Foundations of Computing

Lecture 9

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February 13, 2024

Outline

Midterm 1 – February 22

- Exam 1 will be in class on February 22 (next Thursday)
- It will cover NFA/DFA/regular languages, and PDAs/Context-free grammars

Exam Policies

- The exam will be closed book and closed notes
- You will be allowed two 8.5×11 pieces of paper with notes – anything you choose
- No computers, calculators, or other digital devices – bring a pencil or pen

Important

If you have a conflict with this exam, let me know ASAP!

Next Week

- Lecture and lab next week will be largely for review
- This is your chance to clear things up before the midterm

Bring your questions!

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Lecture 8 Review

- Pushdown Automata
  - Using a stack to recognize non-regular languages
  - Examples of building PDAs

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Today

An alternative formulation for languages accepted by PDAs

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Representing Languages

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We have seen several ways for describing a language L:

- DFA/NFA – the language of strings accepted by M
- Regular expressions
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- A grammar is a set of rules by which strings in L are constructed/derived

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Grammars

- A grammar is a set of rules by which strings in L are constructed/derived
- Today, we will focus on context-free grammars and the languages they represent

Grammar

A grammar G consists of:

- V – finite set of variables (usually Capital Letters)
- Σ – a finite set of symbols called the terminals (usually lower case letters)
- R – finite set of rules how strings in L can be produced
- $S \in V$ – start variable

If no S is specified, can assume it is the variable in the first rule.

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Definition

For a grammar G, the language L_G generated by G is the set of all terminal strings that can be produced by G starting with the start symbol by using a sequence of the production rules.

Example 1

Consider the following grammar G_1:

- $V = \{A, B\}$
- $\Sigma = \{0, 1, \#\}$
- $R =$

$$
\begin{aligned}
A &\rightarrow 0A1 \\
A &\rightarrow B \\
B &\rightarrow \#
\end{aligned}
$$

- $S = A$

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$$L(G_1) = \{0^n\#1^n\}$$

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For a grammar G generating language L, can generate each string $w \in L$ as follows:

1. Write down the start variable
2. Find a written-down variable and a rule starting with that variable. Replace the written variable with the right side of that rule
3. Repeat Step 2 until no variables remain

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- They capture recursive structures common in language (e.g., noun phrases can be made of verb-phrases and vice-versa)
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- Context-free grammars originated in the study of human languages
- They capture recursive structures common in language (e.g., noun phrases can be made of verb-phrases and vice-versa)
  - a girl with a flower likes the boy
- Also, very useful for describing programming languages:
  - Can capture matching, nested brackets:

```
if x > 3 {
    if y < 5 {
        Do something
    }
}
```

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This is Tricky

Designing CFGs is not natural, takes lots of practice

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Question

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$$S_2 \rightarrow 1S_2 0 \mid \epsilon$$

3. Combine the two to give the grammar for the union
$$\begin{aligned} S &\rightarrow S_1 \mid S_2 \\ S_1 &\rightarrow 0S_1 1 \mid \epsilon \\ S_2 &\rightarrow 1S_2 0 \mid \epsilon \end{aligned}$$

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3. Concatenate the two to give the grammar for L

$$\begin{aligned} S &\rightarrow AC \\ C &\rightarrow aCb \mid \epsilon \\ A &\rightarrow aA \mid a \end{aligned}$$

Exercise

Give a CFG for $L = \{a^m b^n \mid m \neq n, m, n \geq 0\}$
Hint: Think of this as the union of two languages

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$$R = A \to 0A1 \mid B, \qquad B \to \#$$

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Why study parse trees?

- Parse trees help us understand the "meaning" of a string
- Also, how parsers can parse a string according to a grammar (e.g., of a programming language)

Parse Trees – An Example

Consider Grammar G_1

$$R = A \rightarrow 0A1 \mid B, \qquad B \rightarrow \#$$

Parse tree for $000\#111$ in grammar G_1

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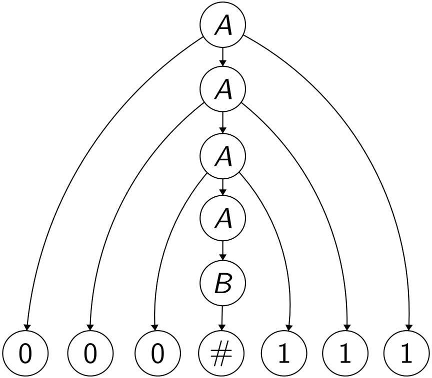
Parse tree for 000#111 in grammar G_1



Another Example

A Grammar G_2 for Arithmetic Statements

- $V = \{\langle EXPR \rangle, \langle TERM \rangle, \langle FACTOR \rangle\}$
- $\Sigma = \{a, +, \times, (,)\}$
- $R =$
$$\begin{array}{rcl} \langle EXPR \rangle & \to & \langle EXPR \rangle + \langle TERM \rangle \mid \langle TERM \rangle \\ \langle TERM \rangle & \to & \langle TERM \rangle \times \langle FACTOR \rangle \mid \langle FACTOR \rangle \\ \langle FACTOR \rangle & \to & (\langle EXPR \rangle) \mid a \end{array}$$

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- What is $L(G_2)$?
- Parse tree for $a + a \times a$

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  Example: The girl touches the boy with the flower

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Is ambiguity a problem?

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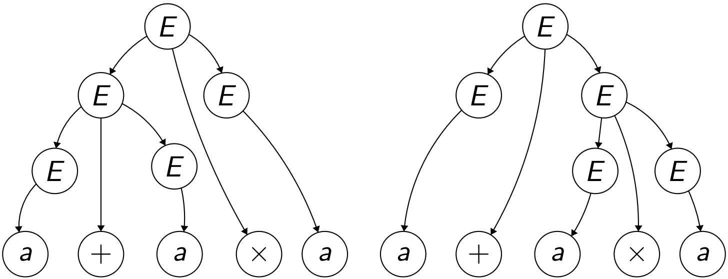
  Example: The girl touches the boy with the flower
- Unfortunately, ambiguous languages cannot be made unambiguous

An Example

Consider the following grammar G_3

$$E \to E + E \mid E \times E \mid (E) \mid a$$



Two parse trees for the string $a + a \times a$

On Thursday

- Equivalence between CFGs and PDAs
- A pumping lemma for CFGs