Foundations of Computing

Lecture 14

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March 4, 2025

Outline

1. Lecture 13 Review
2. Decidable and Turing-recognizable Languages
3. Languages With Machines as Input
4. Preliminaries – Countable and Uncountable Sets

Lecture 13 Review

- More Turing Machines
- Turing Machine Variants
  - Multi-tape Turing Machines
  - Non-deterministic Turing Machines

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- Are there some problems that inherently do not have any algorithmic solution?

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Observations:

- L is Turing-recognizable if can recognize yes instances, L is decidable if can also recognize no instances.
- Every Decidable language is also Turing-recognizable, but the reverse direction may not be true.

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A Second Question

What about Turing-unrecognizable languages?

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- So, can give a description of a machine M to another machine M'
- We already talked about a universal TM which can run any other TM

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2. If simulation ends in an accept, then accept. If it ends in a non-accepting state, then reject

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2. Run TM from previous slide on input $\langle C, w\rangle$
3. Output what this TM outputs

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Algorithm to decide A_{REX}:
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1. Convert R to equivalent DFA M
2. Use TM from previous slide to decide it

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3. If starts symbol is not marked, accept. Otherwise, reject

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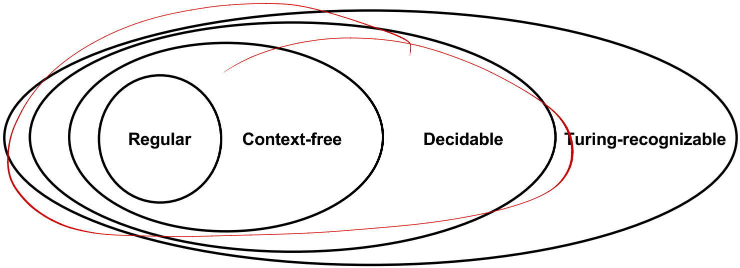
- But, how do we determine this?

Relationships Among Language Classes



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- Example:
  $A = \{0, 1, 2, 3\}$
  $B = \{a, b, c, d\}$
  $f(0) = a, f(1) = b, f(2) = c, f(3) = d$

$$\begin{matrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & 4 & 6 & 8 \end{matrix}$$

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- A set A is countable if it is finite or countably infinite
- A set that is not countable is *uncountable*

Example 1: Evens

Evens

The set of even numbers is countable

Example 2: Rationals

Rationals

The set of rational numbers is

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Rationals

The set of rational numbers is countable

	1	2	3	$\cdots$
1	1/1	1/2	1/3	
2	2/1	2/2	2/3	$\cdots$
3	3/1	3/2	3/3	
4	4/1	4/2	$\vdots$	

Example 3: Strings

Strings

The set of strings in $\{0, 1\}^*$ is countable

Diagonalization

Real Numbers

The set of real numbers ($\mathcal{R}$) is uncountable

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- Then there is a one-to-one and onto mapping f from $\mathcal{N}$ to $\mathcal{R}$

n	f(n)
1	1.234...
2	3.141...
3	5.556...
$\vdots$	$\vdots$

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  Idea: For all $i \in \mathcal{N}$, make $x_i \neq f(i)_i$
- Contradiction – f is not mapping between $\mathcal{R}$ and $\mathcal{N}$