Cryptography

Lecture 13

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Outline

1. Lecture 12 Review
2. Hash Functions (Chapters 5.1, 5.2)
3. Other Applications of Hash Functions (Chapters 5.3, 5.6)

Lecture 12 Review

- Review of MAC domain extension
- Authenticated encryption

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2. Hash Functions (Chapters 5.1, 5.2)
3. Other Applications of Hash Functions (Chapters 5.3, 5.6)

Domain Extension for MAC (Try 4)

Starting Point

- Let $m = m_1||m_2||\cdots||m_\ell$, where each m_i is n bits
- Let $\Pi' = (\text{Gen}', \text{Mac}', \text{Verify}')$ be an n-bit MAC

Include random message identifier in each block:

- Parse m as $m_1||m_2||\cdots||m_{4\ell}$ with each m_i of length $n/4$
- $r \leftarrow \{0,1\}^{n/4}$ - message id
- Compute $t_i = \text{Mac}'_k(r||4\ell||i||m_i)$

The Problem:
This requires

- $|t| = 4\ell n$ bits
- 4ℓ calls to PRF

Question: Can we do domain extension more efficiently?

Another Way to Authenticate Long Messages

What if we could take a *digest* of a long message?

$$m = m_1||m_2|| \quad \dots \quad ||m_\ell$$
$$\downarrow$$
$$H(m)$$

and, then compute $t = \mathrm{Mac}_k(H(m))$

Question

What properties would we need from H for this to be a secure Mac?

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- *Collision resistance*: Hard to find m, m' s.t. $H(m) = H(m')$.

A More Formal Definition

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Definition: A hash function $\Pi = (\text{Gen}, H)$ is *collision resistant* if for all PPT $\mathcal{A}$ it holds that

$$\Pr[\text{Hash} - \text{Coll}_{\mathcal{A},\Pi}(n) = 1] \leq \text{negl}(n)$$

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Comparison to a MAC:

- Given $y = H^s(m)$, hard to find m' that hashes to y
- But, since s is public, any party can produce $(m', y' = H^s(m'))$

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For now, we will stick to the asymptotic definition

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- For a pair, $\Pr[bday_i = bday_j] = 1/365$
- After 365 pairs (28 people), expect a collision
- Generally, $O(2^{\ell/2})$ for output length ℓ – need ℓ large enough

Building a Hash Function

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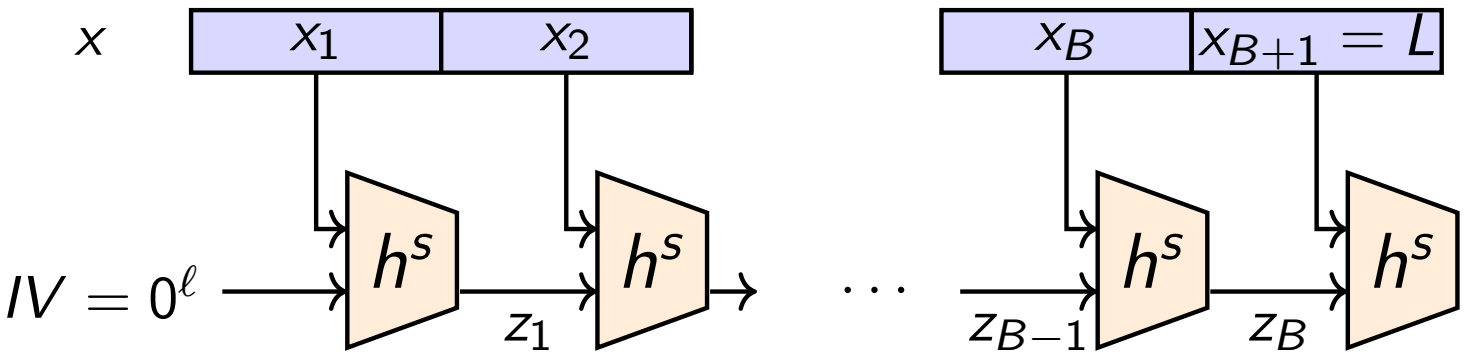
- Extend domain from ℓ'-bit strings to arbitrary bit strings
  - This is what we will do now

Merkle-Damgard Domain Extension



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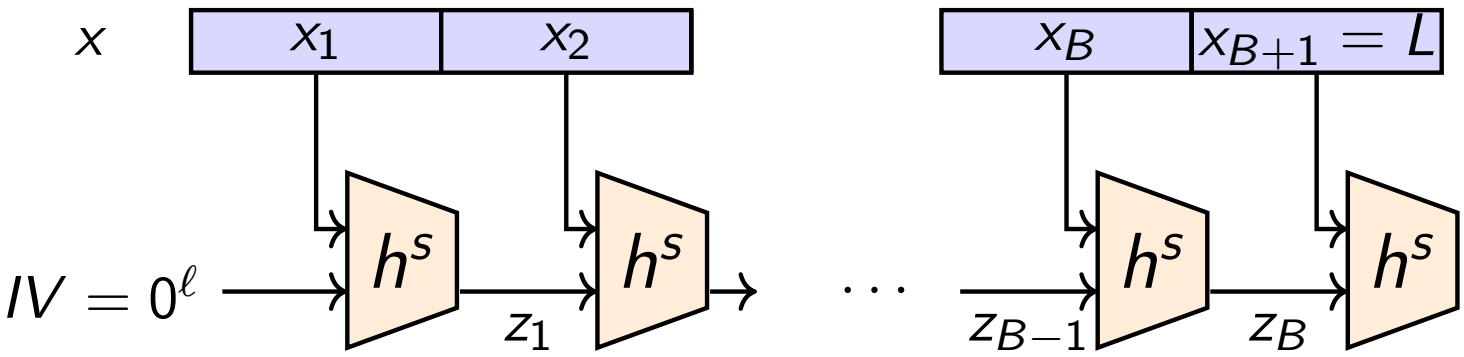

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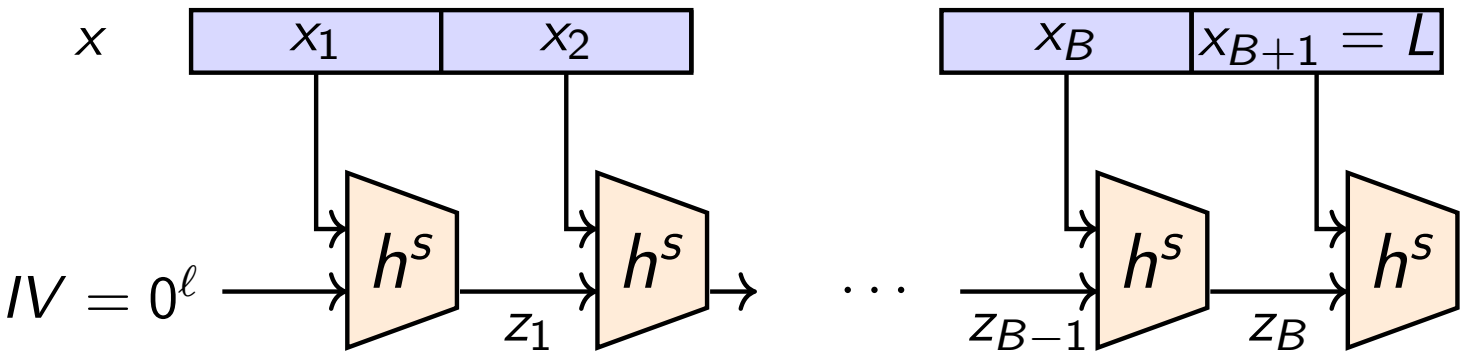

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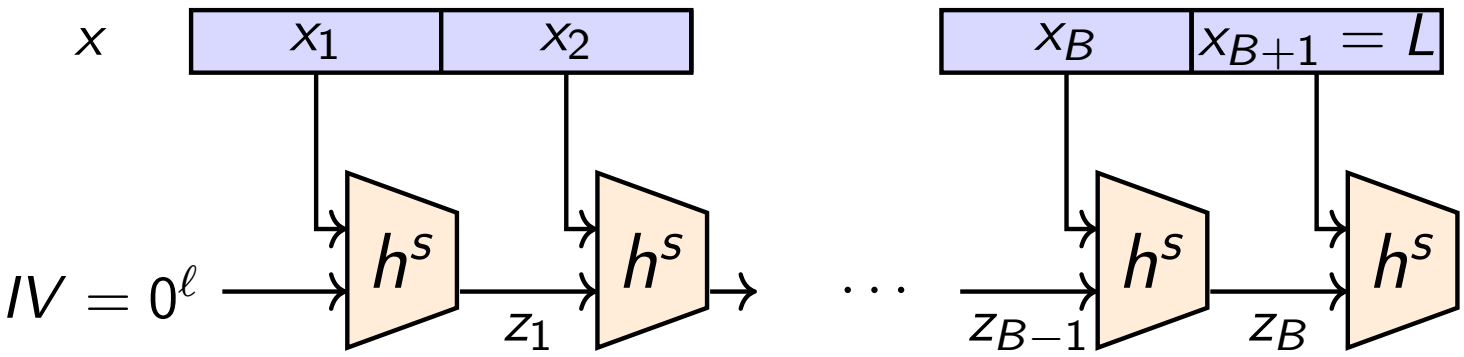

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- Break x into ℓ-bit blocks

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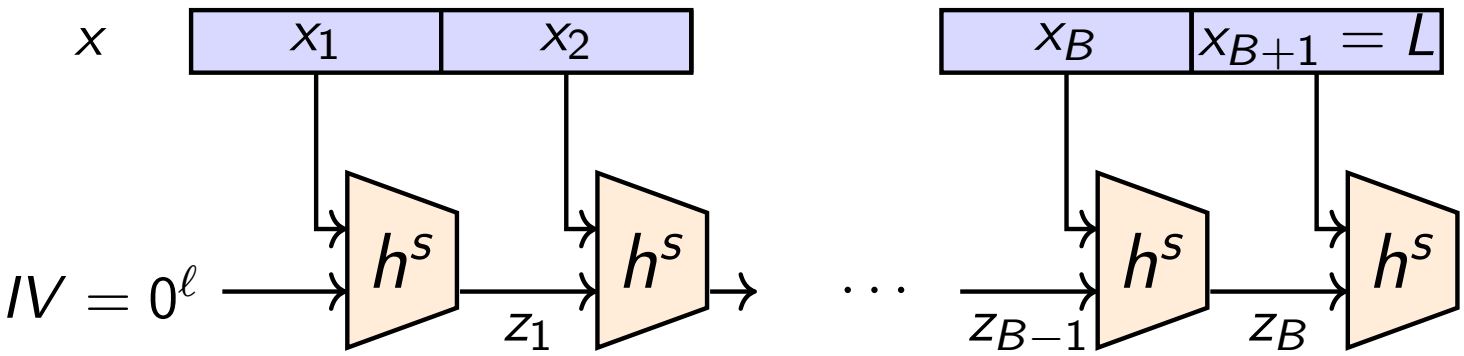

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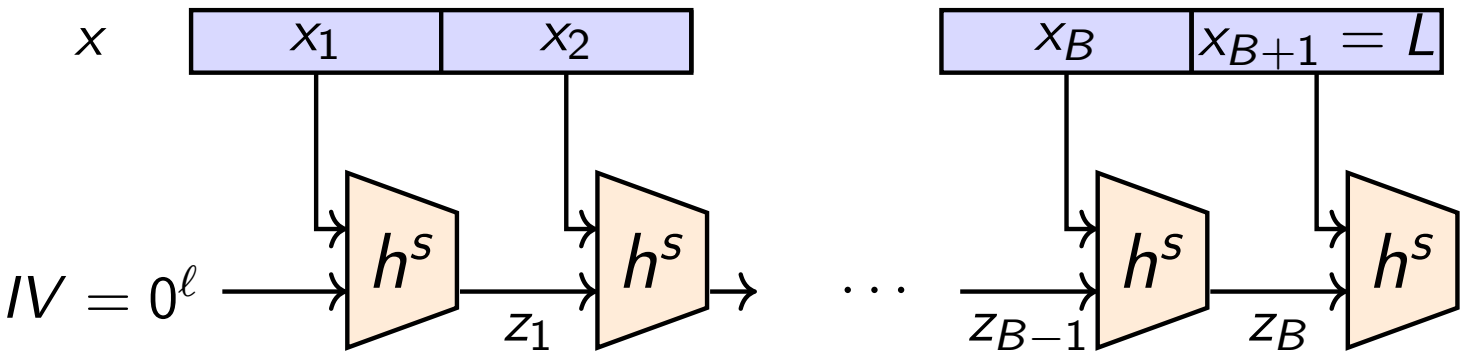

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- Compute $H^s(x)$ as in the figure above

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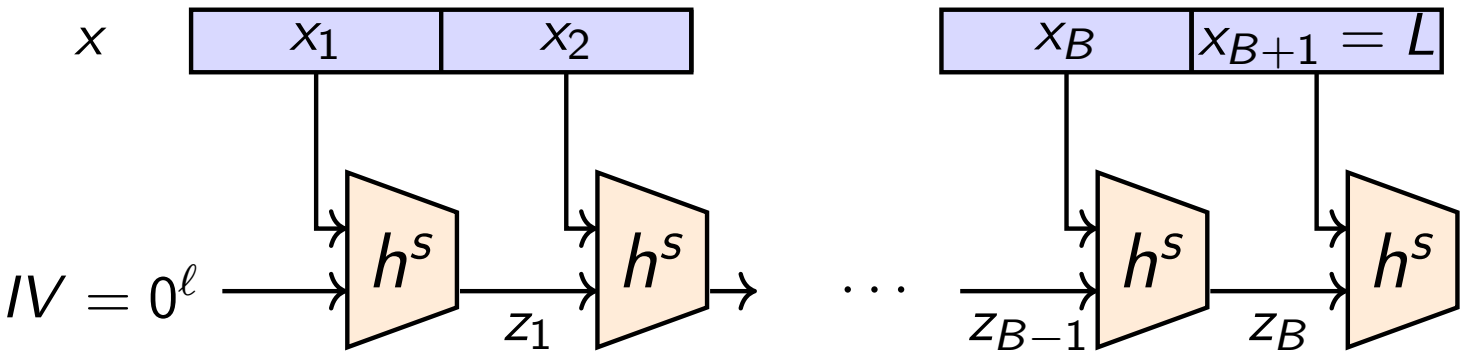

Proof of Collision Resistance:

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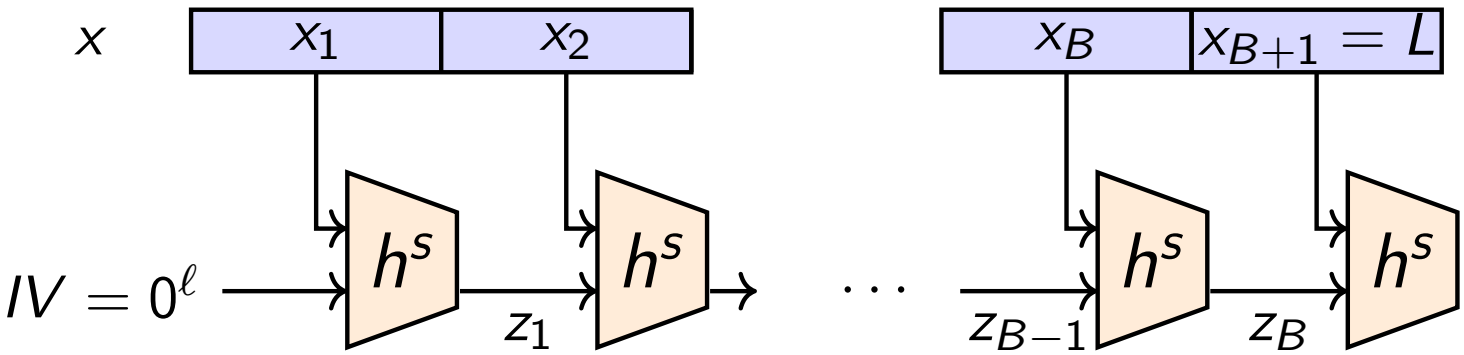

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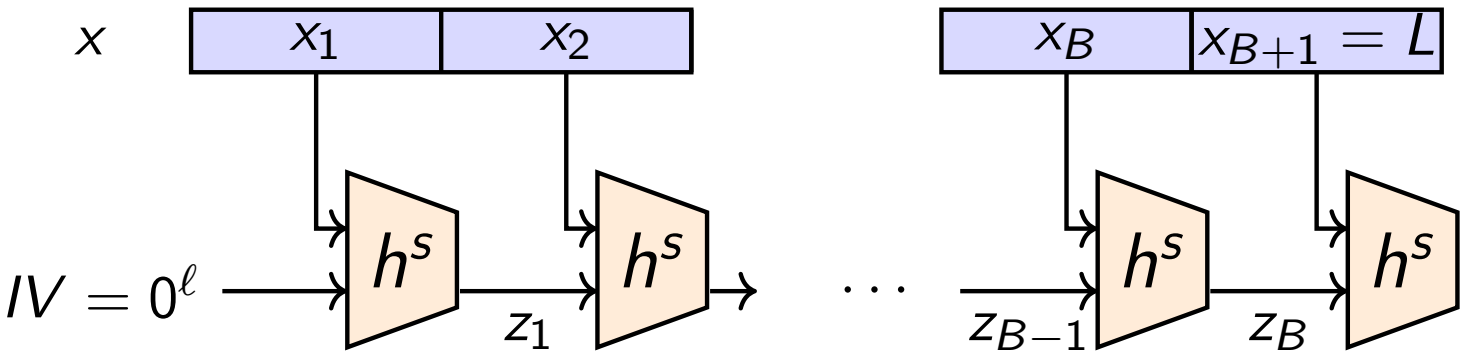

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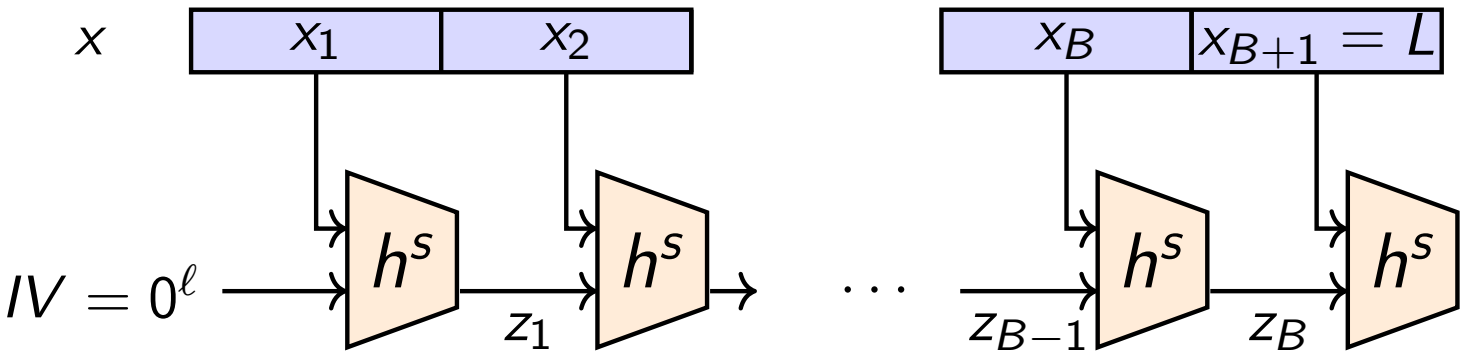

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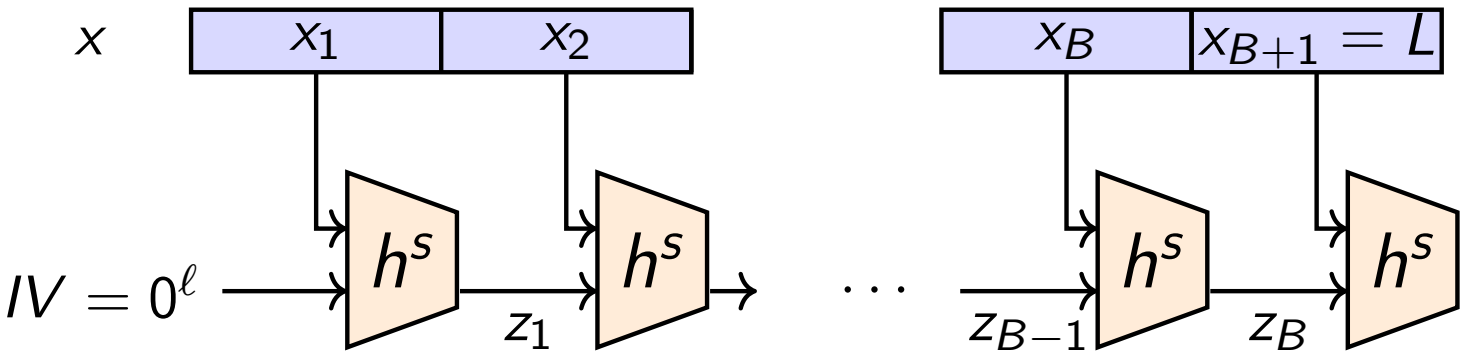

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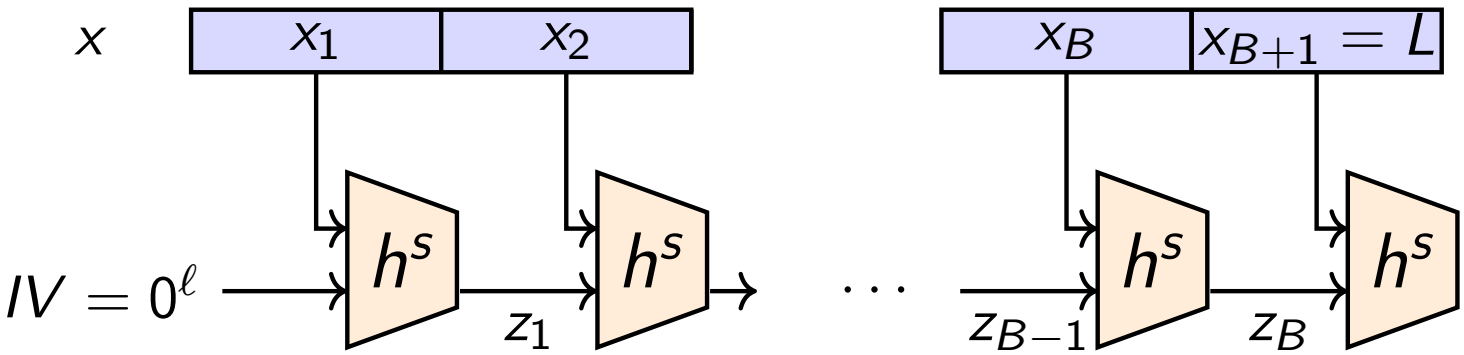

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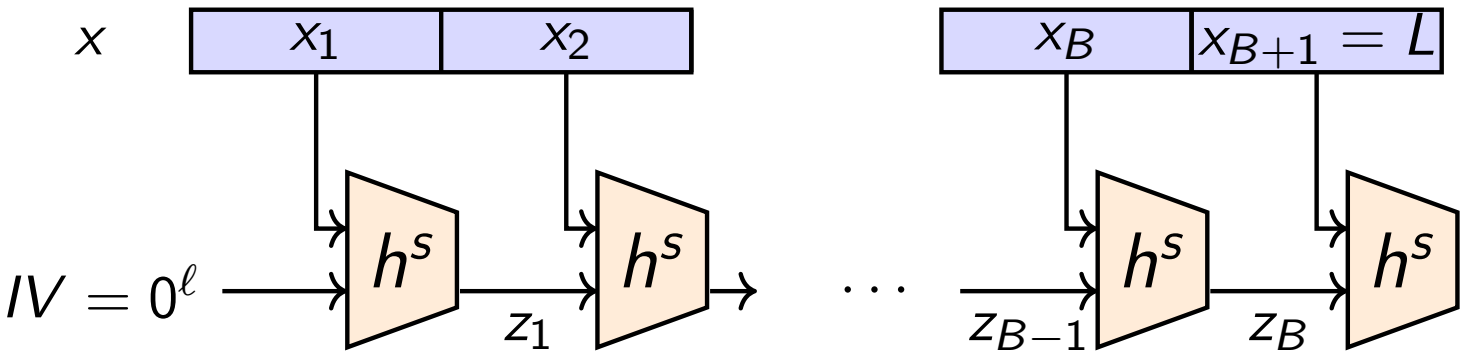

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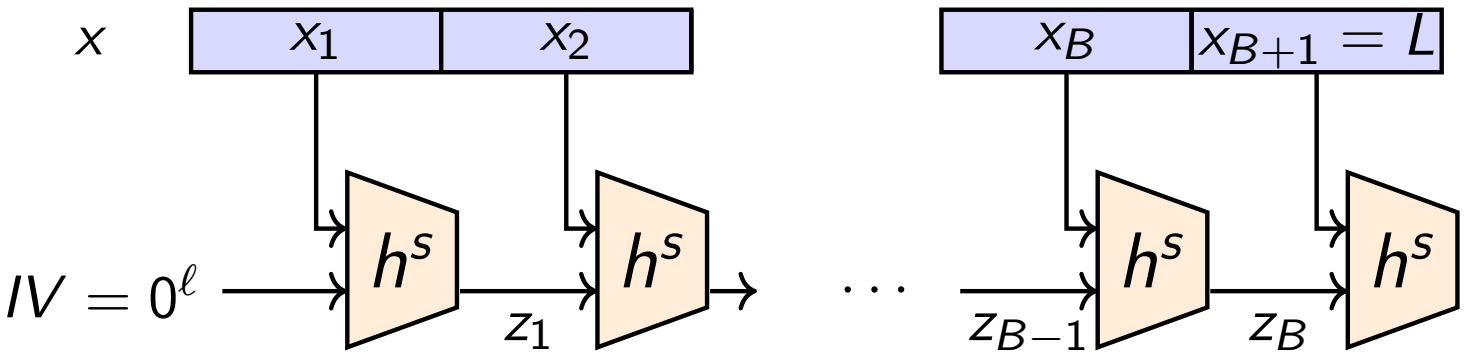

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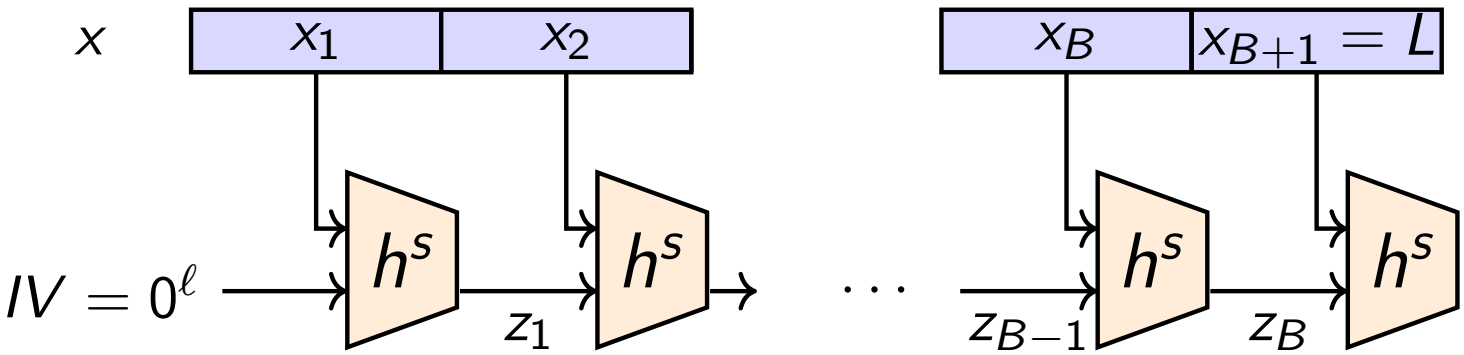

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Domain Extension for MACs (Hash-and-MAC)

Building blocks

- $\Pi = (\mathsf{Gen}, \mathsf{Mac}, \mathsf{Verify})$: Secure MAC for $\ell(n)$-bit messages
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  - But, then $\mathcal{A}$ has forged valid tag on new message $H^s(m^*)$

Outline

1. Lecture 12 Review
2. Hash Functions (Chapters 5.1, 5.2)
3. Other Applications of Hash Functions (Chapters 5.3, 5.6)

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Can we protect passwords even if password file is stolen?

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- Hashing is very fast – Billions of hashes / second

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Takeaway

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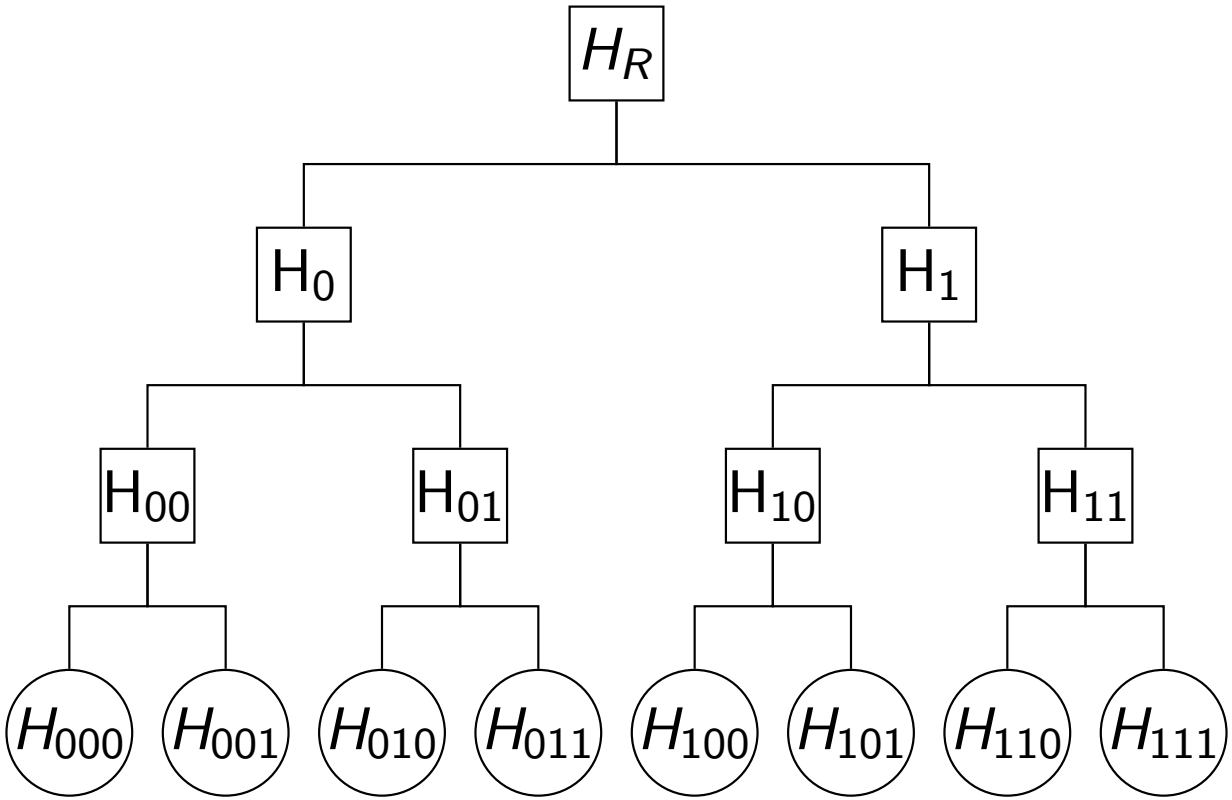
Question: Can we get solution that achieves both?

- Low storage on the client
- Low communication to verify a file is correct

Merkle Tree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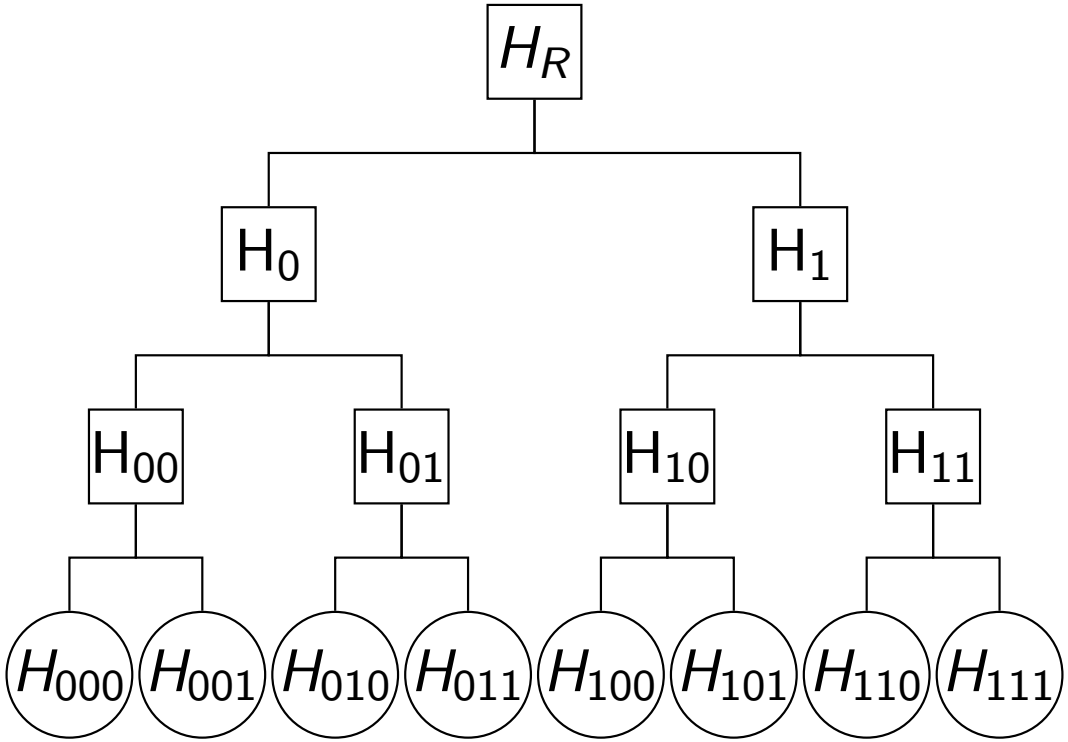

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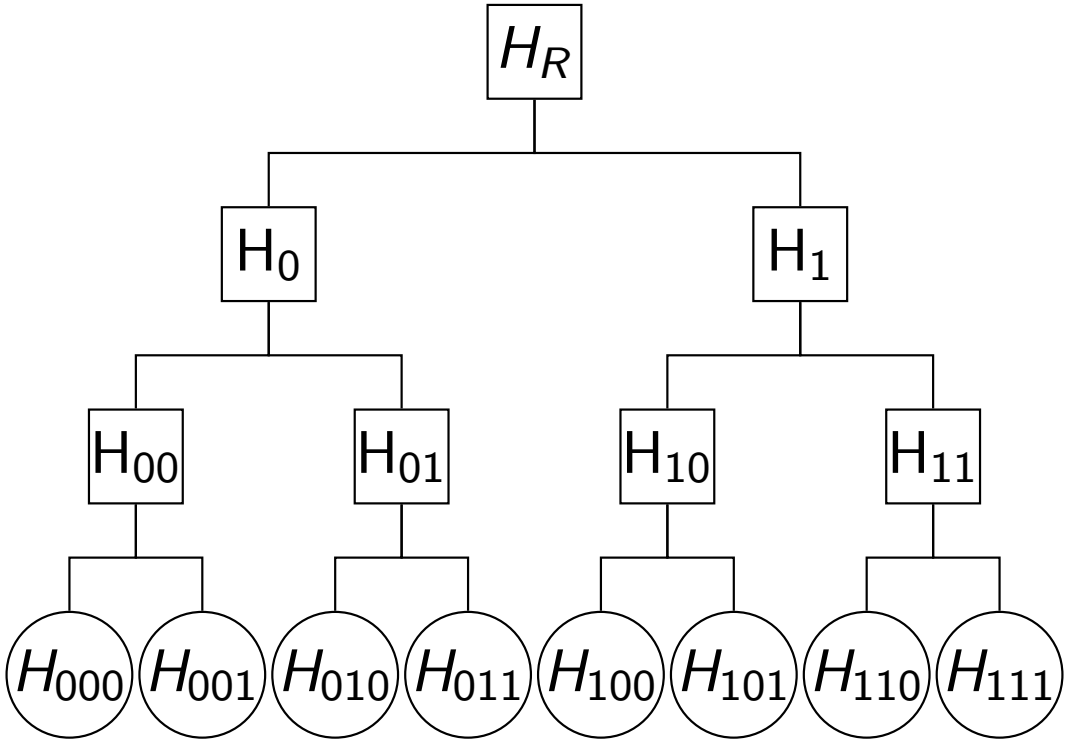

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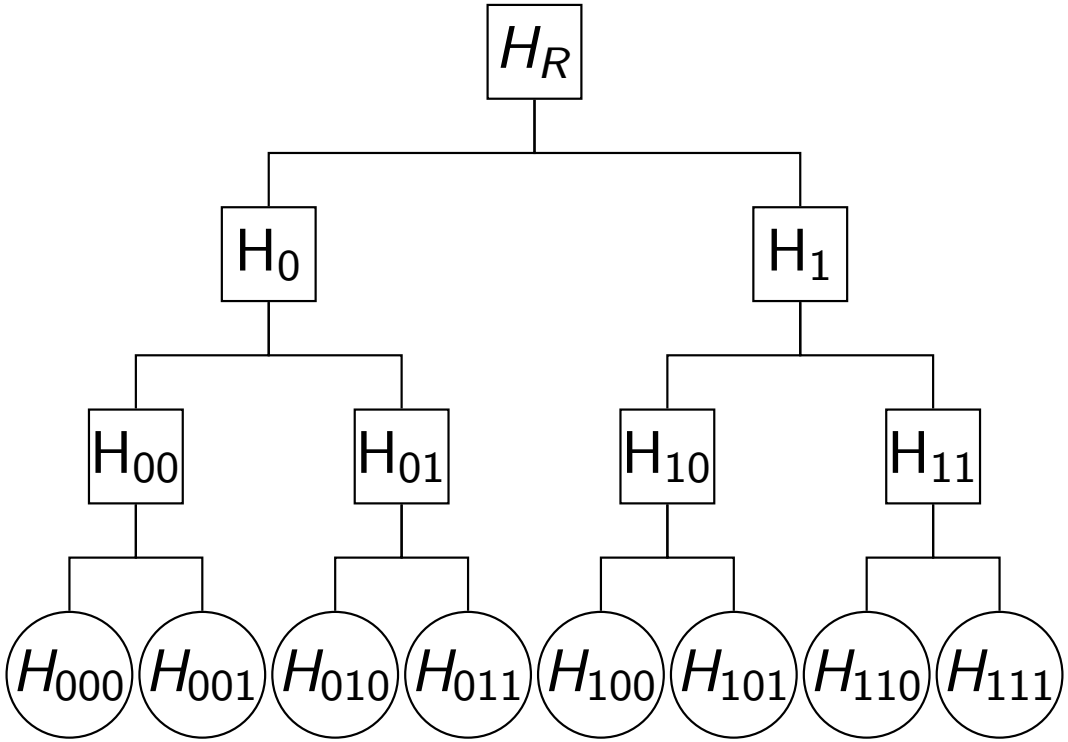
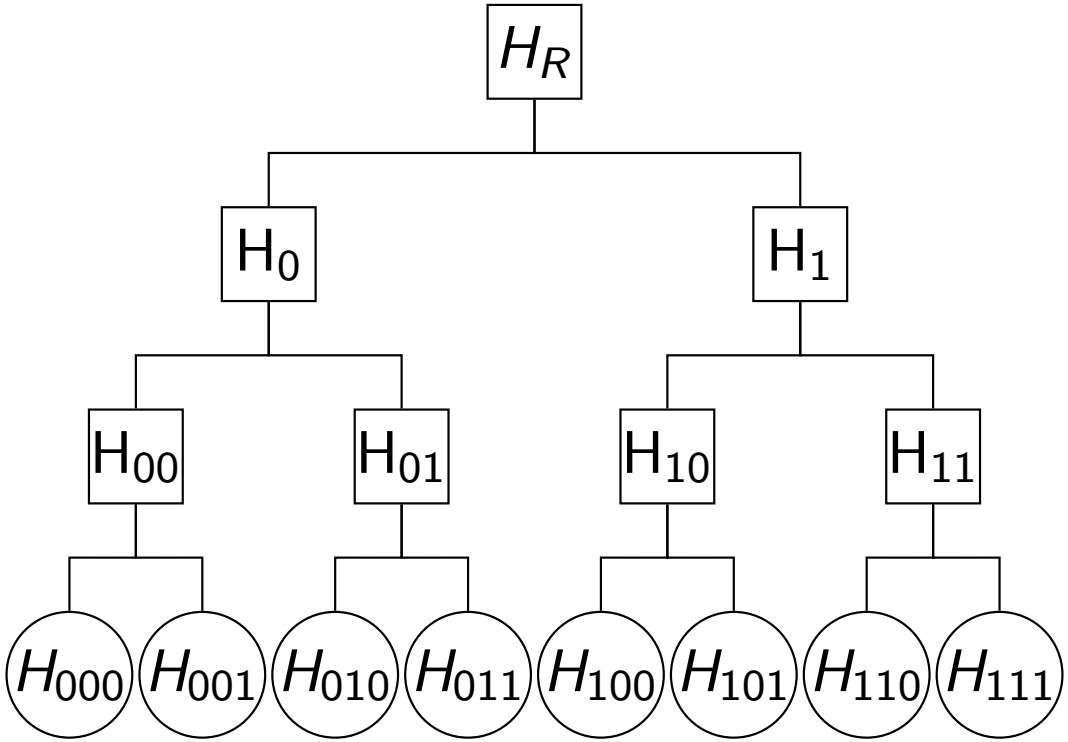
Building a Merkle Tree



- C has files $x_{000}, \ldots, x_{111}$, computes $H_{000} = H(x_{000}), \ldots$
- C computes $H_{00} = H(H_{000}, H_{001}), H_{01} = H(H_{010}, H_{011}), \ldots$
- C does the same for all levels: $H_0 = H(H_{00}, H_{01})$, $H_R = H(H_0, H_1)$

Building a Merkle Tree



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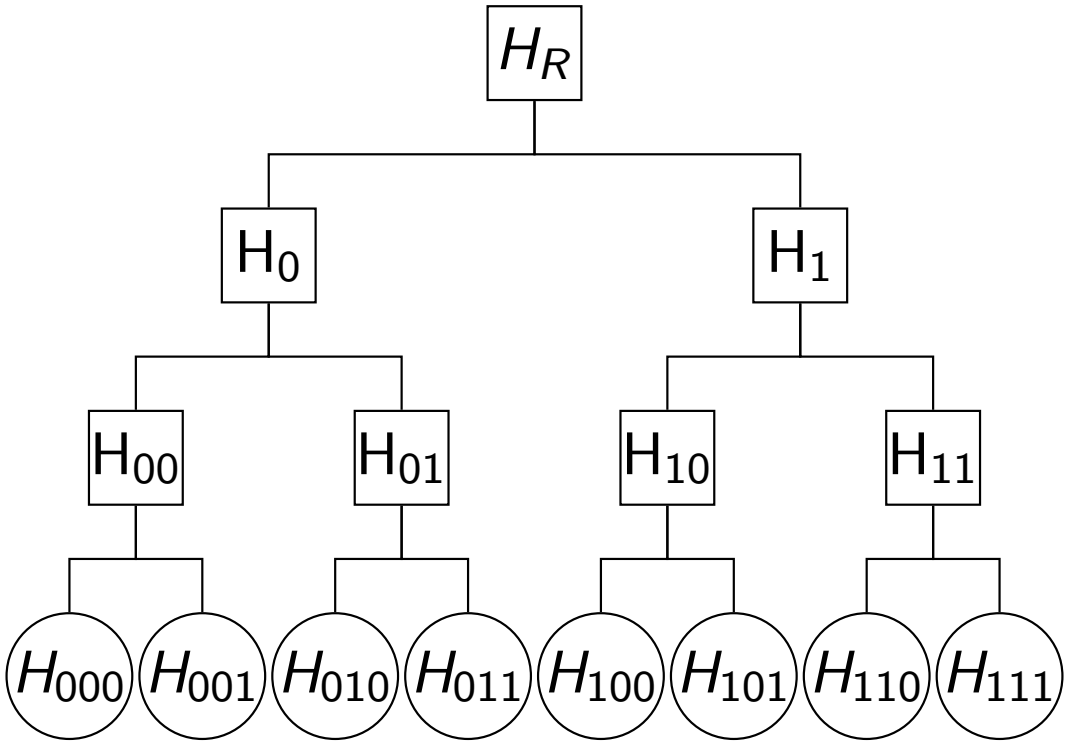
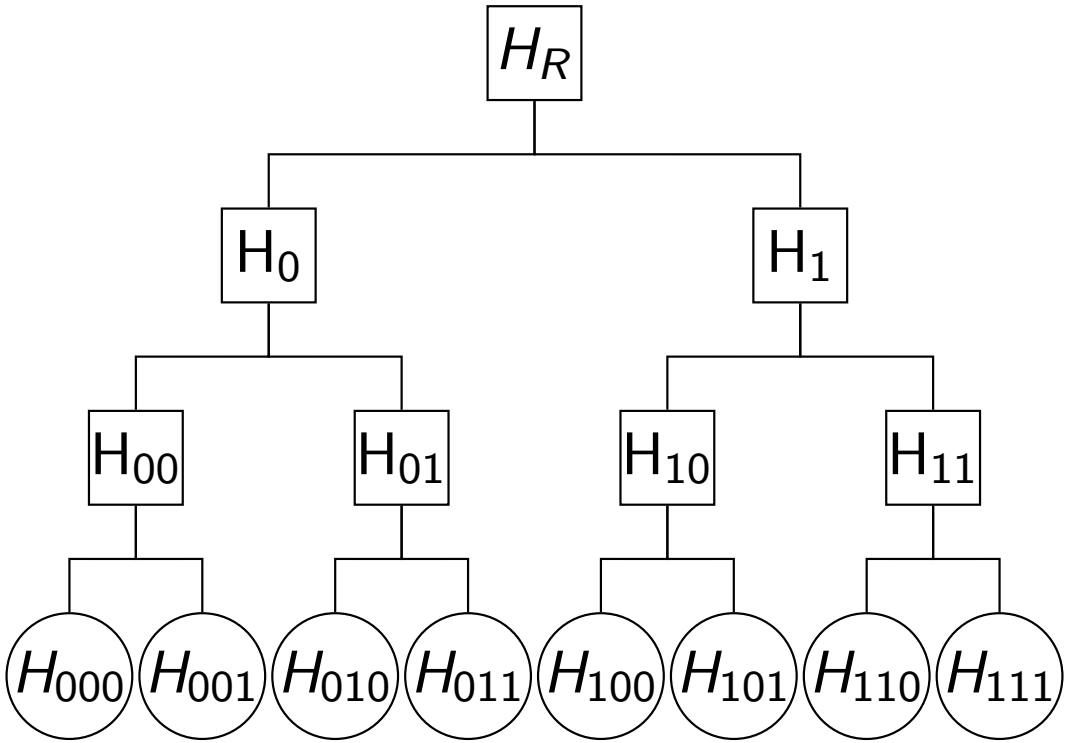
- C does the same for all levels: $H_0 = H(H_{00}, H_{01})$, $H_R = H(H_0, H_1)$
- C stores value H_R at the root and uploads all files to S

Verifying File x_{001}:



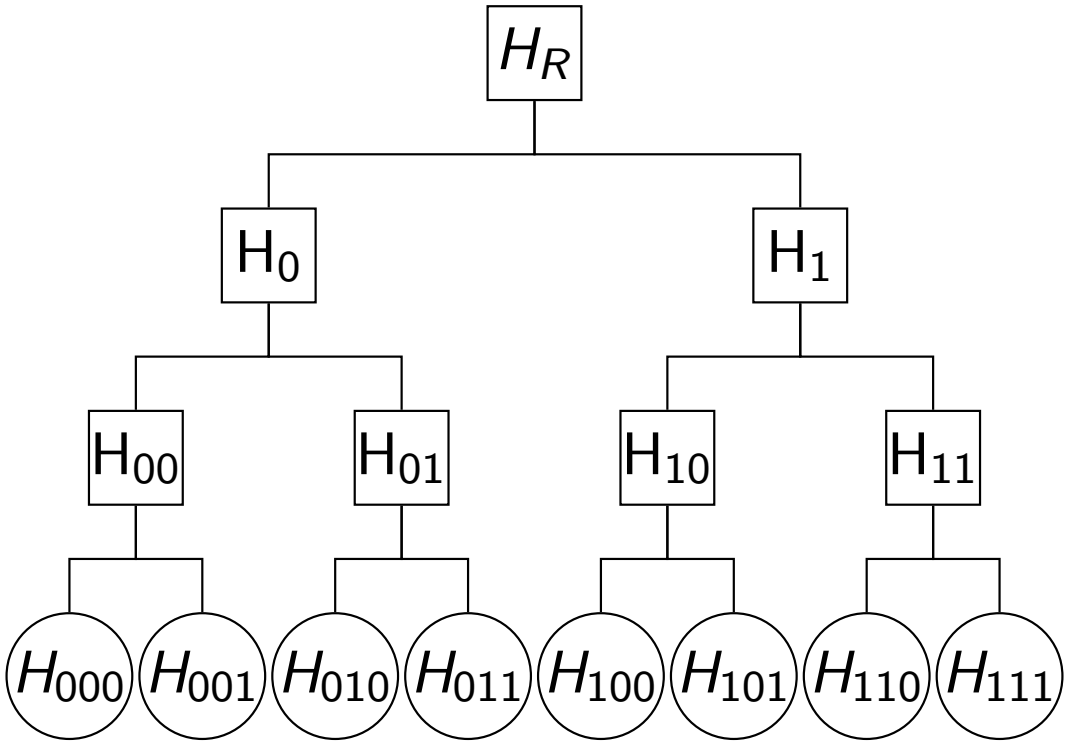
- C has H_R, downloads x_{001} and wants to check it's correct

Verifying File x_{001}:



- C has H_R, downloads x_{001} and wants to check it's correct
- S sends C all sibling hashes along the path to root: H_{000}, H_{01}, H_1

Verifying File x_{001}:



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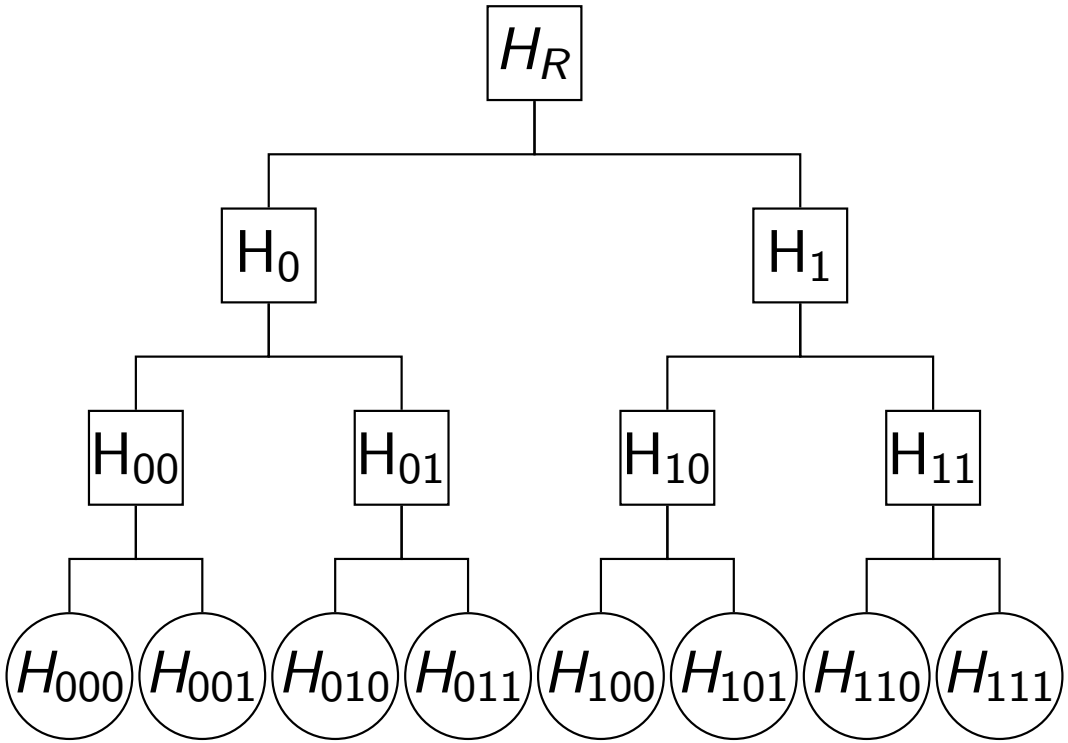
Verifying File x_{001}:



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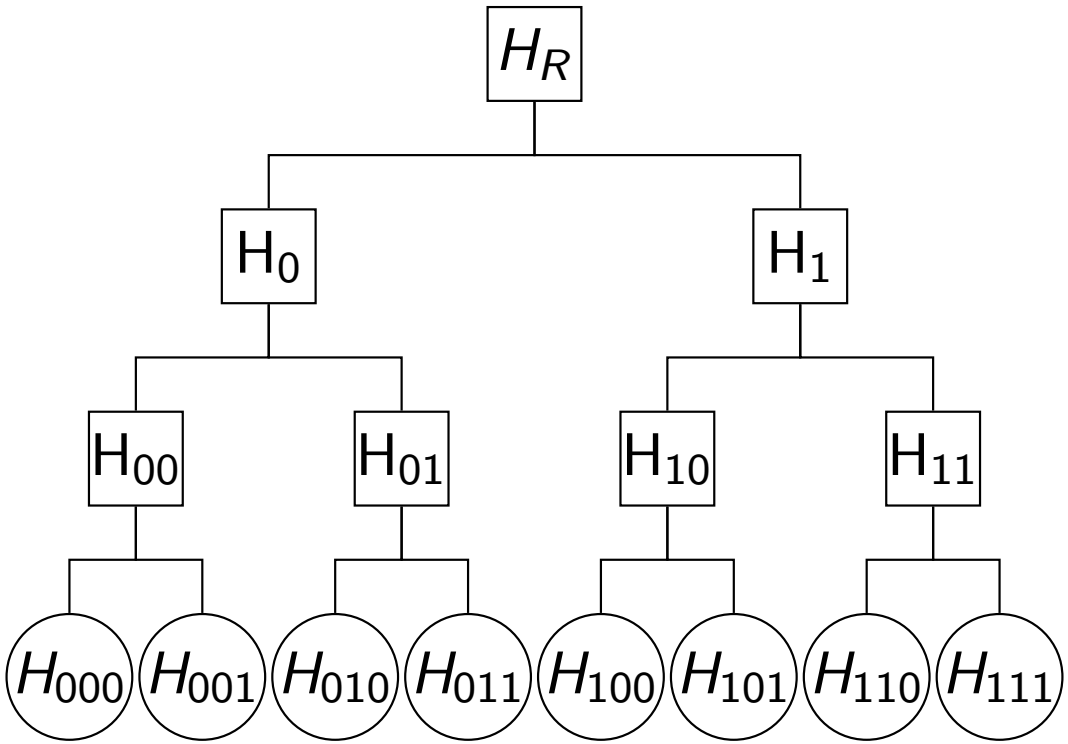
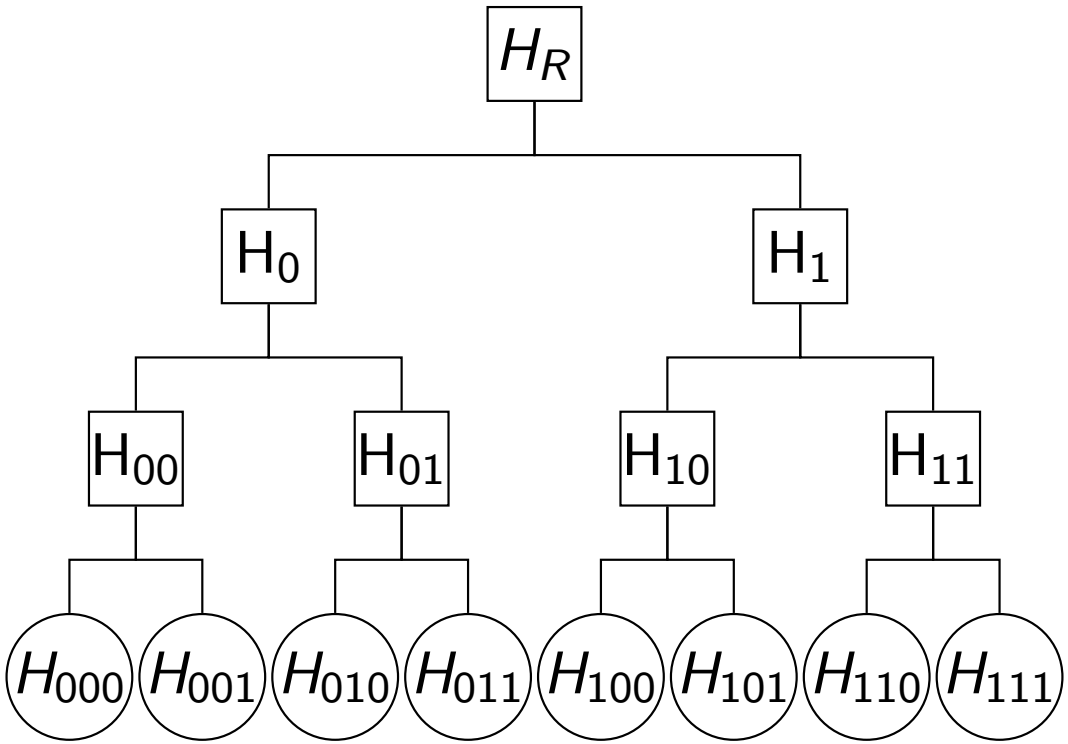
- S sends C all sibling hashes along the path to root: H_{000}, H_{01}, H_1
- C computes H_{001} and can now recompute up to H_R (has both inputs for every node in path)
- C checks that computed value is equal to H_R, accepts if so

Merkle Tree Facts



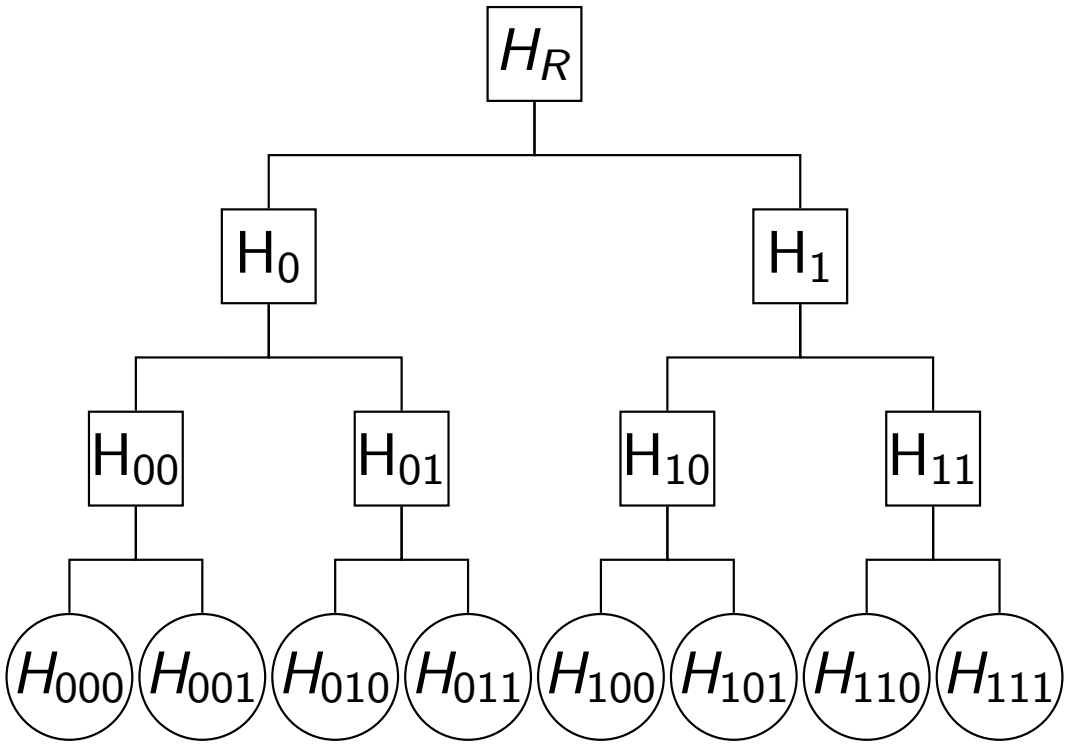
- Allows verified outsourced storage

Merkle Tree Facts



- Allows verified outsourced storage
- C only needs to store a single hash value, H_R

Merkle Tree Facts



- Allows verified outsourced storage
- C only needs to store a single hash value, H_R
- Proof consists of $O(\log n)$ hash values