Cryptography

Lecture 2

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Outline

1. Lecture 1 Review
2. Probability Review (Ch. A.3)
3. Perfectly-Secure Encryption (Ch. 2.1)
4. The One-Time Pad (Ch. 2.2)

Lecture 1 Review

- Syllabus review
- Defining Secure Encryption

Outline

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Let E_1 and E_2 be events: (e.g., a coin toss comes up HEADS)

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Definition

E_1 and E_2 are *independent* if $\Pr[E_1 \wedge E_2] = \Pr[E_1] \cdot \Pr[E_2]$

Some Useful Definitions and Facts

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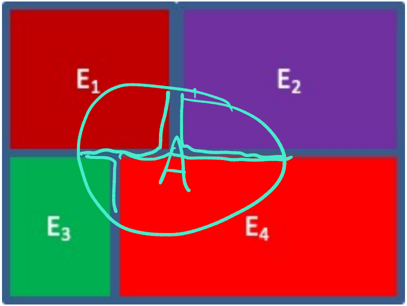
- Proof:
  By definition of conditional probability,
  $\Pr[E_1 \mid E_2] \cdot \Pr[E_2] = \Pr[E_1 \wedge E_2] = \Pr[E_2 \mid E_1] \cdot \Pr[E_1]$.
  So, $\Pr[E_1 \mid E_2] = \frac{\Pr[E_2|E_1] \cdot \Pr[E_1]}{\Pr[E_2]}$

Some Useful Definitions and Facts

- Law of Total Probability: If $E_1, E_2, \ldots, E_n$ are a partition (non-overlapping) of all possibilities. Then, for any event A,

$$\Pr[A] = \sum_{i=1}^{n} \Pr[A \wedge E_i] = \sum_{i=1}^{n} \Pr[A \mid E_i] \cdot \Pr[E_i]$$

- Proof Sketch:



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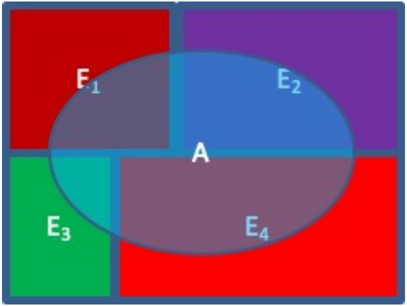

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- Union Bound:

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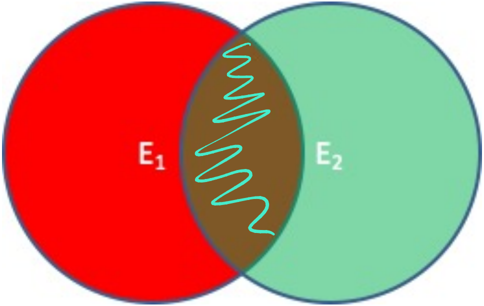

Outline

Private-key encryption

Eve

Alice

c

Bob

Key k
Message m
Encrypt m: $c = \mathrm{Enc}_k(m)$

Key k
Receive ciphertext c
Decrypt c: $m = \mathrm{Dec}_k(c)$

Security

Eve gets to observe c, but can not learn m

Defining Encryption Security

Security Guarantee

What is a successful attack?

- $\mathcal{A}$ learns the key k
- $\mathcal{A}$ learns the message m
- $\mathcal{A}$ learns any character of m
- Semantic security: Regardless of what $\mathcal{A}$ knows about m, she learns no new information

Threat Model

What can an adversary do?

- ciphertext-only
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Probability Distributions:

- Let M be a random variable denoting value of the message
  - M ranges over plaintext space $\mathcal{M}$
  - Distribution of M reflects $\mathcal{A}$'s prior knowledge of message being sent (not all messages are equally likely)

M: 000 0000000 $\underbrace{\text{random}}$ 90-bit string

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- Let C be a random variable (ranging over ciphertext space $\mathcal{C}$) denoting the ciphertext. It's distribution is defined by the following experiment:
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Remember

$\mathcal{M}$ is a space, M is a random variable, m is a value taken on by M
We will often look at $\Pr[M = m]$

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$$\underline{\Pr[M = m \mid C = c]} = \underline{\Pr[M = m]}$$

00 = 1/2
01 1/2
10 0
11 0

$\mathcal{M} =$
00 $\frac{1}{7}$
01 $\frac{1}{7}$
10 1/7
11 4/7

$c = 0^*$

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$$\Pr[M = m \mid C = c] = \Pr[M = m]$$

- For all pairs $m, m' \in \mathcal{M}$, for all $c \in \mathcal{C}$

$$\Pr[\mathsf{Enc}_K(m) = c] = \Pr[\mathsf{Enc}_K(m') = c]$$

A Game-Based Definition

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$\mathsf{PrivK}^{eav}_{\mathcal{A},\Pi}$

- $\mathcal{A}$ outputs two messages $m_0, m_1 \in \mathcal{M}$, s.t. $|m_0| = |m_1|$
- The challenger chooses $k \leftarrow \mathsf{Gen}$, $b \leftarrow \{0,1\}$, computes $c \leftarrow \mathsf{Enc}_k(m_b)$ and gives c to $\mathcal{A}$
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Observation

Note that $\mathcal{A}$ can win with probability $1/2$ by just guessing b' at random. This definition says that this is the best she can do.

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Perfectly Secure Encryption Definition

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The One-Time Pad

XOR

x	y	$x \oplus y$
0	0	0
0	1	1
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One-Time Pad Encryption Scheme

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Correctness: For all $k \in \mathcal{K}$ and all $m \in \mathcal{M}$,

$$\mathsf{Dec}_k(\mathsf{Enc}_k(m)) =$$

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$$\mathsf{Dec}_k(\mathsf{Enc}_k(m)) = k \oplus (k \oplus m) = (k \oplus k) \oplus m = 0^\ell \oplus m = m$$

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The one-time pad has some critical limitations that make it not ideal for real-world use.

- Can only use a key k to encrypt at most one message
- Need the length of the key to be as long as the message (i.e., $|k| \geq |m|$)

Why?