Cryptography

Lecture 24

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Outline

Lecture 23 Review

- Defining digital signatures
- Applications of signatures
- RSA digital signature

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1. Lecture 23 Review
2. Digital Signatures from Private-Key Techniques
3. Digital Signatures from Discrete Log

One-Way Functions

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Observations:

- $\mathcal{A}$ does not necessarily have to recover x to win
- One-way functions are the most basic private-key primitives
- We've seen many examples: CRHFs, PRG, RSA

One-Time Signature Scheme

Let Π be a digital signature scheme. Consider the following game between an adversary $\mathcal{A}$ and a challenger:

$\mathsf{SigForge}_{\mathcal{A},\Pi}(n)$

- Challenger runs $(pk, sk) \leftarrow \mathsf{Gen}(1^n)$ and gives pk to $\mathcal{A}$
- $\mathcal{A}$ gets pk and oracle access to $\mathsf{Sign}_{sk}(\cdot)$ and outputs (m, σ)
  - Let Q denote the set of $\mathsf{Sign}_{sk}(\cdot)$ queries made by $\mathcal{A}$
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- Informally: After seeing one signature, $\mathcal{A}$ can't forge another one
- This is not a very useful notion of security, but we will use it as a building block

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- Hence, $\mathcal{A}$ needs to invert f on the corresponding $y_{i,b}$ in the pk

Assume $\mathcal{A}_c$ breaks OTS security of Lamport, build $\mathcal{A}_r$ that inverts f

Lamport OTS – Proof of Security

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$\mathcal{A}_r(y = f(x))$ – Inverts f on y

- $\mathcal{A}_r$ chooses $i^* \leftarrow \{1, \ldots, \ell\}$, $b^* \leftarrow \{0, 1\}$, and sets $y_{i^*,b^*} = y$

$$pk \begin{pmatrix} y_{1,0} & & & y_{\ell,0} \\ y_{1,1} & \cdots & \boxed{} & y_{\ell,1} \end{pmatrix}$$

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- When $\mathcal{A}_c$ requests signature on a message m

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  - If $m_{i^*} = b^*$, then $\mathcal{A}_r$ aborts
  - Otherwise, return $\sigma = (x_{1,m_1}, \ldots, x_{\ell,m_\ell})$ – $\mathcal{A}_r$ knows all these x_i's

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- When $\mathcal{A}_c$ outputs a forgery (m', σ')

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Lamport OTS – Proof of Security

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Improving on Lamport

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- Include hash key s in public key
- Secure if H is CRHF

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- number of possible signatures, t, bounded at Gen time
- Signature is *stateful* – problematic if state is reset

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- Generate key pairs (pk_i, sk_i) as we go
- Use sk_i to sign $m_i||pk_{i+1}$

pk_1

σ_1

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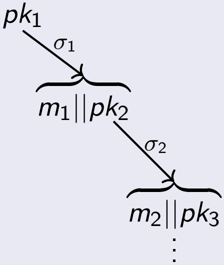

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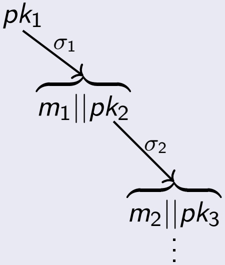

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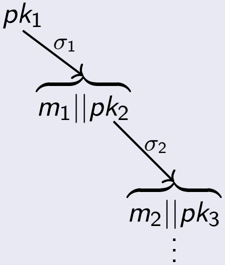

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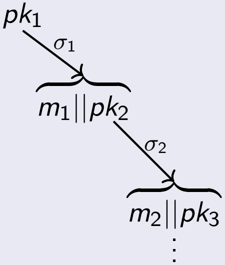

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$$pk_1 \xrightarrow{\sigma_1} \overbrace{m_1||pk_2} \xrightarrow{\sigma_2} \overbrace{m_2||pk_3} \quad \vdots$$

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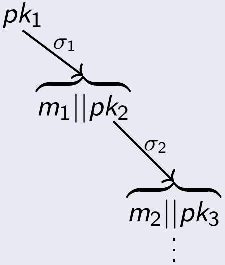

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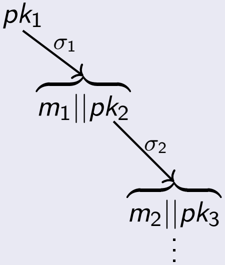

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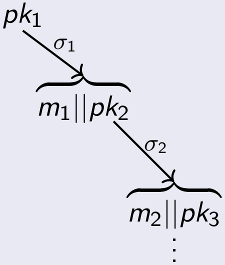

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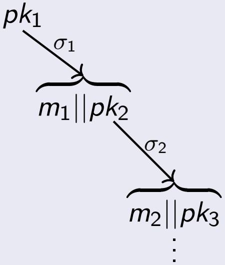

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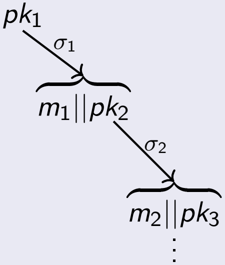

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Limitations:

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- Still have to store state

Going Beyond One Signature

Try 3 – Tree-Based Signatures

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\begin{array}{c}
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\downarrow \\
pk_0 || pk_1
\end{array}
$$

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$$\begin{tikzcd}[column sep=small]
pk_\epsilon \arrow[d] & \\
pk_0 || pk_1 \arrow[dr] & \\
& pk_{10} || pk_{11}
\end{tikzcd}$$

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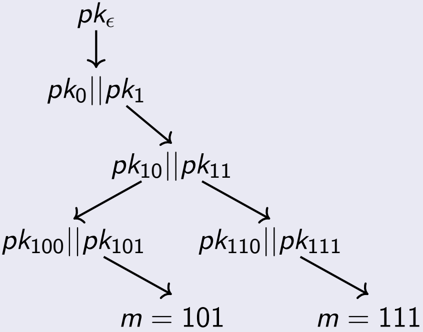

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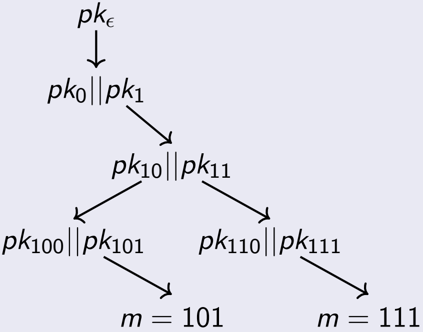

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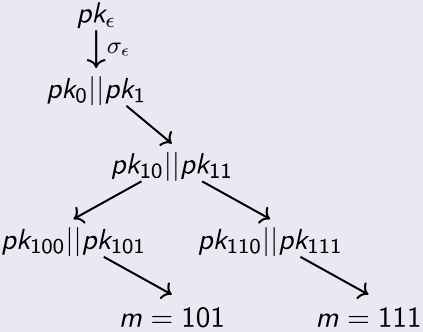

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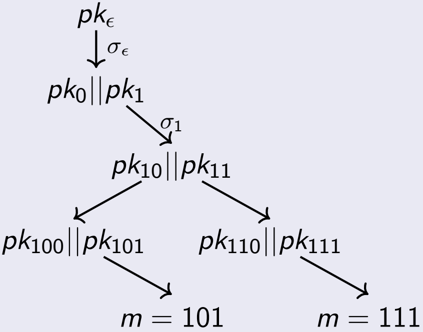

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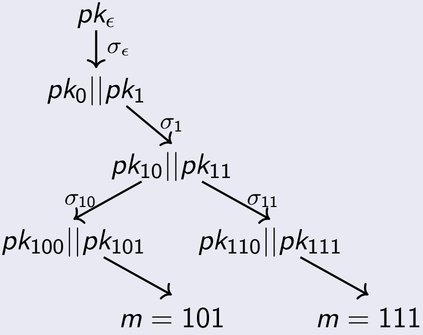

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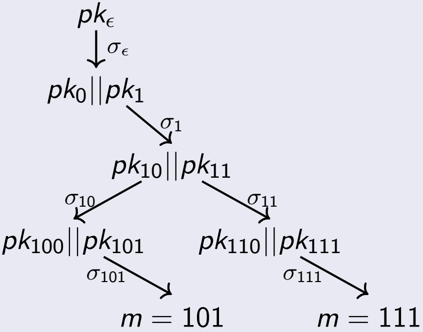

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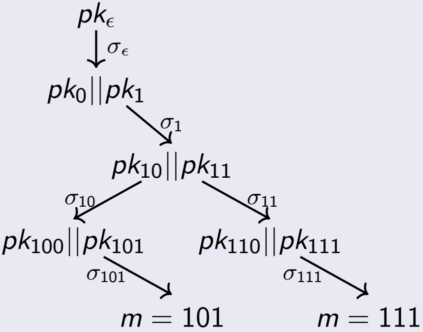

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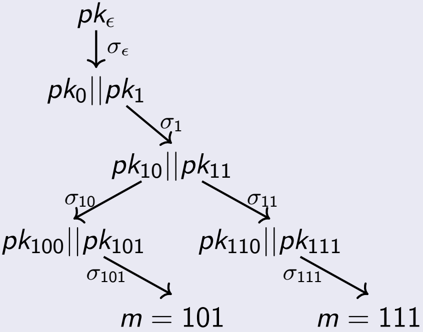

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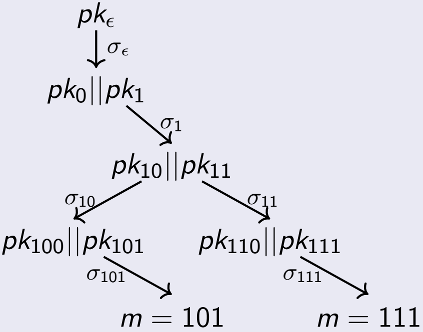

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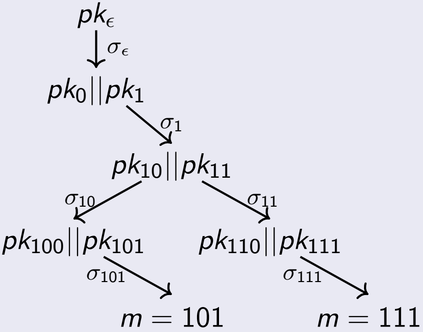

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- Only requires private-key primitives (OWFs, CRHFs)

Comparison to Public-Key Signatures

- Surprisingly, we can build signatures from private-key techniques
- But, public-key based signatures are more efficient (shorter sigs)

Outline

Identification Schemes

Identification Scheme:

- Interactive protocol to allow a party to prove identity

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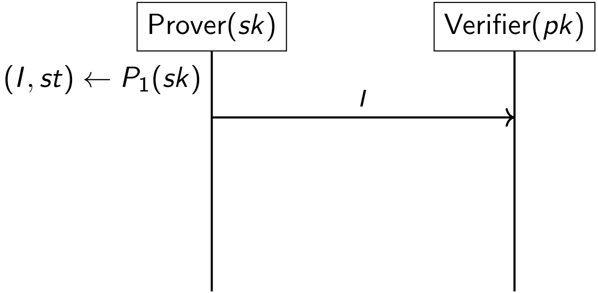

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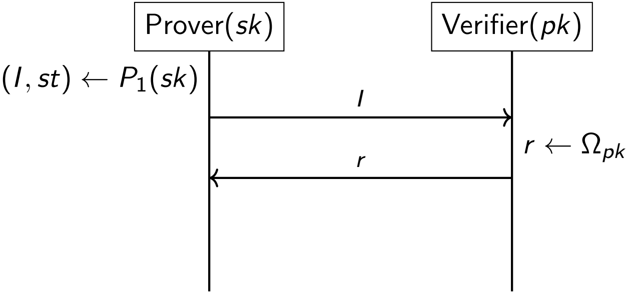

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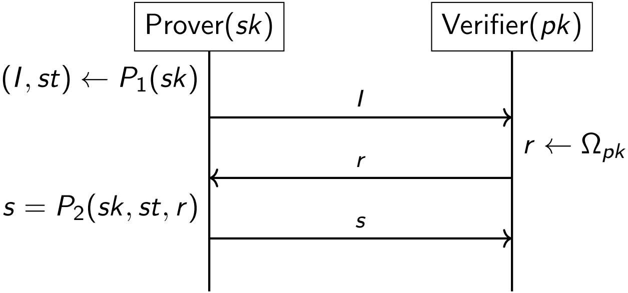

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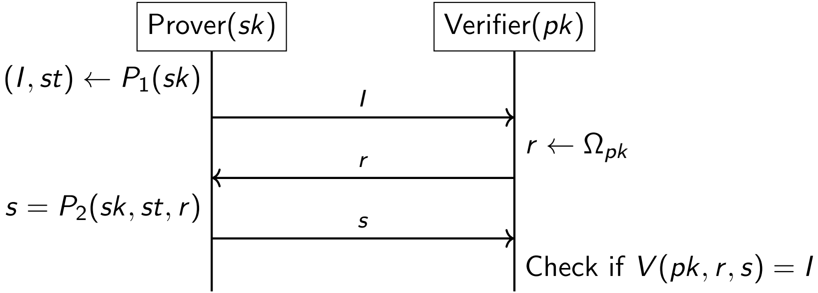

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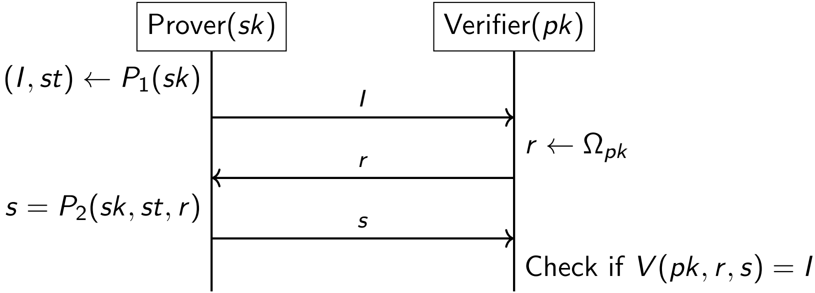

Non-degenerate:

- ID scheme is non-degenerate if for all sk, each I occurs with only negligible probability.

Security of Identification Schemes

Informal

Prover should not be able to get Verifier to accept without knowing sk

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Security of Identification Schemes

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Definition: An ID scheme $\Pi = (\mathsf{Gen}, P_1, P_2, V)$ is *secure* if for all PPT $\mathcal{A}$,

$$\Pr[\mathsf{Ident}_{\mathcal{A},\Pi}(n) = 1] \leq \mathsf{negl}(n)$$

Schnorr ID Scheme

Prover($x \in \mathbb{Z}_q$)

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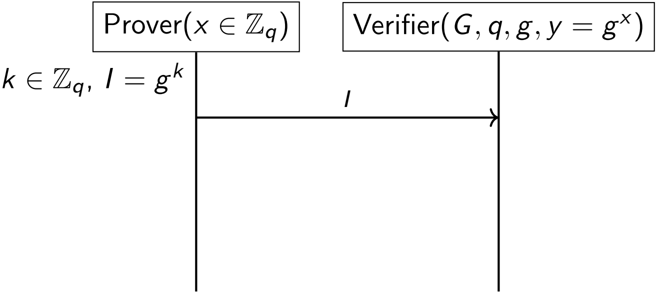
Schnorr ID Scheme



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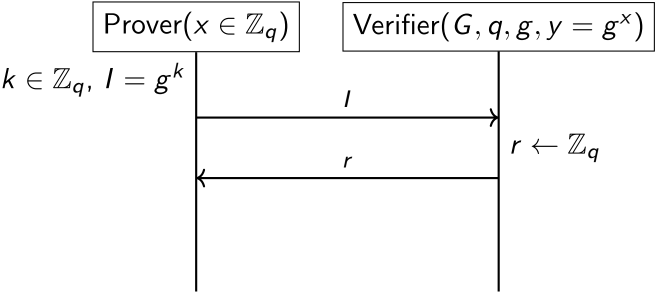

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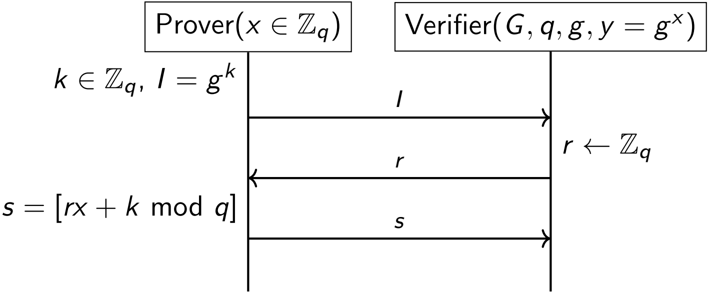

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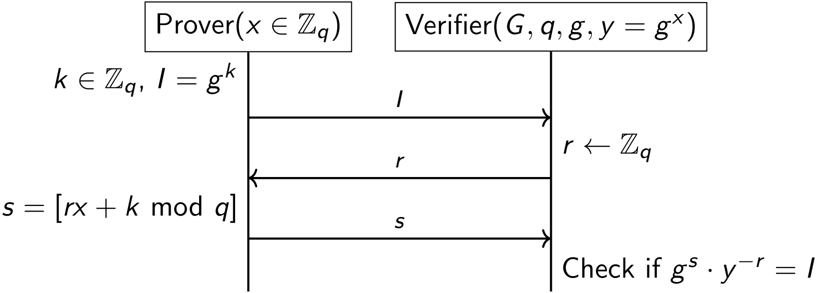

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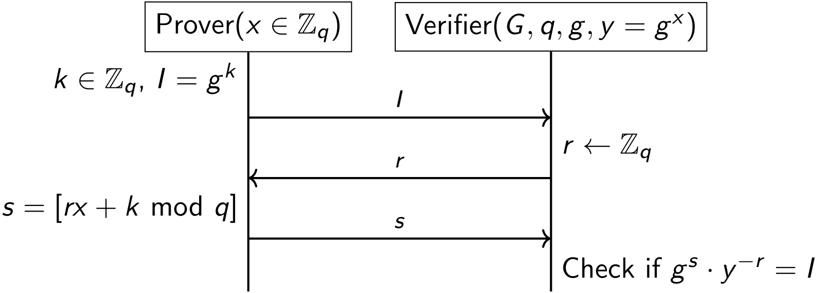

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Security of Schnorr ID Scheme

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  - If $\mathcal{A}_c$ succeeds twice, then break DLOG

Fiat-Shamir Transform

Goal

Convert 3-round ID scheme as earlier to a non-interactive signature

- Key Idea: Have Signer compute r himself using a hash

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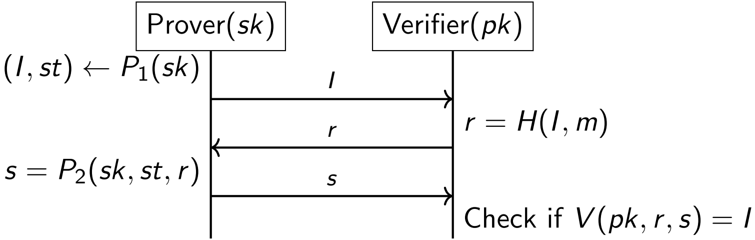

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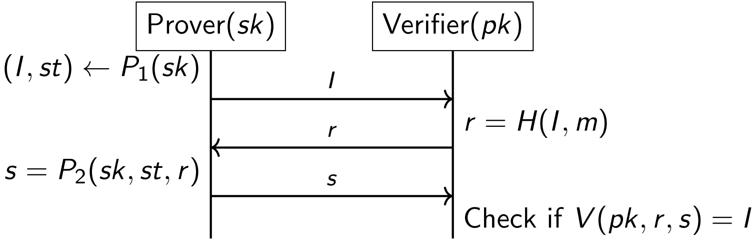

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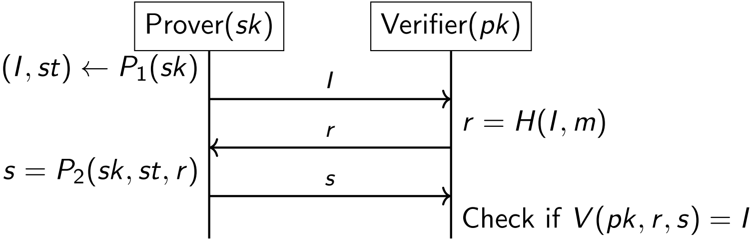

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- Security: Secure if H is modeled as random oracle

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$\text{Prover}(x \in \mathbb{Z}_q)$		$\text{Verifier}(G, q, g, y = g^x)$
$k \in \mathbb{Z}_q,\ I = g^k$	$\xrightarrow{I}$	
	$\xleftarrow{r}$	$r \leftarrow \mathbb{Z}_q$
$s = [rx + k \bmod q]$	$\xrightarrow{s}$	
		Check if $g^s \cdot y^{-r} = I$

Correctness: $g^s \cdot y^{-r} = g^{(rx+k)} \cdot g^{-rx} = g^k = I$

Apply Fiat Shamir

Replace $r \leftarrow \mathbb{Z}_q$ with $H(I, m)$

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Digital Signature Algorithm (DSA)

- Standard DSA algorithm uses similar paradigm, achieves same security