Foundations of Computing

Lecture 8

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Outline

Lecture 7 Review

- Proving languages not regular
  - Using the pumping lemma
  - Using closure properties

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$L = \{0^n 1^n\}$

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Today

Going beyond regular languages.

How Can We Recognize Non-Regular Languages?

Let $L = \{0^n 1^n | n \geq 0\}$

Question

How can we build a machine to recognize this language?

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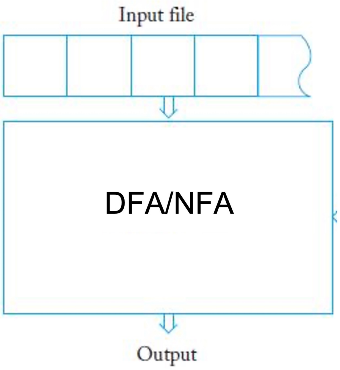
Answer

Add some form of (unbounded) memory to the machine

Outline

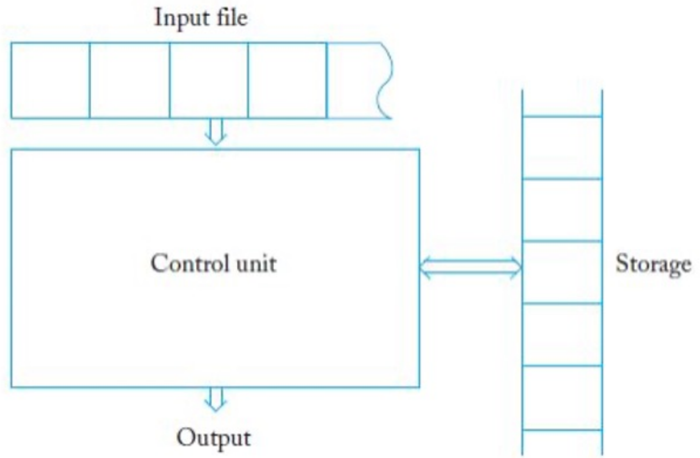
Let's Add Some Storage



Recall:

- An NFA/DFA has no external storage
- Only memory must be encoded in the finite number of states
- Can only recognize regular languages

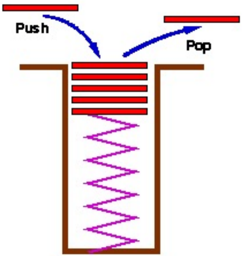
Let's Add Some Storage



Question

What kind of storage should we add?

A Stack



Let's add a Stack for storage

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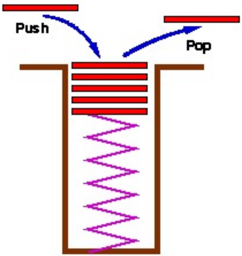

Let's add a Stack for storage

A stack has the following operations:

- push value - push a value onto the top of the stack

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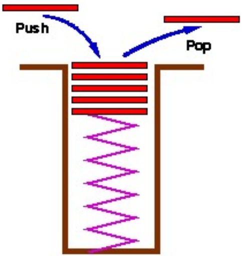

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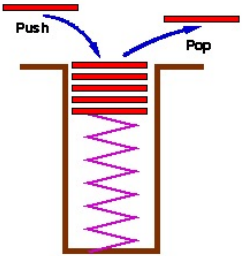

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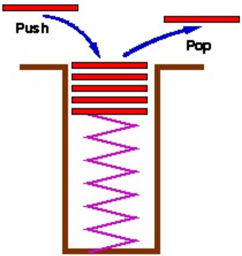

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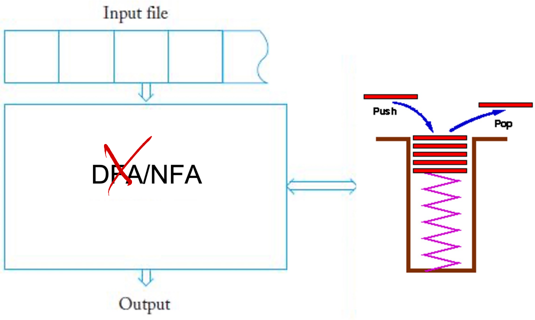
A stack is a Last-In First-Out (LIFO) data structure,
that can hold an infinite amount of information (infinite depth)

Pushdown Automata (PDA)



A PDA consists of:

- An NFA for a control unit
- A Stack for storage

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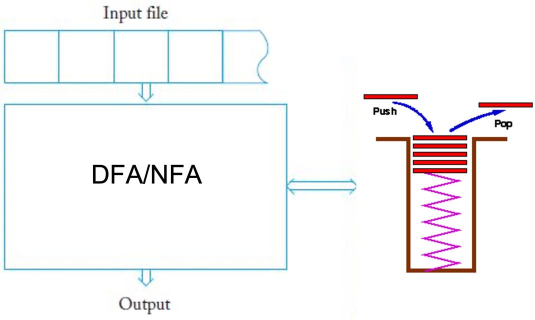

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Is this any more powerful than an NFA?

Computing With a PDA

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Observations:

- Since the control is an NFA, ϵ transitions are allowed
- A PDA may choose not to touch the stack in a particular step
- Unlike the case for DFA/NFA, deterministic PDA's are not equal to non-deterministic ones. We will only study non-deterministic PDAs.

An Example PDA

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1. Read a symbol from the input
2. If it is a 0 and I have not seen any 1s, then push a 0 onto the stack

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1. Read a symbol from the input
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Outline

Formal Definition of PDAs

A PDA M is a 6-tuple $(Q, \Sigma, \Gamma, \delta, q_0, F)$ where

- Q – set of states of the NFA
- Σ – input alphabet
- Γ – Stack alphabet
- $\delta : Q \times \Sigma_\epsilon \times \Gamma_\epsilon \to P(Q \times \Gamma_\epsilon)$ – transition function
- $q_0 \in Q$ – start state
- $F \subseteq Q$ – accept states

Recall that $P(Q \times \Gamma_\epsilon)$ is the power set of the set of pairs $\{(q \in Q, a \in \Gamma_\epsilon)\}$

Computing with a PDA – Formal Notation

A PDA M accepts a string $w = w_1 w_2 \cdots w_m$ with $w_i \in \Sigma_\epsilon$ if there exist

- A sequence of states $q_0, q_1, \ldots q_m \in Q$, and
- A sequence of strings $s_0, s_1, \ldots, s_m \in \Gamma^*$

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2. For $i = 0, \ldots, m-1$, $(q_{i+1}, b) \in \delta(q_i, w_{i+1}, a)$ where $s_i = at$ and $s_{i+1} = bt$ for some $a, b \in \Gamma_\epsilon$ and $t \in \Gamma^*$
   there is a transition in δ s.t. M reads symbol w_{i+1} from the input, pops a from the stack, pushes b back on the stack and moves from q_i to q_{i+1}

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3. $r_m \in F$ – stop in an accept state

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Recall the PDA we described before:

- On input 0, push a 0 on the stack
- On input 1, pop a value from the stack
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- $Q = \{q_0, q_1, q_2, q_3\}$
  - q_0 – start state
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- $q_0 = q_0$
- $F = \{q_3\}$

Transition Function

0011

- q_0,
- q_1, $\boxed{\$}$
- q_1, $\boxed{0\ \$}$
- q_1, $\boxed{0\ 0\ \$}$

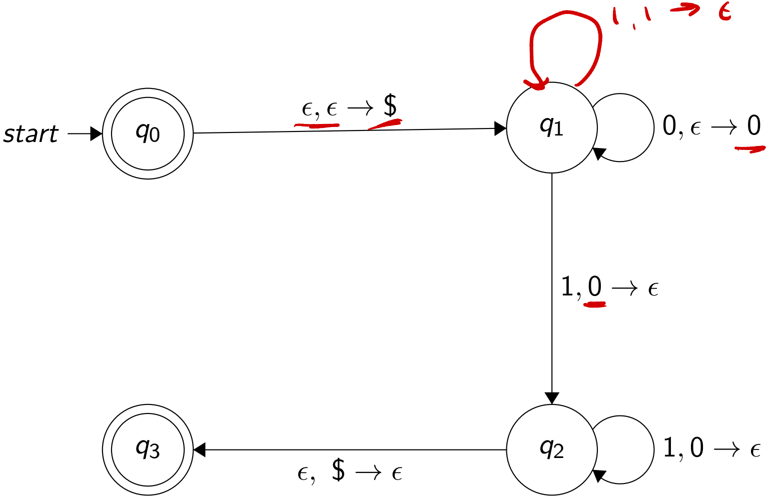
Input:	0			1			ϵ		
Stack:	0	\$	ϵ	0	\$	ϵ	0	\$	ϵ
q_0									$\{(q_1, \$)\}$
q_1			$\{(q_1, 0)\}$	$\{(q_2, \epsilon)\}$					
q_2				$\{(q_2, \epsilon)\}$				$\{(q_3, \epsilon)\}$	
q_3									

Table: Transition Function δ

- q_2, $\boxed{0\ \$}$
- q_2, $\boxed{\$}$

Empty cells correspond to output of $\emptyset$

Example PDA as a Graph



Exercise – Work in Groups

Show a PDA that recognizes the language

$$L = \{w \mid w \text{ has an equal number of 0s and 1s}\}$$

1. Describe a PDA algorithm for this language
2. Write the states and transition function
3. Draw the PDA graph