Foundations of Computing

Lecture 23

Arkady Yerukhimovich

April 16, 2024

Final Exam

Final exam will be on Tuesday, May 7, 10:20-12:20.

Outline

1. Lecture 22 Review
2. Graph Coloring
3. $\mathcal{NP}$-Intermediate Languages
4. co-$\mathcal{NP}$

Lecture 22 Review

- More $\mathcal{NP}$-complete problems
  - SAT
  - 3SAT
  - CLIQUE
  - VERTEX-COVER

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3-Coloring

Definition

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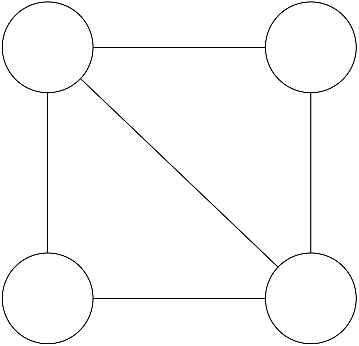

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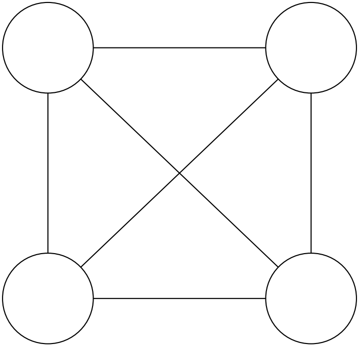

Goal: Prove than 3-Coloring is $\mathcal{NP}$-Complete

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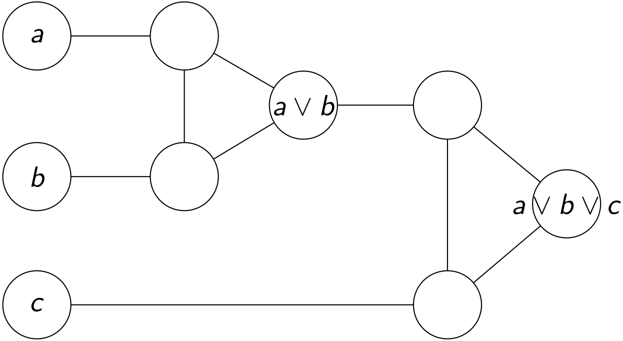
Main Tool

We need gadgets

Clause Gadget

We have 3 colors: T, F, B



Claim

- If a, b, c are all colored F, then $a \vee b \vee c$ is colored F
- If at least one of a, b, c is colored T, then there is a coloring s.t. $a \vee b \vee c$ is colored T

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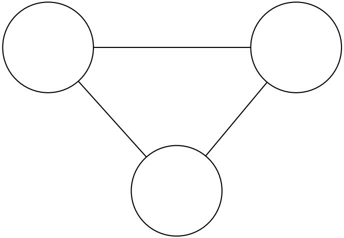

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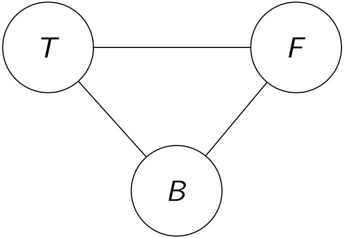

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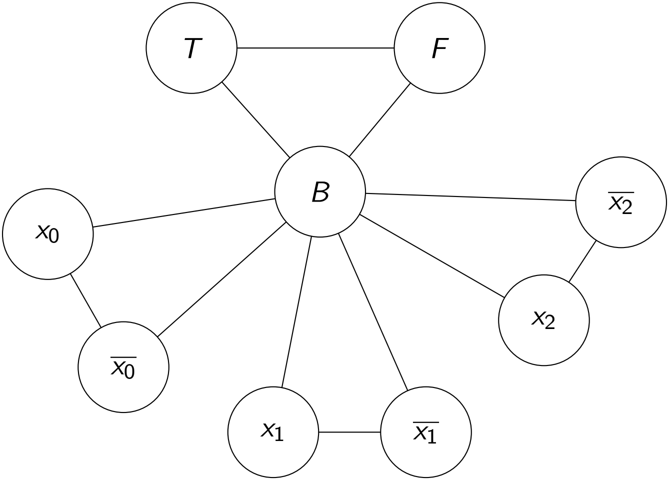
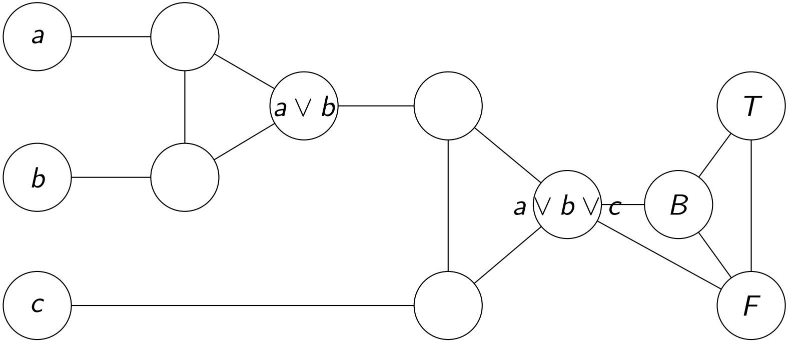


Putting it All Together



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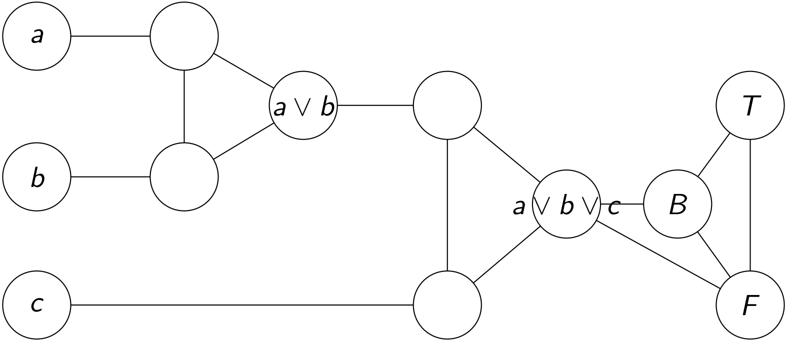

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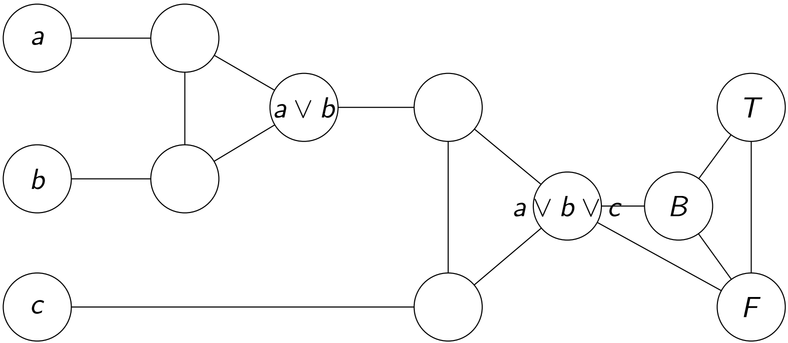

Claim

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2. If G is 3-colorable than ϕ is satisfiable

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Comment: All languages useful for crypto are such $\mathcal{NP}$-intermediate languages

Proof

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1. If $H(n) = n$, then $SAT_H \in \mathcal{P}$
2. If $H(n) \leq c$, then SAT_H is $\mathcal{NP}$-Complete
3. We will define H to be in between these two cases

Defining $H(n)$

Let $M_1, M_2, \ldots$ be an enumeration of all TM's (can do this since TM's are countable)

$H(n)$

- Smallest $i \leq \log\log n$ s.t. for all x, $|x| \leq \log n$, $M_i(x)$ halts in $i|x|^i$ steps and accepts iff $x \in SAT_H$
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   ($\Rightarrow$) By definition of $\mathcal{P}$, there is machine M_k that decides SAT_H in kn^k steps so $H(n) = k$
   ($\Leftarrow$) If $H(n) < c$, then there is infinitely long stretch where $H(x) = i$. But, then M_i decides SAT_H.

Completing the proof

Claim

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1. $SAT_H \notin \mathcal{P}$:
   - Suppose it is in $\mathcal{P}$, then $H(n) < c$
   - Can reduce any SAT formula to SAT_H formula by padding with $H(n)$ 1s
   - But, SAT is $\mathcal{NP}$-Complete, contradiction!

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   - But, SAT is $\mathcal{NP}$-Complete, contradiction!
2. SAT_H is not $\mathcal{NP}$-Complete
   - Assume it is, then $SAT \leq_p SAT_H$
   - Reduction maps ψ of length n to $\phi 01^H(n)$ of length n^c, but $H(n) \to \infty$ so this is super-poly in size of ϕ
   - Hence $|\phi| << n$, so have reduced solving long formula to solving a much shorter one.
   - Repeat this enough times to make $|\phi| = O(1)$ and solve.

Takeaway

If $\mathcal{P} \neq \mathcal{NP}$, then $\mathcal{NP}$-intermediate languages exist!

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$$\text{UNSAT} = \{\langle\phi\rangle \mid \phi \text{ is not satisfiable}\}$$

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- We define complexity class co-$\mathcal{NP}$ to contain all such languages that are complements of languages in $\mathcal{NP}$

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$L \in \mathcal{P}$ if there exists poly-time DTM M s.t $M(x) = [x \in L]$

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Question:

Can you prove that $x \in L$, when $L \in$co-$\mathcal{NP}$?

Proving that $x \in L$ for $L \in$co-$\mathcal{NP}$

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- $\mathcal{NP} \neq$ co-$\mathcal{NP}$

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Complexity Zoo

The complexity zoo (`https://complexityzoo.net/Complexity_Zoo`) now has 547 complexity classes.