Foundations of Computing

Lecture 6

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Outline

Lecture 5 Review

- Regular expressions
- Equivalence of regular expressions and NFAs/DFAs

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What We Know So Far

The following four things are equivalent:

1. Regular languages
2. Languages recognized by a DFA
3. Languages recognized by an NFA
4. Languages described by a regular expression

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Are all languages regular?

Today we will see that there are languages that are not regular

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This means that:

- The number of states is fixed independently of the input size
- An automaton must be able to process strings w s.t. $|w| > |Q|$
- Thus, a finite automaton cannot store its whole input

A Nonregular Language

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$$L = \{0^n 1^n | n \geq 0\}$$

Let's try to build a DFA for L:

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The Problem

We need to count the number of 0s, but this is unbounded so can't have a state for each value

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What we just saw

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Consider the following language:

$L = \{w | w \text{ has an equal number of occurrences of } 01 \text{ and } 10 \text{ as substrings}\}$

General Proof Structure

We will prove that a language L is not regular by contradiction

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2. Pick a string $w \in L$
3. Show that if $M(w) = 1$ then there exists a string $w' \notin L$ s.t. $M(w') = 1$
4. Conclude that L is not regular since any M that accepts all strings in L must also accept strings not in L

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Next steps:

1. Prove the pumping lemma
2. Show how to use the pumping lemma to prove languages nonregular

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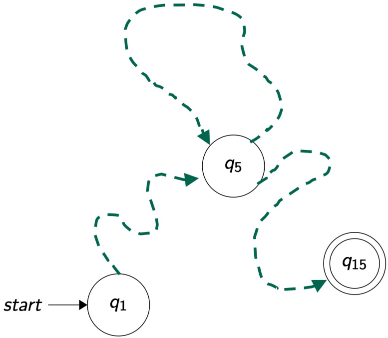
  - Since $n + 1 > p$, there must be some state that is visited twice



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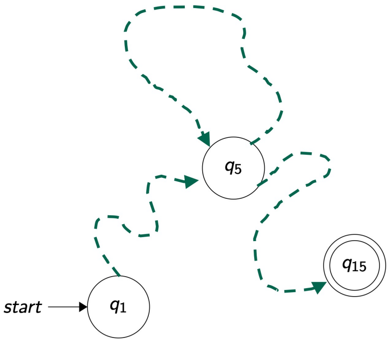

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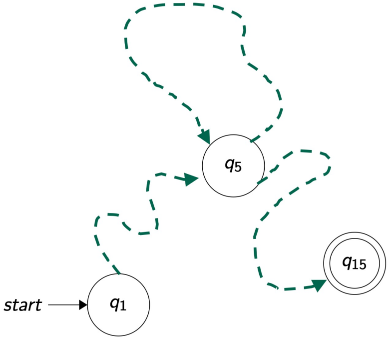

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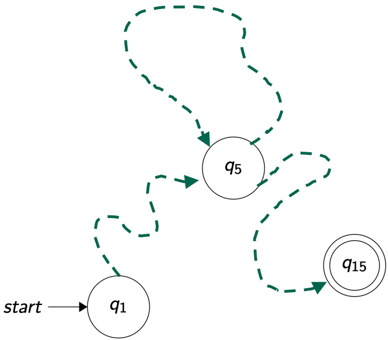

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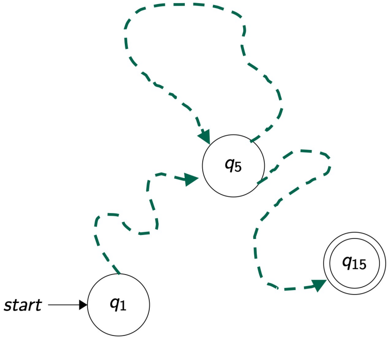

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3. $|xy| \leq p$
   Proof: if q_5 is the first repetition in $M(w)$, then this repetition must occur in the first $p + 1$ states, hence $|xy| \leq p$

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- x takes M from $r_1 = q_1$ to r_j, y takes M from r_j to r_k, and z takes M from r_k to r_{n+1}, which is an accept state. So, M must accept xy^iz for $i \geq 0$

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- $k \leq p + 1$, so $|xy| \leq p$

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4. Since regular languages are closed under $\cap$, if L is regular then L_1 must be regular
5. Since we know L_1 is nonregular, this means that L must be nonregular

Exercise

Prove that the following language is nonregular:

$$L = \{0^i 1^j 2^i 3^j | i, j > 0\}$$

What's Next?

- We will get plenty of practice with proving languages nonregular
- We will add (a small amount of) memory to our machines to recognize a richer class of languages