Heterogeneity & Effective Properties

- Today
  - Heterogeneity
  - Anisotropy
  - Effective Properties

Heterogeneity: flow perpendicular to layers

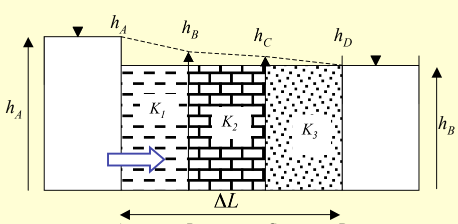
Discharge through each layer [L^3/T]:

$$Q_1 = -K_1 A \frac{h_B - h_A}{b_1}$$

$$Q_2 = -K_2 A \frac{h_C - h_B}{b_2}$$

$$Q_3 = -K_3 A \frac{h_D - h_C}{b_3}$$

What can we say about the relationship between Q_1, Q_2 and Q_3?



b_1 = B - A ,
b_2 = C – B ,
b_3 = D - C .

Heterogeneity: flow perpendicular to layers

Solve for head drops and heads across system

$$Q_1 = -K_1 A \frac{h_B - h_A}{b_1} \qquad \longrightarrow \qquad h_B - h_A =$$

$$Q_2 = -K_2 A \frac{h_C - h_B}{b_2} \qquad \qquad h_C - h_B =$$

$$Q_3 = -K_3 A \frac{h_D - h_C}{b_3} \qquad \qquad h_D - h_C =$$

Layer-head drops all have the same Q.

Layer-head drops $\propto$ layer K. and 1/thickness

Heads:

$$h_B = h_A - \frac{Qb_1}{K_1 A}$$

$$h_C = h_B - \frac{Qb_2}{K_2 A} = h_A - \frac{Qb_1}{K_1 A} - \frac{Qb_2}{K_2 A}$$

$$h_D = h_C - \frac{Qb_3}{K_3 A} = h_A - \frac{Qb_1}{K_1 A} - \frac{Qb_2}{K_2 A} - \frac{Qb_3}{K_3 A}$$

To close the system (find the h's), we need the discharge Q; use effective property concept.

Effective Conductivity: flow perpendicular to layers

Total discharge [L^3/T] written in terms of K_{eff}:

$$h_B - h_A = \frac{-Q\,b_1}{K_1 A}$$

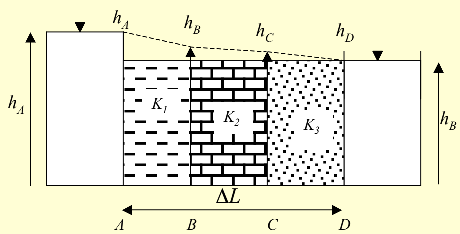
$$h_C - h_B = \frac{-Q\,b_2}{K_2 A}$$

$$h_D - h_C = \frac{-Q\,b_3}{K_3 A}$$

Head drops vary. While gradient within a layer may be constant, it varies from layer to layer, depending on layer conductivity and thickness.

$$h_D - h_A = -$$

$$h_D - h_A = (h_B - h_A) + (h_C - h_B) + (h_D - h_C)$$



b_1 = B - A,
b_2 = C – B,
b_3 = D - C.

Area, A, is the same for all layers

Effective Conductivity: flow perpendicular to layers

Equate the two terms: $h_D - h_A = h_D - h_A$

$$\frac{-Q(b_1 + b_2 + b_3)}{K_{eff} A} = \left(\frac{-Q\, b_1}{K_1 A}\right) + \left(\frac{-Q\, b_2}{K_2 A}\right) + \left(\frac{-Q\, b_3}{K_3 A}\right)$$

Eliminate common elements:

$$\frac{(b_1 + b_2 + b_3)}{K_{eff}} = \left(\frac{b_1}{K_1}\right) + \left(\frac{b_2}{K_2}\right) + \left(\frac{b_3}{K_3}\right)$$

Solve for the effective conductivity : $K_{eff} =$

The "effective conductivity" is the equivalent homogenous K that results in the same discharge. It's the result of "upscaling" or "spatial averaging" over the heterogeneities. In general, for flow perpendicular to layers:

Heterogeneity: flow perpendicular to layers

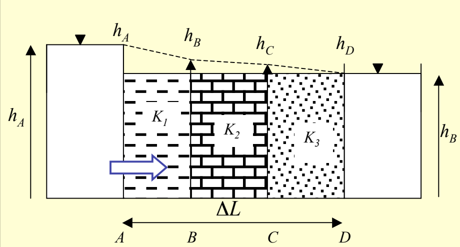
Heads from:

$$K_{eff} = \frac{b_1 + b_2 + b_3}{\left(\frac{b_1}{K_1}\right) + \left(\frac{b_2}{K_2}\right) + \left(\frac{b_3}{K_3}\right)} \longrightarrow Q = -K_{eff} A \frac{h_D - h_A}{b_1 + b_2 + b_3}$$

$$h_B = h_A - \frac{Qb_1}{K_1 A}$$

$$h_C = h_B - \frac{Qb_2}{K_2 A} = h_A - \frac{Qb_1}{K_1 A} - \frac{Qb_2}{K_2 A}$$

$$h_D = h_C - \frac{Qb_3}{K_3 A} = h_A - \frac{Qb_1}{K_1 A} - \frac{Qb_2}{K_2 A} - \frac{Qb_3}{K_3 A}$$



Then linearly interpolate heads between A, B, C, and D to get $h(x)$.

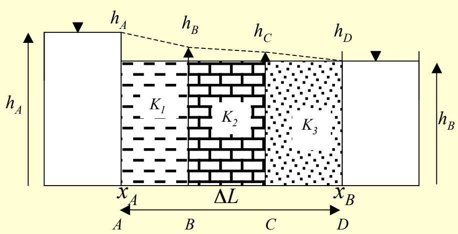
Heterogeneity: flow perpendicular to layers

What is the rate of specific discharge and seepage velocity in each layer?
How long would it take a non-reactive tracer to move from A to B across all layers?

Discharge in layer i [L³/T]:

$$Q_i = Q = -K_{eff} A \frac{h_D - h_A}{b_1 + b_2 + b_3}$$

where $i=1,2,3$:



Specific discharge in layer i [L/T]:

$$q_i = \frac{Q_i}{A} = \frac{Q}{A} = q =$$

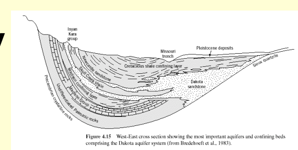
Seepage velocity in layer i [L/T]:

$$v_i =$$

Travel time A to B [T]:

$$t = \int_{x_A}^{x_B} \frac{1}{v_i} dx =$$

Effective Conductivity for layered systems



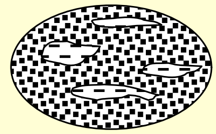
- The "effective conductivity" is the equivalent homogenous K that results in the same discharge.
- It's the result of "upscaling" or "spatial averaging" over the heterogeneities.

- In general,
  - for flow parallel to layers:
    - Use the
    - which is weighted towards
  - for flow perpendicular to layers:
    - use a
    - which is weighted towards the

$$\boxed{K_{eff} = K_A = \frac{\sum Kb}{\sum b}}$$

$$\boxed{K_{eff} = K_H = \frac{\sum b}{\sum \frac{b}{K}}}$$

Effective Conductivity for more complex heterogeneities



When averaging over any spatial arrangement of discrete K values, the upscaling or averaging result <u>depends on orientation of the arrangement relative to the direction of the hydraulic head gradient</u>:

– <u>Arithmetic mean</u>: highest effective value $K_{eff} = K_A = \frac{\sum K_i V_i}{\sum V_i}$

– <u>Harmonic mean</u>: lowest effective value $K_{eff} = K_H = \frac{\sum V_i}{\sum \frac{V_i}{K_i}}$

– <u>Geometric mean</u>: yields an intermediate effective value between arithmetic and harmonic means, $\ln K_{eff} = \ln K_G =$

$K_H \leq$

V_i = volume of ith volume fraction

Averaging or upscaling heterogeneity leads to (upscaled) anisotopy

Snell's Law

- Flow lines refract when crossing an abrupt boundary between two homogeneous units:

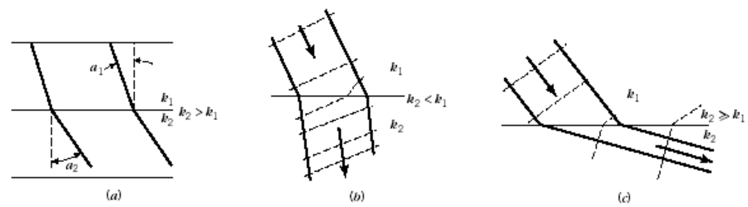
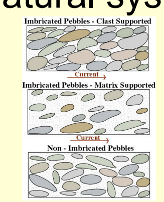


Figure 3.11 Diagrams of flow line refraction and conditions at the boundaries between materials of differing permeability (from Domenico and Schwartz, 1998. *Physical and chemical hydrogeology*). Copyright ©1998 by John Wiley & Sons, Inc. Reprinted by permission of John Wiley & Sons, Inc.

SZ2005

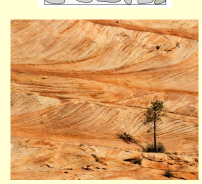
Anisotropy

- Typical causes of anisotropy in natural systems?
  - Imbricated grains
    - Upscaling at the grain, or Darcy scale

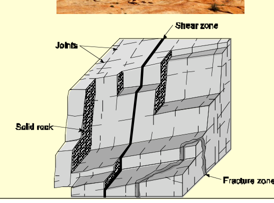


U of Montana, 2005

  - Bedding in sedimentary systems
    - Upscaling over the beds

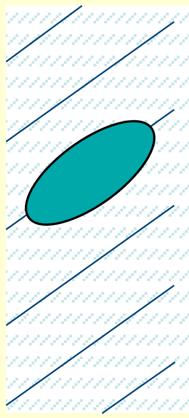


  - Fracture networks
    - With preferred orientation and connectivity of fractures
    - Upscaling over the network



Gale, 1982

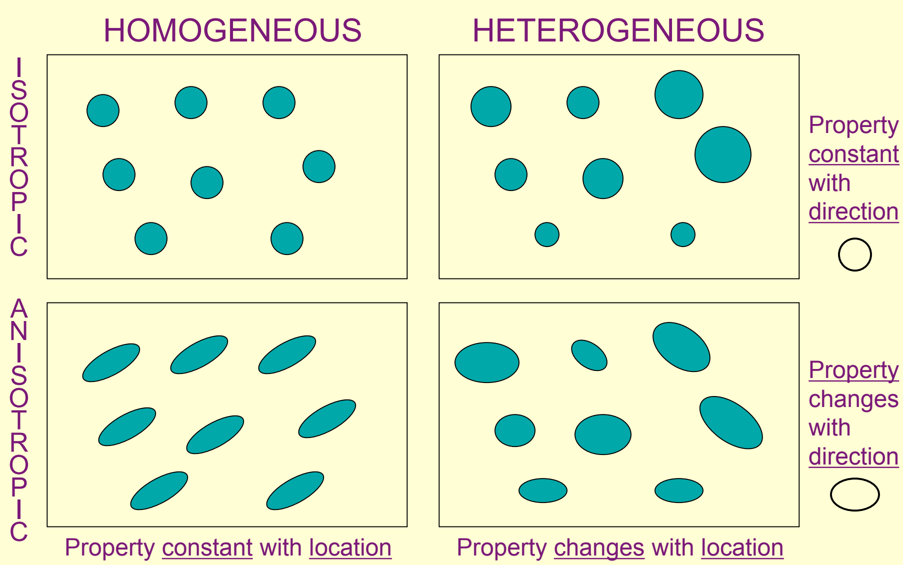
Anisotropy

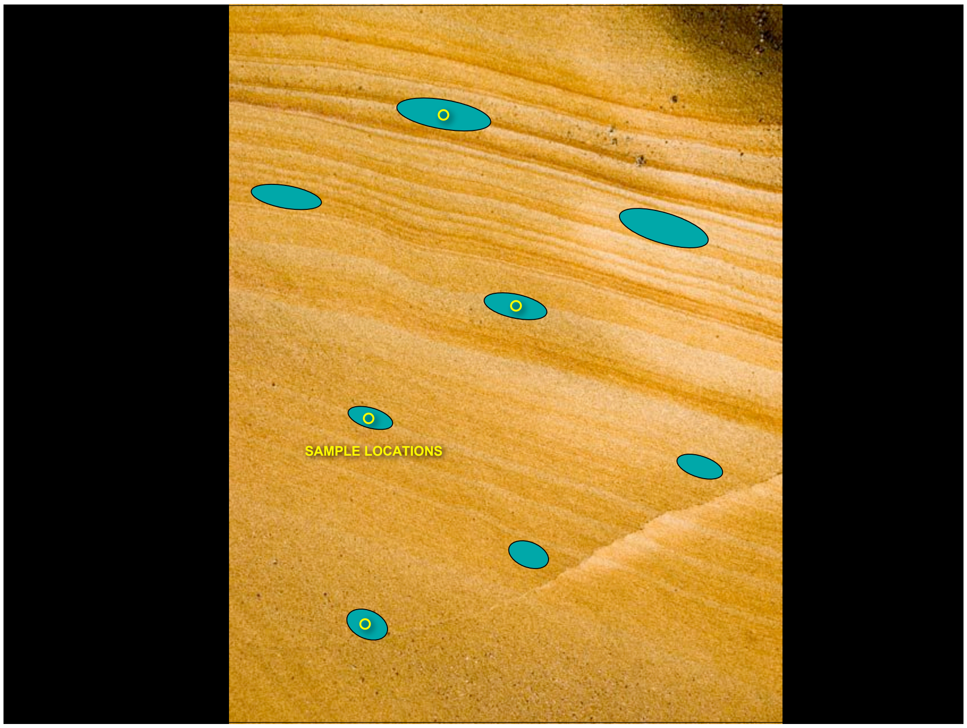


The variation of K with direction forms an ellipse. In 3-D it forms an ellipsoid.

The principal axes of the ellipsoid point in the directions of greatest and least resistance.

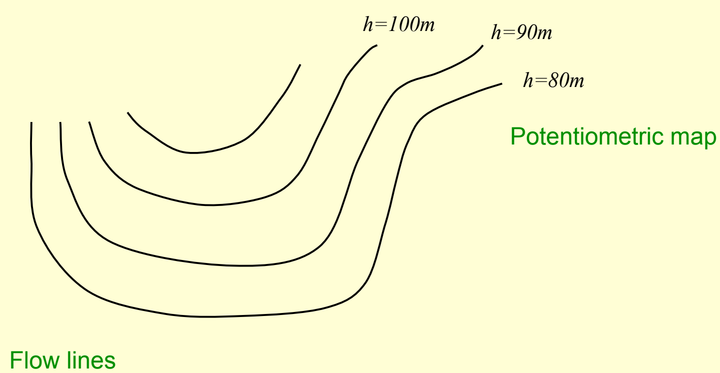
Properties as functions of location and direction





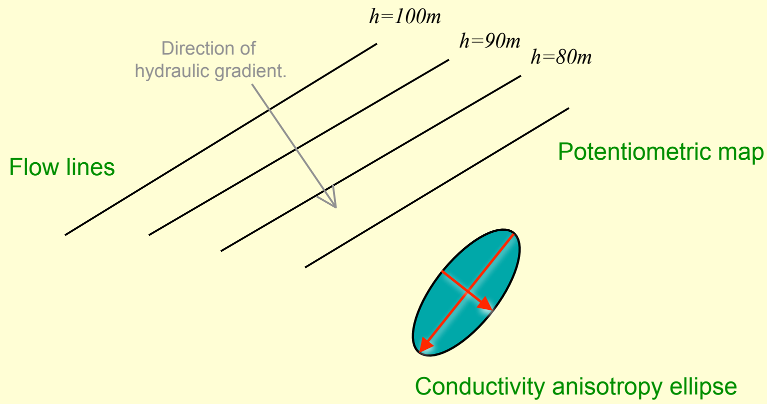
Later we'll see that …

Water always seeks the most efficient way to move, so in an isotropic, homogenous medium it will always flow parallel to the hydraulic gradient, i.e., perpendicular to the potentiometric contours.



However …

In an anisotropic medium flow is at an angle to the potentiometric contours,
an angle that depends on the amount of anisotropy.



Anisotropy and Effective Conductivity

Because of it directional dependence
Hydraulic Conductivity is a tensor quantity:

$$\begin{bmatrix} K_{xx} & K_{xy} & K_{xz} \\ K_{yx} & K_{yy} & K_{yz} \\ K_{zx} & K_{zy} & K_{zz} \end{bmatrix}$$

The tensor is symmetric, with off diagonals:

$K_{yx}=$ $K_{zx}=$ $K_{zy}=$

The tensor is like a matrix.
What does each entry in the tensor represent?

$$\begin{bmatrix} K_{xx} & K_{xy} & K_{xz} \\ K_{yx} & K_{yy} & K_{yz} \\ K_{zx} & K_{zy} & K_{zz} \end{bmatrix}$$

K_{xx} = *specific discharge in the* *due to a unit hydraulic gradient in the*

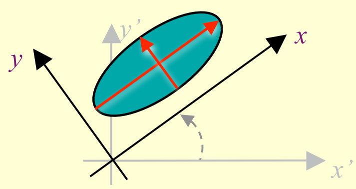
K_{xy} = *specific discharge in the* *ue to a unit hydraulic gradient in the*

Anisotropy and Effective Conductivity

If your coordinate system is selected to lie in the principal directions of anisotropy, the tensor is diagonal

$$\begin{bmatrix} K_{xx} & & \\ & K_{yy} & \\ & & K_{zz} \end{bmatrix}$$

The off diagonals are:



Simply rotate coordinate system to align with principal directions.
Only works if principal directions are homogeneous
(don't change with position).

Storage

- Today
  - Aquifer Storage
  - Storativity
  - Specific Yield

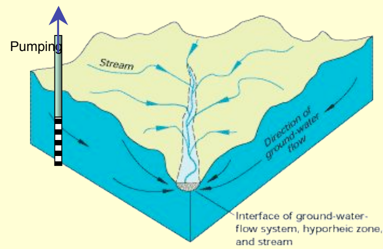
Only two quarts of water in a 2-gallon bucket full of sand? How about only 1/40th of an ounce!

Groundwater Balance

$\boxed{\text{Change in storage}}$

= Recharge – Pumping – GW Discharge

– ET_{GW} ± Underflow = Forcings



Hydraulic Conductivity:

- influences each of the RHS forcings, or how the system responds to these forcings.

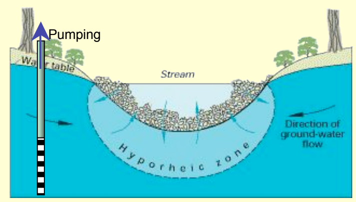
Conductvity doesn't influence the LHS term, involving the rate of change of storage.

Storage Parameters:

- Control the LHS, rate of change of storage
- Two types of storage, due to:

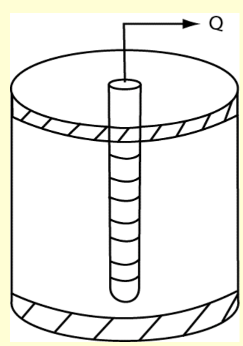
on

- Parameters, respectively



Aqufier Storage

- Consider a pumping well located in an infinite confined aquifer.
  - aquifer is bounded above and below by aquicludes
  - well is fully penetrating
  - well is pumping at a constant volumetic-flow rate, Q



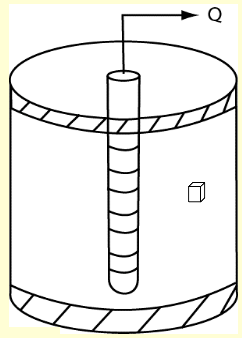
- Where does the water come from?

Pumping = – Change in storage +Recharge – GW Discharge

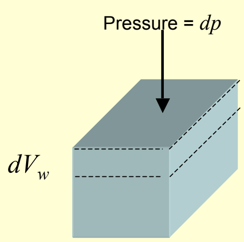
– ET_{GW} ± Underflow

Aqufier Storage

- Where does the water come from?
  - Head drops as well is pumped.
  - Elevation is not changing, so then
  - pressure must be changing.
  - As pressure is released
    - water comes out of storage.
  - How?

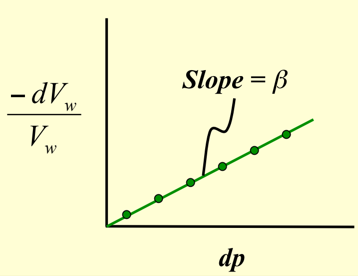


Water Compressibility



Control volume of pure water, volume = V_w

In response to the increase in pressure dp, the volume of water decreases (compresses) by the amount dV_w



Water Compressibility

$$\beta = \frac{1}{V_w}\frac{-dV_w}{dp}$$

- Dividing by V_w normalizes the data

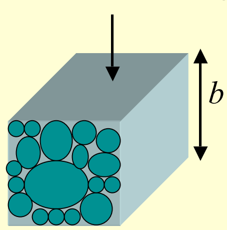
    if we only had $-dV_w$ on the y axis,compressing different volumes of water would yield different slopes

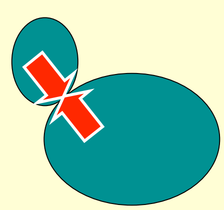
- This is for a cube of pure water, where $V_w = V_{total} = V_t$

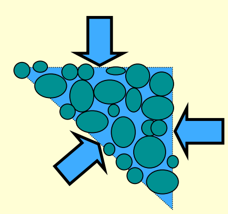
- What happens in an aquifer?

Matrix Compressibility

Overburden Stress = σ







Control volume of aquifer

Total stress σ is balance by:

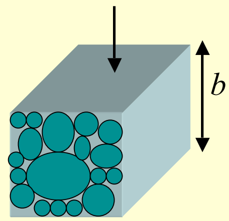
-the transmitted through the matrix (e.g., due to grain-to-grain contacts),

and by

-the

Matrix Compressibility

Overburden Stress = σ



Control volume of aquifer

Total stress σ is balance by:

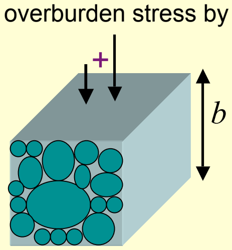
-the effective stress σ_e transmitted through the matrix (e.g., due to grain-to-grain contacts),

and by

-the water pressure, p, transmitted through the pore space

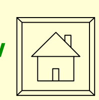
Matrix Compressibility

Add to overburden stress by amount $d\sigma$



Control volume of aquifer

If the total stress σ increases by $d\sigma$, say due to a new building



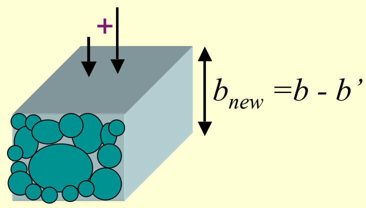
-- initially the water pressure, p, increases

-- if the water is allowed to drain off, the stress is slowly transmitted to the matrix, and effective stress σ_e increases

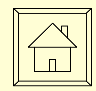
Matrix Compressibility

What happens after the water drains off and dp =0 …

Add to overburden stress by amount $d\sigma$



Control volume of aquifer



If the total stress σ increases by $d\sigma$, say due to a new building

-- initially the water pressure, p, increases

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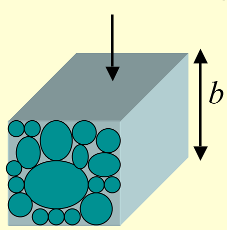
Matrix Compressibility

Overburden Stress = σ



$$\boxed{\sigma = \sigma_e + p}$$

Control volume of aquifer

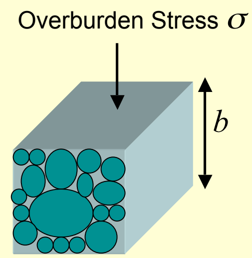
In groundwater hydrology we usually assume that the overburden or total stress σ is fixed: $d\sigma=0$.

Instead we are concerned with what happens to effective stress and thickness b when the water pressure, p, changes

-due to pumping or injection or …

Matrix Compressibility

In groundwater hydrology $d\sigma$ =0 …



Control volume of aquifer

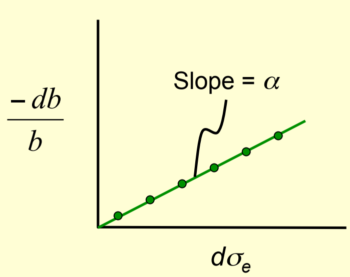
In groundwater hydrology we usually assume that the overburden or total stress σ is fixed: $d\sigma$=0.

Instead we are concerned with what happens to effective stress and thickness b when the water pressure, p, changes
-due to pumping or injection or …

Matrix Compressibility

Suppose effective stress increases
-due to increase in total stress or
-decrease of pressure,
then the control volume will compress.

Measure that compression as a function of effective stress, normalize, and plot. The slope is the matrix compressibility:



Matrix Compressibility

$$\alpha = \frac{-db}{b}\frac{1}{d\sigma_e} =$$

$$\alpha = -\frac{1}{V_t}\frac{dV_t}{d\sigma_e}$$

We usually consider the rock material (eg, grains) incompressible, so changes in the volume of voids (& therefore volume of water) must account for changes in the volume of the aquifer control volume.

Relates change in water volume, due to matrix compressibility, to change in aquifer effective stress.

Matrix Compressibility

Recall that in hydrology applications, $d\sigma \cong 0$:

$$d\sigma_e =$$

$$dV_w =$$

Relates change in water volume, due to matrix compressibility, to change in aquifer water pressure.

The matrix model is due to Karl Terzaghi (1883-1963). It's a principal model of geotechnical engineering and groundwater hydrology. The major assumption is that compression is vertical (no horizontal movement). Secondary assumptions include no compression of grains. Also, to account for irreversible consolidation you need to augment the model or use a more sophisticated model (e.g., Biot's theory). Ralph Peck, one of Teraghi's main disciples and famous in his own right, is almost 90 and lives in Albuquerque. He still travels the world as a geotechnical consultant.