# Cryptography Lecture 19

Arkady Yerukhimovich

November 4, 2024

### Outline

1 Lecture 18 Review

2 Crypto Hardness Assumptions (Chapters 8.2, 8.3)

3 Assumptions in Cyclic Groups (Chapters 8.2, 8.3)

#### Lecture 18 Review

- ullet The Group  $\mathbb{Z}_N^*$
- Chinese Remainder Theorem
- Modular Arithmetic by Hand

#### Modular Arithmetic Without a Calculator

To evaluate exponentiation  $\mod N$  use the following steps:

- If N is not prime, apply the Chinese Remainder Theorem
- Reduce mod  $\phi(N)$  in the exponent
- Reduce mod N in the base

#### Useful Hints:

- Sometimes useful to use negative numbers
- $\bullet$  look for things that are easy to compute (e.g.,  $1^{53}$ )

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# What Are Hardness Assumptions?

- As we've discussed before, all crypto primitives rely on computational hardness
- Thus, we need to assume that some problem is hard to compute
- We have seen such assumptions before: E.g., Existence of PRG, PRF, CRHF

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- As we've discussed before, all crypto primitives rely on computational hardness
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- We have seen such assumptions before: E.g., Existence of PRG, PRF, CRHF
- Going forward, we will instead use hard problems from number theory and mathematics
  - Some of these problems have been studied for 1000s of years
  - Easy to state and widely understood
  - Still believed to be hard for all PPT machines

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Definition: Factoring is hard relative to GenMod if for all PPT  ${\mathcal A}$  it holds that

$$\Pr[\mathsf{Factor}_{\mathcal{A},\mathsf{GenMod}}(n)=1] \leq \operatorname{negl}(n)$$

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  - So, can sample integers at random, and test if they are prime
  - Miller-Rabin primality test efficiently test if a number is prime

Given N=pq, Integer e>1 s.t.  $gcd(e,\phi(N))=1$ , we know that  $f_e(x)=x^e$  is a permutation over  $\mathbb{Z}_N^*$ 

#### **RSA Problem**

Given (N, e) and  $y \in \mathbb{Z}_N^*$ , compute  $[y^{1/e} \mod N]$ 

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- RSA is potentially easier than factoring

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### $GenRSA(1^n)$ :

- $(N, p, q) \leftarrow \mathsf{GenMod}(1^n)$ , let  $\phi(N) = (p-1)(q-1)$
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# Discrete Log Assumption

Let G be a cyclic group of order q with generator g

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Given  $h \in G$ , find  $0 \le x \le q-1$  s.t.  $g^x = h$ . We say  $x = \log_g h$ 

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### Decisional Diffie-Hellman: $DDH_{A,Gen}(n)$

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- If b=0, send  $(G,q,g,g^x,g^y,g^{xy})$  to  $\mathcal{A}$ . If b=1, choose  $z\leftarrow\mathbb{Z}_q$ , and send  $(G,q,g,g^x,g^y,g^z)$  to  $\mathcal{A}$

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Definition: DDH is hard if for all PPT A:  $Pr[A \text{ wins}] \leq 1/2 + negl(n)$ 

Arkady Yerukhimovich Cryptography November 4, 2024

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#### Relative Hardness of The Problems

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DLog > CDH > DDH

DLog vs. CDH

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DLog. Given 
$$5^{\times}$$
 find  $\times$ 

CDH. Given  $5^{\times}$ ,  $3^{\times}$  find  $3^{\times}$ 
 $\times$ 
 $(3^{3})^{\times} = 5^{\times}$ 

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- DLog vs. CDH
  - If  $\mathcal{A}$  can solve DLog, then given  $g^x$ ,  $g^y$ , he can find x, y and compute  $g^{xy}$  (solve CDH)
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### Strength of Assumption

Since DDH is the easiest problem, assuming it is secure is the strongest assumption

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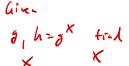
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