

Foundations of Computing

Lecture 5

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Outline

- 1 Lecture 4 Review
- 2 Properties of Regular Languages Using NFAs
- 3 Regular Expressions
- 4 Regular Expressions \equiv Regular Languages
- 5 Properties of Regular Expressions

Lecture 4 Review

- More NFAs
- Equivalence of NFAs and DFAs
- NFAs for union, composition, and star – closure of regular languages

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A Useful Corollary

Recall that:

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A language L is regular if and only if there is a DFA that recognizes it

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We can now use NFAs to argue the properties of regular languages

Closure Under Union

Closure Under Union

If L_1 and L_2 are both regular languages then $L_1 \cup L_2$ is also regular

$L_1 \cup L_2$ is the language consisting of all strings either in L_1 or L_2

Proof:

Closure Under Concatenation

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If L_1 and L_2 are both regular languages then $L_1 \circ L_2$ is also regular

$$L_1 \circ L_2 = \{xy \mid x \in L_1 \text{ and } y \in L_2\}$$

Proof:

Closure Under the Star Operation

Closure Under Star Operation

If L is a regular languages then L^* is also regular

$L^* = \{0 \text{ or more strings from } L\}$

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You've seen this before

Regular expressions very useful in compilers, and string search (e.g., grep)

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- 6 (R_1^*) – 0 or more repetitions of R_1 where R_1 is a regular expression

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- $(0 \cup \epsilon)(1 \cup \epsilon) = \{\epsilon, 0, 1, 01\}$
- $1^*\emptyset = \emptyset$
- $\emptyset^* = \{\epsilon\}$
- $0\Sigma^*0 \cup 1\Sigma^*1 \cup 0 \cup 1 = \{w \mid w \text{ starts and ends with the same symbol}\}$

Languages to Regular Expressions Examples

Consider languages over the alphabet $\{0, 1, 2\}$

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Question:

What does this have to do with FAs and regular languages?

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① $R = a$ for some $a \in \Sigma$

② $R = \epsilon$

③ $R = \emptyset$

④ $R = R_1 \cup R_2$

⑤ $R = R_1 \circ R_2$

⑥ $R = R_1^*$

An Example

Problem: Convert $(ab \cup a)^*$ to an NFA

In English: Either “ab” or “a” repeated 0 or more times

- a :
- b :
- ab :
- $ab \cup a$:
- $(ab \cup a)^*$:

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Enough to show how to build regular expression corresponding to a NFA

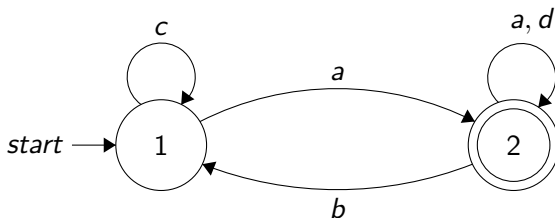
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Question

How do we represent L by a regular expression?

Step 1: NFA \rightarrow generalized NFA

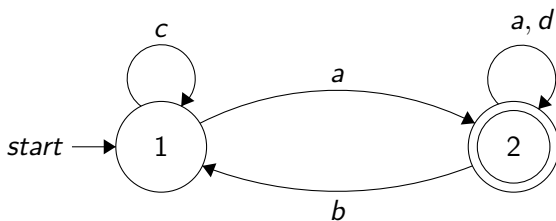
A generalized NFA has 3 important properties:

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- 2 Only one accept state, and it has no outgoing edges
- 3 Edges labeled by regular expressions

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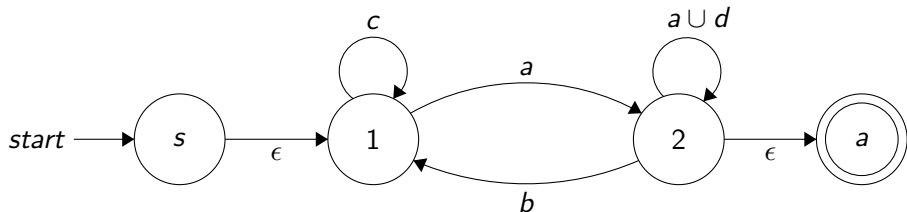
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Step 2: Node Elimination – Remove Node 1

Remove nodes one-by-one (in any order) until only start and accept states left:

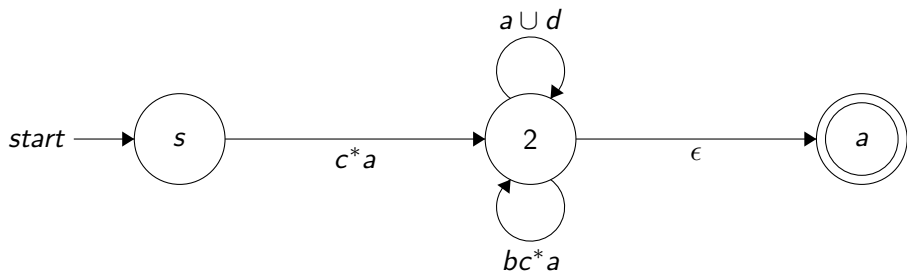
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Step 2: Node Elimination – Remove Node 2

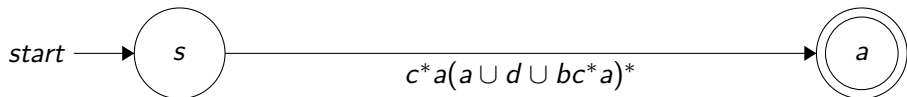
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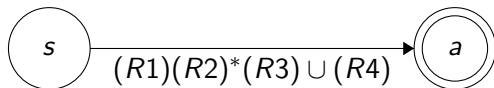
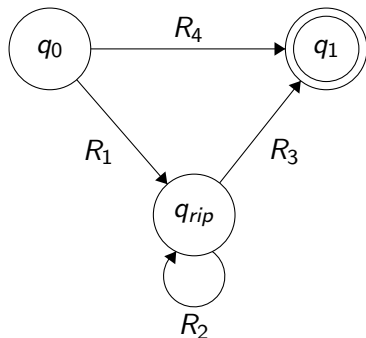


We are Done

Output label of final edge from start to accept state.



Generalized Node Elimination



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Base Case: For $|G| = 2$, G consists of start and accept states and arrow between them. The label on this arrow exactly describes the language of strings accepted by G .

Theorem

For any GNFA G , $G' = \text{NODE-ELIMINATE}(G)$ is equivalent to G

Inductive step: Assume true for $|G| = k - 1$, prove true for $|G| = k$. (i.e., prove that $G' = G$)

- Assume some w s.t. $G(w) = 1$, then on input w , G goes through

$$q_{start}, q_1, q_2, \dots, q_{accept}$$

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 - If it would have gone through q_{rip} then the modified edge accepts w , so there is a path through q_{rip} in G that accepts w .

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 - If the accepting path would not have gone through q_{rip} , then G must also have the same path to accept w

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- Since we already showed how to build NFA to show closure, can convert that to regular expression to prove the claim.