Foundations of Computing

Lecture 22

Arkady Yerukhimovich

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Outline

Lecture 21 Review

- $\mathcal{P}$ and $\mathcal{NP}$
- Polynomial-Time Reductions
- $\mathcal{NP}$-completeness of SAT

Outline

What We Already Know

1. SAT is $\mathcal{NP}$-complete
2. 3-SAT is $\mathcal{NP}$-complete
3. 3-SAT $\leq_P$ CLIQUE
4. CLIQUE $\leq_P$ Independent Set

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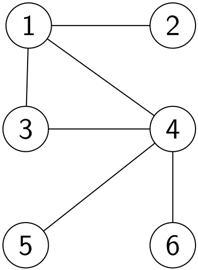
- For example, in proof of 3SAT $\leq_P$ CLIQUE
  - We replaced each variable with a node
  - We replaced each clause with 3 nodes (1 for each variable)
  - Edges capture independent variables between clauses

Vertex Covers

Given a graph $G = (V, E)$, a vertex cover is a subset of the nodes $C \subseteq V$ s.t. each edge in E has an end-point in V.



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2. Show that 3-SAT $\leq_p$ VC

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Goal: Show reduction f from 3-SAT to VC s.t.

- if ϕ is satisfiable, $f(\phi) = \langle G, k \rangle$ s.t. G has VC of size $\leq k$
- if ϕ is not satisfiable, $f(\phi) = \langle G, k \rangle$ s.t. G has no VC of size $\leq k$

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Variable gadget:

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Variable gadget: For every variable x_1, draw pair of nodes

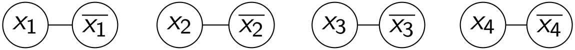
Clause gadget: For every (3-term) clause draw a triangle

Observations:

- For each variable need 1 node in cover
- For each triangle need at least 2 nodes
- Need to connect variables to clauses

3-SAT $\leq_p$ VC Example

$$\phi = (x_1 \vee \overline{x_2} \vee x_3) \wedge (\overline{x_2} \vee \overline{x_3} \vee x_4)$$



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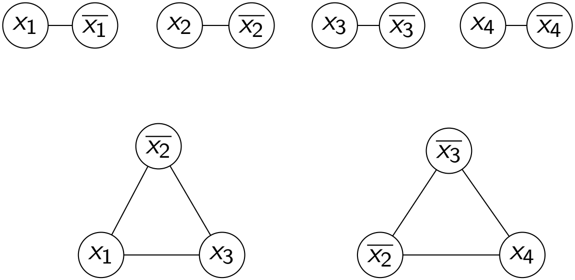

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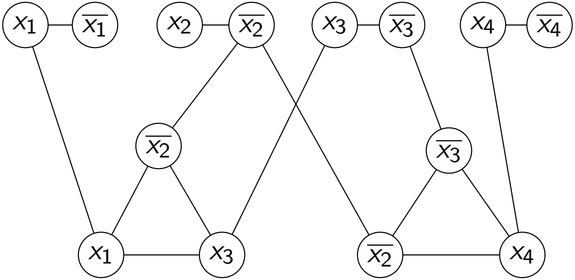

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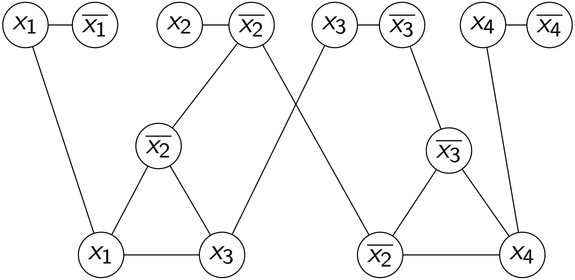

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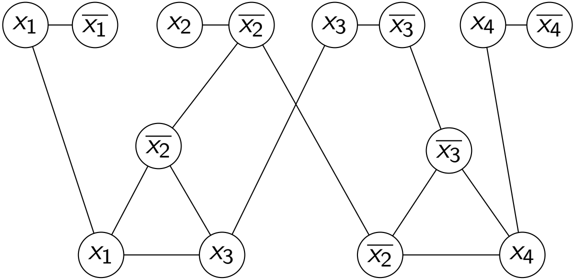

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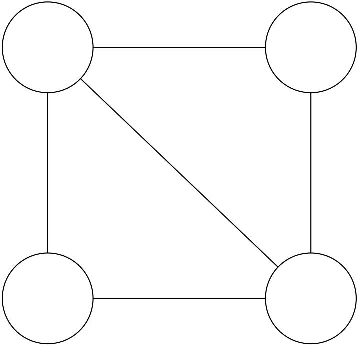

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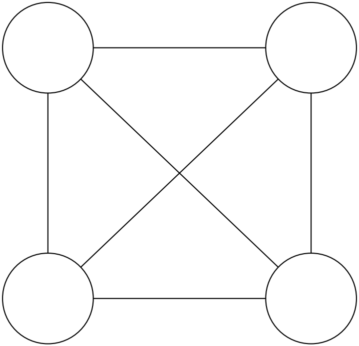

Goal: Prove than 3-Coloring is $\mathcal{NP}$-Complete

NAE-3SAT

NAE-kSAT Problem

$$\text{NAE-kSAT} = \{\langle\phi\rangle \quad | \quad \phi \text{ is in } k\text{-CNF and } \phi \text{ has a satisfying assignment s.t. each clause has at least one 0 and at least one 1}\}$$

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Lemma: If x is NAE-assignment of ϕ then $\overline{x}$ is NAE-assignment of ϕ

Proof:

- x must assign at least one 1 and at least one 0 to every clause
- $\overline{x}$ must also have at least one 1 and one 0 in every clause
- This means every clause is satisfied, and ϕ is satisfied since it's CNF

Goal

Prove that NAE-3SAT is $\mathcal{NP}$-complete: 3SAT $\leq_P$ NAE-3SAT

3SAT $\leq_P$ NAE-3SAT

- Turns out it's not easy to directly prove this
- Instead we take two steps:

$$\text{3SAT} \ \leq_P \ \text{NAE-4SAT} \ \leq_P \ \text{NAE-3SAT}$$

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    - If $S = 0$, then at least one $x_i = 1$, so $(x_1 \vee x_2 \vee x_3) = 1$
    - If $S = 1$, then $(\overline{x_1} \vee \overline{x_2} \vee \overline{x_3} \vee 0)$ is also NAE-assignment. So, $(\overline{x_1} \vee \overline{x_2} \vee \overline{x_3}) = 1$, so just flip all assignments in ϕ'

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- Observations: Starting with $(x_1 \vee x_2 \vee x_3 \vee x_4)$
  - We know that not all x_i have the same value
  - At least one of x_i is a 1 and one is a 0
  - Idea: Let's split the variables into two clauses:
    $$(x_1 \vee x_2 \vee x_3 \vee x_4) \rightarrow (x_1 \vee x_2 \vee z_i) \wedge (x_3 \vee x_4 \vee \overline{z_i})$$
- Why this works:
  - $(\Leftarrow)$ If $(x_1 \vee x_2 \vee z_i)$ and $(x_3 \vee x_4 \vee \overline{z_i})$ are both NAE, then $(x_1 \vee x_2 \vee x_3 \vee x_4)$ is NAE
  - $(\Rightarrow)$ If $(x_1 \vee x_2 \vee x_3 \vee x_4)$ is NAE
    - If $x_1 \neq x_2$: Set $z_i = x_3$
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Theorem

$$\text{3SAT} \leq_P \text{NAE-4SAT} \leq_P \text{NAE-3SAT}$$

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$$\phi = (x_1 \vee \overline{x_2} \vee x_3) \wedge (\overline{x_2} \vee \overline{x_3} \vee x_4)$$

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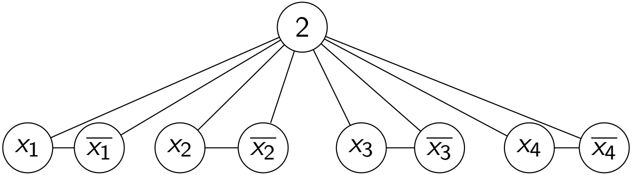

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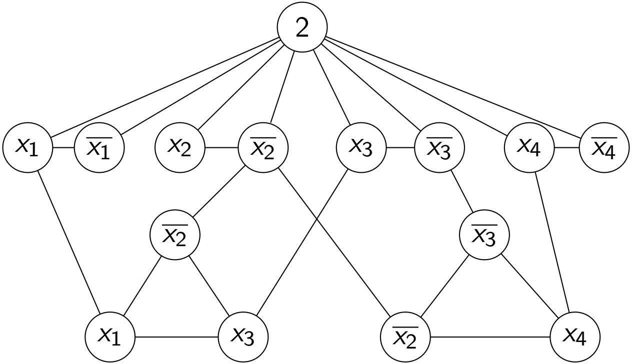

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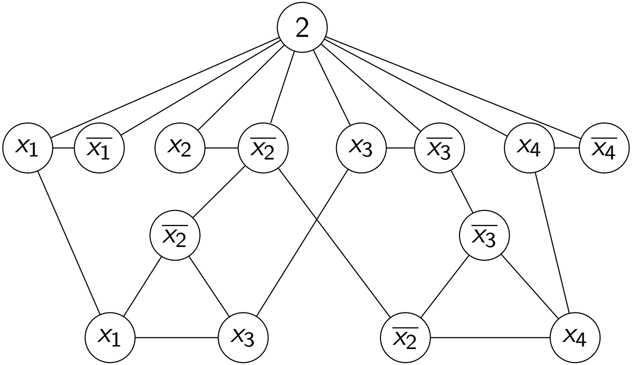

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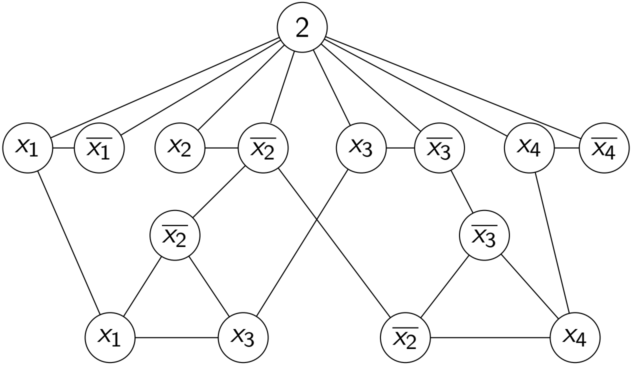

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2. If G is 3-colorable, colors indicate a NAE-SAT assignment

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$$\text{3-Coloring is } \mathcal{NP}\text{-complete, but 2-Coloring} \in \mathcal{P}$$

Outline

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UNSAT

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Problems like UNSAT are in co-$\mathcal{NP}$

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We believe that there are infinitely many levels of the polynomial hierarchy and that $\Pi_i^p \neq \Sigma_i^p$ for $i > 0$, but can't prove it.

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Complexity Zoo

The complexity zoo (`https://complexityzoo.net/Complexity_Zoo`) now has 546 complexity classes.