Cryptography

Lecture 20

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Outline

Lecture 19 Review

- Number-Theoretic Hardness Assumptions
- Assumptions in $\mathbb{Z}_N^*$: Factoring, RSA
- Assumptions in Cyclic Groups: DLOG, CDH, DDH

Assumptions in $\mathbb{Z}_N^*$

Factoring Problem

Given $N = pq$ when p and q are n-bit primes, find p and q

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Factoring Problem

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RSA Problem

Given $(N = pq, e)$ s.t. $gcd(e, \phi(N)) = 1$ and $y \in \mathbb{Z}_N^*$, compute $[y^{1/e} \bmod N]$

Assumptions in Cyclic Groups

Let G be a cyclic group of order q with generator g

Discrete Log Problem

Given $h \in G$, find $0 \leq x \leq q-1$ s.t. $g^x = h$. We say $x = \log_g h$

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Decisional Diffie-Hellman (DDH) Problem

Given $h_1 = g^x$, $h_2 = g^y$, distinguish g^{xy} from g^z for $z \leftarrow \mathbb{Z}_q$

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  - $G(s)$ outputs 3 field elements in G of order q
  - So $|G(s)| > |s|$

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- Pseudorandomness:
  - g^x, g^y are random group elements
  - If $\mathcal{A}_c$ can distinguish (g^x, g^y, g^{xy}) from random, then $\mathcal{A}_r$ just runs $\mathcal{A}_c$ to break DDH

Fixed-length Compression-Function from DLog

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- So, $\mathcal{A}_r$ computes $a = \frac{x-x'}{y'-y}$ breaking DLog of h

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- By DDH, we cannot distinguish such terms from random

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Resulting Challenges:

- Key distribution – how to share the keys in the first place
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- “Open systems” – users don’t know each other (e.g., shopping on Amazon)

Solution 1: Key-distribution Centers (KDC)

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- Does not work with open systems – KDC must know all users

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The Power of Key Exchange

Key agreement allows generation of shared secrets without private communication.

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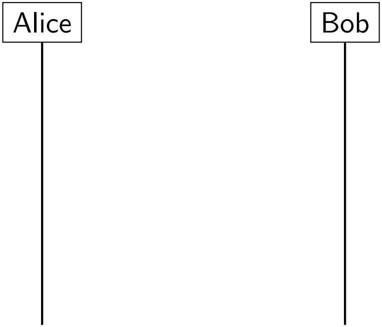
Definition: A key exchange protocol Π is secure against an eavesdropper if for all PPT $\mathcal{A}$ it holds that

$$\Pr[\mathsf{KE}^{eav}_{\mathcal{A},\Pi}(n) = 1] \leq 1/2 + \mathsf{negl}(n)$$

Diffie-Hellman Key Exchange



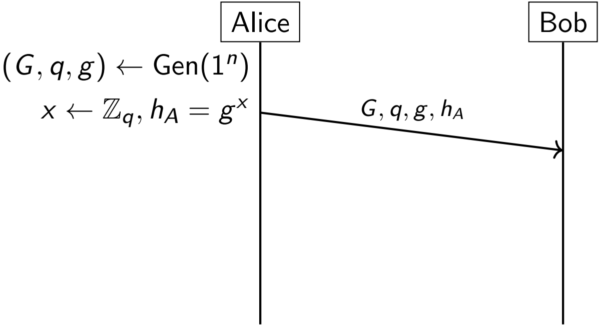
Diffie-Hellman Key Exchange

Alice	Bob

$(G, q, g) \leftarrow \mathsf{Gen}(1^n)$

$x \leftarrow \mathbb{Z}_q, h_A = g^x$

Diffie-Hellman Key Exchange



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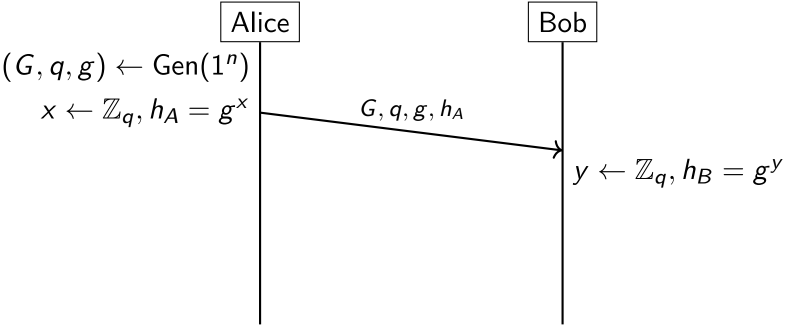

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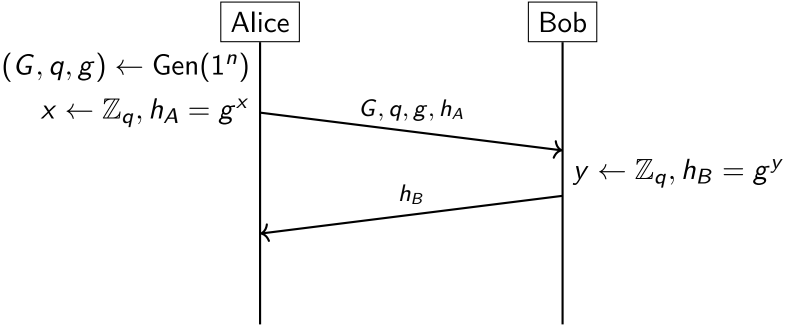

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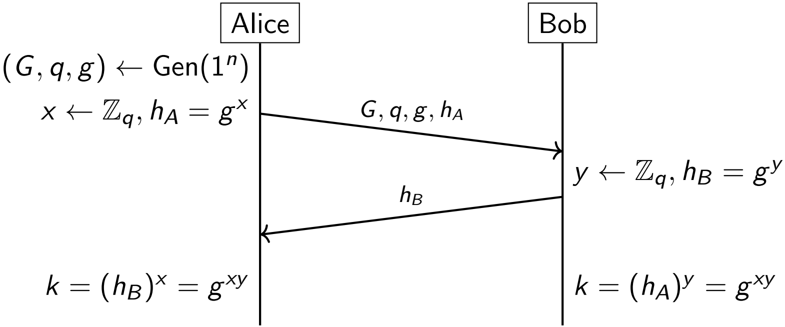

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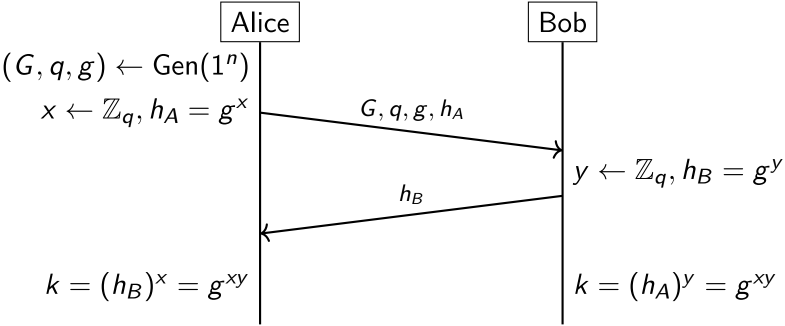

Correctness

Easy to see that A and B output the same key k

Diffie-Hellman Key Exchange – Proof of Security

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- He plays the roles of A and B, producing transcript trans, sets $\hat{k} = c$ and gives both to $\mathcal{A}_c$

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What $\mathcal{A}_c$ sees:

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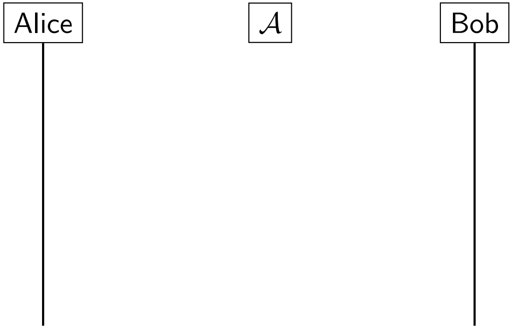
- Can use hash to convert random group element to a random string

Man-in-the-middle Attack



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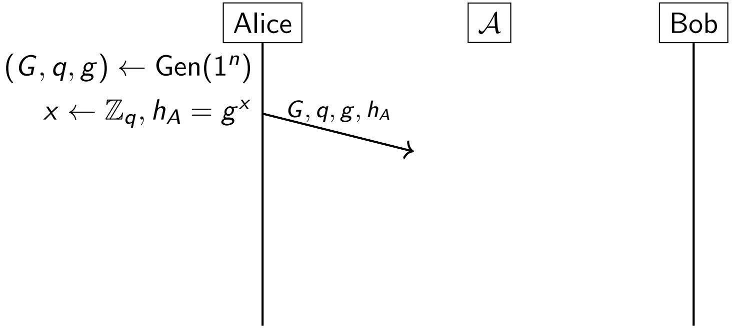

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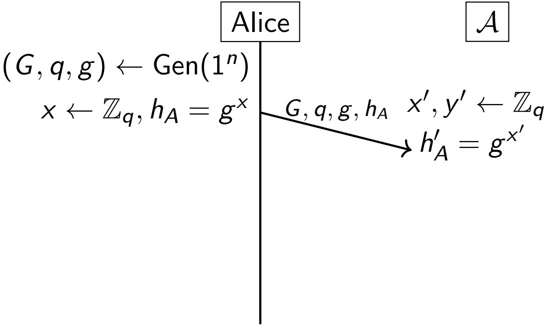

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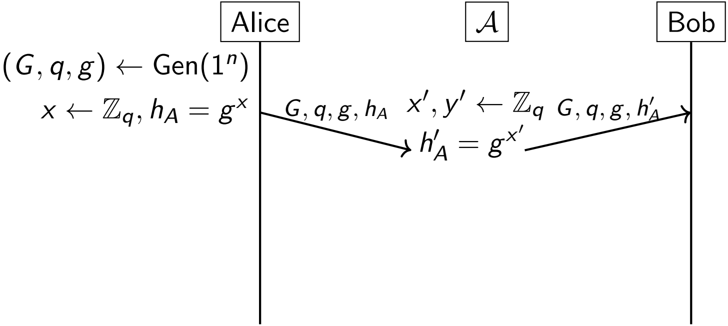

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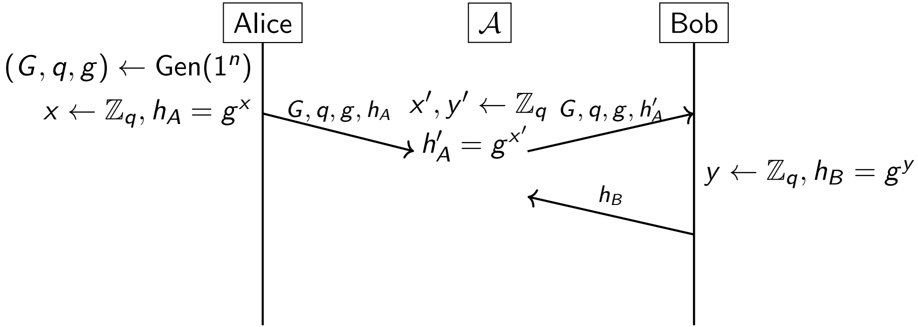

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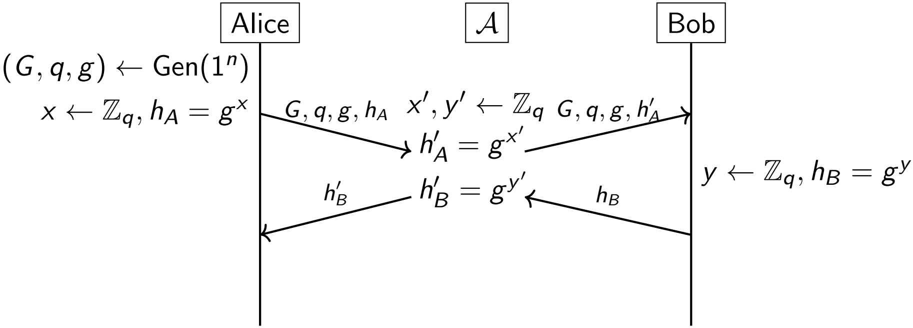

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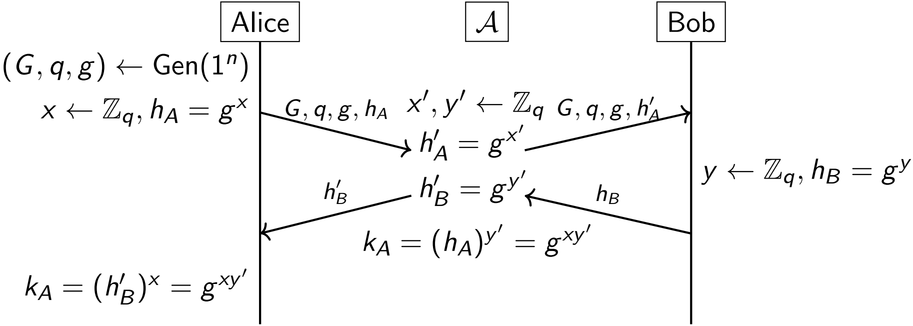

Man-in-the-middle Attack

Alice	$\mathcal{A}$	Bob
$(G, q, g) \leftarrow \mathsf{Gen}(1^n)$		
$x \leftarrow \mathbb{Z}_q, h_A = g^x$	$x', y' \leftarrow \mathbb{Z}_q$	
$\xrightarrow{G, q, g, h_A}$	$h'_A = g^{x'}$	$\xrightarrow{G, q, g, h'_A}$
		$y \leftarrow \mathbb{Z}_q, h_B = g^y$
$\xleftarrow{h'_B}$	$h'_B = g^{y'}$	$\xleftarrow{h_B}$
	$k_A = (h_A)^{y'} = g^{xy'}$	
$k_A = (h'_B)^x = g^{xy'}$	$k_B = (h_B)^{x'} = g^{x'y}$	$k_B = (h'_A)^y = g^{x'y}$

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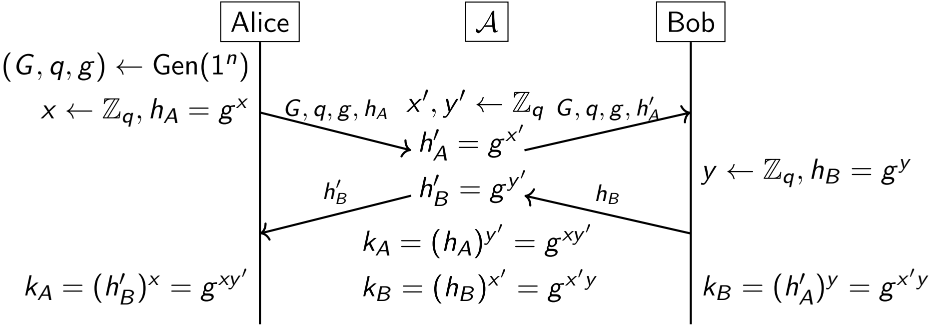

Result

- $k_A \neq k_B$ – A and B fail to agree on a key
- $\mathcal{A}$ has shared keys with both

Going Beyond Key Exchange

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Secrecy	Private-key encryption	Public-key encryption
Integrity	MACs	Digital signatures

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- A has keys (pk_A, sk_A)
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Digital signatures

- A has keys (pk_A, sk_A)
- Secret key sk_A is used by A to sign messages
- Public key pk_A is used to verify A's signatures

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- Only A can sign, anybody can verify

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Public-key crypto today

Public-key cryptography enables today's Internet and more:

- When you buy something on Amazon
- When you surf the web
- . . .