Foundations of Computing

Lecture 12

Arkady Yerukhimovich

February 27, 2024

Outline

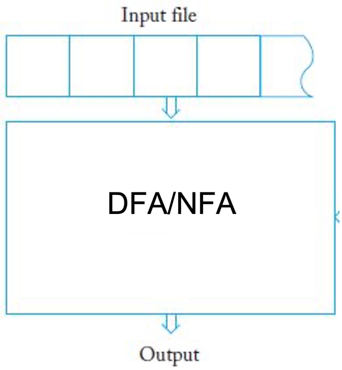
Lecture 10+11 Review

- Equivalence of CFGs and PDAs
- CFL Pumping Lemma
- Using the CFL Pumping Lemma

Outline

1. Lecture 10+11 Review
2. Models of Computation
3. The Turing Machine
4. Formalizing Turing Machines

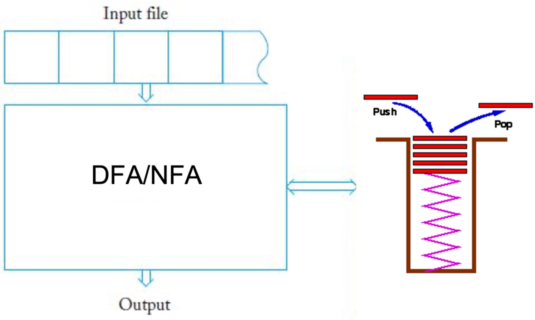
Finite Automata



Recall:

- An NFA/DFA has no external storage
- Only memory must be encoded in the finite number of states
- Can only recognize regular languages

Pushdown Automata (PDA)



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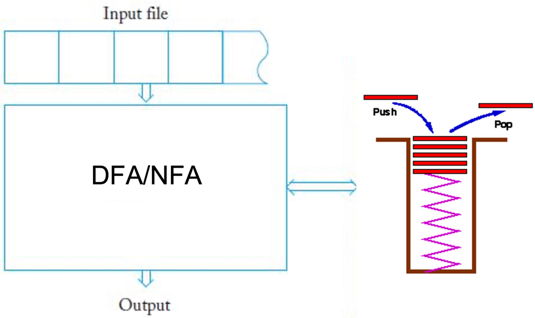

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Recall:

- Can only access memory in LIFO fashion
- Can only recognize context-free languages

A Model for General Computation

Question

All the prior models of computation couldn't recognize some simple languages. Can we develop a computation model that captures all languages that can be computed on any computer?

Our Goal

One model to rule them all!

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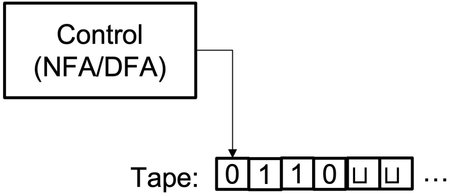
The Turing Machine



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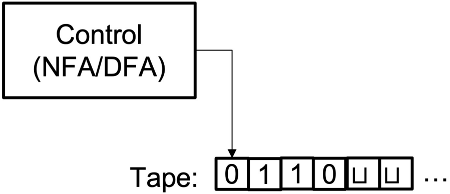

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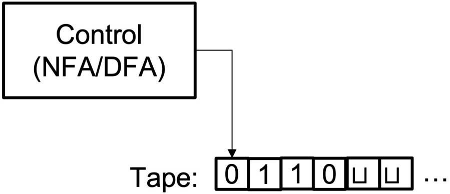

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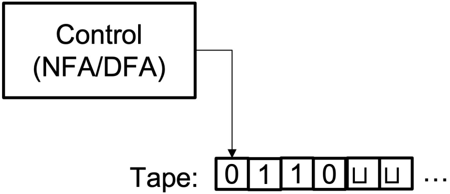

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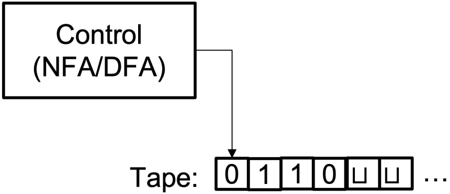

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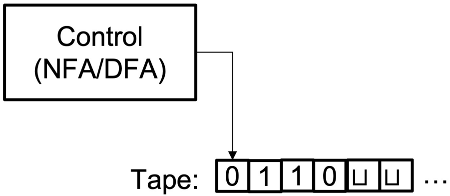

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- The memory tape is infinite
- Control FA has accept and reject states that are immediately output if entered

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On input string s (written on the tape):

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3. When all symbols to the left of $\#$ have been crossed off, check that no uncrossed-off symbols remain to the right of $\#$. If any symbols remain, reject, otherwise accept.

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- To reason about languages accepted by NFA/PDA, we designed algorithms
- How can we reason about the limits of what an algorithm can compute?

Turing Machines and Algorithms

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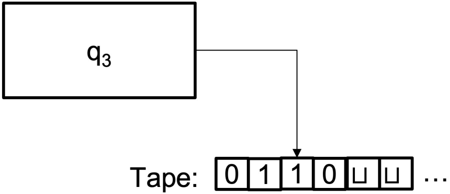

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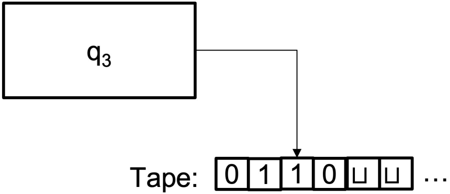

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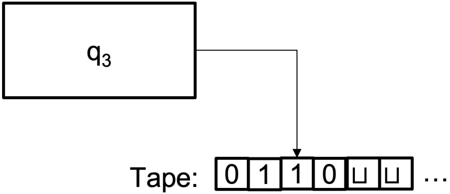

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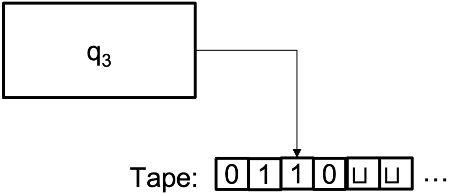

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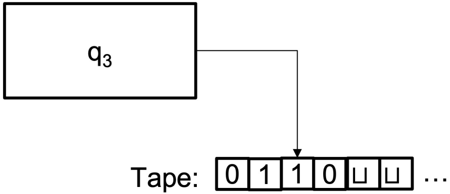

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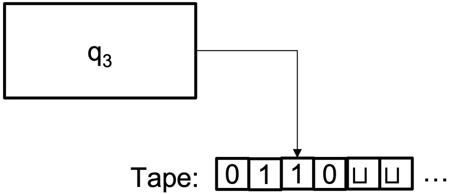

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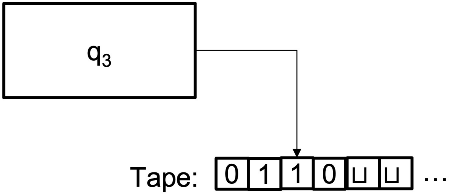

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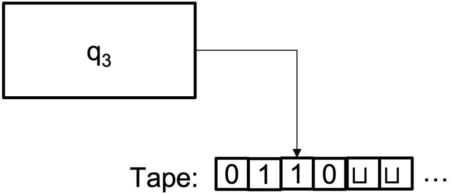

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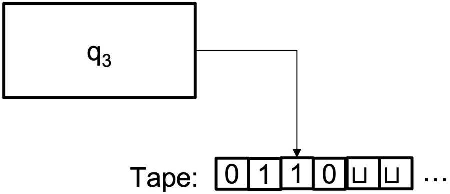

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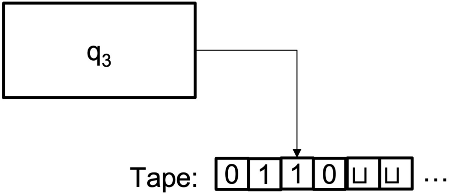

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Language $L(M)$

The collection of strings that M accepts

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- M halts on all inputs, accepting those in L and rejecting those not in L

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Making M Formal

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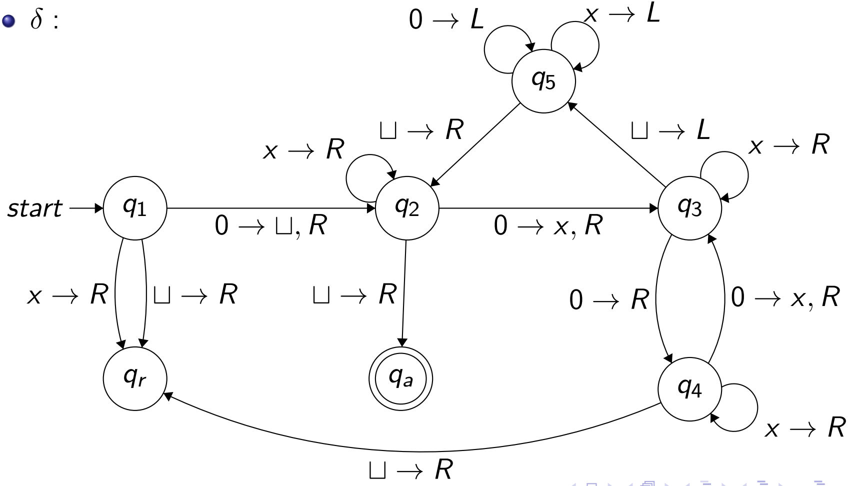
- $\Sigma = \{0\}$
- $\Gamma = \{0, x, \sqcup\}$
- δ :

Making M Formal

- $Q = \{q_1, q_2, q_3, q_4, q_5, q_a, q_r\}$
- $\Sigma = \{0\}$
- $\Gamma = \{0, x, \sqcup\}$
- δ :



Running M on $w = 0000$