# Foundations of Computing

## Lecture 17

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# Outline


1. Lecture 16 Review
2. Where Are We Now?
3. Reduction Types
4. A Computational Definition of Information – Kolmogorov Complexity

# Lecture 16 Review

- Proofs by reduction
- Undecidable languages
  - $HALT_{TM}$
  - $REGULAR_{TM}$

# Exercise

$$EMPTY-STRING_{TM}=\{\langle M\rangle \mid M \text{ is a TM and } M(\epsilon)=1\}$$

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1. To show that a problem is decidable: Give an algorithm that always terminates and outputs the answer
2. To show that a problem is undecidable: Give an algorithm (a reduction) that shows that this problem can be used to solve an undecidable problems

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Recall: $\overline{A_{TM}}$ is Turing-unrecognizable

Problem: $\overline{A_{TM}} \leq A_{TM}$
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Takeaway: General reductions do not work for Turing-unrecognizable languages

**Solution**

We need to restrict what our reductions can do.

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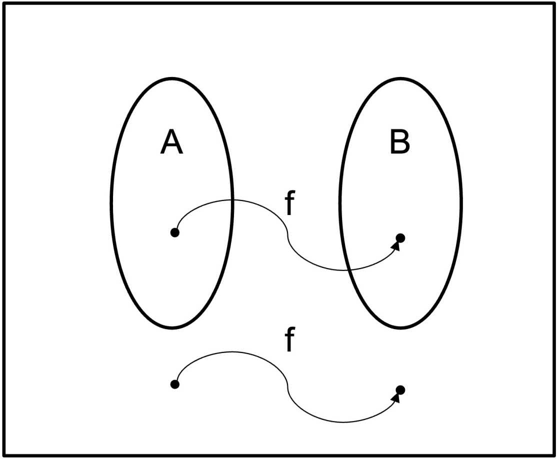
- Works by mapping input $\in A$ to input $\in B$ and vice-versa

# Mapping Reductions



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**Observation:**

Mapping reductions work for both decidability and Turing-recognizability.

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- For example, in the proof that $L_{TM} \leq L_{E_{TM}}$, we flipped the result of $R$ deciding $L_{E_{TM}}$

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3. If $A \leq_T B$
   - If $B$ is decidable then $A$ is decidable
   - If $A$ is not decidable, then $B$ is not decidable

   - If $B$ is Turing-recognizable, $A$ is not necessarily Turing-recognizable
   - If $A$ is not Turing-recognizable, cannot say if $B$ is Turing-recognizable

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# Information in a String

$$A = 010101010101010101010101$$

$$B = 110100100011100010111111$$

**Question**

Which of these strings contains more information?

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- This captures the “amount of information” in $x$

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3. $\forall x, y, K(xy) \leq 2K(x) + K(y) + c$

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   - So, there exists at least one string that is incompressible
2. In fact, incompressible strings look like random strings
3. But, $K(x)$ is not computable, moreover it is undecidable whether a string is incompressible