Cryptography

Lecture 22

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November 13, 2024

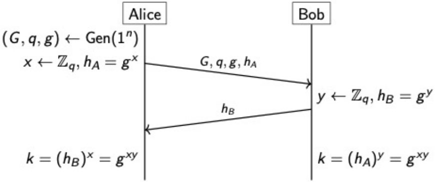
Outline

1. Lecture 21 Review
2. RSA Encryption Scheme (Chapter 11.5)
3. Hybrid Encryption and CCA Security

Lecture 21 Review

- Diffie-Hellman Key Exchange
- Public-key revolution
- From key exchange to public-key encryption

From Key Exchange to Public-Key Encryption



1. To encrypt $m \in G$ using g^{xy}, compute $m \cdot g^{xy}$.
   This is essentially a multiplicative OTP
2. To make encryption non-interactive, A sets pk to be her first message $pk_A = (G, q, g, h_A)$
3. To encrypt a message m, B has to complete the KE and use the resulting key to encrypt
   - Choose $y \leftarrow \mathbb{Z}_q$
   - Compute $g^{xy} \cdot m$
   - To enable A to decrypt, include B's message $h_B = g^y$ in c
4. To decrypt, A computes $(h_B)^x = g^{xy}$ and uses this to unmask m

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Correctness:

$$\hat{m} = c_2 / c_1^x = \frac{h^y \cdot m}{(g^y)^x} = \frac{(g^x)^y \cdot m}{(g^y)^x} = m$$

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  - $c = ([4^2 \mod 11], [9^2 \cdot 3 \mod 11]) = (5, 1)$

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  - So, we recover $m = [1 \cdot 3 \mod 11] = 3$

Defining CPA-Secure Public-Key Encryption

Let $\Pi = (\mathsf{Gen}, \mathsf{Enc}, \mathsf{Dec})$ be an encryption scheme. Consider the following game between an adversary $\mathcal{A}$ and a challenger:

$\mathsf{PrivK}^{cpa}_{\mathcal{A},\Pi}(n)$

- The challenger chooses $k \leftarrow \mathsf{Gen}(1^n)$
- $\mathcal{A}^{\mathsf{Enc}_k(\cdot)}(1^n)$ outputs m_0, m_1 such that $|m_0| = |m_1|$.
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- $\mathcal{A}^{\mathsf{Enc}_k(\cdot)}$ outputs a guess bit b'
- We say that $\mathsf{PrivK}^{cpa}_{\mathcal{A},\Pi}(n) = 1$ (i.e., $\mathcal{A}$ wins) if $b' = b$.

Definition: An encryption scheme $\Pi = (\mathsf{Gen}, \mathsf{Enc}, \mathsf{Dec})$ with message space $\mathcal{M}$ is CPA-secure if for all PPT $\mathcal{A}$ it holds that

$$\Pr[\mathsf{PrivK}^{cpa}_{\mathcal{A},\Pi}(n) = 1] \leq 1/2 + \mathsf{negl}(n)$$

Defining CPA-Secure Public-Key Encryption

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Security: $\Pr[\mathsf{PubK}^{cpa}_{\mathcal{A},\tilde{\Pi}}(n) = 1] = 1/2$ because g^z is random group element

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Contradiction

$\mathcal{A}_r$ can break the DDH assumption with non-negligible advantage

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Correctness:

$$\hat{m} = c_2 / c_1^x = \frac{h^y \cdot m}{(g^y)^x} = \frac{(g^x)^y \cdot m}{(g^y)^x} = m$$

Security: El Gamal is CPA-secure based on DDH

Outline

Plain RSA

Recall:

- GenRSA(1^n): Choose $N = pq$, e, d s.t. $ed = 1 \bmod \phi(N)$
- RSA Assumption: Given $y = x^e \bmod N$, hard to find x

Plain RSA

Recall:

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- $p = 3, q = 7, N = 21, \phi(N) = 12$

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- Suppose $e = 5$, then $d = 5^{-1} \bmod 12 = 5$
- $\text{Enc}_{(21,5)}(11) = [11^5 \bmod 21] = [(16)(16)(11) \bmod 21] = 2$

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Encryptions So Far

El Gamal Encryption: $\Pi = (\mathsf{Gen}, \mathsf{Enc}, \mathsf{Dec})$ with $\mathcal{M} = G$

- $\mathsf{Gen}(1^n)$:
  - $(G, q, g) \leftarrow \mathsf{Gen}(1^n)$
  - $x \leftarrow \mathbb{Z}_q$, $h = g^x$
  - $pk = (G, q, g, h)$ and $sk = x$
- $\mathsf{Enc}_{pk}(m)$:
  - Given $pk = (G, q, g, h)$ and message $m \in G$
  - $y \leftarrow \mathbb{Z}_q$, compute $c = (g^y, h^y \cdot m)$
- $\mathsf{Dec}_{sk}(c)$:
  - Given $sk = x$ and $c = (c_1, c_2)$
  - Compute $\hat{m} = c_2 / c_1^x$

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   - But, $m^3 < N_1 \cdot N_2 \cdot N_3$, so can take e^{th} root over integers

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Padded RSA Encryption Scheme

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Let $\ell(n) \leq 2n - 4$

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Security:

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- Avoids small message and multiple receivers attacks through random padding

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Security:

- Avoids small message and multiple receivers attacks through random padding
- Conjectured to be CPA secure, though not known how to prove this

Outline

1. Lecture 21 Review
2. RSA Encryption Scheme (Chapter 11.5)
3. Hybrid Encryption and CCA Security

Hybrid Encryption

Public-key encryption is 2-3 orders of magnitude slower than private-key
Question: Can we achieve functionality of public-key encryption at cost of private-key encryption?

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- This is how most public-key encryption used in practice

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Malleability

Both El Gamal and RSA are not CCA-secure because they are *malleable*.

Building CCA-Secure Public-Key Encryption – Try 1

Recall that Encrypt-then-authenticate was CCA-secure in the secret-key model

Question

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- Unfortunately, there is a private/public key mismatch
- We want encryption to be done by anyone, but signing key is private
- Without signing key, public cannot encrypt

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- Not known whether you can turn CPA-secure encryption into CCA-secure encryption generically