Foundations of Computing

Lecture 3

Arkady Yerukhimovich

January 21, 2025

Outline

Lecture 2 Review

- Language decided by DFA M
- Building DFAs
- Proving Correctness of DFAs

Outline

From Machines to Languages

- Last lecture we saw how to build DFA M to decide a language L
- Learned to reason about machine M
- Recall that each machine M decides one language $L(M)$

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Let's switch our perspective

Instead of reasoning about machines, let's focus on languages decided by those machines.

Regular Language

Definition

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Observations:

- All languages we have seen thus far are regular
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Something to think about

Are all languages regular?

Closure under Complement

If L is a regular language, then $\overline{L}$ is also regular

$\overline{L}$ is the language that consists of all strings (in alphabet Σ) not in L.

Suppose M decides L

$\overline{M}$ s.t. $\forall x$ if $M(x) = 1$ $\overline{M}(x) = 0$

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Properties of Regular Languages

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Intuition: Swap the accept and reject states

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Proof: Let $M = (Q, \Sigma, \delta, q, F)$ decide L

Construct $M' = (Q', \Sigma', \delta', q', F')$ that decides $\overline{L}$

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1. $Q' = Q$

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3. $\delta' = \delta$
4. $q' = q$
5. $F' = Q \setminus F$

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- If $w \in L \iff w \notin \overline{L}$, then $M(w)$ stops in some $q \in F$, so $q \notin (Q \setminus F)$

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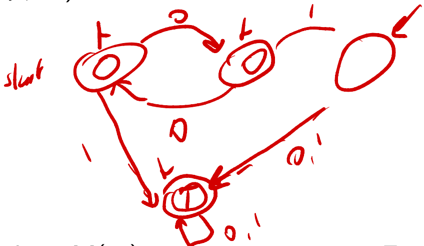

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- If $w \in L \iff w \notin \overline{L}$, then $M(w)$ stops in some $q \in F$, so $q \notin (Q \setminus F)$
- If $w \notin L \iff w \in \overline{L}$, then $M(w)$ stops in some $q \notin F$, so $q \in (Q \setminus F)$

Properties of Regular Languages

Closure Under Union

If L_1 and L_2 are both regular languages then $L_1 \cup L_2$ is also regular

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Intuition: Run both machines in parallel and accept if either of them stops in an accept state

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Proof: Let $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognize L_1, and $M_2 = ((Q_2, \Sigma, \delta_2, q_2, F_2)$ recognize L_2

Construct $M = (Q, \Sigma, \delta, q, F)$ that recognizes $L_1 \cup L_2$

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$$\delta((r_1, r_2), a) = (\delta_1(r_1, a), \delta_2(r_2, a))$$

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5. $F = \{(r_1, r_2) | r_1 \in F_1 \cancel{\text{ or }} \text{ And } r_2 \in F_2\}$

Properties of Regular Languages

Closure Under Intersection

If L_1 and L_2 are both regular languages then $L_1 \cap L_2$ is also regular

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Intuition: Run both machines in parallel (same as for union) and accept if BOTH of them stop in an accept state

Properties of Regular Languages

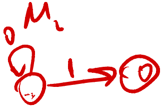
Closure Under Concatenation

If L_1 and L_2 are both regular languages then $L_1 \circ L_2$ is also regular

$L_1 \circ L_2 = \{xy | x \in L_1 \text{ and } y \in L_2\}$

011101

010111



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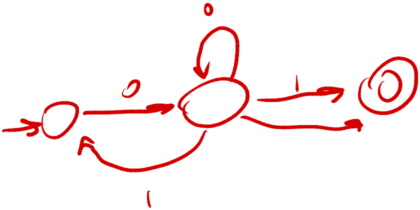
$0 \in L_1$



Outline

Nondeterminism

Deterministic Finite Automaton

- For every state q and every symbol $x \in \Sigma$, exactly one value $\delta(q, x)$ is defined
- State transitions only on an input symbol
- Execution of DFA is fully determined

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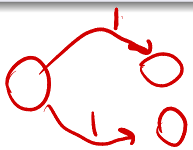
  e.g., $\delta(q1, 1) = \{q2, q3\}$
- Allow empty (ϵ) transitions – transitions not requiring an input



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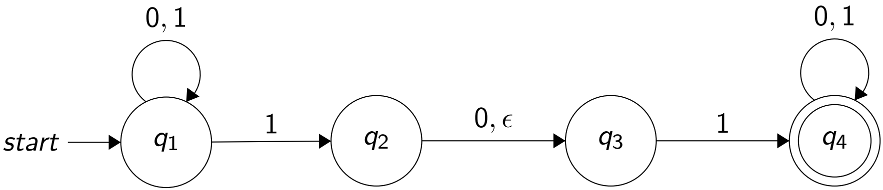
What is going on here?!?

What does non-determinism mean?

An Example NFA



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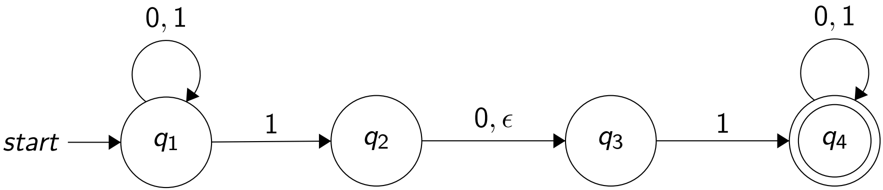

Input: 010

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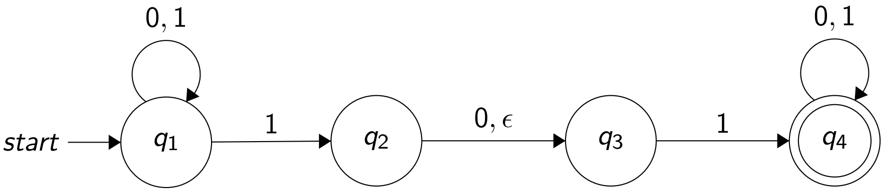

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Input: 010110

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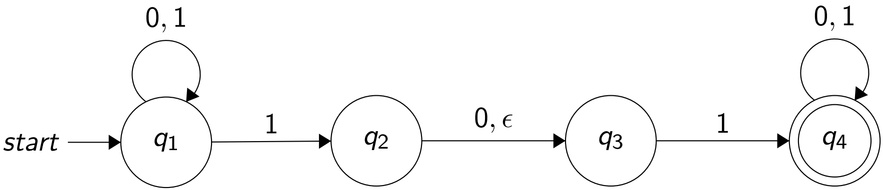
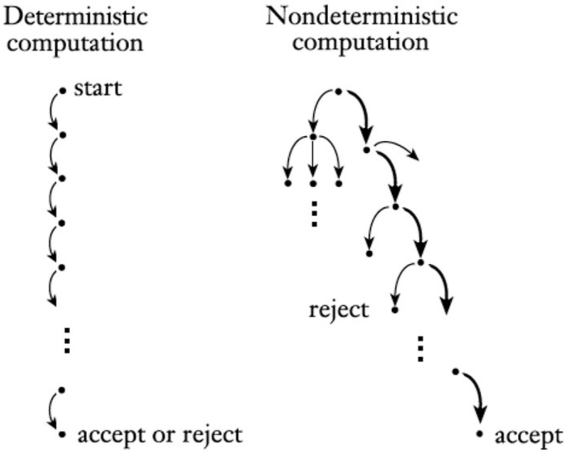


Input: 010
Input: 010110
Question: What language does this recognize?

Understanding Nondeterminism

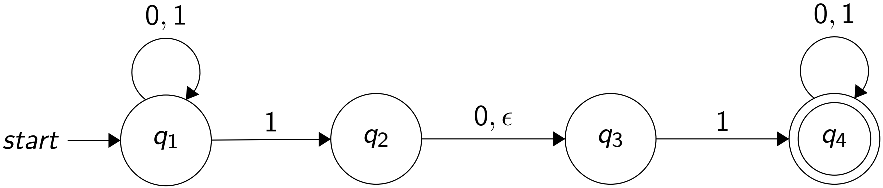
Interpretation 1: Try all paths in parallel



If any path leads to accept then accept

Understanding Nondeterminism

Interpretation 2: Guess and verify



Understanding Nondeterminism

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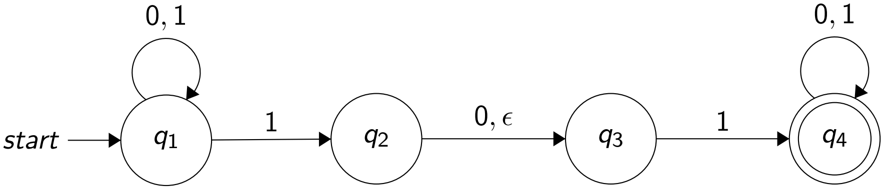

- M stays in q_1 until it "guesses" next input is 11 or 101

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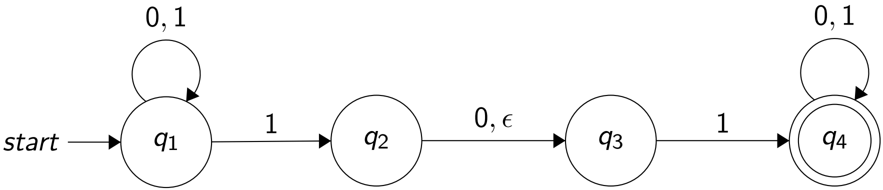

- M stays in q_1 until it "guesses" next input is 11 or 101
- Verifies that this guess was correct on path to q_4

Understanding Nondeterminism

Interpretation 3: Verifying a proof vs. finding a solution

Consider the execution of a finite automaton

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1. DFA execution on input x:
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    - Input x alone does not necessarily take you to an accept state
    - Need to somehow choose which edge to take whenever there is a choice
    - Can view this sequence of nondeterministic choices as a "witness" w that allows you to verify that $x \in L(M)$

Important

For any $x \notin L$, there must be no path to an accepting state – no possible "witness" works