Cryptography

Lecture 17

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October 28, 2024

Outline

1. Lecture 16 Review
2. A Modern Cryptography Approach (Chapter 8.Intro)
3. A Brief Intro to Number Theory (Chapter 8.1)
4. A Brief Intro to Group Theory (Chapter 8.1)

Lecture 16 Review

- AES review
- Feistel Networks and DES
- Davies-Meyer Transform

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Key Question

How can we build crypto on clean mathematical foundations?

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Added Functionality

We will show next week, how this modern crypto approach leads to the development of *public-key* cryptography.

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Fundamental Theorem of Arithmetic

All positive integers $n > 1$ can be expressed uniquely (up to ordering) as $n = \prod p_i^{\ell_i}$ for primes p_i

Greatest Common Divisor (gcd)

Definition

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2. If $a|N$, $b|N$, and $gcd(a, b) = 1$ then $ab|N$

$a = (p_1^{e_1} \cdot p_2^{e_2} \cdots)$

$b = (p_1^{f_1} \cdot p_2^{f_2})$

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3. If $a, b \in \mathbb{Z}^+$, there exist $X, Y \in \mathbb{Z}$ such that $Xa + Yb = gcd(a, b)$
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4. $gcd(a, b) = gcd(b, [a \bmod b])$ if $a, b > 1$ such that $b \nmid a$

Euclidean Algorithm

Goal

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Both of these are poly-time in $||a||$ and $||b||$

Modular Arithmetic

Notation: For Integers a, b, N

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Example: $8 \neq [3 \bmod 5]$, but $8 = 3 \bmod 5$

Modular Arithmetic

Congruence Relation

Congruence mod N is an *equivalence relation* that obeys standard rules of arithmetic.

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Division $\mod N$

Let $b, N \in \mathbb{Z}, b \geq 1, N > 1$, b is invertible $\mod N$ (i.e., can divide by b) if and only if $gcd(b, N) = 1$

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- $\mathbb{Z}_N = \{0, 1, \ldots, N-1\}$ is a group under addition $\mod N$

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$$\begin{aligned} ac = bc \implies (ac)c^{-1} = (bc)c^{-1} &\implies a(cc^{-1}) = b(cc^{-1}) \\ &\implies a \cdot 1_G = b \cdot 1_G \implies a = b \end{aligned}$$

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   - Proof (for abelian groups):
     Consider $(gg_1), (gg_2), \ldots, (gg_m)$ where $g_1, \ldots, g_m \in G$
     Since $(gg_i) = (gg_j)$ iff $g_i = g_j$ (by [1]), each of the (gg_i) is distinct
     Now, we have that

$$g_1 \cdot g_2 \cdots g_m = (gg_1) \cdot (gg_2) \cdots (gg_m) = g^m \cdot (g_1 \cdot g_2 \cdots g_m)$$

     First equality holds because the (gg_i) are all possible values in G.
     So, $g^m = 1$

Important Properties of Groups

1. $\forall a, b, c \in G$, if $ac = bc$, then $a = b$
2. Let $|G| = m$, $\forall g \in G, g^m = 1$
3. Let $|G| = m$, then for any $g \in G$ and any $x \in \mathbb{Z}$, $g^x = g^{[x \bmod m]}$

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   - Proof:
     Let $x = qm + r$ where $q, r \in \mathbb{Z}$ and $r = [x \bmod m]$

$$g^x = g^{qm+r} = g^{qm} \cdot g^r = (g^m)^q \cdot g^r = 1_G^q \cdot g^r = g^r$$

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4. Let $|G| = m$, and let $e > 0 \in \mathbb{Z}$. Define $f_e : G \to G$ by $f_e(g) = g^e$.
   If $gcd(e, m) = 1$, then f_e is a permutation over G.
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   - Proof: Enough to prove that f_d is inverse of f_e
     For any $g \in G$, we have:

$$f_d(f_e(g)) = f_d(g^e) = (g^e)^d = g^{ed} = g^{[ed \bmod m]} = g^1 = g$$

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- $\mathbb{Z}_N =< 1 >$
- $\mathbb{Z}_p^*$ – Not all $g \in \mathbb{Z}_p^*$ are generators: $< 2 >= \{1, 2, 4\} \neq \mathbb{Z}_7^*$
  but, $< 3 >= \{1, 3, 9 = 2, 6, 4, 5\}$ is a generator