CS 3313
Foundations of Computing:

Lab 9 – Asymptotic Notation and Polynomial Time

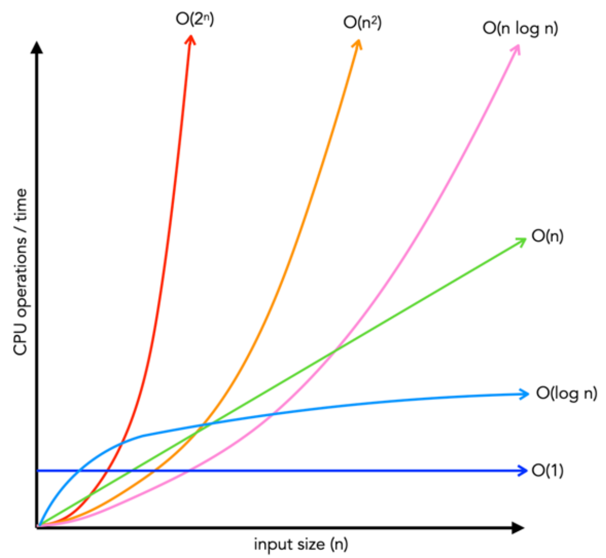
Time Complexity Background

- In programming, we want to minimize the time it takes for our algorithms to run
- By reducing the number of operations we need to compute, we see dramatic decreases in run-time
  - This is *essential!* We want things ASAP!
- For example, consider the following:

n	n^2	n^3
1	1	1
10	100	1,000
100	10,000	1,000,000
1,000	1,000,000	1,000,000,000

Asymptotic Notation

- When we compare programs, we look at them *asymptotically*
- This is because we are concerned with the *growth in time* of our functions as our value *n* (the number of elements we have) increases
- We saw an example of this on the previous slide, where a higher power resulted in a higher growth rate
  - We can demonstrate this visually as well



Note: $O(1) < O(\log n) < O(n) < O(n \log n) < O(n^2) < O(n^3) < O(2^n)$

Big-O Notation

- Big-O Notation helps us describe how long an algorithm takes by setting an *upper bound*
- For example, if we take two functions, *f(n)* and *g(n)*, we can say:
  - *f(n)* = *O(g(n))* if and only if there exists constants $c > 0$ and $n_0 \geq 1$ s.t.

$$f(n) \leq c * g(n) \text{ for all } n \geq n_0$$

- True or False:
  - $5n + 3 = O(n)$?
  - $n^2 + 5n + 3 = O(n)$?
  - $n^2 + 5n + 3 = O(n^2)$?
  - $n^2 + 5n + 3 = O(n^3)$?
  - $3^n = O(3^{n+1})$?

Big-O Notation

- Now that we have defined what Big-O means, how can we show that this holds true as n increases?
- The trick to this is through the induction proof technique

Big-O Notation: Induction Proof

- Let's do the following example from the previous slide:
  - $n^2 + 5n + 3 = O(n^2)$
  - $n^2 + 5n + 3 \leq cn^2$ Now, we pick values for c and n_0
  - $n^2 + 5n + 3 \leq 9n^2 \; \forall \, n \geq n_0 = 1$
  - Base case: $n = 1$ → $(1)^2 + 5(1) + 3 \leq 9(1)^2$
    → $9 \leq 9$
  - Induction Hypothesis: Assume that $k^2 + 5k + 3 \leq 9k^2$ holds for a value n=k
  - Induction: Demonstrate that the equality holds for $k + 1$:

$$
\begin{aligned}
(k+1)^2 + 5(k+1) + 3 &\leq 9(k+1)^2 \\
(k^2+2k+1) + (5k+5) + 3 &\leq 9(k^2+2k+1) \\
k^2 + 7k + 9 &\leq 9k^2 + 18k + 9 \\
0 &\leq 8k^2 + 11k
\end{aligned}
$$

Therefore, this is true, as we know $k \geq n_0 = 1$

Polynomial Time

- We want to measure the number of steps taken by a TM to decide if an input x is in a language L
  - Need to measure runtime on ANY input (i.e., worst-case running time)
  - Runtime measured as a function of |x|
  - Remember that |x| may be much smaller than x (e.g., for numbers)
- We will use big-O notation to define efficient computation

- Polynomial Time:
  $T(n) = O(n^c)$ for any constant c

Exercise 1:

- $L_1 = \{ww^r \mid w \in \{\mathrm{a}, \mathrm{b}\}^*\}$
  - Give a 1-tape TM solution to solve this problem in $O(n^2)$ time
  - Can we improve the time by having a 2-tape TM? If so, give a 2-tape TM solution and describe its time complexity

Big-Omega and Big-Theta (Big-Ω and Big-θ)

- Big-Ω denotes the following relationship between functions *f(n)* and *g(n)*:
  - $f(n) = \Omega\big(g(n)\big)$ if and only if there exists constants $c > 0$ and $n_0 \geq 1$ s.t.

    $$f(n) \geq c * g(n) \text{ for all } n \geq n_0$$
  - For example, $3n^2 = \Omega(n)$

- Big-θ denotes the following relationship between functions *f(n)* and *g(n)*:
  - $f(n) = \theta\big(g(n)\big)$ if and only if there exists constants $c_1, c_2 > 0$ and $n_0 \geq 1$ s.t.

    $$c_1 * g(n) \leq f(n) \leq c_2 * g(n) \text{ for all } n \geq n_0$$
  - For example, $3n = \theta(n)$

Transformations Between Notations

- Here, we see that these relationships are connected
  - For example, if $f(n) = O(g(n))$, then $g(n) = \Omega(f(n))$
    - Why is this?
  - Additionally, if $f(n) = O(g(n))$ and simultaneously $f(n) = \Omega(g(n))$, then $f(n) = \theta(g(n))$
    - Why is this?

Exercise 2:

- Prove or disprove the following:
  - $\frac{n(n+1)}{2} = \Omega(n^2)$
  - $\frac{n(n+1)}{2} = \theta(n^2)$
  - $2^{2n} = O(2^n)$