Foundations of Computing Lecture 25

Arkady Yerukhimovich

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Outline

- 1 Lecture 24 Review
- A New Goal for Proofs
- 3 Defining Knowledge
- 4 Examples of Zero-Knowledge Proofs
- 5 Zero-Knowledge on the Blockchain

Lecture 24 Review

- Interactive Proofs
- Proof for Graph Non-Isomorphism
- Polynomial Identity Testing

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Reviewing the Definition of \mathcal{IP}

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A New Property

We say that a proof is *zero-knowledge* if the verifier learns nothing (other than the truth of the statement) from seeing the proof.

An Example – Where's Waldo



An Example



A Second Example - Puppy and Panda









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What does it mean for a machine to learn nothing from a proof?

Answer: Whatever it can (efficiently) compute after seeing the proof, it could have efficiently computed before seeing the proof.

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- IMPORTANT: $VIEW_V^*$ and S(x) are both distributions, not values. So, equality is of distributions

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- ② Soundness: Suppose G_0 is not isomorphic to G_1 , so there is no such π . Then, if $b \neq b'$, there is no permutation that P can give that V will accept

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Observations:

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- When S stops, he produces a perfect simulation



Graph 3-Coloring

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ZK Proofs enable privacy-preserving transactions on a public Blockchain!