CS 3313
Foundations of Computing:

Closure Properties of RE & Recursive Languages

http://gw-cs3313.github.io

Problems

- Informally, a (decision) "problem" is a yes/no question about an infinite set of possible *instances*.
- Example 1: "Does graph G have a *Hamilton cycle* (cycle that touches each node exactly once)?
  - Each undirected graph is an instance of the "Hamilton-cycle problem."
- Example 2: "Is graph G k-colorable ?
  - Each undirected graph, and value k, is an instance of the "graph coloring problem."
- Example 3: "Does the DFA M accept an infinite set?"

Problems – (2)

- a problem is also a language.
- Each string encodes some instance.
- The string is in the language if and only if the answer to this instance of the problem is “yes.”

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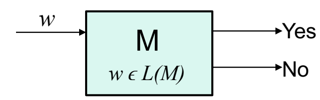
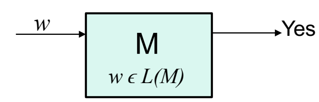
Recall Definitions

- Recursive Language: A language L is recursive language if there is a Turing machine that accepts the language and halts on all inputs



- Recursively Enumerable Language: if there is a Turing machine that accepts the language by halting when the input string is in the language
  - The machine may or may not halt if the string is not in the language



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Decidable Problems

- A problem is *decidable* if there is an algorithm to answer it.
  - Recall: An "algorithm," formally, is a TM that halts on all inputs, accepted or not.
  - Put another way, "decidable problem" = "recursive language."
- Otherwise, the problem is *undecidable*.

- Functions: a function is computable is there is a Turing machine that computes it and halts on all inputs

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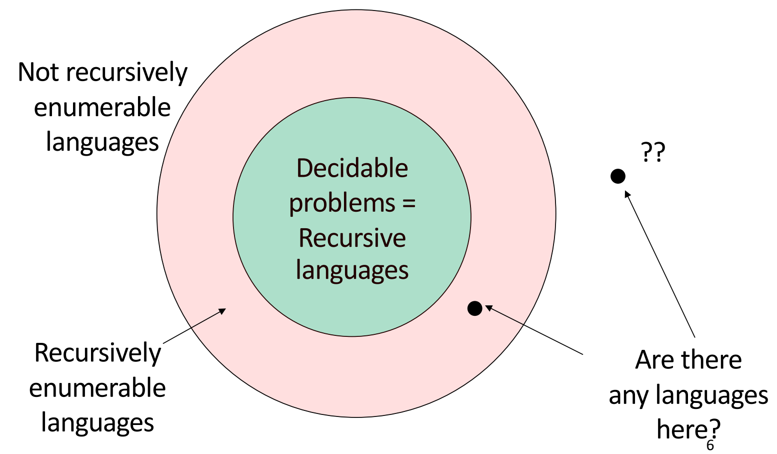
Bullseye Picture



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Closure Properties of Recursive and RE Languages

- Next topic is Decidability
  - Review Lab notes on Math review – diagonalization etc.
- First let's look at closure properties of these classes of languages
- Both closed under union, concatenation, star, reversal, intersection, inverse homomorphism.
- Recursive closed under difference, complementation.
- RE closed under homomorphism.

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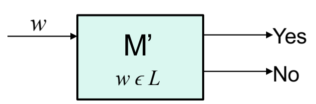
Proving Closure Properties…methodology

- *Observe: To prove the closure properties we have to construct a Turing machine, i.e., an algorithm (!!!), to accept the language*
  - *Construction shown using a flowchart & combining other "algorithms"*
    - *Getting more and more like programming!*
- To prove a language L (constructed from other recursive languages) is recursive, provide an algorithm described by a 'flowchart' below
  - To show it is RE, the machine halts only if w is in the language



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Closure under Union

- Let $L_1 = L(M_1)$ and $L_2 = L(M_2)$.
- Assume M_1 and M_2 are single-semi-infinite-tape TM's.
- Construct 2-tape TM M to copy its input onto the second tape and simulate the two TM's M_1 and M_2 each on one of the two tapes, "in parallel."
- Recursive languages: If M_1 and M_2 are both algorithms, then M will always halt in both simulations.
- RE languages: accept if either accepts, but you may find both TM's run forever without halting or accepting.

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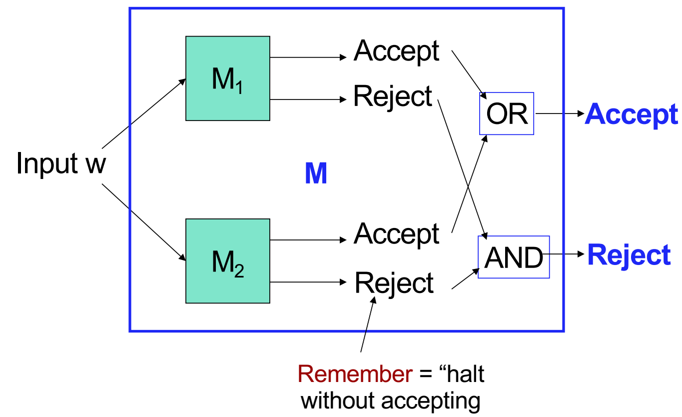
Algorithm/Picture of Union for Recursive Sets

M must halt on all inputs:
accepts if w is in either, rejects if w not in either



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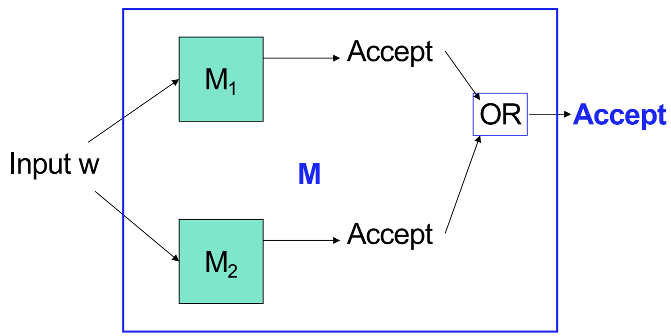
Picture of Union of RE Sets

M must halt and accept if w is in either language, else it may reject and halt or may not halt



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Closure under other set operations

- Recursive Languages are closed under
  - Union, Intersection, Concatenation, Star Closure
  - Complementation. Set difference
  - Reversal
  - Inverse Homomorphism
- Recursively Enumerable (RE) languages are closed under
  - Union, Intersection, Concatenation, Star Closure
  - Reversal
  - Homomorphism
  - Inverse Homomorphism

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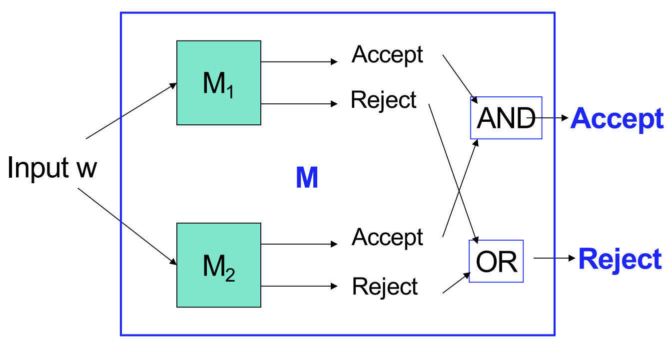
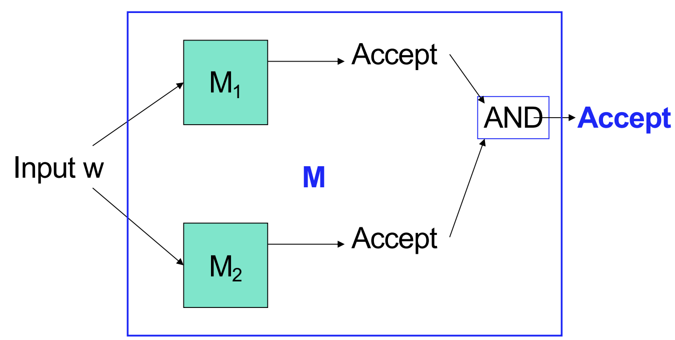
Intersection of Recursive Sets – Same Idea

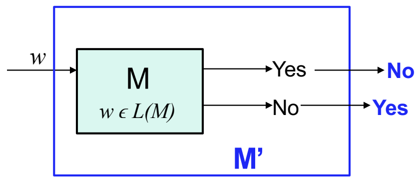


Intersection of RE Sets



Observe: if w is in the intersection then both
machines will accept and halt on w =>
This machine M will halt and accept w

Complement of Recursive Languages



Set Difference

- Recursive languages: both TM's will eventually halt.
- Accept if M_1 accepts and M_2 does not.
  - *Corollary*: Recursive languages are closed under complementation.
- RE Languages: can't do it; M_2 may never halt, so you can't be sure input is in the difference (or complement)

Concatenation of RE Languages

- Let $L_1 = L(M_1)$ and $L_2 = L(M_2)$.
- Assume M_1 and M_2 are single-semi-infinite-tape TM's.
- Construct 2-tape Nondeterministic TM M:
  1. Guess a break in input $w = xy$
  2. Move y to second tape.
  3. Simulate M_1 on x, M_2 on y.
  4. **Accept if both accept**.

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Concatenation of Recursive Languages

- Can't use a NDTM.
- Systematically try each break w = xy.
- M_1 and M_2 will eventually halt for each break.
- Accept if both accept for any one break.
- Reject if all breaks tried and none lead to acceptance.

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Star Closure

- Same ideas work for each case.
- RE: guess many breaks, accept if M_1 accepts each piece.
- Recursive: systematically try all ways to break input into some number of pieces.

Other closure properties: Reversal, Homomorphisms,….

- Reversal:
  - Start by reversing the input.
  - Then simulate TM for L to accept w if and only w^R is in L.
  - Works for either Recursive or RE languages.
- Inverse homomorphism
  - Apply h to input w.
  - Simulate TM for L on h(w).
  - Accept w iff h(w) is in L.
  - Works for Recursive or RE.

Homomorphism/RE

- Let $L = L(M_1)$.
- Design NDTM M to take input w and guess an x such that $h(x) = w$.
- M accepts whenever M_1 accepts x.
- Note: won't work for Recursive languages.

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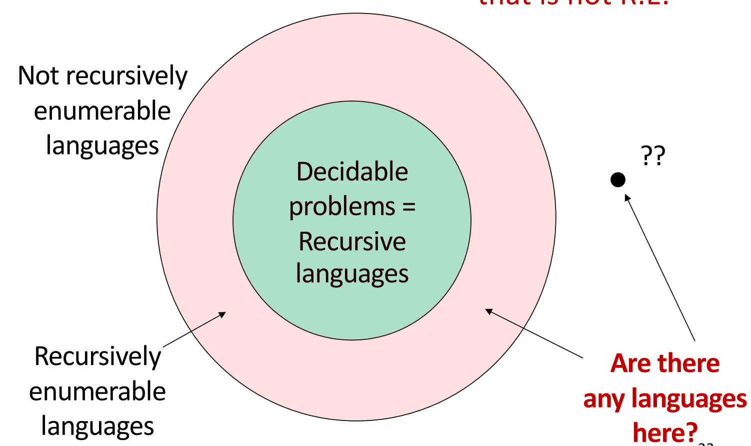
Undecidable Problems

Next: Find a language that is not R.E.



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