Foundations of Computing

Lecture 15

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Outline

Lecture 14 Review

- Decidable and Turing-recognizable languages
- Decidability of regular and context-free languages

Outline

Characterizing Computability of Languages

Definition: Decidable languages

A language L is *decidable* or *recursive* if some TM M decides it

- M halts on ALL inputs, accepts those in L and rejects those not in L
- Seems to match informal definition we wanted before

Definition: Turing-recognizable languages

A language L is *Turing-recognizable* or *recursively enumerable* if some TM M recognizes it

- M halts and accepts all strings in L
- M may not halt on strings not in L – does not necessarily have to reject

Observation

Every Decidable language is also Turing-recognizable, but the reverse direction is not true.

Decidable Languages

We showed the following languages are decidable:

- Languages about Finite Automata
  1. $A_{DFA} = \{\langle B, w\rangle \mid B$ is a DFA that accepts input string $w\}$
  2. $A_{NFA} = \{\langle B, w\rangle \mid B$ is a NFA that accepts input string $w\}$
  3. $A_{REX} = \{\langle R, w\rangle \mid R$ is a reg. exp. that generates the string $w\}$
  4. $E_{DFA} = \{\langle A\rangle \mid A$ is a DFA and $L(A) = \emptyset\}$
  5. $EQ_{DFA} = \{\langle A, B\rangle \mid A, B$ are DFAs and $L(A) = L(B)\}$
- Languages about CFGs
  1. $A_{CFG} = \{\langle G, w\rangle \mid G$ is a CFG that generates $w\}$
  2. $E_{CFG} = \{\langle G\rangle \mid G$ is a CFG and $L(G) = \emptyset\}$

A Language About Turing Machines

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  1. Simulate M on input w
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- Is A_{TM} Decidable?
  - The problem: M may never halt
  - In this case, above algorithm will never output accept or reject
  - If could determine that M will never halt (i.e, it has entered an infinite loop), could reject.

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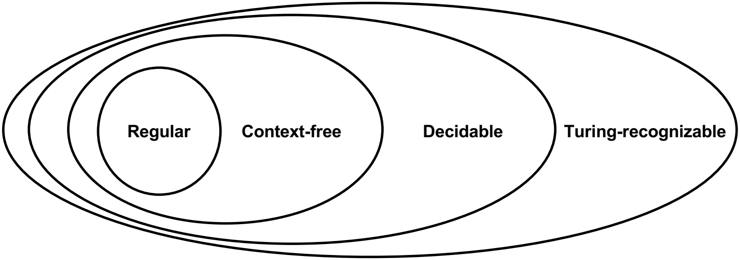
An Undecidable Problem

- We will prove today that A_{TM} is undecidable

Relationships Among Language Classes



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- Example:
  $A = \{0, 1, 2, 3\}$
  $B = \{a, b, c, d\}$
  $f(0) = a, f(1) = b, f(2) = c, f(3) = d$

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- A set A is countable if it is finite or countably infinite
- A set that is not countable is *uncountable*

Example 1: Evens

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The set of even numbers is

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Example 2: Rationals

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Rationals

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Rationals

The set of rational numbers is countable

	1	2	3	$\cdots$
1	1/1	1/2	1/3	
2	2/1	2/2	2/3	$\cdots$
3	3/1	3/2	3/3	
4	4/1	4/2	$\vdots$	

Example 3: Strings

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Diagonalization

Real Numbers

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- Then there is a one-to-one and onto mapping f from $\mathcal{N}$ to $\mathcal{R}$

Diagonalization

Real Numbers

The set of real numbers ($\mathcal{R}$) is uncountable

Proof: By diagonalization

- Assume that $\mathcal{R}$ is countable
- Then there is a one-to-one and onto mapping f from $\mathcal{N}$ to $\mathcal{R}$

n	f(n)
1	1.234...
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  Idea: For all $i \in \mathcal{N}$, make $x_i \neq f(i)_i$

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  Idea: For all $i \in \mathcal{N}$, make $x_i \neq f(i)_i$
- Contradiction – f is not mapping between $\mathcal{R}$ and $\mathcal{N}$

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- But, a TM M can be written as a string $\langle M \rangle \in \Sigma^*$

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- Can similarly show that for any finite alphabet Σ, Σ^* is countable
- But, a TM M can be written as a string $\langle M \rangle \in \Sigma^*$
- Hence, by omitting all strings that are not encodings of valid TMs we get a mapping of TMs to $\mathcal{N}$

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Languages over alphabet Σ

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   - Define the characteristic sequence χ_A of language $A \in \mathcal{L}$

$$
\begin{array}{rclccccccccl}
\Sigma^* & = & \{ & \epsilon & 0 & 1 & 00 & 01 & 11 & 000 & \cdots & \} \\
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   - This is a one-to-one and onto mapping from $\mathcal{L}$ to B, so $|\mathcal{L}| = |B|$
3. Therefore, $\mathcal{L}$ is uncountable

Some Languages are not Turing-recognizable

We have proven:

1. The set of Turing Machines is countable
2. The set of languages is uncountable

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Where are we now?

- We have proven that some languages are not Turing-recognizable
- But, we have not given any examples of such a language

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$$D(\langle M\rangle) = \begin{cases} \mathit{accept} & \text{if } M \text{ does not accept } \langle M\rangle \\ \mathit{reject} & \text{if } M \text{ accepts } \langle M\rangle \end{cases}$$

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- Now consider what happens if we run D on $\langle D\rangle$

$$D(\langle D\rangle) = \begin{cases} \textit{accept} & \text{if } D \text{ does not accept } \langle D\rangle \\ \textit{reject} & \text{if } D \text{ accepts } \langle D\rangle \end{cases}$$

- Contradiction!

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	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\cdots$	$\langle D \rangle$	$\cdots$
M_1	$\underline{\textit{accept}}$	*reject*	*accept*		*accept*	
M_2	*reject*	$\underline{\textit{reject}}$	*reject*	$\ldots$	*accept*	$\ldots$
M_3	*accept*	*accept*	$\underline{\textit{accept}}$		*reject*	
$\vdots$		$\vdots$		$\ddots$		
D	*reject*	*accept*	*reject*		?	

- We have defined D to do the opposite of what M_i does on input $\langle M_i \rangle$
- But what does D do on input $\langle D \rangle$??

A Turing-unrecognizable Language

$\overline{L_{TM}}$

The language

$$\overline{L_{TM}} = \{\langle M, w \rangle \mid M \text{ is a TM and } M(w) \neq 1\}$$

is not Turing-recognizable

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Another Way to Prove Undecidability

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- problem A is no harder than problem B.
- Equivalently, problem B is no easier than problem A

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- Suppose that B is decidable
- Since $A \leq B$, there exists an algorithm (i.e., a reduction) that uses a solution to B to solve A
- But, this means that A is decidable by running the machine for B as needed by the reduction

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On input $\langle M, w\rangle$, S does the following:

Undecidability of HALT

$$HALT_{TM} = \{\langle M, w \rangle \mid M \text{ is a TM and } M \text{ halts on input } w\}$$

Theorem: $HALT$ is undecidable

Proof Sketch:

- We show that $A_{TM} \leq HALT$
- Since we know that A_{TM} is undecidable, this shows that $HALT$ is also undecidable

Proof:

Construct algorithm S that decides A_{TM} given a TM R that decides $HALT$

On input $\langle M, w \rangle$, S does the following:

- Run $R(\langle M, w \rangle)$

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