Cryptography

Lecture 13

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Outline

1. Lecture 12 Review
2. Hash Functions (Chapters 5.1, 5.2)
3. Other Applications of Hash Functions (Chapters 5.3, 5.6)

Lecture 12 Review

- Review of MAC domain extension
- Authenticated encryption

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2. Hash Functions (Chapters 5.1, 5.2)
3. Other Applications of Hash Functions (Chapters 5.3, 5.6)

Domain Extension for MAC (Try 4)

Starting Point

- Let $m = m_1||m_2||\cdots||m_\ell$, where each m_i is n bits
- Let $\Pi' = (\mathsf{Gen}', \mathsf{Mac}', \mathsf{Verify}')$ be an n-bit MAC

Include random message identifier in each block:

- Parse m as $m_1||m_2||\cdots||m_{4\ell}$ with each m_i of length $n/4$
- $r \leftarrow \{0,1\}^{n/4}$ - message id
- Compute $t_i = \mathsf{Mac}'_k(r||4\ell||i||m_i)$

The Problem:

This requires

- $|t| = 4\ell n$ bits
- 4ℓ calls to PRF

Question: Can we do domain extension more efficiently?

Another Way to Authenticate Long Messages

$m = m_1 \| m_2 \| \ldots$ $\quad t = tag(m_1)$

What if we could take a *digest* of a long message?

$t = Mac(m_1 \oplus m_2 \oplus m_3 \ldots)$

$$m = m_1 || m_2 || \quad \ldots \quad || m_\ell$$
$$\downarrow$$
$$H(m)$$

and, then compute $t = \mathrm{Mac}_k(H(m))$

Question

What properties would we need from H for this to be a secure Mac?

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- *Collision resistance*: Hard to find m, m' s.t. $H(m) = H(m')$.

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- $\mathsf{Gen}(1^n)$: Outputs a key s
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Definition: A hash function $\Pi = (\text{Gen}, H)$ is *collision resistant* if for all PPT $\mathcal{A}$ it holds that

$$\Pr[\text{Hash} - \text{Coll}_{\mathcal{A},\Pi}(n) = 1] \leq \text{negl}(n)$$

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Comparison to a MAC:

- Given $y = H^s(m)$, hard to find m' that hashes to y
- But, since s is public, any party can produce $(m', y' = H^s(m'))$

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For now, we will stick to the asymptotic definition

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- After 365 pairs (28 people), expect a collision

$$\sum_{\text{all pairs}} \frac{1}{365} = \frac{n(n-1)}{2(365)} = 1$$

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- For a pair, $\Pr[bday_i = bday_j] = 1/365$
- After 365 pairs (28 people), expect a collision
- Generally, $O(2^{\ell/2})$ for output length ℓ – need ℓ large enough

Building a Hash Function

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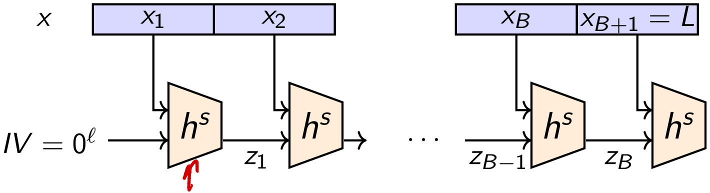
  - Defined only for a fixed $\ell' > \ell$
  - We will see examples later in the course
- Extend domain from ℓ'-bit strings to arbitrary bit strings
  - This is what we will do now

Merkle-Damgard Domain Extension



$h^s : \{0,1\}^{2\ell} \to \{0,1\}^\ell$

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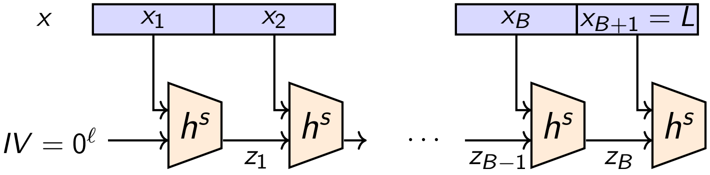

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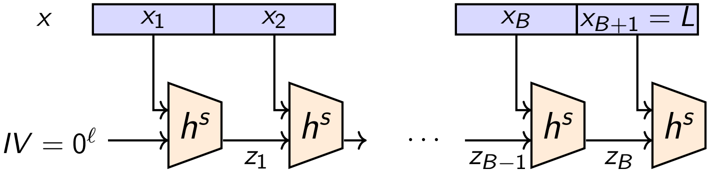

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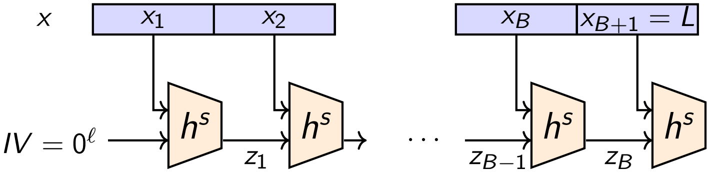

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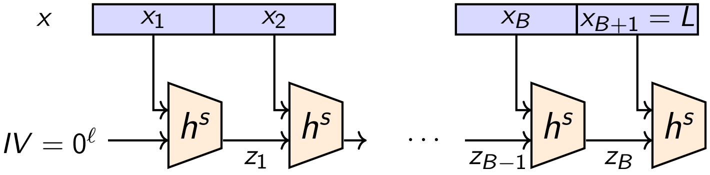

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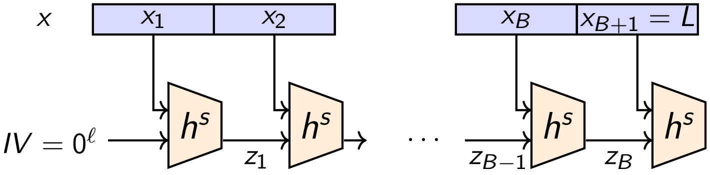

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- Compute $H^s(x)$ as in the figure above

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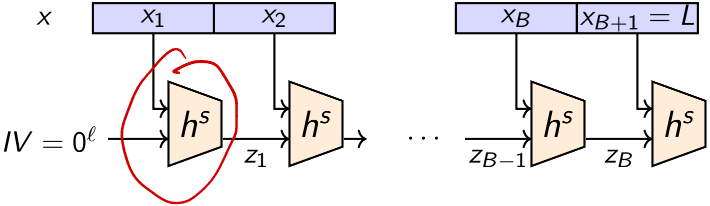

Proof of Collision Resistance:

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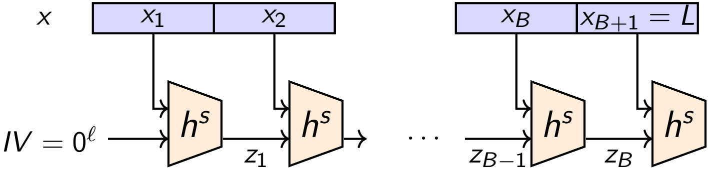

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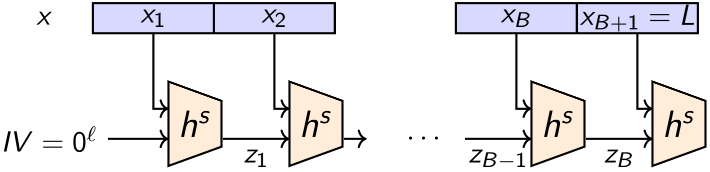

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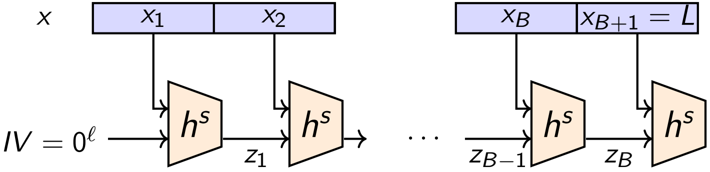

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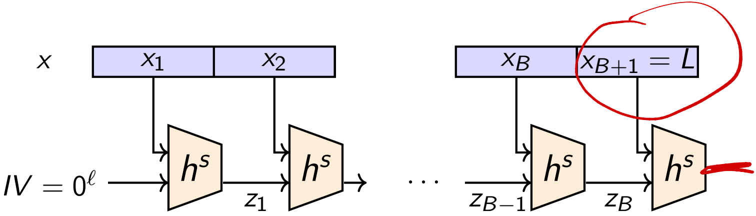

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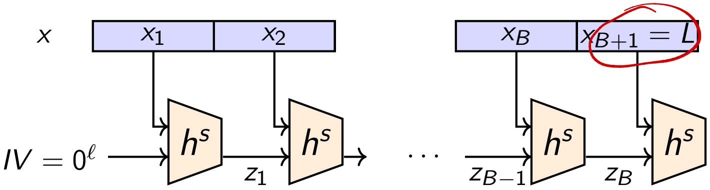

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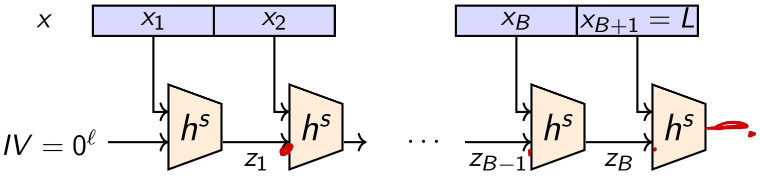

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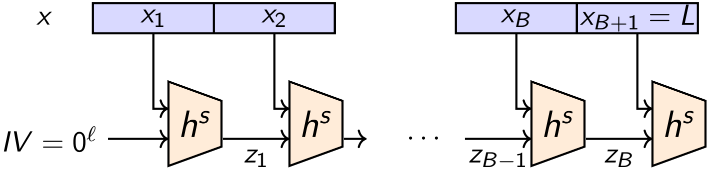

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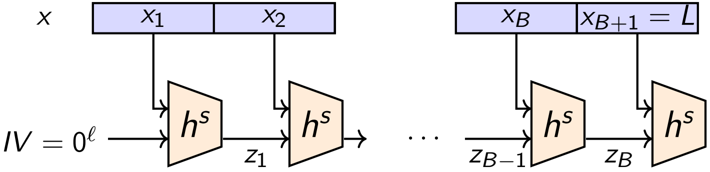

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  - Such index must exist since $x \neq x'$
  - At this index, you have two different inputs to h^s that produce same output – collision

Domain Extension for MACs (Hash-and-MAC)

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  - But, then $\mathcal{A}$ has forged valid tag on new message $H^s(m^*)$

Outline

What Are Hash Functions Good For?

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- Peer-to-peer file sharing – Use $H(\mathit{file})$ as unique identifier

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Can we protect passwords even if password file is stolen?

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- Hashing is very fast – Billions of hashes / second

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Takeaway

Always use a salt

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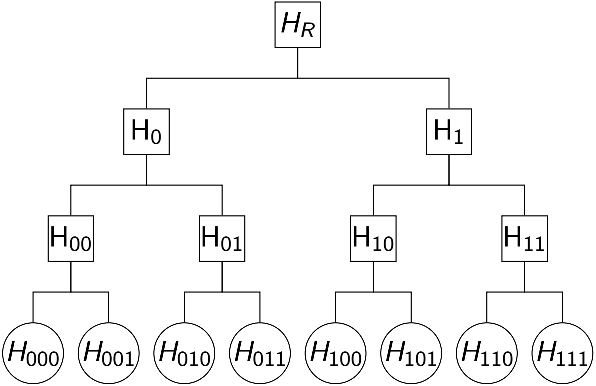
Question: Can we get solution that achieves both?

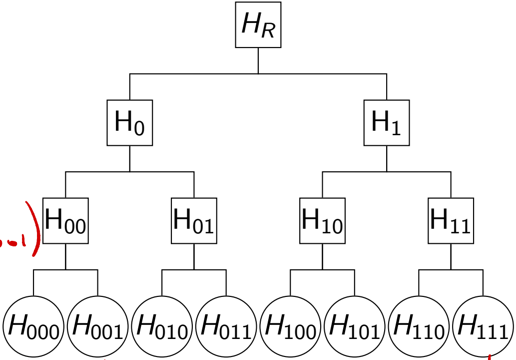
- Low storage on the client
- Low communication to verify a file is correct

Merkle Tree


H_R
H_0
H_1
H_{00}
H_{01}
H_{10}
H_{11}
H_{000}
H_{001}
H_{010}
H_{011}
H_{100}
H_{101}
H_{110}
H_{111}

Building a Merkle Tree

$H_{00} = H(H_{000} \| H_{001})$



x_{000} x_{001} x_{111}

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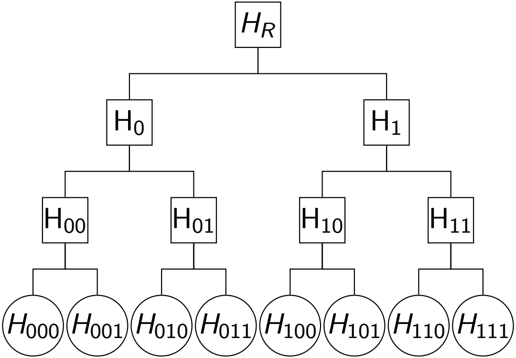

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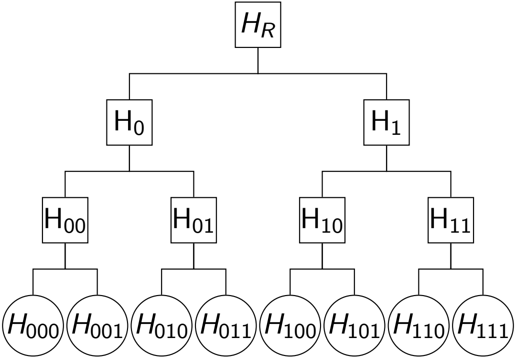

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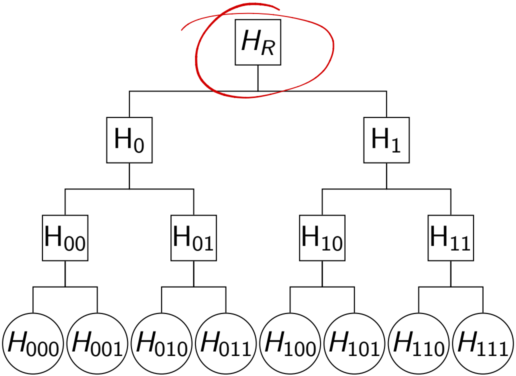

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- C stores value H_R at the root and uploads all files to S

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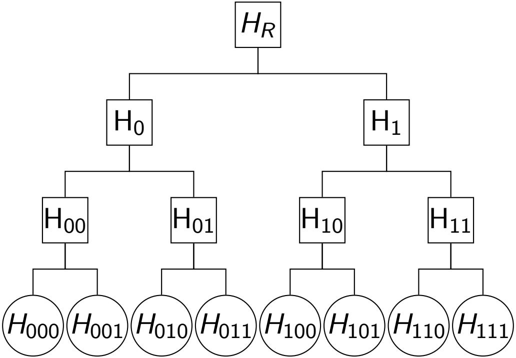

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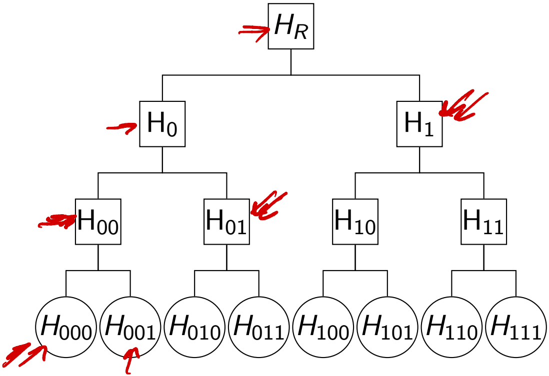

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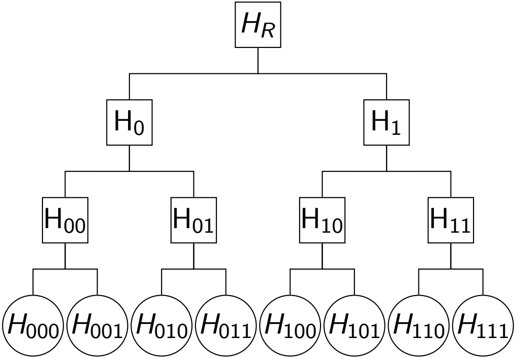

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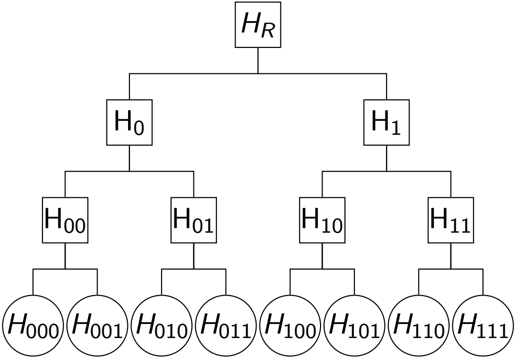

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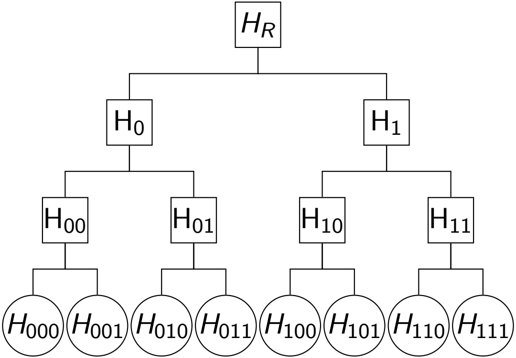
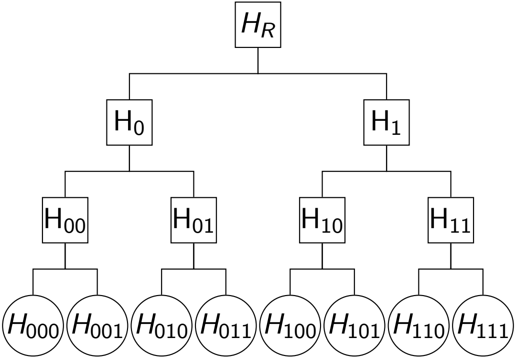
- C checks that computed value is equal to H_R, accepts if so

Merkle Tree Facts



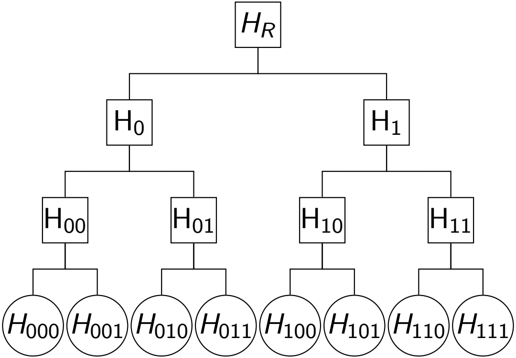
- Allows verified outsourced storage

Merkle Tree Facts



- Allows verified outsourced storage
- C only needs to store a single hash value, H_R

Merkle Tree Facts



- Allows verified outsourced storage
- C only needs to store a single hash value, H_R
- Proof consists of $O(\log n)$ hash values