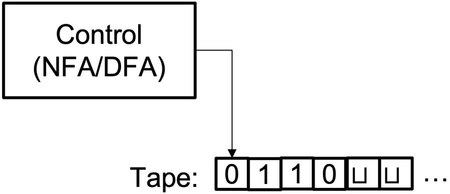

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- The memory tape is infinite
- Control FA has accept and reject states that are immediately output if entered

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On input string s (written on the tape):

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Algorithms are critical to understand solutions / complexity of a problem

- To show how to solve a problem, we design an algorithm
- To reason about languages accepted by NFA/PDA, we designed algorithms
- How can we reason about the limits of what an algorithm can compute?

Outline

Turing Machine – Formal Definition

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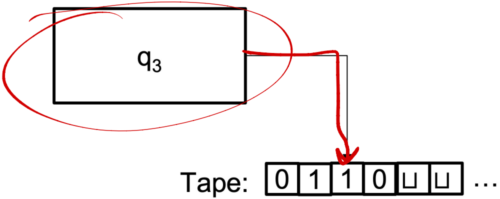
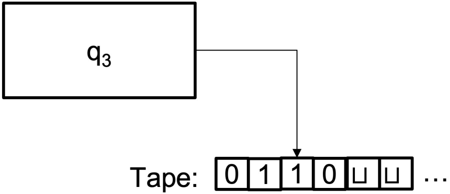
- write γ' to the tape,
- and move the tape head one spot to either Left or Right



Configuration of a TM

Computing on a Turing Machine



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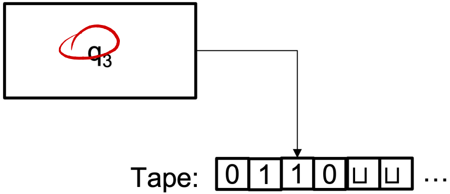

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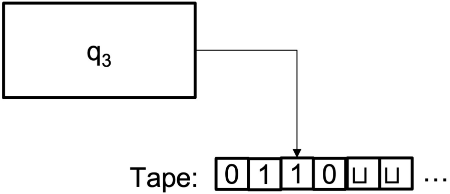

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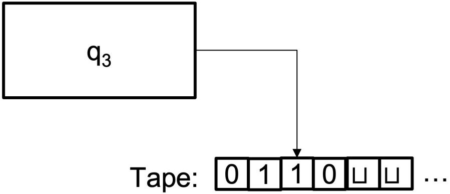

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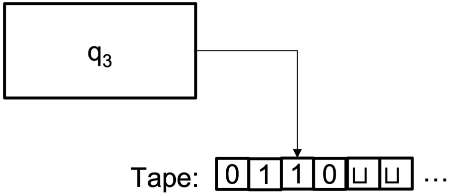

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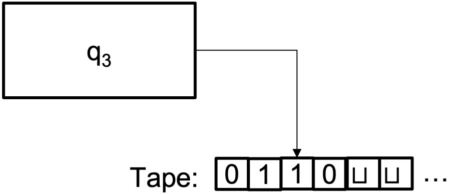

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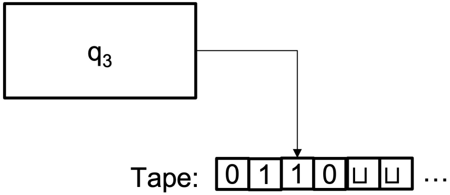

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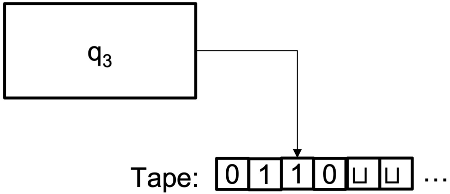

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- halting configuration – accepting or rejecting configuration

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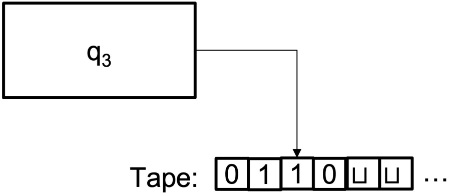

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Language $L(M)$

The collection of strings that M accepts

An Example

L = {0, 00, 0000, 8 0's, 16 0's, ...}

Consider $L = \{0^{2^n} \mid n \geq 0\}$

TM algorithm M for recognizing L:
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Making M Formal

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- $\Sigma = \{0\}$
- $\Gamma = \{0, x, \sqcup\}$
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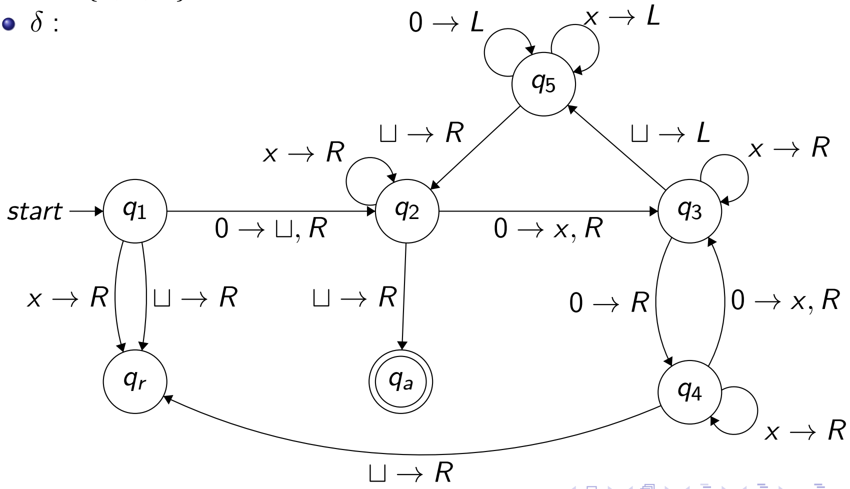
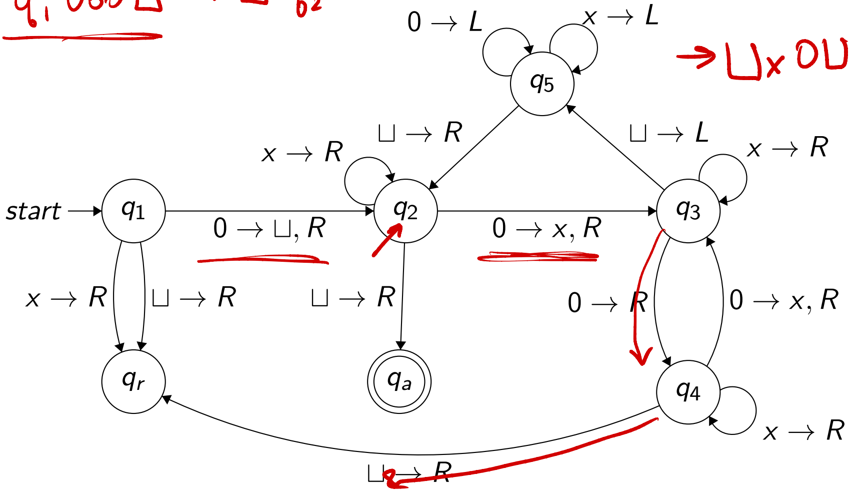


Running M on $w = 0000$

$w = 000$

$q_1 000\sqcup \rightarrow \sqcup q_2 00\sqcup \rightarrow \sqcup x q_3 0 \sqcup \rightarrow \sqcup x 0 q_4 \sqcup \rightarrow \sqcup x 0 \sqcup q_r$



Outline

A Universal Turing Machine

Question

Can we build a single TM that can evaluate all TMs?

$$\hat{M}(M, x) = M(x)$$

$\hat{M}(Q, \delta, \Sigma, \Gamma, q_0, q_a, q_r)^{\Sigma} \times \sqcup)$

q_j

0

A Universal Turing Machine

Question

Can we build a single TM that can evaluate all TMs?

$$\hat{M}(M, x) = M(x)$$

Hint: Recall that $M = (Q, \Sigma, \Gamma, \delta, q_0, q_{accept}, q_{reject})$

Turing Machines and Algorithms

Church-Turing Thesis (1936)

Anything that can be computed by an algorithm can be computed by a Turing Machine

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Observations:

- While unproven, all modern computers satisfy Church-Turing thesis

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- While unproven, all modern computers satisfy Church-Turing thesis
- To prove that some problem cannot be solved by an algorithm, enough to reason about Turing Machines

Turing Machines and Algorithms

Church-Turing Thesis (1936)

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- This means that Turing Machines give an abstraction to capture "feasible computation"

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