Cryptography

Lecture 24

Arkady Yerukhimovich

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Outline

Lecture 23 Review

- Defining digital signatures
- Applications of signatures
- RSA digital signature

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One-Way Functions

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Observations:

- $\mathcal{A}$ does not necessarily have to recover x to win
- One-way functions are the most basic private-key primitives
- We've seen many examples: CRHFs, PRG, RSA

One-Time Signature Scheme

Let Π be a digital signature scheme. Consider the following game between an adversary $\mathcal{A}$ and a challenger:

$\mathsf{SigForge}_{\mathcal{A},\Pi}(n)$

- Challenger runs $(pk, sk) \leftarrow \mathsf{Gen}(1^n)$ and gives pk to $\mathcal{A}$
- $\mathcal{A}$ gets pk and oracle access to $\mathsf{Sign}_{sk}(\cdot)$ and outputs (m, σ)
  - Let Q denote the set of $\mathsf{Sign}_{sk}(\cdot)$ queries made by $\mathcal{A}$
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- In a one-time signature, $\mathcal{A}$ can only query the $\mathsf{Sign}_{sk}(\cdot)$ oracle once.
- Informally: After seeing one signature, $\mathcal{A}$ can't forge another one
- This is not a very useful notion of security, but we will use it as a building block

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- Hence, $\mathcal{A}$ needs to invert f on the corresponding $y_{i,b}$ in the pk

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- When $\mathcal{A}_c$ requests signature on a message m
  - If $m_{i^*} = b^*$, then $\mathcal{A}_r$ aborts
  - Otherwise, return $\sigma = (x_{1,m_1}, \ldots, x_{\ell,m_\ell})$ – $\mathcal{A}_r$ knows all these x_i's

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- When $\mathcal{A}_c$ outputs a forgery (m', σ)

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Lamport OTS – Proof of Security

Assume $\mathcal{A}_c$ breaks OTS security of Lamport, build $\mathcal{A}_r$ that inverts f

$\mathcal{A}_r(y = f(x))$ – Inverts f on y

- $\mathcal{A}_r$ chooses $i^* \leftarrow \{1, \ldots, \ell\}$, $b^* \leftarrow \{0, 1\}$, and sets $y_{i^*,b^*} = y$
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$$\Pr[\mathsf{Invert}_{\mathcal{A}_r,f} = 1] \geq \frac{1}{2\ell} \cdot \Pr[\mathsf{SigForge}_{\mathcal{A},\Pi}(n) = 1] \geq 1/\mathrm{poly}(n)$$

Improving on Lamport

Weaknesses of Lamport:

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- Now, only need to sign ℓ-bit hash outputs
- Include hash key s in public key
- Secure if H is CRHF

Going Beyond One Signature

Try 1 – Many Keys

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- Gen generates t key pairs (pk_i, sk_i)

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Limitations:

- $|pk|$ is large – grows with t
- number of possible signatures, t, bounded at Gen time
- Signature is *stateful* – problematic if state is reset

Going Beyond One Signature

Try 2 – Chain-Based Signatures

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pk_1

Going Beyond One Signature

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$$\vdots$$

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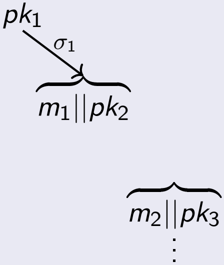

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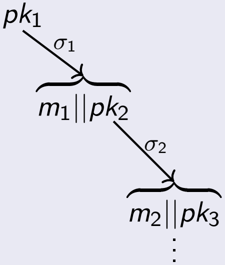

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- Generate key pairs (pk_i, sk_i) as we go
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$$pk_1 \xrightarrow{\sigma_1} m_1||pk_2 \xrightarrow{\sigma_2} m_2||pk_3 \quad \vdots$$

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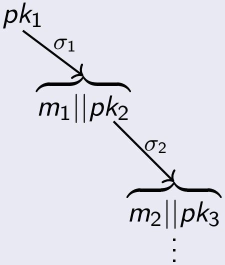

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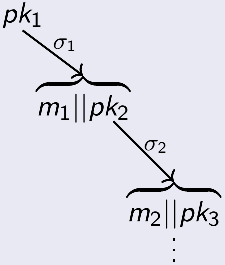

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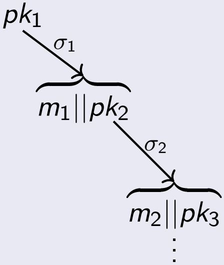

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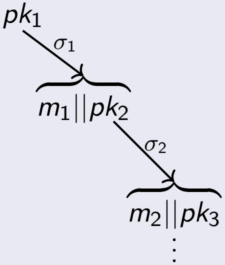

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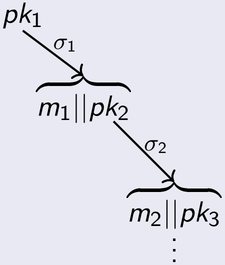

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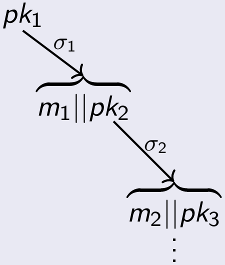

Pros:

- Every pk only used once – OTS security is sufficient
- No bound on number of signatures that can be issued

Limitations:

- $|\sigma|$ grows with the number of signatures issued
- Still have to store state

Going Beyond One Signature

Try 3 – Tree-Based Signatures

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- Use a different pk for each possible message

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- Use a different pk for each possible message
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pk_ϵ

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- Use a different *pk* for each possible message
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$$pk_\epsilon$$
$$\downarrow$$
$$pk_0 || pk_1$$

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$$
\begin{tikzcd}
pk_\epsilon \arrow[d] & \\
pk_0 || pk_1 \arrow[dr] & \\
& pk_{10} || pk_{11}
\end{tikzcd}
$$

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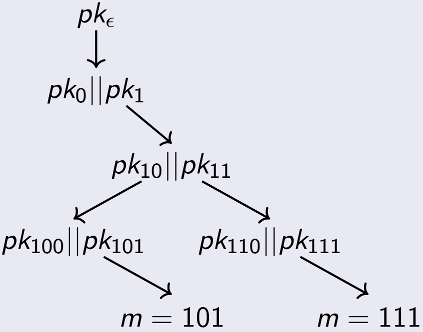

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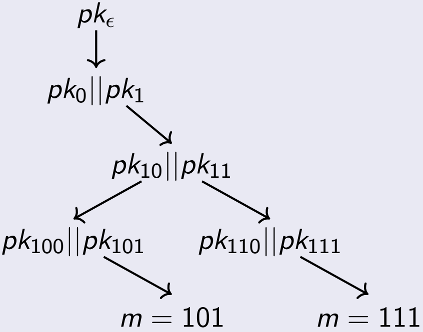

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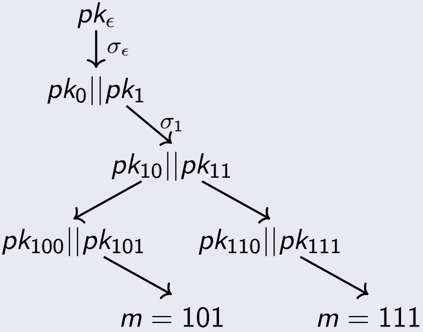

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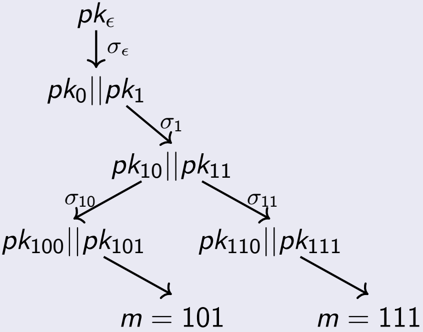

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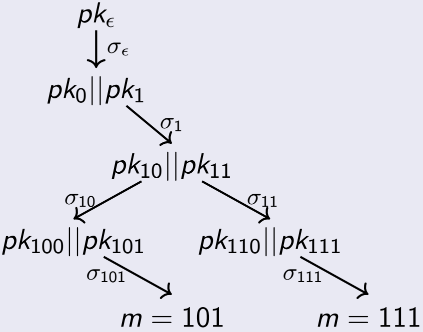

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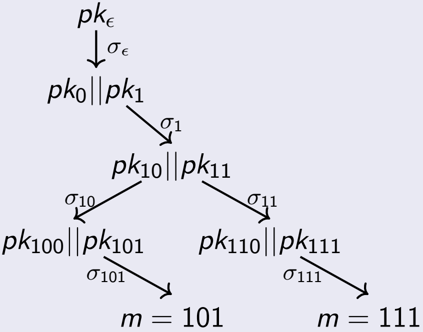

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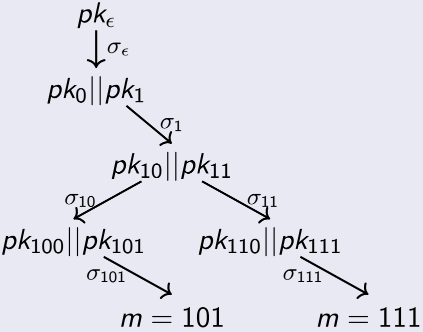

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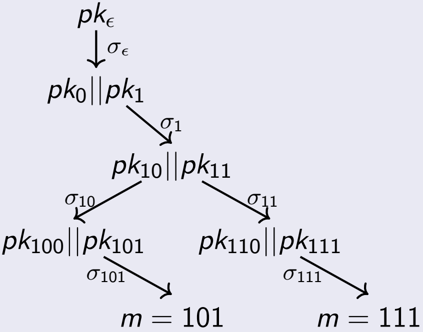

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Limitations:

- Still stateful – Tree is the state

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Idea: Use a PRF to generate randomness for Gen at each node

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Comparison to Public-Key Signatures

- Surprisingly, we can build signatures from private-key techniques
- But, public-key based signatures are more efficient (shorter sigs)

Outline

1. Lecture 23 Review
2. Digital Signatures from Private-Key Techniques
3. Digital Signatures from Discrete Log

Identification Schemes

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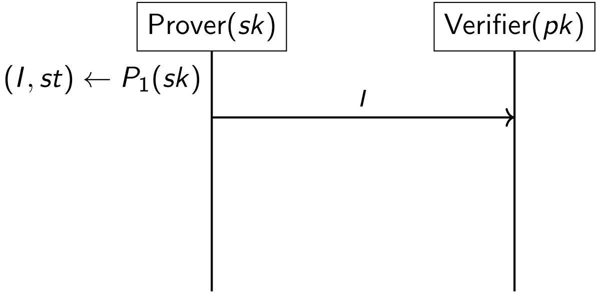

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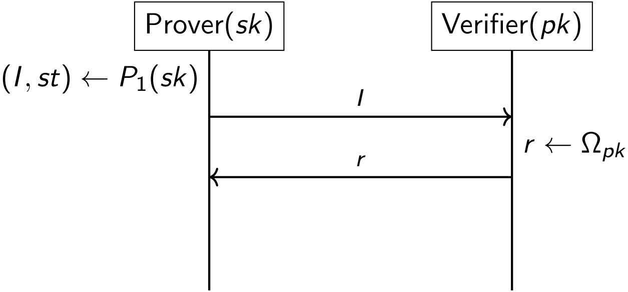

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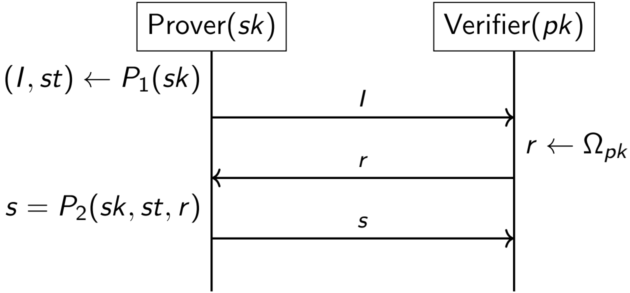

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Prover(sk)		Verifier(pk)
$(I, st) \leftarrow P_1(sk)$	$\xrightarrow{\quad I \quad}$	
	$\xleftarrow{\quad r \quad}$	$r \leftarrow \Omega_{pk}$
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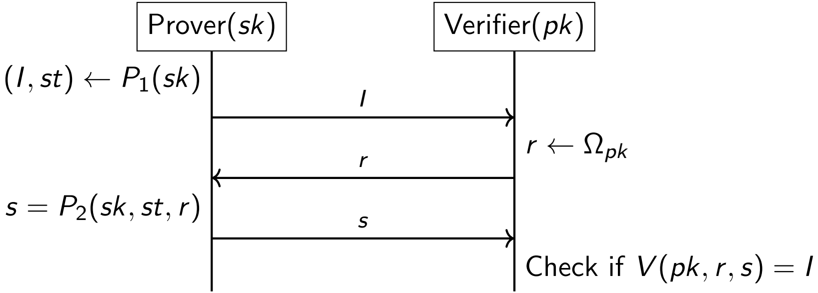

Non-degenerate:

- ID scheme is non-degenerate if for all sk, each I occurs with only negligible probability.

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Definition: An ID scheme $\Pi = (\mathsf{Gen}, P_1, P_2, V)$ is *secure* if for all PPT $\mathcal{A}$,

$$\Pr[\mathsf{Ident}_{\mathcal{A},\Pi}(n) = 1] \leq \mathsf{negl}(n)$$

Schnorr ID Scheme

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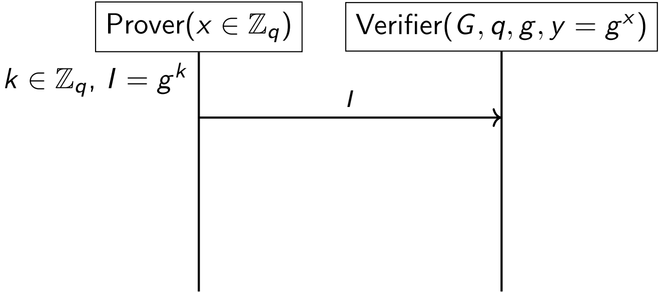

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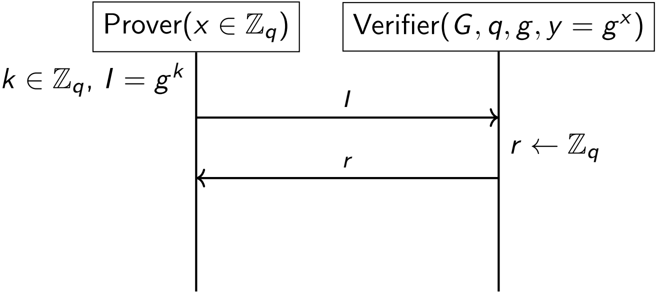

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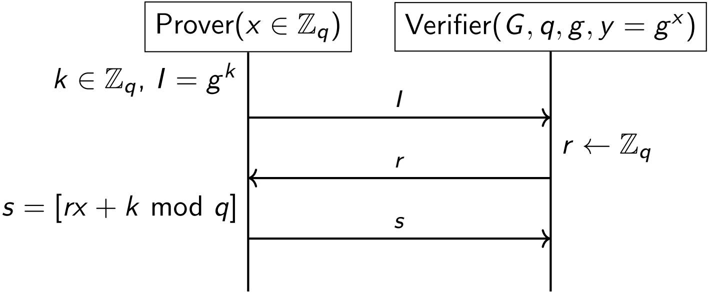

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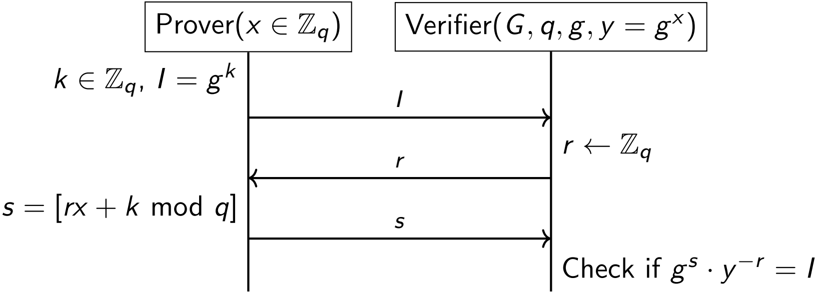

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- Build $\mathcal{A}_r$ that solves DLOG:
  - $\mathcal{A}_r$ runs $\mathcal{A}_c$ twice with the same randomness (producing same I), but gives it two different r
  - If $\mathcal{A}_c$ succeeds twice, then break DLOG

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Convert 3-round ID scheme as earlier to a non-interactive signature

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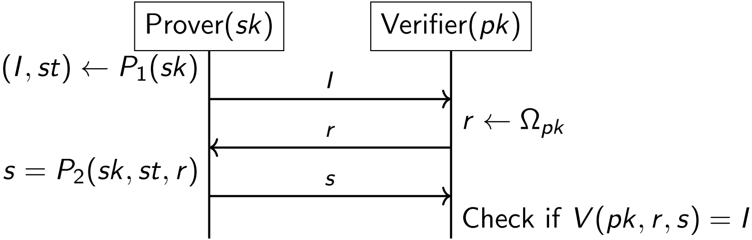

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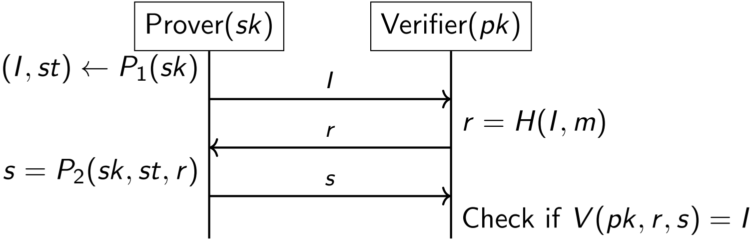

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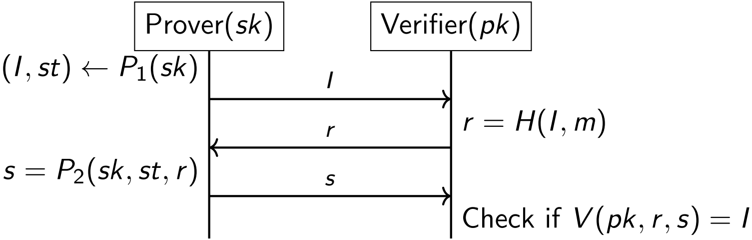

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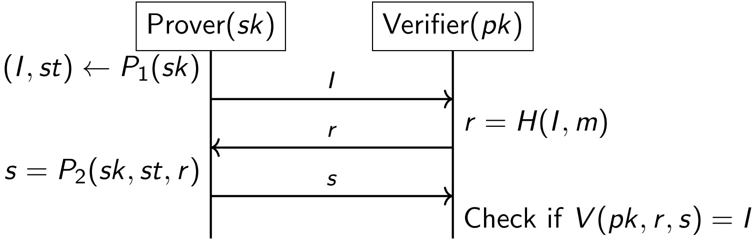

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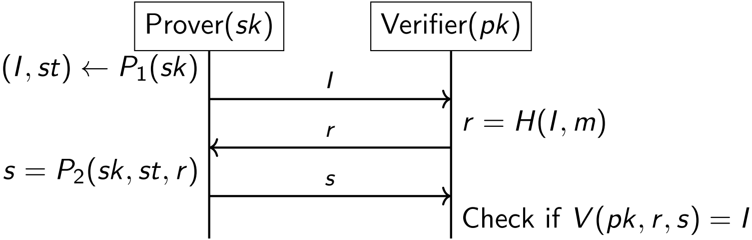

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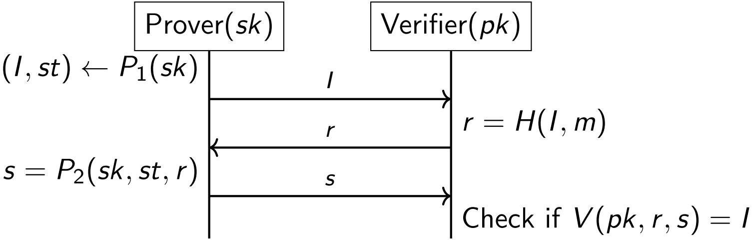

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- Security: Secure if H is modeled as random oracle

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Apply Fiat Shamir

Replace $r \leftarrow \mathbb{Z}_q$ with $H(I, m)$

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Digital Signature Algorithm (DSA)

- Standard DSA algorithm uses similar paradigm, achieves same security