Cryptography

Lecture 9

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Outline

Lecture 8 Review

- Quiz on PRFs
- Started proof of CPA-security for PRF+OTP

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1. Lecture 8 Review
2. Security of PRF+OTP (Chapter 3.5.2)
3. Modes of Operation (Chapter 3.6.2)

CPA-Secure Encryption from a PRF

PRF+OTP Encryption (Π)

- $\mathsf{Gen}(1^n)$: $k \leftarrow \{0,1\}^n$
- $\mathsf{Enc}(k,m)$: Choose $r \leftarrow \{0,1\}^n$, output $c = (r, F_k(r) \oplus m)$
- $\mathsf{Dec}(k,c)$: Parse c as (r, c'), compute $m = F_k(r) \oplus c'$

Theorem

If F is a secure PRF, then PRF+OTP is CPA-secure

Proof Technique

To prove security from a PRF, we often do the following:

1. Consider the scheme where F_k is replaced by a random function f
    - Show by reduction to security of PRF, that $\mathcal{A}$ can't tell we made this change.
    - So, $\mathcal{A}$'s success probability must be (essentially) the same in this and original variant.

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   - So, $\mathcal{A}$'s success probability must be (essentially) the same in this and original variant.
2. Use a probabilistic argument to prove that scheme is unconditionally secure when using a random function f.
   - Random function is essentially a collection of 2^n OTPs
   - Proof is similar to proof of OTP, but need to account for probability of collision in r

Security of PRF+OTP: Step 1

Define the following encryption scheme $\tilde{\Pi}$:

$\tilde{\Pi}$ Encryption Scheme

- $\widetilde{\mathsf{Gen}}(1^n)$: $f \leftarrow \mathcal{F}_n$ (the set of functions $\{0,1\}^n \to \{0,1\}^n$)
- $\widetilde{\mathsf{Enc}}(k, m)$: Choose $r \leftarrow \{0,1\}^n$, output $c = (r, f(r) \oplus m)$
- $\widetilde{\mathsf{Dec}}(k, c)$: Parse c as (r, c'), compute $m = f(r) \oplus c'$

- Observe that this is exactly PRF+OTP with F_k replaced by f
- This encryption is not efficient as we cannot evaluate a random function
- But, it is useful as a "thought experiment" in the proof as it gives us a target for security

Security of PRF+OTP: Step 1

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Lemma: For any PPT $\mathcal{A}$

$$\left|\Pr[PrivK^{cpa}_{\mathcal{A},\Pi}(n) = 1] - \Pr[PrivK^{cpa}_{\mathcal{A},\tilde{\Pi}}(n) = 1]\right| \leq \mathsf{negl}(n)$$

A Story of Two Games

Security of PRF+OTP: Proof of Lemma

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  - $\mathcal{A}_c$ is a CPA-security adversary
  - What we care about is the difference in probability that $\mathcal{A}_c$ wins the CPA-security game when playing with Π vs. $\tilde{\Pi}$.
- Use this to construct $\mathcal{A}_r$ that breaks PRF security of F_k

The Two Adversaries

$PRF_{\mathcal{D},F}(n)$

- The challenger chooses $b \leftarrow \{0,1\}$.
  If $b = 0$, he chooses $f \leftarrow \mathcal{F}_n$ and gives $\mathcal{D}$ an oracle $\mathcal{O} = f$.
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- With access to oracle $\mathcal{O}$, the distinguisher $\mathcal{D}$ outputs a bit b'
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  - We care about the *difference* in $\mathcal{A}_c$'s WIN probability

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- If $\mathcal{A}_c$ WINS, $\mathcal{A}_r$ must use that to win the game against his challenger

Constructing $\mathcal{A}_r^{\mathcal{O}}$

1. Pre-Challenge
   Run $\mathcal{A}_c(1^n)$ and when $\mathcal{A}_c$ asks $\mathsf{Enc}(m)$ query
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3. Post-Challenge
   Continue answering Enc queries until $\mathcal{A}_c$ outputs guess b'
   - Output 1 ("PRF") if $b = b'$, and 0 otherwise.

Observation

- If $\mathcal{O}$ is f, then $\mathcal{A}_r$ is simulating $\tilde{\Pi}$
- If $\mathcal{O}$ if F_k, then $\mathcal{A}_r$ is simulating Π

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    - Since $\mathcal{A}_r$ output 1 when $\mathcal{A}_c$ WINS, we have

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$$\Pr_{k \leftarrow \{0,1\}^n}[\mathcal{A}_r^{F_k(\cdot)}(1^n) = 1] = \Pr[PrivK^{cpa}_{\mathcal{A}_c,\Pi}(n) = 1]$$

- Case 2: $\mathcal{O} = f$ (i.e., $b = 0$ in PRF game)
  - $\mathcal{A}_r$ answers all Enc queries and produces c with $f(r) \oplus m$
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Analysis of $\mathcal{A}_r$'s success

- We assumed that $\mathcal{A}_c$ breaks the lemma – i.e. has different success probability vs. Π and $\tilde{\Pi}$

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- That is, $\mathcal{A}_r$ is able to distinguish between $F_k(\cdot)$ and $f(\cdot)$. But, we know that F_k is a PRF.

Contradiction!

Proof Technique

To prove security from a PRF, we often do the following:

✓ Consider the scheme where F_k is replaced by a random function f
  - Show by reduction to security of PRF, that $\mathcal{A}$ can't tell we made this change.
  - So, $\mathcal{A}$'s success probability must be (essentially) the same in this and original variant.

2. Use a probabilistic argument to prove that scheme is unconditionally secure when using a random function f.
   - Random function is essentially a collection of 2^n OTPs
   - Proof is similar to proof of OTP, but need to account for probability of collision in r

Proving CPA-security of $\tilde{\Pi}$

Lemma

For any $\mathcal{A}$ making at most $q(n)$ queries to $\mathrm{Enc}(\cdot)$

$$\Pr[PrivK^{cpa}_{\mathcal{A},\tilde{\Pi}}(n) = 1] \leq 1/2 + \frac{q(n)}{2^n}$$

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  - $\mathcal{A}$ learns value of $f(r^*)$ (he sees $c = (r^*, c')$, computes $f(r^*) = c' \oplus m$)
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- So,

$$\Pr[r^* \in \{r_1, \ldots, r_{q(n)}\}] \leq \sum_{i=1}^{q(n)} \Pr[r^* = r_i] = \frac{q(n)}{2^n} \leq \mathsf{negl}(n)$$

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Finishing Proof of CPA-security of PRF+OTP

- ✓ Consider the scheme where F_k is replaced by a random function f
  - We showed that any PPT $\mathcal{A}$ has only a $\text{negl}(n)$ advantage in distinguishing the two games
- ✓ Use a probabilistic argument to prove that scheme is unconditionally secure when using a random function f.
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Combining these two statements, we get that for any PPT $\mathcal{A}$,

$$\Pr[PrivK^{cpa}_{\mathcal{A},\text{PRF+OTP}}(n) = 1] \leq 1/2 + \frac{q(n)}{2^n} + \text{negl}(n)$$

Outline

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- Encrypt many messages using the same key k

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Are we done?

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The Question

How do we encrypt long messages without this 2X increase in size

Block-Cipher Modes of Operations

- Modes of operation study how to encrypt many block messages without blow-up in size

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- Combine PRFs, Boolean operations, and randomness to achieve this
- Can think of them as other ways to turn PRF into CPA-secure encryption

Modes of Operations: Important Properties

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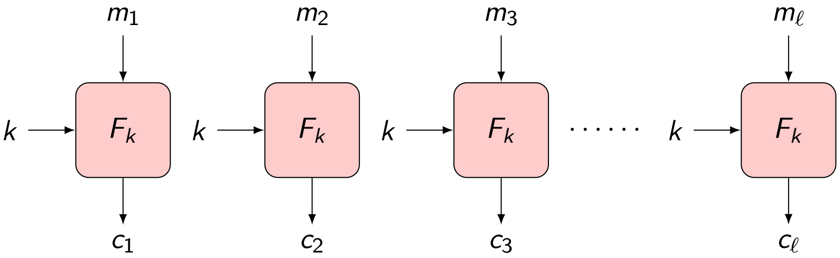
- PRF Inverse – do we need to be able to invert the PRF (i.e., do we need a PRP)
- Security - want at least CPA-security

Electronic Code Book (ECB) Mode



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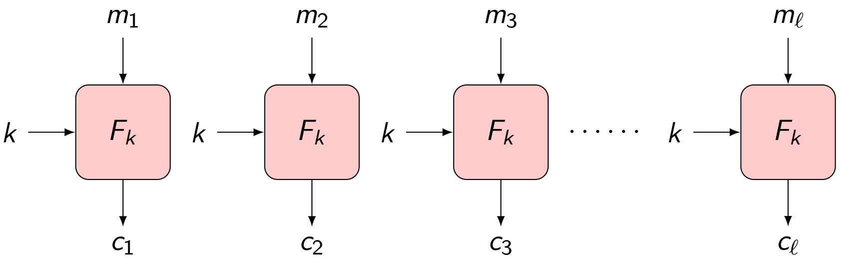

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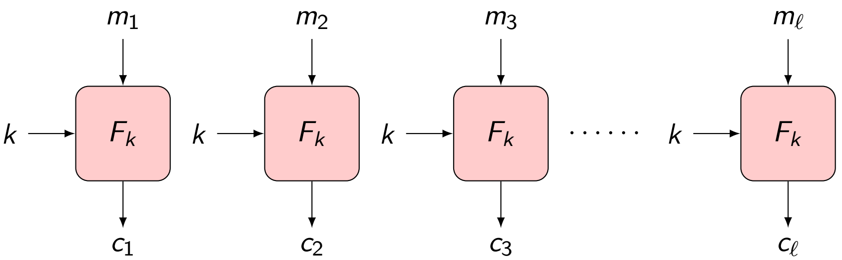

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- Parallelism: Can encrypt/decrypt blocks in parallel
- PRF Inverse: Need to compute inverse to decrypt
- Security: NOT SECURE
  - ECB Mode is deterministic
  - Can tell if two blocks are the same

Electronic Code Book (ECB) Mode: How bad is it?

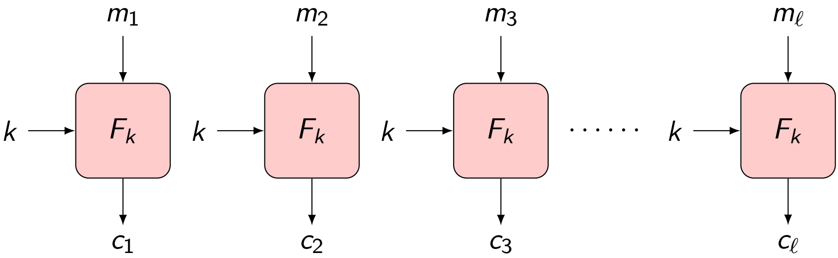

Figure: Original Image

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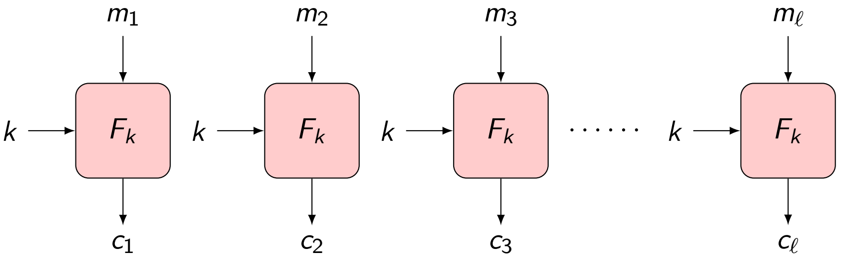

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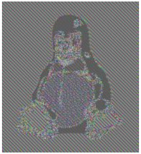

Figure: ECB-encrypted

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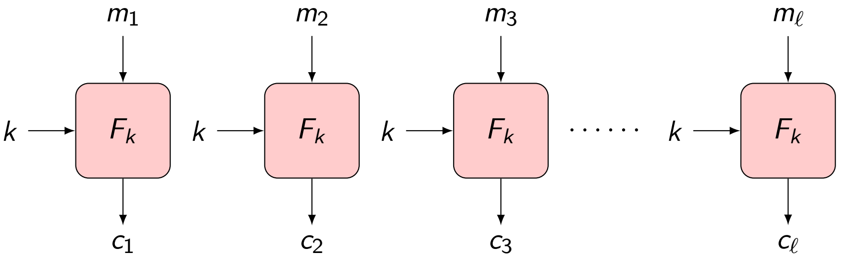

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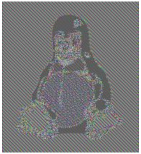
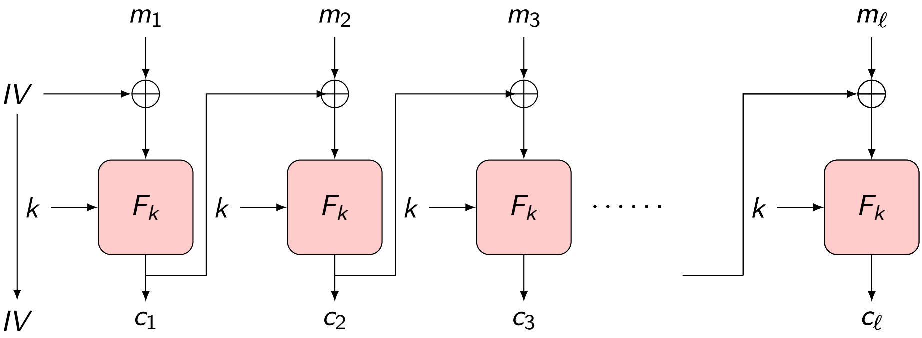


Figure: ECB-encrypted

Warning

Never use ECB mode

Cipher Block Chaining (CBC) Mode



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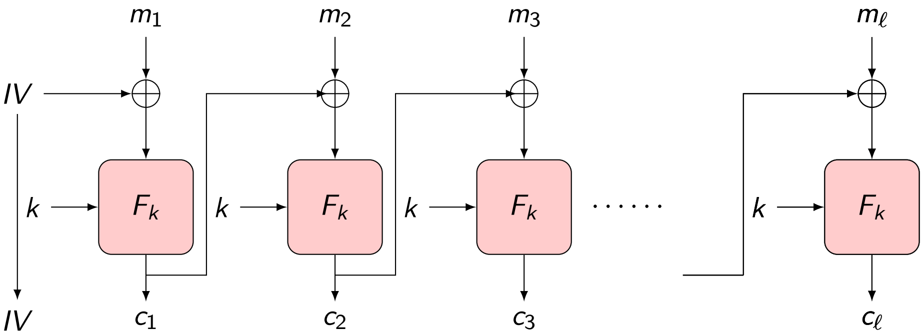
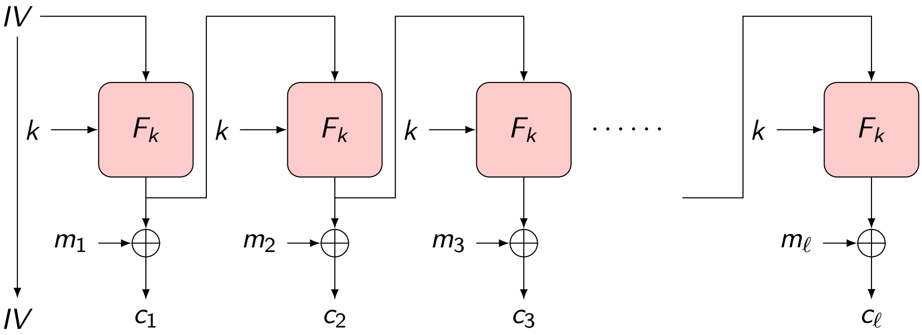
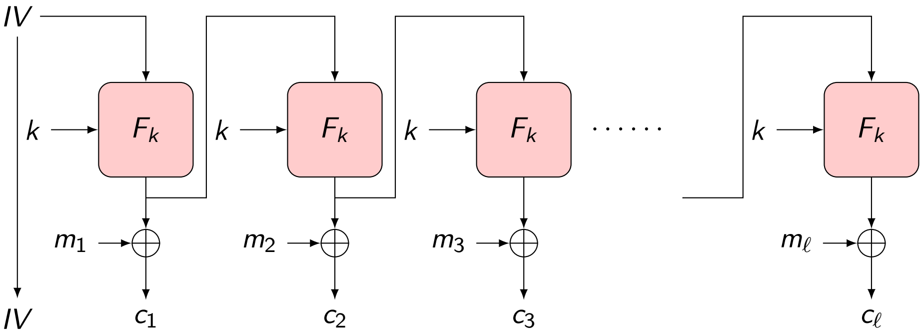


- Ciphertext length: $|c| = |m| + 1$ blocks
- Pre-computation: N/A
- Parallelism: Encryption / Decryption sequential
- PRF inverse: Need to compute inverse to decrypt
- Security: CPA-secure

Output Feedback (OFB) Mode

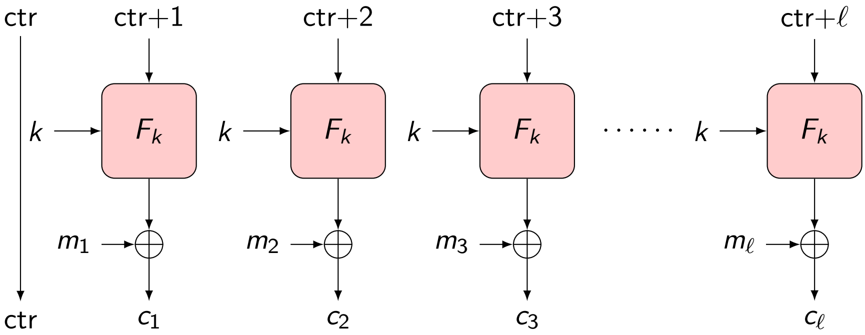


Output Feedback (OFB) Mode

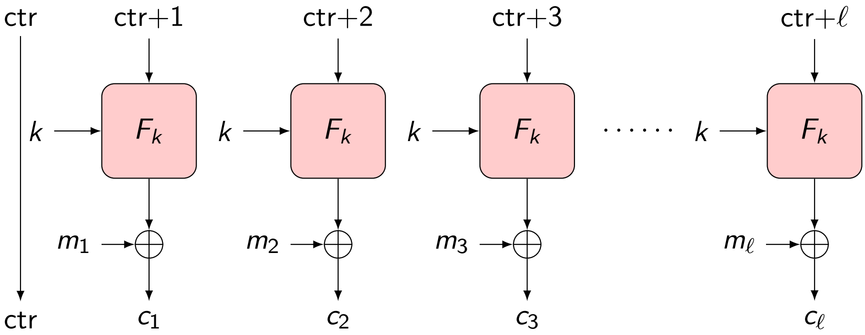


- Ciphertext length: $|c| = |m| + 1$ blocks
- Pre-computation: Can pre-compute entire pad
- Parallelism: If have pad, all encryption/decryption can be parallel
- PRF inverse: No need to compute inverse to decrypt
- Security: CPA-secure

Counter (CTR) Mode



Counter (CTR) Mode



- Ciphertext length: $|c| = |m| + 1$ blocks
- Pre-computation: Can pre-compute entire pad (or any part of pad)
- Parallelism: Can encrypt/decrypt any blocks in parallel
- PRF inverse: No need to compute inverse to decrypt
- Security: CPA-secure