Foundations of Computing

Lecture 22

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Outline

1. Lecture 21 Review
2. More $\mathcal{NP}$-Complete Problems
3. co-$\mathcal{NP}$

Lecture 21 Review

- $\mathcal{P}$ and $\mathcal{NP}$
- Polynomial-Time Reductions
- $\mathcal{NP}$-completeness of SAT

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2 More $\mathcal{NP}$-Complete Problems

3 co-$\mathcal{NP}$

What We Already Know

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6. More on the HW

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- Similarly in proof of 3SAT $\leq_P$ Vertex Cover
  - We replaced each clause with a triangle and
  - each variable with a pair of nodes connected by an edge

NAE-3SAT

NAE-kSAT Problem

$$\text{NAE-kSAT} = \{\langle\phi\rangle \mid \phi \text{ is in } k\text{-CNF and } \phi \text{ has a satisfying assignment s.t. each clause has at least one 0 and at least one 1}\}$$

$(x_1 \vee x_2 \vee x_3 \quad) \wedge (y_1 \vee y_2 \vee y_3)$

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Definition:

- x is an NAE-assignment of ϕ if $\phi(x) = 1$ and x does not assign all the same variables to any clause

Lemma: If x is NAE-assignment

$\overline{x}$ is also NAE-assignment

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Lemma: If x is NAE-assignment of ϕ then $\overline{x}$ is NAE-assignment of ϕ

Proof:

- x must assign at least one 1 and at least one 0 to every clause
- $\overline{x}$ must also have at least one 1 and one 0 in every clause
- This means every clause is satisfied, and ϕ is satisfied since it's CNF

Goal

Prove that NAE-3SAT is $\mathcal{NP}$-complete: 3SAT $\leq_P$ NAE-3SAT

3SAT $\leq_P$ NAE-4SAT

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$A \Rightarrow B$

$\neg B \Rightarrow \neg A$

f is s.t.

if $x \in 3SAT \Rightarrow f(x) \in NAE\text{-}4SAT$

if $x \notin 3SAT \Rightarrow f(x) \notin NAE\text{-}4SAT$

$\Updownarrow$

$\rightarrow$ if $f(x) \in NAE\text{-}4SAT \Rightarrow x \in 3SAT$

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  - $(\Leftarrow)$ If $(x_1 \vee x_2 \vee x_3 \vee S) = 1$
    - If $S = 0$, then at least one $x_i = 1$, so $(x_1 \vee x_2 \vee x_3) = 1$
    - If $S = 1$, then $(\overline{x_1} \vee \overline{x_2} \vee \overline{x_3} \vee 0)$ is also NAE-assignment. So, $(\overline{x_1} \vee \overline{x_2} \vee \overline{x_3}) = 1$

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  - Idea: Let's split the variables into two clauses:

$$(x_1 \vee x_2 \vee z) \wedge (\overset{1}{x_3} \vee \overset{1}{x_4} \vee \bar{z})$$

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Theorem

$$3SAT \leq_P \text{NAE-4SAT} \leq_P \text{NAE-3SAT}$$

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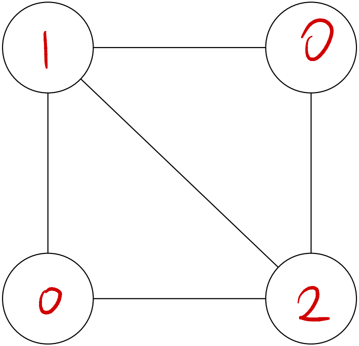

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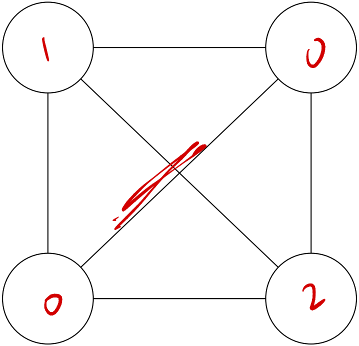
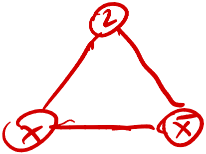


NAE-3SAT $\leq_P$ 3-Coloring



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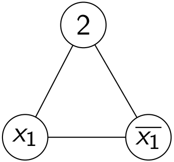
- For each variable x_i: 

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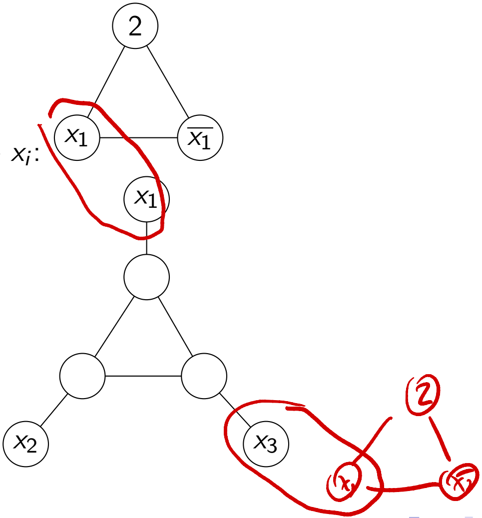

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$$\text{3-Coloring is } \mathcal{NP}\text{-complete, but 2-Coloring} \in \mathcal{P}$$

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2. More $\mathcal{NP}$-Complete Problems
3. co-$\mathcal{NP}$

Are All Problems in $\mathcal{NP}$?

Question

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Are All Problems in $\mathcal{NP}$?

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Consider the following language:

UNSAT

$$\text{UNSAT} = \{\langle\phi\rangle \mid \phi \text{ is not satisfiable}\}$$

Are All Problems in $\mathcal{NP}$?

Question

Do all languages have poly-size proofs?

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Problems like UNSAT are in co-$\mathcal{NP}$

$\mathcal{P}$, $\mathcal{NP}$ and co-$\mathcal{NP}$

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$L \in \mathcal{P}$ if there exists poly-time DTM M s.t $M(x) = [x \in L]$

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Question:

$$\text{Is } \mathcal{NP} = \text{co-}\mathcal{NP}?$$

The Complexity Zoo

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Complexity Zoo

The complexity zoo (`https://complexityzoo.net/Complexity_Zoo`) now has 546 complexity classes.