# Foundations of Computing

## Lecture 24

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April 18, 2024

# Outline

# Lecture 23 Review

- 3-Coloring is $\mathcal{NP}$-complete
- Ladner's Theorem
- The class co-$\mathcal{NP}$

# $\mathcal{NP}$ vs co-$\mathcal{NP}$

**$\mathcal{NP}$ – Yes instances are efficiently verifiable**

$L \in \mathcal{NP}$ if there exists poly-time DTM $V$ s.t. for $x \in L$ there exists a witness $w$ s.t. $V(x, w) = 1$

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Is $SAT \in$ co-$\mathcal{NP}$?

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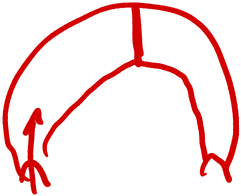
- The verifier (and prover) can use randomness to decide whether to accept

# An Example – Aladdin's Cave



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- Can give proofs for languages not in $\mathcal{NP}$
- Interactive proofs can be much more efficient (e.g., shorter) than non-interactive ones
- Can have additional properties that traditional proofs cannot satisfy.
    - Zero-knowledge

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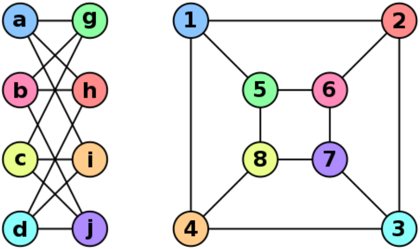
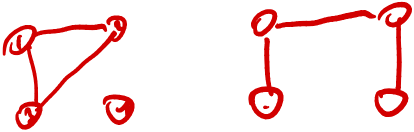
# Another Example – Graph Isomorphism



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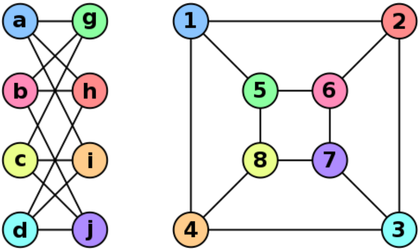


**Claim**

Graph Isomorphism $\in \mathcal{IP}$

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    - Thus, $\Pr[b' = b] = 1/2$

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- It is not believed that there is a short witness $w$ s.t. $V((G_0, G_1), w) = 1$ if $G_0$ and $G_1$ are not isomorphic. I.e., GNI $\notin \mathcal{NP}$
- The power of interaction and randomness has allowed us to do what we couldn't do before

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3. $P^*$ wins with probability $\leq 1/2$ in each run, so

$$\Pr[\langle P^*, V\rangle(x) = 1] \leq 1/2^n$$

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# Another Example – Polynomial Identity Testing

## Polynomial

A polynomial is an equation in one-variable

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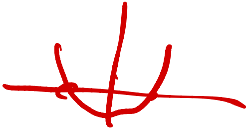
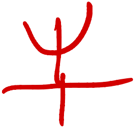
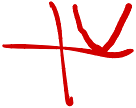

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Question: What should $V$ do? How many queries does he need?

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- Prover $P$ chooses a degree $d$ polynomial $f$ and wants to prove that

$$\forall x, f(x) = 0$$

- Completeness: If $f(x) = 0$, $V$ should accept after interacting with $P$
- Soundness: If $f(x) \neq 0$, $V$ should reject*

$\Pr[f(x)=0] \leq \frac{2}{2^n}$

The rules:

- $V$ is allowed to query $f(x)$ at points $x$ of its choice
- $P$ is required to answer honestly, but
- $P$ knows $V$'s strategy (i.e., how he chooses the points $x$)

Question: What should $V$ do? How many queries does he need?

- Suppose that $V$ is deterministic.
- What if you allow $V$ to be randomized?

$f$ has $\leq 2$ roots

$x \leftarrow [0, 2^n]$

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# Next Week

We have seen the power of interactive proofs in convincing a verifier of the truth of some statement.

**Question:**

What does the verifier learn from seeing the proof?