Cryptography
Lecture 4

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Outline

1. Lecture 3 Review
2. Pseudorandom Generator (PRG) (Ch. 3.3.1)
3. Proofs by Reduction(Ch. 3.3.2)

Lecture 3 Review

- Limitations of OTP and perfect secrecy
- Proof techniques
- Defining computationally-secure encryption

Outline

Constructing Private-Key Encryption for Long Messages

- Recall that we encrypted by computing $\mathsf{Enc}_k(m) = m \oplus k$
- But, if $|k| < |m|$, this is not secure

Key Idea

What if we had a way to stretch key k into something longer that still looked random?

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$s' \leftarrow \{0,1\}^{n+1}$ v/s

$s \leftarrow \{0,1\}^n$, $G(s)$

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  Consider $G : \{0,1\}^n \rightarrow \{0,1\}^{n+1}$, then $G(s)$ can only possibly output at most one half of possible output values

$S \leftarrow \{0,1\}^n$

$\leq 2^n$

G : n-bits $\rightarrow$ $n+1$-bits

2^{n+1} $n+1$-bit strings

$2^{n+1} = 2 \cdot 2^n$

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- $G(s)$ cannot "look random" to someone who knows s
- $G(s)$ is only required to look random to someone who knows nothing about s – i.e., s is uniformly random

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$\mathcal{D}$ cannot distinguish between $G(s)$ and $r \leftarrow \{0,1\}^{\ell(n)}$

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  - If $r = G(s)$, then $r_{n+1} = \oplus_{i=1,\ldots,n} s_i$ with probability 1
  - If $r \leftarrow \{0,1\}^{n+1}$, then $r_{n+1} = \oplus_{i=1,\ldots,n} s_i$ with probability $1/2$
- So, $\mathcal{D}$ will always output 1 when given $G(s)$ and output 0 with probability $1/2$ when given r.

$$\Pr[\mathcal{D}\text{ WINS}] = \Pr[\overset{1/2}{b = 1}] * 1 + \Pr[\overset{1/2}{b = 0}] * 1/2 = 3/4 > 1/2$$

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Encryption from a PRG

P vs. NP

$P \neq NP$

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- Can now encrypt messages longer than key
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Next Step

Prove Security!

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How Do We Prove Security

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A security proof shows that Π is secure if the assumption is true

- We say we show a *reduction* from the security of Π to the assumption.

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- Essentially, this is a proof by contradiction.

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  - If $\mathcal{A}_c$ succeeds in breaking the simulated Π, $\mathcal{A}_r$ uses this to solves X

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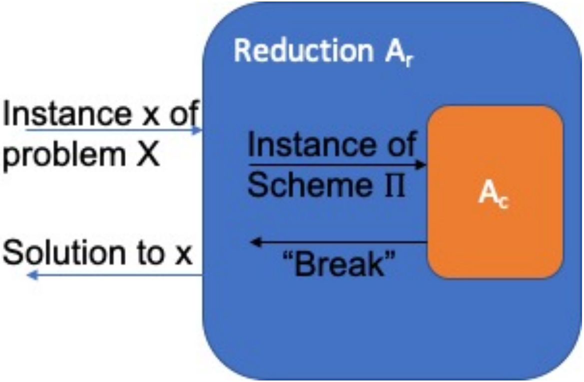

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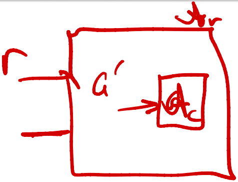

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  - $\mathcal{A}_c$ outputs its guess b' and $\mathcal{A}_r$ outputs this as its guess

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$PRG_{\mathcal{D},G}(n)$

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  If $b = 0$, he chooses $r \leftarrow \{0,1\}^{l(n)}$;
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- Analysis:
  - If $b = 0$, then $r \leftarrow \{0,1\}^{n+2}$ so $r' \leftarrow \{0,1\}^{n+1}$ (same as $b = 0$ for $\mathcal{A}_c$)
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  - If $\mathcal{A}_c$ outputs b=b', then $\mathcal{A}_r$ outputs $b = b'$
  - Since $\Pr[PRG_{\mathcal{A}_C,G'}(n) = 1] > 1/2 + 1/\text{poly}(n)$, we get that $\Pr[PRG_{\mathcal{A}_r,G}(n) = 1] > 1/2 + 1/\text{poly}(n)$

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for any PRG G, G' is a secure PRG

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Fully Black-Box Reductions

- Such reductions are called *fully black-box*
- (Almost) all reductions in cryptography are fully black-box