Foundations of Computing
Lecture 5

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January 31, 2023

Outline

Lecture 4 Review

- Equivalence of NFAs and DFAs
- NFAs for union, composition, and star – closure of regular languages
- Regular expressions

Outline

Regular Expressions

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Σ - alphabet

$1\Sigma^*$

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You've seen this before

Regular expressions very useful in compilers, and string search (e.g., grep)

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5. $(R_1 \circ R_2)$ – R_1 concatenated with R_2 where R_1 and R_2 are regular expressions
6. (R_1^*) – 0 or more repetitions of R_1 where R_1 is a regular expression

$\emptyset^*$

Some More Examples

$\Sigma = \{0, 1\}$

- $(\Sigma\Sigma)^* =$

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- $(\Sigma\Sigma)^* = \{w | w \text{ is a string of even length}\}$
- $(\underline{0 \cup \epsilon})(\underline{1 \cup \epsilon}) =$ $\{01, 0 \cdot \epsilon = 0, \epsilon 1 = 1, \epsilon\epsilon = \epsilon\}$

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- $(0 \cup \epsilon)(1 \cup \epsilon) = \{\epsilon, 0, 1, 01\}$
- $1^*\emptyset =$ $(\epsilon \cup 1 \cup 11 \cup 111 \ldots)\emptyset = \emptyset$

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Consider languages over the alphabet $\{0, 1, 2\}$

1. $L_1 = \{w | w \text{ has 2 consecutive 0's}\}$

   $\Sigma^* 00 \Sigma^*$

2. $L_2 = \{w | w \text{ has a substring 101 and ends in 22}\}$

   $\Sigma^* 101 \Sigma^* 22$

3. $L_3 = \{w | w \in L_1 \text{ or } w \in L_2\}$

   $(\Sigma^* 00 \Sigma^*) \cup (\Sigma^* 101 \Sigma^* 22)$

Languages to Regular Expressions Examples

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Question:

What does this have to do with FAs and regular languages?

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Regular Expressions == Regular Languages == NFA

Theorem

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Enough to show how to construct NFA to recognize L

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1. $R = a$ for some $a \in \Sigma$

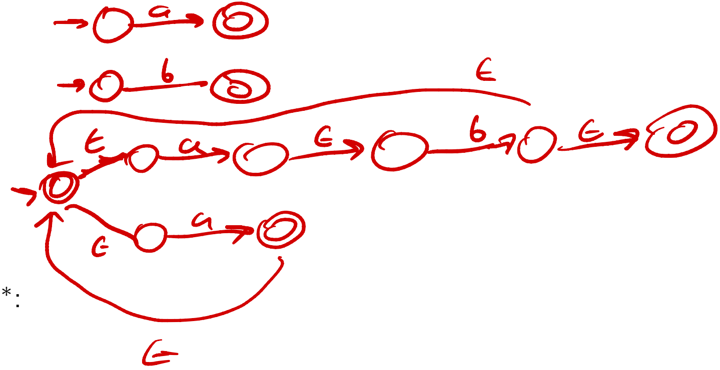
2. $R = \epsilon$

3. $R = \emptyset$

4. $R = R_1 \cup R_2$
5. $R = R_1 \circ R_2$
6. $R = R_1^*$

Problem: Convert $(ab \cup a)^*$ to an NFA
In English: Either "ab" or "a" repeated 0 or more times

- a:
- b:
- ab:
- $ab \cup a$:
- $(ab \cup a)^*$:



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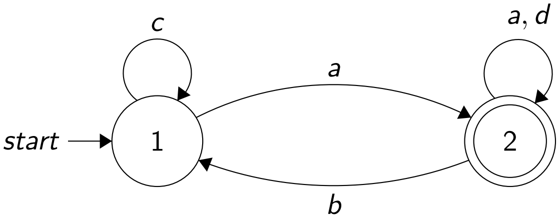


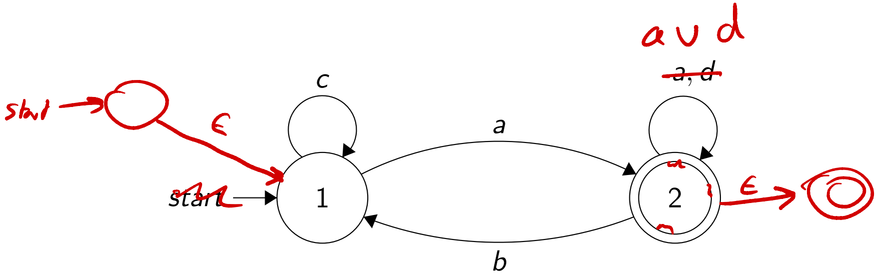
Question

How do we represent L by a regular expression?

Step 1: NFA $\rightarrow$ generalized NFA

A generalized NFA has 3 important properties:

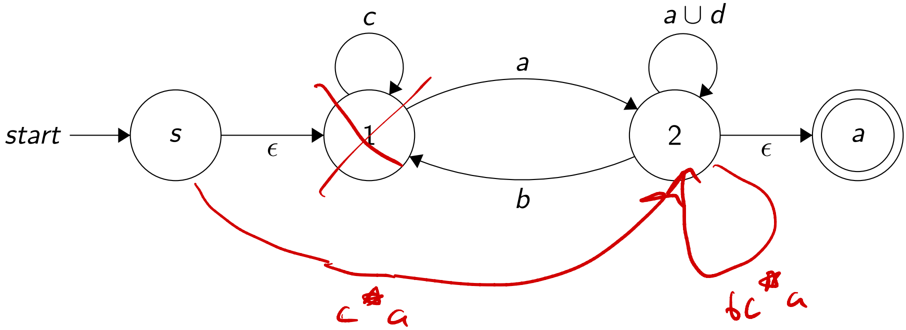
1. Start state has no incoming edges
2. Only one accept state, and it has no outgoing edges
3. Edges labeled by regular expressions



Step 2: Node Elimination – Remove Node 1

Remove nodes one-by-one (in any order) until only start and accept states left:

- Need to update reg. exp.'s for all paths through removed nodes



Step 2: Node Elimination – Remove Node 2

Remove nodes one-by-one (in any order) until only start and accept states left:

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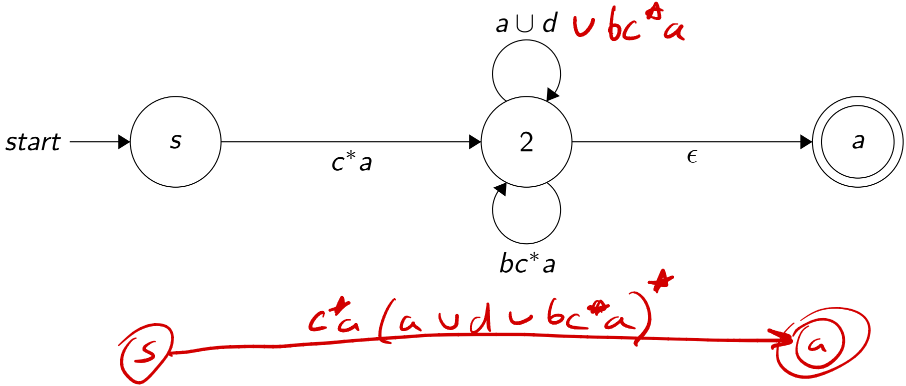
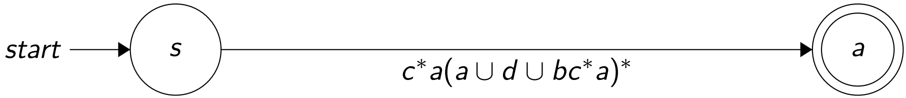
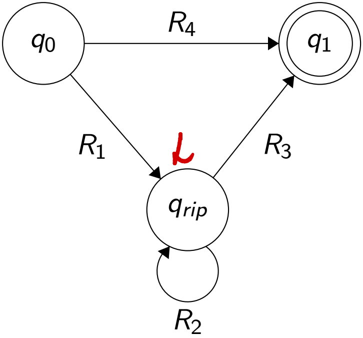


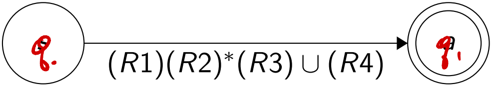
We are Done

Output label of final edge from start to accept state.



Generalized Node Elimination





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Base Case: For $|G| = 2$, G consists of start and accept states and arrow between them. The label on this arrow exactly describes the language of strings accepted by G.

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Inductive step: Assume true for $|G| = k - 1$, prove true for $|G| = k$. (i.e., prove that $G' = G$)

- Assume some w s.t. $G(w) = 1$, then on input w, G goes through

$$q_{start}, q_1, q_2, \ldots, q_{accept}$$

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  - If it would have gone through q_{rip} then the modified edge accepts w, so there is a path through q_{rip} in G that accepts w.

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  - If the accepting path would not have gone through q_{rip}, then G must also have the same path to accept w

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Proof:

- Build NFA M corresponding to each clause
- Since we already showed how to build NFA to show closure, can convert that to regular expression to prove the claim.