Foundations of Computing

Lecture 15

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Outline

Lecture 14 Review

- Decidable and Turing-recognizable languages
- Decidability of regular and context-free languages

Outline

Characterizing Computability of Languages

Definition: Decidable languages

A language L is *decidable* or *recursive* if some TM M decides it

- M halts on ALL inputs, accepts those in L and rejects those not in L
- Seems to match informal definition we wanted before

Definition: Turing-recognizable languages

A language L is *Turing-recognizable* or *recursively enumerable* if some TM M recognizes it

- M halts and accepts all strings in L
- M may not halt on strings not in L – does not necessarily have to reject

Observation

Every Decidable language is also Turing-recognizable, but the reverse direction is not true.

Decidable Languages

We showed the following languages are decidable:

- Languages about Finite Automata
  1. $A_{DFA} = \{\langle B, w\rangle \mid B \text{ is a DFA that accepts input string } w\}$
  2. $A_{NFA} = \{\langle B, w\rangle \mid B \text{ is a NFA that accepts input string } w\}$
  3. $A_{REX} = \{\langle R, w\rangle \mid R \text{ is a reg. exp. that generates the string } w\}$
  4. $E_{DFA} = \{\langle A\rangle \mid A \text{ is a DFA and } L(A) = \emptyset\}$
  5. $EQ_{DFA} = \{\langle A, B\rangle \mid A, B \text{ are DFAs and } L(A) = L(B)\}$
- Languages about CFGs
  1. $E_{CFG} = \{\langle G\rangle \mid G \text{ is a CFG and } L(G) = \emptyset\}$

A Language About Turing Machines

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- Is A_{TM} Decidable?
  - The problem: M may never halt
  - In this case, above algorithm will never output accept or reject
  - If could determine that M will never halt (i.e, it has entered an infinite loop), could reject.

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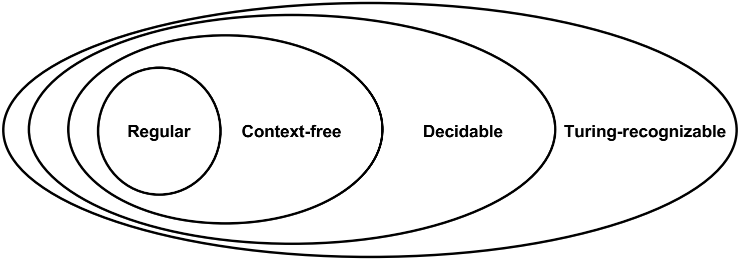
An Undecidable Problem

- We will prove today that A_{TM} is undecidable

Relationships Among Language Classes



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- Example:
  $A = \{0, 1, 2, 3\}$
  $B = \{a, b, c, d\}$
  $f(0) = a, f(1) = b, f(2) = c, f(3) = d$

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- A set A is countable if it is finite or countably infinite
- A set that is not countable is *uncountable*

Outline

The Hotel Setup

- Imagine a hotel with an infinite number of rooms: $1, 2, \ldots$
- All the rooms are occupied, each with a guest.
- The hotel is full, but still, there is room for more guests.

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Conclusion

Even when the hotel is "full," it is still possible to accommodate an additional guest.

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- Now, an infinite number of new guests $(1, 2, \ldots)$ arrive at the hotel.
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- This frees up all the odd-numbered rooms (1, 3, 5, 7, ...), which can be assigned to the new guests.

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Conclusion

Even though there are infinitely many new guests, the hotel can still accommodate them by utilizing the infinite number of even-numbered rooms.

Paradox 3: Accommodating an Infinite Number of Buses

- Suppose there is a countably infinite number of buses, each carrying a countably infinite number of passengers.

Outline

Example 1: Evens

- To show that an infinite set is countable, need to show a 1-to-1 and onto mapping onto the Natural numbers ($\mathcal{N}$): 1,2,...

Evens

The set of even numbers is countable

Example 2: Rationals

Rationals

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$$
\begin{array}{c|cccc}
 & 1 & 2 & 3 & \cdots \\
\hline
1 & 1/1 & 1/2 & 1/3 & \\
2 & 2/1 & 2/2 & 2/3 & \cdots \\
3 & 3/1 & 3/2 & 3/3 & \\
4 & 4/1 & 4/2 & \vdots &
\end{array}
$$

Example 3: Strings

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Real Numbers

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The set of real numbers ($\mathcal{R}$) is uncountable

Proof: By diagonalization

- Assume that $\mathcal{R}$ is countable
- Then there is a one-to-one and onto mapping f from $\mathcal{N}$ to $\mathcal{R}$

n	f(n)
1	1.234...
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- Contradiction – f is not mapping between $\mathcal{R}$ and $\mathcal{N}$

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- But, a TM M can be written as a string $\langle M \rangle \in \Sigma^*$

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The set of all Turing Machines is countable

- We already showed that the set of strings $\{0, 1\}^*$ is countable
- Can similarly show that for any finite alphabet Σ, Σ^* is countable
- But, a TM M can be written as a string $\langle M \rangle \in \Sigma^*$
- Hence, by omitting all strings that are not encodings of valid TMs we get a mapping of TMs to $\mathcal{N}$

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Languages over alphabet Σ

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   - Define the characteristic sequence χ_A of language $A \in \mathcal{L}$

$$
\begin{array}{rcccccccccccc}
\Sigma^* & = & \{ & \epsilon & 0 & 1 & 00 & 01 & 11 & 000 & \cdots & \} \\
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3. Therefore, $\mathcal{L}$ is uncountable

Some Languages are not Turing-recognizable

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Where are we now?

- We have proven that some languages are not Turing-recognizable
- But, we have not given any examples of such a language

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$$D(\langle M\rangle) = \begin{cases} \textit{accept} & \text{if } M \text{ does not accept } \langle M\rangle \\ \textit{reject} & \text{if } M \text{ accepts } \langle M\rangle \end{cases}$$

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- Now consider what happens if we run D on $\langle D\rangle$

$$D(\langle D\rangle) = \begin{cases} \mathit{accept} & \text{if } D \text{ does not accept } \langle D\rangle \\ \mathit{reject} & \text{if } D \text{ accepts } \langle D\rangle \end{cases}$$

- Contradiction!

How Is This a Diagonalization?

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	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\cdots$	$\langle D \rangle$	$\cdots$
M_1	$\underline{accept}$	$reject$	$accept$		$accept$	
M_2	$reject$	$\underline{reject}$	$reject$	$\ldots$	$accept$	$\ldots$
M_3	$accept$	$accept$	$\underline{accept}$		$reject$	
$\vdots$		$\vdots$		$\ddots$		
D	$reject$	$accept$	$reject$		?	

- We have defined D to do the opposite of what M_i does on input $\langle M_i \rangle$
- But what does D do on input $\langle D \rangle$??
- We have found a machine not on the list

A Turing-unrecognizable Language

$\overline{A_{TM}}$

The language

$$\overline{A_{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM and } M(w) \neq 1\}$$

is not Turing-recognizable

Outline

Another Way to Prove Undecidability

Reductions Between Problems

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- Equivalently, problem B is no easier than problem A

Reductions and Undecidability

Main Observation

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Proof: (by contradiction)

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Proof: (by contradiction)

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Reductions and Undecidability

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Suppose that $A \leq B$, then:

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- But, this means that A is decidable by running the machine for B as needed by the reduction

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- Output whatever M output