Foundations of Computing

Lecture 13

Arkady Yerukhimovich

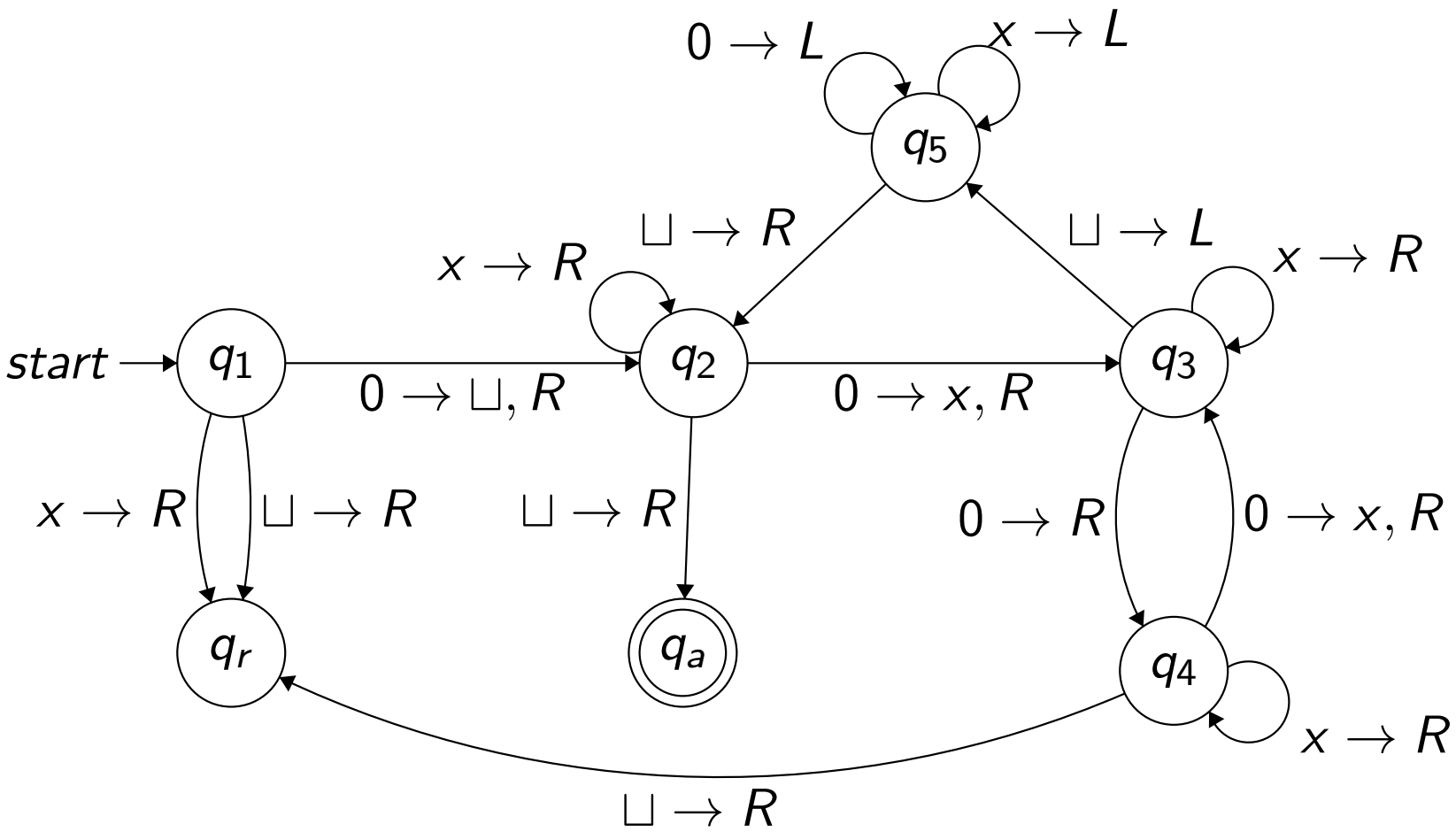
February 29, 2024

Outline

Lecture 12 Review

- Turing Machines
  - Definition
  - Examples
- Church-Turing Thesis
  Informally: Anything that can be computed can be computed by a Turing Machine.

Running M on $w = 00$



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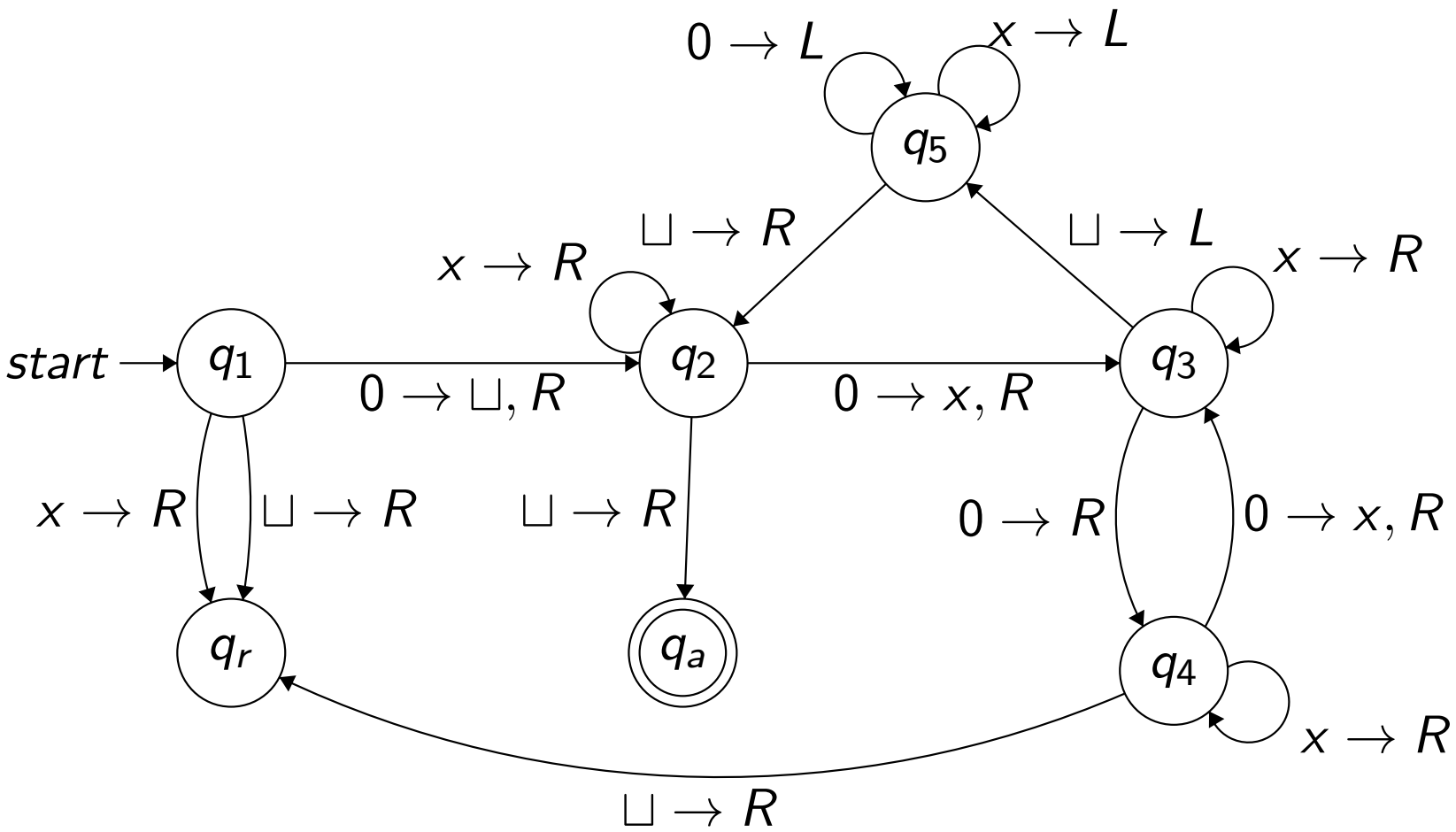

Let's consider an execution on input 00 (as a sequence of configurations)

Outline

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   - Give full detail of transition function δ
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   - Explain algorithmically what happens on the tape
   - For example, scan the tape until you find a #, zig-zag on the tape, etc.
   - Don't bother specifying a DFA for the control state
3. Algorithm specification
   - Give algorithm in pseudocode
   - Don't explicitly spell out what happens on the tape

Example 1: $L = \{a^i b^j c^k \mid i \times j = k \text{ and } i, j, k \geq 1\}$

Machine M deciding L

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5. If all a's are crossed off, check if all c's are crossed off. Accept if yes, reject if no.

Example 2 – Build a TM deciding L Below

$$L = \{\#x_1\#x_2\#\cdots\#x_\ell \mid \text{ each } x_i \in \{0,1\}^* \text{ and } x_i \neq x_j \text{ for all } i \neq j\}$$

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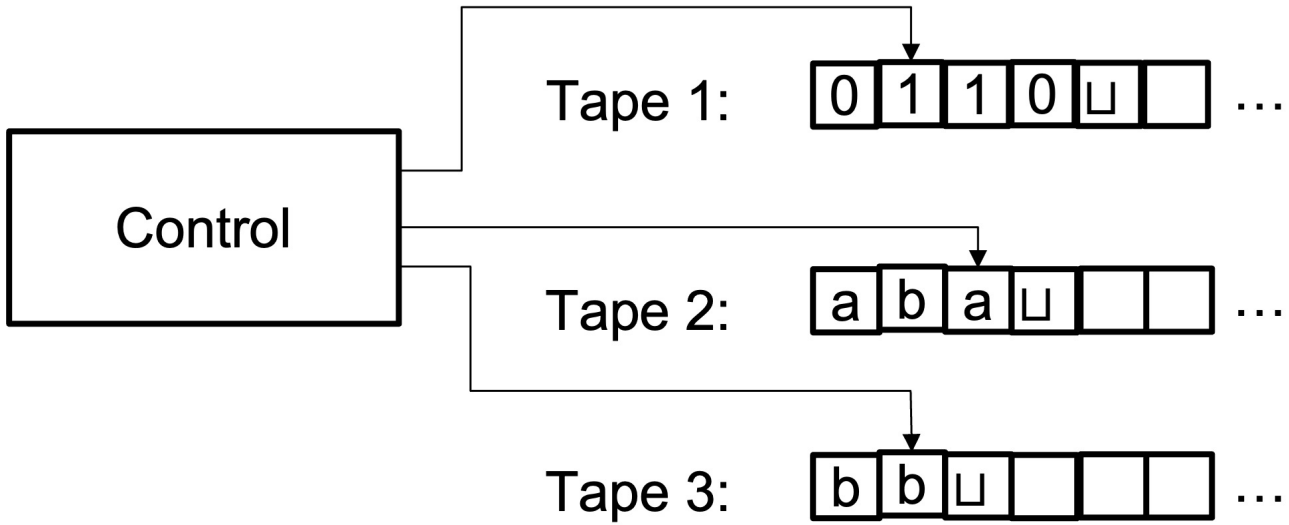
5. Goto step 3

Outline

Multi-Tape Turing Machines



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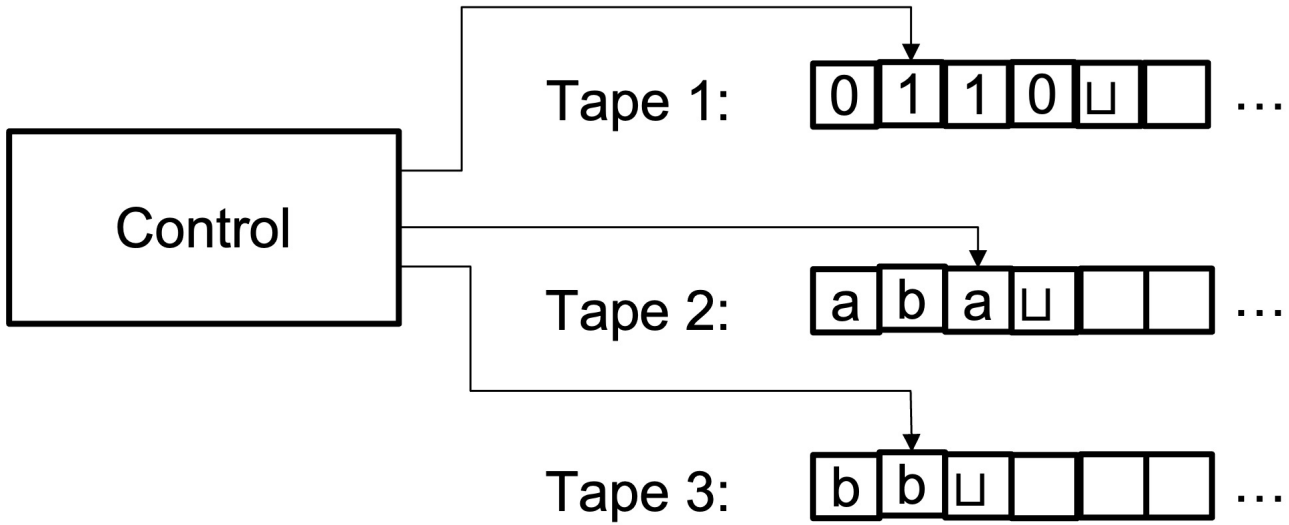
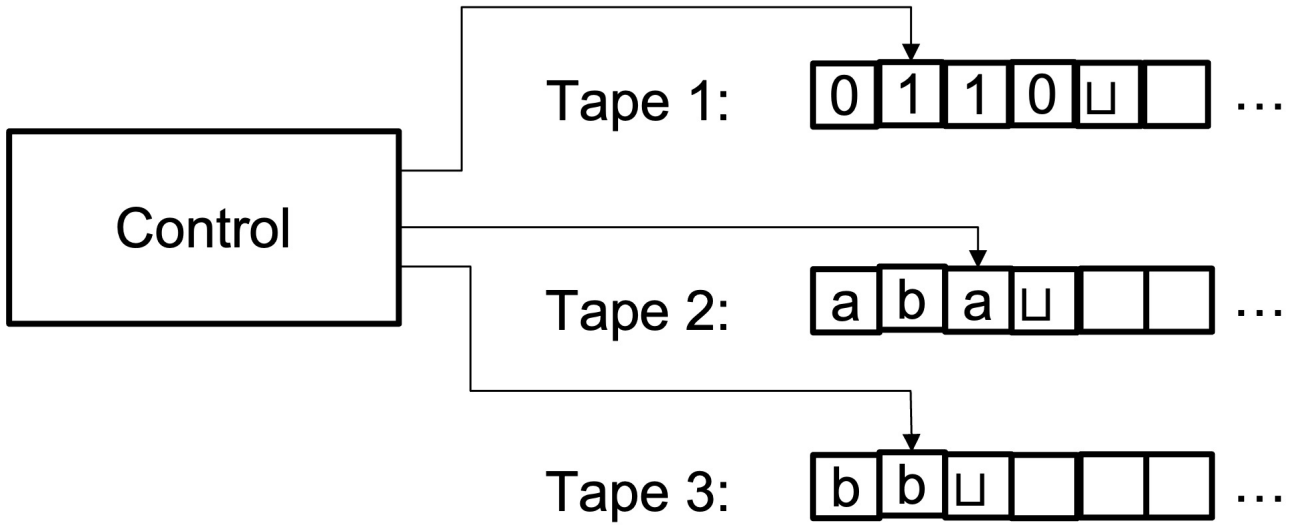


In each step:

- M can read each tape
- M can write to each tape
- M can move each tape head Left or Right

Multi-Tape Turing Machines



In each step:

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Formally, for k tapes

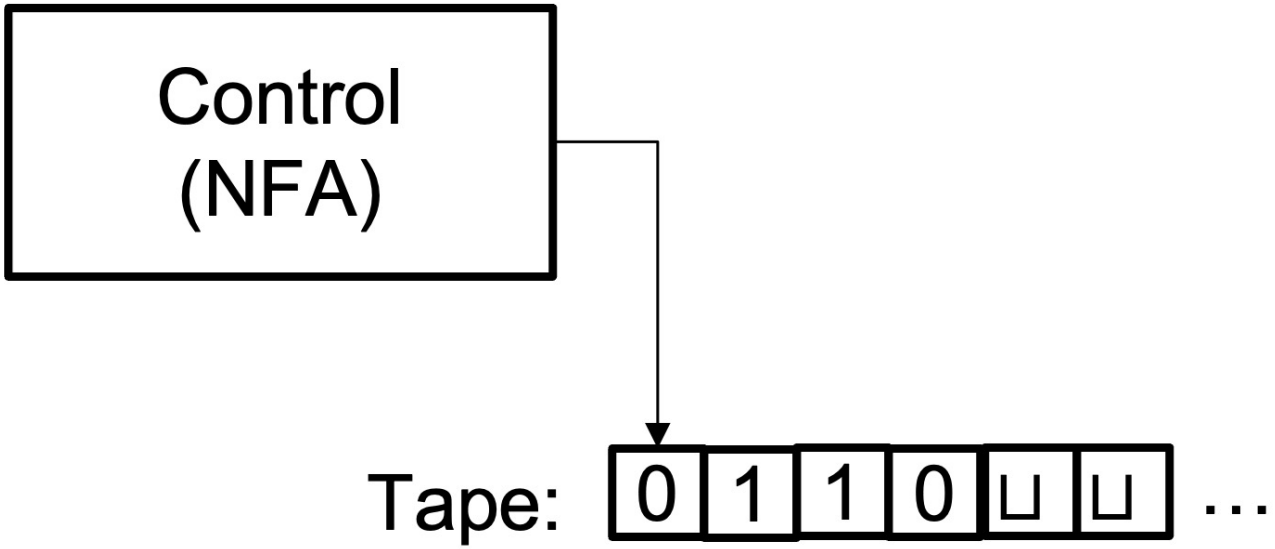
$$\delta : Q \times \Gamma^k \to Q \times \Gamma^k \times \{L, R\}^k$$

Multi-Tape Turing Machines

Theorem

Every multi-tape TM has an equivalent single-tape TM

Nondeterministic Turing Machines



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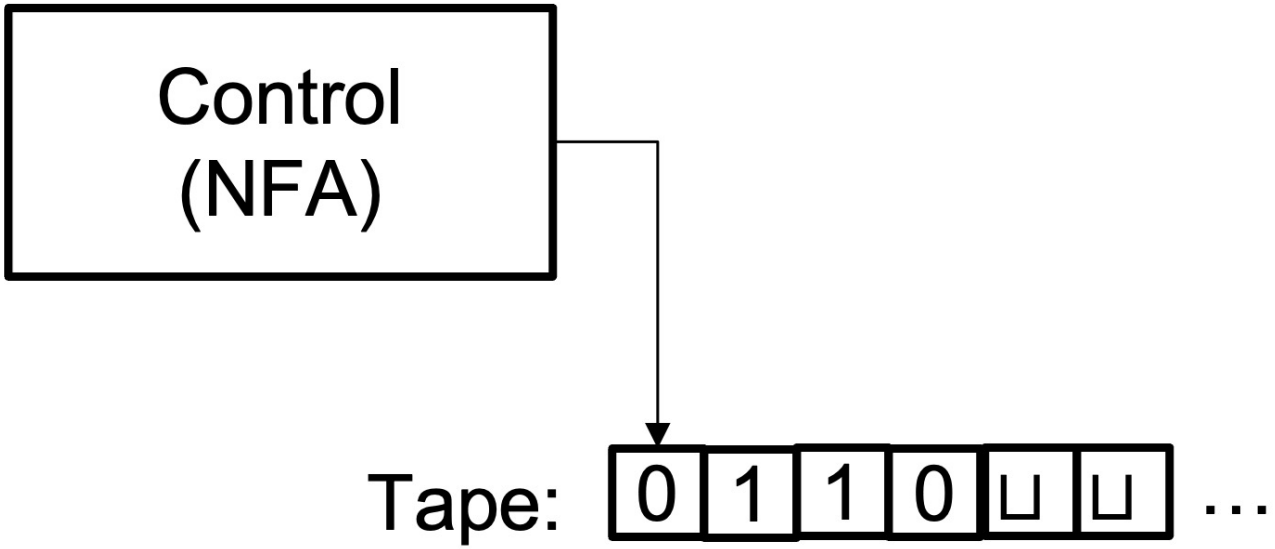

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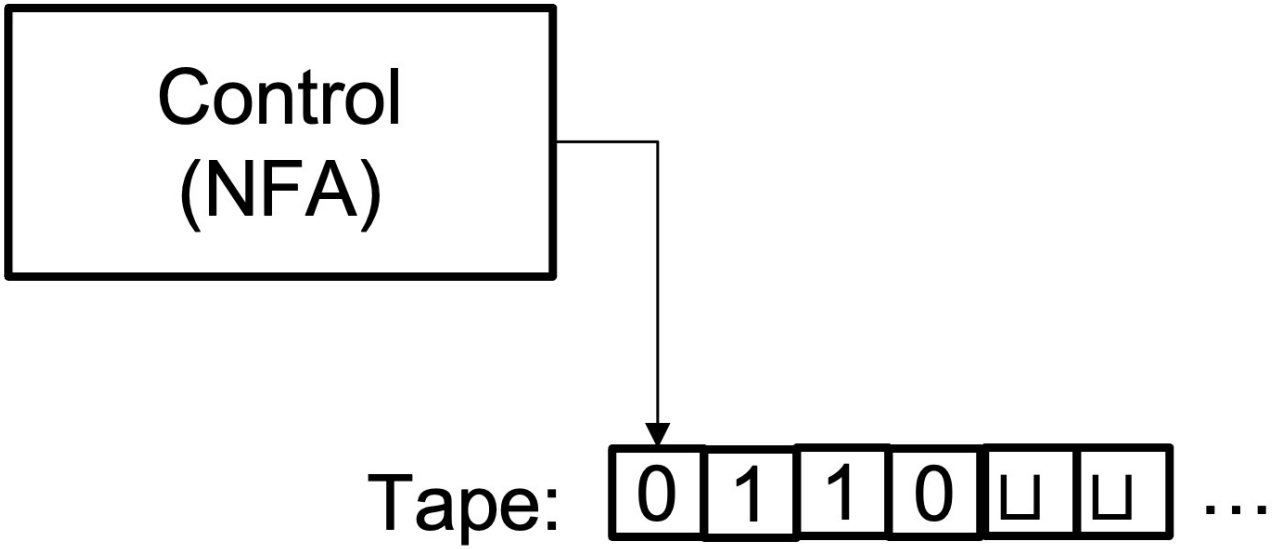

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Intuition:

- The control unit is non-deterministic - many transitions possible on each input
- Execution corresponds to a tree of possible executions
- Accept if any of possible execution leads to accept

Nondeterministic Turing Machine

Theorem

Every nondeterministic TM has an equivalent deterministic TM.

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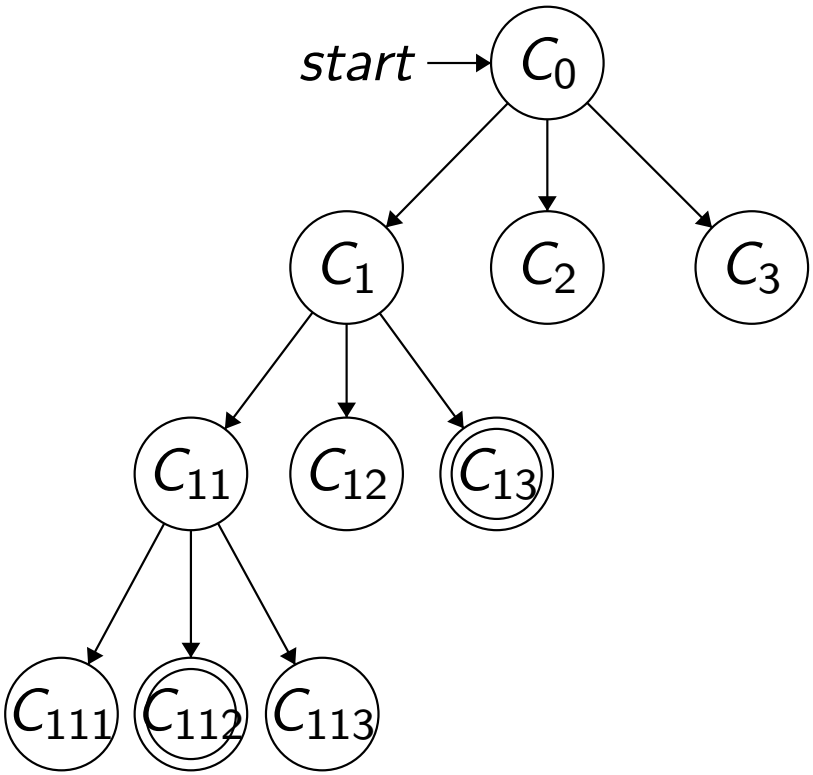

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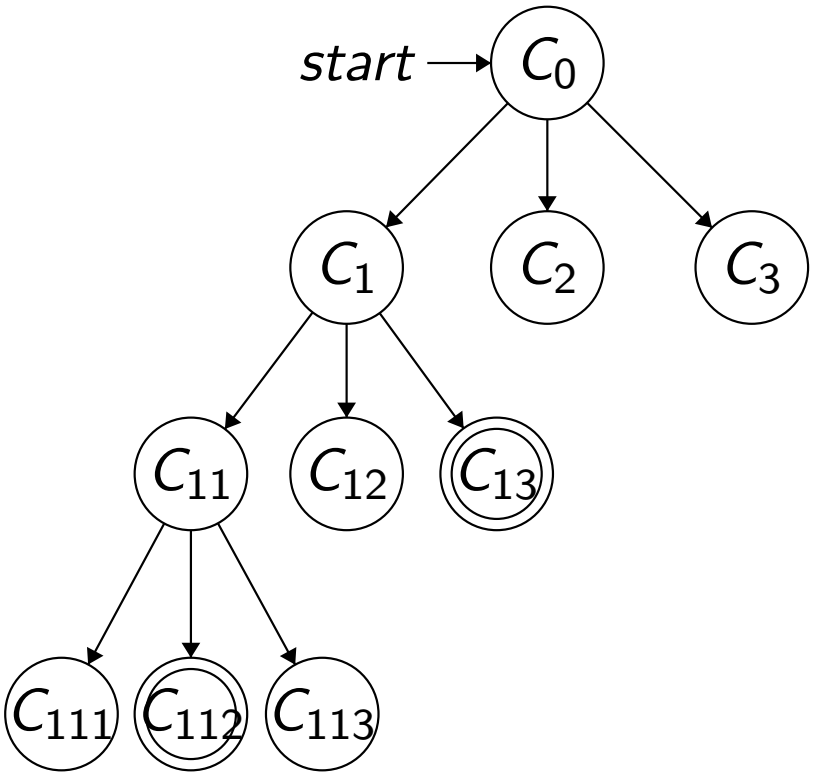

- Recall that an execution of a DTM is a sequence of configurations

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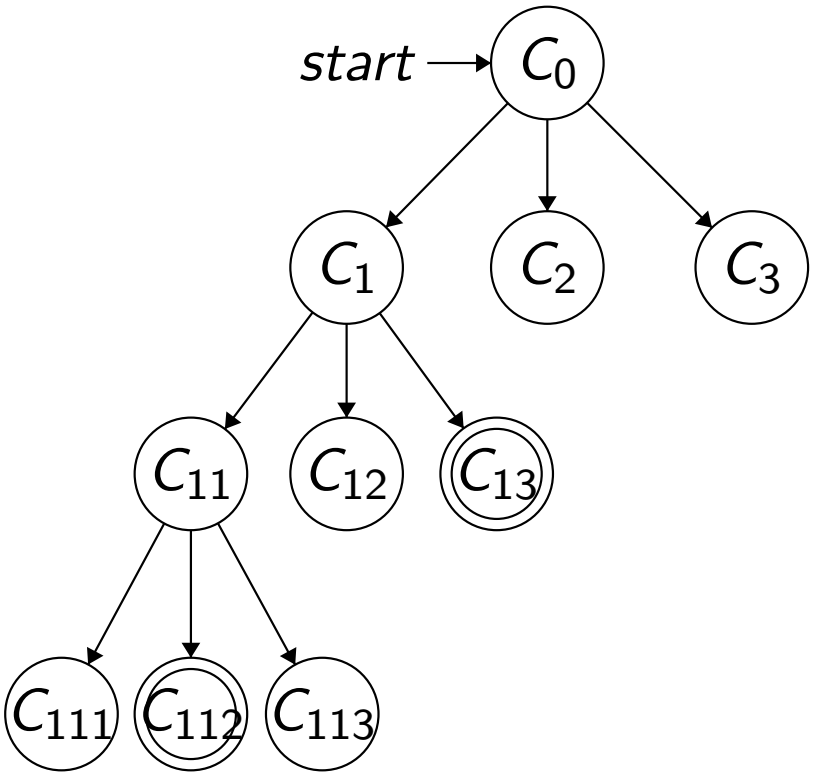

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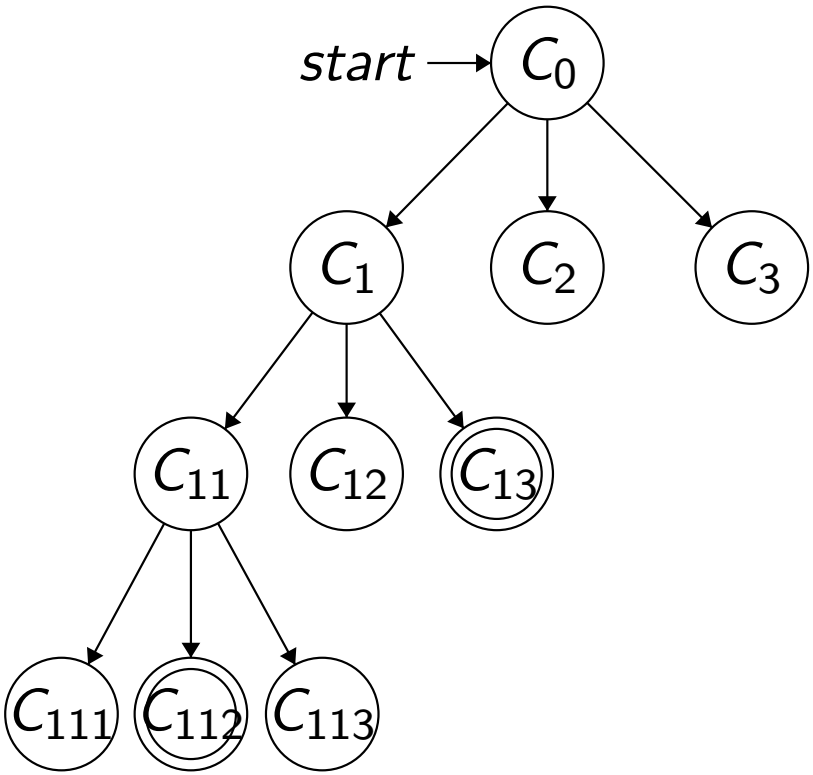

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- If any node in the tree is an accept node, the NTM accepts

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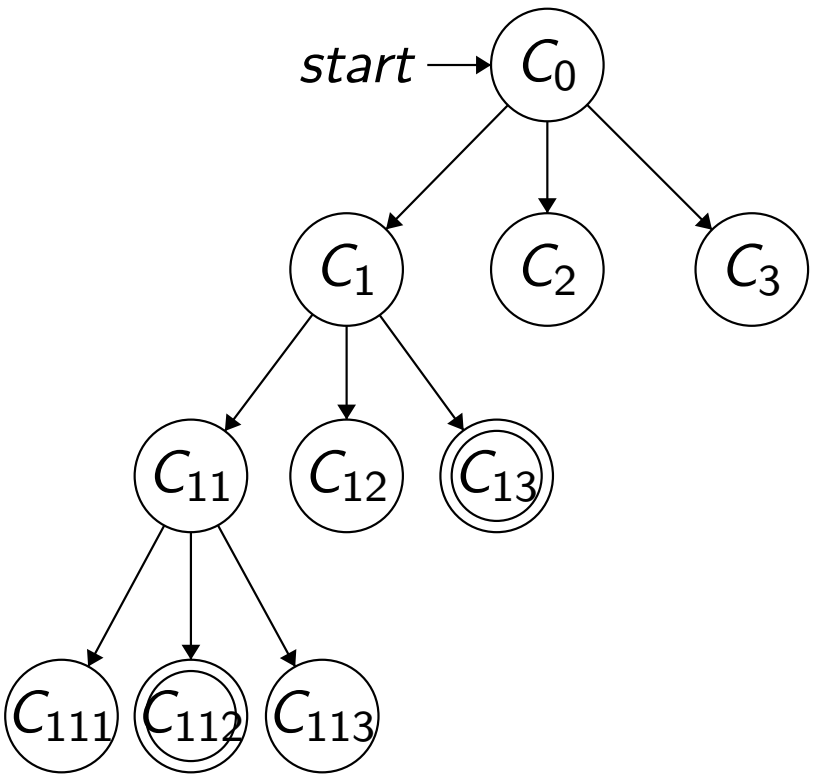
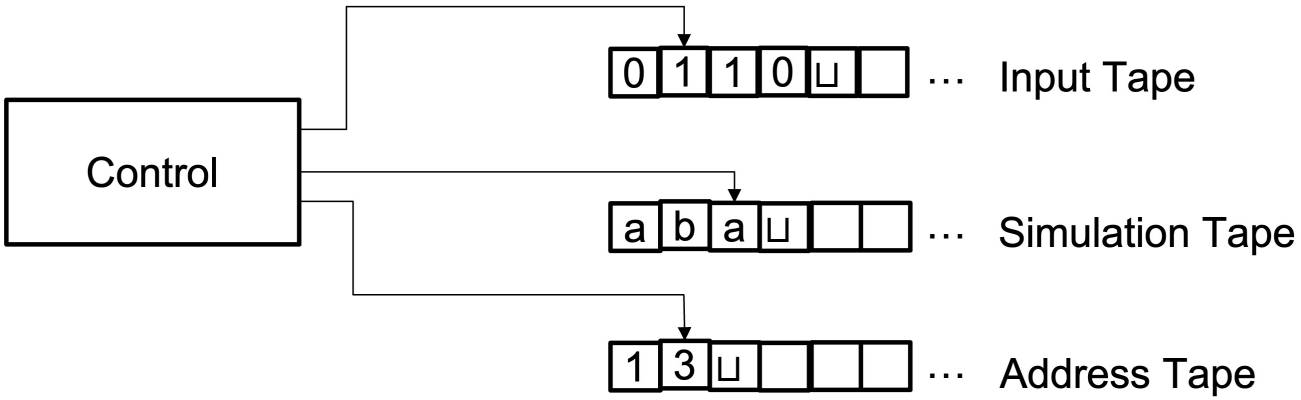


- Recall that an execution of a DTM is a sequence of configurations
- Execution of an NTM is a tree of configurations (branches correspond to non-deterministic choices)
- If any node in the tree is an accept node, the NTM accepts
- To simulate an NTM by a DTM, need to try all configurations in the tree to see if we find an accepting one

Nondeterministic Turing Machines



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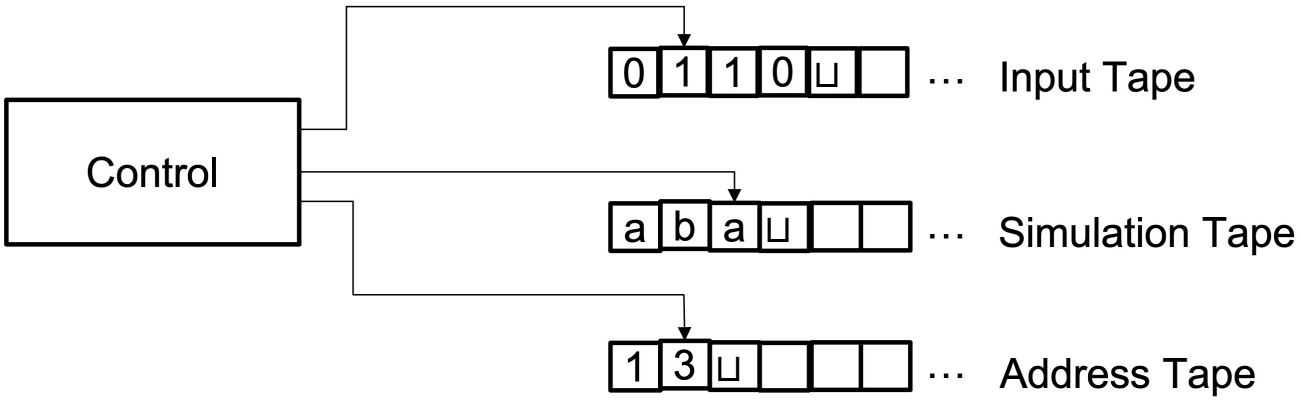

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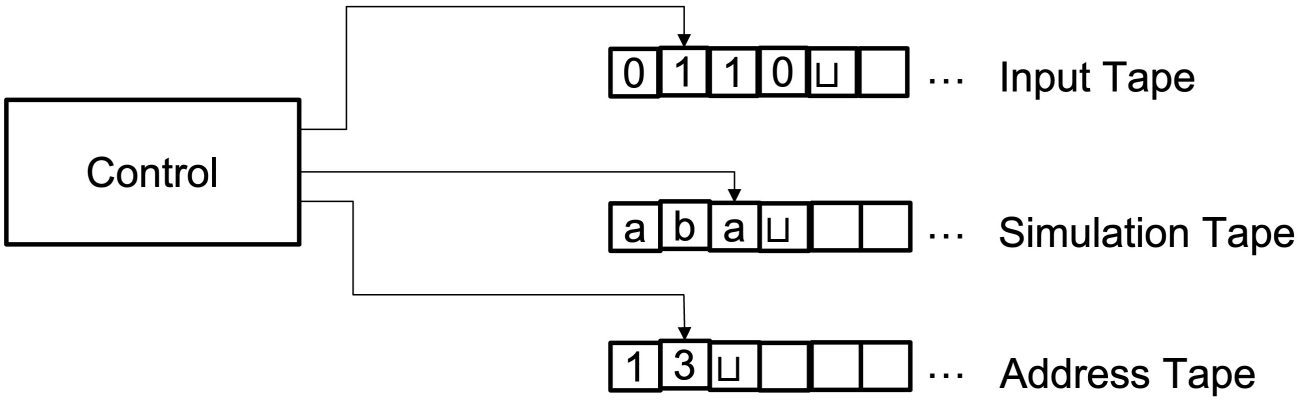

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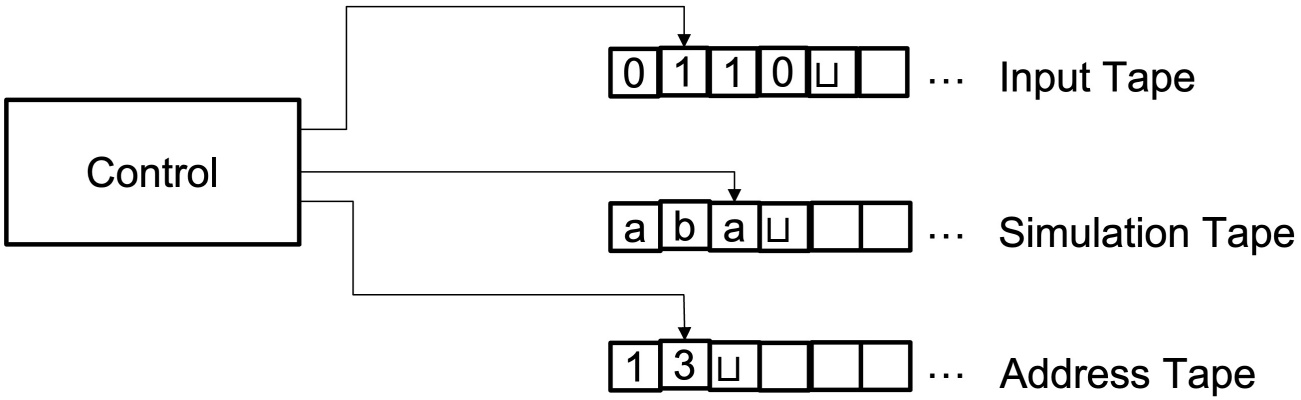

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3. Address tape – use to store which nondeterministic branch you are on

Nondeterministic Turing Machines

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Nondeterministic Turing Machines

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Must traverse NTM tree in breadth-first, not depth-first order

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Important

Must traverse NTM tree in breadth-first, not depth-first order

- Depth-first traversal may get stuck in an infinite loop, and not detect terminating branch

Next Week

- Languages about machines
- Decidable and undecidable languages