Foundations of Computing

Lecture 16

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March 18, 2025

Outline

Lecture 15 Review

- Countable and Uncountable Sets
  - Diagonalization
- Proving A_{TM} is Undecidable

Announcements

Homework 6 is out

- Due at 5PM on Monday, March 24
- Early deadline by midnight on Friday, March 21.

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- You will again be allowed 2 pieces of paper with notes

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- You will again be allowed 2 pieces of paper with notes
- Exam will cover the following topics:
    - Turing Machines
    - Countable and uncountable sets
    - Decidable, Turing-recognizable Languages
    - Proofs by reduction
    - Everything we cover this week
- CFL Pumping Lemma will not be on the exam

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Another Way to Prove Undecidability

Reductions Between Problems

There is a reduction from a problem A to a problem B if we can use a solution to problem B to solve problem A

$$A \leq B$$

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- problem A is no harder than problem B.

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Intuition

$A \leq B$ means that:

- problem A is no harder than problem B.
- Equivalently, problem B is no easier than problem A

Reductions and Undecidability

Main Observation

Suppose that $A \leq B$, then:

- If A is undecidable
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- Suppose that B is decidable – $\exists\ TM_B$ that decides B

TM_B Halts on all inputs

and outputs if

$x \in B$ or not

Reductions and Undecidability

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- Since $A \leq B$, there exists an algorithm (i.e., a reduction) that uses a solution to B to solve A

TM_B decider B

Reductions and Undecidability

I know A_{TM} is undecidable $A_{TM} \le L$

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Suppose that $A \leq B$, then:

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Proof: (by contradiction)

- Suppose that B is decidable
- Since $A \leq B$, there exists an algorithm (i.e., a reduction) that uses a solution to B to solve A
- But, this means that A is decidable by running the reduction using the decider machine for B. TM_B , R

$TM_A(x)$

1. Run $R(x)$
2. whenever R needs to answer if $x' \in B$, run $TM_B(x')$

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Proof Sketch:

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Construct reduction R that decides A_{TM} given a TM D that decides $HALT$

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- Run $D(\langle M, w\rangle)$
- If D rejects – $M(w)$ doesn't halt – halt and reject
- if D accepts – $M(w)$ halts – Simulate $M(w)$ until it halts, and output whatever M outputs

Other Undecidable Languages

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- We show that $A_{TM} \leq REGULAR_{TM}$ (M, w)
- Specifically, reduction builds another TM M' s.t.

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  - If M accepts w, M' recognizes Σ^* – regular language
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- If we can decide whether M' recognizes a regular language or not, can use that to decide whether M accepts w or not

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2. Run D on input $\langle M' \rangle$

$M(w) = 1$, what x does M' accept? Σ^*

$M(w) \neq 1$, $0^n 1^n$

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2. Run D on input $\langle M' \rangle$
    - If $M(w) = 1$, then $M'_{\langle M,w \rangle}$ accepts all $x \in \Sigma^*$ – regular
    - If $M(w) \neq 1$, $M'_{\langle M,w \rangle}$ accepts the language $0^n 1^n$ – not regular
3. Output what D outputs

Other Undecidable Languages – Exercise

$$EMPTY - STRING_{TM} = \{\langle M \rangle \mid M \text{ is a TM and } M(\epsilon) = 1\}$$

Think about:

- What direction should the reduction go?
- What language should the reduction use?